Leon Henkin the Reviewer

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Abstract In this chapter, we intend to look at Henkin's reviews, a total of forty-six. The books and papers reviewed deal with a large variety of subjects that range from the algebraic treatment of logical systems to issues concerning the philosophy of mathematics and, not surprisingly—given his active work in mathematical education—one on the teaching of this subject. Most of them were published in *The Journal of Symbolic Logic* and only one in the *Bulletin of the American Mathematical Society*. We will start by sorting these works into subjects and continue by providing a brief summary of each of them in order to point out those aspects that are originally from Henkin, and what we take to be mistakes. This analysis should disclose Henkin's personal views on some of the most important results and influential books of his time; for instance, Gödel's discovery of the consistency of the *Continuum Hypothesis* with the axioms of set theory or Church's *Introduction to Logic*. It should also provide insight into how various outstanding results in logic and the foundations of mathematics were seen at the time. Finally, we will relate Henkin's reviews to Henkin's major contributions.

Keywords Leon Henkin · Reviews · Logic systems · Type theory · Metalogic · Algebraic logic · Philosophy of logic and mathematics · Mathematical education

1 Introduction

Henkin, at the very least, wrote the 46 reviews we list in Sect. 7. In what follows, we provide a brief abstract of each of those. The reviews have been organized according to subjects, and in many cases, we explain the contributions of each of the reviews falling under each of the epigraphs to the subject in question. In order to track which review is being summarized where, we will write the number that corresponds to the review in our Sect. 7 in boldface and the number of the review in the bibliography between square brackets.

In order to track the relation between Henkin's contributions and the subject of the reviews he was given, we have looked at the works by Henkin himself that he quotes in his reviews. We have also taken into account Monk's description[1] of Henkin's scientific contributions. And we have tried to catalogue the works reviewed according to contemporary areas of knowledge.

[1]See J. Donald Monk [194].

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Note that the Reviews Section in the JSL started with the journal in 1936, and Church was in charge.[2] Henkin's reviews are part of this project. Henkin was designated as "consulting editor"; that means that he "could simply be assigned that publication which [he was] obligated to review." (H.B. Enderton [30, p. 174]) Henkin was consulting editor for volumes 17 to 23 and 25 to 31, which is from 1952 to 1958 and from 1960 to 1966, respectively. Besides, he was editor with A. Church and M. Black for volume 14, 1949, and with A. Church, S.C. Kleene and A.A. Lazerowitz for volume 24 in 1959.

According to H.B. Enderton [30, p. 173], a work was to be reviewed if the work was relevant to symbolic logic independently of its quality. H.B. Enderton quotes different passages by A. Church in which he explains the reasons for his way of understanding the Review Section and its role. According to those passages, Church thought that the situation in mathematical logic was such that results in logic frequently contained errors and absurdities; hence, he thought that competent workers in the field should indicate which works were valuable in order to prevent the field from falling into disrepute. Reviewers were expected to provide an assessment of the work. Henkin's reviews do satisfy this requirement, as we shall see below.

All except Henkin's review of Church's *Introduction to logic* were published in *The Journal of Symbolic Logic*; Henkin's review of Church's book was published in the *Bulletin of the Mathematical Society*. A plausible explanation for this exception could be that since Alonzo Church was responsible for the Reviews Section in *The Journal of Symbolic Logic*, he did not want to ask one of his consulting editors to assess his own work.

2 Logic Systems

Under this heading, we have included those reviews dealing with works on type theory and metalogic.

2.1 Type theory:

- 1 (1942) Review [49] of: Maxwell Herman Alexander Newman and Alan Mathison Turing, "A formal theorem in Church's theory of types" (1942) [201].
- 5 (1949) Review [56] of: Stanislaw Jaśkowski, "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres" (1948) [149].
- 24 (1955) Review [81] of: Evert Willem Beth, "Sur le parallelisme logico-mathematique" (1953) [8].

2.2 Metalogic:

- 4 (1949) Review [57] of: Stanislaw Jaśkowski, "Sur les variables propositionelles dependantes" (1948) [150].
- 6 (1949) Review [55] of: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1948) [232].
- 15 (1952) Review [66] of: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1951) [233].

[2] See H.B. Enderton [30] for a detailed account of Church as a reviewer. All the information we include about the issue is taken from Enderton's work.

18 (1953) Review [69] of: Burton Dreben, "On the completeness of quantification theory" (1952) [28].
19 (1954) Review [74] of: Georg Kreisel, "On a problem of Henkin's" (1953) [157].
23 (1955) Review [83] of: Helena Rasiowa and Roman Sikorski, "On existential theorems in non-classical functional calculi" (1954) [209].
29 (1955) Review [82] of: Gunter Asser, "Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe" (1955) [2].
30 (1956) Review [89] of: Jerzy Łoś, "The algebraic treatment of the methodology of elementary deductive systems" (1955) [167].
31 (1956) Review [90] of: Juliusz Reichbach, "O pełności węższego rachunku funkcyjnego"; Juliusz Reichbach, "O polnoté uzkogo funkcional'nogo isčisléniá" (Russian translation); Juliusz Reichbach, "Completeness of the functional calculus of first-order" (English summary) (1955) [210].

2.1 Type Theory

Type theory was first introduced by Bertrand Russell as a solution to the paradoxes in set theory.

Henkin's contribution to the development of type theory consisted in his applying—to the theory of types—the technique of introducing individual constants in order to eliminate quantifiers and then construct a model[3]—the technique he developed in his proof of the completeness theorem for first-order functional calculi.[4]

Moreover, in his "A theory of propositional types" [111], he studies the theory of types whose only basic type is the propositional type and in which the rest of the types obtain from this basic one by applying the clause that goes: if α and β are types, then so is $\alpha \to \beta$. He establishes certain constants from which all possible functions for each type are definable. This work is connected to B. Russell's logical system in *The Principles of Mathematics* [217], with S. Leśniewski's *Protothetic* [161] and "Introductory remarks to the continuation of my article 'Grundzuge eines neuens systems der Grundlagen der Mathematik' " [162], and with Tarski's paper "O wyrazie pierwotnym logistyki" [226].

Henkin wrote three reviews on type theory; one (1) [49] has to do with axioms of infinity, and both the second (5) [56] and third (24) [81] with how to characterize mathematical theories in simple type theory. Henkin's review on axioms of infinity was his first review [49]; it is a review of the paper published by M.H.A. Newman and A.M. Turing 1942 "A formal theorem in Church's theory of types" [201]. Henkin wrote it in 1942, the same year in which the paper was published. Henkin reports that Newman and Turing present a result that builds on one by Church about his simple type theory [20]. In that paper, Church introduces an axiom of "infinity" Y_ι for type ι and proves—using only axioms 1 to 8—that if the axiom of infinity Y_α for type α holds, then the axiom of infinity $Y_{(\alpha\alpha)(\alpha\alpha)}$ for type $(\alpha\alpha)(\alpha\alpha)$ also holds; and he claims that Y_α does not hold if α contains

[3] L. Henkin: "The completeness of the first-order functional calculus" [52].

[4] "Completeness in the theory of types" [59].

only type $\circ$. Newman and Türing show, using axioms 1 to 10, that if α contains ι, then Y_α holds. Henkin explains the proof they give, clarifies some of the concepts they use, mentions the notational simplifications used by the authors, and claims—mentioning a model (in which axiom 10 holds for type $(\iota\iota)$ and Y_α holds only if α is ι or contains type $(\iota\iota)$)—that axioms 1 to 10 are not sufficient. Henkin raises two problems; namely, whether axiom 9 and axiom 10 are needed in all types. Finally, he lists a number of errata.[5]

The other two works reported, though devoted to the same subject, show different worries. Thus, Henkin [56] reports that Jaśkowski's "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres" [149] demonstrates that natural and real numbers can be defined within the same type[6] as functions that have certain classes of individuals both as values and arguments. The basic concept is that of *dual group* or family of classes of sets of individuals with both a negation and an addition operation. The integer number n is the function that links each element of the dual group with the element that obtains when that element is added n times (and Ø with those that are not in the group). Proving Peano axioms is easy except for the axiom "zero is not the successor of any number," for which an infinity axiom is needed. By defining an order in each dual group, rational and real numbers can be identified with certain functions that work over a dual group with the same order as the continuum. Henkin reports that Jaśkowski shows that "the continuous groups are those families that arise from a measure function f defined over sets of individuals, the elements of the group consisting of classes made up of all sets for which f has a given value [...]." The measure functions f are additive, but they are not of the usual kind: the whole space has measure zero, and complements have measures with opposite signs. Henkin points out that in this way Euclides' idea of using ratios of line segments is expanded to trade with the reals.

According to Henkin [81], Beth starts his book chapter "Sur le parallelisme logico-mathematique" [8] briefly describing type theory and then considers various mathematical theories depending on the number of types needed to formalize them. Beth first considers the possibility of formalizing elemental geometry in a first-order language (Bernays mentions in the discussion that the notion of polygon is not first-order formalizable). He then compares Peano first-order systems with Peano second-order systems. He describes Henkin's nonstandard models [59] and mentions that his construction follows Mostowski's ideas [197]. Beth also remarks that there is a tendency to substitute higher-order mathematical theories for their first-order versions.

2.2 Metalogic

Metalogic can be defined as the study of syntactic and semantic properties of logical systems. As is well known, major developments in the field took place during the first decades of the 20th century, and Henkin himself made (at least) one major contribution to

[5] Note that in 1942 Henkin was Church's student. This review has a strong technical character.

[6] Pre-existent definitions (the usual ones in the foundations of mathematics) placed natural numbers/integers in one type while real numbers were defined as "special sets of integers," hence they/their definition belonged in a higher functional type.

the development of the subject: he provided a proof of the completeness theorem for first-order logic that resulted in the development of a new method of proof (see Henkin L., "The completeness of the first-order functional calculus" [52]).[7] It so happens that his proof, not Gödel's, has been used to teach students how to prove completeness since Henkin's proof is much easier to follow than that of Gödel.

As is well known, the search for new proofs for a given theorem is a common practice in mathematics. The completeness theorem for first-order logic is a clear example of this practice. Thus, Henkin himself reported in 1953 (18) [69] and 1956 (31) on two other proofs of this result: Burton Dreben's proof, published in 1952 "On the Completeness of quantification theory" [28], and Juliusz Reichbach's proof, published in 1955 "Completeness of the functional calculus of first-order" [210]. Reichbach tells us that his proof comes after Gödel's and Henkin's proofs, though he claims it was obtained independently of those by Robinson [213], Rasiowa and Sikorski [206], Beth [7], Rieger [211], and Łoś [167]. Dreben's proof shows, combining results by Herbrand [142] and Gödel [41], that each quantificationally valid scheme in prenex normal form is provable by means of a proof in Herbrand normal form. Henkin's review ends by reporting on Dreben's diagnosis about why Herbrand did not draw from his theorem the conclusion that a formula that cannot be derived in Herbrand normal form is not valid; Dreben claims that Herbrand did not draw this conclusion because the deduction is nonconstructive. Reichbach uses a method of proof in which the set of free variables is disjoint from the set of bound variables. This is what Frege used in his [34] and [36]. Neither the author nor Henkin points out this curiosity, plausibly because it is a curiosity of no consequence for the arguments involved. Reichbach's procedure constructs an interpretation for a nonprovable formula α, an interpretation in which α is false. Henkin does not go into the details of this construction, but gives only its general lines: a maximal set with certain properties is constructed, but the steps in the construction are not accounted for.

