Studies in Universal Logic

María Manzano
Ildikó Sain
Enrique Alonso
Editors

The Life and Work of Leon Henkin

Essays on His Contributions

Birkhäuser

Birkhäuser

Studies in Universal Logic

This series is devoted to the universal approach to logic and the development of a general theory of logics. It covers topics such as global set-ups for fundamental theorems of logic and frameworks for the study of logics, in particular logical matrices, Kripke structures, combination of logics, categorical logic, abstract proof theory, consequence operators, and algebraic logic. It includes also books with historical and philosophical discussions about the nature and scope of logic. Three types of books will appear in the series: graduate textbooks, research monographs, and volumes with contributed papers.

María Manzano • Ildikó Sain • Enrique Alonso
Editors

The Life and Work of Leon Henkin

Essays on His Contributions

Birkhäuser

Editors
María Manzano
Department of Philosophy,
Logic and Aesthetics
University of Salamanca
Salamanca, Spain

Enrique Alonso
Department of Logic and Philosophy
of Science
Autonomous University of Madrid
Madrid, Spain

Ildikó Sain
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary

ISSN 2297-0282 ISSN 2297-0290 (electronic)
Studies in Universal Logic
ISBN 978-3-319-09718-3 ISBN 978-3-319-09719-0 (eBook)
DOI 10.1007/978-3-319-09719-0
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014954444

Mathematics Subject Classification: 03-03, 01A65

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Leon Henkin With permission of © George Bergman, Berkeley, 1973 (source: Bildarchiv des Mathematischen Forschungsinstituts Oberwolfach)

Preface

Leon Henkin (1921–2006) was an extraordinary logician and an excellent teacher. His writings became influential from the very start of his career with his doctoral thesis, *The completeness of formal systems*, defended in 1947 under the direction of Alonzo Church. He then published two papers in the *Journal of Symbolic Logic*, the first, on completeness for first order logic, in 1949, and the second devoted to completeness in type theory, in 1950. He applied the same basic idea to a generalization of ω-consistency and ω-completeness; by similar methods he proved a completeness theorem using Craig's interpolation lemma. Not only Henkin extended his method to other logics, it has become a standard procedure thereafter in all kinds of logics, not just the classical ones.

His main interests were logic and algebra. In particular, Henkin devoted a large portion of his research to algebraic logic. The theory of *Cylindric Algebras* was studied by Leon Henkin, Alfred Tarski, and Donald Monk during an extended period, and their discoveries provided a class of models which are to first-order logic what Boolean algebras are to sentential logic. Further research was taken up by Istvan Németi and Hajnal Andréka.

In some of his publications Henkin went into the philosophy of mathematics, taking a nominalistic position. Another issue with an ample historical precedent is the search for a system that takes the *identity* relation as the sole primitive constant. He also constructed a special theory of propositional types having lambda and equality as the only primitive signs and proved the completeness of that logic.

Several of his publications, dealing with elementary concepts, fall under the category of mathematical education; we believe that his work *On mathematical induction* was the result of his devotion to mathematical education. Between 1957 and 1972, Henkin divided his time between his mathematical research work and enquiries into the teaching of mathematics.

Answering an invitation from Alfred Tarski, Henkin joined the Mathematics Department in Berkeley (University of California) in 1953. When Tarski and Henkin were able to assemble a number of logicians from the Department of Mathematics and Philosophy, they created an interdepartmental agency, the very famous *Group in Logic and the Methodology of Science*. He stayed with the Department until 1991, when he retired and became an Emeritus Professor.

Henkin was often described as a social activist, he labored much of his career to boost the number of women and underrepresented minorities in the upper echelons of mathematics. He was also very aware that we are beings immersed in the crucible of history from which we find it hard to escape.

The editors felt that it was about time for a comprehensive book on the *The Life and Work of Leon Henkin* and we sent a proposal to Springer Basel to be published in *Studies on Universal Logic*. We are grateful to the Series Editor Jean-Yves Béziau for accepting our proposal. It seems that the Logic community shared a similar feeling and the response to our call was incredible. We wish to thank all the contributors for their interesting papers, including personal and historical material. We are sure their contributions provide a foundational logic perspective on Henkin's work.

This book is dedicated to Leon Henkin, a creative and influential logician who has changed the landscape of logic, while also being a highly principled, generous and humble human being. He was and continues to be our mentor.

Henkin's Investigations and Beyond

Two parts of this book are devoted to Henkin's work, where we include chapters on:

Algebraic Logic by Donald Monk, *Leon Henkin and Cylindric Algebras*. Cylindric algebras are abstract algebras which stand in the same relationship to first-order logic as Boolean algebras do to sentential logic. Monk analyzes Henkin's contributions on the subject and also includes a section on *Publications of Henkin Concerning Cylindric Algebras*. Henkin devoted a large portion of his works to this theory, many of them having Tarski and/or Monk as coautors.

Completeness Two chapters are directly devoted to this issue, *Henkin on Completeness* and *Henkin's Theorem in Textbooks*, by María Manzano and Enrique Alonso. In the first one, four of his completeness proofs are analyzed—namely, for first-order logic, type theory, propositional type theory and first-order logic using interpolation—as well as the proofs Henkin used to explain in class; the paper ends with two expository works that Henkin devoted to the same issue (*Truth and Provability* and *Completeness*). The aim in the second paper is to examine the incorporation and acceptance of Henkin's completeness proof in some textbooks on classical logic. All of these textbooks come from the Anglo-Saxon tradition and were published before the beginning of the 1980s.

General Models *Changing a Sematics: Opportunism or Courage?* by Hajnal Andréka, Johan van Benthem, Nick Bezhaishvili, and Ístvan Németi. This paper gives a systematic view of generalized model techniques, discusses what they mean in mathematical and philosophical terms, and presents a few technical themes and results about their role in algebraic representation, calibrating provability, lowering complexity, understanding fixed-point logics, and achieving set-theoretic absoluteness. It is also shown how thinking about Henkin's approach to semantics of logical systems in this generality can yield new results, dispelling the impression of ad-hocness.

Propositional Type Theory Peter Andrews wrote *A Bit of History Related to Equality as a Logical Primitive*. This historical note illuminates how Leon Henkin's work influenced that of the author. It focuses on Henkin's development of a formulation of type theory based on equality, *A Theory of Propositional Types*, and the significance of this contribution.

A Problem Concerning Provability is the title of a problem posed by Henkin in 1952—in the Problem Section of the *JSL*. Volker Halbach and Albert Visser contributed to this volume with *The Henkin Sentence*. They discuss Henkin's question concerning a formula that has been described as expressing its own provability. They analyze Henkin's formulation of the question and the early responses by Kreisel and Löb, and sketch how this discussion led to the development of Provability Logic.

Beyond Henkin's Method The chapter by Robert Goldblatt, *The Countable Henkin Principle*, presents a general result about the existence of finitely consistent theories which encapsulates a key aspect of the "Henkin method". The countable version of the principle is applied here to derive a variety of theorems, including omitting-types theorems, and to strong completeness proofs for first-order logic, omega-logic, countable fragments of languages with infinite conjunctions, and a propositional logic with probabilistic modalities. The paper concludes with a topological approach to the countable principle.

Henkin's Method The chapter by Franco Parlamento, *Henkin's Completeness Proof and Glivenko's Theorem*, also deals with Henkin's method. The author observes that Henkin's argument for the completeness theorem yields also a classical semantic proof of Glivenko's theorem and leads in a straightforward way to the weakest intermediate logic for which that theorem still holds. He also outlines and comments on its application to the logic of partial terms, when "existence" is formulated as equality with a (bound) variable.

Finite Type Theory William Gunther and Richard Statman contributed with *Reflections on a Theorem of Henkin*. In a full type structure with a finite ground domain, say $\mathscr{M}^n$, it is folklore that a member of $\mathscr{M}^n$ is symmetric if and only if it is definable in type theory. A proof follows immediately from an observation due to Leon Henkin in *A Theory of Propositional Types*. In this note the authors generalize the folklore theorem to other calculi and also they provide a straightforward proof of that theorem itself using only simple facts about the symmetric group and its action on equivalence relations.

Extending Completeness in the Theory of Types to Fuzzy Type Theory Vilém Novák contributed with *From Classical to Fuzzy Type Theory*. Mathematical fuzzy logic is a special many-valued logic whose goal is to provide tools for capturing the vagueness phenomenon via degrees. Both propositional and first-order fuzzy logic were proved to be complete, but only in 2005. Also, higher-order fuzzy logic (called the Fuzzy Type Theory, FTT) was developed and its completeness with respect to general models was proved. The proof is based on the ideas of the Henkin's completeness proof for TT.

Many-Sorted Logic In the chapter *April the 19th*, María Manzano credits most of the ideas involved in her translation of a variety of logics into a many-sorted framework to Henkin's paper of 1950 *Completeness in the Theory of Types* and to his paper of 1953, *Banishing the Rule of Substitution for Functional Variables*. April the 19th, 1996, was Henkin's 75 birthdate, but also the date when María was going to give a talk in Berkeley to explain how the ideas of Henkin were applied in her book *Extensions of First-Order Logic*.

Applying Henkin's Method to Hybrid Logic In *Henkin and Hybrid Logic*, by Patrick Blackburn, Antonia Huertas, María Manzano and Klaus Frovin Jørgensen, the authors explain why Henkin's techniques are so important in hybrid logic, an extension of orthodox modal logic in which special proposition symbols are used to name worlds. These symbols, called nominals, allow Henkin's witnessing technique to be applied in modal logic. As a result, in higher-order settings the use of general interpretations and the construction of type hierarchies can be (almost) pure Henkin. The paper ends with the words: *'Hybrid logic? It's a suggestive name. But it could without exaggeration be called: Henkin-style modal logic'*.

Reviews for The Journal of Symbolic Logic Concepción Martínez Vidal and José Pedro Úbeda Rives contribute with *Leon Henkin The Reviewer*. In this chapter they look at the minor, though abundant, works by Henkin; namely his reviews—a total of forty six—and other minor papers. They sort the reviews into four categories and provide a brief summary as well as an analysis of each of them. This analysis reveals Henkin's personal views on some of the most important results and influential books in his time. Finally, they relate these reviews and minor works to Henkin's major contributions.

Mathematics Education Nitsa Movshovitz-Hadar contributes with *Pairing Logical and Pedagogical Foundations for the Theory of Positive Rational Numbers*. Five different ways of "founding" the mathematical theory of positive rational numbers for further logical development are presented. This is done as Leon Henkin outlined in 1979 in the form of notes for a future paper, suggesting that pairing them up with five representation models could possibly lead to further pedagogical development. His wish was to explore how varying modes of deductive development can be mirrored in varying classroom treatments rooted in children's experience and activities. This dream of Henkin never came to full fruition.

Leon Henkin Up Close and Personally

Some of the contributions to this volume fall under the category of very personal writings, describing the relationship the authors have maintained with Leon Henkin. We also include a biographical chapter on Henkin's life and work with some suggestions on future research.

Arithmetization of Metamathematics Solomon Feferman contributes with a very personal paper, *A Fortuitous Year with Leon Henkin*, in which he tells us how his doctoral thesis changed when his advisor Tarski went on sabbatical leave and Henkin acted as supervisor. *'What if Henkin had not been in Berkeley to act in his place? In fact, none of the "what ifs" held and I am eternally grateful to Leon Henkin for his being there for me at the right place at the right time'*.

Henkin and the Suit Albert Visser tell us a funny story concerning a young obedient respectful student (himself) during a short visit Leon and Ginette paid to Amsterdam. The story ends with these words: *'This is all well and good, but what does the story teach us about the Henkins? The story makes clear that the Henkins were very open and friendly* *people. They easily gained someone's trust. The fact that I told them the story during the dinner is the best possible testimony of that.'*

Lessons from Leon Diane Resek, one of his doctoral students, describes Leon as a human being. In her words, *'Leon was the least intimidating of the professors I met at Berkeley'*. She highlights his love for good writing and his strength as a lecturer, and she concludes saying: *'In general, Leon's recognition of values in life beyond pushing for the best and strongest theorems set him apart from many mathematicians at Berkeley.'*

Tracing Back 'Logic in Wonderland' to My Work with Leon Henkin Nitsa Movshovitz-Hadar describes in her personal relationship with Leon, who was her thesis advisor, defended in 1975. She ends the contribution with these words: *'I feel very fortunate to have been his graduate student as I learned from him much more than logic. It is his humanity that conquered my heart.'*

Leon Henkin and a Life of Service The contribution by Benjamin Wells is entirely of biographical character. The author includes several letters from Henkin and some personal communications held between them. He analizes some historical events surrounding some classical discoveries, including the controversy around some of the Mal'tsev's contributions.

Biographical Note *Leon Henkin*, by Maria Manzano and Enrique Alonso, provides some key biographical notes to Henkin's life. It guides readers through Henkin's formative period and his most outstanding logical contributions, but also discusses his other interests, most notably his dedication to mathematical pedagogy and minority-group education.

Salamanca, Spain — María Manzano
Madrid, Spain — Enrique Alonso
Budapest, Hungary — Ildikó Sain
July 2014

Contents

Contributors

Enrique Alonso Department of Logic and Philosophy of Science, Faculty of Philosophy, Autonomous University of Madrid, Madrid, Spain

Peter B. Andrews Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

Hajnal Andréka Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

Nick Bezhanishvili Institute for Logic, Language and Computation (ILLC), University of Amsterdam, Amsterdam, The Netherlands

Patrick Blackburn Roskilde Universitet, Roskilde, Denmark

Solomon Feferman Department of Mathematics, Stanford University, Stanford, CA, USA

Robert Goldblatt School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, Wellington, New Zealand

William Gunther Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

Volker Halbach New College, Oxford, England

Antonia Huertas Universitat Oberta de Catalunya, Barcelona, Spain

Klaus Frovin Jørgensen Roskilde Universitet, Roskilde, Denmark

María Manzano Department of Philosophy, Logic and Aesthetics, Faculty of Philosophy, Edificio FES, University of Salamanca, Salamanca, Spain

Concha Martínez Vidal Department of Logic and Moral Philosophy, Faculty of Philosophy, University of Santiago de Compostela, Santiago de Compostela, Spain

J. Donald Monk Mathematics Department, University of Colorado, Boulder, CO, USA

Nitsa Movshovitz-Hadar Technion, Israel Institute of Technology, Haifa, Israel

István Németi Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

Vilém Novák Institute for Research and Applications of Fuzzy Modelling, National Supercomputing Centre IT4 Innovations, University of Ostrava, Ostrava 1, Czech Republic

Franco Parlamento Department of Mathematics and Computer Science, University of Udine, Udine, Italy

Diane Resek Department of Mathematics, San Francisco State University, San Francisco, CA, USA

Richard Statman Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

José Pedro Úbeda Rives Department of Logic and Philosophy of Science, Faculty of Philosophy and Education Sciences, University of Valencia, Valencia, Spain

Johan van Benthem Institute for Logic, Language and Computation (ILLC), University of Amsterdam, Amsterdam, The Netherlands

Albert Visser Philosophy, Faculty of Humanities, Utrecht University, Utrecht, The Netherlands; Philosophy, Faculty of Humanities, Utrecht University, Utrecht, The Netherlands

Benjamin Wells Departments of Mathematics and of Computer Science, College of Arts and Sciences, University of San Francisco, San Francisco, CA, USA

Part I
Biographical Studies

Leon Henkin

María Manzano and Enrique Alonso

Abstract Leon Henkin was born in 1921 in Brooklyn, New York, in the heart of a Jewish family that originally came from Russia. He died at the beginning of November in 2006. He was an extraordinary logician, an excellent teacher, a dedicated professor, and an exceptional person overall. He had a huge heart and he was passionately devoted to his ideas of pacifism and socialism (in the sense of belonging to the left). He not only believed in equality, but also worked actively to see that it was brought about.

Keywords History of logic · Biographical studies of mathematicians · Foundational studies in logic · Henkin

1 Life[1]

Leon Henkin was born in 1921 in Brooklyn, one of the five boroughs of New York City, in the heart of a Jewish family that originally came from Russia. His second name was Albert, but he never used it in publications. In a footnote to one of his biographical papers, *The Discovery of my Completeness Proofs*, Henkin tells the story of his middle name:[2]

> In fact, he (father) had shown his high expectations for me at the time of my birth by choosing my middle name to be "Albert". He once told me that at that time (April 1921) the *New York Times* had run a series of articles publicizing Einstein's revolutionary theory of relativity, so my father decided to borrow Einstein's first name for his newborn son.

Leon's oldest uncle Abraham was the first in the family to emigrate from Gomel, in White Russia, and became a doctor in the USA. After saving some money, he was able to bring Leon's father to the States and later he brought Leon's grandmother and aunts as well.

In 2009, during a short visit to California, I (María Manzano) had the happy opportunity to interview Ginette Henkin, Leon's wife. Steve Givant and I visited her at her Oakland apartment and she drove us to a restaurant where we had lunch. Ginette told us that Leon belongs to a very close-knit family: in the first place because they belong to a tradition where family relationships were important, but also because they were immigrants and the new families they formed were also related by family links, as two brothers

[1] Information for writing this section was mainly provided by Leon's family: his wife Ginette, his oldest son Paul, and his youngest son Julian. Unfortunately, Ginette passed away in June 2012. Other sources are [1] and [39].

[2] In [29, p. 150, footnote 30].

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_1

married three sisters. Leon's oldest uncle married one sister and Leon's father married another sister; when Leon's mother passed away, his father married a third sister. These women were also immigrants from White Russia whose father was a doctor. In Leon's son Paul words [32] *'Perhaps all these doctors explain his lifelong fascination with health and dentistry. Well into his eighties, he would exercise…'*.

He lost his mother shortly after the birth of her daughter Estelle, almost five years younger than him. Leon was very close to his cousins, they lived nearby, and grew up together in Brooklyn. There are several stories in the memorial documents ([32] and [10]) referring to playing in the streets with other kids. His son Paul told us [32], that *'In Brooklyn, he was nearly run over by a milk truck while playing stickball in the street.'* Most of his early education was in the NY public schools; he went to Lincoln High School, which produced a lot of mathematicians that year.

Leon's family was socialist, they were all supporters of Norman Thomas, an American minister who achieved fame as a socialist, pacifist, and six-time presidential candidate for the *Socialist Party of America*. The family was not religious, and even the grandfather (who was a Hebrew scholar) was not very observant, but they strongly kept the Jewish tradition. It seems that the family rigorously observed the Passover Seder, probably because the Seder is conducted in the family home, while other Jewish holidays revolve around the synagogue. On such occasions, it is customary to invite guests, in particular strangers and the needy.

In the aforementioned interview, Ginette gave us a nice headline concerning how she met Leon the first time: *Two couples met at same weekend.* The two couples were Leon and Ginette Henkin and Harold and Estelle Kuhn. The story goes like this: Leon and Estelle travelled to Montreal for the September Labor Day weekend (1948); Leon was driving a friend (Harold Kuhn) from gradschool who wanted a lift and so he introduced Harold to Estelle. The three of them and Maurice L'Abbé went to Montreal; during their stay there Leon met Ginette. Estelle, L'Abbé, Ginette, Leon and several others had dinner together and went dancing afterwards. In December that year, Ginette took a first plane ride to NY (Leon invited her) and she stayed in father's and Estelle's place. Ginette was 25 when they married in 1950; she was a Catholic with a western style education. According to David Gale [10], a colleague and co-student at Princeton University, *'In graduate school he was well known for being "anti-marriage". "I oppose it in all forms" he told me when I asked him about his views on the subject.'* However, after the mentioned trip to Montreal he had a life changing experience. *'One day shortly after returning to Princeton we were talking about this and then to my surprise Leon said "Maybe I'll get married and see what that looks like". As we know, he did and from all indicators it turned out to look pretty good!'* Ginette was a wonderful person and an excellent cook, as we can attest. Ginette and Leon had two adopted sons, Paul and Julian, in 1957 and 1959. Julian said: '*I would characterize my father as the most supportive parents and persons I have known.* […] *My father was a very good driver. He could handle long driving shifts on family trips.*' Julian also said that Leon forced them to learn, as he always was in a teacher role, thus '*It would not surprise me to know that all of his students have that Leon-instilled confidence*' [10].

On several occasions Leon attended his children's school to '*introduce us six-year-olds to negative numbers*' or '*subtraction by addition*'. One classmate of Leon's son Julian told us: *'The first thing Mr. Henkin asked was whether it was difficult for us to remember to carry a number in addition and to borrow a number in subtraction. We second graders, responded, "Yes". Mr. Henkin went on to say that we should not worry about that because* *he was going to teach us a way to do subtraction without borrowing. It was doing subtraction by addition.'* Henkin's enthusiasm and understanding of just how much to challenge younger age groups was a rare gift.

Paul, in his *Proposed Memorial Oration*, also told us about a terrible experience the family had to face: *'In 1969, I had a brain tumor removed and all sorts of complications, and barely survived, but my parents poured time, resources and energy into my care. Meanwhile, my mother supported my father's career in all sorts of ways, from giving dinners and parties for academics and students alike to working on faculty groups distributing supplies to foreign and minority students.'*[3]

When Leon had his own family, they used to celebrate Passover. Bill Henkin (Jr.), his first cousin once removed said: *'I actually learned a great deal about the purpose of religion from his approach to Pesach because his attention at this holiday was not on a capital G—God—of any sort, his concern was clearly for the people around him at those times, as well as for people who could not be present, and he enrolled us all in his approach and thoughts as he enrolled students in mathematics in his other, better-known roles as an educator.'* According to Cliff Kuhn, Leon's nephew, the eldest son of his sister Estelle, *'He was THE central bridge in our family, reaching across continents and generations to make and strengthen family connections, and serving as custodian of the family's history.'* In Nicholas Kuhn's, the middle son of Estelle, words: *'I own two books given to me by my uncle Leon. The first is "Geometric Algebra" by Emil Artin.* [...] *The second book is "The Apted Book of Country Dances"*.[4]

Leon and Ginette were always happy to invite students and colleagues to their home. Their broad interests in politics, education, and arts, plus good wine and excellent cuisine always made the stay a memorable one. They first lived in an apartment on Virginia Street and later on moved to a house on Maybeck Twin Dr. in Berkeley. As some colleagues attested: '[...] *in 1955, the Henkins were among the first people in the math department to invite us to their home* [...]. *I was much impressed with the all-white décor of their livingroom and diningroom, and with the charm and sophistication of Leon and Ginette.'* I (María Manzano) was invited on two unforgettable occasions to his simple, vanguard, and minimalist home in Berkeley Hills, decorated with exquisite *objects d'art*, among which those from American Indian cultures were outstanding. The view from his living room was breathtaking, with a gorgeous sight of the Golden Gate of San Francisco. Henkin actually died in Oakland, since some years previously he had to move there with his wife Ginette, leaving behind his beautiful home.

I could also appreciate the couple's good taste during the preparation of their visit to Barcelona. The *Hotel Colón* was chosen by me as a venue and the couple agreed to this since Ginette and Leon preferred a local touch: *'Both of us share your preference for a place with distinctive Spanish character, rather than the ugly, commonplace international style that infects big cities'*.

Leon's fondness for dancing, both ballroom and modern, is well known. According to Ginette, Leon was extremely shy when he was young. At that time boys and girls only met on weekends to dance; as there were more girls than boys, he was forced to dance. When he connected music and steps he became a dance lover and remained one all his

[3]See [32]. The *Faculty wives equipment loan center* is a very useful organization. In 1977, when María Manzano was in Berkeley with a Fulbright grant, she was able to loan dishes and sheets from them.

[4]In this paragraph, all the quotes are taken from [10].

life. In some of the anecdotes compiled at his death, *Many views of Leon Henkin*, some people refer to this. His son Paul ended his *Proposed Memorial Oration* with these words: *'In closing, I must make mention of his love of dance. Music too. One of my last memories of him was his feet trying to dance under the covers.'*

Ginette told us *'He used to go solo camping for a week, started this around age 60. But one year he got lost around Mammoth Lakes and he never went solo camping again after that.'*

Henkin was also interested in literature. In 1983, in a personal letter to me, in reply to me praising the novel *Rabbit is Rich* by the newly awarded Pullitzer Prize winner John Updike, Henkin writes: *'I am glad you enjoyed the language of Updike's writing. I took a volume of his stories on my camping trip last summer, and I admired his writing very much. I still have my school-boy dream of becoming a story writer, but I'm aware I could never achieve such a master style as Updike.'* Ginette gave me a short story written by Leon, *A Walk*, in which a fifteen-year old Leon takes a memorable walk for the first time in his life. *'So far as he remembered he had never done such a thing before. Until the age of twelve, he had played in the street, after school, with the other boys.'* During the walk he saw three girls, the blonde hair intrigued him, she and the picture of a certain tree remained in his mind. *'No, he did not sleep. He lay in bed, with his eyes wide open, thinking, staring, far into the night.'*

Leon was always on the chess team in high school and college. He avoided games of chance, preferring to encourage thinking. He liked blindfold chess, go, and scrabble. He had speech trouble with r, so while he was at Columbia the family sent him to speech therapy. The difficulty disappeared, but he had a slow and peculiar pronunciation. Anyway, he was such an eloquent, creative, and persuasive teacher that this slight oddity became unnoticeable as soon as he went on talking.

Henkin was a politically committed person, he was passionately devoted to his ideas of pacifism and socialism (in the sense of belonging to the left). Paul told us that early in the fifties he was participating in political groups: *'Next year, he became a professor of math at USC, and he stayed in California thereafter. There, he also became involved in a political club where he was beaten for president by Jess Unruh, the future Speaker of the Assembly.'*[5]

Henkin did not wish to visit Spain during the Franco era, but after November of 1975 this was fortunately no longer the situation. His first trip was in 1982 and he visited several Spanish Universities, such as Barcelona, Madrid and Seville. He wished to know all about Spain, especially its social and political developments, and in 1982 he commented: *'Yes, we too were very pleased with the clear and strong victory of the Socialist Party in Spain.* [...] *we were worried about the role of the military, and relieved that they did not interfere'*. In turn, I (María) was always informed about the developments in his country: *'Our own election in California was a disaster, with a new Governor and Senator, each more conservative than the other. The State is in a fiscal crisis and I'm afraid the University will be in for a rough time...'*. He bewailed the absence of international news in the American press and in July 1984 he wrote: *'I recall your Spanish election in the week following our visit in 1982. I hope that your new government is working out well...we get little news of Spain in our journals.'* In the same letter, he told me that for the first time in American history a woman

[5]See [32].

had been nominated as Vice-President of the Democratic Party. However, he was saddened by the fact that Reagan might win the election again, as in fact happened. Today he would possibly have been thrilled by the role played by Barack Obama and Hillary Clinton.

According to Ginette, Leon's life was the university, it was his baby. In fact, when he died he left some money to the university. Ginette also said that he wanted to be an academic early in his life, partially because he didn't want to think about money.

He died at the beginning of November in 2006, as we are told by mutual friends, from the same cause as the great mathematician Eratosthenes of Cyrene.

In the obituary in the *San Francisco Chronicle* (Monday, November 20, 2006 [6]), the logician J.W. Addison, at that time an emeritus professor in the same department of the University of California at Berkeley, we read: *'You could say he was an academic triple threat—very strong in teaching, very strong in research, very strong in administration.'*

What kind of effect did Leon Henkin have on the world? That was the main question we asked ourselves. To answer it, we have investigated not only his research, but also his teaching, as well as his contribution to the development of logic in a broader sense. He made great efforts to disseminate mathematics and logic to a wider circle of students. As far as we are beings immersed in history, we have tried to reproduce his mathematical and social environment in order to contextualize his contributions.

In the following sections we discuss Henkin's academic life as well as his social activism.

2 Graduate Studies at Columbia University

Between 1937 and 1941, Henkin pursued his undergraduate studies at Columbia College in New York City obtaining a diploma in mathematics and philosophy. His father then moved uptown because he didn't want Leon to commute. Henkin lived in this new home for four years before going to Princeton, and it was during that time when he became interested in logic. Let us quote the first two paragraphs of his 1962 paper, *Are Logic and Mathematics Identical?*, which won the prestigious Chauvenet Prize in 1964.

> It was 24 years ago that I entered Columbia College as a freshman and discovered the subject of logic. I can recall the particular circumstance which led to this discovery.
> One day I was browsing in the library and came across a little volume by Bertrand Russell entitled *Mysticism and Logic*. At that time, barely 16, I fancied myself something of a mystic.
> [. . .]
> Having heard that Russell was a logician I inferred from the title of his work that his purpose was to contrast mysticism with logic in order to exalt the latter at the expense of the former, and I determined to read the essay in order to refute it.[6]

While at Columbia, Henkin took a course in logic with Ernest Nagel in the Dept. of Philosophy, and this led him to become interested in the field to the point where he even read Russell's *Principles of Mathematics*. It was in that book that he first have read about the axiom of choice, and he tells us that he was impressed by the amusing and intimate way Russell used to explain it, contrasting how easy is to choose a shoe from an infinite collection of pairs of shoes, with how difficult it is to choose a sock from an

[6] See [21].

infinite collection of pairs of socks. Russell's book inspired him to read the *Principia Mathematica* of Russell and Whitehead [38], and he became infatuated with the theory of types and the axiom of reducibility. He then took another course on logic with Nagel, during which he read an article by Quine, containing a proof of the completeness theorem for propositional logic. Let us quote him:[7] *'As to the concept of completeness which was the focus of Quine's paper, it did not get through to me. I simply noted that the aim of the paper was to show that every tautology had a formal proof* [...] *the result was that I failed to get "the idea of the proof", the essential ingredient needed for discovery.'*

He also had the chance to listen to a lecture by Tarski concerning Gödel's work on undecidable propositions in type theory. (Tarski travelled to America in August of 1939 to attend the Unity of Science Congress being held at Harvard that year. The German invasion of Poland on September 1 prevented him from returning home.) However, Henkin's first deeper contact with the work of Gödel arose from a sort of reading seminar he took with F.J. Murray—a collaborator of von Neumann—to study Gödel's results on the consistency of the continuum hypothesis. Henkin said:[8] *'As far as I can recall, Murray and I had one or two meetings to discuss the scope and the beginning of the work, and then he found himself too busy with other projects and I was left to work through Gödel's monograph on my own.'* At that time, Henkin had still not graduated from Columbia, but he was the only student who seemed to be interested in these issues, and who was prepared to invest time and energy to study them.

Upon finishing his studies at Columbia, he applied for admission to several universities where logic was well established: Harvard (where Quine was), Princeton (where Church was) and Columbia. Nevertheless, as he stated during an interview held much later (on May 18, 1984) in Berkeley.

> Those are the places [Harvard, Princeton, and Columbia] to which I made application. I was accepted at all places, but Harvard did not offer any financial help. Both Princeton and Columbia did. My Columbia professor said, "Well, you've been around here. You know, you've learned from us. Here is this exceptional logician, Alonzo Church, at Princeton. Why don't you go there?" So I did. Happily ever after, as they say.[9]

3 Doctorate Studies with Alonzo Church

The Completeness of Formal Systems[10] is the title of the doctoral dissertation that Henkin wrote at Princeton in 1947 under the direction of Alonzo Church. Chapter "Henkin on Completeness" of this book is devoted to this topic, so we will limit ourselves here to a discussion of his previous training in logic under the supervision of Church.

It was at Princeton University that Henkin followed his master's and doctorate studies, although between the two, he worked on the famous Manhattan project, an initiative of the United States government with the collaboration of Canada and the United Kingdom.

---

[7] In [29, pp. 129–130].

[8] In [29, p. 131].

[9] See [36].

[10] The thesis was never published, but the results were published in several papers; the more directly related are [11], and [12].

During his initial term at Princeton, he followed a course in logic given by Church in which both propositional and first-order calculi were studied and normal form and completeness theorems were proved for them, and the Löwenheim–Skolem results were analyzed. The completeness proof was that of Gödel (Henkin's, obviously, still did not exist) and the *reductive nature* of the proof was remarked on—namely, that the problem of completeness for first-order logic was reduced to that of completeness for sentential logic using Skolem's normal form. During the second semester, a second-order language was studied and, in particular, Peano Arithmetic was introduced in great detail, and the results of incompleteness were proved, both for arithmetic and second-order logic. To prove incompleteness, recursive functions were introduced, although only the primitive ones. General recursive functions were not studied, but the role they play in the proof of certain results of undecidability was mentioned. Henkin tells the reader in [29] that although the content of the course was not, in the least surprising, what was striking was the style of Church, the way he had of transmitting his conception of logic. It appears that he would make frequent halts in his discourse to clarify the idea that he was following the *"logistic method"*: clearly delimiting what was language and what was metalanguage; stressing how the formal language should be established in a completely effective way, and why the metalanguage (English, in his case) should be limited. One can gain an idea of Church's style by reading his book *Introduction to Mathematical Logic, vol I*.[11]

The Frege theory about the notions of *sense and denotation*[12] were the topic of a seminar that Henkin followed with Church when he (Henkin) returned from his four years of service. Henkin affirms that in that seminar Church convincingly defended the notion that in addition to the formal language and the universe of mathematical objects in which we interpret the formulae of the language, there is a third dimension of abstract objects, namely concepts or senses. A sentence expresses a proposition but also names a truth-value. Henkin writes: *'Under this theory a symbolic expression functioning as a name denotes an object of the universe of discourse, and expresses some sense of that object; a sentence is construed as a name of its truth value, and the sense it expresses is called a proposition.'*[13]

3.1 The Princeton Mathematics Community in the 1930s

In an interview from the series entitled *"The Princeton Mathematics Community in the 1930s"* [36], Henkin recounts an amusing anecdote about his years at Princeton. Professor Alfred Tucker asked him to create a sort of disturbance on the last day of the academic year: *'Like every great teacher he wanted some dramatic incident to imprint the course on the minds of the young students."*

So Henkin started dancing around and contorting himself before his classmates, whose eyes bulged because they were unaware of the theatrical nature of the event and were

[11] The second volume, whose index appears in the first, was never published, although some of its chapters were circulated among his students.

[12] Probably Church explained in class what was to become his paper [5] on that subject.

[13] See [29, p. 142].

unable to understand his lack of respect towards his teacher. He ended up by removing his waistcoat without taking off his jacket!

There are other interesting anecdotes in these interviews ([36] and [37]). Here is one that had to do with a conflict he felt about two different approaches to teaching: presenting the topics clearly to students or forcing them to make an effort: '*That effortless way in which the ideas came made them too easy to slip away. I probably learned more densely packed material from what we called the "baby seminar", in point set topology conducted by Arthur Stone. I learned more because he made us do all the work.*'

In relation to that same seminar, he describes the following exchange with Professor Lefschetz, which reveals how self-confident Henkin was, even as a graduate student:

> I was giving my solution to one of the problems that Arthur Stone has set to me before, and being a logician I wanted to make all the details very clear and Lefschetz became impatient. As I got into some of those details he said, "Well, that's all obvious. Just go on toward the end." I was a very brash young man. I said, "Professor Lefschetz, it may be obvious to you, but I have come from an environment where a proof requires us to give all the details." And I just went ahead.

3.2 The Second World War

In an interview from the above mentioned series, Henkin recounts how the atmosphere of relative calm with respect to WWII changed radically towards the end of 1941, when America entered the conflict after the bombing of Pearl Harbor on December 7, 1941. He tells us that he came upon Mrs Eisenhart in the street, and we learn that in their conversation she said: *'We must all do our duty and get on with it.'*

Henkin says that very soon everybody was expressing similar opinions. He also mentions that his professor, Herman Weyl, decided not to change his work schedule and that he (Henkin) was positively impressed by this:

> I also remember that I had a lecture by Hermann Weyl that same morning, Monday the 8th. It was 9:00. He said, "I know that all of you are very excited and upset and cannot let go of these great world events that have engulfed us". But, he said, "I've learned from my experience that in the most tempestuous of times, there is a great value in giving some of your attention and your energy to your continuing work". Therefore, he said, "I am just going to give the regular lecture now that I planned with you last week." So he did, and I think there is something of real value in those opening remarks.[14]

Like many scientists, Henkin felt that he had to be committed and he worked for four years on the Manhattan project. The project gathered together a large number of eminent scientists—including exiled Jews, pacifists and people on the Left—, many of whom joined the cause against fascism and contributed to the mission of developing an atomic bomb before the Germans. As he tells us: *'During the period May, 1942–March, 1946, I worked as a mathematician, first on radar problems and then, beginning January 1943, on the design of a plan to separate uranium isotopes. Most of my work involved numerical analysis to obtain solutions of certain partial-differential equations.'*[15]

[14] See [36].

[15] In [29, p. 132, footnote 11].

4 Service to the Department and to the University

Tarski arranged for Henkin to be offered a tenured position in the Mathematics Department of the University of California at Berkeley in 1952, but Henkin refused the offer because of the controversial loyalty oath that was required of all professors at the University of California at that time.[16] When the oath was no longer required in 1953, Henkin accepted Tarski's offer and went to Berkeley. He valued the role of Tarski very highly as regards his own decision to come to Berkeley. On 30 October 1983, in a personal communication to María Manzano he wrote: *'I write to tell you that Alfred Tarski, who came to Berkeley in 1942 and founded our great center for the study of logic and foundations, died Wednesday night (Oct. 26), at age 82. All through this year he has been getting weaker; his wife Maria worked heroically to comfort and protect him, but finally he gave up his life* [...] *It was he who brought me to Berkeley in 1953, so I owe much to him personally as well as scientifically.'*

Feferman [7] explains how Tarski's general conception of logic, as the quintessential interdisciplinary subject, urged him to campaign on behalf of it from his base at the University of California in Berkeley.

> The first order of business was to build up a school in logic bridging the university's Mathematics and Philosophy Departments, and the opening wedge in that was the hiring of Leon Henkin as Professor of Mathematics in 1953. From then on, Henkin was Tarski's right-hand man in the logic campaigns, locally, nationally and internationally, but he had other allies, both in Mathematics and in Philosophy. The first goal was to increase the proportion of logicians on the mathematics faculty to 10 % of the whole; that took a number of years, eventually achieved with the appointment of Addison, Vaught, Solovay, Scott, Silver, Harrington and McKenzie. Through his influence in Philosophy, he succeeded in recruiting Myhill, Craig, Chihara and Sluga.[17]

Assembling together a number of logicians, mathematicians, and philosophers from the departments of mathematics and philosophy, Tarski and Henkin created an interdisciplinary group called the *Group in Logic and the Methodology of Science*. In 1957, they initiated a pioneering interdisciplinary graduate program leading to the degree of Ph.D. in *Logic and the Methodology of Science*. They were able to organize several very important meetings on logic in the Bay Area: the conference on *The Axiomatic Method in Mathematics and Physics* at U.C. Berkeley in 1957; the *First International Congress for Logic, Methodology and Philosophy of Science* that Tarski presided over at Stanford University, in 1960; and the very important *Theory of Models* conference at Berkeley in 1963. Henkin was the driving force behind the organization of the Tarski Symposium at Berkeley in 1971, honoring Tarski on the occasion of his 70th birthday. In 1983, a meeting of the Association of Symbolic Logic was held at Berkeley, and Alonzo Church gave an invited talk on intensional logic: *'a subject he was beginning to study as I was finishing my studies with him in Princeton, some 35 years ago'*, Henkin told María Manzano in a private letter. Feferman says: *'In a sense the Logic and Methodology congresses are an intellectual descendant of the Unity of Science movement, but now with logic at the center stage, very much in tune with Tarski's conception of logic as a common basis for the whole of human knowledge'*.

---

[16] In his contribution to this volume (Chap. "A Fortuitous Year with Leon Henkin"), Solomon Feferman explains a little bit The Loyalty Oath, whose main requirement was to force people to declare they were not members of the Communist Party.

[17] See [7, pp. 5–6].

Henkin was chairman or acting chairman of the Department of Mathematics at the University of California at Berkeley on at least two occasions: during 1966–1968 and during 1983–1985. Although it seems strange, the second period was more complicated than the first, in 1983 he wrote:[18] *'It is much harder now than when I served during 66–68. One big difference is that the University budget has suffered greatly through a combination of political and economic conditions.* [...] *I send a clipping from our student newspaper—on the first day of our academic year—describing some of the problems. (The Dean was unhappy, but the Chancellor gave us $20,000 more to open 2 new courses!).'*

In that clipping from the *The Daily Californian* [35], the situation is explained and the writer tells us that Henkin and the Vice-Dean David Goldschmidt had sent a letter to the Republican Governor George Deukmeijian in which they set forth their demands. The situation deteriorated to the extent that a year and a half later Henkin said he wanted to resign, although he was eventually convinced to carry one for a further year.

5 His Research

Leon Henkin has left behind as his heritage an important collection of papers, filled with exciting results and very original methods. One example is the paper containing his innovative and highly versatile method for proving the completeness theorem—both for the theory of types and for first-order logic—a method that was applied later to many other logics, even non-classical ones. For some of his results we know the process of discovery,[19] what observed facts he was trying to explain, and why he ended up discovering things that were not originally the target of his enquiries. In these cases we do not have to engage in risky business of trying to explain the origins of his ideas merely on the basis of the final scientific papers that presented them. It is well known that the *logic of discovery* is often hidden in the final exposition of our research through their different propositions, lemmas, theorems and corollaries.

5.1 Completeness

If you take a look at the list of publications Leon Henkin left us (Appendix of this book), the first published paper, *The completeness of first order logic* [11], corresponds to his well known result, while the last, *The discovery of my completeness proof* [29], is an extremely interesting autobiographical one, the two papers forming a sort of fascinating loop around his career.

For a countable first-order language the completeness proof proceeds in two steps. First, every consistent set of formulas is extended to a maximal consistent set with witnesses. Second, once we have the maximal consistent set with witnesses, we use it as a guide to build the precise model that the formulas of this set are describing; the universe of the model being the set of witnesses or a set of equivalent classes. According to Monk [33]: *'He also used the above basic idea to a generalization of* ω*-consistency* [15]

[18]Personal letter to María Manzano.

[19]Explained in his [29].

and ω-completeness [18]*, to results concerning the interpolation theorem* [23]*, and the representation theorems for cylindric algebras* [17]*'.*

Several chapters of this book deal with Henkin's completeness theorems (Chaps. "Henkin's Theorem in Textbooks" and "Henkin on Completeness") and its extensions to different logical systems (Chaps. "Henkin's Completeness Proof and Glivenko's Theorem", "From Classical to Fuzzy Type Theory" and "Henkin and Hybrid Logic"). Henkin's method, as well as an abstract setting of Henkin's proof, are also treated in this book (Chaps. "The Countable Henkin Principle" and "Changing a Semantics: Opportunism or Courage?").