Henkin reported (19) [74], on a paper by Kreisel "On a problem of Henkin's" [157], also closely related to completeness issues. Henkin himself [219] posed the problem that Kreisel tried to solve: "If Σ is any standard formal system adequate for recursive number theory, a formula (having a certain integer q as its Gödel number) can be constructed that expresses the proposition that the formula with Gödel number q is provable in Σ. Is this formula provable or independent in Σ?" Löb solved the problem and published the solution in 1955 (see his [169]). According to Henkin, Kreisel's contribution consisted of showing that the concept that is expressed by the formula can be interpreted in various ways, and depending on how it is interpreted, the answer to Henkin's question will vary. Henkin notes that "[a] clear explication of the concept of that which is expressed by a formula must be based on an axiomatic treatment of this notion" and further suggests that such treatment can be done by following what Church said in [22].[8]

[7] Works by Henkin that are contributions to the subject are: "Completeness in the theory of types" [59]; "On the primitive symbols of Quine's 'Mathematical Logic' " [68]; "A generalization of the concept of ω-consistency" [73]; "On the definitions of 'formal deduction' " (with R. Montague) [195]; "A generalization of the concept of ω-completeness" [92]; "Some remarks on infinitely long formulas" [104]; "An extension of the Craig–Lyndon interpolation theorem" [107]; *Logical Systems Containing Only a Finite Number of Symbols* [114], and "Relativization with respect to formulas and its use in proofs of independence" [116].

[8] Note that Church wrote a second version of this paper, in which he revised this proposal. Moreover, Church explains that along the years he doubted the viability of his initial proposal in his 1973 paper

Henkin recapitulates four papers on another important metalogical issue, the decision problem. Following Church [24, p. 100, n. 184], we understand the decision problem "as a general name for problems to find an effective criterion (a decision procedure) for something, and to distinguish different decision problems by means of qualifying adjectives or phrases." Hence, in relation to logical systems, we have the decision problem for provability, the decision problem for validity, and the decision problem for satisfiability.

A solution for a particular decision problem is an algorithm that applied to a particular instance of the problem, results in a "yes" or "no" answer, depending whether the specific instance of the considered decision problem has or fails to have the considered property. For instance, an algorithm that decides about provability would be an algorithm that results in a "yes" if the considered formula is a theorem or in a "no" if the considered formula is not a theorem. This algorithm is, of course, nothing but the definition of a recursive or computable function whose value will be one if there is a solution for the considered argument and zero otherwise.

Providing an algorithm that gives a yes or no answer for any instance of a problem—whatever the decision problem is (whether it is the decision problem for provability, validity, satisfiability, etc.)—counts as a *direct* way of solving any of these decision problems. Alternatively, a solution obtains if an algorithm is provided that transforms each instance of a unsolved decision problem, into an instance of another decision problem for which there is a known solution. Hence, "[a] *reduction of the decision problem* (of the pure first-order functional calculus) consists in a special class Γ of wffs and an effective procedure by which, when an arbitrary wff A is given, a corresponding wff A_Γ of the class Γ can be found such that A is a theorem if and only if a proof of A_Γ is known" ([24, p. 270]). Hence, in those cases for which there is no general solution, a clear way to go to is look for some subset B of A for which there is a solution. In other words, logicians try to find special cases for which there is a solution.

The four works accounted for by Henkin deal with the decision problem for one area or another. In particular, the first, Jaśkowski, S. "Sur les variables propositionelles dependantes" [150], offers a reduction to a particular case for first-order decidability. The second and third, respectively the first and second editions of Tarski's *A Decision Method for Elementary Algebra and Geometry* [232], offer a procedure to decide whether a statement of algebra and elementary geometry is or is not true by using the technique of quantifier elimination. Finally, the fourth work by H. Rasiowa and R. Sikorski—"On existential theorems in non-classical functional calculi" [209]—presents a reduction of the decision problem for a set of statements of intuitionistic functional logic into the decision problem for intuitionistic propositional logic.

Henkin (4) [57] summarizes Jaśkowski's method to reduce the decidability problem for first-order logic to the decidability problem for certain systems I and II when certain conditions are obtained. Those conditions are that there are only three variables and a propositional variable that play the role of a predicate with three arguments in those systems. Henkin points out that Jaśkowski's results are obtained only if the theorems in I are defined as the theorems that are obtained from theorems in the first-order calculus, but not from formulas that are not first-order theorems. In this case, I and II are equivalent.

"Outline of a revised formulation of the logic of sense and denotation (Part I)" [25] what he takes to be an adequate proposal. The next year, Church published the second part [26], and in 1993, he published an alternative formulation [27].

Henkin claims that if theorems in system I are obtained—as Jaśkowski sustains—from homogeneous theorems[9] in first-order logic, then a formula in system I is a theorem in I if and only if a tautology is obtained when quantifiers are eliminated. But, in that case, theorems in I would be decidable, contrary to what Jaśkowski claims when he says that the decision problem for first-order logic reduces to the decision problem in I. That is why Henkin modifies the definition in system I claiming that its theorems are obtainable from theorems in first-order logic and that there is no first-order formula that is not a theorem from which a formula in system I is obtained. We believe that Henkin is wrong. Take, for instance, the formula $\exists x_0 P x_0 \to \forall x_0 P x_0$ from which the formula $\exists x_0 p \to \forall x_0 p$ is obtained; the latter, according to Henkin's criterion, would be a theorem in I because after removing quantifiers and variables a tautology, $p \to p$, is obtained. Henkin mentions that the author defines other systems, one of which he intends to use for modal logic; but Henkin considers this system, which is not really a logic since no effective criterion for deducibility is given.

According to Henkin (6) [55] and (15) [66], Tarski's very well edited book [232] precisely and profusely describes, and proves the validity of a decision method obtained by Tarski himself in 1930. The decision method is applied to a system of elementary algebra: its sentences are constructed with connectives and quantifiers from equations and inequations of polynomials in real variables whose coefficients are 1, -1, and 0. The method decides whether a given statement expresses a proposition that is true according to its standard interpretation. A method for axiomatizing the system is given, and it is related to the decision method. Indications are provided about how to extend it to elemental theories in Euclidean and projective geometry. He uses the method of quantifier elimination that was introduced by Löwenheim [170]. The review of the first edition ends listing a number of errata. The review for the second edition [233] reports that: (i) errata in the first edition have been corrected; (ii) contents have been enlarged and applications expanded.

Henkin's review (23) [83] of Rasiowa and Sikorski's [209] "On existential theorems in non-classical functional calculi" notes that they build on a previous result by McKinsey and Tarski [189]; the latter proved a fact that Gödel had established though not proved [43]. The fact is that if a disjunction $\sigma \vee \tau$ is provable in an intuitionistic propositional calculus S, then σ is provable in S or τ is provable in S. Rasiowa and Sikorski, using their algebraic treatment of the notion of satisfaction [208] and McKinsey and Tarski's line of proof, extend the result showing that if $\exists x\sigma$ is provable in a first-order intuitionistic system S^*, then some formula σ'—obtained from σ by substituting for all occurrences of x occurrences of a free variable y—is also provable in S^*. They also establish that a formula in S^* without quantifiers is provable in S^* if and only if it is an instance of some theorem of S. Then, since there is a decision method for S, they infer that there is a decision method for the class of formulas that can be put in prenex normal form. They claim that this last result can be applied more generally to the class of formulas in which no quantifier occurs under the scope of a negation or implication. Finally, Henkin claims that analogous results are obtainable for positive, minimal, and Lewis' calculus and also for the rest of the calculi addressed by the authors in [208].[10]

[9]Homogeneous theorems are homogeneous formulas that are derivable from axioms using the inference rules in the system, and a formula is *homogeneous* if only predicates with the same number of variables and in the same order occur in it.

[10]This review could have also been included in Sect. 3.1 "Algebraic treatment of logic systems."

Finally, Henkin assesses two papers devoted to the analysis, at the abstract level, of logical consequence operations: (29) [82] Asser's "Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe" [2] and (30) [89] Łoś's "The algebraic treatment of the methodology of elementary deductive systems" [167].

Asser analyzes the logical consequence formal relation that holds between sets of formulas X and certain formulas H. Henkin remarks that Asser fails to quote other works in which the same issue is dealt with (for instance, Henkin [58] and Rasiowa–Sikorski [208]). The consequence relation, **Abl**, is such that $X\mathbf{Abl}H$ complies with the following conditions:

1. If $H \in X$, then $X\mathbf{Abl}H$;
2. If $X_1 \subseteq X_2$, and $X_1\mathbf{Abl}H$, then $X_2\mathbf{Abl}H$;
3. If $\mathbf{Cn}(X) = \{H|X\mathbf{Abl}H\}$ and $\mathbf{Cn}(X)\mathbf{Abl}H_1$, then $X\mathbf{Abl}H_1$;
4. If H_1 is a substitution instance for bound variables, free individual variables, and predicate variables in H and $X\mathbf{Abl}H$, then $X\mathbf{Abl}H_1$.

Moreover, Asser considers a very general semantic interpretation for well-formed formulas. The models used by Asser for formulas H are formed by a set M of truth values, a set $M_0 \subset M$ of designated values, functions associated with connectives, functions that assign an element in M for each subset of M associated with each quantifier, an arbitrary set J over which individual variables range, and for each n-ary predicate nonempty set B_n as a range for n-predicate variables whose elements are functions that assign an element in M to each n-tuple of elements in J. The main result is that for each set of well-formed formula X that is closed under the relation **Abl**, there exists a model μ such that X is precisely the class of valid formulas with respect to μ. The construction of μ is one by Lindenbaum.

Henkin says that Łoś provides "a comprehensive account of the principal results concerning elementary deductive systems" since (1) he studies axiomatically the formal consequence relation introducing it in Tarski's way in [227] and [228]; (2) Łos proves Gödel's theorem [41]—each consistent set of formulas is simultaneously satisfiable—by means of maximally O-consistent sets and forming the corresponding Lindenbaum algebra (see Tarski [228]); (3) he extends this result to the first-order case by applying the quantifier elimination method through introducing new functional signs (something that had previously been done by Skolem [221] and Hilbert and Bernays [146]); (4) by introducing nondenumerable sets of constants (compare to Henkin [52]), Łos deduces Tarski's generalization of the Skölem–Löwenheim theorem [220] and [222]; (5) using Robinson's "complete diagrams" [213] or the complete description of a model by means of constants (Henkin [71]), he deals with the extension of models establishing that "each ordered set can be extended to a densely ordered one"; (6) he introduces systems that consist of the consequences of a set of universal statements whose models constitute what Tarski [235] called a *universal class* and independently shows the theorem by Tarski "that ... an arithmetical class is universal if and only it is closed under formation of submodels"; (7) he treats the notion of categoricity in power that he had previously studied in his [166] and gives a proof of Vaught's theorem [237]—a system without finite models that is categorical in some (infinite) power has to be complete; (8) he proves Skolem's result [222]—the system of all true statements of elemental number theory is not categorical in $\aleph_0$—and constructs, by a process similar to Henkin's [59], a nonstandard model pointing out that the order relation over this model is not well ordered (it is of type $\omega + (\omega^* + \omega)\eta$ (see Henkin [59]).

3 Algebraic Logic

According to what Andréka, Németi, and Sain[11] say, we organize this section as follows.

3.1 Algebraic treatment of logic systems:

3 (1948) Review [50] of: John Charles Chenoweth McKinsey and Alfred Tarski, "Some theorems about the sentential calculi of Lewis and Heyting" (1948) [189].
7 (1949) Review [54] of: Andre Chauvin, "Structures logiques" (1949) [19].
8 (1949) Review [53] of: Andre Chauvin, "Généralisation du théorème de Gödel" (1949) [18].
9 (1950) Review [60] of: László Kalmár, "Une forme du théorème de Gödel sous des hypothèses minimales" (1949) [153]; László Kalmár, "Quelques formes générales du théorème de Gödel" (1949) [152].
17 (1953) Review [70] of: Helena Rasiowa, "Algebraic treatment of the functional calculi of Heyting and Lewis" (1951, pub. 1952) [205].
22 (1955) Review [84] of: Helena Rasiowa and Roman Sikorski, "Algebraic treatment of the notion of satisfiability" (1953) [208].
28 (1955) [85]: Ladislav Rieger, "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem" (1951) [211].
39 (1959) Review [99] of: Louis Nolin, "Sur l'algebre des predicats"; Andrzej Mostowski, Jean Porte, Alfred Tarski, Jacques Riguet, "Interventions" (1958) [202].
42 (1963) Review [110] of: Antonio Monteiro, "Matrices de Morgan caractéristiques pour le calcul propositionnel classique" (1960) [196].
44 (1967) Review [115] of: Marshall Harvey Stone, "Free Boolean Rings and Algebras" (1954) [223].
45 (1971) Review [118] of: Maurice L'Abbé, "Structures algébriques suggérées par la logique mathématique" (1958) [158].
46 (1971) Review [117] of: Marc Krasner, "Les algèbres cylindriques" (1958) [156].

3.2 Applications of logic to algebra:

14 (1952) Review [65] of: Abraham Robinson, *On the Metamathematics of Algebra* (1951) [213].

[11] "Algebraic logic can be divided into two main parts. Part I studies algebras which are relevant to logic(s), e.g. algebras which were obtained from logics (one way or another). Since Part I studies algebras, its methods are, basically, algebraic. One could say that Part I belongs to 'Algebra Country'. Continuing this metaphor, Part II deals with studying and building the bridge between Algebra Country and Logic Country. Part II deals with the methodology of solving logic problems by (i) translating them to algebra (the process of algebraization), (ii) solving the algebraic problem (this really belongs to Part I), and (iii) translating the result back to logic. There is an emphasis here on step (iii), because without such a methodological emphasis one could be tempted to play the 'enjoyable games' (i) and (ii), and then forget about the 'boring duty' of (iii). Of course, this bridge can also be used backwards, to solve algebraic problems with logical methods." (Hajnal Andréka, Istvan Németi, and Ildikó Sain [1, p. 133].)

25 (1955) Review [79] of: Abraham Robinson, “Les rapports entre le calcul déductif et l’interprétation sémantique d’un système axiomatique”; Evert Willem Beth, Luitzen Egbertus Jan Brouwer, Abraham Robinson, “Discussion” (1953) [215].
27 (1955) Review [80] of: A. Chatelet, “Allocution d’ouverture” (1953); (3) Luitzen Egbertus Jan Brouwer, “Discours final” (1953); (4) Abraham Robinson, “On axiomatic systems which possess finite models” (1951) [212].
33 (1957) Review [94] of: Kaarlo Jaakko Hintikka, “An application of logic to algebra” (1954) [147].
37 (with Andrzej Mostowski) (1959) Review [136] of: Anatolĭ Ivanovič Mal’cév, “Ob odnom obščém métodé polučéniá lokal’nyh téorém téorii grupp” (On a general method for obtaining local theorems in group theory) (1941) [181]; Anatolĭ Ivanovič Mal’cév, “O prédstavléniáh modéléj” (On representations of models) (1956) [182].

3.1 Algebraic Treatment of Logic Systems

Henkin himself made major contributions to the development of the algebraic treatment of logic systems. Namely, his results on cylindric algebras (see below) and his “The completeness of the first-order functional calculus” [52], “An algebraic characterization of quantifiers” [58], “Completeness in the theory of types” [59], and “Some interconnections between modern algebra and mathematical logic” [71].

Henkin wrote thirteen reviews of various works in this field: three (3), (17), (22) about works dealing with the algebraic treatment for intuitionistic calculus and modal calculi; five on algebras related to classical logic (polyadic algebras (39); two on cylindric algebras (45), (46); two on Boolean algebras (44), (28)); one on logical matrices (42), and three approach algebraic structures suggested by Gödel’s incompleteness theorems (7), (8), (9).

“Over a period of three decades or so from the early 1930’s there evolved two kinds of mathematical semantics for modal logic. *Algebraic* semantics interprets modal connectives as operators on Boolean algebras. *Relational* semantics uses relational structures, often called Kripke *models*” (R. Goldblatt [39, p. 1]).

The first substantial algebraic analysis of modalized statements was carried out by Hugh MacColl, in a series of papers that appeared in *Mind* between 1880 and 1906 under the title *Symbolical Reasoning*.[12] Some time later, in 1938, the first topological interpretations for modal and intuitionistic logics developed by Tsa-Chen Tang [225] and Tarski [231], respectively, saw the light. The three works reported by Henkin we will summarize next belong to this area.

In 1948 (3) [50], Henkin reviews a 1948 paper “Some theorems about the sentential calculi of Lewis and Heyting” by McKinsey and Tarski [189]. In this paper, starting from the syntactic similarity between the axioms in certain logics and the axioms in the theory of point topology, the authors apply the techniques developed by them in [185–187], and [188] to the analysis of logic S4 and its Lewis’ extensions (S5 among them) [164] and to Heyting’s intuitionistic logic [145]. Henkin enhances the following results: a new axiomatization for S4 without a substitution rule is given; a matrix with a designated

[12] See [171–177], and [178].

element satisfies the axioms in S4 and modus ponens if and only if it is a closure algebra with operations $\cdot$, $-$, and C corresponding, respectively, to $\wedge$, $\neg$, and $\Diamond$, with 1 as a designated element; if α is deducible in S4, so is $\neg\Diamond\neg\alpha$ (Gödel [43] just stated it without proof). They also establish that there are extensions for S4—S5 for instance that satisfy this property, whereas the characteristic matrix for those extensions that do not satisfy the property is to have two designated elements; if $\Box\alpha \vee \Box\beta$ is derivable in S4, then either α or β is derivable in S4 (it does not hold in S5); they introduce the notion of *reducibility* and prove that neither S4 nor S5 is reducible (they generalize a result by Dugundji [29]); the class of matrices that satisfy the axioms in Heyting's calculus and detachment (logic **I**) and have just one designated element is the class of Brouwerian algebras isomorphic to a system of closed sets of some topological space; logic **I** satisfies: if $\alpha \vee \beta$ is derivable, then so is α or β; there are infinite nonequivalent formulae that contain a unique variable; and the logic is not reducible. Finally, they describe effective applications f from intuitionistic formulas to formulas in S4 in such a way that a formula φ is derivable in **I** if and only if $f(\varphi)$ is derivable in S4. Henkin considers the paper to be too condensed, without indications for proofs and with cumbersome symbolism. This paper is a key for the next two reviewed by Henkin.

In 1953, Henkin reviews (17) [70] "Algebraic treatment of the functional calculi of Heyting and Lewis" by Rasiowa [205]. In her paper, Rasiowa generalizes results obtained by McKinsey and Tarski [189] on modal and intuitionistic propositional logic, and she solves a problem posed by Mostowski [198]. Mostowski showed that Heyting's intuitionistic functional calculus [143] can be interpreted in a model $(I, \mathbf{B})$, where I is a set, and $\mathbf{B}$ is a complete Brouwerian lattice. Then, he asked whether there is a particular model $(I_0, \mathbf{B}_0)$ in which all nonderivable formulas take a value different from zero, while derivable ones take 0. This question has a positive answer for classical calculus as Gödel showed [41] and was partially answered by Henkin [59]. But Rasiowa provides the definite answer for intuitionistic calculus by using McKinsey and Tarski's results [188], and MacNeille's [179] and McKinsey's [185] methods. She also shows that system S4 by Lewis and Langford [164] is complete with relation to models $(I, \mathbf{C})$, where $\mathbf{C}$ is a closure algebra. Henkin ends the review pointing out that the relation between the modal calculus considered by Rasiowa and those considered by Barcan [3] and Carnap [16] is not clarified.

Henkin (22) [84] comments on "Algebraic treatment of the notion of satisfiability" by Rasiowa and Sikorski [208]. The authors study the algebraic interpretation for first-order calculi proposed by Mostowski [198] as a generalization of the algebraic interpretation of propositional calculus developed by Tarski and McKinsey [189]; part of this work presents previous results by them [205, 206], and [207]. In it, they consider systems S^* that obtain from adding axiom schema and quantification rules to a propositional calculus S, where S contains "positive logic" and, possibly, more connectives, and which is closed under modus ponens and replacement. An algebraic structure or S-algebra that includes an ordering relation (that corresponds to the conditional) is associated (as an interpretation) with each system S, and, if the logic is complete in relation to the order defined, then the algebra is called "S^*-algebra." An S^*-algebra together with a set J of "individuals" is a model for the system. The main result presented is that for systems S that comply with property E (that is, systems that can be embedded in an S^*-algebra in a way in which an arbitrarily given sequence of sums and products is preserved), the following requisites on a formula α in S^* are equivalent: α is provable in S^*; α is valid (it takes as a value only the unit element of the S^*-algebra A for all interpretations (J, A)); α is valid for all interpretations $(\mathbb{N}, A)$; α is valid in the model $(\mathbb{N}, L^*)$, where L^* is any complete extension of the "Lindenbaum algebra" L of S^*. (Feferman in his revision [31] of [206] claims that this name is not appropriate because it had been used previously by Tarski [228].) Later on, they apply all these results to the classical functional calculus—as they had done in [206] and [207]—then to intuitionistic and Lewis' logics (analogously to what they had done in [205] but this time using Heyting algebras instead of Brouwerian algebras). They end by describing a transformation ψ of intuitionistic formulae into modal formulae; they also show that a formula α is provable if and only if $\psi(\alpha)$ is provable (analogous to the result that Tarski and McKinsey [189] obtained for the propositional calculus). Henkin considers the authors have given up a possible generalization without advantage; this can be seen after analyzing his treatment of *implicational* logic [58]. Finally, Henkin poses an unresolved problem: do the S-algebras considered by the authors satisfy condition E? Henkin believes that it will be a difficult question to answer.

Henkin (39) [99] reviews in 1959 Nolin's algebraic version of first-order logic [202] "Sur l'algebre des predicats" (1958). Nolin introduces a calculus that is similar to Halmos' polyadic algebras,[13] except that it divides the universe of discourse into separate types and uses nonindependent primitive notions.

Henkin devoted many of his works to the study of cylindric algebras;[14] in fact, he can be considered as one of the authors that contributed most to the development of the field. These algebras, together with polyadic algebras, are to first-order logic like Boolean algebras are to propositional logic.[15]

Henkin published two reviews of works on Cylindric Algebras in 1971: L'Abbé's [158] "Structures algébriques suggérées par la logique mathématique" and Krasner's [156] "Les algèbres cylindriques".