5.2 Algebraic Logic

Henkin devoted a large portion of his research to algebraic logic. Tarski and Henkin believed that logic investigation was interesting in and of itself, since it could provide unifying principles for mathematics, *'In fact we would go so far as to venture a prediction that through logical research there may emerge important unifying principles which will help to give coherence to a mathematics which sometimes seems in danger of becoming infinitely divisible.'*[20] The promotion of research in logic (algebraic logic in particular) also had the aim of attracting the interesting of mathematicians. Let us quote Feferman [7] *'I believe that it was because Tarski thought that formal syntax and metamathematics were principal obstacles to mathematicians' appreciation of, and interest in, logic that he was led to work on the algebraic reformulations of logic in terms of Boolean algebras, relation algebras and, finally and most extensively, cylindric algebras.'* Feferman also cites other interesting quotes *'from a 1955 letter that Tarski wrote, with Leon Henkin, to the committee for summer institutes of the American Mathematical Society urging support of the 1957 Cornell Institute in Symbolic Logic.'*

In particular, Feferman includes the following quote from a letter in which Tarski and Henkin identify the source of mathematicians' disaffection with logic as being its philosophical flavor: '*[many] mathematicians have the feeling that logic is concerned exclusively with those foundation problems which originally gave impetus to the subject; they feel that logic is isolated from the main body of mathematics, perhaps even classify it as principally philosophical in character.*'[21]

Algebraic notions are to be found in logic from the very beginning, starting with the work of Boole. This tendency to establish relationships between certain fields of mathematics and algebra, which increased with time, proved particularly fruitful in the case of propositional logic. We know that every Boolean algebra is isomorphic to a quotient algebra of formulae. By the Stone Representation Theorem we also know that every Boolean algebra is isomorphic to an algebra of sets. The theory of cylindric algebras [31] provides a class of algebras that are to first-order logic what Boolean algebra is to sentential logic. The subject was extensively developed by Leon Henkin, Alfred Tarski and Donald Monk over an extended period. Further research in this area was carried out by Istvan Németi and Hajnal Andréka. Henkin's contributions to this field of cylindric algebras are treated

[20] Feferman in [7, pp. 5–6], cites this 1955 letter that Tarski wrote, with Leon Henkin.

[21] See [7, pp. 5–6].

in this book (Chaps. “Leon Henkin and Cylindric Algebras” and “Changing a Semantics: Opportunism or Courage?”).

5.3 Identity

The search for a system of logic that takes the *identity* relation as the sole primitive constant has an ample historical precedent. Henkin studied this matter in depth in at least two of his contributions, namely, in *“A theory of Propositional Types”* [24] and in *“Identity as a Logical Primitive”* [26]. Two chapters of this book deal with his definition of a theory of propositional types based on identity, and in which all types are finite (Chaps. “A Bit of History Related to Logic Based on Equality” and “Reflections on a Theorem of Henkin”).

5.4 Nominalism

The work of Henkin that is most directly related to philosophy is an article entitled: *“Some Notes on Nominalism”* [14] which appeared in the *Journal of Symbolic Logic* in 1953 and to which there was to be a sequel, in 1955, in the form of a lecture [16]. As Henkin himself was eager to clarify, this was a reply to two papers that appeared in that journal over a short period of time in 1947. The first, *“On universals”* [34] was written by Quine and the second one, *“Steps toward a constructive nominalism”* [8], by Quine and Nelson Goodman jointly. These works, including that of Henkin, are part of a *foundations sensitive* tradition that is perhaps not as popular today. In this book there are no chapters directly devoted to this topic, but in Chap. “Henkin on Completeness” we highlight that the models he builds in all his completeness proofs are in accordance with a nominalistic position.

5.5 Mathematical Induction

We believe that his work on mathematical induction was the result of his devotion to mathematical education. We will comment later on the film about this subject that was part of the *Mathematics Today series*. He also wrote a paper, *On mathematical induction* [19] that he liked very much: ‘[...] *but my little paper on induction models from 1960, which has always been my favorite among my expository papers’*.[22] In it, the relationship between the induction axiom and recursive definitions was studied in depth. Henkin defined induction models as the ones in which the induction axiom holds, and he was able to prove that not all recursive operations can be defined in these models. For instance, exponentiation fails. Induction models are straightforward mathematical structures; there is the standard model that is isomorphic to the natural numbers, and there are non-standard models. The latter fall into two categories: cycles—namely the integers modulo a natural number n—and what Henkin termed *“spoons”*, having a cycle and a handle. The reason there are two categories of non-standard models is that the induction axiom by itself always implies the validity of either the first or the second of Peano’s axioms for arithmetic.

[22] Personal communication to María Manzano.

5.6 Mathematical Education

Several of his publications, dealing with elementary concepts, fall under mathematical education ([20, 22, 25] and [27]). Between 1957 and 1972, Henkin shared his work in research into mathematics with enquiries into its teaching. As from 1972, he devoted himself mainly to investigating the teaching of the subject. In fact, in 1979, with a Fulbright fellowship, he spent time in Israel devoted to looking into the teaching of mathematics. He was then at the *Department of Education in Science* at *Technion University* in Haifa. On this occasion he also visited two universities in Egypt.

5.7 A Problem Concerning Provability

Henkin was also very good at posing problems.[23] Chapter "The Henkin Sentence" in this book is devoted to the most famous one, *A problem concerning provability*. The problem deals with a formula similar to the one Gödel used in his incompleteness results but this formula is expressing its own provability while Gödel's expressed of its own truth. The question Henkin posed was whether the formula is provable or not. Halbach and Visser analyze *'Henkin's formulation of the question and the early responses by Kreisel and Löb and sketch how this discussion led to the development of Provability Logic.'*

5.8 Language with Only a Finite Number of Variables

To finish this section, let us tell you an interesting story Wilfrid Hodges told María:

> First, if my memory is accurate, Henkin played a role you might not know about in computer science logic. I think it was in 1970 or 1971 that he came to Bedford College London and gave a talk to Kneebone's seminar. He talked about proofs in which only some finite number of variables occur, and reported some results of Don Monk in that area. At the end of the talk I asked him whether there is a first-order sentence with no function symbols and only two variables that has models but no finite ones. He said he didn't know. Alex Wilkie remembered the question and passed it on a year or so later to my student Mike Mortimer, who proved that the answer was no. After he'd proved that, we found that Dana Scott had already published the result; but Scott's proof relied on a dubious theorem of Gödel which is now known to be false. So Mortimer has the theorem, and Mortimer's Theorem is the first in a line of theorems about what can be said using only a fixed number of variables.[24]

6 Henkin the Teacher

The story behind this is that of María Manzano, who during the academic year 1977–1978 attended his classes in *Algebra* for students in the first years of the degree course, and of

[23] The whole reference of these problems is included in the Appendix *Curriculum vitae* of this book.

[24] Personal e-mail to María Manzano.

Metamathematics (225A) for doctorate students. According to her impression during that year, Henkin was an extraordinary insightful professor in the clarity of his expositions and was well loved by his students, who on his last day of class in the academic years would applaud his efforts with great emotion. Indeed, Henkin always wondered whether his classes should be easy to follow or whether they should force his students to make important efforts. In the above-commented interview [37] he stresses that things learned with effort are less easy to forget.

Proof of this internal debate are the following words that formed part of the summary of his course on *Metamathematics*.

> Many bright students find my lectures a little slow, and they consider my concern with the machinery of logic (as distinct from the results) as pedantic. Concerning the first of these judgments, it is valid, but since many of the students are as slow-thinking as I, and the quick-thinking ones can always skip lectures and study the references, the pace as a whole is not bad—and indeed, the poorly-prepared students may find themselves struggling a bit to keep up. Concerning the second charge, however, I think it can be at least partially turned aside by adverting to pedagogical principles—which I am quite willing to explicate and discuss in office hours, or even in class if demand warrants.

Many of us who knew him believe that he reached a perfect balance and that he would never be obscure on purpose.

The textbook used in the algebra course Birkhoff and Mac Lane's, *A Survey of Modern Algebra*, but we did not follow the order of the book. The topics were the usual ones in an algebra course: Rings, Fields, Polynomials, Homomorphisms, Vector Spaces, etc. Before each class Henkin would give us a text of some 4–5 pages that summarized what was to be addressed in the class. The texts were in purple ink, printed with the old multicopiers that we called *"Vietnamese copiers"* and that were so often used to (illegally) print pamphlets in our revolutionary days in Spain. Before starting the sessions, and handing out the printed texts with the topics to be addressed, he would give us a sheet explaining the tasks of the week: revision of class notes and of the corresponding sections of the book (indeed, *exactly* which ones), and some 8 problems to solve. In addition to giving us back the exercises corrected, he would give us a copy with exercises containing problems solved by him. Detailed information of everything about the course that might be of interest to us was announced a long time before strictly necessary: the dates for handing in the various tasks, the dates of our exams (to be done in class with our books and notes), and his tutorial schedule. In the courses given in the first years of the degree, he was always very enthusiastic, even jovial, in class and he transmitted a feeling of confidence. His tutorials were always well attended.

In the summary of the metamathematics course Henkin defined metamathematics as the mathematical theory of mathematical theories, and he introduces the latter as the study of structures and their interrelations. *'In metamathematics, we study three classes of structures and their close interrelations: formal grammars, deductive calculus, and interpreted languages.'* The languages of sentential logic, of first-order and equational logic, and of many-sorted logic and higher-order logic were dealt with in a unified way in the course. In Henkin's words, *'we may attempt to deal comprehensively with all of these classes through the study of many-sorted grammars.'* In each case, we studied the relationship between the semantic notion of consequence and the syntactical notion of derivability, with proofs of the soundness and completeness theorems.

It is surprising that the first-order completeness proof that Henkin explained in class was not his own, but was developed by using Herbrand's theorem and the completeness of propositional logic. Another completeness proof he also developed in class was his result based on Craig's interpolation theorem.[25]

The issue of implicit and explicit definability was addressed in detail and the Beth/Padoa theorems relating them, as well as the interpolation theorems of Craig and Lyndon were proved. The Löwenheim–Skolem and compactness theorems were proved and commented upon. Naturally, the notions of universal algebra were introduced to relate structures: substructures, homomorphisms, direct products and also ultraproducts. In particular, the ultraproduct construction was used to prove compactness. Henkin did not forget classic themes such as quantifier elimination and categoricity. The theory of types and Gödel's incompleteness theorem were important parts of the course; indeed, they accounted for 2/5th of the whole. The language of the theory of types introduced by Henkin was that based on identity, very similar to that in his works ([24] and [26]), which contained a selector operator that allows the axiom of choice to be expressed. The recursive functions, the arithmetization of formal language, the Gödelization and self-reference inevitably led to Gödel's theorems of incompleteness. The last topic was that of general recursive functions and relations.

Even though the topics of this course were more or less standard, the whole organization of the material and the presentation of each were highly original. We agree with Steve Givant statement that:[26] *'Henkin never seemed to teach a "standard" course, but rather he presented what one can only describe as his own creations.'*

7 The Roles of Action and Thought in Mathematics Education

Henkin was often described as a social activist. He labored much of his career to boost the number of women and underrepresented minorities in the upper echelons of mathematics. He was also very aware that we are beings immersed in the crucible of history from which we find it hard to escape. This is in fact reflected in what he wrote in the beginning of his interesting article about the teaching of mathematics [28]:

> Waves of history wash over our nation, stirring up our society and our institutions. Soon we see changes in the way that all of us do things, including our mathematics and our teaching. These changes form themselves into rivulets and streams that merge at various angles with those arising in parts of our society quite different from education, mathematics, or science. Rivers are formed, contributing powerful currents that will produce future waves of history.
> The Great Depression and World War II formed the background of my years of study; the Cold War and the Civil Rights Movement were the backdrop against which I began my career as a research mathematicians, and later began to involve myself with mathematics education.[27]

In that paper, Henkin explains how during the first 15 years of his career the path of his research trajectory and his mathematics education path *'had only trivial points of contact (i.e. in my teaching).* [...] *Then, around 1972/73, there was a change. Something new entered my educational work: First, I began to incorporate a few basic elements*

[25] In this book we have devoted the chapter *Henkin on Completeness* to this issue. Therein, we include a section on the completeness proofs Henkin told us in class.

[26] Personal communication to María Manzano.

[27] See [28, p. 3].

of the methodology I employed in doing mathematics; second, I even found that I could employ results! [...] *Best of all, it has allowed me to integrate somehow two streams of my mathematical experience, deepening the satisfaction I derive from both.* [...] *Now another wave of history is welling up in the tide of our country's mathematicians, leading many of them to consider taking some work in mathematics education beyond the teaching of courses* [...].

In this paper he gave both a short outline of the variety of educational programs he created and/or participated in, and very interesting details of some of them; in particular, of the following six:

1. 1957–1959, *NSF* SUMMER INSTITUTES. Henkin relates this initiative to historical facts: *'The launching of Sputnik demonstrated superiority in space travel, and our country responded in a variety of ways to improve capacity for scientific and technical developments'*. These programs were directed at improving high school and college mathematics instruction and addressed math teachers. As a result of his teaching on the axiomatic foundations of number systems, he collaborated at a book, *Retracing elementary mathematics* [20].
2. 1959–1963, *MAA* MATH FILMS. At that time internet resources were not available, so movies were produced. *'Sensing a potential infusion of technology into mathematics instruction, MAA set up a committee to make a few experimental films.* [...] *the committee approached me in 1959–1960 with a request to make a filmed lecture on mathematical induction which could be shown at the high-school-senior/college-freshman level. I readily agreed.'* The film was part of the Mathematics Today series, and was shown on public television in New York City and in high schools. A 20 page manual to supplement the film was also published, containing two parts, one appendix, and some problems. Therein Henkin asks: *'Of what good is this principle anyhow? you may ask.* [...] *there are very few direct applications of mathematical induction* [...] *And yet, to me, the true significance of mathematical induction does not lie in its importance for practical applications. Rather I see it as a creation of man's intellect which symbolizes his ability to transcend the confines of his environment.'*
3. 1961–1964 CUPM. The *Committee on the Undergraduate Program in Mathematics* proposed courses to be taken by elementary teachers. *'Some of my colleagues and I began, for the first time, to have classroom contact with prospective elementary teachers, and that led, in turn, to in-service programs for current teachers. I learned a great deal from teaching teachers-students; I hope they learned at least half as much as I!'*
4. 1964–. ACTIVITIES TO BROADEN OPPORTUNITY. In the sixties, Berkeley students were taken energic actions against segregation in Southeastern USA as well as against military actions in Vietnam. *'In the midst of this turmoil I joined in forming two committees at Berkeley which enlarged the opportunity of minority ethnic groups for studying mathematics and related subjects.* [...] *We noted that while there was a substantial black population in Berkeley and the surrounding Bay Area, our own university student body was almost "lily white" and the plan to undertake action through the Senate was initiated'*. The committee recruited promising student and offered them summer programs to study math and English. If they persisted in the program, they were offered special scholarships. Henkin also collaborated with Bill Johntz, a very engaging and efficient Berkeley High School math teacher, *'He asked if I could steer university math students to him who might be interested in working in parallel classes of elementary students'*. Over the years, many graduated students participated, together with faculty

spouses who had studied mathematics in college, and also some engineers, for several hours a week. The program was called Project *SEED*—Special Elementary Education for the Disadvantaged.

5. 1960–1968. TEACHING TEACHERS, TEACHING KIDS. Therein Henkin described several Conferences on School Mathematics as well as several projects and courses he was involved in. The following paragraph caught my eye: *'After I began visiting elementary school classes in connection with CTFO, I came to believe that the emotional response of the teachers to mathematics was of more importance to the learning process of the students than the teacher's ability to relate the algorithms of arithmetic to the axioms of ring theory'*. The National Council of Teachers of Mathematics (NCTM) invited Henkin to participate in the elaboration of several films, and accompanying text, dealing with Whole-Number Systems as well as Rational-Number Systems.
6. 1968–1970. OPEN SESAME: THE LAWRENCE HALL OF SCIENCE. The Lawrence Hall of Science, created in honor of and named after the Nobel prize winner, was originally a science museum in Berkeley. In 1968 Alan Portis transformed the museum into a live center of science and mathematics education when he was appointed as new director. He gathered a group of faculty members from a variety of science departments interested in science education. *'These faculty members proposed a new, interdisciplinary Ph.D program under the acronym SESAME—Special Excellence in Science and Mathematics Education. Entering students were required to have a masters degree in mathematics or in one of the sciences. Courses and seminars in theories of learning, cognitive science, and experimental design were either identified in various departments, or created'*. Nitsa Hadar, a student from the Technion in Israel, was admitted in the SESAME program; she has contributed for this volume Chaps. "Tracing Back "Logic in Wonderland" to My Work with Leon Henkin" and "Pairing Logical and Pedagogical Foundations for the Theory of Positive Rational Numbers—Henkin's Unfinished Work".

8 The Never Ending Story

Leon Henkin has been one of the most original and brilliant minds in the history of Logic. One of those that beyond solving problems posed by others, open new ways to keep looking for the truth. What better way to finish than to talk about some hot historical issues concerning *The Life and Work of Leon Henkin*?

8.1 Bertrand Russell's Request

On April 1, 1963, Henkin received a very interesting letter from Bertrand Russell. In it, Russell thanks Henkin for *'your letter of March 26 and for the very interesting paper which you enclosed.'* He is talking about Henkin's paper entitled *Are Logic and Mathematics Identical?*

Right at the beginning Russell declares *'It is fifty years since I worked seriously at mathematical logic and almost the only work that I have read since that date is Gödel's. I realized, of course, that Gödel's work is of fundamental importance, but I was puzzled* *by it. It made me glad that I was no longer working at mathematical logic.*' It seems that Russell understood Gödel's theorem as implying the inconsistency of Principia: '*If a given set of axioms leads to a contradiction, it is clear that at least one of the axioms must be false. Does this apply to school-boys*' arithmetic, and, if so, can we believe anything that we are taught in youth? Are we to think that $2+2$ is not 4, but 4.001?'[28] He then goes on explaining his *'state of mind'* while Whitehead and him were writing the Principia.

> What I was attempting to prove was, not the truth of the propositions demonstrated, but their deducibility from the axioms. And, apart from proofs, what struck us as important was the definitions.
> You note that we were indifferent to attempts to prove that our axioms could not lead to contradictions. In this Gödel showed that we had been mistaken.
> [...]
> Both Whitehead and I were dissapointed that the Principia was almost wholly considered in connection with the question whether mathematics is logic.[29]

Russell ended the letter with a request: *'If you can spare the time, I should like to know, roughly, how, in your opinion, ordinary mathematics—or, indeed, any deductive system—is affected by Gödel's work.'*[30]

Unfortunately, Leon's answer did not freed Russell of his misunderstanding, as Grattan-Guinness affirms that *'Russell was still struggling with the theorem at the end of his life.'*[31]

According to Annellis: *'Henkin replied to Russell at length with an explanation of Gödel's incompleteness results, in a letter of 17 July 1963, specifically explaining that Gödel showed, not the inconsistency, but the incompleteness, of the [Principia] system.'*[32]

8.2 Henkin's Boolean Models: Peter Andrews' Proposed Homework

In 1975 Henkin published a paper with the title *Identity as a logical primitive* [26]. This paper is included in a volume of *Philosophia. Philosophical Quarterly of Israel*, completely devoted to identity. As Peter Andrews pointed out to us by e-mail: *'This expository paper concludes with a brief discussion of Boolean-valued (B-valued) models for type theory, and ends with a footnote which says, "Proofs for the results on B-models will be given in a forthcoming paper.* [...] *Boolean models of type theory were described in my paper 'Models of Type Theory' at the Symposium on Theory of Models held in Berkeley in 1963, but the paper was not published. (Cf. The Theory of Models, North-Holland, 1965, p. vii)."'* In his e-mail Peter asked: *'Did Henkin ever publish his paper on Boolean-valued Models of Type Theory? If so, it should be added to the bibliography page. If not, I wonder if his heirs have a manuscript which would be worth publishing even if it does not contain everything that Henkin wanted to include in it. If this were the case, perhaps it would be appropriate to publish it in the volume you are preparing.'*

---

[28] See [9, p. 592].

[29] See [9, pp. 592–593].

[30] See [9, p. 593].

[31] See [9, p. 593].

[32] See [2, p. 89, footnote 3]. See also Anellis [3].

Knowing that The Bancroft Library, UC Berkeley, contains important documents of Leon Henkin, we contacted them.[33] The reply was that the collection record is unarranged and therefore unavailable for use.

Acknowledgements The authors wish to thank several colleagues and staff, including Steve Givant, Peter Andrews, Wilfrid Hodges, John Addison, Antonia Huertas and David Kessler. Special thanks to Henkin's family: Ginette Henkin, Paul Henkin and Julian Henkin.

References

1. Addison, J., et al.:. In Memoriam: Leon Albert Henkin. University of California. Available via http://senate.universityofcalifornia.edu/inmemoriam/leonalberthenkin.html
2. Anellis, I.: Review of Bertrand Russell, Towards the Principles of Mathematics, 1900–2002, edited by Gregory H. Moore, and Bertrand Russell, Foundations of Logic, 1903–2005 edited by Alasdair Urquhart with the assistance of Albert C. Lewis. Mod. Log. **8**(34), 57–93 (2001). http://projecteuclid.org/euclid.rml/1081173771
3. Anellis, I.: Evaluating Bertrand Russell: The Logician and His Work (2012). Docent Press
4. Church, A.: The Calculi of Lambda-Conversion. Annals of Mathematical Studies, vol. 6. Princeton University Press, Princeton (1941)
5. Church, A.: A formulation of the logic of sense and denotation. In: Structure, Methods and Meaning, Essays in Honor of Henry M. Sheffer, New York. pp. 3–24. (1951). Revised in NOUS: vol. 7 (1973), pp. 24–33; vol. 8 (1974), pp. 135–156; and vol. 27 (1993), pp. 141–157
6. DelVecchio, R.: Leon A. Henkin—Cal math educator. SFGate *SF Chronicle*. Available via http://www.sfgate.com/default/article/Leon-A-Henkin-Cal-math-educator-2466402.php (2006)
7. Feferman, S.: Tarski's conception of logic. Ann. Pure Appl. Log. **126**, 5–13 (2004)
8. Goodman, N., Quine, W.V.: Steps toward a constructive nominalism. J. Symb. Log. **12**(4), 105–122 (1947)
9. Grattan-Guinness, I.: The Search for Mathematical Roots, 1870–1940: Logics, Set Theories, and the Foundations of Mathematics from Cantor Through Russell to Gödel. Princeton University Press, Princeton (2000)
10. Henkin, J., Henkin, P., et al.: Many views of Leon Henkin. Manuscript (2006)
11. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**(3), 159–166 (1949)
12. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**(2), 81–91 (1950)
13. Henkin, L.: Banishing the rule of substitution for functional variables. J. Symb. Log. **18**(3), 201–208 (1953)
14. Henkin, L.: Some notes on nominalism. J. Symb. Log. **18**(1), 19–29 (1953)
15. Henkin, L.: A generalization of the concept of ω-consistency. J. Symb. Log. **19**(3), 183–196 (1954)
16. Henkin, L.: The nominalistic interpretation of mathematical language. Bull. Soc. Math. Belg. **7**, 137–141 (1955)
17. Henkin, L.: The representation theorem for cylindrical algebras. In: Skolem, Th., Hasenjaeger, G., Kreisel, G., Robinson, A. (eds.) Mathematical Interpretation of Formal Systems, pp. 85–97. North-Holland, Amsterdam (1955)
18. Henkin, L.: A generalization of the concept of ω-completeness. J. Symb. Log. **22**(1), 1–14 (1957)
19. Henkin, L.: On mathematical induction. Am. Math. Mon. **67**(4), 323–338 (1960)
20. Henkin, L., Smith, W.N., Varineau, V.J., Walsh, M.J.: Retracing Elementary Mathematics. Macmillan, New York (1962)
21. Henkin, L.: Are logic and mathematics identical? Science **138**, 788–794 (1962)
22. Henkin, L.: New directions in secondary school mathematics. In: Ritchie, R.W. (ed.) New Directions in Mathematics, pp. 1–6. Prentice Hall, New York (1963)

[33] We interchanged several e-mails.

23. Henkin, L.: An extension of the Craig–Lyndon interpolation theorem. J. Symb. Log. **28**(3), 201–216 (1963)
24. Henkin, L.: A theory of propositional types. Fundam. Math. **52**, 323–344 (1963)
25. Henkin, L.: Mathematical foundations for mathematics. Am. Math. Mon. **78**(5), 463–487 (1971)
26. Henkin, L.: Identity as a logical primitive. Philosophia. Philos. Q. Isr. **5**(1–2), 31–45 (1975)
27. Henkin, L.: The logic of equality. Am. Math. Mon. **84**(8), 597–612 (1977)
28. Henkin, L.: The roles of action and of thought in mathematics education—one mathematician's passage. In: Fisher, N.D., Keynes, H.B., Wagreich, Ph.D. (eds.) Changing the Culture: Mathematics Education in the Research Community. CBMS Issues in Mathematics Education, vol. 5, pp. 3–16. Am. Math. Soc. in cooperation with Math. Assoc. of America, Providence (1995)
29. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**(2), 127–158 (1996)
30. Henkin, L., Tarski, A.: Cylindric algebras. In: Dilworth, R.P. (ed.) Lattice Theory. Proceedings of Symposia in Pure Mathematics, vol. 2, pp. 83–113. Am. Math. Soc., Providence (1961). [Reprint, Givant, S.R. and McKenzie, R.N. (eds.) Alfred Tarski. Collected Papers, vol. 4, pp. 647–655, Birkhäuser, Basel (1986)]
31. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras. Part I. Studies in Logic and the Foundations of Mathematics, vol. 64. North-Holland, Amsterdam (1971)
32. Henkin, P.: Proposed memorial oration. Manuscript (2006)
33. Monk, J.D.: In memoriam: Leon Albert Henkin. Bull. Symb. Log. **15**(3), 326–331 (2009)
34. Quine, W.V.: On universals. J. Symb. Log. **12**(3), 75–84 (1947)
35. Duke's cuts crowd math dept. The Daily Californian, Monday, August 29, 1983
36. The Princeton Mathematics Community in the 1930s (PMC14)
37. The Princeton Mathematics Community in the 1930s (PMC19)
38. Russell, B., Whitehead, A.: Principia Mathematica, vols. 1–3. Cambridge University Press, Cambridge (1910–1913)
39. Sanders, R.: Leon Henkin, advocate for diversity in math & sciences, has died. UC Berkeley Press Release. 09 November 2006. Available via http://www.berkeley.edu/news/media/releases/2006/11/09_henkin.shtml

M. Manzano (✉)
Department of Philosophy, Logic and Aesthetics, Faculty of Philosophy, Edificio FES,
University of Salamanca, Campus Unamuno, 37007 Salamanca, Spain
e-mail: mara@usal.es

E. Alonso
Department of Logic and Philosophy of Science, Faculty of Philosophy, Autonomous University of Madrid, Avda. Tomás y Valiente s/n, Campus Canto Blanco, 28049 Madrid, Spain
e-mail: enrique.alonso@uam.es

Lessons from Leon

Diane Resek

Abstract This paper describes some lesson learned from Leon Henkin while the author wrote a dissertation on Cylindric Algebras and collaborated with Henkin on several projects in math education. The lessons were about writing, lecturing, relating to administrators, and making decisions.

Keywords Algebraic logic

Leon Henkin was one of my professors in my first semester of graduate school at Berkeley in Fall 1962. I came to Berkeley from a small women's college and was overwhelmed by the faculty and other students in the Mathematics Department. During my entire time in graduate school, I never spoke out in class. If I had a question, I would have the male (and it always was a male) next to me, ask the question. Leon was the least intimidating of the professors I met at Berkeley. So, when it came time to ask someone to work with me on a thesis, he was the first one I approached (and also the last). I worked (and did not work) on my thesis for nine years. During that time, I was involved in various mathematics education projects. In most of those projects, I also worked with Leon. He was the principal investigator, and I was the director. I learned a great deal from Leon in a number of areas in those years. One important lesson I learned from Leon was the role of being a professor, in particular, how a professor should relate to administrators. Once when I was admiring his technique, he said "But they are only administrators. WE are professors. They should be supporting us." Before that university administrators intimidated me. But after that exchange, I looked at them as people who should be facilitating my work. In part to avoid working on my thesis, I submitted grant proposals with Leon in the area of mathematics education. A perceptive dean wrote to Leon about my lack of progress toward completing my Ph.D.: "Can you suggest any reason why Ms. Resek should not be given a termination for lack of progress? I am afraid that you have done nothing to deter her from yielding to a situation that is likely to generate long-range handicaps for her."

From one of the letters I saved, Leon's artful response was:

> It is quite true that Diane's interest in and work at a variety of projects in mathematics education during the last few years is largely responsible for the delay in completing her doctoral work. I plead guilty to having encouraged her natural inclinations in this direction. I believe that she is laying the groundwork for an extremely fruitful career which will combine her mathematical training with her experience in schools and other educational centers (such as LHS). In these times when our traditionally trained mathematics Ph.D.'s are finding rough going in the marketplace, it seems to me that we on the faculty should particularly seek new realms wherein mathematics training can make a substantial contribution to the basic aims of society.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_2

Needless to say, Leon's response bought me a few more years. Leon was an elegant writer. I loved reading his mathematical writings. One of the most important lessons I learned from him was in the area of writing. Alas! I never became an elegant writer, but I did become a clear and concise one—at least I improved in those aspects under Leon's harsh tutelage. His method of instruction was for me to first submit drafts. Then, to my embarrassment, no matter how hard I had tried to do good work, he totally covered my page with red marks. Most of those marks were rewritings of my work. This instruction occurred not only with my mathematical writing, but also with proposals and reports for the math education projects. Besides the rewritings, Leon gave me some specific tips. I do not remember them all since it was almost 50 years ago, but a few have been indelibly imprinted on my memory. The one that stands out most clearly was not to use symbols where they could be avoided in writing mathematics. For example, Leon thought that instead of writing

$$\forall x \exists y (y > x)$$

one should write something like:

For every x, we can find a y such that y is greater than x.

His reasoning was that no matter how familiar readers were with the symbols, they needed to use a tiny part of their brain to translate the symbols into words. A writer should want his or her readers to be able to apply every bit of their brainpower to processing the ideas being written rather than to the interpretation of symbols.

Another writing principle was "Do not add in superfluous phrases." One should get to the point. For example, I remember writing something like "I think ...", and Leon saying, you are writing this, your reader knows that you think what you are saying is true. You do not have to spend time telling them that, just say it! Leon was very exact in his writing, and he delighted in using mathematical or technical terms in ordinary discourse. I recall him telling his wife on the phone "I will meet you at 6 pm in the canonical place." In a letter to a dean on my behalf, he stated: "Your letter of October 26 deals with several related questions to which I shall address myself seriatum." I was a heavy smoker at that time, which Leon hated. Once coming into my office, I remember him shaking his head and saying: "Diane, you are going to asphyxiate your students."

Leon impressed me with the importance of working hard on the exposition of any writing to try to make it as elegant as possible. It did not work well with my dissertation, but he kept pushing. I never could find the silver bullet to make my thesis sparkle, but at each meeting he would have new mathematical tacks for me to try: congruence relations? ultra products?

Another strength of Leon was as a lecturer. Part of his magic was his elegant expression of the mathematics, but he also worked hard to engage his audience in conjecturing and seeing the next step or in being surprised by it. He certainly captured the interest of his audiences. At one point, he entranced a group of elementary school teachers with a discussion on whether two people could have the same number of hairs on their head.

Leon was committed to work toward equity in society. He was able to see that professional mathematicians could make a difference, particularly regarding racial inequities in the United States. He was one of the first people to say that one thing holding back racial minorities and poorer people in America was their low participation rates in math/science careers. He believed that there were ways of teaching and new programs that could correct this problem.

I joined other math graduate students at Berkeley, who needed little encouragement to work with him in these endeavors. At that time, the drive to be a change agent in American society was strong among young people in the United States and particularly around Berkeley. Leon gave us a means to use our training to make a real difference. It was his expertise in working with administrators, and his eloquence as an expositor that established programs to support our work.

Leon was a leader in establishing special summer programs for minority high school students to prepare them for work at the university. He was the leader in setting up support for minority and women students in calculus courses to allow them not only to pass courses, but also to excel. A major coup of Leon was to persuade the Berkeley mathematics faculty to allow mathematics graduate students financial support for working in elementary school mathematics classrooms equal to what they received as teaching assistants in university classes. And he was able to establish this program statewide in the University of California.

In general, Leon's recognition of values in life beyond pushing for the best and strongest theorems set him apart from many mathematicians at Berkeley. After finishing my degree, I was deciding which job to accept: a tenure track position with a reduced course load at a research institution in Chicago or a lectureship with a four course teaching load at a State University in San Francisco. Leon told me that there were many roads to happiness, not just one right one. But when I decided to stay in San Francisco, I felt he would be disappointed because I did not choose the research position. However, he smiled and said that Chicago was no fit place to live.

Of course, I learned much more from Leon than I recounted here, but these are the easiest for me to articulate.

D. Resek (✉)
Department of Mathematics, San Francisco State University, Thornton Hall 937, 1600 Holloway Ave, San Francisco, CA 94132, USA
e-mail: resek@sfsu.edu

Tracing Back "Logic in Wonderland" to My Work with Leon Henkin

Nitsa Movshovitz-Hadar

Abstract The book *Logic in Wonderland*, I co-authored with Atara Shriki, was published in early 2013, about 40 years after I started my Ph.D. thesis with Leon Henkin. The time gap did not diminish his influence. A few anecdotes from my early days with Leon and two sample tasks from the book recently published illustrate it.

Keywords Leon Henkin · Thesis advisor · Mathematical humor · Logic in Wonderland · The teaching and learning of logic

1 How Did I End Up Having Leon for a Thesis Advisor?

In the early 1970s, I decided to go to the US for graduate studies in mathematics education because there was none available at this time in Israel. I sent letters to about 20 universities asking for information. Shlomo Libeskind, who was then a visiting scholar at Technion—Israel Institute of Technology, said to me: "If you are admitted to Berkeley—go there." It was quite surprising that the very first reply I received was from Berkeley. Professor Fred Reif, then the chairman of a fairly new interdisciplinary program called SESAME, referred me to Prof. Leon Henkin as a potential supervisor. Leon and I carried out an intensive exchange for several months in which he clarified all the difficulties involved in becoming a foreign student at SESAME. All his letters were carefully typed; later on I understood that they were dictated to his faithful secretary. But the last letter, dated June 2nd, 1972, was handwritten and personal. He said: "…I should say that I don't expect ours to be a normal student–teacher relation. I've already confessed to you that I'm quite an amateur in mathematics education—but you are a professional! I expect to learn at least as much from you as you from me. Perhaps most accurate: We'll explore the field together as fellow students, or co-workers…" This is what a distinguished logician, who had already contributed a lot to mathematics education, wrote to an Israeli high school teacher, 20 years younger than him, who wished to pursue Doctoral studies with him. Needless to say, I was astounded.

2 Leon the Tactful Advisor

In my first meeting with Leon at Berkeley, he took me around to see the campus to make me feel "at home." His kindness was striking. I was unable to express my gratitude since

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_3

my English was very poor. He said that we would meet once a week and I was supposed to come the next week with a list of courses I wanted to take for the first semester. The following week I came with a list of 12 courses. He looked at my list, and then straight in my eyes examining the naive student sitting there, and said in a fatherly tone “three or four courses are the maximum in our quarter system.” He then started to cross off those courses he thought I should not take. The last one was a lab in English for foreign students. He said: “You don’t need it. I’ll be your English teacher.” Indeed he was. The course load was heavy, and in addition I had to cope with the adjustment of my family to the new circumstances. Towards the end of the first quarter, I was very close to giving up. I had a nightly dream of going back home to Israel and waked up at Berkeley to attend classes, submit homework, take care of my children (then of age 3 and 8) and deal with our family survival on a limited budget (as my husband was an attorney unlicensed to work In California, who took a leave of absence from his permanent job for my sake). As I entered Leon’s office, he asked me, as usual, how I was. That was too difficult to answer casually, and I burst in tears. Leon did not patronize, did not push. He very quietly got up of his chair, slowly went to the door and left the room, letting me come back to myself. After five minutes he entered with a cup of tea, and I felt it was the noblest way of dealing with my embarrassing behavior.

3 Hanukah at 9 Maybeck Twin Drive

Living away of home was a daily struggle. There was no e-mail in those days, of course, and international phone calls were pretty expensive. I found comfort in writing long detailed letters to my parents twice a week. Weekends were more difficult than weekdays. Thanksgiving Day and Christmas had no meaning for us. But then came Hanukah, the first time for us to be away of home on a traditionally celebrated holiday. We felt very lonely. Leon invited me with my family for dinner in his house. He was careful to give me directions. Driving up to 9 Maybeck Twin Drive was an unforgettable experience. The view from the living room was breathtaking, and the table was elegantly set. Leon poured wine, and we had a lovely Hanukah celebration. The food was delicious, Julian and Paul were marvelous with my kids, and I felt so deeply grateful for everything. Then it was Ginette Henkin, Leon’s faithful wife, always thoughtful, who really surprised us by giving the kids chocolate coins for Hanukah Gelt. We went back to the married students’ residency at Albany village heart warmed.

4 Leon—A Logician in Every Way

Leon and I had weekly meetings to discuss my progress. One day he stopped the discussion and asked me to write his name in Hebrew on the office blackboard. With full confidence I stood up and wrote HENKIN in Hebrew. As soon as I put the chalk down, he said: This can’t be right. I was astonished. Here I am, a native Israeli, with Hebrew as my mother tongue, and he is telling me I was wrong. How come?—I asked him. “Well” he said “in my name, there are two identical consonants **n**, and I cannot see two identical letters in your Hebrew spelling of my name.” I was amused by the logical argument, and he was happy to accept the resolution that in Hebrew there are two different ways of writing this consonant, the ordinary one and the other one for a suffix.

5 Leon's Mathematical Humor and Warm Correspondence

Since I got my Ph.D. in 1975, Leon and I corresponded occasionally. I visited the Henkins almost every summer, and they traveled to Israel as well.

His letters were always cheerful. Here are a few quotes:

22 Aug 1995:
Dear Nitsa, Your message indicates that I should not reply because you'll seldom get to e-mail before your return home on Sept 16, but I can't wait.
The urgency of my desire to communicate with you arises from a combination of personal impulse and social obligation... both of which are operative concerning two communications. I only hesitate because I see no way to select one of these messages to put first. The mathematical solution to this dilemma is to repeat both messages alternately in a doubly infinite sequence (i.e., one having neither first nor last elements):

```
..., x^-(n+1), x^-n, ..., x^0, ..., x^n, x^(n+1), ...
```

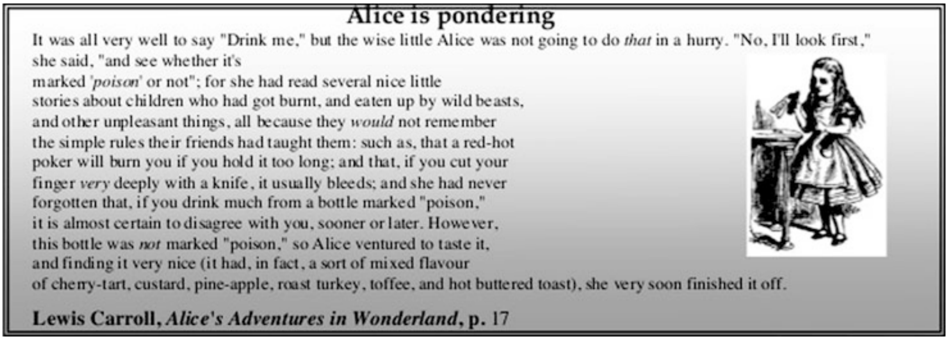
for all positive integers n.
With this understanding, here are the messages `{x^even}` and `{x^odd}` that I wish to convey without further delay.
`{x^even}` = hearty congratulations on your receipt of the Lester Ford award,
`{x^odd}` = thank you VERY much for the weighty photographic memory.

Tue, 8 Oct 1996
Subject: Re: Your new papers
So what do you think, Nitsa—shall we form a Partnership for Mutual Admiration?
Love, Leon

From: henkin@math.berkeley.edu (Leon Henkin)
Date: Wed, 16 Oct 1996
To: nitsa@techunix.technion.ac.il
Well OF COURSE I don't mind your sending (copies of) my two papers to colleagues, Nitsa...
Mind? I'm DELIGHTED!
After all, when someone publishes an article (or book), (s)he wants people to read it!
Anyway, THANKS for sharing my papers with others.
And I'm glad I made you laugh the other day just before you went to give that talk about which you were tense.... I've been tense for some weeks, and will be tense for a few more, about a talk I have to give in Montreal on Nov 5. Part of the aging process for me has been an increasing concern about whether I can produce writing, talks, or teaching, at a level that will satisfy me.
Well, it will take you quite a while to start aging, so don't start worrying now!
Leon

6 Tracing Back Logic in Wonderland to My Ph.D. Studies Days

Years of experience in teaching a course in introductory Logic and Set Theory yield repeated frustration about prospective elementary school teachers' attitudes towards these topics. These student teachers regularly regard the course as not useful, dry, uninteresting (if not awfully boring), and definitely not directly applicable to the school curricula. Over and again, as school year starts we wonder—How can the teaching of logic and set theory


Alice is pondering

It was all very well to say "Drink me," but the wise little Alice was not going to do *that* in a hurry. "No, I'll look first," she said, "and see whether it's marked '*poison*' or not"; for she had read several nice little stories about children who had got burnt, and eaten up by wild beasts, and other unpleasant things, all because they *would* not remember the simple rules their friends had taught them: such as, that a red-hot poker will burn you if you hold it too long; and that, if you cut your finger *very* deeply with a knife, it usually bleeds; and she had never forgotten that, if you drink much from a bottle marked "poison," it is almost certain to disagree with you, sooner or later. However, this bottle was *not* marked "poison," so Alice ventured to taste it, and finding it very nice (it had, in fact, a sort of mixed flavour of cherry-tart, custard, pine-apple, roast turkey, toffee, and hot buttered toast), she very soon finished it off.

Lewis Carroll, *Alice's Adventures in Wonderland*, p. 17

Fig. 1 From [1, p. 96]

be made more "juicy"? Is there an intriguing way to expose students to "the game" of drawing conclusions? to the set-theoretical symbolic language? It occurred to us that it might be useful to employ Lewis Carroll's masterpiece for this purpose. After all, he was a logician and surely could not ignore it while writing his beautiful stories. In a paper published in 2006 with my former student and present colleague Atara Shriki, we described the process of developing such a program and its outlines. Indeed, we found 75 places along the book which we were able to use as leverage for teaching sentential logic and some first-order logic too. During the whole process, I found myself listening to Leon's voice echoing in my mind while I was working on Childern's Conditional Reasoning for my dissertation under his supervision. Here are two sample handouts from the book entitled *Logic in Wonderland* recently published (in Hebrew; English version is in progress).

6.1 Sample Handout 1

Based upon the quote from Alice in Fig. 1, answer the questions below:

1. List all the expressions in the text that describe a stipulation. Do they have a common structure?
2. Consider Alice's decision to drink from the bottle. What was it based upon? Would you approve of it? Why?

6.2 Sample Handout 2

During her walk in Wonderland, Alice met a cat. Alice asked for the cat's assistance. A part of their conversation appears in Fig. 2.

Consider again the conversation between Alice and the Cat. Which of the following statements summarizes the Cat's claim:

1. In order to be mad, it is necessary to come to Wonderland;
2. In order to be mad, it is sufficient to come to Wonderland;
3. In order to come to Wonderland, it is necessary to be mad;
4. In order to come to Wonderland, it is sufficient to be mad.