According to Henkin (45) [118], because L'Abbé believes that "the two fundamental instruments in mathematical logic" are the theory of recursive functions and modern algebra, he describes various classes of algebraic structures whose characterization has come

[13] In his book *Algebraic logic* [47], Halmos publishes his 10 main papers on polyadic algebra.

[14] Cylindric algebras generalize Boolean algebras for each ordinal α by adding "distinguished elements" (the so called "diagonal elements" $\mathbf{d}_{\kappa,\lambda}$ where κ and λ are less than α) to the elements of the Boolean algebra 0 and 1, and unary operations called "cylindrifications" ($\mathbf{c}_\kappa$ where $\kappa < \alpha$).

[15] A hopefully complete list of Henkin's works on the subject is: "The representation theorem for cylindric algebras" [78]; "La structure algébrique des théories mathématiques" [87]; "Cylindrical algebras" (with A. Tarski) [140]; the abstract "Cylindrical algebras of dimension 2. Preliminary report" [91]; *Cylindric Algebras. Lectures presented at the 1961 Seminar of the Canadian Mathematical Congress* [103]; "Cylindric algebras" (with A. Tarski) [141]; *Cylindric Algebras, Part I* (1971 with J.D. Monk and A. Tarski) [132]; "Cylindric algebras and related structures" (with J.D. Monk) [131]; "Relativization of cylindric algebras" (with D. Resek) [137]; "Cylindric set algebras and related structures" (with J.D. Monk and A. Tarski) [133]; *Cylindric Algebras, Part II* (with J.D. Monk and A. Tarski) [134], and "Representable cylindric algebras" (with J.D. Monk and A. Tarski) [135].

from logical studies. He establishes the following correspondences:

Classical propositional logic	Boolean algebras
Many-valued logics	Post algebras
Intuitionistic logic	Brouwerian and Heyting algebras
Modal logics	Closure algebras

With a certain amount of detail, L'Abbé describes Boolean algebras that are relational, projective and cylindric, and monadic algebras. Henkin mentions that L'Abbé gives a wrong definition of cylindrification for cylindric set algebras. Henkin also points out that L'Abbé quotes Lyndon [168] for supposedly having proved that the class of representable relational algebras cannot be characterized by equational identities, but he does not quote Tarski [235], who succeeded in proving just the opposite.

In his work "Les algèbres cylindriques" (1958), Krasner contends that the theory of cylindric algebras does not start with Tarski's 1945 work, as L'Abbé asserts [158], but with several 1938 works about Galois theory, such as his own [155].[16] Henkin reports (46) [117], following Krasner, that in it, Krasner considers cylindric algebras whose Boolean component is complete and that Krasner describes it in relation to two other works.

Henkin writes two reports of works on Boolean algebras. In 1967, in one of them, he reports on a work by Stone [223] "Free Boolean rings and algebras," in which free Boolean algebras are generalized. The other was published in 1955 and deals with a work by Rieger written in 1951 [211], "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem," in which σ-algebras are generalized.

Henkin (44) [115] tells us that Stone ([223] "Free Boolean Rings and Algebras" (1954)) obtains a class of algebraic structures of which free Boolean rings are a special subclass. Stone obtains them by generalizing the notion of group-algebra A of a group G over a field Φ. Henkin also notes that Stone gives a more classical description of free Boolean rings, and that at the end of the paper, he outlines the significance of these rings for propositional logic. He ends by pointing out that Stone follows Boole's path [15] since he, like Boole, uses algebraic methods to study logic. Henkin also remarks that Stone's presentation follows Tarski' *Systemenkalkül* [228].

In 1955, Henkin reviews (28) [85] the work [211] "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem," in which Rieger generalizes σ-algebras by defining "Boolean algebras with marked sequences." In these algebras, only marked sequences have to have sums and products. Marked sequences constitute a set of multiple sequences that contain all "constant" sequences, and the set is closed in relation to some operations over sequences. This concept comes up when a certain model that derives from considering first-order functional calculi of symbolic logic is examined. In particular, if we associate with each formula A, the set $|A|$ of all formulas B such that $A \equiv B$ is provable, then the class of all sets $|A|$ is a Boolean algebra for the following operations: $-|A| = |\neg A|$, $|A| + |B| = |A \vee B|$, and $|A| \cdot |B| = |A \wedge B|$. With each formula A and finite sequence of integers $j_1, \ldots, j_n$, we consider the sequence a of elements whose general term $a_{i_1}, \ldots, a_{i_n}$ is $|B|$, where B is obtained from A by replacing free occurrences of variable x_{j_k} for free occurrences of x_{i_k} for $k = 1, \ldots, n$; in that case, $\Sigma a = |\exists x_{j_1} \cdots \exists x_{j_n} A|$,

[16] Krasner in [156] dates that work in 1958, not in 1938. Henkin does not provide the explicit reference for [155].

$\Pi a = |\forall x_{j_1} \cdots \forall x_{j_n} A|$ (see Mostowski [198], Henkin [58], and Rasiowa [167, 205]). The set Φ of those sequences satisfies the axioms for marked sequences. Using the result by Loomis [165]—according to which there are σ-algebras that are not σ-isomorphic to any σ-field of sets, but such that each algebra can be σ-represented as the quotient algebra of a σ-field of sets modulo an appropriate σ-ideal—he proves that each Boolean algebra in a denumerable family of marked sequences is isomorphic to a field of sets in such a way that the sums and products of marked sequences respectively become unions and intersections. From this Gödel's result follows: each formula of a first-order calculus is either a theorem or false in the domain of the integers for an appropriate interpretation for predicates. His result is related to Rasiowa and Sikorski's proof [207] of Gödel's theorem. Moreover, the strongest form of Gödel's completeness theorem [41]—each set Γ of formally consistent closed formulas is simultaneously satisfiable—can be obtained by a slight modification of Rieger's argument. Rieger argues that it applies to calculi with a denumerable number of symbols, but other proofs apply also when the number of symbols is nondenumerable (Henkin [52], Robinson [213], Rasiowa [207], and Beth [7]). Henkin finishes by posing the problem of characterizing those Boolean algebras with marked sequences that are σ-isomorphic to Tarski–Lindenbaum algebras and listing a number of errata.

In 1963, Henkin reviews (42) [110] a work by Monteiro [196] "Matrices de Morgan caractéristiques pour le calcul propositionnel classique," in which he describes an interesting class of *characteristic matrices for classical propositional logic* (ccpl)[17] that are *irregular*[18] and includes those described in Church [23]. When Church described such matrices, he asked which other such matrices existed. *De Morgan lattices* are distributive lattices with a monadic operation satisfying $-(-x) = x$ and $-(x \vee y) = (-x) \wedge (-y)$. They were studied by Bialynicki-Birula and Rasiowa [12] as *quasi-Boolean algebras* and by Kalman [151] as *i-lattices*. Monteiro considers matrices (M, D) where M is a De Morgan lattice and D a proper filter of M, establishing that such a matrix is ccpl if and only if $(x \vee -x) \in D$ for each $x \in M$. And, a ccpl matrix is regular if and only if it is an intersection of filters such that, for each $x \in M$, it contains exactly one of the elements x or $-x$. Monteiro constructs an irregular matrix with 12 elements not belonging to Church's class that consists of the subalgebras of direct products of linearly ordered De Morgan matrices, each of which satisfies $(x \wedge -x) \leq (y \vee -y)$.

Also in 1963, Henkin gives a lecture at the Twenty-Eighth Annual Meeting of the Association for Symbolic Logic; in this lecture, he presents a result that generalizes the result of Monteiro.

Henkin reports in one of his 1949 reviews (7) [54] that Chauvin [19] "Structures logiques" defines in an abstract way logical structures to which Gödel's incompleteness theorem applies. Chauvin characterizes *deductive* structures as those that are like the standard ones but for which no recursivity criteria are established; *classical* are those that add certain elements in type theory; *Peanian* those that contain numerical variables, zero, and successor, and *Gödelian* (in relation to a one-to-one application φ from symbols to numbers) are those in which the necessary predicates (relative to formal derivability) for Gödel's construction can be represented.

[17] Each tautology adopts only designated values, while non-tautologies don't.

[18] There are designated elements x and $x \to y$ such that y is not a designated element.

Henkin reports (8) [53] that Chauvin [18] “Généralisation du théorème de Gödel” uses a one-to-one function ψ that assigns a value to each value of a function φ, where φ is a one-to-one function that assigns to each formula its Gödel number. Then, if Δ is the class of systems such that ψ and $R(x, y)$ are representable, then $R(\psi x, \psi y)$ is also representable in those systems. For each Gödelian logic in respect to φ and belonging to Δ, Chauvin gives a procedure to construct an undecidable proposition corresponding to each ψ. He ends by proposing a program: to look for functions ψ such that the undecidable proposition associated with it expresses unsolved conjectures in number theory. He also asks whether each undecidable proposition can be constructed from such a ψ. Henkin observes that the answer is negative as can be seen from the results obtained by Kleene in 1943 [154].

Henkin reviews in 1950 (9) Kalmar’s “Une forme du théorème de Gödel sous des hypothèses minimales” [153] and “Quelques formes générales du théorème de Gödel” [152]). In those works, Kalmar presents formalizations of systems for which incompleteness results in Gödel’s style [42] can be established that are more abstract than those by Chauvin [19] and [18]. Henkin considers that both the works by Kalmar and those by Chauvin lack any interest for logic; he argues that they raise neither interesting particular new cases that are important in relation to Gödel’s theorems nor cases that are mathematically interesting given that no new methods of proof are needed.

3.2 Applications of Logic to Algebra

Henkin reviewed several works whose contents fall under this heading. The first (14) [65] is a review of A. Robinson’s book [213] *On the Metamathematics of Algebra*, in which he applies the techniques of symbolic logic to obtain different results in various algebraic theories (groups, rings, and fields). Among them, Henkin points out the following: (i) for each first-order formula ψ that is true in all fields of characteristic zero, there is a prime number p such that ψ is true in all fields of characteristic greater than p; (ii) a formula ψ true in all ordered non-Archimedean fields must be true in all Archimedean fields; and (iii) a formula true in one algebraically closed field must be true in every algebraically closed field of the same characteristic. Some of these results had already been obtained; for instance, the first of them had been obtained by Tarski in 1946 and by Henkin in 1947 (in his Ph.D.), but they had not been published. These results can be obtained by applying the theorem—closely connected to the completeness proof for first-order calculus—that establishes that each formally consistent set of sentences has a model. Robinson’s proofs use the techniques developed by Gödel [41] and Henkin [52] for calculi whose languages may contain a nondenumerable number of primitive symbols (Mal’cév was the first to apply the method to nondenumerable sets in 1936 [180]). Robinson uses infinite conjunctions and disjunctions, though he does not describe them clearly. He also generalizes several concepts; one of them reminds one of Tarski’s “relative systems” [230]. In relation to those generalizations, Henkin asks whether generalizing provides a unified treatment for pre-existent and independently developed theories and whether it leads to new and deeper results than the original theory. Henkin thinks that Robinson’s book does not provide an affirmative answer to these questions. Nevertheless, his assessment of the book is positive, though he also notices some errata and drawbacks. For instance, he mentions that the use by Robinson of predicates instead of the functions commonly used by mathematicians makes the reading more difficult and that the logical theorem used by Robinson also applies to calculi with function symbols (see Henkin [59]).

In 1955, Henkin reviews (25) [79] a work by Robinson [215] "Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique," in which deals essentially with the application to abstract algebra of the metatheorem according to which a set of first-order sentences is satisfiable if each of its finite subsets is satisfiable. These kinds of applications had been given before by Tarski [234], Henkin [71], and Robinson himself [213, 214]. Now Robinson gives new and deeper applications. He ends his paper with speculative passages, some of which show clearly, according to Henkin, that he is confusing use and mention.

Also in 1955, Henkin (27) [80] speaks about another work by Robinson [212] "On axiomatic systems which possess finite models," in which the author deals with models for first-order statement systems (he dealt with the same subject in his [213]). He establishes a theorem about "arithmetic algebraic structures," a concept defined by Tarski [234], for which he establishes the existence of finite models. Henkin remarks that the application of Robinson's result would be maximized if it would establish a feature of those first-order sentences that defined persistent arithmetical classes; Henkin then ends pointing out a series of errata.

This is one of the three reviews Henkin signs in 1959 (37) [136], this time in collaboration with Mostowski. The review reports on Mal'cév [181] "On a general method for obtaining local theorems in group theory" and [182] "On representations of models." Both papers deal with the theorem we nowadays call "compactness theorem" (if every finite subset of a given, possibly nondenumerable, set of first-order sentences is satisfiable, then so is the whole set). The first paper by Mal'cév establishes without proof the so called *general local theorem*: "If every finite subset of a given (possibly nondenumerable) set of first-order sentences is satisfiable, then so is the whole set"; also, several applications of it "to special problems of group theory are made which do not properly fall within the scope of this Journal." The second is devoted to local properties of models (relational structures) $M = \langle A; O_1, O_2, \ldots \rangle$, where A is a set, and O_i are n_i-ary relations. He defines the notion of "local property" as "a property which is necessarily possessed by M if it is possessed by all its finite submodels," the general notion of "representation" for a model, and the more specific ones of prime and predicative representation. Then, he proves two theorems that establish that if every finite submodel M^* of M possesses a representation of a given type (prime or predicative) in a model N^* of a class K of models characterized for a set of first-order axioms, then M possesses a representation (prime, predicative) in a model N of K. A third theorem has to do with second-order properties of models. Henkin and Mostowski believe that the papers have been written carelessly. The reviewers point out that both the formulation and proof of the local general theorem is due to Gödel [41] (denumerable case) and Mal'cév [180] (the nondenumerable case); the latter uses the Skolem normal form in his proof, and because of that, his proof has been found unsatisfactory. They also note that there is a satisfactory proof by Henkin [52] and Robinson [213].

Henkin reviews (33) [94] Hintikka's work [147] "An application of logic to algebra." Henkin starts providing a panoramic view of the completeness theorem and its applications. Henkin tells us that in recent years the first-order completeness result by Gödel [41] has been extended to systems with a nondenumerable number of symbols (Henkin

[52]) and that this result has been applied to establish theorems in algebra by Beth [7], Henkin [71], Łos [167], Robinson [213] and [215], Tarski [234], and, before that, Mal'cév [181]. In all cases, the theorem is used in order to establish that if every finite subset of a set Γ of first-order sentences is satisfiable, then there exists a model that satisfies all sentences in Γ. Hintikka uses a result by Dilworth (for which Henkin does not give the bibliographic reference) and a process of completing by cuts by Birkhoff [13] to obtain a result that, according to Henkin, could have been obtained by less sophisticated methods.

4 Philosophy of Logic and Mathematics

Both the philosophy of logic and mathematics underwent many interesting developments in Henkin's days; in Leon Horsten' words, "...it has turned out that to some extent it is possible to bring mathematical methods to bear on philosophical questions concerning mathematics. The setting in which this has been done is that of mathematical logic when it is broadly conceived as comprising proof theory, model theory, set theory, and computability theory as subfields. Thus the twentieth century has witnessed the mathematical investigation of the consequences of what are at bottom philosophical theories concerning the nature of mathematics" [148].

Henkin does not quote any of his works when he reports on issues in the philosophy of logic.[19] Most probably, because his main contributions do not belong in this area. Yet, he did write several papers dealing with diverse matters in the philosophy of logic: namely, his "Identity as a logical primitive" [121], where he explores the possibility of defining other logical expressions in terms of identity. Moreover, in his "The foundations of Mathematics I" [95], he clearly introduces basic notions in the philosophy of logic such as deduction, proof, consistency, and so forth.

Henkin also wrote a couple of works dealing with the relations between logic and mathematics "Are logic and mathematics identical?" [105], "Mathematics and logic" [108], whereas his commitment to improve the teaching of mathematics was so strong that he not only reviewed a paper on the issue but also made his own contribution to the subject.[20]

Henkin himself had a nominalist position in what is still one of the main issues in the philosophy of mathematics: the ontology and epistemology of mathematical entities. In fact, he at least wrote the following three works on the topic: "Some notes on nominalism" [72], "Nominalistic analysis of mathematical language" [106], and "The nominalistic interpretation of mathematical language" [77].

[19] Well, in fact, he quotes his other review of another work by Menger, when he discusses Menger's book *Calculus. A Modern Approach*.

[20] "On mathematical induction" [100]; with W.N. Smith, V.J. Varineau, and M.J. Walsh *Retracing Elementary Mathematics* [138]; "New directions in secondary school mathematics" [109]; "The axiomatic method in mathematics courses at the secondary level" [113]; "Linguistic aspects of mathematical education" [119]; "The logic of equality" [122]; with Nitsa Hadar "Children's conditional reasoning, Part II: Towards a Reliable Test of Conditional Reasoning Ability" [129]; with Robert B. Davis "Aspects of mathematics learning that should be the subject of testing" [127]; with Robert B. Davis "Inadequately tested aspects of mathematics learning" [128]; with Shmuel Avital "On equations that hold identically in the system of real numbers" [126]; "The roles of action and of thought in mathematics education—One mathematician's passage" [123].

Finally, Henkin published a series of reviews having to do with various problems in these areas that we have systematized as follows.

4.1 Basic logical notions

20 (1954) Review [76] of: Karl Menger, "The ideas of variable and function" (1953) [193].
21 (1954) Review [75] of: Karl Menger, *Calculus. A Modern Approach* (1953) [192].

4.2 Semantic notions and semantic issues in logic

10 (1951) Review [63] of: Heinrich Scholz, "Zur Erhellung des Verstehens" (1942) [218].
13 (1951) Review [61] of: Hugues Leblanc, "On definitions" (1950) [160].

4.3 Alternative logics

26 (1955) Review [86] of: Paul Bernays, Evert Willem Beth, Luitzen Egbertus Jan Brouwer, Jean-Louis Destouches, Robert Feys, "Discussion générale" (1953) [6].[21]
35 (1958) Review [96] of: Arend Heyting, "Logique et intuitionnisme" (1954) [144].
40 (1960) Review [102] of: Jerome Rothstein, *Communication, Organization and Science* with a foreword by C.A. Muses (1958) [216].
41 (1960) Review [101] of: Hilary Putnam, "Three-valued logic" (1957) [203]; Paul Feyerabend, "Reichenbach's interpretation of quantum-mechanics" (1958) [33]; Isaac Levi, "Putnam's three truth values" (1959) [163].

4.4 The Foundations of mathematics

2 (1948) Review [51] of: Jean Cavailles, *Transfini et continu* (1947) [17].
11 (1951) Review [64] of: James Kern Feibleman, "Class-membership and the ontological problem" (1950) [32].
12 (1951) Review [62] of: Hugues Leblanc, "The semiotic function of predicates" (1949) [159].
16 (1952) Review [67] of: Kurt Gödel, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory* (1951) [46].
32 (1956) Review [88] of: Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, Roman Sikorski, *The Present State of Investigations on the Foundations of Mathematics* (1955) [200].
38 (1959) Review [98] of: Andrzej Mostowski, "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques"; Daniel Lacombe, Andrzej Mostowski, "Interventions" (1958) [199].

4.5 Mathematical education and the foundations of mathematics

36 (1958) Review [97] of: Evert Willem Beth, "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" (1955) [9].

[21] This is a really short review. Its text goes: "This is a brief discussion of the following questions. Should semantics be considered as a part of, or as complementary to, symbolic logic? Does the formalization of theoretical physics require the introduction of new logical systems?"

4.1 Basic Logical Notions

The two works by Menger ([193] "The ideas of variable and function" and [192] *Calculus. A Modern Approach*) Henkin reviews ((20) [76] and (21) [75]) try to get a better understanding of the use of variables given both in logic and physics. In the second work, Menger also addresses the problem of functional notation.

Henkin says [76] that in his paper "The ideas of variable and function" [193], Menger distinguishes two different uses of the term "variable." The logical use of the term in which "variable" refers to the symbol employed in building sentences about a certain set that is the *range* of the variable; the other type variables that he calls *Weierstrass* variables refer—depending on the occasion—to pressure, weight, time, etc.; this second use is typical in science. In this last case, there is a class Σ whose elements are *observable*, and an equivalence relation $\cong$ is defined over Σ. Then, a variable v is a function defined over Σ such that $v\sigma_1 = v\sigma_2$ if $\sigma_1 \cong \sigma_2$. Henkin suggests an improvement over Menger's treatment when a law that involves variables in different domains occurs.

In his book, Menger [192] presents the theory of variable quantities (see [76]) and, according to Henkin, it is the first book in which there is given an explicit analysis of what is implicit when the theory of functions is applied to physics. Henkin considers the book as written in a lively and thought-provoking style and remarks that Menger distinguishes the notation for numbers from the notation for functions. A notation for functions had been developed by Menger in two previous works, *Algebra of analysis* [190] and "Are variables necessary in calculus?" [191], which Church—in his review of Menger's works—considers to follow the ideas in Schönfinkel–Curry combinatory logic. Yet, Henkin comments that Menger does not apply the ideas of combinatory logic to the end since those would have allowed him to do away with all variables occurring in function names. Moreover, Henkin reports that the distinction between use and mention is lacking, even though the author aims to use unambiguous notation. Moreover, an inappropriate use of the identity symbol is made in the common use of $\lim_\theta I^{-2} = \infty$. The author establishes a set of principles for a correct notation, but some of them are questionable; for instance, "unnecessary symbols or names should not be created."

4.2 Semantic Notions and Semantic Issues in Logic

The two earliest reviews by Henkin published in 1951 deal with notions such as interpretation, understanding, and definition.

In the first one (10) [63], Henkin reports that Scholz' "Zur Erhellung des Verstehens" [218] tries to explain how research in mathematical foundations can help clarify the notion of "understanding" by considering a particular formal system and explaining how it can be interpreted in two different ways. The system is the propositional calculus by Frege; its connectives are $\neg$ and $\rightarrow$, and its theorems are the tautologies. Henkin explains that, following Frege [35] and [37], Scholz describes a proposition as the sense expressed by a sentence. In the first interpretation, in the domain, there are arithmetical propositions, and the notion of "satisfaction" is defined recursively à la Tarski [229]. In the second interpretation, the elements of the domain are the numbers 0 and 1, and the satisfaction definition assigns the corresponding truth-tables to ¬ and →. Henkin believes that Scholz' presentation is correct, clear, appropriate, and illuminating. But he objects that Scholz should have pointed out that the first interpretation ranging over the domain of mathematical propositions is not the usual one and that he should have clarified what the relation between both interpretations is.