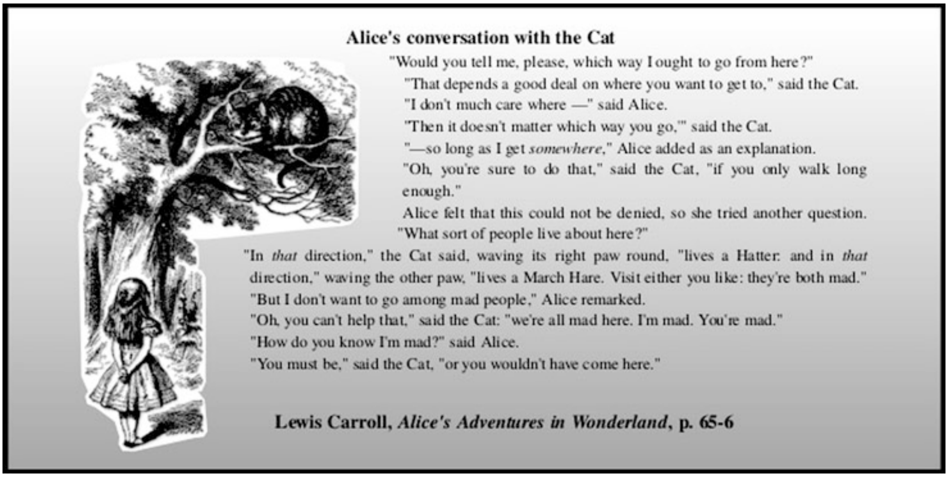


Fig. 2 From [1, p. 99]

7 In Summary

Leon was one of the most serious logicians of the 20th century. His work is mentioned in many places, but my favorite one is the way Hofstater describes it in his famous book: Gödel, Escher, Bach (pp. 541–543, 709). I feel very fortunate to have been his graduate student since I learned from him much more than logic. It is his humanity that conquered my heart. I always wish I am not less kind to my graduate students and no less eager to follow their professional growth after graduation than he was to me.

Reference

1. Movshovitz-Hadar, N., Shriki, A.: Logic in wonderland: Alice's adventures in wonderland as the context of a course in logic for future elementary teachers. In: Clarke, B., Grevholm, B., Millman, R. (eds.) Tasks in Primary Mathematics Teacher Education, pp. 85–103. Springer, Berlin (2009)
2. Hofstadter, D.R.: Gödel, Escher, Bach: An Eternal Golden Braid, pp. 541–543, 709. Vintage Books, New York (1980)

N. Movshovitz-Hadar (✉)
Technion, Israel Institute of Technology, Technion City, Haifa 3200003, Israel
e-mail: nitsa@technion.ac.il

Henkin and the Suit

Albert Visser

The events described here took place in 1979.

At this time, I was a Ph.D. student in Utrecht under Dirk van Dalen. I was working on a thesis concerning Diagonalization & Provability. To put the story into context, the reader must imagine that I was a person who was not much concerned with clothing. I had long hair and always wore an old sweater and old trousers.

At a given moment, my promotor Dirk van Dalen called me to him speaking the words *By the way, Albert . . . .* These words are not translated from Dutch, Dirk actually spoke them in English, including the pronunciation of my name. These words usually signaled that either Dirk was going to utter some point of criticism or had some assignment for me. In the case at hand, it was both. Dirk told me (in Dutch) that the famous Leon Henkin was visiting but that he, Dirk, and his wife Dokie would be abroad at the moment Henkin and his wife would arrive. Would I function as his replacement to be the host as long as he, Dirk, was away? Of course, I would. It was an honor to do this. I should realize, Dirk continued, that Leon Henkin was an old-fashioned man, a professor of the old school. He was used to being received by someone properly dressed, more specifically by someone wearing *a suit* with a tie. So would I be so good to wear a suit when receiving Henkin?

Well, I could hardly say *no*, but there was a difficulty. I did not possess anything even remotely resembling a suit, and I had no idea how to select a good one. Fortunately, a solution was easily found: A female friend of impeccable taste volunteered to help me find a suit. We went to a number of shops, and, following her advice, I bought a truly splendid grey suit with accompanying tie. Additionally, I had my hair cut by a high quality hairdresser.

So when the Henkins arrived, I met them in proper attire. Everything went well, and Henkin's visit proceeded without a wrinkle.

At the end of the visit, there was a farewell dinner at which I sat opposite Henkin and his wife. We immediately engaged in companionable conversation. The Henkins were very easy to talk to, so before I knew it, I told them the prequel concerning the suit. Mrs. Henkin looked properly amused. She told me: *Well, after you received us wearing that suit, I said to Leon that apparently they are quite formal here. So, the next days you better wear a suit.* And, thus, Leon Henkin appeared each day wearing a suit.

This is all well and good, but what does the story teach us about the Henkins? The story makes clear that the Henkins were very open and friendly people. They easily gained

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_4

someone's trust. The fact that I told them the story during the dinner is the best possible testimony of that.

A. Visser (✉)
Philosophy, Faculty of Humanities, Utrecht University, Janskerkhof 13, 3512BL Utrecht,
The Netherlands
e-mail: a.visser@uu.nl

A Fortuitous Year with Leon Henkin

Solomon Feferman

Abstract This is a personal reminiscence about the work I did under the direction of Leon Henkin during the last year of my graduate studies, work that proved to be fortuitous in the absence of Alfred Tarski, my thesis advisor.

Keywords Completeness of predicate calculus · Henkin's proof of completeness · Incompleteness theorems · Formal consistency statements · Arithmetization of metamathematics · Interpretability of theories

In September 1955, I returned from a two-year stint in the US Army to continue and hopefully complete my graduate studies in mathematics at UC Berkeley. When I was drafted in 1953, I had been working on a thesis with Alfred Tarski under considerable strain and with only partial results. As it happened, Tarski was on sabbatical in Europe the year I returned, and he had asked Leon Henkin to act as my supervisor in his absence, to which Henkin agreed. This would prove to be crucial in helping me bring my doctoral work to a successful conclusion. Unlike my often late night discussions with Tarski, Henkin requested that we meet every Thursday afternoon for an hour or so, easy enough to do and even enjoy; I recall standing at the blackboard with him in his light-filled office in Dwinelle Hall. He also said that he wanted to hear something new from me each time; that was more challenging and a powerful constant prod. And finally, Henkin readily accepted my proposal for a major change in my thesis topic, a change that Tarski might well have resisted. So all that proved to be fortuitous, but there were also deeper connections with Henkin's own work.

I first met Leon Henkin in 1952 when he came to Berkeley to consider an offer for a tenured position in the Mathematics Department. He was then teaching in Los Angeles at the University of Southern California (USC), where he was working in comparative isolation in the field of logic. One reason for the offer was that he had made some of the earliest applications of model theory to algebra, a direction that had great appeal to Tarski. In those days, model theory, set theory, and algebraic logic were the main topics of research in the group of students and faculty surrounding Tarski. Being part of such a center of activity had great appeal to Henkin, but he refused to come while the infamous Loyalty Oath was still in force. This special and quite controversial oath was a requirement that had been laid down in 1950 by the Board of Regents for employees of the University of California system. It declared that one was not a member of the Communist Party or any other organization dedicated to the overthrow of the United States government. With it the university had aimed to forestall the McCarthy-era threats of investigation by such

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_5

entities as the House Un-American Activities Committee. The institution of the Loyalty Oath had thrown the faculty into an uproar, among other things on the grounds that one already had to swear to uphold the US Constitution and that it was a clear violation of academic freedom. A number of distinguished faculty members who refused to sign on reasons of principle were fired, some left, while others stayed on but objected strenuously. Despite the great attractions of the Berkeley offer, Henkin sided with those who opposed the Loyalty Oath, and he decided to bide his time while the matter played out in the courts.

When the oath requirement was overturned in 1953, Henkin accepted the offer and came to Berkeley. By then I had already left for the army, but the personal contact I had made with him in 1952 laid the ground for our later work together. In fact, our encounter during his initial visit had been very friendly, and Leon had encouraged me to get in touch with him if I happened to be in Los Angeles. On the next occasion when my wife, Anita, and I were there visiting family and friends, I did just that, and he and his wife, Ginette, invited us to a casual dinner at their small apartment and made us feel at ease. With a difference in age of seven years, Leon was much closer to me than Tarski, and our similar ethnic and cultural backgrounds—both of us descendants of Eastern European Jewish immigrants—was a common touchstone that was understood without needing to be discussed. Also, he had grown up in Brooklyn, whereas I had grown up in the Bronx before my family moved to Los Angeles in the latter part of the 1930s.

Though at the height of the Cold War, my period of service, 1953–1955, in the US Army fortunately fell between the hot wars of Korea and Vietnam. And thanks to my mathematical background, I ended up being stationed in a Signal Corps research lab at Ft. Monmouth, New Jersey, where we mainly spent the time calculating "kill" probabilities of defensive Nike missile batteries; that did not exclude one from being assigned KP ("Kitchen Police") or night guard duty from time to time. I lived off base with my wife in a tiny house where our first daughter was born; finances were more than tight, and there was much to do at home to help out. Still I managed to keep my logical studies alive (when sleep deprivation and breathing space allowed) by reading Kleene's Introduction to Metamathematics (see [14]) in order to get a better understanding of recursion theory and Gödel's theorems than I had obtained in my Berkeley courses. As it happened, out of the blue one day when I was well advanced in that work, I received a postcard from Alonzo Church asking if I would review for The Journal of Symbolic Logic an article by Hao Wang [22] on the arithmetization of the completeness theorem for the classical first-order predicate calculus. (I can still visualize what turned out to be the characteristic card from Church, meticulously handwritten in multicolored ink with wavy, straight, and double-straight underlines.) I do not know what led Church to me, since we had had no previous contact, and I was not known for expertise in that area; perhaps my name had been recommended to him by Dana Scott, who had left Berkeley to study with Church in Princeton. Quite fortuitously, my work on that review led me directly down the path to my dissertation. I have described that in some detail in an article "My route to arithmetization" (see [6]) and so will only give an idea of some of the main points here.

The completeness theorem for the first-order predicate calculus is a simple consequence of the statement that if a sentence φ is consistent, then it has a model and in fact a countable one. Actually, Gödel had shown that this holds for any set of sentences T, from which the compactness theorem follows directly. A theorem due to Paul Bernays in Hilbert and Bernays [13] tells us that any sentence φ can be formally modeled in the natural numbers if one adjoins the statement of the consistency of φ to PA, the Peano

Axioms; in other words, φ is interpretable[1] in $PA + Con(\ulcorner\varphi\urcorner)$, where $\ulcorner\varphi\urcorner$ is the numeral corresponding to the Gödel number of φ, and $Con(\ulcorner\varphi\urcorner)$ expresses the logical consistency of φ. Wang generalized this to the statement that if T is any recursive set of sentences, then T is interpretable in $PA + Con_T$, where Con_T expresses the logical consistency of T in arithmetic. Wang's somewhat sketchy proof more or less followed the lines of Gödel's original proof of the completeness theorem. In my review, I noted that his argument could be simplified considerably by following Henkin's proof (see [11]) instead, by then much preferred in expositions.[2] But in addition I criticized Wang's statement on the grounds that it contained an essential ambiguity. Namely, there is no canonical number theoretical statement expressing the consistency of an infinite recursive set of sentences T since there are infinitely many ways $\tau(x)$ in which membership in T (or more precisely, the set of Gödel numbers of sentences in T) can be defined in arithmetic, and the associated statements Con_τ of consistency of T need not be equivalent. So that led me to ask what conditions should be placed on the way that the formula τ defines T in PA in order to obtain a precise version of Wang's theorem. Moreover, the same question could be raised about formulations of Gödel's second incompleteness theorem for arbitrary recursive theories T. By the time I was ready to return to Berkeley, I was determined to strike out on my own and deal with these issues as the subject of my thesis. There are several reasons why this would have been a problem if Tarski were not away that year.

My dissertation efforts under Tarski's direction prior to 1953 had been decidedly mixed. He had been sufficiently impressed with me in my course and seminar work with him to make me an assistant in graduate courses on metamathematics and set theory and then a research assistant on several of his projects. I had demonstrated that I could meet his exacting standards of rigor and clarity of presentation. When I came to him for a research topic for a thesis, he proposed that I establish his conjecture that the first-order theory of ordinals under addition is decidable. This would strengthen his earlier result with Mostowski that the theory of the simply ordered structure of ordinals is decidable. Moreover, it would be best possible since adjunction of multiplication would lead to an undecidable theory. I worked long and hard on this problem without arriving at the desired result, but I was able to show that the decidability of the theory of ordinals under addition was reducible to that of the weak second-order theory of the ordered structure of ordinals, using a new notion of generalized powers of structures. This was definite progress, but not by itself enough for a thesis.[3]

Meanwhile, Tarski proposed another problem to me, namely to demonstrate the representation theorem for locally finite cylindric algebras, in other words, that every nontrivial such algebra is isomorphically embeddable in the cylindric algebra of essentially finitary relations on an infinite set. That is an algebraic version of the completeness theorem for

[1] Throughout, I use "interpretable" here to mean relatively interpretable in the sense of Tarski, Mostowski, and Robinson; see [20].

[2] Actually, a further simplification of Henkin's argument due to Hasenjaeger (see [10]) became the preferred mode of presentation.

[3] I was pleased to learn in 1957 that Andrzej Ehrenfeucht obtained the sought-for decidability results; he applied back-and-forth methods rather than the elimination of quantifier methods that Tarski had expected. And, as it later turned out, the basic idea of generalized powers that I had introduced to reduce the decision problem could be combined in a fruitful way with the work of my fellow student Bob Vaught on sentences preserved under Cartesian products, leading to the paper Feferman and Vaught [7] on generalized products of theories.

first-order logic. And that led me to establish the desired representation theorem by transforming Henkin's proof of the completeness theorem into algebraic terms. I proposed to Tarski that I combine this with my reductive work on the theory of ordinals to constitute a thesis in two parts. But Tarski was not satisfied with my proof of the representation theorem; he wanted something that was more intrinsically algebraic.[4] My prospects of making further substantial improvements on either topic did not look promising, nor did they any longer hold any appeal to me, so I decided to tackle something new. Tarski might have resisted my choice to work on the problem of arithmetization of the completeness and incompleteness theorems rather than on a problem that he had proposed because it related instead to questions that basically went back to the work of Gödel, his chief rival for the honorific, "most important logician of the 20th century."[5]

On the other hand, Henkin was sympathetic because he had already raised another problem concerning arithmetization in "A problem concerning provability" (see [12]), namely whether or not the sentence ψ that expresses of itself that it is provable in *PA*, that is, for which $PA \vdash \psi \longleftrightarrow Prov_{PA}(\ulcorner \psi \urcorner)$ is provable in *PA*. This was by contrast with Gödel's sentence γ that expresses of itself that it is not provable in *PA* in the sense of $PA \vdash \gamma \longleftrightarrow \neg Prov_{PA}(\ulcorner \gamma \urcorner)$. In the latter case, all we need to know about the formula $Prov_{PA}(x)$ used to state this is that it numeralwise defines the set of provable sentences of arithmetic in *PA*. However, Kreisel [15] showed that the same condition is not sufficient to give a definite answer to Henkin's question: under one choice of the numeralwise representation of provability, the associated sentence ψ is provable in *PA*, whereas under another choice, it is not provable in *PA*.[6]

In my thesis work with Henkin I decided to move as closely as possible to canonical arithmetization by taking $Prov_\tau(x)$ to express in a standard way that x is the number of a sentence that is provable in the first-order predicate calculus from the set of sentences represented by $\tau(x)$. But even so I was able to show that that is not determinative. On the one hand, by taking Con_τ to be the negation of the sentence $Prov_\tau(\ulcorner 0 = 1 \urcorner)$, I was able to obtain a precise generalization of Gödel's second incompleteness theorem to arbitrary recursively enumerable consistent extensions T of *PA*.[7] Namely, if T is represented in *PA* by an RE (i.e., Σ_1) formula $\tau(x)$, then Con_τ is not provable in T. In particular, for the standard (bi-)numeration π of *PA* in *PA*, Con_π is not provable in *PA*. But, on the other hand, I was able to construct a bi-numeration π^* of *PA* in *PA* for which Con_{π^*} is provable in *PA*; $\pi^*(x)$ expresses that x belongs to the "longest" consistent initial segment of the axioms of *PA*.

These results turned out to have novel consequences for the relation of interpretability between theories. On the one hand, my generalization of Gödel's second incompleteness theorem could be further strengthened to show that if T is a recursively enumerable consistent extension of *PA* and $\tau(x)$ is any RE formula numeralwise representing T in *PA*

[4] Years later, I learned from Steve Givant that this was the standard route for proving the representation theorem, but I have not checked the literature to see exactly how it is usually presented.

[5] Tarski told John Corcoran that he considered himself to be "the greatest living sane logician"; cf. [1, p. 5]. My frustrations working with Tarski as a student were by no means unique as is testified to in the many stories in that biography.

[6] See Halbach and Visser [9]. Löb [18] proved that for the standard formalization of the provability predicate, the Henkin sentence is provable in *PA*.

[7] As is well known nowadays, this can be improved to arbitrary recursively enumerable extensions of the fragment Σ_1-*IA* of *PA* and even weaker theories.

then $T + Con_\tau$ is not interpretable in T. On the other hand, I obtained a precise general version of the Bernays–Wang arithmetized completeness theorem in the following form: if $\tau(x)$ is any formula that numeralwise represents T in *PA*, then T is interpretable in $PA + Con_\tau$. Moreover, it turned out by use of the formula $\pi'(x) = \pi^*(x) \vee x = \ulcorner \neg Con_\pi \urcorner$ that one has $PA \vdash Con_{\pi'}$, so $PA + (\neg Con_\pi)$ is interpretable in *PA*. This "interpretability of inconsistency" is thus a foil to the "noninterpretability of consistency."[8] With further related results obtained during that crucial year (1955–1956), Henkin was satisfied that I had enough for a thesis, and he enthusiastically recommended that to Tarski, who was still my principal advisor. But it was only when Tarski obtained a further corroboration from Andrzej Mostowski that he eventually agreed to accept it for such.

To cap off my fortuitous year with Leon Henkin, he heard from Patrick Suppes of an opening for an instructorship at Stanford to teach logic and mathematics. The subject of logic was there based in the Philosophy Department since Mathematics was a bastion of classical analysis in those days. After a personal visit to meet Suppes, an appointment with a joint position in Mathematics and Philosophy was made, and I came to Stanford in 1956; I have been there ever since, except for a number of fellowships and visiting positions elsewhere over the years.[9]

1 Coda

The results of my thesis were published in the paper "Arithmetization of metamathematics in a general setting" (see [3]), which has been cited frequently for that subject in the subsequent literature.[10] I also made use there of a further result due to Steven Orey, who realized that nonstandard representations like π^* could be used to arithmetize the compactness theorem. The following is a special case of Orey's theorem: If T is a recursively enumerable theory and each finite subset of T is interpretable in *PA*, then T is interpretable in *PA*.[11] One extensive direction of work that was fruitfully opened up by the 1960 paper but that I did not myself pursue any further concerned the general lattice structure of the interpretability relationship between theories; cf., e.g., Hájek and Pudlák [8, Chap. III.4], Lindström [16] and [17], and Visser [23]. However, I did go on to use RE representations in an essential systematic way in Feferman [4] to obtain a precise uniform formulation of the transfinite iteration of consistency statements that allowed me to extend the results of Turing [21]. And that, at the important suggestion of Georg Kreisel, opened the way for me to go on to characterize predicative provability in analysis via an autonomous progression of theories; see [5]. The rest, as they say, is history.

---

[8] See Visser [24] for a full exploration of the phenomenon of interpretability of inconsistency.

[9] I spent the first year at Stanford writing up my thesis (see [2]) and received the Ph.D. at UC Berkeley in 1957.

[10] Cf., e.g., Hájek and Pudlák [8, p. 2]. Incidentally, see Feferman [6, p. 177] for an explanation of why the ongoing plans to combine my thesis work with that of my fellow student, Richard Montague, in the form of a monograph were never completed.

[11] Orey had heard me talk about my thesis work at the Institute for Symbolic Logic held at Cornell in the summer of 1957, and that led him to his theorem, which he kindly let me include in my 1960 publication; cf. also Orey [19].

I am not in general one for "what ifs," but here goes. What if my work with Tarski prior to 1953 on the problems he proposed had been successful? What if I had not been drafted? What if I had not been asked by Church to review the Wang paper? What if Tarski had not been away the year I had returned from the Army? What if Henkin had not been in Berkeley to act in his place? In fact, none of the "what ifs" held, and I am eternally grateful to Leon Henkin for his being there for me at the right place at the right time.

Acknowledgements I wish to thank Anita Burdman Feferman and Albert Visser for their helpful comments on a draft of this article.

References

1. Feferman, A.B., Feferman, S.: Alfred Tarski. Life and Logic. Cambridge University Press, New York (2004)
2. Feferman, S.: Formal Consistency Proofs and Interpretability of Theories. Dissertation, Univ. of California at Berkeley, Berkeley (1957)
3. Feferman, S.: Arithmetization of metamathematics in a general setting. Fundam. Math. **49**, 35–92 (1960)
4. Feferman, S.: Transfinite recursive progressions of axiomatic theories. J. Symb. Log. **27**, 259–316 (1962)
5. Feferman, S.: Systems of predicative analysis. J. Symb. Log. **29**, 1–30 (1964)
6. Feferman, S.: My route to arithmetization. Theoria **63**(3), 168–181 (1997)
7. Feferman, S., Vaught, L.: The first order properties of products of algebraic systems. Fundam. Math. **47**, 57–103 (1959)
8. Hájek, P., Pudlák, P.: Metamathematics of First Order Arithmetic. Springer, Berlin (1993)
9. Halbach, V., Visser, A.: The Henkin sentence. (This volume)
10. Hasenjaeger, G.: Eine Bemerkung zu Henkin's Beweis für die Vollständigkeit des Prädikatenkalküls der ersten Stufe. J. Symb. Log. **18**, 42–48 (1953)
11. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**, 159–166 (1949)
12. Henkin, L.: A problem concerning provability. J. Symb. Log. **17**, 160 (1952)
13. Hilbert, D., Bernays, P.: Grundlagen der Mathematik, Vol. II. Springer, Berlin (1939)
14. Kleene, S.C.: Introduction to Metamathematics. North-Holland, Amsterdam (1952)
15. Kreisel, G.: On a problem of Henkin's. Indag. Math. **15**, 405–406 (1953)
16. Lindström, P.: Some results on interpretability. In: Jensen, F.V., et al. (eds.) Proceedings of the 5th Scandinavian Logic Symposium, pp. 329–361. Aalborg Univ. Press, Aalborg (1979)
17. Lindström, P.: Interpretability in reflexive theories—A survey. Theoria **63**(3), 182–209 (1997)
18. Löb, M.H.: Solution of a problem of Leon Henkin. J. Symb. Log. **20**, 115–118 (1955)
19. Orey, S.: Relative interpretations. Z. Math. Log. Grundl. Math. **7**, 146–153 (1961)
20. Tarski, A., Mostowski, A., Robinson, R.M.: Undecidable Theories. North-Holland, Amsterdam (1953)
21. Turing, A.: Systems of logic based on ordinals. Proc. Lond. Math. Soc. **245**, 161–228 (1939)
22. Wang, H.: Arithmetical models for formal systems. Methodos **3**, 217–232 (1951)
23. Visser, A.: Interpretability logic. In: Petkov, P. (ed.) Mathematical Logic, pp. 175–208. Plenum Press, New York (1990)
24. Visser, A.: The interpretability of inconsistency. In: Logic Group Preprint Series, Preprint Series of the Department of Philosophy, vol. 318. Universiteit Utrecht (2014). Available via http://www.phil.uu.nl/preprints/lgps/authors/visser/

S. Feferman (✉)
Department of Mathematics, Stanford University, Stanford CA 94305-2125, USA
e-mail: feferman@stanford.edu

Leon Henkin and a Life of Service

Benjamin Wells

Abstract For 45 years, Leon Henkin provided dedicated, unstinting service to people learning mathematics. During most of that time, the author had personal contact with him. Henkin's seminars, projects, letters, and advice influenced the author's career path on many control points.

Keywords Henkin · Logic · Mathematics education · Service · Personal recollection

1 Seminars in Berkeley and Montreal with Leon Henkin

My first impressions of Leon Henkin were permanent. He had little hair, slightly impeded speech, and constant good cheer signaled by an instant yet lasting smile—indeed, he beamed. Even when serious or taking one to task, he conveyed sympathy, warmth, and genuine interest. Although our relationship was neither social nor personal, there was great fondness and appreciation on both sides.

In 1962, I began graduate work at the University of California in Berkeley having completed three courses in topology (and audited two more) and two courses in logic at MIT as an undergraduate. All of these except the first topology course were listed as graduate courses. At Berkeley, I enrolled in Metamathematics from John Addison, Topological Groups from Gerhard Hochschild, General Algebraic Systems from Alfred Tarski, and a seminar on Algebraic Geometry from Claude Chevalley. I considered topology at the top of the list, but there were no advisors in the field, so I chose Shiing-Shen Chern's geometry listing as the closest thing. He remained unfailingly interested in whatever I was doing for decades after advising me that first year. I signed up for Tarski's course because of a partial knowledge of and fascination with the Banach–Tarski paradox; I wanted to meet him.

I also audited (not for credit) a seminar on Metamathematics and Field Theory jointly offered by Leon Henkin and Tarski. Although Henkin and Tarski played equal roles in the conversation of the seminar, Henkin was the structural leader. He made the schedule of talks and kept the course organized. This would be the pattern in future Henkin–Tarski seminars I attended. I never took a course lectured by Leon, but I read cover to cover his printed notes for the undergraduate course in mathematical logic and found them lucid beyond measure. This was also my assessment of his talks in seminars and conferences.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_6

> Wells was a Ph.D. student in our Department and I first got to know him in 1962, when he enrolled in a seminar on the metamathematics of field theory which Alfred Tarski and I conducted jointly. In the next two years I saw him in action in further seminars on universal algebra and on cylindric algebras, and he also worked in a seminar on equational logic with Tarski in 1965.[1]

For the 1963–1964 Henkin–Tarski seminar on algebraic logic, Henkin suggested I try to rewrite the Daigneault–Monk proof of the representability of infinite-dimensional polyadic algebras "in the style of the seminar." He outlined the approach, and I painstakingly worked through the conversion. But there was a snag, reported in [19, p. 228]. I still think Henkin's impression of the possibility and worthiness of this alternate proof are correct, but it was not mine to give.

In spring 1966, Henkin and Tarski offered a seminar on metamathematics and algebra. Both were happy that I was willing to translate and discuss Yuri Yershov's result on the decidability of the elementary theory of p-adic fields. That summer in Montreal, at an institute sponsored by Séminaire de Mathématiques Supérieures, I presented Paul Cohen's sketch of a simplified proof of that result to Simon Kochen, coauthor with James Ax of a published independent and lengthy proof of this and much more.

> Tarski and I considered him a very promising student, and were well pleased with the seminar talks that he gave. As a result, we appointed him to work on an NSF research project in metamathematics and foundations; he served as a Research Assistant and Postgraduate Research Associate during 1965–1967, and again in 1971. Between 1967 and 1971 Wells visited Poland as a Stanford Exchange Fellow. This visit came about because Wells met the world-famous logician Andrzej Mostowski at a Séminaire de Mathématique Supérieure that was held in Montreal for a period of several weeks in Summer, 1966; I gave a series of talks there, subsequently published as a monograph [5], and Wells's dissertation is related to some of those ideas.[2]

Leon's lectures introduced predicate logic with a finite number of variables. This novel development raised many issues that were solved collaboratively in and beyond class. More than any other seminar that summer, his talks stimulated a continuous interactive discussion. My dissertation [18] studied equational logic with finite numbers of variables, based on an issue arising from my translation of a Mal'tsev paper appearing as [15, Chap. 29]. Equational theories are constructed that are not recursive despite every fragment with a bounded finite number of variables being recursive. Although unrelated in substance, I have often mused how this was foreshadowed in tone by Henkin's topic in Montreal.

Leon correctly recalls that my wish to visit Poland arose from my meeting Mostowski (and his student Janusz Onyszkiewicz, later to be the foreign press spokesman for Solidarity) in Montreal. A corrective outcome from our first encounter is related in [22, p. 422].

2 Leon Henkin's Help with a Visit to Poland

Both Henkin and Mostowski were consulted in Montreal about whether it made sense for me to visit Warsaw in 1966–1967. Both were encouraging. Next, I wrote Tarski for his view.

---

[1] See [9].

[2] See [9].

July 23, 1966

Dear sir,

Montréal has proved to be a delightful experience: mathematically, culturally, gastronomically, and a bit socially. I have been writing lecture notes for Kochen. He is speaking on the relations among completeness proofs by model completeness, elimination of quantifiers, and ultraproducts. Unfortunately he will not get to talk about p-adic fields in class, but we are discussing such problems on the sly. Also I have written the Mal'cev review and clarified some of my work relating to it—and some new problems. I have also been reading and thinking about the recursive real numbers and trying to write up an explanation of Cohen's p-adic number decision procedure. There are fine restaurants and beautiful girls.

I want to go to Warsaw this fall. Before you count me totally mad, read on, please.

(1) The situation in Berkeley. [...] I have committed myself to a parttime RA appointment as you know. Mr. Henkin says that as far as he is concerned I would not be bound to it if you too released me. [...] At this point let me state that while I do want to convince you of the wisdom of this project, I do not intend to undertake it without your blessing and am prepared to consider alternatives with an open mind. Roughly speaking then, there are no strong reasons of an "administrative" sort why I could not be away from Berkeley for at least 6–9 months.

(2) The situation in Poland. [...] A seminar will be given throughout the year on model theory for infinitary logics. Mr. Mostowski assures me that I will be well-received. I am already making plans to go climbing in the Tatra.

(3) The wider context. [...] If I stay in Berkeley I may or may not finish, and if not it will be another two years (from now) before I can go anywhere. The last few months in Berkeley were very bad; there are too many diversions; I tried to shut myself off from the world, but could not. Even though it may seem paradoxical, I think the change of scene would stimulate my mathematics rather than my procrastination. In evidence of this I might point to the situation here—while not great it must be seen in the context of 10–15 lectures per week, weekends often elsewhere, and mild climate. And yet here I can sit down and work steadily for hours. In Berkeley it happens, but it's difficult to control. [...]

(4) The arrangements. I have discussed this matter with Mostowski and Henkin. [...] I have asked Dana Scott to get Stanford to send me information on the exchange program. [...] As far as finances go, I have talked to Mostowski's student Janusz Onyszkiewicz, and he thinks one can live well on about \$50/month. [...] University rules permitting, Henkin thinks I could hold my RA in Warsaw, if, of course, you both approve. [...] The prospect of spending a fruitful season in Warsaw is an exciting one. But a year in Berkeley is not without hope and promise. [...]

Here is Tarski's response in a letter imperfectly recalled in [3, p. 325] and not rediscovered until after that book was published. Part of it appeared in [21].

July 29, 1966

Mr. Benjamin F. Wells III, c/o Seminaire de Mathematiques Superieures, Universite de Montreal, Montreal 3, Quebec CANADA

Dear Pete,

I must say frankly that your letters have caught me somewhat by surprise. My reaction to their contents is rather mixed.

First let me say that there is certainly no "administrative" obstacle to your desire to go to Warsaw for the next year. I am ready to release you from your commitment. The only thing I wish is that you inform me of your final decision as soon as feasible so that we can appoint somebody else in your place. As regards the possibility of your retaining the research assistantship and the connected salary for the period of your studies in Poland the matter seems to me rather dubious. [...] Also the idea could not be realized without an explicit agreement on the part of the National Science Foundation. I shall discuss this matter with Professor Henkin on his return to Berkeley.

You will certainly have inspiring scientific contacts in Warsaw. Let me say however that you really could have had more such contacts in the Bay Area than you actually had if you tried hard enough.

This brings us to the heart of the whole problem. Both you and I realize and have realized for a long time that your greatest weakness is the lack of ability to "shut yourself off from the world",

> to resist diversions and distractions coming from outside. This single factor probably accounts for the insignificant progress which you have made so far in your work on the thesis. I wonder whether you will find in Warsaw a more favorable atmosphere from this point of view. I have learned from your letter with interest that there are fine restaurants and beautiful girls in Montreal. This information may be quite important for me if I ever get an invitation to teach there. I can assure you however that Warsaw is not a monastery and at any rate not a nunnery. There are fine restaurants there [. . .] and as regards the amount of beautiful girls Warsaw claims to be second to none among the cities of the world. If after your return from Warsaw you do not lend wholehearted support to this claim you will hurt my deepest feelings.
>
> To speak now seriously, if I am to evaluate the problem from a purely rational point of view I do not see much point in your going to Warsaw, at any rate not at this stage of the game. I might feel differently about this matter if you had now just finished the work on your thesis or at least were very close to its completion. I know however that reactions of human beings are usually not rational and that their actions are not motivated by rational factors. People in that part of the world are claimed to have various secret weapons. Maybe they will make some of them available to you and you will learn how to beat down the intrusion of the outside world.
>
> As you see this letter is something less than blessing but I am equally far from wishing to keep you by force in Berkeley. If the decision were entirely up to me I would probably suppress all my misgivings and have you go to Warsaw, treating the whole venture as a calculated risk. [. . .]
>
> Sincerely, Alfred Tarski

In fact, I did not visit Poland until 1967–1968, and then indeed as a Stanford Exchange Student. In the meantime, I learned some Polish and worked as a Research Assistant in the Group in Logic and the Methodology of Science (L&M) for the Tarski–Henkin NSF project as planned. Because my NSF Graduate Fellowship was not renewed, I also taught several UC Berkeley undergraduate math sections, including a Fortran class that would ultimately lead me to Meher Baba and India on my return from Europe.

In Warsaw, Leon's student Diane Resek and I shared an office in the Palace of Culture and Science, Stalin's wedding-cake building given to the Polish people. It dominated the Warsaw skyline. According to Dana Scott [17], S.C. Kleene was very pleased to tell this story about the Palace. When Adlai Stevenson visited Warsaw, he was shown the Palace and asked how he liked it. He replied: "Small, but in perfect style." Diane and I had many conversations about mathematics and life with our thesis advisors. Our paths crossed often after that.

3 Some Conferences; the Political Situation in Poland

In 1963, Addison, Henkin, and Tarski organized the very important International Symposium on the Theory of Models at Berkeley. The focus on model-theoretic applications in algebra and set theory was innovative and trend-setting. The three organizers edited the proceedings [2].

In 1968, the third International Congress for Logic, Methodology, and Philosophy of Science was held in Amsterdam shortly after I arrived in Poland. Mostowski and several of his students, including me, came from Warsaw. The proceedings do not list Henkin as an organizer or speaker, but I have a sense that he was there. He is not mentioned as a participant, and neither am I.

During my stay in Poland, there was much political unrest. There was an anti-Soviet atmosphere, but also government-stimulated antisemitism. In fall 1967, protesting students were arrested after the banning and closing of an anti-Soviet production of Mickiewicz's anti-Russian play “Dziady”; I was among them. Student demonstrations led to beatings by police and mass arrests at the University of Warsaw starting on 8 March 1968. Protests and strikes followed, accompanied by governmental claims that Zionists were the fomenters. Official promotion of antisemitism increased. There were consequent coerced departures—call them exiles—from Poland of many Jews during the spring and summer.

For summer 1968, Mostowski organized a Conference on the Construction of Models for Axiomatic Systems in Warsaw. In spring 1968, Leon wrote Abraham Robinson, then president of the Association for Symbolic Logic (ASL), expressing dismay at the Polish government’s behavior and requesting ASL to consider withdrawing sponsorship of the meeting—according to Robinson’s reply [16] to Leon.

ASSOCIATION FOR SYMBOLIC LOGIC

ABRAHAM ROBINSON
President

DEPARTMENT OF MATHEMATICS
YALE UNIVERSITY
NEW HAVEN, CONNECTICUT 06520

May 14, 1968
Professor Leon Henkin, Department of Mathematics University of California Berkeley, California 94720

Dear Leon:

Thank you very much for your letter of May 7.

I wrote to Mostowski about a month ago telling him that I had decided not to attend the Warsaw meeting. I offered to give my reasons for doing so but have not heard from him since. Of course it did not escape me that if several people in this country came to the same decision, the phenomenon might be noticed in Poland. At the same time I did not want to harm our friends there.

However, the question of taking official action in this matter on behalf of the Association for Symbolic Logic is of an entirely different nature. To withdraw our sponsorship at this date would as far as I know be without precedent and might in the long run work against the return of sanity to Poland rather than for it. Although I agree that the behavior of the Polish authorities is quite obnoxious, many other governments make mistakes now and then, and I think that official bodies like the ASL have to react more prudently than individuals in such cases.

The situation will be changed if it appears that the Polish authorities actually refuse to admit prospective participants in the meeting for reasons of race, creed, etc. If such cases come to my knowledge I should consider it my duty to take action.

Yours sincerely,
Abraham Robinson
c.c. Prof. Martin Davis Prof. Dana Scott Prof. Alfred Tarski

I was planning to revisit Warsaw for this meeting. As I hitchhiked with a friend to the Austrian–Czechoslovakian border on my way to Prague to pick up a visa and then Warsaw, we learned that the Soviet Union had invaded Czechoslovakia the night before. Austrian border guards let us pass, and we met confused, worried, scared Czech guards in the buffer zone. They turned us back. Our attempt to cross was shown on national Austrian television that night. By the time I could return to Warsaw, the conference had ended, and there were no direct observations of any effects from the political situation or this letter. After a final interview with Mostowski, I left Poland for 33 years.

I have learned [17] that there were other cancelations besides Robinson’s (and presumably Henkin’s). Mostowski’s feelings were very hurt by people’s reactions. He said “Why me?” to Dana Scott after he canceled, meaning that people were somehow blaming

Mostowski or making him partly guilty. According to Dana, he meant that his whole history before, during, and after the war had nothing to do with anti-Jewish actions in Poland. And he also suffered under communism very much. So Robinson's sensitivity did not outweigh the insensitivity and the pain that Mostowski felt and some who canceled continue to feel. I choose to read courage on both sides.

The next (and last) time I would see Mostowski would be at the 1971 Tarski Symposium that celebrated Alfred's 70th birthday. Henkin participated, offering a talk and paper on cylindric algebras jointly with Don Monk. He is also listed as the principal editor of the published proceedings [11]. He and John Addison hired me as translator of Yershov's contribution and technical copy editor for several other chapters. But his major work was behind the scenes in the organization of this complex conference with visitors from seven countries and many states.

4 The Mal'tsev Book

> One of the aftermaths of Wells's sojourn in Europe was his translation of A.I. Malcev's papers on the metamathematics of algebraic systems, from Russian. Published in the prestigious Studies in Logic series, the volume represents an important contribution by Wells to research on the border between algebra and logic.[3]

The story of how Mostowski guided me to translate and compile this book is recounted in [22, p. 426]. The first three chapters of the Mal'tsev book [15] are of particular interest here because of controversy raised by the joint Henkin–Mostowski review [13] of the original papers in Russian for the second and third chapters and their historical note on the German paper [14] I translated as the first chapter. The controversy involves whether Henkin (and Mostowski) treated the Mal'tsev papers fairly and even whether Henkin should have given more, and more appropriate, credit to Mal'tsev for a compactness theorem anticipating in some sense his own completeness result in his dissertation. Here are some recent comments [23] I wrote to Sol Feferman on this matter.

> Dear Sol,
>
> It seems there are three issues: (1) Did Henkin fail to give credit (or at least acknowledgment) to Mal'tsev? (2) Did Henkin have an on-the-face conflict of interest in the joint Henkin–Mostowski JSL review? (3) Did that bias the review? My Mal'tsev translation of the 1936 German "Untersuchungen" paper ([14] and Chapter 1 of the Mal'tsev book) reveals nothing relevant and merely states there are issues with Skolemization, deferring to the 1959 Henkin–Mostowski review, which is NOT a review of the 1936 paper but two later ones—translated as my Chapters 2, 3. But it includes a historical note dealing with the early paper. (There is also a 1937 JSL review by Rosser of the 1936 paper.) I'm not sure Henkin should be commenting (and that, strongly) on 1936, but he was not reviewing it, very strictly speaking. On the other hand, I am disturbed that he sends the readers to his own proof (or A. Robinson's) for a "satisfactory proof." Maybe one could say that was Mostowski's voice.
>
> In an email exchange with you before the publication of ATL&L [3], you sent me an early version of this passage on p. 306:
>
>> A student of Kolmogorov's in Moscow during the 1930s, Mal'cev had done pioneering work on the applications of logic to algebra. His 1941 paper (in Russian) made use of a compactness theorem for first-order languages allowing uncountably many symbols, which

[3] See [9].

> anticipated much that was done after the war by Leon Henkin and Abraham Robinson. This work was largely unknown in the West until the 1950s; and only in 1971 would Mal'cev's papers become widely available with the publication of a volume of English translations by Tarski's student, Benjamin ("Pete") Wells.

In response, I wrote:

> I have reread my note (2) on page 14 of my book [15] and I have reread my notes and corrections written on the reviews of the 1936 German paper [15, Chap. 1] by Rosser and of the 1941 and 1956 papers [15, Chaps. 2, 3] by Henkin and Mostowski. First, Rosser is plain wrong: he misunderstood "equivalent." I think it is clear that Henkin and Mostowski are giving a generous (and not unjustified) interpretation when they reflect on the nearness of proof in 1936 but I think they are being picky when they say it is not formulated there. It takes no great leap to put results of Section 4 and Theorem 1 together, as I point out in the note (2). They don't flat out deny that Mal'cev proved compactness with arbitrary symbol sets in 1936, but they refer the reader to a "satisfactory" proof in Henkin or Robinson. My own introduction by Tarski to reviewing for JSL (and further experience) tells me there is a little turfing going on here. Who cares?
>
> My impression is that Tarski thought a compactness result was there in Mal'cev. My note (2) refers to "some difficulty" and specifically two types of difficulty. I gather from my notes on the Henkin–Mostowski review that that was my understanding of why the stronger unstated/unproved form of the SNF was required.

[...] —— cites the footnote in Henkin's 1950 JSL article [4, p. 90] on "Completeness in the Theory of Types":

> A similar result for formulations of arithmetic within the first order functional calculus was established by A. Malcev [14]. Malcev's method of proof bears a certain resemblance to the method used above. I am indebted to Professor Church for bringing this paper to my attention. (Added February 14, 1950.)

as if that means he knew about the 1936 paper in advance of his thesis. Of course he knew about it in advance of his 1996 historical article [10] on completeness. [...] Apparently it does not mention Mal'tsev, and that seems to be where the most definite finger can be pointed—given his 1950 footnote and his 1959 review, Leon should have said SOMETHING about the Mal'tsev result in a 30-page paper.

It is worth noting that the Rosser review antedates everything but its subject, Mal'tsev's paper. Although German would be more generally readable, the paper appeared in a Russian journal. But here is a very early reference in JSL. It may be that this review (or more probably arranging it) alerted Church, who was of course reviews editor from 1936 to 1979. [...]

Henkin was a friend, an incredibly valuable and valued advisor on several matters, and a politician. I am surprised to learn that his behavior is questioned so boldly [...], but I cannot defend him specifically on this matter. I do not recall if we ever discussed the Mal'tsev 1936 paper and the reviews, but I am sure Alfred and I did and maybe I spoke to Leon. I am sure I was alerted to the joint review by someone other than my own digging, although I did dig up all reviews (and all other translations) of papers in the book. Unfortunately, that sheds no light on these issues.