Leblanc's project [160] is, according to Henkin (13) [61], completely mistaken. He tells us that in "On definitions," Leblanc introduces the notion of absolute definition in order to explain synonymy. An *absolute definition* is, according to Leblanc, the kind of definition for a term when there is a finite system of axioms that characterizes the term, and the corresponding formula by means of which the term is defined is provable in the system in question. Henkin mentions that the author acknowledges that there are some shortcomings for the proposal, but Henkin thinks that the proposal is untenable given that "even the most complete axiom system determines its models only to isomorphism, so that no significant concept can ever be given an absolute definition!" Henkin considers that the analysis of the problem should rather be done following Church [21].

4.3 Alternative Logics

In 1958, Henkin reviews (35) [96] a contribution by Heyting [144] "Logique et intuitionnisme," in which the author offers a challenging series of notes about the role of logic from an intuitionistic viewpoint. According to Henkin, Heyting contends that logic does not provide a criterion to decide on the validity of mathematical reasoning (mathematical reasoning is acceptable only if it is "immediately clear"); Heyting believes that logical laws have been used to justify reasoning because our language is not appropriate to express mental constructions. The problem is, Heyting claims, that mathematical language lacks an adequate grammar for imperatives; thus, it proceeds as if doing mathematics were about discovering facts, whereas doing mathematics is about constructing. Heyting gives the intuitionistic meaning for some logical laws and claims that logic is a part of mathematics and that its task is not to provide foundations for it since it is highly abstract; rather, logic is "at the end of mathematics," and it is as formal as any other part of mathematics. Finally, Heyting rejects the possibility of a unique formal system in which all theorems in intuitionistic mathematics are obtainable.

Henkin assesses (40) [102] what seems to be quite strange in the book by Rothstein [216]: *Communication, Organization and Science*. Most of the content of the book is certainly nonstandard from our contemporary viewpoint. The author—as Henkin tells us—intends to show that the notion of *entropy* can be used to give a unifying perspective on many phenomena in the philosophy of science and language since measuring "may be regarded as a form of communication between a scientist and the things he studies, and the amount of information gained by a measurement may be indicated by a number expressed as a quantity of entropy." Henkin points out that discussed notions are not given in detail by the author but just suggested and that a lot of what is claimed is based on analogy. In one chapter, the rules for the calculus of classes are given, and it is discussed whether they can be applied to empirical sciences. Henkin also reports that Rothstein mentions the attempts to apply alternative logic to physical theory and suggests that they can be adequate if it is necessary to consider things that are operationally undefined. He also sets out the possibility of a universal language based in symbolic logic that allows for the use of computers, but he does not analyze whether it is really possible. In the prologue, Muses both criticizes and praises Rothstein' work. Henkin mentions that Rothstein told him that he did not hear of the prologue until the book was published. Henkin does not give any personal assessment of the work in his review.

Henkin (41) [101] reviews three works on quantum logic. Henkin reports that Putnam [203] proposes a three-valued interpretation for sentential connectives according to which any sentence involving only the standard True and False values will obtain a standard truth value. He says that Putnam suggests that truth values have to do with the verification status of the sentence; hence, the nonclassical values for logical expressions might be useful in domains in which there are sentences that cannot be verified or falsified. Putnam also draws an analogy between logic and geometry to conclude that classical logic cannot have a privileged position any more than Euclidean geometry does. He also argues that the laws of classical logic cannot be those that underlie physics since, when two-valued logic is used, "the laws of quantum mechanics are incompatible with the principle of contact action." Henkin reports Feyeraberd [33] as disagreeing about this last argument because it "would violate one of the most fundamental principles of scientific methodology, namely, the principle to take refutations seriously." He says that Feyerabend also criticizes Reichenbach's proposal in favor of three-valued logic by claiming that "while statements expressing anomaly should have truth-value indeterminate, laws of quantum-mechanics should have only values T or F," but Feyerabend lists a number of quantum laws he contends can only be indeterminate. Feyerabend also considers the Copenhagen interpretation for quantum physics to reject Reichenbach's and Putnam's arguments against it. Levi [163] argues that "three-valued logic has a dim future" on the basis of Putnam's contention, according to which to ask for an interpretation of the third value lacks any sense; Putnam asking for an interpretation for the third value is tantamount to asking for a translation of the sentences of three-valued logic into sentences of two-valued logic.

Levi counterargues: (a) that not providing a translation makes sense if statements in one language have greater expressive power than those in the other, but he claims that this does not apply in our case; (b) that if an analogy between geometry and logic is to be established, as Putnam contends, then it is relevant to say that Euclidean geometry and Lobachevskian geometry are intertranslatable. (c) Finally, Levi also considers Reichenbach's possible interpretations for quantum mechanics to conclude "that a formulation in terms of three-valued logic should be intertranslatable with the two-valued formulation which admits 'causal anomalies' " (Henkin points out that Levi thinks it would not be intertranslatable if we contemplate the two-valued interpretations that consider that some sentences are meaningless.) Henkin ends his review with some critical comments: (1) He paraphrases Putnam as claiming that three-valued logic should not be applied to ordinary discourse because "if a sentence is ever shown to be verified or falsified, the claim that it has "middle" value will be shown to be unfounded" (Henkin's literal restatement of Putnam's view). Henkin objects to it, because "it would be 'dangerous' to make any statement of empirical content whatever!" But it seems to us that Henkin is not getting Putnam right. Putnam is some kind of a realist at this point; hence, he intends truth-values for sentences to obtain independently of us. That is why—we take—he contemplates as possible a third-value for quantum physics but not for ordinary discourse; that third indeterminate value in quantum physics is obtained independently of us, whereas our failure to establish the truth value of a statement, like in the example Henkin gives, is dependent of us, it is epistemic. (2) He rejects Feyerabend's use of the contention that a theory from which a false prediction follows must be modified. Henkin assumes a wholist structure for justification to sustain that a theory can be changed while leaving logical axioms untouched. (3) He rejects Levi's arguments for the dimness of three-valued logic because he takes them to depend on the assumption that the translation of the theory has to be into the two-valued theory that admits "causal anomalies," and he considers this assumption has not been argued for.

4.4 The Foundations of Mathematics

Henkin reviewed papers dealing with the hottest issues at the time; namely, problems in the philosophy of set theory (such as the consistency of the axiom of choice and the generalized continuum hypothesis) and in the foundations of mathematics. Henkin also reviews two papers in which the ontological status of the entities to which expressions in a logical language refer is discussed.

Henkin published his review (2) [51] of Cavailles' posthumous book [17] *Transfini et Continu* in 1948, six years after the publication of Henkin's first review in 1942. Henkin did not publish anything during these six years. This five-years void, 1942–1947, is due to the fact that he was first working on his Ph.D. (he presented it in 1947 under Church's supervision) and then, during World War II, on the Manhattan Project (Monk [194]).[22] Cavaillès book was written in 1940 or 1941, but, so Henkin goes, its publication at the time was forbidden by occupation authorities; hence, it was not published until 1947, well after Cavaillès was put to death in 1944. Henkin reports that the author presents the following topics dynamically and quite accurately: a discussion of the role of the axiom of choice and the continuum hypothesis, Gödel's proof of their relative consistency (of the preliminary version of the proof [44], not its final version [46]), Gentzen's proof of the consistency of arithmetic—though he does not mention Gentzen's proof in [38] but that of Bernay—and the theory of recursive functions and the theory of constructible ordinals by Kleene and Church. Henkin comments on Cavaillès' discussion of the proof of the relative consistency of the axiom of choice and of the generalized continuum hypothesis because he thinks that Cavaillès is wrong when he claims that Gódel's proof is intuitionistically acceptable, arguing that it does not use impredicative definitions. The rejection of impredicative definitions would be the defining feature of intuitionism, according to

[22]The Manhattan project was a US government project that produced the first atomic bombs. According to the obituary published on the web page of the University of Berkeley (http://www.berkeley.edu/news/media/releases/2006/11/09_henkin.shtml), "During World War II, he worked in industry for the Manhattan project, first as a mathematician for the Signal Corps Radar Laboratory in Belmar, New Jersey; then in New York City on the design of an isotope diffusion plant; and finally as head of the separation performance group at Union Carbide and Carbon Corp. in Oak Ridge, Tenn." In fact, Henkin himself explains the circumstances in [124, pp. 133–134, note 11]:

> "During the period May 1942–March 1946 I worked as a mathematician, first on radar problems and then, beginning January 1943, on the design of a plant to separate uranium isotopes. Most of my work involved numerical analysis to obtain solutions of certain partial difference-differential equations. During this period I neither read, nor thought about, logic."

Cavaillès. Henkin also notes that Cavaillès and Gödel [45] differ in their interpretation of the role Gödel's hypothesis (all sets are constructible) plays in general set theory. Henkin also claims that Cavaillès' "doubts as to whether the Cantor–Dedekind program to eliminate the geometric from mathematical reasoning by means of the notion of set" succeeds seems to confuse the use of geometrical intuition with its use as a criterion of correctness.

In his "Class-membership and the ontological problem" [32], Feibleman criticizes Quine's [204] and Quine–Goodman's [40] nominalist position in relation to logic and, as Henkin (11) [64] thinks, introduces a series of unsupported views about the development and future of symbolic logic. Henkin claims that the author's attack on Quine relies on a lack of understanding of Quine's ideas, in particular, given that the author believes that extensional logic is nominalistic and that "the consistent nominalist must eschew all words except names of particular concrete individuals."

Leblanc's paper "The semiotic function of predicates" [159], as Henkin narrates (12) [62], describes three ways of seeing the role of predicates in a symbolic language. According to the Platonic interpretation, predicates denote abstract classes; the nominalist contends that they are not names but that they "combine with names of concrete objects to form sentences that express assertions about those objects." And finally, according to the Aristotelian interpretation of their role, predicates denote "components" of particular entities (Henkin points out that the author does not either justify the attribution of this view to Aristotle or explain the view clearly). Henkin stresses that the author agrees with Quine that any of the three above-mentioned interpretations works if no quantification over predicates is involved. Otherwise, a nominalist interpretation is not possible.

Henkin reports (16) [67] on Gödel's second edition of the 1951 paper *The Consistency of the Axiom of Choice and the Generalized Continuum-Hypothesis with the Axioms of Set Theory* [46]. The first edition was reviewed by Paul Bernays [5]. This review is radically different from the one above on the same subject (Cavaillès') in that it is a technical review and it contains no philosophical discussion. Because Henkin's review is a review of the second edition of the work, he concentrates on the added material, namely, the 10 notes added by Gödel. In particular, Henkin emphasizes three of them. The first goes that $V = L$ implies the existence of a well-ordering of the reals that has as its graph in the Euclidean plane a projective set of points (the well-ordering is formalizable by means of a formula whose quantifiers range only over the real numbers). The eighth note states that the notion of "normality" is extensional. And finally, note 10 claims that the proof can be extended to systems with strong infinity axioms.

In 1956, Henkin publishes a review (32) [88] of the work by Mostowski et al. [200] *The Present State of Investigations on the Foundations of Mathematics*. In this work, a unified view of the foundational developments obtained by the Polish school, in particular, by Mostowski and his collaborators, is given. According to the authors, Henkin states, there are two main problems at the root of these: (A) to come to a better understanding of the nature of mathematical notions in the sense of clarifying whether mathematical notions are the result of human construction or forced upon us and also to explain how it is possible for us to get to know them; (B) to elucidate the nature of mathematical proofs and to provide criteria that allow us to distinguish between correct and "false" proofs.[23]

[23]Mostowski himself uses the word "false" applied to proofs, but "incorrect" seems to be the right word to use; it is sentences, statements, or propositions that are true or false, and hence the conclusion of an intended proof can be false but not the proof itself.