My initial answers to the questions above are: (1) Yes, but did he owe credit? In the dissertation probably not, but in the history paper, he should have offered recognition of Mal'tsev's work if only to say he felt it was too flawed to count. Even if he felt that had been clearly put in the joint review, it should have been revisited. (2) Yes. (3) Maybe, but it may still all be true and all in accord with Mostowski's views.

This may contribute to resolving the controversy, but it is not a solution. With the passing of the players, that may prove unsolvable.

5 Mathematics Education and Leon Henkin

Leon spent over 45 years supporting and improving mathematics education through a variety of initiatives, organizations, courses, and projects. My own involvement with Project SEED and Project APT (Alternatives through Peer Teaching) stemmed from a letter Leon wrote graduate students in February 1971.

DEPARTMENT OF MATHEMATICS University of California, Berkeley

TO: Selected graduate students
FROM: Leon Henkin
RE: Part-time positions teaching mathematics in the schools

I am writing to call to your attention the existence of four programs which from time to time have available part-time positions for which graduate students in the mathematical sciences are eligible. All of these positions involve teaching mathematics to pre-college students.

Most graduate students recall their early mathematics training as dull, and perhaps even painful. I think it is fair to say that this widespread feeling, together with the slow pace of mathematics learning in the early grades, is at least in large part due to the fact that teachers in the early grades know very little mathematics, and often dislike—sometimes even fear—grappling with this part of their teaching duties. On the other hand, there is every reason to believe that during those early years school children have an enormous capacity to learn, and can easily be interested in mathematical ideas when presented enthusiastically by competent people. In an attempt to apply these observations, the following programs have been developed to provide supplemental mathematics education for pre-college students at various grade levels by specialized mathematics teachers, over and above the normal mathematics courses for those grade levels taught by the regular teacher.

(1) Project SEED. This program operates in various school districts throughout California (and in a few other states). Instructors must have the equivalent of an A.B. degree with a major in mathematics or a closely related field. Instructors currently earn about $2,000 for teaching one class 40 minutes per day 5 days a week through the school year. Often two classes are available. Most job opportunities start in September, with applicants chosen in the spring or summer; a few jobs open up during the year. Instructors are employees of individual school districts, but beginning March 1, 1971, will be recommended through a University office.

This program was developed by William Johntz ... Instruction is by the so-called “discovery method”, wherein no text or syllabus is prescribed, but education proceeds by oral questions and answers. After visiting such a class several times, a potential instructor is invited to try his hand at it. It is normally limited to grades 2–6 (ages 7–12). [...]

(3) The Peer Teaching Program. [Alternatives through Peer Teaching, or Project APT] This has developed on a very small scale as an offshoot of Project SEED; the NSF has just funded a project to begin next fall on the Berkeley campus, through the Special Scholarship Committee of the Academic Senate. Louis Schell [...] will be the Director.

In this program school students normally in the junior high school grades (7–9) are given small-group and individual help to bring them to the point where they can teach mathematics by the discovery method to a group of students either at their own grade level or perhaps one or two years younger. Instructors for this program require qualifications similar to that of Project SEED; they can earn up to $ 5,000 during the next academic year, although a heavier commitment of time is called for. Each instructor will work with 8 “peer teachers” of junior high school age, who will be divided into two groups of 4. The instructor will meet each group for one period each day, and will have to put in a certain amount of additional time with individual peer teachers. Instructors who have not had experience in using the group discovery method of instruction (as in Project SEED) will be required to gain a limited amount of such experience this spring, before they can be employed in this project.[4]

[4]See [6].

I immediately requested interview appointments for both of these positions. Louis Schell conducted the two interviews as one. I began training that spring and was hired for both projects, based on observations of my teaching and reports I wrote. It is likely that Henkin played a role in the hiring, especially with Project SEED. It is worth noting that the Special Opportunity Scholarships office, which was the first home of APT at UC Berkeley, administered a program that served as a model for the federal Upward Bound Program launched a year or so after Henkin founded the former [1].

I taught with Project SEED for a year and with Project APT for nine years.

> At this point in his career, Wells suddenly turned his interest from mathematical research to mathematics education. He was employed on a university project, directed by Louis Schell, which involved him in training ninth-grade "peer teachers" to conduct a complete mathematics class of seventh-grade students twice a week. This project was funded by the NSF [National Science Foundation] during 1971–1973, and then there was a follow-up evaluation year funded by NIE [National Institute of Education]; I was the Principal Investigator for both projects and know that Wells did fine work, and gained invaluable experience, through his involvement. I was then teaching a course in our Department for prospective elementary-school teachers, and I several times asked Wells to bring a peer-teacher and a group of seventh-graders to the University, so that my students—the future teachers—could gain some appreciation for the potential of this unusual mode of instruction.[5]

In particular, Leon was PI for the NIE study [12] on the effectiveness of our peer-teaching program. This showed that under numerous measures, there was no significant loss of performance when 8th and 9th grade peer teachers took over entire scheduled 7th or 8th grade math classes for two of five periods per week. In many of the experiments, there was significant gain. This is why Leon was enthusiastic about showing the method to the future teachers in his classes. He was also a frequent visitor to our peer-teacher-taught classes in the junior high schools.

In an effort to communicate across the entire peer-teaching project, we invited all peer teachers and trainers to a day at UC Berkeley, with a talk by Leon on logic and a swim at Strawberry Canyon. It is interesting that the peer teachers found Leon to be a clear and eloquent explainer but could not grasp why he did not ask more questions, the basis of their own teaching style. I would frequently recall this later in my own college classes when I talked more than I asked. Leon stayed in touch with the program after the first two years of federal support as the school district picked up the funding.

6 Leon's Help with University Administrations

Leon Henkin eagerly provided graduate students his advice and help with administrative issues. When it came to getting anything done, it was always Leon who wrote the letters. For a number of years, he documented that graduate assistants did not need to pay income taxes because he affirmed that their services were those expected of all degree candidates (US income tax guidelines specifically exclude this reasoning now). He held the post of NSF project co-PI and administrator, first chair of L&M, chair of the Department of Mathematics, and president of Association for Symbolic Logic (ASL) at various times.

[5] See [9].

Not only was he a great help with university administrations, he was himself a great administrator.

Leon was the chair or cochair for fifteen students awarded Ph.D.s. Of these, he was sole dissertation committee chair for only two Ph.D. students in mathematics and one in mathematics education, three women named Carol Karp, Diane Resek, and Nitsa Hadar.

> From about 1976, without slackening his work with school mathematics, Wells began to resume work on his doctoral dissertation with Professor Tarski. This is an enormously difficult thing to do—I've seen many dozens of advanced mathematical students drop out of a Ph.D. program with the intent of completing their research while doing other kinds of work, but I know only one other, besides Wells, who succeeded.[6]

Leon and Alfred served as joint chairs for Haragauri Narayan Gupta's dissertation committee. He is not the other successful returnee, because he finished quickly and continuously, but he may be one of the strongest examples of Tarski's tendency to overwork students. Wanda Szmielew's student Zenon Piesyk was also researching geometry based on Tarski's axioms. Tarski and Szmielew drove their two students crazy by repeatedly telling them that one had surpassed the other's results. The consequence was that Gupta's thesis approached a ream in length, well over 400 pages.

> In the fall of 1981 his [Wells's] research was largely completed, and I attended a presentation of his work that he made at our Logic Colloquium. I was very well impressed, both with the quality of his mathematical ideas and findings, and with his very clear and able presentation.[7]

The colloquium date was actually 8 May 1981. After the talk, Henkin thanked me, and Bob Solovay told me that it was very good work. I said I wished Tarski saw it that way, but he wanted much more. Bob said, "I would give you a Ph.D. for your first theorem." I was astounded, grateful, encouraged, and somewhat tempted to jump ship to the Solovay fleet.

By fall 1981, the thesis was coming along but had a way to go yet. In addition, there were several bureaucratic obstacles, primarily the lapsing of my candidacy for the doctoral degree. When the Department tried to have me reinstated, Dean Geoghegan refused, saying I needed to take the oral qualifying exams plus two language tests again. Alfred's attitude was: "I don't see why it is a difficulty to recertify the qualifying exams. I see no need for you to be present." At about the same time, I mentioned my relief that Berkeley did not require oral thesis defenses. Tarski said, "I always liked oral defenses. I will be happy to arrange one for you … if you like."

In spring 1982, Dean Geoghagen went on leave, and Leon joined the fray by engaging his replacement, Dean Simmons.

> May 17, 1982
>
> Benjamin Wells September 1, 2014 9:08 PM
> TO: Dean William Simmons, Graduate Division
>
> At your suggestion, I am following up our phone conversation concerning the reinstatement of Mr. Wells into candidacy for the Ph.D. degree. He was last a registered student in 1967. About one year ago, Professors McKenzie and Lam of this Department wrote to Dean Geoghegan to request reinstatement of Mr. Wells. I enclose a copy of their letter. Subsequently, Dean Geoghegan replied

[6] See [9].

[7] See [9].

> to Mr. Lam, declining to reinstate Mr. Wells. He stated that it would be necessary for Wells to take new qualifying exams, have his languages recertified, and file a new advancement to candidacy.
>
> In our conversation, I inquired as to the reason for requiring the (former) student, and a group of faculty examiners, to undertake the time and effort necessary to follow through the steps outlined by Dean Geoghegan. You replied that the purpose was to ensure that the candidate is still as knowledgeable and capable mathematically as he was at the time of his original examination. I then pointed out that the letter from McKenzie and Lam gave strong evidence, in the form of a list of the continuing mathematical activities and publications by Mr. Wells, to show that his earlier level of attainment had not fallen into neglect. However, Dean Geoghegan's letter continued with the sentence, "No matter what Mr. Wells has been doing in the interim, sixteen years is just too much." As I pointed out, this statement is not consistent with the explanation you gave for requiring a former student to retake examinations.
>
> Mr. Wells' dissertation adviser is Professor Emeritus Alfred Tarski, whose exacting standards are well-known in our Department. Professor Tarski has informed me that during the year Wells continued to make further improvements on his dissertation, and Tarski now considers that it is completely in order and deserving of the degree. For this reason, we are anxious to proceed to have the degree issued as expeditiously as possible.[8]

On 28 May, Simmons reinstated my candidacy, thanks wholly to Leon's intervention. The candidacy was valid until the fall thesis-filing deadline of 12 November 1982. Geoghegan returned to duty in fall 1982, so it became clear that this deadline was indeed drop dead. Simmons required me to register for fall classes, and although Tarski reluctantly supported a filing-fee waiver of registration on 30 September, Louise Kerr in the Graduate Division office pointed out that Geoghegan would have to approve it, so if I was smart I would register. That bought me 12 units of credit, which I asked Manuel Blum of Computer Science to grant me for time served on Turing machines. Manuel had been a fellow student in a topology course at MIT. He agreed; he was about to become much more involved.

The Logic and Methodology of Science Group as well as those associated with logic in the Department of Philosophy and the Department of Mathematics participated enthusiastically in the annual Logic Picnic held at a Berkeley or regional park, frequently Codornices Park across Euclid Avenue from Berkeley's Rose Garden. Leon was usually a gregarious host at this communal affair. It was at the picnic in fall 1982 that I learned Bill Craig was on sabbatical in England, possibly from Leon. That was a big surprise because Craig was the "outside" member of my dissertation committee, and he was due to get a near-final draft in a few days. I knew how to reconstitute the committee because I had just arranged for Ralph McKenzie to replace Gerhard Hochschild, who had no memory of even being on it after 18 years. I wrote Tarski on 3 October, developed a short list of candidates, reviewed it with him, and happily agreed with his proposal of Manuel Blum, who joined the reformed committee!

Leon was involved in many of these adjustments. His greatest gift came with the filing of the dissertation and is discussed in the next section.

During my three sabbatical leaves from the University of San Francisco, I was a Visiting Scholar in Mathematics at UC Berkeley (1989–1991, 1997–1998, 2004–2006). In each case, Leon served as the official sponsor or as a supporter of the appointment. The first sabbatical spanned an ASL meeting at UC Berkeley. As usual, Leon helped with the organization of the conference. He also attended my talk there, "Infinity on Purpose."

[8] See [7].

He thought it was humorous, a need at such meetings. I thought both judgments were positive.

Around that time he served on the dissertation committee for Art Quaife, chaired by John Addison. Quaife had arrived at UC Berkeley the year after John and I did. He quickly acquired a pet lion and then wrote a theorem prover for Stephen Cook. He soon dropped out in favor of joining the cryonics growth industry. Many years later, he consulted Addison on the state of automated theorem proving, suggesting he could start a prover on the Goldbach Hypothesis, be on ice for several centuries, and have the proof after a reconstitution. Addison guided him discreetly and indirectly toward doctoral-level results. John made a point of telling me at the ASL conference's social event, "Your record of taking nearly 20 years to finish your doctorate is broken. Art Quaife has completed a thesis to my satisfaction. I'm telling you before I tell him, because he has taken 26 years!"

I was Alfred Tarski's last student, in the sense that my dissertation was the last he signed. He did not write me a reference letter largely because of scientific projects he devoted his waning energy to completing before his death.

> Professor Tarski, his dissertation supervisor, was known as a teacher who holds his students to the very highest standards. Despite his expressed satisfaction with Wells's work [in spring 1982], he asked Wells to polish some of the writing, and try to resolve one or two related questions for inclusion in the final dissertation. When this was completed to specifications, the degree was finally awarded in December 1982. Meanwhile, Professor Tarski's health began to weaken seriously and he passed away in October 1983. Throughout the previous year he had great difficulty in speaking with people, or in writing. For this reason, he was unable to write on behalf of Wells, as I know he would otherwise have done.[9]

Leon wrote me many letters, and this one, substituting for the traditional Professor's Letter, was the kindest.

7 The Henkin Plan and the Dissertation

Henkin's most critical, most helpful contribution to my scientific life was what came to be called the Henkin Plan for completing my degree requirements by filing a dissertation.

> 11 November 1982
>
> To: Dean, Graduate Division
> Fr: Leon Henkin
> Re: Benjamin F. Wells, III.
>
> Mr. Wells is a candidate for the Ph.D. degree in this Department. His work, and he as an individual, are well known to me.
>
> His dissertation supervisor is Professor Emeritus Alfred Tarski. The latter told me, as long ago as last summer, that he was satisfied with Wells's dissertation and its writing, although he had asked the candidate to make small changes in the Introduction and in the Abstract. This Wells did, approximately two months ago. However, Professor Tarski has not been able to look at the work he requested of Wells, or to sign the title page of the dissertation. The reason is that Professor Tarski is desperately ill.
>
> I use the word "desperately" advisedly. In mid-September, just before leaving for Europe, I visited Tarski and found him very frail. At that time his wife told me that he found it impossible

[9] See [9].

> to sleep, except for dozing in a chair for 20 minutes at a time, several times in each 24 hours. Since my return from Europe 10 days ago I have learned that Tarski was hospitalized for 2 days in October and 4 days last week, because in addition to being unable to sleep, he has the greatest difficulty in eating. I have tried to talk to him on the phone each day since my return, but each time his wife returns to tell me that he feels too weak to speak on the phone.
>
> Under these circumstances, I recommend that Mr. Wells's Ph.D. Committee be re-constituted, so as to replace Prof. Tarski. Indubitably the person most appropriate to serve as Chairman would be Professor Ralph McKenzie, already a member of the Committee. Other members of the Department who are familiar with Wells's work and could serve on the Committee include Professor Julia Robinson, and myself. Tomorrow is the last day before Wells's candidacy lapses, and I feel that in the light of all the circumstances of his case, it would be grossly unfair to delay his degree because of Professor Tarski's illness.[10]

Only changes in the abstract and introduction needed to be verified, but Tarski had delayed this for weeks. Leon had recently returned from Yugoslavia. He was outraged to hear how Tarski was dragging this out toward the deadline of my candidacy lapse, Friday, 12 November 1982. On Wednesday, 10 November, he proposed what I called the Henkin Plan: (1) he would personally urge Alfred to sign the dissertation immediately; (2) if Tarski were too ill to consider that, then Henkin would sign Tarski's name, initialed LH, as long Tarski approved this; if both of these failed, then (3) Tarski would be removed as a member of the committee, Ralph McKenzie would succeed as chair, and Julia Robinson would join the committee, with Manuel Blum of Computer Science as outside member in all cases. While changing the committee members required only an advisor's signature, changing the chair was more complex—hence the letter above. Adding Henkin to the committee was not considered because Julia had already read a recent draft, and Leon's familiarity with the results dated from the Logic Colloquium talk in May 1981.

Before Julia would sign (3), she told me that I had to guarantee her that someone on the new committee had read every page—she knew it had grown longer, and she said her scientific reputation was at stake. I went straight to Ralph and told him that I had added about 30 pages at Tarski's direction and he must read them immediately, to make my promise to Julia good. At first refusing, he reluctantly agreed to do it within three hours. Later he said the material looked fine, but there were several typos—I never found them.

The next 24 hours are hard to relive. Some reference to them appears in [3, p. 375], but the actual events were more bizarre than discernible there. The amazing story of Tarski's eventually signing the title page I brought to his house early on 12 November 1982, but his writing the date as 11 November is recounted in [20]. One aspect was absent there. I had Tarski's signature at the last moment, but Blum had refused to sign until Tarski did, a deference that did not extend to the pages with Leon's "Alfred Tarski LH" or Ralph's signatures, which he gladly signed ahead of time. Manuel and Lenore were going to a movie, and I asked him what I should do if Tarski signed and they had already left. I followed Manuel's advice (not part of the Henkin Plan), and—omitting details—it eventually involved my headlight.

Louise Kerr at Graduate Division kept the office open past 5 pm till I could turn in the completed dissertation and multiple copies. In fact, her building, California Hall, was locked up, but one massive metal door was jogged out a quarter inch. I grabbed that edge, prayed and pulled it open, and entered. By then I had signed dissertation title pages for

[10] See [8].

all three committees, but of course only the first was used. Henkin's plan had succeeded in the best way!

It seemed to me very likely that Leon's intercession at (1) was crucial, that essentially and maybe solely because of Leon Henkin, Tarski signed on time. My gratitude to both of them was boundless. But this impression is not final or complete. After all, the letter above indicates Alfred would not take Leon's earlier calls. My friend Jim, who advised me and recommended implementation of the Henkin plan, said on 11 November, "We have done all that is humanly possible." I took it as a sign that greater assistance might be needed. and might be given—it always may be, kindly, justly. How else to explain the 11 November dating?

The day after Alfred signed the thesis, he told Leon, "I did not know Wells had such a strict deadline." (I had told him often.) Tarski told Steve Givant then, "If I had felt better, I would have made Wells work longer."

8 Conclusion

Leon Henkin led a lifetime of loving service to mathematics and education, to students and teachers. The Leon Henkin Citation for Distinguished Service is awarded by the Committee on Student Diversity and Academic Development of the Berkeley Division of the Academic Senate. It is given in recognition of an "exceptional commitment to the educational development of students from groups who are underrepresented in the academy." This award was first given in 2000, and the first laureate was Leon Henkin [1].

The near decade I spent working in school programs created and shepherded by him taught me the value of this contribution, beyond his staggering influence on mathematical logic and algebraic logic. This review of connections with Henkin has shown me how vital, how free, how generous was his help in so many ways for so many years.

Thousands of children, young college students, teachers in training, women, minority members, doctoral candidates, and many others received opportunities for better education and better mathematics education through Leon's decades of effort—a magnificent benefit!

References

1. Addison, J., et al.: In memoriam: Leon Albert Henkin (2006). http://senate.universityofcalifornia.edu/inmemoriam/leonalberthenkin.html
2. Addison, J., Henkin, L., Tarski, A.: The Theory of Models. North-Holland, Amsterdam (1965)
3. Feferman, A.B., Feferman, S.: Alfred Tarski: Life and Logic. Cambridge University Press, Cambridge (2004)
4. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**, 81–91 (1950)
5. Henkin, L.: Logical systems containing only a finite number of symbols. Séminaire Mathématiques Supérieures, Université de Montréal (1967)
6. Henkin, L.: Letter to selected graduate students (February 1971)
7. Henkin, L.: Letter to Dean William Simmons, Graduate Division, University of California (17 May 1982)
8. Henkin, L.: Letter to the Dean, Graduate Division; dictated by phone to Jim's friend Shireen Kaul and signed in Henkin's name and his own by Morris Hirsch, Chair of Mathematics, University of California, Berkeley (11 November 1982)

9. Henkin, L.: Letter to whom it may concern. Reference letter for Benjamin Wells (22 January 1984)
10. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**, 127–158 (1996)
11. Henkin, L., et al.: Proceedings of the Tarski Symposium. Proceedings of Symposia in Pure Mathematics, vol. 25 (1974)
12. Henkin, L., Amster, H., Schell, L.: Experimental investigation of peer teaching (1975). Final Report. http://eric.ed.gov/?id=ED139753
13. Henkin, L., Mostowski, A.: Review of A. Mal'cev, On a general method for obtaining local theorems in group theory (Russian), and A.I. Mal'cev, On representations of models (Russian). J. Symb. Log. **24**, 55–57 (1959). Includes an historical note concerning [14]
14. Mal'tsev, A.I.: Untersuchungen aus dem Gebiete der mathematischen Logik. Mat. Sb. N.S. **1**(43), 323–336 (1936)
15. Mal'tsev, A.I.: The Metamathematics of Algebraic Systems—Collected Papers: 1936–1967. North-Holland, Amsterdam (1971)
16. Robinson, A.: Letter to Leon Henkin on Association for Symbolic Logic letterhead with copies to Martin Davis, Dana Scott, and Alfred Tarski (14 May 1968). BANC MSS 84/69, Alfred Tarski papers, courtesy of The Bancroft Library, University of California, Berkeley; also by permission of Martin Davis and Dana Scott
17. Scott, D.: Private communication (2014)
18. Wells, B.: Pseudorecursive varieties and their implications for word problems. Ph.D. Dissertation, Univ. of California, Berkeley, CA (1982)
19. Wells, B.: Applying, extending, and specializing pseudorecursiveness. Ann. Pure Appl. Log. **126**, 225–254 (2004)
20. Wells, B.: The triumph of the One. In: Sarhangi, R., Sequin, C. (eds.) Bridges: Mathematical Connections in Art, Music, and Science; Bridges 2004 Conference Proceedings, pp. 103–108 (2004)
21. Wells, B.: Alfred Tarski, friend and daemon. AMS Not. Am. Math. Soc. **54**, 982–984 (2007)
22. Wells, B.: The prince of logic. In: Ehrenfeucht, A., Marek, V.W., Srebrny, M. (eds.) Andrzej Mostowski and Foundational Studies, pp. 421–428. IOS Press, Amsterdam (2008)
23. Wells, B.: E-mail to Sol Feferman (15 February 2014)

B. Wells (✉)
Departments of Mathematics and of Computer Science, College of Arts and Sciences,
University of San Francisco, 2130 Fulton St., San Francisco, CA 94117, USA
e-mail: wells@usfca.edu

Part II
Henkin's Contribution to XX Century Logic

Leon Henkin and Cylindric Algebras

J. Donald Monk

Abstract This is a description of the contributions of Leon Henkin to the theory of cylindric algebras.

Keywords Cylindric algebras · Permutation models · Dilation · Twisting

1 Introduction

Cylindric algebras are abstract algebras that stand in the same relationship to first-order logic as Boolean algebras do to sentential logic. There are two ways of passing from logic to cylindric algebras. For the first, we are given a first-order language L and a set Γ of sentences in L. We assume that L has the sequence $v_0, v_1, \ldots$ of individual variables. We define an equivalence relation $\equiv$ on the set of formulas of L by defining $\varphi \equiv \psi$ iff $\Gamma \vdash \varphi \leftrightarrow \psi$. Then it is easy to see that there are the following operations on the set A of $\equiv$-classes:

$$[\varphi] + [\psi] = [\varphi \vee \psi]; \qquad [\varphi] \cdot [\psi] = [\varphi \wedge \psi];$$
$$-[\varphi] = [\neg\varphi]; \qquad c_i[\varphi] = [\exists v_i \varphi].$$

Then the following structure is a cylindric algebra:

$$\langle A, +, \cdot, -, \bigl[\neg(v_0 = v_0)\bigr], [v_0 = v_0], c_i, [v_i = v_j]\rangle_{i,j\in\omega}.$$

For the second method of obtaining a cylindric algebra, we suppose that a set A is given. We consider the following unary operations C_i of cylindrification acting upon subsets of ${}^{\omega}A$ (the set of infinite sequences of elements of A):

$$C_i X = \bigl\{a \in {}^{\omega}A : \exists b \in X[a_j = b_j \text{ for all } j \neq i]\bigr\}.$$

Then the following is a cylindric set algebra: $\langle B, \cup, \cap, -, \emptyset, {}^{\omega}A, C_i, D_{ij}\rangle_{i,j\in\omega}$, where B is a collection of subsets of ${}^{\omega}A$ closed under the operations $\cup$, $\cap$, $-$ (with $-X = {}^{\omega}A \backslash X$ for any $X \subseteq {}^{\omega}A$), and with $\emptyset$, ${}^{\omega}A$, and D_{ij} as members, where $D_{ij} = \{a \in {}^{\omega}A : a_i = a_j\}$.

Tarski and his students F.B. Thompson and L.H. Chin introduced an abstract notion of cylindric algebra that encompasses both of these cases. For any ordinal number α, a cylindric algebra of dimension α is an algebra of the form $\overline{A} = (A, +, \cdot, -, 0, 1, c_\xi, d_{\xi\eta})_{\xi,\eta<\alpha}$ such that the following conditions hold:

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_7

(1) $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra.
(2) $c_\xi 0 = 0$.
(3) $x + c_\xi x = c_\xi x$.
(4) $c_\xi(x \cdot c_\xi y) = c_\xi x \cdot c_\xi y$.
(5) $c_\xi c_\eta x = c_\eta c_\xi x$.
(6) $d_{\xi\xi} = 1$.
(7) If $\xi \neq \eta, \rho$, then $d_{\eta\rho} = c_\xi(d_{\eta\xi} \cdot d_{\xi\rho})$.
(8) If $\xi \neq h$, then $c_\xi(d_{\xi\eta} \cdot x) \cdot c_\xi(d_{\xi\eta} \cdot -x) = 0$.

Historical remarks on the development of cylindric algebras up to the time of Henkin's work on them can be found in [5]. The development went via the relation algebras of Tarski and the projective algebras of Everett and Ulam.

The work of Leon Henkin concerning cylindric algebra can be divided into these parts: on the algebraic theory of them, the theory of set algebras, representation theorems, construction of nonrepresentable algebra, and applications to logic. Many of the publications of Henkin concerning cylindric algebras are devoted to exposition without proofs. Detailed proofs of most of his results are found in the two volumes [9] and [16], written jointly with Monk and Tarski.

2 Algebraic Theory

The purely algebraic theory of cylindric algebras, exclusive of set algebras and representation theory, is fully developed in [9]. The parts of this theory developed mainly by Henkin are as follows.

If $\overline{A}$ is a CA_α and $\Gamma = \{\xi(0), \ldots, \xi(m-1)\}$ is a finite subset of α, then we define $c_{(\Gamma)}a = c_{\xi(0)} \cdots c_{\xi(m-1)}a$. This does not depend on the order of $\xi(0), \ldots, \xi(m-1)$, by axiom (5). An element a is *rectangular* iff $c_{(\Gamma)}a \cdot c_{(\Delta)}a = c_{(\Gamma\cap\Delta)}a$ for any finite subsets Γ, Δ of α. This notion was first introduced in [3]. Elementary properties of the notion are given in Sect. 1.10 of [9]. Their use in representation theory is described below.

The dimension set Δx of an element x of a CA_α is the collection of all $\xi < \alpha$ such that $c_\xi x \neq x$. The CA_ω obtained from first-order theories as above have the property that the dimension sets are always finite. A CA_α is *locally finite* iff Δx is always finite. This notion is due to Tarski. Henkin introduced the following generalization. A CA_α is *dimension complemented* iff $\Delta x \neq \alpha$ for all x. Algebraic properties of these two notions are worked on in Sect. 1.11 of [9]. Both notions are important in representation theory. In [10], Henkin proved that every locally finite CA_α is isomorphic to one of the cylindric algebras described at the beginning of this article, an algebra of formulas modulo some theory in the language.

If $\overline{A}$ is a CA_α and $a \in A$, then the *relativization* of $\overline{A}$ to a is the structure

$$\overline{A}|a = \langle A \mid a, +', \cdot', -', 0', 1', c'_\xi, d'_{\xi\eta}\rangle_{\xi,\eta<\alpha},$$

where $A|a = \{x \in A : x \leq a\}$, $x +' y = x + y$, and $x \cdot' y = x \cdot y$ for any $x, y \in A|a$, $-'x = a \cdot -x$ for any $x \in A|a$, $0' = 0$, $1' = a$, $c'_\xi x = c_\xi x \cdot a$ for any $x \in A|a$ and any $\xi < \alpha$, and $d'_{\xi\eta} = d_{\xi\eta} \cdot a$ for any $\xi, \eta < \alpha$. In general, the relativization is not itself a CA_α. Algebraic properties of relativizations are developed in Sect. 2.2 of [9]. This is a notion that Henkin worked on thoroughly. It is interesting in its own right and is also useful in constructing nonrepresentable cylindric algebras. In [12], written jointly by Henkin and his student Diane Resek, some simple equations are shown to characterize the class Cr_2 of two-dimensional relativized cylindric algebras. It is also shown there that the class Cr_3 is not closed under subalgebras. Additional results are stated without proof.

Given a CA_α $\overline{A}$ and an ordinal $\beta < \alpha$, we can associate the *β-reduct* of $\overline{A}$, which is the algebra $\langle A, +, \cdot, -, 0, 1, c_\xi, d_{\xi\eta}\rangle_{\xi,\eta<\beta}$. We say that $\overline{B}$ is *neatly embedded* in $\overline{A}$, provided that $\overline{B}$ is a subalgebra of the β-reduct of $\overline{A}$, and $c_\xi b = b$ for all $b \in B$ and $\xi \in [\beta, \alpha)$. Algebraic properties of reducts and neat embeddings are explored in Sect. 2.6 of [9]. Neat embeddings play a prominent role in representation theory.

The duality theory of Boolean algebras can be adapted to cylindric algebras as follows. For a CA_α $\overline{A}$, we associate the following structure, called the *canonical embedding algebra* $\mathrm{Em}A$, where M is the collection of all maximal ideals of the Boolean part of $\overline{A}$:

$$\left\langle \mathrm{P}(M), \cup, \cap, -, \emptyset, M, c_\xi, d_{\xi\eta}\right\rangle_{\xi,\eta<\alpha},$$

where $-x = M\backslash x$ for any $x \subseteq M$, for any $x \subseteq M$ and $\xi < \alpha$, we define

$$c_\xi x = \left\{J \in M : c_\xi^{-1}[J] = \emptyset\right\} \cup \bigcup_{I\in x}\left\{J \in M : c_\xi^{-1}[J] \subseteq I\right\},$$

and for any $\xi, \eta < \alpha$, we define

$$d_{\xi\eta} = \{I \in M : d_{\xi\eta} \notin I\}.$$

The *canonical embedding function* $\mathrm{em}_{\overline{A}}$ is defined by $\mathrm{em}_{\overline{A}}(a) = \{I \in M : a \notin I\}$. The algebraic theory of canonical embedding algebras is developed in Sect. 2.7 of [9]. In particular, $\mathrm{Em}\overline{A}$ is always a CA_α, and *rmem*$_{\overline{A}}$ is an isomorphic embedding of $\overline{A}$ into $\mathrm{Em}\overline{A}$. To show that $\mathrm{Em}\overline{A}$ is a CA_α, one of course has to check the axioms for a CA_α. This procedure can be generalized to Boolean algebras with operators, and this has been carried out by Jónsson and Tarski. The question of extending equations valid in a Boolean algebra with operators to its canonical embedding algebra is difficult. Henkin in [8] contributed to answering this question.

Every CA_α $\mathrm{Em}\overline{A}$ is complete and atomic. This gives rise to another way of defining the class of cylindric algebras. A *cylindric atom structure* is a relational structure $\langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ such that the following conditions hold:

(1) T_ξ is an equivalence relation on B.
(2) $T\xi|T_\eta = T_\eta|T_\xi$.
(3) $E_{\xi\eta} = T_\mu[E_{\xi\mu} \cap E_{\mu\eta}]$ if $\mu \neq \xi, \eta$.
(4) If $\xi \neq \eta$ and $b, c \in E_{\xi\eta}$, then $bT_\xi c$ iff $b = c$.

Given a cylindric atom structure $\overline{B} = \langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$, we define its *complex algebra* $Cm(\overline{B})$ to be the algebra

$$\left\langle \mathrm{P}(B), \cup, \cap, -, \emptyset, B, c_\xi, E_{\xi\eta}\right\rangle_{\xi,\eta<\alpha},$$

where $-x = B\backslash x$ for any $x \subseteq B$ and $c_\xi x = T_\xi[x]$ for any $\xi < \alpha$. Then $Cm(\overline{B})$ is a complete atomic CA_α, and every complete and atomic CA_α can be obtained in this way. Hence, any CA_α is a subalgebra of $Cm(\overline{B})$ for some cylindric atom structure $\overline{B}$. The details of this correspondence are worked out in Sect. 2.7 of [9].

3 Set Algebras

The notion of a cylindric set algebra given in the introduction can be generalized as follows. An algebra $\overline{A}$ is a *cylindric-relativized set algebra of dimension* α iff there is a nonempty set U and a set $V \subseteq {}^{\alpha}U$ such that $\overline{A}$ has the form

$$\langle A, \cup, \cap, -, \emptyset, V, c_\xi, d_{\xi\eta}\rangle_{\xi,\eta<\alpha},$$

where A is a collection of subsets of V closed under $\cup$, $\cap$, $-$ (with $-a = V\backslash a$), c_ξ, with

$$c_\xi a = \left\{y \in V : \exists x \in a[x_\eta = y_\eta \text{ for all } \eta \neq \xi]\right\}$$

and with $d_{\xi\eta} \in A$, where $d_{\xi\eta} = \{y \in V : y_\xi = y_\eta\}$.

In general, such algebras do not satisfy the axioms for cylindric algebras. However, the following special cases do.

With $V = {}^{\alpha}U$, giving *cylindric set algebras*.

With $V = \bigcup_{i\in I} {}^{\alpha}Z_i$, where $\langle Z_i : i \in I\rangle$ is a system of nonempty pairwise disjoint sets, giving *generalized cylindric set algebras*. These set algebras were first introduced in [1].

With $V = {}^{\alpha}W^{(p)}$, where $p \in {}^{\alpha}U$ and ${}^{\alpha}W^{(p)} = \{x \in {}^{\alpha}W : \{\xi < \alpha : x_\xi \neq p_\xi\}\}$ is finite, giving *weak cylindric set algebras*.

With $V = \bigcup_{i\in I} {}^{\alpha}W_i^{(p_i)}$ with ${}^{\alpha}W_i^{(p_i)} \cap {}^{\alpha}W_j^{(p_j)} = \emptyset$ for $i \neq j$, giving *generalized weak cylindric set algebras.*

It turns out that generalized cylindric set algebras and generalized weak cylindric set algebras are the natural algebras for representation theory; a CA_α is *representable* iff it is isomorphic to one of these. The theory of the various kinds of set algebras is described in Sect. 3.1 of [16]; see also [14]. Some of the results were proved earlier in [11].

4 Representation Theorems

That every infinite-dimensional locally finite CA_α is representable is due to Tarski. In [16], this theorem is proved by an algebraic adaptation of Henkin's proof of the completeness theorem for first-order logic. In fact, call an element x of a CA_α ξ-thin iff there is an $\eta \neq \xi$ such that $x \cdot c_\xi(d_{\xi\eta} \cdot x) \leq d_{\xi\eta}$ and $c_\xi x = 1$ and call a CA_α $\overline{A}$ *rich* iff for every nonzero $y \in A$ such that $\Delta y \subseteq 1$, there is a 0-thin element x such that $x \cdot c_0 y \leq y$. The main technical lemma which implies the above representation theorem runs as follows:

If $2 \leq \alpha$ *and* $\overline{A}$ *is a simple rich locally finite* CA_α *satisfying the equality*

$$c_\xi\big(x \cdot y \cdot c_\eta(x \cdot -y)\big) \cdot -c_\eta(c_\xi x \cdot -d_{\xi\eta}) = 0$$

for all distinct $\xi, \eta < \alpha$ *and all* $x, y \in A$*, then* $\overline{A}$ *is representable.*

From this lemma, using algebraic results and facts about set algebras, one can derive the following additional representation theorems due to Henkin:

For $\alpha \geq 2$*, a* CA_α *is representable iff it can be neatly embedded in a* $\mathrm{CA}_{\alpha+\omega}$*. This was first stated, for finite* α*, in* [3]. *For arbitrary* α*, it was stated in* [5].

Every dimension-complemented CA_α *of infinite dimension is representable. This was first stated without proof in* [2].

By a direct proof given in [16] we have the following representation theorem of Henkin, first stated in [3]:

For $\alpha \geq 2$*, a* CA_α *is representable iff it can be embedded in an atomic* CA_α *in which all atoms are rectangular.*

A special representation theorem due to Henkin runs as follows.

Suppose that $\overline{A}$ *is a* CA_α *and the subalgebra of* $\overline{A}$ *generated by* $\{d_{\xi,\eta} : \xi, \eta < \alpha\}$ *is simple. Suppose that there is a positive integer* m *such that*

$$c_0 \cdots c_{m-1}\left(\prod_{i,j<m} -d_{ij}\right) = 0.$$

Then $\overline{A}$ *is representable.*

It is easy to prove that all CA_0 and CA_1 are representable. This is no longer true for CA_2, but Henkin proved that one only needs to add two equations to obtain representability:

A CA_2 $\overline{A}$ *is representable iff the following two equations hold in* $\overline{A}$:

$$c_1\big(x \cdot y \cdot c_0(x \cdot -y)\big) \cdot -c_0(c_1 x \cdot -d_{01}) = 0;$$
$$c_0\big(x \cdot y \cdot c_1(x \cdot -y)\big) \cdot -c_1(c_0 x \cdot -d_{10}) = 1.$$

Many of these representation theorems can be found in [17].

5 Nonrepresentable Cylindric Algebras

It turns out that not every cylindric algebra is representable. Henkin invented three methods of constructing nonrepresentable cylindric algebras, described in Sect. 3.2 of [16].

5.1 Permutation Models

Let U be a nonempty set, and consider the cylindric algebra $\overline{A}$ of all subsets of ${}^\alpha U$. Every permutation f of U extends in a natural way to an automorphism $\tilde{f}$ of $\overline{A}$. If H is a subgroup of the group of all permutations of U, then one can consider the set $\{a \in A : \tilde{f}(a) = a$ for all $f \in H\}$. This set forms a subalgebra $\mathrm{fix}_H(\overline{A})$ of $\overline{A}$. By choosing U and H suitably and taking a relativization of $\mathrm{fix}_H(\overline{A})$ one can obtain a nonrepresentable cylindric algebra. This is carried out in [16] to show that the following inequality (which can be written as an equation) holds in every representable CA_α with $\alpha \geq 3$ but fails in a permutation model:

$$c_0 x \cdot c_1 y \cdot c_2 z \leq c_0 c_1 c_2\big[c_2(c_1 x \cdot c_0 y) \cdot c_1(c_2 x \cdot c_0 z) \cdot c_0(c_2 y \cdot c_1 z)\big].$$

5.2 Dilation

Whereas permutation models take a relativization of a subalgebra of some cylindric algebra, dilation does the opposite: starting with an algebra, one adds atoms. More precisely, let $\overline{B} = \langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ be a cylindric atom structure. Suppose that $a \in {}^\alpha B$ satisfies the following conditions:

$$[a_\xi]_{Y_\eta} \cap [a_\eta]_{T_\xi} \neq 0 \quad \text{for all } \xi, \eta < \alpha.$$

$$a_\mu \notin E_{\xi\eta} \quad \text{for distinct ordinals } \xi, \eta, \mu.$$

Suppose that u is some object not in B. Then we form a new relational structure $\overline{B}' = \langle B', T'_\xi, E'_{\xi\eta}\rangle_{\xi,\eta<\alpha}$ by setting

$$B' = B \cup \{u\}.$$

For any $\xi < \alpha$, T'_ξ is an equivalence relation on B' such that $T'_\xi \cap (B \times B) = T_\xi$, and for any $b \in B$, $bT'_\xi u$ iff $bT_\xi a_\xi$.

$$E'_{\xi\eta} = E_{\xi\eta} \quad \text{for distinct } \xi, \eta < \alpha, \text{ and } E_{\xi\xi} = B' \text{ for any } \xi < \alpha.$$

By a suitable choice of a one obtains in this way a cylindric atom structure whose associated cylindric algebra is nonrepresentable. This is done in [16] to show that the following equation holds in every representable CA_α with $\alpha \geq 3$ but fails in some dilation model: $x; (y; z) = (x; y); z$, where, in general, $u; v = c_2(c_1(d_{12} \cdot c_2 u) \cdot c_0(d_{02} \cdot c_2 v))$.

5.3 Twisting

Roughly speaking, this method consists of selecting two members x, y of a cylindric atom structure together with an index $\xi < \alpha$ and redefining the equivalence relation E_ξ using x and y. Formally, we are given a cylindric atom structure $\overline{B} = \langle B, T_\xi, E_{\xi\eta}\rangle_{\xi,\eta<\alpha}$, two elements $x, y \in B$, an index $\xi < \alpha$ such that not $(xT_\xi y)$, and two partitions $[x]_{T_\xi} = X_0 \cup X_1$ and $[y]_{T_\xi} = Y_0 \cup Y_1$ such that the following conditions hold, where $M = [x]_{T_\xi} \cup [y]_{T_\xi}$:

(1) If $\eta \neq \xi$ and $(a, b) \in (M \times M) \cap T_\eta$ and $a \neq b$, then $(a, b) \in (X_0 \times Y_0) \cup (Y_0 \times X_0) \cup (X_1 \times Y_1) \cup (Y_1 \times X_1)$.
(2) If $\eta \neq \xi$ and $a \in M$, then there is a $b \in M \backslash \{a\}$ such that $aT_\eta b$.
(3) If $i \in \{0, 1\}$ and $\eta, \nu < \alpha$, then $X_i \cap E_{\xi\eta} \cap E_{\xi\nu} \neq 0$ iff $Y_i \cap E_{\xi\eta} \cap E_{\xi\nu} \neq 0$.

Then a new structure $\overline{B}' = \langle B, T'_\eta, E_{\eta\nu}\rangle_{\eta,\nu<\alpha}$ is defined as follows: $T'_\eta = T_\eta$ if $\eta \neq \xi$, whereas T_ξ is the equivalence relation on B with equivalence classes $[z]_{T_\xi}$ for $z \notin M$, along with $X_0 \cup Y_1$ and $X_1 \cup Y_0$.