The authors admit, though Henkin does not report on that, that these are not purely mathematical problems, but also philosophical. They seem to contend that two lines of work in foundations have contributed to answer the first problem. The development of the axiomatic method, A1, and the study of the a priori operations needed in order to account for mathematics, A2, have contributed to a better understanding of the first, while the axiomatization of logic and the completeness proofs for nonclassical logics, B1, and decision problems, B2—especially essentially undecidable systems (a notion due to Tarski) and Kleene's hierarchy—seem to have helped clarify the second. In relation to the axiomatic method A1, they explain that systems are classified into elementary and nonelementary. Concerning elementary systems, they mention their applications to abstract algebra, the characterization of special types of arithmetic classes and multivalued systems. Concerning nonelementary ones, they remark that their interpretation is ambiguous because it depends on the underlying set theory; they also mention the fact that their models are nonabsolute results in incompleteness. From these difficulties they conclude that the axiomatic method does not provide an adequate foundation for mathematics. In relation to problem A2, they point out that Gödel's constructible sets are important. Finally, they mention what then were two new lines of work: the theory of recursive functions and the increasing association with algebra. The paper contains a broad bibliography (108 items) and their philosophical conception of mathematics. With regards to the latter, Henkin quotes the authors: "negative results obtained by the mathematical method confirm the assertion of materialistic philosophy that mathematics is in the last resort a natural science, that its notions and methods are rooted in experience, and that attempts at establishing the foundations of mathematics without taking into account its originating in natural sciences are bound to fail."

Mostowski [199] "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques" analyzes several metamathematical results obtained by using nonfinitistic techniques. Note that, at the time, the use of finitistic methods was prevalent due to Hilbert's authority and position on the issue. Some of the analyzed outcomes are: the famous results by Löwenheim and Skolem, the proof of the undecidability of a sentence Δ that says that there exists a certain class of models for axiomatic set theory, and the construction of $2^{\aleph_0}$ essentially different models for an arbitrary system that includes Peano arithmetic. The author also considers his work on automorphisms and on generalized quantifiers. Henkin's verdict on the paper (38) [98] is that even if many of the results presented in it are not new, the paper is still valuable because the fact that all these results are presented together allows us to see the relevance of nonfinitistic methods.

4.5 Mathematical Education and the Foundations of Mathematics

Henkin's review (36) [97] of Beth's "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" [9] is the second review he publishes in 1958. Henkin tells us that Beth's chapter tries to establish a connection between mathematical logic and research in the foundations of mathematics with problems in the teaching of mathematics. Beth points out that almost all of the mathematics taught in secondary school is formalizable in first-order logic and that psychology has failed to contribute in a relevant way to the teaching of mathematics; Henkin states that Beth nevertheless acknowledges that psychology might contribute to a better understanding of how students learn facts. Beth's disputable claim (though Henkin remains neutral and just paraphrases Beth's views without further comment) has to do with what he takes to be the main purpose of mathematical education: namely to make the student familiar with the deductive method. And to that purpose, as Beth claims, the use of (meta)logical results should prove effective.

5 Manuals

Henkin reviews two logic manuals, one by his advisor, Church, and another by Beth.[24]

Henkin's review (34) [93] of Church's *Introduction to Logic* [24] was published in 1957. Henkin stresses the unusual features of the book: its sixty-eight pages introduction divided into 10 sections like the five other chapters in the book and its 590 footnotes, which are to be added to the historical notes in each chapter. However, the distinction between footnotes and historical notes is not sharp since many footnotes include data with a historical interest. Among the strengths of the book, Henkin mentions that it provides plenty of exercises with various levels of difficulty, which, contrary to what is usually the case in symbolic logic manuals, can be of interest for the student of mathematics. He also emphasizes that the introduction provides a conceptual frame for a general theory of linguistic systems with an analysis of Frege's distinction between sense and reference. And that, in spite of its nonmathematical character, the introduction is of interest for the mathematician since it illuminates many basic notions that are applicable to mathematical language. For instance, when Church defines "logistic method," he explains how to use English as a metalanguage, emphasizing the convenience of constraining the use of English to the level that is "just sufficient to enable us to give general directions for the manipulation of concrete physical objects," in particular, "those additional portions of English are excluded which would be used in order to treat of infinite classes," advice that, according to Henkin, the author himself does not follow and one that Henkin doubts can be followed. The first chapter deals with a particular propositional system and includes the deduction theorem, the decision problem, duality, consistency, completeness, and independence. In the second chapter, the author introduces up to 40 different propositional systems, something that Henkin does not welcome because he thinks that it can have the wrong effect on beginners. Chapters 3 and 4 present many systems for first-order functional calculus, substitution rules, prenex and Skolem normal forms are studied in detail, and so are Gödel's completeness theorem, Löwenheim–Skölem's result and decision problems (solution for special classes and reductions for the general problem). Henkin objects that Church's presentation of first-order calculi

[24] Henkin quotes none of his works in these reviews of the two manuals mentioned. In his review of Church's manual, he quotes no other work, whereas in his review of Beth's, he quotes Beth's work "The Foundations of Mathematics. A Study of the Philosophy of Science" [10].

does not include systems that have operational symbols in addition to functional ones (relational, as we would say); he believes that this is important because functional symbols are really adequate for the formalization of many mathematical theories. The last chapter deals with predicative and ramified second-order calculi establishing Henkin's completeness theorem for the former; it also includes the study of the infinity axioms and well-ordering. Other topics are mentioned, and Church informs that they are to be included in a second volume; the topics Church intended to present in a second volume are: higher-order functional calculi, second-order arithmetic, Gödel's incompleteness theorems, recursive arithmetic, simple type theory, axiomatic set theory, and mathematical intuitionism. According to Henkin, "[t]he appearance of this volume promises to complete a work of great usefulness both for students and scholars, and it is so hoped that a way can be found to shorten its publication in time." As is well known, it never was.

Beth's book [11] *Formal Methods. An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic* is a book in which Beth, as Henkin says (43) [112], tries to explain the principles, foundations, and methods of contemporary logic "with contemporary theoretic insight...." Henkin believes that it includes unusual topics for an introductory manual and that Beth's style might discourage the beginner. Henkin claims that it should be useful as a manual if the teacher is ready to provide additional material and that professional logicians will find it interesting "for its wide-ranging and provocative comments as well as for the new ways of presenting familiar material." Beth presents from three different perspectives, deductive, semantic (he includes his semantic tableaux [10]), and axiomatic, a propositional logic system with a single connective, the conditional; he then extends it to complete propositional logic, quantificational logic, and a system with functional symbols. He presents consistency and completeness proofs and, according to Henkin, uses the name *strong completeness theorem* for results that do not deserve the adjective "strong" since they apply only to finite sets of formulas. Other topics he deals with are: the formalization of arithmetic (based on the notions zero, successor, addition, multiplication, and exponentiation), *Church's thesis*, the *theory of definition*, a description of Padoa's method, incompleteness, and to close the chapter, "On machines which prove theorems." The appendix is a potpourri of subjects.

6 Conclusion

Leon Henkin's scientific production starts in 1942 with his review of M.H.A. Newman and A.M. Turing, "A formal theorem in Church's theory of types" [201] and ends in 1995 with the paper "The discovery of my completeness proofs" [124]. This wide period includes fifteen years during which Henkin was "unproductive."[25] His scientific results include the

[25] The period from 1943 to 1947 in which he worked in his Ph.D. and in the Manhattan project (see note 22 above), years 1969, 1982, 1984, 1987, 1988, and the period from 1990 to 1994.

edition of four congress proceedings,[26] seven books,[27] 54 papers,[28] 46 reviews,[29] and 17 minor works.

Table 1 contains a synopsis of his production, and Table 2 summarizes the results in Table 1. It is easy to see that 1955 is his most productive year and that more than half his production was produced between 1950 and 1960.

Table 3 shows the time elapsed between the publication of the reviewed work and the review. It is worth noting that Henkin published more than half of his reviews only one year after the corresponding work had seen the light. This clearly shows Henkin's commitment to Church's endeavor. It is also salient that most of the works reviewed by Henkin have to do with logical systems and algebraic logic—no doubt, the subjects to which he contributed most (see Table 4). His expertise in these matters is also shown in the fact that the number of papers and books he quotes in his reviews of publications on these subjects is larger than the number of works he quotes in his reviews of publications on other topics.

There is a close correspondence between Henkin's areas of expertise and the topics of the works reviewed by Henkin. Hence, according to Mathematics Subject Classification 2010,[30] the area of mathematical logic and foundations divides into: philosophical aspects of logic and foundations, general logic, model theory, computability and recursion theory, set theory, proof theory and constructive mathematics, algebraic logic, and nonstandard models. Practically, all of Henkin's reviews fall into three of these areas: philosophical aspects of logic and foundations, general logic, and algebraic logic. He does not report on any works on computability and recursion theory, most plausibly because Church, Kleene, McKinsey, Vaughan, and Ribeiro, among others, were in charge. He appraises only two works in set theory *Transfini et Continu* by Cavailles and the second edition

[26] *The Axiomatic Method* [139] with P. Suppes and A. Tarski; *The Theory of Models* [125] with J.W. Addison and A. Tarski; *Logic, Methodology and Philosophy of Science IV* [224] with P. Suppes, A. Joja, and Gr.C. Moisil; and *Proceedings of the Tarski Symposium* [120].

[27] *La structure algébrique des theories mathématiques* [87]; *Cylindric Algebras. Lectures presented at the 1961 Seminar of the Canadian Mathematical Congress* [103]; *Retracing Elementary Mathematics* [138] with W.N. Smith, V.J. Varineau, and M.J. Walsh; *Logical Systems Containing Only a Finite Number of Symbols* [114]; *Cylindric Algebras, Part I* [132] with J.D. Monk and A. Tarski; *Cylindric Algebras, Part II* [134] with J.D. Monk and A. Tarski; and *Mathematics-Report of the Project 2061 Phase I Mathematics Panel* [14] with D. Blackwell.

[28] 12 of those papers were written in cooperation with someone else: "On the definition of 'formal deduction' " [195] with R. Montague; "Cylindrical Algebras" [140] and "Cylindric Algebras" [141] with A. Tarski; "Cylindric algebras and related structures" [131] with J.D. Monk; "Relativization of cylindric algebras" [137] with D. Resek; "A Euclidean construction?" [130] with W. Leonard; "Children's conditional reasoning, Part II: Towards a reliable test of conditional reasoning ability" [129] with Nitsa Hadar; "Aspects of mathematics learning that should be the subject of testing" [127] and "Inadequately tested aspects of mathematics learning" [128] with Robert B. Davis; "Cylindric set algebras and related structures" [133] and "Representable cylindric algebras" [135] with J.D. Monk and A. Tarski; and "On equations that hold identically in the system of real numbers" [126] with Shmuel Avital.

[29] See Appendix.

[30] This classification coincides with that provided in 2000. Barwise's [4, p. vii] "Mathematical logic is traditionally divided into four parts: model theory, set theory, recursion theory, and proof theory" is not that complete.

Table 1 Under the column "Minor," we write the number of reviews Henkin wrote in that year, and, in the same column, after the "+" symbol, we write the number of minor works published in that year that are not reviews. Under "Major," we include the number of books, papers, and so forth. After "+" and under "major," we indicate the number of books edited

Year	Minor	Major	Total	Year	Minor	Major	Total
1942	1		1	1967	1	2	3
1948	2 + 1		3	1968	+1	1	2
1949	5	2	7	1970		1	1
1950	1	3	4	1971	2	2	4
1951	4		4	1972		1	1
1952	3 + 2		5	1973		1 + 1	2
1953	2	4	6	1974		1 + 1	2
1954	3 + 3	2	8	1975		2	2
1955	8 + 1	5	14	1977		2	2
1956	3 + 2	3	8	1978		2	2
1957	2	3	5	1979		2	2
1958	2	1	3	1980	+1		1
1959	3	+1	4	1981		2	2
1960	2 + 1	1	4	1983	+2		2
1961		3	3	1985		1	1
1962		4	4	1986		1	1
1963	1 + 1	4	6	1989		1	1
1964	+1		1	1995		2	2
1965	1	+1	2	1996		1	1
1966	+1	1	2	Total	46 + 17	61 + 4	128

Table 2 Synopsis of the results in Table 1

Period	Reviews	Minors	Majors	Editor	Total	%
1955	8	1	5	0	**14**	10.94
1954–1956	14	6	10	0	**30**	23.44
1953–1957	18	6	17	0	**41**	32.03
1952–1958	23	8	18	1	**49**	38.28
1951–1959	30	9	18	0	**57**	44.53
1950–1960	32	9	22	0	**65**	50.78
1949–1961	37	9	27	0	**75**	58.59
1948–1962	39	10	31	0	**82**	64.06
1948–1963	40	11	35	0	**88**	68.75

of *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory* by Gödel, and only one in proof theory and constructive mathematics, Kreisel's "On a problem of Henkin's".

Table 3 The columns called "Δ" indicate the years passed between the year of publication of the reviewed work and the year of publication of the review, and the columns called "No." denote the number of works revised. The total number of works examined by Henkin is 49, even though he published only 46 reviews. This is because some of the published reviews include revisions of more than one work

Δ	No.	Δ	No.
0	5	4	2
1	23	9	1
2	8	13	3
3	6	18	1

Table 4 In the third column, we can see the number of reviews made by Henkin, whereas the fourth gives the average number of quotes in the different reviews

Topic	Subtopic	No.	Quotes
Logic systems		**12**	3.16
	Type theory	3	1.00
	Metalogic	9	3.89
Algebraic logic		**17**	4.12
	Algebraic treatment of logic	12	4.08
	Applications of logic to algebra	5	4.20
Philosophy of logic & math.		**15**	0.93
	Basic logical notions	2	1.50
	Semantic notions	2	2.00
	Alternative logics	4	0.00
	The foundations of mathematics	6	1.47
	Mathematical education	1	0
Manuals		**2**	0.50

Finally, he gives an account of only one paper on mathematical education, even though he published up to 10 major works on the topic,[31] one of his dearest.

7 Reviews by Leon Henkin

The list that follows comprises the papers and books reviewed by Henkin; they have been put in order according to year of publication; we have also included the subjects

[31] "The roles of action and of thought in mathematics education—One mathematician's passage" [77]; with W.N. Smith, V.J. Varineau and M.J. Walsh *Retracing Elementary Mathematics* [138]; "New directions in secondary school mathematics" [109]; "The axiomatic method in mathematics courses at the secondary level" [113]; "Linguistic aspects of mathematical education" [119]; with Nitsa Hadar "Children's conditional reasoning, Part II: Towards a Reliable Test of Conditional Reasoning Ability" [129]; with Robert B. Davis "Aspects of mathematics learning that should be the subject of testing" [127]; with Robert B. Davis "Inadequately tested aspects of mathematics learning" [128]; with D. Blackwell *Mathematics-Report of the Project 2061 Phase I Mathematics Panel* [14] and "The roles of action and of thought in mathematics education—One mathematician's passage" [123].

under which they were listed in the *Journal of Symbolic Logic* (JSL from now on)—volumes 26 (1961) and 45 (1985)—whereas our own classification goes in bold type. The classification in the JSL was elaborated by Church.[32]

Complete references of the works reviewed by Henkin are provided in the bibliography. Our listing of the reviews includes the cross-referencing system that was used in the JSL: "The reviews used a system of cross-referencing by volume number (in roman numerals) and page number (in arabic numerals). For example, Turing's classic paper on computability was II 42—it was reviewed in volume 2, beginning on p. 42" (Enderton [30, p. 175]). Moreover, we have added the publication year and, in square brackets, the number of the corresponding reference in this chapter. Also in square brackets, but after the title of the reviewed work, we include the number for that reviewed work in Sect. 7.

List of Reviews

1. VII 122(1) (1942) [49]: Maxwell Herman Alexander Newman and Alan Mathison Turing, "A formal theorem in Church's theory of types" (1942) [201].
2. XIII 143(1) (1948) [51]: Jean Cavailles, *Transfini et Continu* (1947) [17].
3. XIII 171(2) (1948) [50]: John Charles Chenoweth McKinsey and Alfred Tarski, "Some theorems about the sentential calculi of Lewis and Heyting" (1948) [189].
4. XIV 65(1) (1949) [57]: Stanislaw Jaśkowski, "Sur les variables propositionelles dépendantes" (1948) [150].
5. XIV 66(1) (1949) [56]: Stanislaw Jaśkowski, "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres" (1948) [149].
6. XIV 188(1) (1949) [55]: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1948) [232].
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9. XV 230(1) (1950) [60]: László Kalmár, "Une forme du théorème de Godel sous des hypothèses minimales" (1949) [153]; László Kalmár, "Quelques formes générales du théorème de Gödel" (1949) [152].
10. XVI 53(1) (1951) [63]: Heinrich Scholz, "Zur Erhellung des Verstehens" (1942) [218].
11. XVI 213(1) (1951) [64]: James Kern Feibleman, "Class-membership and the ontological problem" (1950) [32].
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13. XVI 213(3) (1951) [61]: Hugues Leblanc, "On definitions" (1950) [160].
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16. XVII 207(2) (1952) [67]: Kurt Gödel, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory* (1951) [46].

[32] See Enderton [30].

17. XVIII 72(2) (1953) [70]: Helena Rasiowa, “Algebraic treatment of the functional calculi of Heyting and Lewis” (1951, pub. 1952) [205].
18. XVIII 339(3) (1953) [69]: Burton Dreben, “On the completeness of quantification theory” (1952) [28].
19. XIX 219 (1954) [74]: Georg Kreisel, “On a problem of Henkin’s” (1953) [157].
20. XIX 227(1) (1954) [76]: Karl Menger, “The ideas of variable and function” (1953) [193].
21. XIX 227(2) (1954) [75]: Karl Menger, *Calculus. A modern Approach* (1953) [192].
22. XX 78(2) (1955) [84]: Helena Rasiowa and Roman Sikorski, “Algebraic treatment of the notion of satisfiability” (1953) [208].
23. XX 80(1) (1955) [83]: Helena Rasiowa and Roman Sikorski, “On existential theorems in non-classical functional calculi” (1954) [209].
24. XX 184(2) (1955) [81]: Evert Willem Beth, “Sur le parallelisme logico-mathematique” (1953) [8].
25. XX 185 (1955) [79]: Abraham Robinson, “Les rapports entre le calcul déductif et l’interprétation sémantique d’un système axiomatique”; Evert Willem Beth, Luitzen Egbertus Jan Brouwer, and Abraham Robinson, “Discussion” (1953) [215].
26. XX 186(1) (1955) [86]: Paul Bernays, Evert Willem Beth, Luitzen Egbertus Jan Brouwer, Jean-Louis Destouches, and Robert Feys, “Discussion générale” (1953) [6].[33]
27. XX 186(2) (1955) [80]: A. Chatelet, “Allocution d’ouverture” (1953); (3) Luitzen Egbertus Jan Brouwer, “Discours final” (1953); (4) Abraham Robinson, “On axiomatic systems which possess finite models” (1951) [212].[34]
28. XX 281(1) (1955) [85]: Ladislav Rieger, “On countable generalised σ-algebras, with a new proof of Gödel’s completeness theorem” (1951) [211].
29. XX 282(1) (1955) [82]: Gunter Asser, “Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe” (1955) [2].
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31. XXI 194(1) (1956) [90]: Juliusz Reichbach, “O pełności węższego rachunku funkcyjnego”; Juliusz Reichbach, “O polnoté uzkogo funkcional’nogo isčisléniá” (Russian translation); Juliusz Reichbach, “Completeness of the functional calculus of first-order” (English summary) (1955) [210].
32. XXI 372(2) (1956) [88]: Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, and Roman Sikorski, *Der gegenwartige Stand der Grundlagenforschung in der Mathematik* (1955); Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, and Roman Sikorski, *The Present State of Investigations on the Foundations of Mathematics* (1955); Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, and Roman Sikorski, *Sovréménnoé sostoánié isslédovanij po osnovaniám matématiki* (1954) [200].

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[33] This is a really short review. Its text goes: “This is a brief discussion of the following questions. Should semantics be considered as a part of, or as complementary to, symbolic logic? Does the formalization of theoretical physics require the introduction of new logical systems?”

[34] In fact, Henkin lists the three works, but he only reviews Robinson’s.

33. XXII 216 (1957) [94]: Kaarlo Jaakko Hintikka, "An application of logic to algebra" (1954) [147].
34. (1957) [93]: Alonzo Church, *Introduction to Mathematical Logic, Vol. I* (1956) [24].
35. XXIII 33(2) (1958) [96]: Arend Heyting, "Logique et intuitionnisme" (1954) [144].
36. XXIII 34(1) (1958) [97]: Evert Willem Beth, "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" (1955) [9].
37. XXIV 55 (with Andrzej Mostowski) (1959) [136]: Anatolĭ Ivanovič Mal'cév, "Ob odnom obščém métodé polučéniá lokal'nyh téorém téorii grupp" (On a general method for obtaining local theorems in group theory) (1941) [181]; Anatolĭ Ivanovič Mal'cév, "O prédstavléniáh modéléj" (On representations of models) (1956) [182].
38. XXIV 234(2) (1959) [98]: Andrzej Mostowski, "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques"; Daniel Lacombe and Andrzej Mostowski, "Interventions" (1958) [199].
39. XXIV 235 (1959) [99]: Louis Nolin, "Sur l'algebre des predicats"; Andrzej Mostowski, Jean Porte, Alfred Tarski, and Jacques Riguet, "Interventions" (1958) [202].
40. XXV 256(1) (1960) [102]: Jerome Rothstein, *Communication, Organization and Science* with a foreword by C.A. Muses (1958) [216].
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42. XXVIII 174(3) (1963) [110]: Antonio Monteiro, "Matrices de Morgan caractéristiques pour le calcul propositionnel classique" (1960) [196].
43. XXX 235 (1965) [112]: Evert Willem Beth, *Formal Methods. An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic* (1962) [11].
44. XXXII 415(1) (1967) [115]: Marshall Harvey Stone, "Free Boolean rings and algebras" (1954) [223].
45. XXXVI 337(1) (1971) [118]: Maurice L'Abbé, "Structures algébriques suggérées par la logique mathématique" (1958) [158].
46. XXXVI 337(2) (1971) [117]: Marc Krasner, "Les algèbres cylindriques" (1958) [156].

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