It is shown in [16] that $\overline{B}'$ is a cylindric atom structure. This is used to show that the following equation holds in every representable CA_α but fails in some twisting model:

$$c_2\big(d_{20} \cdot c_0\big(d_{01} \cdot c_1(d_{12} \cdot x)\big)\big) = c_2\big(d_{21} \cdot c_1\big(d_{01} \cdot c_0(d_{02} \cdot x)\big)\big).$$

6 Applications to Logic

In [6], Henkin considers first-order logic with only finitely many variables. In the case of just two variables x and y, he proves that the formula

$$\exists x(x = y \wedge \exists y Gxy) \to \forall x(x = y \to \exists y Gxy)$$

is universally valid but not derivable from the natural axioms (restricted to two variables). Here G is a binary relation symbol. The nonderivability is proved using a modified cylindric set algebra. This example suggests adding all formulas of the following forms to the axioms for two-variable logic:

$$\exists x(x = y \wedge \varphi) \to \forall x(x = y \to \varphi),$$
$$\exists y(x = y \wedge \varphi) \to \forall y(x = y \to \varphi).$$

Henkin shows, again using a modified cylindric set algebra, that this axiom system is also incomplete; the following universally valid formula is not provable in the expanded axiom system:

$$\exists x Fx \wedge \forall x \forall y[Fx \wedge Fy \to x = y] \to \big[\exists x(Fx \wedge Gxy) \leftrightarrow \forall x(Fx \leftrightarrow Gxy)\big].$$

An analysis of this situation leads to adding the following formulas to the axioms: $\exists x \forall y(\varphi \leftrightarrow y = x) \to [\exists y(\varphi \wedge \psi) \leftrightarrow \forall y(\varphi \to \psi)]$ with x not free in φ; $\exists y \forall x(\varphi \leftrightarrow x = y) \to [\exists x(\varphi \wedge \psi) \leftrightarrow \forall y(\varphi \to \psi)]$ with y not free in φ. But again the resulting axiom system is not complete. By another modified cylindric set algebra Henkin shows that the following formula is universally valid but not derivable in this axiom system:

$$\exists x Gxy \leftrightarrow \exists x(x = y \wedge \exists y Gyx).$$

Finally, adding the following axioms results in a complete axiom system:

$$\exists x \varphi \leftrightarrow \exists x\big(x = y \wedge \exists y \varphi^r\big),$$
$$\exists y \varphi \leftrightarrow \exists y\big(y = x \wedge \exists y \varphi^r\big),$$

where φ^r is recursively defined by interchanging x and y if φ is atomic, with $(\neg\varphi)^r = \neg\varphi^r$, $(\varphi \vee \psi)^r = \varphi^r \vee \psi^r$, $(\exists x \varphi)^r = \exists y(x = y \wedge \exists x \varphi)$, and $(\exists y \varphi)^r = \exists x(x = y \wedge \exists y \varphi)$. The proof of completeness of the resulting axiom system is rather involved but is completely carried out.

It is shown that the above axioms do not suffice for logic with three variables.

In [7], Henkin translates the notion of relativization of a cylindric algebra, described above, into a purely logical framework. Namely, given a first-order language L and a formula π of L (with no restriction on the number of free variables of π), one associates with each formula φ of L its relativization φ^π as follows: $\varphi^\pi = \varphi$ for φ atomic; $(\neg\varphi)^\pi = \neg\varphi^\pi$; $(\varphi \wedge \psi)^\pi = \varphi^\pi \wedge \psi^\pi$, and $(\exists x \varphi)^\pi = \exists x(\pi \wedge \varphi^\pi)$. The main theorem of [7] is as follows. Consider first-order logic with variables in the list $v_0, \ldots, v_n, \ldots$ with $n < \alpha$, where $\alpha \leq \omega$. A natural set of axioms for logic with these variables is explicitly described. Let π be a formula with free variables among $v_0, \ldots, v_{n-1}$. Then a set Δ of formulas having to do with relativization are described, namely all formulas of one of the following two forms:

$$\forall v_0 \dots \forall v_m \big[(\exists v_i \exists v_j \varphi)^\pi \to (\exists v_j \exists v_i \varphi)^\pi\big],$$

where all free variables of φ are among $v_0, \dots, v_m$;

$$\forall v_0 \dots \forall v_m \big(\pi \to \pi'\big),$$

where there are $i, j < \alpha$ such that π' is obtained from π by replacing all free occurrences of v_i in π by free occurrences of v_j, and the free variables of $\pi \to \pi'$ are among $v_0, \dots, v_m$.

The theorem is then that $\Gamma \vdash \varphi$ implies that $\Delta \cup \Gamma \vdash \varphi^\pi$.

This theorem is used, along with a suitable (ordinary) model to show that a certain sentence involving only three variables cannot be proved from the logical axioms. The sentence expresses that if a function has at most two elements in its domain, then it has at most two elements in its range.

References: Publications of Henkin Concerning Cylindric Algebras

1. The algebraic structure of mathematical theories. Bull. Soc. Math. Belg. **7**, 137–141 (1955)
2. The representation theorem for cylindric algebras. In: Mathematical Interpretation of Formal Systems, pp. 85–97. North-Holland, Amsterdam (1955)
3. La structure algébrique des théories mathématiques. In: Collection de Logique Mathématique, Series A, vol. XI. Gauthier-Villars, Paris (1956)
4. (with A. Tarski) Cylindrical algebras. Summer Inst. Symb. Log. **3**, 332–340 (1957)
5. (with A. Tarski) Cylindrical algebras. Proc. Symp. Pure Math., Am. Math. Soc. **2**, 83–113 (1961)
6. Logical systems containing only a finite number of symbols. Sém. Math. Sup., 21. Univ., Montréal (1967)
7. Relativization with respect to formulas and its use in proofs of independence. Compos. Math. **20**, 88–106 (1968)
8. Extending Boolean operations. Pac. J. Math. **32**, 723–752 (1970)
9. (with J.D. Monk and A. Tarski) Cylindric Algebras, Part I. North-Holland, Amsterdam (1971)
10. Internal semantics and algebraic logic. In: Truth, Syntax, and Modality Studies in Logic and the Foundations of Mathematics, vol. 68 pp. 111–127. North-Holland, Amsterdam (1973)
11. (with J.D. Monk) Cylindric algebras and related structures. Proc. Tarski Symp., Proc. Symp. Pure Math., vol. XXV, Univ. Calif, Berkeley, pp. 105–121 (1974)
12. (with D. Resek) Relativization of cylindric algebras. Fund. Math. **82**, 363–383 (1975)
13. Algebraic aspects of logic: past, present, future. Colloq. Internat. de Logique (Clermont-Ferrand 1975), pp. 89–106, Colloq. Internat. CNRS (1977)
14. (with J.D. Monk, A. Tarski, H. Andréka, and I. Németi) Cylindric Set Algebras. Lecture Notes in Mathematics, vol. 883. Springer, Berlin (1981)
15. Notes for Lectures on the Theory of Cylindric Algebras. Beograd, Kragujevac, Novi Sad (1982)
16. (with J.D. Monk and A. Tarski) Cylindric Algebras, Part II. North-Holland, Amsterdam (1985)
17. (with J.D. Monk and A. Tarski) Representable cylindric algebras. Ann. Pure Appl. Log. **31**, 23–60 (1986)

J.D. Monk (✉)
Mathematics Department, University of Colorado, Boulder, CO 80309-0395, USA
e-mail: don.monk@colorado.edu

A Bit of History Related to Logic Based on Equality

Peter B. Andrews

Abstract This historical note illuminates how Leon Henkin's work influenced that of the author. It focuses on Henkin's development of a formulation of type theory based on equality, and the significance of this contribution.

Keywords Type theory · Equality · Henkin · Axiom · Extensionality

Leon Henkin and I were both students of Alonzo Church, but he finished his Ph.D. thesis in 1947, and I did not arrive at Princeton for graduate work until 1959. However, Henkin was at the Institute for Advanced Study in Princeton on a Guggenheim Fellowship during the 1961–1962 academic year. I was working on questions related to Church's type theory [8] and was familiar with Henkin's groundbreaking paper [11], so I was delighted to have the opportunity to get to know him. We both attended logic seminars, and we had a few meetings. He was present at the seminars in which I discussed the gap in Herbrand's proof of Herbrand's theorem[1] in May 1962.

In February 1962, I copied the following material from [18, p. 350] (or perhaps from [19, p. 17]) into my journal:

> The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called 'equations', for which I would prefer to substitute 'identities'. . . . (It) is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found it was faced with what seemed to me insuperable difficulties.

I was very interested in this problem, and about 9 April, I entered a note in my journal showing how quantifiers and connectives could be defined in terms of equality and the abstraction operator λ in the context of Church's type theory. By June I had seen at least a reference to Quine's abstract [17], which shows how these things can be done, but I do not remember whether I made the entry in my journal before seeing Quine's solution to the problem. My definition of conjunction was fundamentally different from that used by Quine.

In June 1962, Henkin mentioned that he was finishing work on a paper (published the following year as [12]) that gave a complete axiomatic treatment of type theory based on equality and abstraction in the context of propositional types.[2]

[1] See [6, 9].

[2] In both *propositional type theory* and *full type theory* (as we shall use these terms), the types are generated inductively from basic types by the condition that if α and β are types, then $(\alpha\beta)$ is the type of

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_8

I told Henkin that I had seen some reference in the logical literature to defining quantifiers as well as propositional connectives in terms of equality, though I no longer remembered exactly where. Henkin was very interested to hear this, and together we searched my card file of bibliographic references (which I happened to have with me at the moment) and found a note I had made about this on the card for Quine's abstract [17]. Later Henkin found the papers [16] and [15]. In his later paper [13], Henkin noted on p. 33 that Quine was the first to observe that quantifiers could be defined in this context. It is clear that Henkin made this discovery independently, since his paper [12] was already written when I brought Quine's abstract to his attention. Quine described how to make these definitions in the short final section of [16], but Henkin developed this topic much further in [12], introducing an axiomatic system and establishing its soundness and completeness. Indeed, the decidability of Henkin's axiomatic system for propositional types follows directly from the results in his paper.

I was very interested in seeing Henkin's paper, and he was very busy, so he agreed to loan me his handwritten copy of the paper and the typed copy, which still did not have the formulas written in, and I agreed to write in the formulas while I read the paper. We were both doing some traveling, but by 13 July I was back in Princeton, and Henkin was in California, and he sent me the paper.

The axioms of Henkin's system, which are given below, were chosen to express basic properties of equality. α and β stand for arbitrary type symbols; A_α, B_α, and C_α stand for arbitrary formulas of type α; and X_β stands for an arbitrary variable of type β. T^n and F^n are formulas that denote truth and falsehood, respectively.

Henkin's axioms in [12]:

(H1) $A_\alpha \equiv A_\alpha$.
(H2) $(A_0 \equiv T^n) \equiv A_0$.
(H3) $(T^n \wedge F^n) \equiv F^n$.
(H4) $(g_{00}T^n \wedge g_{00}F^n) \equiv (\forall X_0(g_{00}X_0))$.
(H5) $(x_\beta \equiv y_\beta) \rightarrow .(f_{\alpha\beta} \equiv g_{\alpha\beta}) \rightarrow .(f_{\alpha\beta}x_\beta) \equiv (g_{\alpha\beta}y_\beta)$.
(H6) $(\forall X_\beta(f_{\alpha\beta}X_\beta \equiv g_{\alpha\beta}X_\beta)) \rightarrow (f_{\alpha\beta} \equiv g_{\alpha\beta})$.
(H7) $((\lambda X_\beta B_\alpha)A_\beta) \equiv C_\alpha$, where C_α is obtained from B_α by replacing each free occurrence of X_β in B_α by an occurrence of A_β (with a restriction).

A few days after I returned the manuscript to Henkin, I noticed that Axiom H3 was derivable from the other axioms. Other simplifications of the axiom system followed in the next two months, stimulated by many letters back and forth. We were both busy with other matters, but we managed to exchange several letters every week, sometimes writing two letters a day as we discussed new ideas. At one point, I remarked that the mere action of putting a letter to Henkin in the mailbox seemed to stimulate new ideas. (Of course, there was no email at that time.) Henkin started his letter of 8 August with the comment "This two-letters-at-a-time is infectious!". Bit by bit Axioms H1, H2, and H3 were all eliminated, and H4, H5, and H6 were simplified somewhat. The result was that the axiom system above was replaced by the following:

---

functions with arguments of type β and values of type α. In propositional type theory, the only basic type is the type 0 of truth values, but in full type theory, the basic types are 0 and a type ι of individuals. Thus, propositional type theory may be regarded as higher-order propositional calculus, while full type theory includes nth-order logic for each positive integer n.

Simplified axioms as presented in [1]:

(A1) $(g_{oo}T^n \wedge g_{oo}F^n) \equiv \forall x_o(g_{oo}x_o)$.
(A2) $(f_{\alpha o} \equiv g_{\alpha o}) \rightarrow (h_{o(\alpha o)} f_{\alpha o} \equiv h_{o(\alpha o)} g_{\alpha o})$.
(A3) $(\forall x_\beta(f_{\alpha\beta}x_\beta \equiv g_{\alpha\beta}x_\beta)) \equiv (f_{\alpha\beta} \equiv g_{\alpha\beta})$.
(A4) $((\lambda X_\beta B_\alpha)A_\beta) \equiv C_\alpha$, where C_α is obtained from B_α by replacing each free occurrence of X_β in B_α by an occurrence of A_β (with a restriction).

Axiom schema A2 can be replaced by the superficially simpler schema

(A2′) $(f_\beta \equiv g_\beta) \rightarrow (h_{o\beta} f_\beta \equiv h_{o\beta} g_\beta)$,

but the formulation of (A2) shows that one does not need to assume (A2′) for all type symbols β.

On several occasions, I suggested to Henkin that he simply incorporate my proofs into his paper, but he insisted that I publish a separate paper presenting these proofs, and he wrote a very complimentary letter to Andrzej Mostowski (the editor of *Fundamenta Mathematicae*) recommending that my paper be published immediately following his own paper. He was very concerned that my paper be easy to read as well as technically correct, and made a number of suggestions about it. After we had discussed a number of ideas related to Axiom H2, Henkin found a way to derive it, but he did not want to write an addendum to my addendum to his paper, so he told me to simply include his proof of Axiom H2 in my paper.

The idea of formalizing type theory by using equality as a logical primitive can be used for the full theory of types as well as for propositional types, but I was concerned that some readers might not be sure of this and would therefore not understand the full significance of Henkin's paper. At my urging, Henkin added a discussion of this to the end of his paper, including a discussion of the need for an Axiom of Descriptions for the full theory of types.

As I think back to my interactions with Henkin, I realize how fortunate I was that he was so kind, generous, helpful, and wise.

Henkin's work played a decisive role in my life. Of course, like many other logicians, every time I taught a logic course I benefited from his work on completeness [10, 11] and his clarification of the notion of a nonstandard model. Questions about the nature of the general models of [11] provided the impetus for my paper [4]. Henkin's work developing a formulation of Church's type theory with equality (identity) as the sole logical primitive was particularly important for me. I used such a formulation of full type theory, called Q_0, in my Ph.D. thesis [2] and the textbook [5].

As noted in [2], it is easy to see that Axioms A1, A3, and A4 are independent. Henkin and I discussed the problem of proving the independence of Axiom A2 several times in 1962, and I mentioned it regularly in my course on type theory, but this remained an open problem until 2013, when Richard Statman showed that Axiom A2 is indeed independent.[3]

It is important to realize the significance of Henkin's contribution in developing a formulation of type theory based on equality. It is a real improvement of the system $\mathscr{C}$ discussed in [11], which has primitive constants for propositional connectives and quantifiers, and in which the formula $\mathscr{Q}_{o\alpha\alpha}$ for equality is defined in terms of these as

[3]The proof has not yet been published.

$[\lambda x_\alpha \lambda y_\alpha \forall f_{o\alpha} \,.\, f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha]$. (Except for some axiomatic differences, $\mathscr{C}$ is the system introduced by Church [8]). The formulation based on equality does far more than provide a cute and elegant formulation of type theory; it cleans up a subtle semantic problem, which we now explain.

As shown in [3], there is a nonstandard general model for $\mathscr{C}$ in which the Axiom of Extensionality $\forall x_\beta (f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta) \supset (f_{\alpha\beta} = g_{\alpha\beta})$ is not valid, since the sets in this model are so sparse that the denotation of the defined equality formula $\mathscr{Q}_{o\alpha\alpha}$ is not the actual equality relation. Thus, Theorem 2 of [11] (which asserts the completeness and soundness of $\mathscr{C}$) is technically incorrect. The apparently trivial soundness assertion is false.

However, this problem does not arise for the system Q_0 of full type theory based on equality, since in models of Q_0 the denotation of each equality symbol is the actual equality relation for that type in the model. (A detailed proof of the completeness and soundness of Q_0 may be found in [5].) Thus, in contexts where one wants to assume extensionality and discuss general models, a formulation of full type theory based on equality such as Q_0 is more appropriate than $\mathscr{C}$.

References

1. Andrews, P.B.: A reduction of the axioms for the theory of propositional types. Fundam. Math. **52**, 345–350 (1963)
2. Andrews, P.B.: Transfinite Type Theory with Type Variables. Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam (1965)
3. Andrews, P.B.: General models and extensionality. J. Symb. Log. **37**, 395–397 (1972). Reprinted in [7]
4. Andrews, P.B.: General models, descriptions, and choice in type theory. J. Symb. Log. **37**, 385–394 (1972). Reprinted in [7]
5. Andrews, P.B.: An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof, 2nd edn. Kluwer Academic, Dordrecht (2002). Now published by Springer
6. Andrews, P.B.: Some historical reflections. In: [7], pp. 3–34
7. Benzmüller, Ch., Brown, Ch.E., Siekmann, J., Statman, R.: Reasoning in Simple Type Theory. Festschrift in Honour of Peter B. Andrews on His 70th Birthday. College Publications, King's College, London (2008). Available via http://www.collegepublications.co.uk/logic/mlf/?00010
8. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**, 56–68 (1940). Reprinted in [7]
9. Burton Dreben, B., Andrews, P., Aanderaa, S.: False lemmas in Herbrand. Bull. Am. Math. Soc. **69**, 699–706 (1963)
10. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**, 159–166 (1949). Reprinted in [14]
11. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**, 81–91 (1950). Reprinted in [14] and in [7]
12. Henkin, L.: A theory of propositional types. Fundam. Math. **52**, 323–344 (1963). Errata Ibid., **53**, 119 (1964)
13. Henkin, L.: Identity as a logical primitive. Philosophia. Philos. Q. Isr. **5**(1–2), 31–45 (1975)
14. Hintikka, J. (ed.): The Philosophy of Mathematics. Oxford University Press, Oxford (1969)
15. Quine, W.V.: Logic based on inclusion and abstraction. J. Symb. Log. **2**, 145–152 (1937)
16. Quine, W.V.: Unification of universes in set theory. J. Symb. Log. **21**, 267–279 (1956)
17. Quine, W.V.: Unification of universes in set theory (abstract). J. Symb. Log. **21**, 216 (1956)

18. Ramsey, F.P.: The foundations of mathematics. In: Proceedings of the London Mathematical Society, Series 2, vol. 25, pp. 338–384 (1926). Reprinted in [19]
19. Ramsey, F.P.: The Foundations of Mathematics and Other Logical Essays. Routledge, London (1931). Reprinted 2001

P.B. Andrews (✉)
Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA
e-mail: andrews@cmu.edu

Pairing Logical and Pedagogical Foundations for the Theory of Positive Rational Numbers—Henkin's Unfinished Work

Nitsa Movshovitz-Hadar

Abstract Five different ways of "founding" the mathematical theory of positive rational numbers for further logical development are presented as Leon Henkin outlined in 1979 in the form of notes for a future paper, suggesting that pairing them up with five representation models could possibly yield further pedagogical development. In turn, he thought that this would indicate a large variety of ways in which this number system, as any other, can be introduced in a classroom setting—many of them quite different from traditional ways. His wish was to explore how varying modes of deductive development can be mirrored in varying classroom treatments rooted in children's experience and activities. This dream never came to full fruition. Preliminary ideas about corresponding pedagogical embodiments for each logical development are briefly presented here for the next generation of researchers in mathematics-education and curriculum developers to contemplate with, bearing in mind that Henkin's work itself may need additional polishing.

Keywords Logical foundation · Pedagogical foundation · Rational numbers · The teaching and learning of rational numbers · Leon Henkin

1 Introduction

In education, as in other realms of human activity, we tend to do things the way they have been done before. Teachers at one time were students at an earlier time, and hence inertial component in teaching is far from negligible. Still, change does take place.

It is a mind-freeing exercise for us to ask ourselves, sometimes: "How would I teach this subject if I were doing it for the first time, without any examples to follow?" A reasonable way to arrive at an answer would be to ask ourselves first: "How *could* I teach this subject?" That is, we could examine several different possible ways of approaching the subject, trying to weigh advantages and disadvantages of each. In the end, being aware of several alternatives, one might conclude that there is no "best" way. If that is the case, then we would perhaps decide to try using different combinations of methods with various students in varying circumstances. Of course, in devising such combinations, we must be careful to avoid any that may confuse the student.

How, then, can we devise different methods of teaching a particular mathematical subject? One way is by looking at various logical structures within which we can formulate that particular subject, trying to find corresponding pedagogical embodiments for each. Further, we will explore this approach to find different ways to introduce the study of rational numbers. Traditionally, this is accomplished after several years of school work on the *whole* numbers.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_9

At a certain point in the middle of the 20th century, it was suggested to start students toward a consideration of the new kind of numbers by introducing the concept of "equivalent number pairs," referring to pairs of whole numbers such as $(6, 4)$ and $(9, 6)$. Such a pedagogical approach already derives from a particular foundation for the deductive theory of rational numbers. In this theory, two pairs (m, n) and (p, q) of whole numbers m, n, p, q are defined to be *equivalent* if $mq = np$; and then a *rational number* is defined to be a set of number pairs any two of which are equivalent, which contains *all* the pairs equivalent to each of its members.

A development of the deductive theory, founded in this way, was published in Germany in 1897 by the famous mathematician Edmund Georg Hermann Landau (1877–1938) and achieved very wide use in German university courses, including those intended for secondary teachers. Landau's book was later translated into English and published in 1951 in the U.S. under the title *Foundations of Analysis*.[1] It served as the standard treatment in American university courses throughout the first half of the 20th century. Small wonder, then, that when American mathematicians began to influence the content of elementary-school textbooks through the "new math" projects in the 1950s, Landau's treatment percolated into the pedagogical presentation.

But this treatment of Landau is by no means the only correct way of founding the deductive theory of rational numbers, and it was certainly not devised, originally to influence the elementary-school presentation of this part of mathematics; it only came to have such an influence through events that we might term a "historical accident." Thus, it seems appropriate to examine other foundations for the deductive theory of rational numbers, in order to see what they might suggest by way of alternative pedagogical foundations for this basic part of school mathematics.

What, then, is a foundation for a deductive theory? They come in two kinds, definitional foundations and axiomatic ones.

First there are *definitional* foundations, such as Landau's, in which the set of rational numbers, sum and product of such numbers, and ordering relation for such numbers are defined in terms of the concepts of some *prior* theory (whole numbers), and then the basic facts about the new concepts are proved as theorems that are logically derived from the facts or theorems of the prior theory.

Second, there are *axiomatic* foundations, such as Euclid's treatment of geometry. In this kind of treatment, the basic concepts of the theory (e.g., point, line, plane, the relation of betweenness) are taken as undefined, and the basic facts about them are logically derived from axioms, which are sentences involving the undefined terms that are stated without proof, along with the list of primitive (undefined) terms, at the beginning of the theoretical development. Any deductive mathematical theory—whether it deals with a number system or a geometric space—admits a variety of foundations of both of these kinds.

In the case of a definitional foundation for a deductive theory, it is clear that some other theory must be presupposed, namely, the prior theory in terms of which the concepts of the new theory are defined, and whose theorems serve as the starting point for deriving

[1]Landau, Edmund. G.H. (1951, 3rd revised edition 2001): Foundations of Analysis; The Arithmetic of Whole, Rational, Irrational, and Complex Numbers. A Supplement to Text-Books on the Differential and Integral Calculus. English Translation by F. Steinhardt. AMS Chelsea Publications Co., N.Y. (The original was published in German in 1948 under the title: Grundlangen der Analysis. ISBN-13: 978-0821826935.)

theorems in the new theory. But even in an axiomatic foundation for a given theory, as a rule, there are prior mathematical theories that are presupposed. For example, Euclid's axiomatic theory of geometry presupposes at least a rudimentary theory of whole numbers. We can see this, for example, in the definition of a polygon, which involves not only such geometric concepts as points and lines, but refers also to an arbitrary natural number (the number of sides or vertices of the polygon).

The only axiomatic theories in which *no* prior theory is presupposed are theories of logic. Any other mathematical theory *must* presuppose logic since it is logic that supplies the basic rules of sentence-formation and of derivation, and the basic vocabulary of words such as "not", "and", "if", "all", which enter into axioms of geometry, number systems, and other mathematical theories.

In all of the axiomatic and definitional foundations for the theory of rational numbers that will be further considered, we shall presuppose not only the basic parts of logic referred to above, but also certain rudimentary parts of set theory. This is necessary because among the basic things we discuss in the theory of rational numbers, there are such things as the operation of addition and the relation "less-than"; and the general notions of operations and relations belong to the theory of sets.

In addition to logic and set theory, *certain* of our axiomatic foundations for the theory of rational numbers will presuppose a theory of whole numbers, but others will *not*. Within a theory of the later type, the development of the theory from its foundation must provide for a *definition* of the whole numbers, as a special kind of rational numbers, in terms of the primitive (undefined) concepts that enter into the foundation. When we pass from *such* a foundation for the deductive theory to a corresponding pedagogical presentation of rational numbers at the elementary-school level, we are dealing with a presentation in which rational numbers are treated *before* whole numbers. Of course, this will seem shocking, and indeed may even seem a priori to be impossible, when compared with the traditional process by which school children are introduced to numbers. But it *is* possible, at least in principle, as we further suggest.

And while we do not expect, or even advocate, that such a radical approach be rushed into practice, we hope that the reader will agree that its consideration and comparison with the other foundations to be further presented, provide a worthwhile perspective from which ideas can be profitably borrowed by those developing elementary mathematics curricula.

One final word before we proceed to the five logical foundations. We have been speaking before about a theory of rational numbers. In fact, the traditional school curriculum most often begins the study of fractions before negative numbers are introduced. For this reason and for simplicity, all five deductive foundations and their pedagogical counterparts are formulated in terms of a theory of *positive* rational numbers.

In the rest of this paper, five different ways of "founding" the mathematical *theory of positive rational numbers* for further logical development are presented. These were notes by Leon Henkin for a future paper. Some terms are used and not defined. Some theorems do not follow from the axioms or definitions and need some tweaking. However, the general directions of five different ways to ground positive rational numbers in logic are sound, original, and interesting. The readers are invited to solidify the approaches that need work.

Henkin's wish (private communication 1979–1999) was that each of the five different ways of "founding" the mathematical *theory of positive rational numbers* would be paired with a representation model of these numbers for further *pedagogical development*. He stated the aim as twofold:

i. to indicate the large variety of ways in which this number system, as any other, can be introduced in a classroom setting—many of them quite different from traditional ways;
ii. to explore how varying modes of deductive development can be mirrored in varying classroom treatments rooted in children's experience and activities.

The author of this paper was Henkin's graduate student in the 1970s (the first one specializing in mathematics education) and afterwards kept corresponding with him regularly and visiting him occasionally. This topic was brought up over and over again, but it never came to fruition, although several prominent mathematics educators (such as Pearla Nesher and Diane Resek) found it challenging.

The five mathematical foundations Henkin outlined are each followed by a few pedagogical comments. In putting these together, I feel that I paid partial dues to Henkin's vision, as it is now at the hands of the mathematics educators' community in collaboration with the logicians' community for in-depth analysis and consideration.

2 Five Pairs of Deductive-Didactical Presentations

The following holds for all five logical foundations.
Notation:

$P = (P, +, \cdot, 1) =$ system of positive integers;
$Q = (Q, +_Q, \cdot_Q, 1_Q) =$ system of positive rationals.

Terminology: Whenever we consider a *system* of numbers with four components, we will understand that the first component is a set, the last one is an element of the first, and the remaining components are binary operations under which the first component is closed.

Proofs of the theorems stated below are left for the interested reader. Many of them can be found in the literature.

2.1 Pair I: Axiomatic Method 1 and Pedagogical Comments

Axiomatic Method 1

Here we assume the system of positive integers with the regular binary operations of addition and multiplication and develop the rationals with operations on them through axioms based upon having an undefined set of numbers that includes a unit and two binary operations on its elements.

Presupposed: Logic, Sets, Theory of **P** (system of positive integers).
Undefined: Set Q with binary operations $+_Q$ and $\cdot_Q$ and the element 1_Q.

1. **Axioms:**
   i. The system $Q = (Q, +_Q, \cdot_Q, 1_Q)$ is an extension of the system $P = (P, +, \cdot, 1)$. This means that $P \subset Q$, $1_Q = 1$, and for all $a, b \in P$, we have
   $$a \cdot_Q b = a \cdot b \quad \text{and} \quad a +_Q b = a + b.$$
   ii. For all $a \in P$ and $x, y \in Q$, we have
   $$(a \cdot_Q x) \cdot_Q y = a \cdot_Q (x \cdot_Q y).$$
   (This is a special case of the associative law for $\cdot_Q$ in which the first element is restricted to come from P. The general law can be inferred from this special case as we see in 6.ii below.)
   iii. For all $a \in P$ and $x, y \in Q$, we have
   $$a \cdot_Q (x +_Q y) = (a \cdot_Q x) +_Q (a \cdot_Q y).$$
   (This is a special case of the distributive law.)
   iv. For any $a, b \in P$, there is a unique $x \in Q$ such that $a \cdot_Q x = b$. (Solvability of multiplicative equations with coefficient in P.)
   v. For any $x \in Q$, there exist $a, b \in P$ such that $a \cdot_Q x = b$. (This says that every element of Q is a solution of a multiplicative equation with coefficient in P.)
2. **Definition:** For any $a, b \in P$, we let b/a be the unique $x \in Q$ such that $a \cdot_Q x = b$. Thus, $a \cdot_Q (b/a) = b$. (This definition introduces fractional notation for solutions of multiplicative equations.)
3. **Theorem:** For any $a \in P$, $a/1 = a$ and $a/a = 1$. (By Axiom 1.i we have $1_Q = 1$, so we use the notation 1 for the unit element of Q.)
4. **Theorem:** For any $a, b, c, d \in P$, we have $b/a = d/c$ iff (if and only if) $b \cdot c = d \cdot a$.
5. **Theorem:** For any $a, b, c, d \in P$, we have
   $$(b/a) \cdot_Q (d/c) = (b \cdot d)/(a \cdot c)$$
   and
   $$(b/a) +_Q (d/c) = (a \cdot d + b \cdot c)/(a \cdot c).$$
   (These are the usual rules for adding and multiplying fractions. Note that they are consequences of the axioms and Definition 2 and so do not have to be prescribed independently.)
6. **Theorems:** For all $x, y, z \in Q$,
   i. $x \cdot_Q y = y \cdot_Q x$ and $x +_Q y = y +_Q x$. (These are the commutative laws.)
   ii. $x \cdot_Q (y \cdot_Q z) = (x \cdot_Q y) \cdot_Q z$; $x +_Q (y +_Q z) = (x +_Q y) +_Q z$. (Extension of Axiom 1.ii in case of $\cdot_Q$, whereas for $+_Q$, this theorem is "new".)
   iii. $x \cdot_Q (y +_Q z) = x \cdot_Q y +_Q x \cdot_Q z$. (Distributive law, extension of Axiom 1.iii.)
   iv. $x \cdot_Q 1 = x$. (This expresses that 1, which by Axiom 1.i is the unit element of Q, is an identity element for $\cdot_Q$.)

v. $x +_Q y \neq x$.
(There is no identity element for $+_Q$ in Q.)
vi. If $x \cdot_Q z = y \cdot_Q z$, then $x = y$ and similarly for $+_Q$.
(Cancelation laws; the one for $\cdot_Q$ is true because we are only creating a theory of positive numbers, no zero.)

7. **Theorem:** For all $x, y \in Q$, there is one and only one $z \in Q$ such that $x \cdot_Q z = y$. (This strengthens Axiom 1.iv, showing that all multiplicative equations have unique solutions.)
8. **Definition:** For any $x, y \in Q$, let $y//x$ be the unique $z \in Q$ such that $x \cdot_Q z = y$; thus, $x \cdot_Q (y//x) = y$.
(Remark: // is an operation on Q that is an extension of the operation / introduced in Definition 2. See terminology in Axiom 1.i.)
9. **Theorem:** For any $x, y, z, t \in Q$,
   i. $x//y = x/y$ in case $x, y \in P$.
   ii. $(x//y) \cdot_Q (z//t) = (x \cdot_Q z)//(y \cdot_Q t)$.
   iii. $(x//y) +_Q (z//t) = (x \cdot_Q t +_Q y \cdot_Q z)//(y \cdot_Q t)$.
   iv. $x//x = 1, x//1 = x$, and $(x \cdot_Q z)//(y \cdot_Q z) = x//y$.
   v. $(x//z) +_Q (y//z) = (x +_Q y)//z$ and $(x//y) \cdot_Q (y//x) = 1$.
10. **Theorem:** If G is any nonempty subset of Q that is closed under // and $+_Q$, then $G = Q$.
(This theorem can be viewed as a kind of induction principle.)
11. **Metatheorem:** If Q and Q' are any two systems each satisfying Axioms 1.i–1.v, then Q is isomorphic to Q'.
(The significance of this meta-theorem is that it establishes the uniqueness of the mathematical structure laid out by Axioms 1.i–1.v.)

Pedagogical Comments to Axiomatic Method 1

The essence of the above axiomatic approach is in extending P, the system of natural numbers (positive whole numbers) to Q, the system of positive rational numbers, through the requirement of closure under division. This is embedded in Axioms 1.iv and 1.v, where the existence of a unique solution to any equation of the form $a \cdot x = b$ is assumed.

In conjunction with the logical presupposition of logic, set theory, and the theory of natural numbers, it is assumed here that:

1. Children can use simple sentences to communicate about the natural numbers (and the number zero).
2. They are familiar with the decimal notation for these numbers.
3. They are familiar with the computation of sums, products, and differences of these numbers up to 1000 and with division by numbers from 1 to 9.
Furthermore,
4. Experience with a variety of simple applications of these concepts is presupposed.
5. Students having finished third grade at age about 10 are familiar with the use of a few special fractions, say 1/2, 1/4, and 1/3 to deal with portions of objects or of sets of objects.

The fact that we attempt paralleling an axiomatic, rather than a definitional theory of positive numbers means that there is no room for introducing a *definition* of the basic concepts (numbers and operations) in class. Instead, we assume that it will seem intuitively natural to students to have numbers similar to, but other than, $1/2$, $1/4$, and $1/3$ and to operate on these numbers with addition and multiplication as was previously done with positive whole numbers. Thus, we focus our efforts on how to use the new numbers and operate on them. In order to construct a pedagogical approach that is paired with the above logical construct, a design of a didactic development should be carried out in such a way that the principles embodied as axioms in the logical theory will seem believable to the students at a fairly early stage, even though we never formulate the statements themselves as abstract, universal propositions. It remains at the hands of the potential user to consider the suitability of the pedagogical approach suggested here to any specific target audience.

One of the common pedagogical problems in beginning courses on rational numbers is the multiplicity of names used to denote each rational number, although each rational number has a "standard name" (in decimal notation), which plays an exclusive role. This problem will be encountered when we try to get students to appreciate the uniqueness of the x postulated to exist in Axiom 1.iv, whereas we want them to appreciate that the pairs (a, b) whose existence we asserted in Axiom 1.v are not unique.

The basic means of dealing with this problem is to identify each rational number with a fixed, single, physical object, displayed in the classroom, *without* a standard name (except, of course, for the whole numbers), and to have a system for attaching names to these objects in such a way that given names can be removed and replaced by others or that several names can be viewed simultaneously.

The basic model proposed here is to identify the rational numbers with marked points on a large fixed line. (A larger model can be printed on a strip of wood or plastic and is to remain at the front of the classroom throughout this "unit" of instruction.) Above equally spaced points that are marked on that line, the numerals $0, 1, \ldots, 12$ should be marked. Between each successive pair of the points thus numbered, there should be 120 marked points (equally spaced), unnumbered. (These 120 points will become useful in the activities to be designed when a detailed curriculum is developed.) In addition to this fixed number line, students can use clear plastic strips printed with various fractions having denominator d, for $d = 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20$; each of these strips may be placed on pegs, by the large number line in such a way that the fractions indicate the appropriate marked points on the large fixed number line. (For $d = 1, \ldots, 5$, all fractions up to $12d/d$ can be indicated; for $d > 5$, some regular selection of all fractions should be made to avoid clutter. It is desirable that two (or possibly more) of these "naming strips" can be simultaneously adjoined in such a way that two (or more) names for certain points can be viewed together.) Figure 1 illustrates the basic fixed line and a few strips. A good supply is needed of unmarked strips of various lengths that can be fitted (possibly into a slot attached to the number line) in such a way that both of the strips lie opposite marked points on the basic line. (Smaller models of some of this equipment should be available in the form of kits for use by individual students).

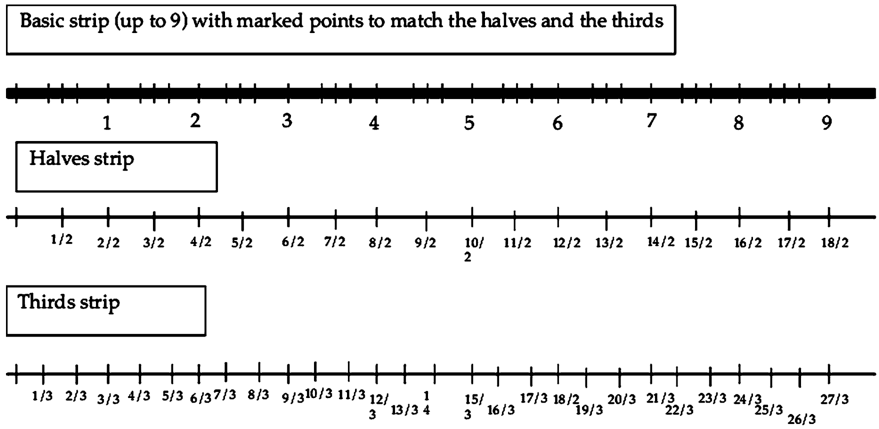


Fig. 1 Basic fixed number line and a few strips

2.2 Pair II: Axiomatic Method 2 and Pedagogical Comments

Axiomatic Method 2

Presupposed: Logic, sets, theory of P.
Undefined: Set Q, with binary operation /.

1. **Axioms:**
   i. $P \subset Q$, and / maps $P \times P$ into Q (i.e., Q is closed under /.)
   ii. For all $a \in P$, $a/1 = a$.
   iii. For each $x \in Q$, there exist $a, b \in P$ such that $x = a/b$. (All elements of Q are in the range of operation /.)
   iv. For any $a, b, c, d \in P$, we have $a/b = c/d$ iff $a \cdot d = b \cdot c$.
2. **Lemma:** If $a, b, c, d, a', b', c', d' \in P$, $b/a = b'/a'$, and $d/c = d'/c'$, then also $(b \cdot d)/(a \cdot c) = (b' \cdot d')/(a' \cdot c')$ and $(b \cdot c + a \cdot d)/(a \cdot c) = (b' \cdot c' + a' \cdot d')/(a' \cdot c')$. (This lemma is needed for the following definition. This point is clarified below.)
3. **Definition:** We define the binary operations $+_Q$, and $\cdot_Q$ on Q by the rule that for any $x, y \in Q$, we find $a, b, c, d \in P$ with $x = b/a$ and $y = d/c$ (by Axiom 1.iii) and set

$$x \cdot_Q y = (b \cdot d)/(a \cdot c)$$

and

$$x +_Q y = (b \cdot c + a \cdot d)/(a \cdot c),$$

which is justified by Lemma 2.
There is a difficulty with this definition. We could have $x = b'/a'$ and $y = d'/c'$, as well as having $x = b/a$ and $y = d/c$, and could have $a' \neq a$, $b' \neq b$, $c' \neq c$, $d' \neq d$. What if we then found that $(b' \cdot d')/(a' \cdot c') \neq (b \cdot d)(a \cdot c)$? We would not know which of these two different elements is supposed to be $x \cdot_Q y$. Our Definition 3 would not be logically satisfactory. How do we overcome this difficulty? Lemma 2 is the remedy.
Further, we set $1_Q = 1$, $Q = (Q, +_Q, \cdot Q, 1_Q)$.

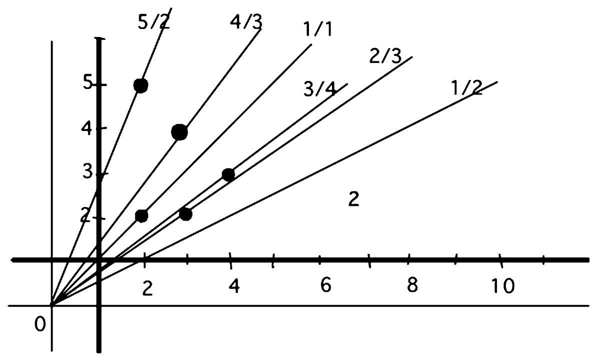


Fig. 2 A model reflecting the 4 axioms of Axiomatic Method 2. Note: The origin is at (1, 1) to reflect Axiom 1.ii

4. **Theorem:** Q is an extension of P. (Cf. Pair I, Axiom 1a.) Let $P \subset Q$ and for all $x, y \in P$, $x +_Q y = x + y$; $x \cdot_Q y = x \cdot y$ (notice that this result was taken as Axiom 1a in Pair I, and in this pair, we can prove it).
5. **Further development:** We could demonstrate as theorems each of the laws ii–v that were used as Axioms in Pair I. Thereafter, the development indicated in that the system can naturally be copied in this one.

Pedagogical Comments to Axiomatic Method 2

The essence of Axiomatic Method 2 is a transformation of $P \times P$ to Q. This has the most significant bearing on the didactical approach proposed here.

In conjunction with the logical presupposition of logic, set theory, and the theory of P, it is assumed here that students are capable of:

a. Matching a point with an ordered pair of numbers in a right-angle coordinate system.
b. Adding of two natural numbers on the number line (climbing a ladder, translation).
c. Multiplication of two natural numbers on the number line (possibly by the model of stretching).

We seek a pedagogical model in which all four axioms are reflected. Naturally, the first quadrant comes to mind. However, we need the "origin" at (1, 1) in order to reflect properly Axiom 1.ii. (For all $a \in P$, $a/1 = a$.) (See Fig. 2.)

2.3 Pair III: Axiomatic Method 3 and Pedagogical Comments

Axiomatics Method 3

This is a Radical Version of Axiomatic Method 2.

Presupposed: Logic, sets. (Theory of P not presupposed.)
Undefined: Nonempty set Q with operations $+_Q$, $\cdot_Q$.

1. **Axioms (partial list, completed in 7 below):**
   i. For all $x, y, z \in Q$:

$$(x +_Q y) +_Q z = x +_Q (y +_Q z);$$
$$(x \cdot_Q y) \cdot_Q z = x \cdot_Q (y \cdot_Q z);$$
$$x \cdot_Q (y +_Q z) = (x \cdot_Q y) +_Q (x \cdot_Q z); \quad \text{and}$$
$$(y +_Q z) \cdot_Q x = (y \cdot_Q x) +_Q (z \cdot_Q x).$$

   (The latter two are the left and right distributive laws. Of course, we could get one from the other by using the commutative law for $\cdot_Q$, but we are not taking that law as an axiom. We will get that law as a theorem (Theorem 8) but only by taking both distributive laws as axioms.)

   ii. For all $x, y \in Q$, there exist $u, v \in Q$ such that $x \cdot_Q u = y$ and $v \cdot_Q x = y$.
2. **Theorem:** There is one and only one $u \in Q$ such that $u \cdot_Q y = y$ for every $y \in Q$; for this u, also $y \cdot_Q u = y$.
3. **Definition:** Let 1_Q be the unique u of Theorem 2.
4. **Theorem:** For each $x \in Q$, there is one and only one $y \in Q$ such that $x \cdot_Q y = 1_Q$; for this y, we also have $y \cdot_Q x = 1_Q$.
5. **Definition:** For each $x \in Q$, we let x^{-1} be the unique y of Theorem 4. For all $x, z \in Q$, we set $x/z = x \cdot_Q z^{-1}$.
6. **Theorem:** For all $x, y, z \in Q$:
   i. $(x^{-1})^{-1} = x$;
   ii. $(xy) - 1 = y - 1x - 1$;
   iii. $x/x = 1$;
   iv. $(x/y)^{-1} = y/x$;
   v. $(x \cdot_Q z)/(y \cdot_Q z) = x/y$;
   vi. $(x/y) \cdot_Q (y/z) = x/z$;
   vii. $(x/z) + Q(y/z) = (x + Qy)/z$.

   The parts of this theorem express familiar rules for dealing with fractions. The last one, for example, tells us how to add two fractions with the same denominator. But where is the general rule for adding any two fractions? Answer: It cannot be proven from the axioms so far given. At a later point we shall indicate how this can be shown. We need further axioms.
7. **Axioms (completed):**
   a. If G is any nonempty subset of Q that is closed under $+_Q$ and/, then $G = Q$.
   b. Whenever $x, y \in Q$ and $x +_Q 1_Q = y +_Q 1_Q$, then $x = y$. (A special case of the conclusion for $+Q$.)
   c. There exist $x, y \in Q$ with $x \neq y$. (That is, there exist at least two distinct elements in our set Q.)
8. **Theorem:** For all $x, y, u, v \in Q$:
   i. $x \cdot_Q y = y \cdot_Q x$;
   ii. $(x/u) \cdot_Q (y/v) = (x \cdot_Q y)/(u \cdot_Q v)$;
   iii. $(x/u) +_Q (y/v) = (x \cdot_Q v +_Q u \cdot_Q y)/u \cdot_Q v$; and $x/u = y/v$ iff $x \cdot_Q v = u \cdot_Q y$.

   Parts ii and iii give the general rules mentioned in the note following Theorem 6.
9. **Theorem:** Whenever $x +_Q z = y +_Q z$ in Q, then also $x = y$.
10. **Theorem:** For all $x, y \in Q$, $x +_Q y = y +_Q x$.

11. **Theorem:** For all $x, y \in Q$, $x +_Q y \neq x$.
    (The general cancelation law, extending Axiom 7.b. This law expresses the fact that there is no additive identity element—i.e., no number zero—in this number system. The theorem allows us to define the order relation $<$ in our system and to develop its properties.)
    Now we come to the definition of the set P of integers, which of course will be a certain subset of Q.
12. **Definition:** A subset W of Q is called inductive iff
    i. $1_Q \in W$, and
    ii. whenever $x \in W$, then also $x +_Q 1_Q \in W$.

    We define P to be the intersection of all inductive sets, that is, for any $x \in Q$, we put $x \in P$ iff we find that $x \in W$ for *every* inductive set W.
13. **Theorem:** P is itself inductive.
14. **Theorem:** P is closed under $+_Q$ and $\cdot_Q$.
15. **Definition:** We take $+$ and $\cdot$ to be the restrictions to P of $+_Q$ and $._Q$ respectively. We set $1 = 1_Q$ and $P = (P, +, \cdot, 1)$.
16. **Theorem:** For every $x \in Q$, there exist $a, b \in P$ with $x = a/b$.
17. **Metatheorem:** If $(Q, +_Q, \cdot_Q)$ and $(Q', +_{Q'}, \cdot_{Q'})$ are any systems satisfying Axioms 1.i–1.ii and 7.a–7.c, then they are isomorphic.

Pedagogical Comments to Axiomatic Method 3

The essence of this Radical Version is that without presupposing P, we create Q through five axioms as a field with an inductive minimal set P.

Presupposed knowledge: Logic & sets only.

Since the knowledge of natural numbers is not presupposed, one could consider teaching this unit at a very early age (1st grade, for example) as a first introduction of numbers in a more general mode of presentation than usually done. The following pedagogical discussion is taking a *different stand*. A unit can be aimed at the 4th or 5th grade. That is to say, when the children had already mastered the four operations in natural numbers $(+, -, \cdot, \div)$, and these will serve as a background knowledge. However, this knowledge will not be directly used in the presentation of the positive rational numbers.

One can use the number line as a model for presenting the rational numbers. Here is a suggestion as to how to place them on the number line, and how to present the operations with those numbers, whose presentation was associated with the number line.

For placing the numbers on the line, an apparatus is to be built, comprised of the following (see Fig. 3):

- Three identical number lines (the standard number-lines), in the following manner: The x-scale which is horizontal; the y-scale which is perpendicular to the x-scale at the starting point 0; and the U-scale which is perpendicular to the x-scale at a special point U (later on to be known as 1);
- A movable pointer attached to the starting point 0, which will represent $\cdot_Q u$ ("multiplied by u");
- A rectangular "reader" that will assist to read the corresponding numbers on the x-scale and on the y-scale.

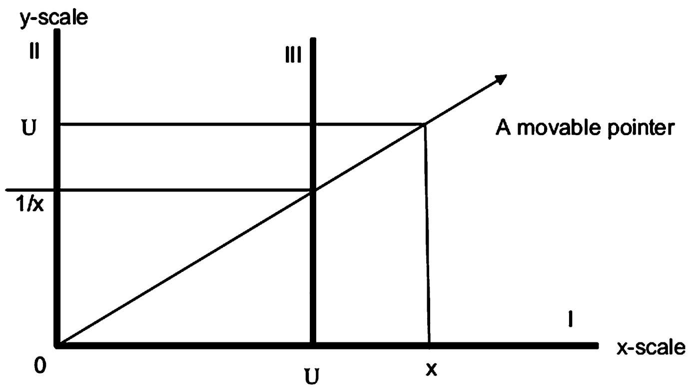


Fig. 3 An apparatus for learning about the positive rational numbers via reflecting Axiomatic Method 3

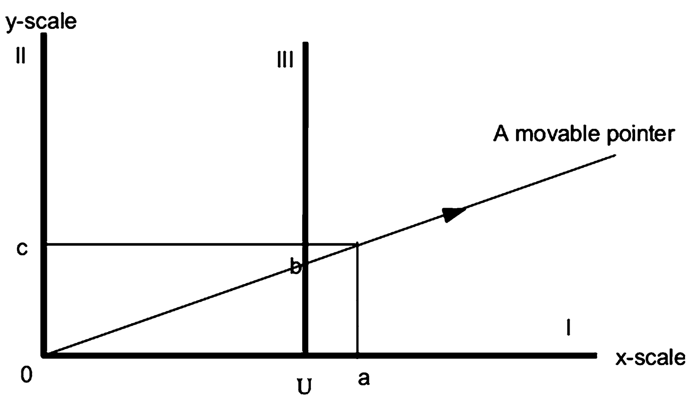


Fig. 4 The two steps procedure for finding the product of $a \cdot_Q b$

For every point x on the x-scale, we allocate y on the y-scale for which $x \cdot y = 1$ using the unit rectangle that is formed by drawing a line parallel to the unit line and the y-scale. The diagonal of this rectangle crosses the unit line at the desired value of y (marked $1/x$) (see Fig. 3).

The same apparatus can be used to illustrate the product $x._Q u = y$. Figure 4 illustrates the two-step procedure for finding c, the product of $a \cdot_Q b$:

i. Find the point b on the U-scale and adjust the movable pointer to pass through b. (You have thus received the presentation of $\cdot_Q b$.)
ii. Take the rectangular reader and read, on the y-scale, the point that corresponds to point a on the x-scale. This point is c such that $a \cdot_Q b = c$.

For addition ($+_Q$), two parallel number lines can be used, with identical scale, performing translation of one of them in relation to the other, in order to arrive at the number c that is the sum of any two numbers a and b (see Fig. 5).

Thus, the interpretation (reading) of the scheme $x +_Q y = z$ using this model will go as follows: to add b to a, put scale II so that its 0 (starting point) will coincide with a.

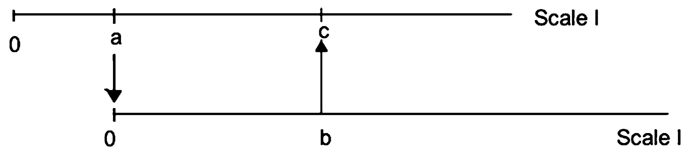


Fig. 5 Sliding identical scales for finding the sum of $a +_Q b$

Then, read the sum on scale I as the dot that corresponds to b on scale II. This method can be used for any $x, y \in Q$.

2.4 Pair IV: Definitional Method 1 (Classical) and Pedagogical Comments

Definitional Method 1

Outside of logic, sets, theory of P, there are no undefined terms in this theory of Q since we shall define every term we need. That is why we call this a definitional method. Undefined terms are needed when we give an axiomatic foundation for a theory.

Presupposed: Logic, sets, theory of P.

1. **Definition:** Let X be the set of all ordered pairs (a, b) such that $a, b \in P$. Let E be the binary relation on X such that $(a, b)E(c, d)$ iff $a\cdot d = b\cdot c$.
2. **Lemma:** E is an equivalence relation on X. (This equivalence relation E generates the "equivalent number pairs.")
3. **Definition:** For any $(a, b) \in X$, let a/b be the equivalence class of (a, b) under the relation E.

   Let Q be the set of all equivalence classes a/b.
   Let 1_Q be the element $1/1$ of Q.

4. **Lemma:** Suppose that $(a, b)E(a', b')$ and $(c, d)E(c', d')$. Then also $(a\cdot c, b\cdot d)$ $E(a'\cdot c', b'\cdot d')$ and $(a\cdot d + b\cdot c, b\cdot d)E(a'\cdot d' + b'\cdot c', b'\cdot d')$.
5. **Definition:** For any a/b and c/d of Q:

   Let $(a/b)\cdot_Q (c/d) = (a\cdot c)/(b\cdot d)$ and $(a/b) +_Q (c/d) = (a\cdot d + b\cdot c)/(b\cdot d)$.
   Let $Q = (Q, +_Q, \cdot_Q, 1_Q)$.

6. **Theorem:** For all x, y, z of Q:
   i. $x\cdot_Q y = y\cdot_Q x$ and $x + Qy = y +_Q x$.
   ii. Associative laws for $+_Q$, and $\cdot_Q$
   iii. $x\cdot_Q (y +_Q z) = (x\cdot_Q y) +_Q (x\cdot_Q z)$.
   iv. $x\cdot_Q 1_Q = x$.
   v. $x +_Q y \neq x$.
   vi. If $x\cdot_Q z = y\cdot_Q z$, then $x = y$, and similarly for $+_Q$.
7. **Theorem:** For all x, y of Q, there is one and only one z of Q, such that

$$x\cdot_Q z = y.$$

8. **Definition:** For any two equivalent classes x, y of Q, let $y//x$ be the unique z of Q such that $x \cdot_Q z = y$. Thus, $x \cdot_Q (y//x) = y$. (We thereby generate a new equivalence class of pairs of elements of Q.)
9. **Theorem:** For any $x, y, z, t \in Q$:
   i. $(x//y) \cdot_Q (z//t) = (x \cdot_Q z)//(y \cdot_Q t)$.
   ii. $(x//y) +_Q (z//t) = (x \cdot_Q t +_Q y \cdot_Q z)//(y \cdot_Q t)$.
   iii. $x//x = 1_Q$ and $x//1 = x$.
   iv. $(x \cdot_Q z)//(y \cdot_Q z) = x//y$.
   v. $(x//z) +_Q (y//z) = (x +_Q y)//z$.
   vi. $(x//y) \cdot_Q (y//x) = 1_Q$.
10. **Theorem:** If G is any nonempty subset of Q that is closed under $//$ and $+_Q$, then $G = Q$.
11. **Definition:** Let h be the function mapping P into Q such that, for any $a \in P$, $h(a) = a/1$.
    This definition and the theorems that follow are needed because unlike the situation in the axiomatic foundations of Pairs 1, 2, and 3, in this treatment, Q is *not* an extension of P. In particular, P is not a subset of Q. The situation is more complicated: Q has a subset that is an isomorphic *copy* of P. Thus, in Q, we have the equivalence class of all pairs (x, x)—for all $x \in P$—*representing* the number 1. The mathematician would say: "We 'identify' this equivalence class with the integer 1." To be precise, we are using the isomorphism h of Definition 11 as this "identification" operation. In schoolrooms, the distinction between the integer 1 and the rational number 1 is usually overlooked. This makes everything easier—even if, strictly speaking, it is not logically correct in the Definitional Foundation of Pair IV. But it is logically correct in the preceding axiomatic foundations, where the set P is literally a subset of Q, and the number system Q is an extension of the number system P.
12. **Theorem:** h is an isomorphism of P into Q, that is, h is one-one, $h(l) = 1_Q$, and for all $a, b \in P$, $h(a \cdot b) = h(a) \cdot_Q h(b)$ and $h(a + b) = h(a) +_Q h(b)$.
13. **Theorem:** For all $a, b \in P$, $h(a)//h(b) = a/b$.

Pedagogical Comments to Definitional Method 1

The classical definitional method is essentially based upon forming equivalence classes of quotients in P, where quotients are ordered pairs of elements in P, including those having 1 as "denominator". In Definition 11, h is the function that identifies the positive integers with the (positive) rationals possessing 1 as a denominator (and their equivalent-rationals).

Cuisenaire Rods are suggested as the concrete materials to exemplify the notions of ordered pairs. In many cases, students had previous experience with Rods while learning about P. An ordered pair of numbers can be defined following C. Gategno[2] as the arrangement of two Rods one on top of the other, end to end.

[2]For example, Molly Silha (1997) Learn Fractions with Cuisenaire Rods. http://teachertech.rice.edu/Participants/silha/Lessons/teacher.html (retrieved June 2013).

|__the_measured_rod_|
|__________ the_Measure________|

Fig. 6 A pair of rods can exemplify a relation in which the one on top in the pair is measured in length in comparison to the bottom one

|__2_| |___4___| |______6___|

|_1| |__2_| |___3__|

Fig. 7 The relation "to be twice the length of" can be fulfilled by many pairs of rods

In order to distinguish between the two positions, one can assign different roles (and names) to the different positions. Let the top position of the pair be "The measured Rod" in that binary relation, and let the bottom position be "The Measure Rod," and the pair therein is (measured, Measure) or (m, M). The idea behind this naming is that one should refer to that ordered pair as a relation. The two numbers (or rods) stand in some relation to each other.

Thus, a pair of rods will exemplify a relation in which the one on top in the pair is measured in length in comparison to the bottom one (see Fig. 6). Clearly, the order in (m, M) must be maintained so that we know which one serves as a Measure, and which one is going to be measured.

To reflect Definition 1, notice that the relation "to be twice the length of" can be fulfilled by many pairs of rods, for example, see Fig. 7.

It is then interesting to find a decision rule that will help us see which ordered pairs (a, b) fulfill "a is twice than b." This will be of more interest for pairs in which we cannot intuitively grasp the number that states the relation between the measured m and the measure M as a whole number. For example, do (7, 8) and (9, 10) belong to the same relation?

Let us return to the (a, b) relation

"a is twice the length of b" (I)

in contrast to (c, d), where the relation is

"c is three times the length of d" (II).

Students will be asked to provide numerical examples for ordered pairs (a, b) as defined in (I) (Answers: $(2, 1), (12, 6), (100, 50), \ldots$) and also examples for ordered pairs (c, d) as defined in (II) (answers: $(3, 1), (90, 30), (12, 4), \ldots$). This will reflect Lemma 2 (see above) since there are many ordered pairs that belong to a given relation. Students need to realize that these are equivalent ordered pairs.

This also leads to an exemplification of Definition 3 for E as the binary relation on X such that $(a, b)E(c, d)$ iff $ad = bc$. Starting with two ordered pairs that are known to belong to the same relation, for example, (3, 1) and (6, 2) or (3, 1) and (12, 4), students can try to multiply The Measured of one pair with The Measure of the other pair and "see" the equality ($3 \times 2 = 1 \times 6$ or $1 \times 12 = 3 \times 4$). This can be generalized and demonstrated (see Fig. 8): If two ordered pairs $(m_1 M_1)$ and (m_2, M_2) are equivalent, then they imply the following equivalence: $m_1 M_2 = M_1 m_2$.

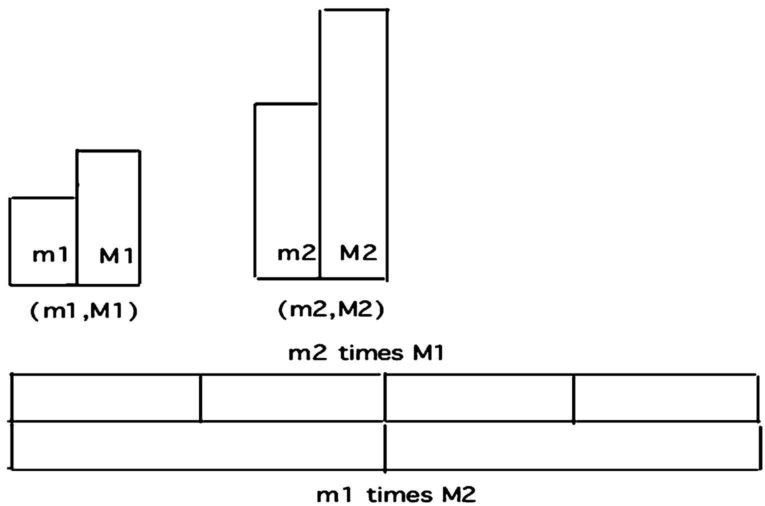


Fig. 8 Rod demonstration of the statement: if two ordered pairs (m_1, M_1) and (m_2, M_2) are equivalent, then they imply: $m_1 M_2 = M_1 m_2$

2.5 Pair V: Definitional Method 2 (Nonclassical) and Pedagogical Comments

Definitional Method 2

Presupposed: Logic, sets, theory of P.

1. **Definition:** Let F (for Fractions) be the set of all ordered pairs (a, b) such that $a, b \in P$, $b \neq 1$, and a, b are relatively prime. Let $Q = P \cup F$.
   Notation: We write a/b for (a, b) when the latter is in F.
   Assumption: Our theory of P is such that $P \cap F = \emptyset$.
   This assumption may look strange. The point is that when we say that we are presupposing the theory of P, we mean that we assume knowledge of the correct mathematical statements in the theory of positive integers, such as commutative laws, $1 + 1 = 2$, etc. But this does not tell us exactly what the elements of P are. If, in fact, we are presupposing an axiomatic development of P, so that the elements of P are not defined in that treatment. So, it is consistent with the theory that one element of P could be a horse, another a building, and a third one could be the ordered pair of that horse and that building. It is this that we wish to exclude by making this assumption.
2. **Definition:** Let $a, b \in P$. In case b is a factor of a, define red $(a, b) = a \div b$ ("red" is a shortcut for "reduced form of," namely quotient). In case b is not a factor of a, let $g = gcd(a, b)$ and define $red(a, b) = (a \div g)(b \div g)$. Thus, in all cases $red(a, b) \in Q$.
3. **Definition:** We define operations $\cdot_Q$ and $+_Q$ on Q by the following rules (for any $a, b, c, d \in P$):

i. $a \cdot_Q c = a.c$ and $a +_Q c = a + c$.
ii. $a \cdot_Q (c/d) = (c/d) \cdot_Q a = red(a \cdot c, d)$ and
$a +_Q (c/d) = (c/d) +_Q a = red(a \cdot d + c, d)$.
iii. $(a/b) \cdot_Q (c/d) = red(a \cdot c, b \cdot d)$ and
iv. $(a/b) +_Q (c/d) = red(a \cdot d + b \cdot c, b \cdot d)$
v. Let 1_Q be the element $1/1$ in Q.

4. **Theorem:** For all x, y, z of Q,
   i. $x \cdot_Q y = y \cdot_Q x$ and $x +_Q y = y +_Q x$.
   ii. Associative laws for $+_Q$, and $\cdot_Q$.
   iii. $x \cdot_Q (y +_Q z) = (x \cdot_Q y) +_Q (x \cdot_Q z)$.
   vi. $x \cdot_Q 1Q = x$.
   v. $x +_Q y \neq x$.
   vi. If $x \cdot_Q z = y \cdot_Q z$, then $x = y$, and similarly for $+_Q$.

5. **Theorem:** For all x, y of Q, there is one and only one z of Q such that

$$x \cdot_Q z = y.$$

6. **Definition:** For any two equivalent classes x, y of Q, let $y//x$ be the unique z of Q such that $x \cdot_Q z = y$. Thus, $x \cdot_Q (y//x) = y$. (We thereby generate a new equivalence class of pairs of elements of Q.)
7. **Theorem:** For all $x, y, z, t \in Q$,
   i. $(x//y) \cdot_Q (z//t) = (x \cdot_Q z)//(y \cdot_Q t)$.
   ii. $(x//y) +_Q (z//t) = (x \cdot_Q t +_Q y \cdot_Q z)//(y \cdot_Q t)$.
   iii. $x//x = 1Q$ and $x//1 = x$.
   iv. $(x \cdot_Q z)//(y \cdot_Q z) = x//y$.
   v. $(x//z) +_Q (y//z) = (x +_Q y)//z$.
   iv. $(x//y) \cdot_Q (y//x) = 1Q$.

8. **Theorem:** If G is a nonempty subset of Q that is closed under $//$ and $+_Q$, then $G = Q$.
9. **Theorem:** $a//b = a/b$ whenever $(a, b) \in F$.

Pedagogical Comments to Definitional Method 2

The difference between this logical foundation and the preceding one is that here we separate order pairs with 1 as a second component from order pairs in which the two components are relatively prime and the second component (later to become the denominator) $\neq 1$. The union of the two sets of order pairs form Q (their intersection is presumed to be empty). This deductive structure starts with a definition of a set F disjoint from the presupposed set P (Definition 1). Objects for a pedagogical model should be chosen such that they can reflect this structure. A possible choice is the unit-"pie" (unit-circle) model. The theory of P is presupposed, and thus it is advisable to assume that students are familiar with adding whole unit-"pies" and multiplying unit-"pies" by a positive integer. Sections of the unit-"pie" will represent ordered pairs in F where $a = 1$ and b is any integer greater than 1. For example, we define the ordered pair $(1, 7)$ as one section of a

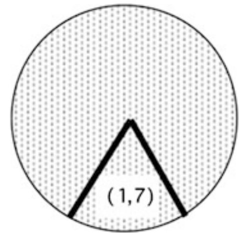


Fig. 9 The ordered pair (1, 7) as one section of a unit-"pie" cut into seven equal parts

unit-"pie" that was cut into seven equal parts. The visual model is that of a full unit-circle with a shaded section, the size of which is determined by b (see Fig. 9).

Ordered pairs (a, b) in F for which $a \neq 1$ will be constructed through duplication of the unit fractions; for example, (3, 7) will be perceived as three unit-circles, each with a marked section of (1, 7) to reflect the 3 divided by 7 (see Fig. 10).

This is consistent with the definition of multiplication in the logical foundation (see Definition 3ii), where $3 \cdot_Q (1, 7) = (3.1, 7) = (3, 7)$. Now, naming nonreduced fraction can be taught through arithmetical considerations based upon the presupposition of full knowledge of the theory of P. That is, for instance, the ordered pair (6, 8) equals (3, 4) because previous knowledge shows that 2 is the g.c.d. of 6 and 8. Thus, $(6, 8) = (6 \div 2, 8 \div 2)$ (see Definition 2). This can also be demonstrated by the unit-pie model using transparent (1, 8) sections and superimposing six of them in three pairs on three (1, 4) sections. To demonstrate how (2, 3) is multiplied by 4, one can duplicate four times two (1, 3) sections of a unit-pie. Obviously, there are eight pies each with a one third section that is (8, 3). A crucial point is the identification of (8, 4) with $8 \div 4$ (see Definition 2, first part). This should also be based upon the presupposed knowledge of basic arithmetic, but it can very elegantly be demonstrated using the unit-"pie" model, superimposing eight transparent (1, 4) in two groups to form two full unit-"pies" (circles).

Notice that full circles and sections are two disjoint sets (Definition 1 in the mathematical model), which are combined into the set Q.

As for addition, $3 + (4, 7) = (3 \times 7 + 4, 7) = (25, 7)$, a preparatory exercise should include the transition from 3 to $(3 \times 7, 7)$ through a decomposition procedure in a reversed order to that described before for the case of (8, 4).

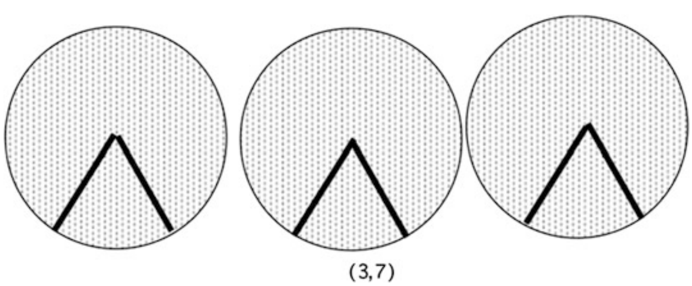


Fig. 10 The ordered pair (3, 7) can be perceived as three unitcircles, each with a marked section of (1, 7) to reflect 3 divided by 7

3 Closing Remarks

Mathematicians organize mathematical knowledge into deductive theories. Educators organize it into grade curricula.

The proper relation between these two modes of organization is in dispute. Mathematicians tend to ignore it, and in practice they follow the deductive structure in their college teaching. Educators range from those who feel that curriculum should closely follow deductive organization to those who feel, at least, with regards to the theory of positive rational numbers, that recent curriculum development has been strongly popularized at university level and then worked their way down to the secondary and elementary level. What is not generally realized, either by mathematicians or educators, is that there is a wide variety of ways of organizing the theory of positive rational numbers, some of them very different from those familiar from the literature.

In this paper, five different foundations for this theory are mentioned, some definitional and others axiomatic, with different "presupposed theories." Then we consider their relevance to possible ways of developing curricula at several levels.

Hopefully, by examining several ways of introducing ideas related to a given mathematical theory teachers will acquire a basis for critically examining classroom textbooks. This will allow them to separate out ideas that are often mixed together in such texts. Curriculum developers can also benefit from it since clarifying the foundations and pairing each way with pedagogical ideas provide curriculum developers with a rich background and serve as an idea resource. Finally, mathematics educators and cognitive scientists who do research in this area may be interested in assessing the worth of each of the five system-pairs or any of their various potential combinations. But first of all let us not forget that this paper is based upon an unfinished work by Leon Henkin and the pedagogical ideas expressed in it should be subject to critical consideration and further elaboration by professionals in both logic and mathematics-education. Undoubtedly, carrying on such work would have given Leon Henkin an utmost pleasure.

N. Movshovitz-Hadar (✉)
Technion, Israel Institute of Technology, Technion City, Haifa 3200003, Israel
e-mail: nitsa@technion.ac.il

Leon Henkin the Reviewer

Concha Martínez Vidal and José Pedro Úbeda Rives

Abstract In this chapter, we intend to look at Henkin's reviews, a total of forty-six. The books and papers reviewed deal with a large variety of subjects that range from the algebraic treatment of logical systems to issues concerning the philosophy of mathematics and, not surprisingly—given his active work in mathematical education—one on the teaching of this subject. Most of them were published in *The Journal of Symbolic Logic* and only one in the *Bulletin of the American Mathematical Society*. We will start by sorting these works into subjects and continue by providing a brief summary of each of them in order to point out those aspects that are originally from Henkin, and what we take to be mistakes. This analysis should disclose Henkin's personal views on some of the most important results and influential books of his time; for instance, Gödel's discovery of the consistency of the *Continuum Hypothesis* with the axioms of set theory or Church's *Introduction to Logic*. It should also provide insight into how various outstanding results in logic and the foundations of mathematics were seen at the time. Finally, we will relate Henkin's reviews to Henkin's major contributions.

Keywords Leon Henkin · Reviews · Logic systems · Type theory · Metalogic · Algebraic logic · Philosophy of logic and mathematics · Mathematical education

1 Introduction

Henkin, at the very least, wrote the 46 reviews we list in Sect. 7. In what follows, we provide a brief abstract of each of those. The reviews have been organized according to subjects, and in many cases, we explain the contributions of each of the reviews falling under each of the epigraphs to the subject in question. In order to track which review is being summarized where, we will write the number that corresponds to the review in our Sect. 7 in boldface and the number of the review in the bibliography between square brackets.

In order to track the relation between Henkin's contributions and the subject of the reviews he was given, we have looked at the works by Henkin himself that he quotes in his reviews. We have also taken into account Monk's description[1] of Henkin's scientific contributions. And we have tried to catalogue the works reviewed according to contemporary areas of knowledge.

[1] See J. Donald Monk [194].

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_10

Note that the Reviews Section in the JSL started with the journal in 1936, and Church was in charge.[2] Henkin's reviews are part of this project. Henkin was designated as "consulting editor"; that means that he "could simply be assigned that publication which [he was] obligated to review." (H.B. Enderton [30, p. 174]) Henkin was consulting editor for volumes 17 to 23 and 25 to 31, which is from 1952 to 1958 and from 1960 to 1966, respectively. Besides, he was editor with A. Church and M. Black for volume 14, 1949, and with A. Church, S.C. Kleene and A.A. Lazerowitz for volume 24 in 1959.

According to H.B. Enderton [30, p. 173], a work was to be reviewed if the work was relevant to symbolic logic independently of its quality. H.B. Enderton quotes different passages by A. Church in which he explains the reasons for his way of understanding the Review Section and its role. According to those passages, Church thought that the situation in mathematical logic was such that results in logic frequently contained errors and absurdities; hence, he thought that competent workers in the field should indicate which works were valuable in order to prevent the field from falling into disrepute. Reviewers were expected to provide an assessment of the work. Henkin's reviews do satisfy this requirement, as we shall see below.

All except Henkin's review of Church's *Introduction to logic* were published in *The Journal of Symbolic Logic*; Henkin's review of Church's book was published in the *Bulletin of the Mathematical Society*. A plausible explanation for this exception could be that since Alonzo Church was responsible for the Reviews Section in *The Journal of Symbolic Logic*, he did not want to ask one of his consulting editors to assess his own work.

2 Logic Systems

Under this heading, we have included those reviews dealing with works on type theory and metalogic.

2.1 Type theory:

- 1 (1942) Review [49] of: Maxwell Herman Alexander Newman and Alan Mathison Turing, "A formal theorem in Church's theory of types" (1942) [201].
- 5 (1949) Review [56] of: Stanislaw Jaśkowski, "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres" (1948) [149].
- 24 (1955) Review [81] of: Evert Willem Beth, "Sur le parallelisme logico-mathematique" (1953) [8].

2.2 Metalogic:

- 4 (1949) Review [57] of: Stanislaw Jaśkowski, "Sur les variables propositionelles dependantes" (1948) [150].
- 6 (1949) Review [55] of: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1948) [232].
- 15 (1952) Review [66] of: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1951) [233].

[2] See H.B. Enderton [30] for a detailed account of Church as a reviewer. All the information we include about the issue is taken from Enderton's work.

18 (1953) Review [69] of: Burton Dreben, "On the completeness of quantification theory" (1952) [28].
19 (1954) Review [74] of: Georg Kreisel, "On a problem of Henkin's" (1953) [157].
23 (1955) Review [83] of: Helena Rasiowa and Roman Sikorski, "On existential theorems in non-classical functional calculi" (1954) [209].
29 (1955) Review [82] of: Gunter Asser, "Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe" (1955) [2].
30 (1956) Review [89] of: Jerzy Łoś, "The algebraic treatment of the methodology of elementary deductive systems" (1955) [167].
31 (1956) Review [90] of: Juliusz Reichbach, "O pełności węższego rachunku funkcyjnego"; Juliusz Reichbach, "O polnoté uzkogo funkcional'nogo isčisléniá" (Russian translation); Juliusz Reichbach, "Completeness of the functional calculus of first-order" (English summary) (1955) [210].

2.1 Type Theory

Type theory was first introduced by Bertrand Russell as a solution to the paradoxes in set theory.

Henkin's contribution to the development of type theory consisted in his applying—to the theory of types—the technique of introducing individual constants in order to eliminate quantifiers and then construct a model[3]—the technique he developed in his proof of the completeness theorem for first-order functional calculi.[4]

Moreover, in his "A theory of propositional types" [111], he studies the theory of types whose only basic type is the propositional type and in which the rest of the types obtain from this basic one by applying the clause that goes: if α and β are types, then so is $\alpha \to \beta$. He establishes certain constants from which all possible functions for each type are definable. This work is connected to B. Russell's logical system in *The Principles of Mathematics* [217], with S. Leśniewski's *Protothetic* [161] and "Introductory remarks to the continuation of my article 'Grundzuge eines neuens systems der Grundlagen der Mathematik' " [162], and with Tarski's paper "O wyrazie pierwotnym logistyki" [226].

Henkin wrote three reviews on type theory; one (1) [49] has to do with axioms of infinity, and both the second (5) [56] and third (24) [81] with how to characterize mathematical theories in simple type theory. Henkin's review on axioms of infinity was his first review [49]; it is a review of the paper published by M.H.A. Newman and A.M. Turing 1942 "A formal theorem in Church's theory of types" [201]. Henkin wrote it in 1942, the same year in which the paper was published. Henkin reports that Newman and Turing present a result that builds on one by Church about his simple type theory [20]. In that paper, Church introduces an axiom of "infinity" Y_ι for type ι and proves—using only axioms 1 to 8—that if the axiom of infinity Y_α for type α holds, then the axiom of infinity $Y_{(\alpha\alpha)(\alpha\alpha)}$ for type $(\alpha\alpha)(\alpha\alpha)$ also holds; and he claims that Y_α does not hold if α contains

[3] L. Henkin: "The completeness of the first-order functional calculus" [52].

[4] "Completeness in the theory of types" [59].

only type $\circ$. Newman and Türing show, using axioms 1 to 10, that if α contains ι, then Y_α holds. Henkin explains the proof they give, clarifies some of the concepts they use, mentions the notational simplifications used by the authors, and claims—mentioning a model (in which axiom 10 holds for type $(\iota\iota)$ and Y_α holds only if α is ι or contains type $(\iota\iota)$)—that axioms 1 to 10 are not sufficient. Henkin raises two problems; namely, whether axiom 9 and axiom 10 are needed in all types. Finally, he lists a number of errata.[5]

The other two works reported, though devoted to the same subject, show different worries. Thus, Henkin [56] reports that Jaśkowski's "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres" [149] demonstrates that natural and real numbers can be defined within the same type[6] as functions that have certain classes of individuals both as values and arguments. The basic concept is that of *dual group* or family of classes of sets of individuals with both a negation and an addition operation. The integer number n is the function that links each element of the dual group with the element that obtains when that element is added n times (and Ø with those that are not in the group). Proving Peano axioms is easy except for the axiom "zero is not the successor of any number," for which an infinity axiom is needed. By defining an order in each dual group, rational and real numbers can be identified with certain functions that work over a dual group with the same order as the continuum. Henkin reports that Jaśkowski shows that "the continuous groups are those families that arise from a measure function f defined over sets of individuals, the elements of the group consisting of classes made up of all sets for which f has a given value [...]." The measure functions f are additive, but they are not of the usual kind: the whole space has measure zero, and complements have measures with opposite signs. Henkin points out that in this way Euclides' idea of using ratios of line segments is expanded to trade with the reals.

According to Henkin [81], Beth starts his book chapter "Sur le parallelisme logico-mathematique" [8] briefly describing type theory and then considers various mathematical theories depending on the number of types needed to formalize them. Beth first considers the possibility of formalizing elemental geometry in a first-order language (Bernays mentions in the discussion that the notion of polygon is not first-order formalizable). He then compares Peano first-order systems with Peano second-order systems. He describes Henkin's nonstandard models [59] and mentions that his construction follows Mostowski's ideas [197]. Beth also remarks that there is a tendency to substitute higher-order mathematical theories for their first-order versions.

2.2 Metalogic

Metalogic can be defined as the study of syntactic and semantic properties of logical systems. As is well known, major developments in the field took place during the first decades of the 20th century, and Henkin himself made (at least) one major contribution to

[5]Note that in 1942 Henkin was Church's student. This review has a strong technical character.

[6]Pre-existent definitions (the usual ones in the foundations of mathematics) placed natural numbers/integers in one type while real numbers were defined as "special sets of integers," hence they/their definition belonged in a higher functional type.

the development of the subject: he provided a proof of the completeness theorem for first-order logic that resulted in the development of a new method of proof (see Henkin L., "The completeness of the first-order functional calculus" [52]).[7] It so happens that his proof, not Gödel's, has been used to teach students how to prove completeness since Henkin's proof is much easier to follow than that of Gödel.

As is well known, the search for new proofs for a given theorem is a common practice in mathematics. The completeness theorem for first-order logic is a clear example of this practice. Thus, Henkin himself reported in 1953 (18) [69] and 1956 (31) on two other proofs of this result: Burton Dreben's proof, published in 1952 "On the Completeness of quantification theory" [28], and Juliusz Reichbach's proof, published in 1955 "Completeness of the functional calculus of first-order" [210]. Reichbach tells us that his proof comes after Gödel's and Henkin's proofs, though he claims it was obtained independently of those by Robinson [213], Rasiowa and Sikorski [206], Beth [7], Rieger [211], and Łoś [167]. Dreben's proof shows, combining results by Herbrand [142] and Gödel [41], that each quantificationally valid scheme in prenex normal form is provable by means of a proof in Herbrand normal form. Henkin's review ends by reporting on Dreben's diagnosis about why Herbrand did not draw from his theorem the conclusion that a formula that cannot be derived in Herbrand normal form is not valid; Dreben claims that Herbrand did not draw this conclusion because the deduction is nonconstructive. Reichbach uses a method of proof in which the set of free variables is disjoint from the set of bound variables. This is what Frege used in his [34] and [36]. Neither the author nor Henkin points out this curiosity, plausibly because it is a curiosity of no consequence for the arguments involved. Reichbach's procedure constructs an interpretation for a nonprovable formula α, an interpretation in which α is false. Henkin does not go into the details of this construction, but gives only its general lines: a maximal set with certain properties is constructed, but the steps in the construction are not accounted for.

Henkin reported (19) [74], on a paper by Kreisel "On a problem of Henkin's" [157], also closely related to completeness issues. Henkin himself [219] posed the problem that Kreisel tried to solve: "If Σ is any standard formal system adequate for recursive number theory, a formula (having a certain integer q as its Gödel number) can be constructed that expresses the proposition that the formula with Gödel number q is provable in Σ. Is this formula provable or independent in Σ?" Löb solved the problem and published the solution in 1955 (see his [169]). According to Henkin, Kreisel's contribution consisted of showing that the concept that is expressed by the formula can be interpreted in various ways, and depending on how it is interpreted, the answer to Henkin's question will vary. Henkin notes that "[a] clear explication of the concept of that which is expressed by a formula must be based on an axiomatic treatment of this notion" and further suggests that such treatment can be done by following what Church said in [22].[8]

[7] Works by Henkin that are contributions to the subject are: "Completeness in the theory of types" [59]; "On the primitive symbols of Quine's 'Mathematical Logic' " [68]; "A generalization of the concept of ω-consistency" [73]; "On the definitions of 'formal deduction' " (with R. Montague) [195]; "A generalization of the concept of ω-completeness" [92]; "Some remarks on infinitely long formulas" [104]; "An extension of the Craig–Lyndon interpolation theorem" [107]; *Logical Systems Containing Only a Finite Number of Symbols* [114], and "Relativization with respect to formulas and its use in proofs of independence" [116].

[8] Note that Church wrote a second version of this paper, in which he revised this proposal. Moreover, Church explains that along the years he doubted the viability of his initial proposal in his 1973 paper

Henkin recapitulates four papers on another important metalogical issue, the decision problem. Following Church [24, p. 100, n. 184], we understand the decision problem "as a general name for problems to find an effective criterion (a decision procedure) for something, and to distinguish different decision problems by means of qualifying adjectives or phrases." Hence, in relation to logical systems, we have the decision problem for provability, the decision problem for validity, and the decision problem for satisfiability.

A solution for a particular decision problem is an algorithm that applied to a particular instance of the problem, results in a "yes" or "no" answer, depending whether the specific instance of the considered decision problem has or fails to have the considered property. For instance, an algorithm that decides about provability would be an algorithm that results in a "yes" if the considered formula is a theorem or in a "no" if the considered formula is not a theorem. This algorithm is, of course, nothing but the definition of a recursive or computable function whose value will be one if there is a solution for the considered argument and zero otherwise.

Providing an algorithm that gives a yes or no answer for any instance of a problem—whatever the decision problem is (whether it is the decision problem for provability, validity, satisfiability, etc.)—counts as a *direct* way of solving any of these decision problems. Alternatively, a solution obtains if an algorithm is provided that transforms each instance of a unsolved decision problem, into an instance of another decision problem for which there is a known solution. Hence, "[a] *reduction of the decision problem* (of the pure first-order functional calculus) consists in a special class Γ of wffs and an effective procedure by which, when an arbitrary wff A is given, a corresponding wff A_Γ of the class Γ can be found such that A is a theorem if and only if a proof of A_Γ is known" ([24, p. 270]). Hence, in those cases for which there is no general solution, a clear way to go to is look for some subset B of A for which there is a solution. In other words, logicians try to find special cases for which there is a solution.

The four works accounted for by Henkin deal with the decision problem for one area or another. In particular, the first, Jaśkowski, S. "Sur les variables propositionelles dependantes" [150], offers a reduction to a particular case for first-order decidability. The second and third, respectively the first and second editions of Tarski's *A Decision Method for Elementary Algebra and Geometry* [232], offer a procedure to decide whether a statement of algebra and elementary geometry is or is not true by using the technique of quantifier elimination. Finally, the fourth work by H. Rasiowa and R. Sikorski—"On existential theorems in non-classical functional calculi" [209]—presents a reduction of the decision problem for a set of statements of intuitionistic functional logic into the decision problem for intuitionistic propositional logic.

Henkin (4) [57] summarizes Jaśkowski's method to reduce the decidability problem for first-order logic to the decidability problem for certain systems I and II when certain conditions are obtained. Those conditions are that there are only three variables and a propositional variable that play the role of a predicate with three arguments in those systems. Henkin points out that Jaśkowski's results are obtained only if the theorems in I are defined as the theorems that are obtained from theorems in the first-order calculus, but not from formulas that are not first-order theorems. In this case, I and II are equivalent.

"Outline of a revised formulation of the logic of sense and denotation (Part I)" [25] what he takes to be an adequate proposal. The next year, Church published the second part [26], and in 1993, he published an alternative formulation [27].

Henkin claims that if theorems in system I are obtained—as Jaśkowski sustains—from homogeneous theorems[9] in first-order logic, then a formula in system I is a theorem in I if and only if a tautology is obtained when quantifiers are eliminated. But, in that case, theorems in I would be decidable, contrary to what Jaśkowski claims when he says that the decision problem for first-order logic reduces to the decision problem in I. That is why Henkin modifies the definition in system I claiming that its theorems are obtainable from theorems in first-order logic and that there is no first-order formula that is not a theorem from which a formula in system I is obtained. We believe that Henkin is wrong. Take, for instance, the formula $\exists x_0 P x_0 \rightarrow \forall x_0 P x_0$ from which the formula $\exists x_0 p \rightarrow \forall x_0 p$ is obtained; the latter, according to Henkin's criterion, would be a theorem in I because after removing quantifiers and variables a tautology, $p \rightarrow p$, is obtained. Henkin mentions that the author defines other systems, one of which he intends to use for modal logic; but Henkin considers this system, which is not really a logic since no effective criterion for deducibility is given.

According to Henkin (6) [55] and (15) [66], Tarski's very well edited book [232] precisely and profusely describes, and proves the validity of a decision method obtained by Tarski himself in 1930. The decision method is applied to a system of elementary algebra: its sentences are constructed with connectives and quantifiers from equations and inequations of polynomials in real variables whose coefficients are 1, -1, and 0. The method decides whether a given statement expresses a proposition that is true according to its standard interpretation. A method for axiomatizing the system is given, and it is related to the decision method. Indications are provided about how to extend it to elemental theories in Euclidean and projective geometry. He uses the method of quantifier elimination that was introduced by Löwenheim [170]. The review of the first edition ends listing a number of errata. The review for the second edition [233] reports that: (i) errata in the first edition have been corrected; (ii) contents have been enlarged and applications expanded.

Henkin's review (23) [83] of Rasiowa and Sikorski's [209] "On existential theorems in non-classical functional calculi" notes that they build on a previous result by McKinsey and Tarski [189]; the latter proved a fact that Gödel had established though not proved [43]. The fact is that if a disjunction $\sigma \vee \tau$ is provable in an intuitionistic propositional calculus S, then σ is provable in S or τ is provable in S. Rasiowa and Sikorski, using their algebraic treatment of the notion of satisfaction [208] and McKinsey and Tarski's line of proof, extend the result showing that if $\exists x\sigma$ is provable in a first-order intuitionistic system S^*, then some formula σ'—obtained from σ by substituting for all occurrences of x occurrences of a free variable y—is also provable in S^*. They also establish that a formula in S^* without quantifiers is provable in S^* if and only if it is an instance of some theorem of S. Then, since there is a decision method for S, they infer that there is a decision method for the class of formulas that can be put in prenex normal form. They claim that this last result can be applied more generally to the class of formulas in which no quantifier occurs under the scope of a negation or implication. Finally, Henkin claims that analogous results are obtainable for positive, minimal, and Lewis' calculus and also for the rest of the calculi addressed by the authors in [208].[10]

[9]Homogeneous theorems are homogeneous formulas that are derivable from axioms using the inference rules in the system, and a formula is *homogeneous* if only predicates with the same number of variables and in the same order occur in it.

[10]This review could have also been included in Sect. 3.1 "Algebraic treatment of logic systems."

Finally, Henkin assesses two papers devoted to the analysis, at the abstract level, of logical consequence operations: (29) [82] Asser's "Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe" [2] and (30) [89] Łoś's "The algebraic treatment of the methodology of elementary deductive systems" [167].

Asser analyzes the logical consequence formal relation that holds between sets of formulas X and certain formulas H. Henkin remarks that Asser fails to quote other works in which the same issue is dealt with (for instance, Henkin [58] and Rasiowa–Sikorski [208]). The consequence relation, **Abl**, is such that $X\mathbf{Abl}H$ complies with the following conditions:

1. If $H \in X$, then $X\mathbf{Abl}H$;
2. If $X_1 \subseteq X_2$, and $X_1\mathbf{Abl}H$, then $X_2\mathbf{Abl}H$;
3. If $\mathbf{Cn}(X) = \{H|X\mathbf{Abl}H\}$ and $\mathbf{Cn}(X)\mathbf{Abl}H_1$, then $X\mathbf{Abl}H_1$;
4. If H_1 is a substitution instance for bound variables, free individual variables, and predicate variables in H and $X\mathbf{Abl}H$, then $X\mathbf{Abl}H_1$.

Moreover, Asser considers a very general semantic interpretation for well-formed formulas. The models used by Asser for formulas H are formed by a set M of truth values, a set $M_0 \subset M$ of designated values, functions associated with connectives, functions that assign an element in M for each subset of M associated with each quantifier, an arbitrary set J over which individual variables range, and for each n-ary predicate nonempty set B_n as a range for n-predicate variables whose elements are functions that assign an element in M to each n-tuple of elements in J. The main result is that for each set of well-formed formula X that is closed under the relation **Abl**, there exists a model μ such that X is precisely the class of valid formulas with respect to μ. The construction of μ is one by Lindenbaum.

Henkin says that Łoś provides "a comprehensive account of the principal results concerning elementary deductive systems" since (1) he studies axiomatically the formal consequence relation introducing it in Tarski's way in [227] and [228]; (2) Łos proves Gödel's theorem [41]—each consistent set of formulas is simultaneously satisfiable—by means of maximally O-consistent sets and forming the corresponding Lindenbaum algebra (see Tarski [228]); (3) he extends this result to the first-order case by applying the quantifier elimination method through introducing new functional signs (something that had previously been done by Skolem [221] and Hilbert and Bernays [146]); (4) by introducing nondenumerable sets of constants (compare to Henkin [52]), Łos deduces Tarski's generalization of the Skölem–Löwenheim theorem [220] and [222]; (5) using Robinson's "complete diagrams" [213] or the complete description of a model by means of constants (Henkin [71]), he deals with the extension of models establishing that "each ordered set can be extended to a densely ordered one"; (6) he introduces systems that consist of the consequences of a set of universal statements whose models constitute what Tarski [235] called a *universal class* and independently shows the theorem by Tarski "that ... an arithmetical class is universal if and only it is closed under formation of submodels"; (7) he treats the notion of categoricity in power that he had previously studied in his [166] and gives a proof of Vaught's theorem [237]—a system without finite models that is categorical in some (infinite) power has to be complete; (8) he proves Skolem's result [222]—the system of all true statements of elemental number theory is not categorical in $\aleph_0$—and constructs, by a process similar to Henkin's [59], a nonstandard model pointing out that the order relation over this model is not well ordered (it is of type $\omega + (\omega^* + \omega)\eta$ (see Henkin [59]).

3 Algebraic Logic

According to what Andréka, Németi, and Sain[11] say, we organize this section as follows.

3.1 Algebraic treatment of logic systems:

3 (1948) Review [50] of: John Charles Chenoweth McKinsey and Alfred Tarski, "Some theorems about the sentential calculi of Lewis and Heyting" (1948) [189].
7 (1949) Review [54] of: Andre Chauvin, "Structures logiques" (1949) [19].
8 (1949) Review [53] of: Andre Chauvin, "Généralisation du théorème de Gödel" (1949) [18].
9 (1950) Review [60] of: László Kalmár, "Une forme du théorème de Gödel sous des hypothèses minimales" (1949) [153]; László Kalmár, "Quelques formes générales du théorème de Gödel" (1949) [152].
17 (1953) Review [70] of: Helena Rasiowa, "Algebraic treatment of the functional calculi of Heyting and Lewis" (1951, pub. 1952) [205].
22 (1955) Review [84] of: Helena Rasiowa and Roman Sikorski, "Algebraic treatment of the notion of satisfiability" (1953) [208].
28 (1955) [85]: Ladislav Rieger, "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem" (1951) [211].
39 (1959) Review [99] of: Louis Nolin, "Sur l'algebre des predicats"; Andrzej Mostowski, Jean Porte, Alfred Tarski, Jacques Riguet, "Interventions" (1958) [202].
42 (1963) Review [110] of: Antonio Monteiro, "Matrices de Morgan caractéristiques pour le calcul propositionnel classique" (1960) [196].
44 (1967) Review [115] of: Marshall Harvey Stone, "Free Boolean Rings and Algebras" (1954) [223].
45 (1971) Review [118] of: Maurice L'Abbé, "Structures algébriques suggérées par la logique mathématique" (1958) [158].
46 (1971) Review [117] of: Marc Krasner, "Les algèbres cylindriques" (1958) [156].

3.2 Applications of logic to algebra:

14 (1952) Review [65] of: Abraham Robinson, *On the Metamathematics of Algebra* (1951) [213].

[11] "Algebraic logic can be divided into two main parts. Part I studies algebras which are relevant to logic(s), e.g. algebras which were obtained from logics (one way or another). Since Part I studies algebras, its methods are, basically, algebraic. One could say that Part I belongs to 'Algebra Country'. Continuing this metaphor, Part II deals with studying and building the bridge between Algebra Country and Logic Country. Part II deals with the methodology of solving logic problems by (i) translating them to algebra (the process of algebraization), (ii) solving the algebraic problem (this really belongs to Part I), and (iii) translating the result back to logic. There is an emphasis here on step (iii), because without such a methodological emphasis one could be tempted to play the 'enjoyable games' (i) and (ii), and then forget about the 'boring duty' of (iii). Of course, this bridge can also be used backwards, to solve algebraic problems with logical methods." (Hajnal Andréka, Istvan Németi, and Ildikó Sain [1, p. 133].)

25 (1955) Review [79] of: Abraham Robinson, "Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique"; Evert Willem Beth, Luitzen Egbertus Jan Brouwer, Abraham Robinson, "Discussion" (1953) [215].
27 (1955) Review [80] of: A. Chatelet, "Allocution d'ouverture" (1953); (3) Luitzen Egbertus Jan Brouwer, "Discours final" (1953); (4) Abraham Robinson, "On axiomatic systems which possess finite models" (1951) [212].
33 (1957) Review [94] of: Kaarlo Jaakko Hintikka, "An application of logic to algebra" (1954) [147].
37 (with Andrzej Mostowski) (1959) Review [136] of: Anatolĭ Ivanovič Mal'cév, "Ob odnom obščém métodé polučéniá lokal'nyh téorém téorii grupp" (On a general method for obtaining local theorems in group theory) (1941) [181]; Anatolĭ Ivanovič Mal'cév, "O prédstavléniáh modéléj" (On representations of models) (1956) [182].

3.1 Algebraic Treatment of Logic Systems

Henkin himself made major contributions to the development of the algebraic treatment of logic systems. Namely, his results on cylindric algebras (see below) and his "The completeness of the first-order functional calculus" [52], "An algebraic characterization of quantifiers" [58], "Completeness in the theory of types" [59], and "Some interconnections between modern algebra and mathematical logic" [71].

Henkin wrote thirteen reviews of various works in this field: three (3), (17), (22) about works dealing with the algebraic treatment for intuitionistic calculus and modal calculi; five on algebras related to classical logic (polyadic algebras (39); two on cylindric algebras (45), (46); two on Boolean algebras (44), (28)); one on logical matrices (42), and three approach algebraic structures suggested by Gödel's incompleteness theorems (7), (8), (9).

"Over a period of three decades or so from the early 1930's there evolved two kinds of mathematical semantics for modal logic. *Algebraic* semantics interprets modal connectives as operators on Boolean algebras. *Relational* semantics uses relational structures, often called Kripke *models*" (R. Goldblatt [39, p. 1]).

The first substantial algebraic analysis of modalized statements was carried out by Hugh MacColl, in a series of papers that appeared in *Mind* between 1880 and 1906 under the title *Symbolical Reasoning*.[12] Some time later, in 1938, the first topological interpretations for modal and intuitionistic logics developed by Tsa-Chen Tang [225] and Tarski [231], respectively, saw the light. The three works reported by Henkin we will summarize next belong to this area.

In 1948 (3) [50], Henkin reviews a 1948 paper "Some theorems about the sentential calculi of Lewis and Heyting" by McKinsey and Tarski [189]. In this paper, starting from the syntactic similarity between the axioms in certain logics and the axioms in the theory of point topology, the authors apply the techniques developed by them in [185–187], and [188] to the analysis of logic S4 and its Lewis' extensions (S5 among them) [164] and to Heyting's intuitionistic logic [145]. Henkin enhances the following results: a new axiomatization for S4 without a substitution rule is given; a matrix with a designated

[12]See [171–177], and [178].

element satisfies the axioms in S4 and modus ponens if and only if it is a closure algebra with operations $\cdot$, $-$, and C corresponding, respectively, to $\wedge$, $\neg$, and $\Diamond$, with 1 as a designated element; if α is deducible in S4, so is $\neg\Diamond\neg\alpha$ (Gödel [43] just stated it without proof). They also establish that there are extensions for S4—S5 for instance that satisfy this property, whereas the characteristic matrix for those extensions that do not satisfy the property is to have two designated elements; if $\Box\alpha \vee \Box\beta$ is derivable in S4, then either α or β is derivable in S4 (it does not hold in S5); they introduce the notion of *reducibility* and prove that neither S4 nor S5 is reducible (they generalize a result by Dugundji [29]); the class of matrices that satisfy the axioms in Heyting's calculus and detachment (logic **I**) and have just one designated element is the class of Brouwerian algebras isomorphic to a system of closed sets of some topological space; logic **I** satisfies: if $\alpha \vee \beta$ is derivable, then so is α or β; there are infinite nonequivalent formulae that contain a unique variable; and the logic is not reducible. Finally, they describe effective applications f from intuitionistic formulas to formulas in S4 in such a way that a formula φ is derivable in **I** if and only if $f(\varphi)$ is derivable in S4. Henkin considers the paper to be too condensed, without indications for proofs and with cumbersome symbolism. This paper is a key for the next two reviewed by Henkin.

In 1953, Henkin reviews (17) [70] "Algebraic treatment of the functional calculi of Heyting and Lewis" by Rasiowa [205]. In her paper, Rasiowa generalizes results obtained by McKinsey and Tarski [189] on modal and intuitionistic propositional logic, and she solves a problem posed by Mostowski [198]. Mostowski showed that Heyting's intuitionistic functional calculus [143] can be interpreted in a model $(I, \mathbf{B})$, where I is a set, and $\mathbf{B}$ is a complete Brouwerian lattice. Then, he asked whether there is a particular model $(I_0, \mathbf{B}_0)$ in which all nonderivable formulas take a value different from zero, while derivable ones take 0. This question has a positive answer for classical calculus as Gödel showed [41] and was partially answered by Henkin [59]. But Rasiowa provides the definite answer for intuitionistic calculus by using McKinsey and Tarski's results [188], and MacNeille's [179] and McKinsey's [185] methods. She also shows that system S4 by Lewis and Langford [164] is complete with relation to models $(I, \mathbf{C})$, where $\mathbf{C}$ is a closure algebra. Henkin ends the review pointing out that the relation between the modal calculus considered by Rasiowa and those considered by Barcan [3] and Carnap [16] is not clarified.

Henkin (22) [84] comments on "Algebraic treatment of the notion of satisfiability" by Rasiowa and Sikorski [208]. The authors study the algebraic interpretation for first-order calculi proposed by Mostowski [198] as a generalization of the algebraic interpretation of propositional calculus developed by Tarski and McKinsey [189]; part of this work presents previous results by them [205, 206], and [207]. In it, they consider systems S^* that obtain from adding axiom schema and quantification rules to a propositional calculus S, where S contains "positive logic" and, possibly, more connectives, and which is closed under modus ponens and replacement. An algebraic structure or S-algebra that includes an ordering relation (that corresponds to the conditional) is associated (as an interpretation) with each system S, and, if the logic is complete in relation to the order defined, then the algebra is called "S^*-algebra." An S^*-algebra together with a set J of "individuals" is a model for the system. The main result presented is that for systems S that comply with property E (that is, systems that can be embedded in an S^*-algebra in a way in which an arbitrarily given sequence of sums and products is preserved), the following requisites on a formula α in S^* are equivalent: α is provable in S^*; α is valid (it takes as a value only the unit element of the S^*-algebra A for all interpretations (J, A)); α is valid for all interpretations $(\mathbb{N}, A)$; α is valid in the model $(\mathbb{N}, L^*)$, where L^* is any complete extension of the "Lindenbaum algebra" L of S^*. (Feferman in his revision [31] of [206] claims that this name is not appropriate because it had been used previously by Tarski [228].) Later on, they apply all these results to the classical functional calculus—as they had done in [206] and [207]—then to intuitionistic and Lewis' logics (analogously to what they had done in [205] but this time using Heyting algebras instead of Brouwerian algebras). They end by describing a transformation ψ of intuitionistic formulae into modal formulae; they also show that a formula α is provable if and only if $\psi(\alpha)$ is provable (analogous to the result that Tarski and McKinsey [189] obtained for the propositional calculus). Henkin considers the authors have given up a possible generalization without advantage; this can be seen after analyzing his treatment of *implicational* logic [58]. Finally, Henkin poses an unresolved problem: do the S-algebras considered by the authors satisfy condition E? Henkin believes that it will be a difficult question to answer.

Henkin (39) [99] reviews in 1959 Nolin's algebraic version of first-order logic [202] "Sur l'algebre des predicats" (1958). Nolin introduces a calculus that is similar to Halmos' polyadic algebras,[13] except that it divides the universe of discourse into separate types and uses nonindependent primitive notions.

Henkin devoted many of his works to the study of cylindric algebras;[14] in fact, he can be considered as one of the authors that contributed most to the development of the field. These algebras, together with polyadic algebras, are to first-order logic like Boolean algebras are to propositional logic.[15]

Henkin published two reviews of works on Cylindric Algebras in 1971: L'Abbé's [158] "Structures algébriques suggérées par la logique mathématique" and Krasner's [156] "Les algèbres cylindriques".

According to Henkin (45) [118], because L'Abbé believes that "the two fundamental instruments in mathematical logic" are the theory of recursive functions and modern algebra, he describes various classes of algebraic structures whose characterization has come

[13]In his book *Algebraic logic* [47], Halmos publishes his 10 main papers on polyadic algebra.

[14]Cylindric algebras generalize Boolean algebras for each ordinal α by adding "distinguished elements" (the so called "diagonal elements" $\mathbf{d}_{\kappa,\lambda}$ where κ and λ are less than α) to the elements of the Boolean algebra 0 and 1, and unary operations called "cylindrifications" ($\mathbf{c}_\kappa$ where $\kappa < \alpha$).

[15]A hopefully complete list of Henkin's works on the subject is: "The representation theorem for cylindric algebras" [78]; "La structure algébrique des théories mathématiques" [87]; "Cylindrical algebras" (with A. Tarski) [140]; the abstract "Cylindrical algebras of dimension 2. Preliminary report" [91]; *Cylindric Algebras. Lectures presented at the 1961 Seminar of the Canadian Mathematical Congress* [103]; "Cylindric algebras" (with A. Tarski) [141]; *Cylindric Algebras, Part I* (1971 with J.D. Monk and A. Tarski) [132]; "Cylindric algebras and related structures" (with J.D. Monk) [131]; "Relativization of cylindric algebras" (with D. Resek) [137]; "Cylindric set algebras and related structures" (with J.D. Monk and A. Tarski) [133]; *Cylindric Algebras, Part II* (with J.D. Monk and A. Tarski) [134], and "Representable cylindric algebras" (with J.D. Monk and A. Tarski) [135].

from logical studies. He establishes the following correspondences:

Classical propositional logic	Boolean algebras
Many-valued logics	Post algebras
Intuitionistic logic	Brouwerian and Heyting algebras
Modal logics	Closure algebras

With a certain amount of detail, L'Abbé describes Boolean algebras that are relational, projective and cylindric, and monadic algebras. Henkin mentions that L'Abbé gives a wrong definition of cylindrification for cylindric set algebras. Henkin also points out that L'Abbé quotes Lyndon [168] for supposedly having proved that the class of representable relational algebras cannot be characterized by equational identities, but he does not quote Tarski [235], who succeeded in proving just the opposite.

In his work "Les algèbres cylindriques" (1958), Krasner contends that the theory of cylindric algebras does not start with Tarski's 1945 work, as L'Abbé asserts [158], but with several 1938 works about Galois theory, such as his own [155].[16] Henkin reports (46) [117], following Krasner, that in it, Krasner considers cylindric algebras whose Boolean component is complete and that Krasner describes it in relation to two other works.

Henkin writes two reports of works on Boolean algebras. In 1967, in one of them, he reports on a work by Stone [223] "Free Boolean rings and algebras," in which free Boolean algebras are generalized. The other was published in 1955 and deals with a work by Rieger written in 1951 [211], "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem," in which σ-algebras are generalized.

Henkin (44) [115] tells us that Stone ([223] "Free Boolean Rings and Algebras" (1954)) obtains a class of algebraic structures of which free Boolean rings are a special subclass. Stone obtains them by generalizing the notion of group-algebra A of a group G over a field Φ. Henkin also notes that Stone gives a more classical description of free Boolean rings, and that at the end of the paper, he outlines the significance of these rings for propositional logic. He ends by pointing out that Stone follows Boole's path [15] since he, like Boole, uses algebraic methods to study logic. Henkin also remarks that Stone's presentation follows Tarski' *Systemenkalkül* [228].

In 1955, Henkin reviews (28) [85] the work [211] "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem," in which Rieger generalizes σ-algebras by defining "Boolean algebras with marked sequences." In these algebras, only marked sequences have to have sums and products. Marked sequences constitute a set of multiple sequences that contain all "constant" sequences, and the set is closed in relation to some operations over sequences. This concept comes up when a certain model that derives from considering first-order functional calculi of symbolic logic is examined. In particular, if we associate with each formula A, the set $|A|$ of all formulas B such that $A \equiv B$ is provable, then the class of all sets $|A|$ is a Boolean algebra for the following operations: $-|A| = |\neg A|$, $|A| + |B| = |A \vee B|$, and $|A| \cdot |B| = |A \wedge B|$. With each formula A and finite sequence of integers $j_1, \ldots, j_n$, we consider the sequence a of elements whose general term $a_{i_1}, \ldots, a_{i_n}$ is $|B|$, where B is obtained from A by replacing free occurrences of variable x_{j_k} for free occurrences of x_{i_k} for $k = 1, \ldots, n$; in that case, $\Sigma a = |\exists x_{j_1} \cdots \exists x_{j_n} A|$,

[16] Krasner in [156] dates that work in 1958, not in 1938. Henkin does not provide the explicit reference for [155].

$\Pi a = |\forall x_{j_1} \cdots \forall x_{j_n} A|$ (see Mostowski [198], Henkin [58], and Rasiowa [167, 205]). The set Φ of those sequences satisfies the axioms for marked sequences. Using the result by Loomis [165]—according to which there are σ-algebras that are not σ-isomorphic to any σ-field of sets, but such that each algebra can be σ-represented as the quotient algebra of a σ-field of sets modulo an appropriate σ-ideal—he proves that each Boolean algebra in a denumerable family of marked sequences is isomorphic to a field of sets in such a way that the sums and products of marked sequences respectively become unions and intersections. From this Gödel's result follows: each formula of a first-order calculus is either a theorem or false in the domain of the integers for an appropriate interpretation for predicates. His result is related to Rasiowa and Sikorski's proof [207] of Gödel's theorem. Moreover, the strongest form of Gödel's completeness theorem [41]—each set Γ of formally consistent closed formulas is simultaneously satisfiable—can be obtained by a slight modification of Rieger's argument. Rieger argues that it applies to calculi with a denumerable number of symbols, but other proofs apply also when the number of symbols is nondenumerable (Henkin [52], Robinson [213], Rasiowa [207], and Beth [7]). Henkin finishes by posing the problem of characterizing those Boolean algebras with marked sequences that are σ-isomorphic to Tarski–Lindenbaum algebras and listing a number of errata.

In 1963, Henkin reviews (42) [110] a work by Monteiro [196] "Matrices de Morgan caractéristiques pour le calcul propositionnel classique," in which he describes an interesting class of *characteristic matrices for classical propositional logic* (ccpl)[17] that are *irregular*[18] and includes those described in Church [23]. When Church described such matrices, he asked which other such matrices existed. *De Morgan lattices* are distributive lattices with a monadic operation satisfying $-(-x) = x$ and $-(x \vee y) = (-x) \wedge (-y)$. They were studied by Bialynicki-Birula and Rasiowa [12] as *quasi-Boolean algebras* and by Kalman [151] as *i-lattices*. Monteiro considers matrices (M, D) where M is a De Morgan lattice and D a proper filter of M, establishing that such a matrix is ccpl if and only if $(x \vee -x) \in D$ for each $x \in M$. And, a ccpl matrix is regular if and only if it is an intersection of filters such that, for each $x \in M$, it contains exactly one of the elements x or $-x$. Monteiro constructs an irregular matrix with 12 elements not belonging to Church's class that consists of the subalgebras of direct products of linearly ordered De Morgan matrices, each of which satisfies $(x \wedge -x) \leq (y \vee -y)$.

Also in 1963, Henkin gives a lecture at the Twenty-Eighth Annual Meeting of the Association for Symbolic Logic; in this lecture, he presents a result that generalizes the result of Monteiro.

Henkin reports in one of his 1949 reviews (7) [54] that Chauvin [19] "Structures logiques" defines in an abstract way logical structures to which Gödel's incompleteness theorem applies. Chauvin characterizes *deductive* structures as those that are like the standard ones but for which no recursivity criteria are established; *classical* are those that add certain elements in type theory; *Peanian* those that contain numerical variables, zero, and successor, and *Gödelian* (in relation to a one-to-one application φ from symbols to numbers) are those in which the necessary predicates (relative to formal derivability) for Gödel's construction can be represented.

[17] Each tautology adopts only designated values, while non-tautologies don't.

[18] There are designated elements x and $x \to y$ such that y is not a designated element.

Henkin reports (8) [53] that Chauvin [18] "Généralisation du théorème de Gödel" uses a one-to-one function ψ that assigns a value to each value of a function φ, where φ is a one-to-one function that assigns to each formula its Gödel number. Then, if Δ is the class of systems such that ψ and $R(x, y)$ are representable, then $R(\psi x, \psi y)$ is also representable in those systems. For each Gödelian logic in respect to φ and belonging to Δ, Chauvin gives a procedure to construct an undecidable proposition corresponding to each ψ. He ends by proposing a program: to look for functions ψ such that the undecidable proposition associated with it expresses unsolved conjectures in number theory. He also asks whether each undecidable proposition can be constructed from such a ψ. Henkin observes that the answer is negative as can be seen from the results obtained by Kleene in 1943 [154].

Henkin reviews in 1950 (9) Kalmar's "Une forme du théorème de Gödel sous des hypothèses minimales" [153] and "Quelques formes générales du théorème de Gödel" [152]). In those works, Kalmar presents formalizations of systems for which incompleteness results in Gödel's style [42] can be established that are more abstract than those by Chauvin [19] and [18]. Henkin considers that both the works by Kalmar and those by Chauvin lack any interest for logic; he argues that they raise neither interesting particular new cases that are important in relation to Gödel's theorems nor cases that are mathematically interesting given that no new methods of proof are needed.

3.2 Applications of Logic to Algebra

Henkin reviewed several works whose contents fall under this heading. The first (14) [65] is a review of A. Robinson's book [213] *On the Metamathematics of Algebra*, in which he applies the techniques of symbolic logic to obtain different results in various algebraic theories (groups, rings, and fields). Among them, Henkin points out the following: (i) for each first-order formula ψ that is true in all fields of characteristic zero, there is a prime number p such that ψ is true in all fields of characteristic greater than p; (ii) a formula ψ true in all ordered non-Archimedean fields must be true in all Archimedean fields; and (iii) a formula true in one algebraically closed field must be true in every algebraically closed field of the same characteristic. Some of these results had already been obtained; for instance, the first of them had been obtained by Tarski in 1946 and by Henkin in 1947 (in his Ph.D.), but they had not been published. These results can be obtained by applying the theorem—closely connected to the completeness proof for first-order calculus—that establishes that each formally consistent set of sentences has a model. Robinson's proofs use the techniques developed by Gödel [41] and Henkin [52] for calculi whose languages may contain a nondenumerable number of primitive symbols (Mal'cév was the first to apply the method to nondenumerable sets in 1936 [180]). Robinson uses infinite conjunctions and disjunctions, though he does not describe them clearly. He also generalizes several concepts; one of them reminds one of Tarski's "relative systems" [230]. In relation to those generalizations, Henkin asks whether generalizing provides a unified treatment for pre-existent and independently developed theories and whether it leads to new and deeper results than the original theory. Henkin thinks that Robinson's book does not provide an affirmative answer to these questions. Nevertheless, his assessment of the book is positive, though he also notices some errata and drawbacks. For instance, he mentions that the use by Robinson of predicates instead of the functions commonly used by mathematicians makes the reading more difficult and that the logical theorem used by Robinson also applies to calculi with function symbols (see Henkin [59]).

In 1955, Henkin reviews (25) [79] a work by Robinson [215] "Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique," in which deals essentially with the application to abstract algebra of the metatheorem according to which a set of first-order sentences is satisfiable if each of its finite subsets is satisfiable. These kinds of applications had been given before by Tarski [234], Henkin [71], and Robinson himself [213, 214]. Now Robinson gives new and deeper applications. He ends his paper with speculative passages, some of which show clearly, according to Henkin, that he is confusing use and mention.

Also in 1955, Henkin (27) [80] speaks about another work by Robinson [212] "On axiomatic systems which possess finite models," in which the author deals with models for first-order statement systems (he dealt with the same subject in his [213]). He establishes a theorem about "arithmetic algebraic structures," a concept defined by Tarski [234], for which he establishes the existence of finite models. Henkin remarks that the application of Robinson's result would be maximized if it would establish a feature of those first-order sentences that defined persistent arithmetical classes; Henkin then ends pointing out a series of errata.

This is one of the three reviews Henkin signs in 1959 (37) [136], this time in collaboration with Mostowski. The review reports on Mal'cév [181] "On a general method for obtaining local theorems in group theory" and [182] "On representations of models." Both papers deal with the theorem we nowadays call "compactness theorem" (if every finite subset of a given, possibly nondenumerable, set of first-order sentences is satisfiable, then so is the whole set). The first paper by Mal'cév establishes without proof the so called *general local theorem*: "If every finite subset of a given (possibly nondenumerable) set of first-order sentences is satisfiable, then so is the whole set"; also, several applications of it "to special problems of group theory are made which do not properly fall within the scope of this Journal." The second is devoted to local properties of models (relational structures) $M = \langle A; O_1, O_2, \ldots \rangle$, where A is a set, and O_i are n_i-ary relations. He defines the notion of "local property" as "a property which is necessarily possessed by M if it is possessed by all its finite submodels," the general notion of "representation" for a model, and the more specific ones of prime and predicative representation. Then, he proves two theorems that establish that if every finite submodel M^* of M possesses a representation of a given type (prime or predicative) in a model N^* of a class K of models characterized for a set of first-order axioms, then M possesses a representation (prime, predicative) in a model N of K. A third theorem has to do with second-order properties of models. Henkin and Mostowski believe that the papers have been written carelessly. The reviewers point out that both the formulation and proof of the local general theorem is due to Gödel [41] (denumerable case) and Mal'cév [180] (the nondenumerable case); the latter uses the Skolem normal form in his proof, and because of that, his proof has been found unsatisfactory. They also note that there is a satisfactory proof by Henkin [52] and Robinson [213].

Henkin reviews (33) [94] Hintikka's work [147] "An application of logic to algebra." Henkin starts providing a panoramic view of the completeness theorem and its applications. Henkin tells us that in recent years the first-order completeness result by Gödel [41] has been extended to systems with a nondenumerable number of symbols (Henkin

[52]) and that this result has been applied to establish theorems in algebra by Beth [7], Henkin [71], Łos [167], Robinson [213] and [215], Tarski [234], and, before that, Mal'cév [181]. In all cases, the theorem is used in order to establish that if every finite subset of a set Γ of first-order sentences is satisfiable, then there exists a model that satisfies all sentences in Γ. Hintikka uses a result by Dilworth (for which Henkin does not give the bibliographic reference) and a process of completing by cuts by Birkhoff [13] to obtain a result that, according to Henkin, could have been obtained by less sophisticated methods.

4 Philosophy of Logic and Mathematics

Both the philosophy of logic and mathematics underwent many interesting developments in Henkin's days; in Leon Horsten' words, "...it has turned out that to some extent it is possible to bring mathematical methods to bear on philosophical questions concerning mathematics. The setting in which this has been done is that of mathematical logic when it is broadly conceived as comprising proof theory, model theory, set theory, and computability theory as subfields. Thus the twentieth century has witnessed the mathematical investigation of the consequences of what are at bottom philosophical theories concerning the nature of mathematics" [148].

Henkin does not quote any of his works when he reports on issues in the philosophy of logic.[19] Most probably, because his main contributions do not belong in this area. Yet, he did write several papers dealing with diverse matters in the philosophy of logic: namely, his "Identity as a logical primitive" [121], where he explores the possibility of defining other logical expressions in terms of identity. Moreover, in his "The foundations of Mathematics I" [95], he clearly introduces basic notions in the philosophy of logic such as deduction, proof, consistency, and so forth.

Henkin also wrote a couple of works dealing with the relations between logic and mathematics "Are logic and mathematics identical?" [105], "Mathematics and logic" [108], whereas his commitment to improve the teaching of mathematics was so strong that he not only reviewed a paper on the issue but also made his own contribution to the subject.[20]

Henkin himself had a nominalist position in what is still one of the main issues in the philosophy of mathematics: the ontology and epistemology of mathematical entities. In fact, he at least wrote the following three works on the topic: "Some notes on nominalism" [72], "Nominalistic analysis of mathematical language" [106], and "The nominalistic interpretation of mathematical language" [77].

[19] Well, in fact, he quotes his other review of another work by Menger, when he discusses Menger's book *Calculus. A Modern Approach*.

[20] "On mathematical induction" [100]; with W.N. Smith, V.J. Varineau, and M.J. Walsh *Retracing Elementary Mathematics* [138]; "New directions in secondary school mathematics" [109]; "The axiomatic method in mathematics courses at the secondary level" [113]; "Linguistic aspects of mathematical education" [119]; "The logic of equality" [122]; with Nitsa Hadar "Children's conditional reasoning, Part II: Towards a Reliable Test of Conditional Reasoning Ability" [129]; with Robert B. Davis "Aspects of mathematics learning that should be the subject of testing" [127]; with Robert B. Davis "Inadequately tested aspects of mathematics learning" [128]; with Shmuel Avital "On equations that hold identically in the system of real numbers" [126]; "The roles of action and of thought in mathematics education—One mathematician's passage" [123].

Finally, Henkin published a series of reviews having to do with various problems in these areas that we have systematized as follows.

4.1 Basic logical notions

20 (1954) Review [76] of: Karl Menger, "The ideas of variable and function" (1953) [193].
21 (1954) Review [75] of: Karl Menger, *Calculus. A Modern Approach* (1953) [192].

4.2 Semantic notions and semantic issues in logic

10 (1951) Review [63] of: Heinrich Scholz, "Zur Erhellung des Verstehens" (1942) [218].
13 (1951) Review [61] of: Hugues Leblanc, "On definitions" (1950) [160].

4.3 Alternative logics

26 (1955) Review [86] of: Paul Bernays, Evert Willem Beth, Luitzen Egbertus Jan Brouwer, Jean-Louis Destouches, Robert Feys, "Discussion générale" (1953) [6].[21]
35 (1958) Review [96] of: Arend Heyting, "Logique et intuitionnisme" (1954) [144].
40 (1960) Review [102] of: Jerome Rothstein, *Communication, Organization and Science* with a foreword by C.A. Muses (1958) [216].
41 (1960) Review [101] of: Hilary Putnam, "Three-valued logic" (1957) [203]; Paul Feyerabend, "Reichenbach's interpretation of quantum-mechanics" (1958) [33]; Isaac Levi, "Putnam's three truth values" (1959) [163].

4.4 The Foundations of mathematics

2 (1948) Review [51] of: Jean Cavailles, *Transfini et continu* (1947) [17].
11 (1951) Review [64] of: James Kern Feibleman, "Class-membership and the ontological problem" (1950) [32].
12 (1951) Review [62] of: Hugues Leblanc, "The semiotic function of predicates" (1949) [159].
16 (1952) Review [67] of: Kurt Gödel, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory* (1951) [46].
32 (1956) Review [88] of: Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, Roman Sikorski, *The Present State of Investigations on the Foundations of Mathematics* (1955) [200].
38 (1959) Review [98] of: Andrzej Mostowski, "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques"; Daniel Lacombe, Andrzej Mostowski, "Interventions" (1958) [199].

4.5 Mathematical education and the foundations of mathematics

36 (1958) Review [97] of: Evert Willem Beth, "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" (1955) [9].

[21] This is a really short review. Its text goes: "This is a brief discussion of the following questions. Should semantics be considered as a part of, or as complementary to, symbolic logic? Does the formalization of theoretical physics require the introduction of new logical systems?"

4.1 Basic Logical Notions

The two works by Menger ([193] "The ideas of variable and function" and [192] *Calculus. A Modern Approach*) Henkin reviews ((20) [76] and (21) [75]) try to get a better understanding of the use of variables given both in logic and physics. In the second work, Menger also addresses the problem of functional notation.

Henkin says [76] that in his paper "The ideas of variable and function" [193], Menger distinguishes two different uses of the term "variable." The logical use of the term in which "variable" refers to the symbol employed in building sentences about a certain set that is the *range* of the variable; the other type variables that he calls *Weierstrass* variables refer—depending on the occasion—to pressure, weight, time, etc.; this second use is typical in science. In this last case, there is a class Σ whose elements are *observable*, and an equivalence relation $\cong$ is defined over Σ. Then, a variable v is a function defined over Σ such that $v\sigma_1 = v\sigma_2$ if $\sigma_1 \cong \sigma_2$. Henkin suggests an improvement over Menger's treatment when a law that involves variables in different domains occurs.

In his book, Menger [192] presents the theory of variable quantities (see [76]) and, according to Henkin, it is the first book in which there is given an explicit analysis of what is implicit when the theory of functions is applied to physics. Henkin considers the book as written in a lively and thought-provoking style and remarks that Menger distinguishes the notation for numbers from the notation for functions. A notation for functions had been developed by Menger in two previous works, *Algebra of analysis* [190] and "Are variables necessary in calculus?" [191], which Church—in his review of Menger's works—considers to follow the ideas in Schönfinkel–Curry combinatory logic. Yet, Henkin comments that Menger does not apply the ideas of combinatory logic to the end since those would have allowed him to do away with all variables occurring in function names. Moreover, Henkin reports that the distinction between use and mention is lacking, even though the author aims to use unambiguous notation. Moreover, an inappropriate use of the identity symbol is made in the common use of $\lim_\theta I^{-2} = \infty$. The author establishes a set of principles for a correct notation, but some of them are questionable; for instance, "unnecessary symbols or names should not be created."

4.2 Semantic Notions and Semantic Issues in Logic

The two earliest reviews by Henkin published in 1951 deal with notions such as interpretation, understanding, and definition.

In the first one (10) [63], Henkin reports that Scholz' "Zur Erhellung des Verstehens" [218] tries to explain how research in mathematical foundations can help clarify the notion of "understanding" by considering a particular formal system and explaining how it can be interpreted in two different ways. The system is the propositional calculus by Frege; its connectives are $\neg$ and $\rightarrow$, and its theorems are the tautologies. Henkin explains that, following Frege [35] and [37], Scholz describes a proposition as the sense expressed by a sentence. In the first interpretation, in the domain, there are arithmetical propositions, and the notion of "satisfaction" is defined recursively à la Tarski [229]. In the second interpretation, the elements of the domain are the numbers 0 and 1, and the satisfaction definition assigns the corresponding truth-tables to $\neg$ and $\rightarrow$. Henkin believes that Scholz' presentation is correct, clear, appropriate, and illuminating. But he objects that Scholz should have pointed out that the first interpretation ranging over the domain of mathematical propositions is not the usual one and that he should have clarified what the relation between both interpretations is.

Leblanc's project [160] is, according to Henkin (13) [61], completely mistaken. He tells us that in "On definitions," Leblanc introduces the notion of absolute definition in order to explain synonymy. An *absolute definition* is, according to Leblanc, the kind of definition for a term when there is a finite system of axioms that characterizes the term, and the corresponding formula by means of which the term is defined is provable in the system in question. Henkin mentions that the author acknowledges that there are some shortcomings for the proposal, but Henkin thinks that the proposal is untenable given that "even the most complete axiom system determines its models only to isomorphism, so that no significant concept can ever be given an absolute definition!" Henkin considers that the analysis of the problem should rather be done following Church [21].

4.3 Alternative Logics

In 1958, Henkin reviews (35) [96] a contribution by Heyting [144] "Logique et intuitionnisme," in which the author offers a challenging series of notes about the role of logic from an intuitionistic viewpoint. According to Henkin, Heyting contends that logic does not provide a criterion to decide on the validity of mathematical reasoning (mathematical reasoning is acceptable only if it is "immediately clear"); Heyting believes that logical laws have been used to justify reasoning because our language is not appropriate to express mental constructions. The problem is, Heyting claims, that mathematical language lacks an adequate grammar for imperatives; thus, it proceeds as if doing mathematics were about discovering facts, whereas doing mathematics is about constructing. Heyting gives the intuitionistic meaning for some logical laws and claims that logic is a part of mathematics and that its task is not to provide foundations for it since it is highly abstract; rather, logic is "at the end of mathematics," and it is as formal as any other part of mathematics. Finally, Heyting rejects the possibility of a unique formal system in which all theorems in intuitionistic mathematics are obtainable.

Henkin assesses (40) [102] what seems to be quite strange in the book by Rothstein [216]: *Communication, Organization and Science*. Most of the content of the book is certainly nonstandard from our contemporary viewpoint. The author—as Henkin tells us—intends to show that the notion of *entropy* can be used to give a unifying perspective on many phenomena in the philosophy of science and language since measuring "may be regarded as a form of communication between a scientist and the things he studies, and the amount of information gained by a measurement may be indicated by a number expressed as a quantity of entropy." Henkin points out that discussed notions are not given in detail by the author but just suggested and that a lot of what is claimed is based on analogy. In one chapter, the rules for the calculus of classes are given, and it is discussed whether they can be applied to empirical sciences. Henkin also reports that Rothstein mentions the attempts to apply alternative logic to physical theory and suggests that they can be adequate if it is necessary to consider things that are operationally undefined. He also sets out the possibility of a universal language based in symbolic logic that allows for the use of computers, but he does not analyze whether it is really possible. In the prologue, Muses both criticizes and praises Rothstein' work. Henkin mentions that Rothstein told him that he did not hear of the prologue until the book was published. Henkin does not give any personal assessment of the work in his review.

Henkin (41) [101] reviews three works on quantum logic. Henkin reports that Putnam [203] proposes a three-valued interpretation for sentential connectives according to which any sentence involving only the standard True and False values will obtain a standard truth value. He says that Putnam suggests that truth values have to do with the verification status of the sentence; hence, the nonclassical values for logical expressions might be useful in domains in which there are sentences that cannot be verified or falsified. Putnam also draws an analogy between logic and geometry to conclude that classical logic cannot have a privileged position any more than Euclidean geometry does. He also argues that the laws of classical logic cannot be those that underlie physics since, when two-valued logic is used, "the laws of quantum mechanics are incompatible with the principle of contact action." Henkin reports Feyeraberd [33] as disagreeing about this last argument because it "would violate one of the most fundamental principles of scientific methodology, namely, the principle to take refutations seriously." He says that Feyerabend also criticizes Reichenbach's proposal in favor of three-valued logic by claiming that "while statements expressing anomaly should have truth-value indeterminate, laws of quantum-mechanics should have only values T or F," but Feyerabend lists a number of quantum laws he contends can only be indeterminate. Feyerabend also considers the Copenhagen interpretation for quantum physics to reject Reichenbach's and Putnam's arguments against it. Levi [163] argues that "three-valued logic has a dim future" on the basis of Putnam's contention, according to which to ask for an interpretation of the third value lacks any sense; Putnam asking for an interpretation for the third value is tantamount to asking for a translation of the sentences of three-valued logic into sentences of two-valued logic.

Levi counterargues: (a) that not providing a translation makes sense if statements in one language have greater expressive power than those in the other, but he claims that this does not apply in our case; (b) that if an analogy between geometry and logic is to be established, as Putnam contends, then it is relevant to say that Euclidean geometry and Lobachevskian geometry are intertranslatable. (c) Finally, Levi also considers Reichenbach's possible interpretations for quantum mechanics to conclude "that a formulation in terms of three-valued logic should be intertranslatable with the two-valued formulation which admits 'causal anomalies' " (Henkin points out that Levi thinks it would not be intertranslatable if we contemplate the two-valued interpretations that consider that some sentences are meaningless.) Henkin ends his review with some critical comments: (1) He paraphrases Putnam as claiming that three-valued logic should not be applied to ordinary discourse because "if a sentence is ever shown to be verified or falsified, the claim that it has "middle" value will be shown to be unfounded" (Henkin's literal restatement of Putnam's view). Henkin objects to it, because "it would be 'dangerous' to make any statement of empirical content whatever!" But it seems to us that Henkin is not getting Putnam right. Putnam is some kind of a realist at this point; hence, he intends truth-values for sentences to obtain independently of us. That is why—we take—he contemplates as possible a third-value for quantum physics but not for ordinary discourse; that third indeterminate value in quantum physics is obtained independently of us, whereas our failure to establish the truth value of a statement, like in the example Henkin gives, is dependent of us, it is epistemic. (2) He rejects Feyerabend's use of the contention that a theory from which a false prediction follows must be modified. Henkin assumes a wholist structure for justification to sustain that a theory can be changed while leaving logical axioms untouched. (3) He rejects Levi's arguments for the dimness of three-valued logic because he takes them to depend on the assumption that the translation of the theory has to be into the two-valued theory that admits "causal anomalies," and he considers this assumption has not been argued for.

4.4 The Foundations of Mathematics

Henkin reviewed papers dealing with the hottest issues at the time; namely, problems in the philosophy of set theory (such as the consistency of the axiom of choice and the generalized continuum hypothesis) and in the foundations of mathematics. Henkin also reviews two papers in which the ontological status of the entities to which expressions in a logical language refer is discussed.

Henkin published his review (2) [51] of Cavailles' posthumous book [17] *Transfini et Continu* in 1948, six years after the publication of Henkin's first review in 1942. Henkin did not publish anything during these six years. This five-years void, 1942–1947, is due to the fact that he was first working on his Ph.D. (he presented it in 1947 under Church's supervision) and then, during World War II, on the Manhattan Project (Monk [194]).[22] Cavaillès book was written in 1940 or 1941, but, so Henkin goes, its publication at the time was forbidden by occupation authorities; hence, it was not published until 1947, well after Cavaillès was put to death in 1944. Henkin reports that the author presents the following topics dynamically and quite accurately: a discussion of the role of the axiom of choice and the continuum hypothesis, Gödel's proof of their relative consistency (of the preliminary version of the proof [44], not its final version [46]), Gentzen's proof of the consistency of arithmetic—though he does not mention Gentzen's proof in [38] but that of Bernay—and the theory of recursive functions and the theory of constructible ordinals by Kleene and Church. Henkin comments on Cavaillès' discussion of the proof of the relative consistency of the axiom of choice and of the generalized continuum hypothesis because he thinks that Cavaillès is wrong when he claims that Gódel's proof is intuitionistically acceptable, arguing that it does not use impredicative definitions. The rejection of impredicative definitions would be the defining feature of intuitionism, according to

[22]The Manhattan project was a US government project that produced the first atomic bombs. According to the obituary published on the web page of the University of Berkeley (http://www.berkeley.edu/news/media/releases/2006/11/09_henkin.shtml), "During World War II, he worked in industry for the Manhattan project, first as a mathematician for the Signal Corps Radar Laboratory in Belmar, New Jersey; then in New York City on the design of an isotope diffusion plant; and finally as head of the separation performance group at Union Carbide and Carbon Corp. in Oak Ridge, Tenn." In fact, Henkin himself explains the circumstances in [124, pp. 133–134, note 11]:

> "During the period May 1942–March 1946 I worked as a mathematician, first on radar problems and then, beginning January 1943, on the design of a plant to separate uranium isotopes. Most of my work involved numerical analysis to obtain solutions of certain partial difference-differential equations. During this period I neither read, nor thought about, logic."

Cavaillès. Henkin also notes that Cavaillès and Gödel [45] differ in their interpretation of the role Gödel's hypothesis (all sets are constructible) plays in general set theory. Henkin also claims that Cavaillès' "doubts as to whether the Cantor–Dedekind program to eliminate the geometric from mathematical reasoning by means of the notion of set" succeeds seems to confuse the use of geometrical intuition with its use as a criterion of correctness.

In his "Class-membership and the ontological problem" [32], Feibleman criticizes Quine's [204] and Quine–Goodman's [40] nominalist position in relation to logic and, as Henkin (11) [64] thinks, introduces a series of unsupported views about the development and future of symbolic logic. Henkin claims that the author's attack on Quine relies on a lack of understanding of Quine's ideas, in particular, given that the author believes that extensional logic is nominalistic and that "the consistent nominalist must eschew all words except names of particular concrete individuals."

Leblanc's paper "The semiotic function of predicates" [159], as Henkin narrates (12) [62], describes three ways of seeing the role of predicates in a symbolic language. According to the Platonic interpretation, predicates denote abstract classes; the nominalist contends that they are not names but that they "combine with names of concrete objects to form sentences that express assertions about those objects." And finally, according to the Aristotelian interpretation of their role, predicates denote "components" of particular entities (Henkin points out that the author does not either justify the attribution of this view to Aristotle or explain the view clearly). Henkin stresses that the author agrees with Quine that any of the three above-mentioned interpretations works if no quantification over predicates is involved. Otherwise, a nominalist interpretation is not possible.

Henkin reports (16) [67] on Gödel's second edition of the 1951 paper *The Consistency of the Axiom of Choice and the Generalized Continuum-Hypothesis with the Axioms of Set Theory* [46]. The first edition was reviewed by Paul Bernays [5]. This review is radically different from the one above on the same subject (Cavaillès') in that it is a technical review and it contains no philosophical discussion. Because Henkin's review is a review of the second edition of the work, he concentrates on the added material, namely, the 10 notes added by Gödel. In particular, Henkin emphasizes three of them. The first goes that $V = L$ implies the existence of a well-ordering of the reals that has as its graph in the Euclidean plane a projective set of points (the well-ordering is formalizable by means of a formula whose quantifiers range only over the real numbers). The eighth note states that the notion of "normality" is extensional. And finally, note 10 claims that the proof can be extended to systems with strong infinity axioms.

In 1956, Henkin publishes a review (32) [88] of the work by Mostowski et al. [200] *The Present State of Investigations on the Foundations of Mathematics*. In this work, a unified view of the foundational developments obtained by the Polish school, in particular, by Mostowski and his collaborators, is given. According to the authors, Henkin states, there are two main problems at the root of these: (A) to come to a better understanding of the nature of mathematical notions in the sense of clarifying whether mathematical notions are the result of human construction or forced upon us and also to explain how it is possible for us to get to know them; (B) to elucidate the nature of mathematical proofs and to provide criteria that allow us to distinguish between correct and "false" proofs.[23]

[23] Mostowski himself uses the word "false" applied to proofs, but "incorrect" seems to be the right word to use; it is sentences, statements, or propositions that are true or false, and hence the conclusion of an intended proof can be false but not the proof itself.

The authors admit, though Henkin does not report on that, that these are not purely mathematical problems, but also philosophical. They seem to contend that two lines of work in foundations have contributed to answer the first problem. The development of the axiomatic method, A1, and the study of the a priori operations needed in order to account for mathematics, A2, have contributed to a better understanding of the first, while the axiomatization of logic and the completeness proofs for nonclassical logics, B1, and decision problems, B2—especially essentially undecidable systems (a notion due to Tarski) and Kleene's hierarchy—seem to have helped clarify the second. In relation to the axiomatic method A1, they explain that systems are classified into elementary and nonelementary. Concerning elementary systems, they mention their applications to abstract algebra, the characterization of special types of arithmetic classes and multivalued systems. Concerning nonelementary ones, they remark that their interpretation is ambiguous because it depends on the underlying set theory; they also mention the fact that their models are nonabsolute results in incompleteness. From these difficulties they conclude that the axiomatic method does not provide an adequate foundation for mathematics. In relation to problem A2, they point out that Gödel's constructible sets are important. Finally, they mention what then were two new lines of work: the theory of recursive functions and the increasing association with algebra. The paper contains a broad bibliography (108 items) and their philosophical conception of mathematics. With regards to the latter, Henkin quotes the authors: "negative results obtained by the mathematical method confirm the assertion of materialistic philosophy that mathematics is in the last resort a natural science, that its notions and methods are rooted in experience, and that attempts at establishing the foundations of mathematics without taking into account its originating in natural sciences are bound to fail."

Mostowski [199] "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques" analyzes several metamathematical results obtained by using nonfinitistic techniques. Note that, at the time, the use of finitistic methods was prevalent due to Hilbert's authority and position on the issue. Some of the analyzed outcomes are: the famous results by Löwenheim and Skolem, the proof of the undecidability of a sentence Δ that says that there exists a certain class of models for axiomatic set theory, and the construction of $2^{\aleph_0}$ essentially different models for an arbitrary system that includes Peano arithmetic. The author also considers his work on automorphisms and on generalized quantifiers. Henkin's verdict on the paper (38) [98] is that even if many of the results presented in it are not new, the paper is still valuable because the fact that all these results are presented together allows us to see the relevance of nonfinitistic methods.

4.5 Mathematical Education and the Foundations of Mathematics

Henkin's review (36) [97] of Beth's "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" [9] is the second review he publishes in 1958. Henkin tells us that Beth's chapter tries to establish a connection between mathematical logic and research in the foundations of mathematics with problems in the teaching of mathematics. Beth points out that almost all of the mathematics taught in secondary school is formalizable in first-order logic and that psychology has failed to contribute in a relevant way to the teaching of mathematics; Henkin states that Beth nevertheless acknowledges that psychology might contribute to a better understanding of how students learn facts. Beth's disputable claim (though Henkin remains neutral and just paraphrases Beth's views without further comment) has to do with what he takes to be the main purpose of mathematical education: namely to make the student familiar with the deductive method. And to that purpose, as Beth claims, the use of (meta)logical results should prove effective.

5 Manuals

Henkin reviews two logic manuals, one by his advisor, Church, and another by Beth.[24]

Henkin's review (34) [93] of Church's *Introduction to Logic* [24] was published in 1957. Henkin stresses the unusual features of the book: its sixty-eight pages introduction divided into 10 sections like the five other chapters in the book and its 590 footnotes, which are to be added to the historical notes in each chapter. However, the distinction between footnotes and historical notes is not sharp since many footnotes include data with a historical interest. Among the strengths of the book, Henkin mentions that it provides plenty of exercises with various levels of difficulty, which, contrary to what is usually the case in symbolic logic manuals, can be of interest for the student of mathematics. He also emphasizes that the introduction provides a conceptual frame for a general theory of linguistic systems with an analysis of Frege's distinction between sense and reference. And that, in spite of its nonmathematical character, the introduction is of interest for the mathematician since it illuminates many basic notions that are applicable to mathematical language. For instance, when Church defines "logistic method," he explains how to use English as a metalanguage, emphasizing the convenience of constraining the use of English to the level that is "just sufficient to enable us to give general directions for the manipulation of concrete physical objects," in particular, "those additional portions of English are excluded which would be used in order to treat of infinite classes," advice that, according to Henkin, the author himself does not follow and one that Henkin doubts can be followed. The first chapter deals with a particular propositional system and includes the deduction theorem, the decision problem, duality, consistency, completeness, and independence. In the second chapter, the author introduces up to 40 different propositional systems, something that Henkin does not welcome because he thinks that it can have the wrong effect on beginners. Chapters 3 and 4 present many systems for first-order functional calculus, substitution rules, prenex and Skolem normal forms are studied in detail, and so are Gödel's completeness theorem, Löwenheim–Skölem's result and decision problems (solution for special classes and reductions for the general problem). Henkin objects that Church's presentation of first-order calculi

[24]Henkin quotes none of his works in these reviews of the two manuals mentioned. In his review of Church's manual, he quotes no other work, whereas in his review of Beth's, he quotes Beth's work "The Foundations of Mathematics. A Study of the Philosophy of Science" [10].

does not include systems that have operational symbols in addition to functional ones (relational, as we would say); he believes that this is important because functional symbols are really adequate for the formalization of many mathematical theories. The last chapter deals with predicative and ramified second-order calculi establishing Henkin's completeness theorem for the former; it also includes the study of the infinity axioms and well-ordering. Other topics are mentioned, and Church informs that they are to be included in a second volume; the topics Church intended to present in a second volume are: higher-order functional calculi, second-order arithmetic, Gödel's incompleteness theorems, recursive arithmetic, simple type theory, axiomatic set theory, and mathematical intuitionism. According to Henkin, "[t]he appearance of this volume promises to complete a work of great usefulness both for students and scholars, and it is so hoped that a way can be found to shorten its publication in time." As is well known, it never was.

Beth's book [11] *Formal Methods. An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic* is a book in which Beth, as Henkin says (43) [112], tries to explain the principles, foundations, and methods of contemporary logic "with contemporary theoretic insight. . . ." Henkin believes that it includes unusual topics for an introductory manual and that Beth's style might discourage the beginner. Henkin claims that it should be useful as a manual if the teacher is ready to provide additional material and that professional logicians will find it interesting "for its wide-ranging and provocative comments as well as for the new ways of presenting familiar material." Beth presents from three different perspectives, deductive, semantic (he includes his semantic tableaux [10]), and axiomatic, a propositional logic system with a single connective, the conditional; he then extends it to complete propositional logic, quantificational logic, and a system with functional symbols. He presents consistency and completeness proofs and, according to Henkin, uses the name *strong completeness theorem* for results that do not deserve the adjective "strong" since they apply only to finite sets of formulas. Other topics he deals with are: the formalization of arithmetic (based on the notions zero, successor, addition, multiplication, and exponentiation), *Church's thesis*, the *theory of definition*, a description of Padoa's method, incompleteness, and to close the chapter, "On machines which prove theorems." The appendix is a potpourri of subjects.

6 Conclusion

Leon Henkin's scientific production starts in 1942 with his review of M.H.A. Newman and A.M. Turing, "A formal theorem in Church's theory of types" [201] and ends in 1995 with the paper "The discovery of my completeness proofs" [124]. This wide period includes fifteen years during which Henkin was "unproductive."[25] His scientific results include the

[25]The period from 1943 to 1947 in which he worked in his Ph.D. and in the Manhattan project (see note 22 above), years 1969, 1982, 1984, 1987, 1988, and the period from 1990 to 1994.

edition of four congress proceedings,[26] seven books,[27] 54 papers,[28] 46 reviews,[29] and 17 minor works.

Table 1 contains a synopsis of his production, and Table 2 summarizes the results in Table 1. It is easy to see that 1955 is his most productive year and that more than half his production was produced between 1950 and 1960.

Table 3 shows the time elapsed between the publication of the reviewed work and the review. It is worth noting that Henkin published more than half of his reviews only one year after the corresponding work had seen the light. This clearly shows Henkin's commitment to Church's endeavor. It is also salient that most of the works reviewed by Henkin have to do with logical systems and algebraic logic—no doubt, the subjects to which he contributed most (see Table 4). His expertise in these matters is also shown in the fact that the number of papers and books he quotes in his reviews of publications on these subjects is larger than the number of works he quotes in his reviews of publications on other topics.

There is a close correspondence between Henkin's areas of expertise and the topics of the works reviewed by Henkin. Hence, according to Mathematics Subject Classification 2010,[30] the area of mathematical logic and foundations divides into: philosophical aspects of logic and foundations, general logic, model theory, computability and recursion theory, set theory, proof theory and constructive mathematics, algebraic logic, and nonstandard models. Practically, all of Henkin's reviews fall into three of these areas: philosophical aspects of logic and foundations, general logic, and algebraic logic. He does not report on any works on computability and recursion theory, most plausibly because Church, Kleene, McKinsey, Vaughan, and Ribeiro, among others, were in charge. He appraises only two works in set theory *Transfini et Continu* by Cavailles and the second edition

[26] *The Axiomatic Method* [139] with P. Suppes and A. Tarski; *The Theory of Models* [125] with J.W. Addison and A. Tarski; *Logic, Methodology and Philosophy of Science IV* [224] with P. Suppes, A. Joja, and Gr.C. Moisil; and *Proceedings of the Tarski Symposium* [120].

[27] *La structure algébrique des theories mathématiques* [87]; *Cylindric Algebras. Lectures presented at the 1961 Seminar of the Canadian Mathematical Congress* [103]; *Retracing Elementary Mathematics* [138] with W.N. Smith, V.J. Varineau, and M.J. Walsh; *Logical Systems Containing Only a Finite Number of Symbols* [114]; *Cylindric Algebras, Part I* [132] with J.D. Monk and A. Tarski; *Cylindric Algebras, Part II* [134] with J.D. Monk and A. Tarski; and *Mathematics-Report of the Project 2061 Phase I Mathematics Panel* [14] with D. Blackwell.

[28] 12 of those papers were written in cooperation with someone else: "On the definition of 'formal deduction' " [195] with R. Montague; "Cylindrical Algebras" [140] and "Cylindric Algebras" [141] with A. Tarski; "Cylindric algebras and related structures" [131] with J.D. Monk; "Relativization of cylindric algebras" [137] with D. Resek; "A Euclidean construction?" [130] with W. Leonard; "Children's conditional reasoning, Part II: Towards a reliable test of conditional reasoning ability" [129] with Nitsa Hadar; "Aspects of mathematics learning that should be the subject of testing" [127] and "Inadequately tested aspects of mathematics learning" [128] with Robert B. Davis; "Cylindric set algebras and related structures" [133] and "Representable cylindric algebras" [135] with J.D. Monk and A. Tarski; and "On equations that hold identically in the system of real numbers" [126] with Shmuel Avital.

[29] See Appendix.

[30] This classification coincides with that provided in 2000. Barwise's [4, p. vii] "Mathematical logic is traditionally divided into four parts: model theory, set theory, recursion theory, and proof theory" is not that complete.

Table 1 Under the column "Minor," we write the number of reviews Henkin wrote in that year, and, in the same column, after the "+" symbol, we write the number of minor works published in that year that are not reviews. Under "Major," we include the number of books, papers, and so forth. After "+" and under "major," we indicate the number of books edited

Year	Minor	Major	Total
1942	1		1
1948	2 + 1		3
1949	5	2	7
1950	1	3	4
1951	4		4
1952	3 + 2		5
1953	2	4	6
1954	3 + 3	2	8
1955	8 + 1	5	14
1956	3 + 2	3	8
1957	2	3	5
1958	2	1	3
1959	3	+1	4
1960	2 + 1	1	4
1961		3	3
1962		4	4
1963	1 + 1	4	6
1964	+1		1
1965	1	+1	2
1966	+1	1	2
1967	1	2	3
1968	+1	1	2
1970		1	1
1971	2	2	4
1972		1	1
1973		1 + 1	2
1974		1 + 1	2
1975		2	2
1977		2	2
1978		2	2
1979		2	2
1980	+1		1
1981		2	2
1983	+2		2
1985		1	1
1986		1	1
1989		1	1
1995		2	2
1996		1	1
Total	46 + 17	61 + 4	128

Table 2 Synopsis of the results in Table 1

Period	Reviews	Minors	Majors	Editor	Total	%
1955	8	1	5	0	**14**	10.94
1954–1956	14	6	10	0	**30**	23.44
1953–1957	18	6	17	0	**41**	32.03
1952–1958	23	8	18	1	**49**	38.28
1951–1959	30	9	18	0	**57**	44.53
1950–1960	32	9	22	0	**65**	50.78
1949–1961	37	9	27	0	**75**	58.59
1948–1962	39	10	31	0	**82**	64.06
1948–1963	40	11	35	0	**88**	68.75

of *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory* by Gödel, and only one in proof theory and constructive mathematics, Kreisel's "On a problem of Henkin's".

Table 3 The columns called "Δ" indicate the years passed between the year of publication of the reviewed work and the year of publication of the review, and the columns called "No." denote the number of works revised. The total number of works examined by Henkin is 49, even though he published only 46 reviews. This is because some of the published reviews include revisions of more than one work

Δ	No.	Δ	No.
0	5	4	2
1	23	9	1
2	8	13	3
3	6	18	1

Table 4 In the third column, we can see the number of reviews made by Henkin, whereas the fourth gives the average number of quotes in the different reviews

Topic	Subtopic	No.	Quotes
Logic systems		**12**	3.16
	Type theory	3	1.00
	Metalogic	9	3.89
Algebraic logic		**17**	4.12
	Algebraic treatment of logic	12	4.08
	Applications of logic to algebra	5	4.20
Philosophy of logic & math.		**15**	0.93
	Basic logical notions	2	1.50
	Semantic notions	2	2.00
	Alternative logics	4	0.00
	The foundations of mathematics	6	1.47
	Mathematical education	1	0
Manuals		**2**	0.50

Finally, he gives an account of only one paper on mathematical education, even though he published up to 10 major works on the topic,[31] one of his dearest.

7 Reviews by Leon Henkin

The list that follows comprises the papers and books reviewed by Henkin; they have been put in order according to year of publication; we have also included the subjects

[31] "The roles of action and of thought in mathematics education—One mathematician's passage" [77]; with W.N. Smith, V.J. Varineau and M.J. Walsh *Retracing Elementary Mathematics* [138]; "New directions in secondary school mathematics" [109]; "The axiomatic method in mathematics courses at the secondary level" [113]; "Linguistic aspects of mathematical education" [119]; with Nitsa Hadar "Children's conditional reasoning, Part II: Towards a Reliable Test of Conditional Reasoning Ability" [129]; with Robert B. Davis "Aspects of mathematics learning that should be the subject of testing" [127]; with Robert B. Davis "Inadequately tested aspects of mathematics learning" [128]; with D. Blackwell *Mathematics-Report of the Project 2061 Phase I Mathematics Panel* [14] and "The roles of action and of thought in mathematics education—One mathematician's passage" [123].

under which they were listed in the *Journal of Symbolic Logic* (JSL from now on)—volumes 26 (1961) and 45 (1985)—whereas our own classification goes in bold type. The classification in the JSL was elaborated by Church.[32]

Complete references of the works reviewed by Henkin are provided in the bibliography. Our listing of the reviews includes the cross-referencing system that was used in the JSL: "The reviews used a system of cross-referencing by volume number (in roman numerals) and page number (in arabic numerals). For example, Turing's classic paper on computability was II 42—it was reviewed in volume 2, beginning on p. 42" (Enderton [30, p. 175]). Moreover, we have added the publication year and, in square brackets, the number of the corresponding reference in this chapter. Also in square brackets, but after the title of the reviewed work, we include the number for that reviewed work in Sect. 7.

List of Reviews

1. VII 122(1) (1942) [49]: Maxwell Herman Alexander Newman and Alan Mathison Turing, "A formal theorem in Church's theory of types" (1942) [201].
2. XIII 143(1) (1948) [51]: Jean Cavailles, *Transfini et Continu* (1947) [17].
3. XIII 171(2) (1948) [50]: John Charles Chenoweth McKinsey and Alfred Tarski, "Some theorems about the sentential calculi of Lewis and Heyting" (1948) [189].
4. XIV 65(1) (1949) [57]: Stanislaw Jaśkowski, "Sur les variables propositionelles dépendantes" (1948) [150].
5. XIV 66(1) (1949) [56]: Stanislaw Jaśkowski, "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres" (1948) [149].
6. XIV 188(1) (1949) [55]: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1948) [232].
7. XIV 193(1) (1949) [54]: Andre Chauvin, "Structures logiques" (1949) [19].
8. XIV 193(2) (1949) [53]: Andre Chauvin, "Généralisation du théorème de Gödel" (1949) [18].
9. XV 230(1) (1950) [60]: László Kalmár, "Une forme du théorème de Godel sous des hypothèses minimales" (1949) [153]; László Kalmár, "Quelques formes générales du théorème de Gödel" (1949) [152].
10. XVI 53(1) (1951) [63]: Heinrich Scholz, "Zur Erhellung des Verstehens" (1942) [218].
11. XVI 213(1) (1951) [64]: James Kern Feibleman, "Class-membership and the ontological problem" (1950) [32].
12. XVI 213(2) (1951) [62]: Hugues Leblanc, "The semiotic function of predicates" (1949) [159].
13. XVI 213(3) (1951) [61]: Hugues Leblanc, "On definitions" (1950) [160].
14. XVII 205(2) (1952) [65]: Abraham Robinson, *On the Metamathematics of Algebra* (1951) [213].
15. XVII 207(1) (1952) [66]: Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry* (1951) [233].
16. XVII 207(2) (1952) [67]: Kurt Gödel, *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory* (1951) [46].

[32] See Enderton [30].

17. XVIII 72(2) (1953) [70]: Helena Rasiowa, "Algebraic treatment of the functional calculi of Heyting and Lewis" (1951, pub. 1952) [205].
18. XVIII 339(3) (1953) [69]: Burton Dreben, "On the completeness of quantification theory" (1952) [28].
19. XIX 219 (1954) [74]: Georg Kreisel, "On a problem of Henkin's" (1953) [157].
20. XIX 227(1) (1954) [76]: Karl Menger, "The ideas of variable and function" (1953) [193].
21. XIX 227(2) (1954) [75]: Karl Menger, *Calculus. A modern Approach* (1953) [192].
22. XX 78(2) (1955) [84]: Helena Rasiowa and Roman Sikorski, "Algebraic treatment of the notion of satisfiability" (1953) [208].
23. XX 80(1) (1955) [83]: Helena Rasiowa and Roman Sikorski, "On existential theorems in non-classical functional calculi" (1954) [209].
24. XX 184(2) (1955) [81]: Evert Willem Beth, "Sur le parallelisme logico-mathematique" (1953) [8].
25. XX 185 (1955) [79]: Abraham Robinson, "Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique"; Evert Willem Beth, Luitzen Egbertus Jan Brouwer, and Abraham Robinson, "Discussion" (1953) [215].
26. XX 186(1) (1955) [86]: Paul Bernays, Evert Willem Beth, Luitzen Egbertus Jan Brouwer, Jean-Louis Destouches, and Robert Feys, "Discussion générale" (1953) [6].[33]
27. XX 186(2) (1955) [80]: A. Chatelet, "Allocution d'ouverture" (1953); (3) Luitzen Egbertus Jan Brouwer, "Discours final" (1953); (4) Abraham Robinson, "On axiomatic systems which possess finite models" (1951) [212].[34]
28. XX 281(1) (1955) [85]: Ladislav Rieger, "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem" (1951) [211].
29. XX 282(1) (1955) [82]: Gunter Asser, "Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe" (1955) [2].
30. XXI 193(4) (1956) [89]: Jerzy Łoś, "The algebraic treatment of the methodology of elementary deductive systems" (1955) [167].
31. XXI 194(1) (1956) [90]: Juliusz Reichbach, "O pełności węższego rachunku funkcyjnego"; Juliusz Reichbach, "O polnoté uzkogo funkcional'nogo isčisléniá" (Russian translation); Juliusz Reichbach, "Completeness of the functional calculus of first-order" (English summary) (1955) [210].
32. XXI 372(2) (1956) [88]: Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, and Roman Sikorski, *Der gegenwartige Stand der Grundlagenforschung in der Mathematik* (1955); Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, and Roman Sikorski, *The Present State of Investigations on the Foundations of Mathematics* (1955); Andrzej Mostowski, Andrzej Grzegorczyk, Stanisław Jaśkowski, Jerzy Łos, Stanisław Mazur, Helena Rasiowa, and Roman Sikorski, *Sovréménnoé sostoánié isslédovanij po osnovaniám matématiki* (1954) [200].

[33] This is a really short review. Its text goes: "This is a brief discussion of the following questions. Should semantics be considered as a part of, or as complementary to, symbolic logic? Does the formalization of theoretical physics require the introduction of new logical systems?"

[34] In fact, Henkin lists the three works, but he only reviews Robinson's.

33. XXII 216 (1957) [94]: Kaarlo Jaakko Hintikka, "An application of logic to algebra" (1954) [147].
34. (1957) [93]: Alonzo Church, *Introduction to Mathematical Logic, Vol. I* (1956) [24].
35. XXIII 33(2) (1958) [96]: Arend Heyting, "Logique et intuitionnisme" (1954) [144].
36. XXIII 34(1) (1958) [97]: Evert Willem Beth, "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" (1955) [9].
37. XXIV 55 (with Andrzej Mostowski) (1959) [136]: Anatolĭ Ivanovič Mal'cév, "Ob odnom obščém métodé polučéniá lokal'nyh téorém téorii grupp" (On a general method for obtaining local theorems in group theory) (1941) [181]; Anatolĭ Ivanovič Mal'cév, "O prédstavléniáh modéléj" (On representations of models) (1956) [182].
38. XXIV 234(2) (1959) [98]: Andrzej Mostowski, "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques"; Daniel Lacombe and Andrzej Mostowski, "Interventions" (1958) [199].
39. XXIV 235 (1959) [99]: Louis Nolin, "Sur l'algebre des predicats"; Andrzej Mostowski, Jean Porte, Alfred Tarski, and Jacques Riguet, "Interventions" (1958) [202].
40. XXV 256(1) (1960) [102]: Jerome Rothstein, *Communication, Organization and Science* with a foreword by C.A. Muses (1958) [216].
41. XXV 289 (1960) [101]: Hilary Putnam, "Three-valued logic" (1957) [203]; Paul Feyerabend, "Reichenbach's interpretation of quantum-mechanics" (1958) [33]; Isaac Levi, "Putnam's three truth values" (1959) [163].
42. XXVIII 174(3) (1963) [110]: Antonio Monteiro, "Matrices de Morgan caractéristiques pour le calcul propositionnel classique" (1960) [196].
43. XXX 235 (1965) [112]: Evert Willem Beth, *Formal Methods. An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic* (1962) [11].
44. XXXII 415(1) (1967) [115]: Marshall Harvey Stone, "Free Boolean rings and algebras" (1954) [223].
45. XXXVI 337(1) (1971) [118]: Maurice L'Abbé, "Structures algébriques suggérées par la logique mathématique" (1958) [158].
46. XXXVI 337(2) (1971) [117]: Marc Krasner, "Les algèbres cylindriques" (1958) [156].

References

1. Andréka, H., Németi, I., Sain, I.: Algebraic logic. In: Gabbay, D.M., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn. vol. II, pp. 133–247. Kluwer Academic, Dordrecht (2001)
2. Asser, G.: Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe. Z. Math. Log. Grundl. Math. **1**, 3–28 (1955)
3. Barcan, R.C.: A functional calculus of first order based on strict implication. J. Symb. Log. **11**, 1–16 (1946)
4. Barwise, J. (ed.): Handbook of Mathematical Logic. North-Holland, Elsevier, Amsterdam (1977)
5. Bernays, P.: Review: K. Gödel [46, 1 edn.]. J. Symb. Log. **6**(3), 112–114 (1941)
6. Bernays, P., Beth, E.W., Brouwer, J.L., Destouches, L.E.J., Feys, R.: Discussion générale. In: Colloques Internationaux du Centre Nacional de la Recherche Scientifique, Editor, les Méthodes Formelles en Axiomatique, Paris, Décembre 1950, vol. 36, pp. 73–74 (1953)
7. Beth, B.W.: A topological proof of the theorem of Löwenheim–Skolem–Gödel. In: Proceedings Koninklijke Nederlandse Akademie van Wetenschappen, series A, vol. 54, pp. 436–444 (1951). Also Indag. Math. **13**, 436–444 (1951)

8. Beth, B.W.: Sur le parallélisme logico-mathématique. In: Colloques Internationaux du Centre Nacional de la Recherche Scientifique, Editor, les Méthodes Formelles en Axiomatique, Paris, Décembre 1950, vol. 36, pp. 27–32 (1953)
9. Beth, E.W.: Réflexions sur l'organisation et la méthode de l'ensignement mathématique. In: Piaget, J., Beth, B.W., Choquet, G., Dieudonné, J., Lichnerowicz, A., Gattegno, C. (eds.) L'Enseignement des Mathématiques, pp. 35–46 (1955). Delachaux & Niestlé
10. Beth, E.W.: The Foundations of Mathematics. A Study in the Philosophy of Science. North-Holland, Amsterdam (1959)
11. Beth, E.W.: Formal Methods. An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic. Reidel, Dordrecht (1962)
12. Białynicki-Birula, A., Rasiowa, H.: On the representation of quasi-Boolean algebras. Bull. Acad. Pol. Sci., Cl. III **5**, 259–261 (1957)
13. Birkhoff, G.: Lattice Theory. American Mathematical Society Colloquium Publications, vol. 25. Am. Math. Soc., Providence (1940)
14. Blackwell, D., Henkin, L.: Mathematics-Report of the Project 2061 Phase I Mathematics Pannel. Am. Assoc. Adv. Sci., Washington (1989)
15. Boole, G.: The Mathematical Analysis of Logic, Being an Essay Toward a Calculus of Deductive Reasoning (1847). London and Cambridge. Reprinted in George Boole's' collected logical works, ed. by P.E.B. Jourdain, vol. 1, Chicago and London (1916)
16. Carnap, R.: Modalities and quantification. J. Symb. Log. **11**, 33–64 (1946)
17. Cavaillès, J.: Transfini et continu. Actualités scientifiques et industrielles 1020. Hermann, Paris (1947). In: Cavaillès, J.: Philosophie Mathématique, pp. 253–274. Hermann, Paris (1961)
18. Chauvin, A.: Généralisation du théorème de Gödel. C. R. Hebd. Séances Acad. Sci. **228**, 1179–1180 (1949)
19. Chauvin, A.: Structures logiques. C. R. Hebd. Séances Acad. Sci. **228**, 1085–1087 (1949)
20. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**(2), 56–68 (1940)
21. Church, A.: A formulation of the logic of sense and denotation. In: Eighth Meeting of the Association for Symbolic Logic, pp. 27–28 (1945)
22. Church, A.: A formulation of the logic of sense and denotation. In: Henle, P., Kallen, H.M., Langer, S.K. (eds.) Structure, Method and Meaning, Essays in Honor of Henry M. Sheffer, pp. 3–24. The Liberal Arts Press, New York (1951)
23. Church, A.: Non-normal truth-tables for the propositional calculus. Bol. Soc. Mat. Mex. **10**(1–2), 41–52 (1953)
24. Church, A.: Introduction to Mathematical Logic. Princeton University Press, Princeton (1956)
25. Church, A.: Outline of a revised formulation of the logic of sense and denotation (Part I). Noûs **7**(1), 24–33 (1973)
26. Church, A.: Outline of a revised formulation of the logic of sense and denotation (Part II). Noûs **8**(2), 135–156 (1974)
27. Church, A.: A revised formulation of the logic of sense and denotation. Alternative (1). Noûs **27**(2), 141–157 (1993)
28. Dreben, B.: On the completeness of quantification theory. In: Proceedings of the National Academy of Sciences of the United States of America, vol. 38, pp. 1047–1052 (1952)
29. Dugundji, J.: Note on a property of matrices for Lewis and Langford's calculi of propositions. J. Symb. Log. **5**, 150–151 (1940)
30. Enderton, H.B.: Alonzo Church and the reviews. Bull. Symb. Log. **4**(2), 172–180 (1998)
31. Feferman, S.: Review: H. Rasiowa, and R. Sikorski. "A proof of the completeness theorem of Godel" [206]. J. Symb. Log. **17**(1), 72 (1952)
32. Feibleman, J.K.: Class-membership and the ontological problem. Philos. Sci. **17**, 254–259 (1950)
33. Feyerabend, P.: Reichenbach's interpretation of quantum-mechanics. Philos. Stud. (Minneapolis) **9**, 49–59 (1958)
34. Frege, G.: Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens. Verlag von Louis Nebert, Halle (1879). Reeditada en Begriffsschrift und andere Aufsätze, Georg Holms, Hidesheim (1964) por I. Angelelli. Tradución inglesa en [48, pp. 1–82]. Traducción castellana de H. Padilla, Conceptografía. Los fundamentos de la aritmética. Otros estudios filosóficos, Universidad Nacional Autónoma de México (1972)

35. Frege, G.: Über Sinn und Bedeutung. Z. Philos. Philos. Krit. **100**, 25–50 (1892). Traducción inglesa de M. Black: A translation of Frege's Ueber Sinn und Bedeutung. Introductory note. Philos. Rev. **57**, 207–208 (1948). G. Frege. Sense and reference. Ibid., pp. 209–230
36. Frege, G.: Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet, Jena (1893) and (1903). Reeditada por Georg Olms, Hildesheim (1962)
37. Frege, G.: Aritmetica e Logica. Biblioteca di Cultura Scientifica XVIII. Giulio Einaudi, Turin (1948). Translation and notes by L. Geymonat
38. Gentzen, G.: Neue Fassung des Widerspruchsfreits beweises für die reine Zalhentheorie. Forsch. Log. Grundlegung Exakten Wiss. **4**, 19–44 (1938). Traducción inglesa en M.E. Szabo: The Collected Papers of Gerhard Gentzen, North-Holland (1969)
39. Goldblatt, R.: Mathematical modal logic: a view of its evolution. In: Gabbay, D.M., Woods, J. (eds.) Handbook of the History of Logic, Logic and the Modalities in the Twentieth Century, vol. 7, pp. 1–98. Elsevier, Amsterdam (2006). Chap. 1
40. Goodman, N., Quine, W.V.: Steps toward a constructive nominalism. J. Symb. Log. **12**, 105–122 (1947)
41. Gödel, K.: Die Vollständigkeit der Axiome des logischen Funktionenkalküls. Monatshefte Math. Phys. **37**, 349–360 (1930)
42. Gödel, K.: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I. Monatshefte Math. Phys. **38**, 173–198 (1931). Translations into English: van Heijenoort, J.: From Frege to Gödel: A Source Book on Mathematical Logic, Harvard University Press (1967), pp. 596–616. Hirzel, M.: On formally undecidable propositions of Principia Mathematica and related systems I. Translation into Spanish: Mosterín, J.: Kurt Gödel: Obras completas, Alianza Editorial, Madrid (1981)
43. Gödel, K.: Eine Interpretation des intuitionistischen Aussagenkalkuls. Ergebnisse eines mathematischen. Kolloquiums **4**, 39–40 (1933)
44. Gödel, K.: The consistency of the axiom of choice and of the generalized continuum hypothesis. In: Proceedings of the National Academy of Sciences, vol. 24, pp. 556–557 (1938)
45. Gödel, K.: What is Cantor's continuum problem? Am. Math. Mon. **54**, 515–525 (1947)
46. Gödel, K.: The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory. Annals of Mathematics Studies, vol. 3. Princeton University Press, Princeton (1951). Lectures delivered at the Institute for Advanced Study 1938–1939; notes by George W. Brown. Lithoprinted, Princeton University Press, Princeton (1940)
47. Halmos, P.R.: Algebraic Logic. Chelsea, New York (1962)
48. Van Heijenoort, J. (ed.): From Frege to Gödel. A Source Book in Mathematical Logic (1967). Cambridge
49. Henkin, L.: Review: M.H.A. Newman and A.M. Turing, "A formal theorem in Church's theory of types" [201]. J. Symb. Log. **7**(3), 122 (1942)
50. Henkin, L.: Review: J.C.C. McKinsey and A. Tarski, "Some theorems about the sentential calculi of Lewis and Heyting" [189]. J. Symb. Log. **13**(3), 171–172 (1948)
51. Henkin, L.: Review: Jean Cavaillès, Transfini et continu [17]. J. Symb. Log. **13**(3), 143–144 (1948)
52. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**, 159–166 (1949)
53. Henkin, L.: Review: A. Chauvin, "Généralisation du théorème de Gödel" [18]. J. Symb. Log. **14**(3), 193 (1949)
54. Henkin, L.: Review: A. Chauvin, "Structure logiques" [19]. J. Symb. Log. **14**(3), 193 (1949)
55. Henkin, L.: Review: A. Tarski, A decision method for elementary algebra and geometry [232]. J. Symb. Log. **14**(3), 188 (1949)
56. Henkin, L.: Review: S. Jaśkowski, "Sur certains groupes formès de classes d'ensembles et leur application aux définitions des nombres" [149]. J. Symb. Log. **14**(1), 66 (1949)
57. Henkin, L.: Review: S. Jaśkowski: "Sur les variables propositionelles dépendantes" [150]. J. Symb. Log. **14**(1), 65 (1949)
58. Henkin, L.: An algebraic characterization of quantifiers. Fundam. Math. **37**, 63–74 (1950)
59. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**, 81–91 (1950)
60. Henkin, L.: Review: L. Kalmar, "Une forme du théorème de Godel sous des hypothèses minimales" [153] and "Quelques formes générales du théorème de Gödel" [152]. J. Symb. Log. **15**(3), 230 (1950)

61. Henkin, L.: Review: H. Leblanc, “On definitions” [160]. J. Symb. Log. **16**(3), 213–214 (1951)
62. Henkin, L.: Review: H. Leblanc, “The semiotic function of predicates” [159]. J. Symb. Log. **3**(3), 213 (1951)
63. Henkin, L.: Review: H. Scholtz, “Zur Erhellung des Verstehens” [218]. J. Symb. Log. **16**(1), 53 (1951)
64. Henkin, L.: Review: J. K. Feibleman, “Class-membership and the ontological problem” [32]. J. Symb. Log. **16**(3), 213 (1951)
65. Henkin, L.: Review: A. Robinson, “On the metamathematics of algebra” [213]. J. Symb. Log. **17**(3), 205–207 (1952)
66. Henkin, L.: Review: A. Tarski, “A decision method for elementary algebra and geometry” [233]. J. Symb. Log. **17**(3), 207 (1952)
67. Henkin, L.: Review: K. Gödel, “The consistency of the axiom of choice and the generalized continuum hypothesis with the axioms of set theory” [46]. J. Symb. Log. **17**(3), 207–208 (1952)
68. Henkin, L.: On the primitive symbols of Quine’s ‘Mathematical Logic’. Rev. Philos. Louvain, Trois. Sér. **51**(32), 591–593 (1953)
69. Henkin, L.: Review: B. Dreben, “On the completeness of quantification theory” [28]. J. Symb. Log. **18**(4), 339 (1953)
70. Henkin, L.: Review: H. Rasowa, “Algebraic treatment of the functional calculi of Heyting and Lewis” [205]. J. Symb. Log. **18**(1), 72–73 (1953)
71. Henkin, L.: Some interconnections between modern algebra and mathematical logic. Trans. Am. Math. Soc. **74**, 410–427 (1953)
72. Henkin, L.: Some notes on nominalism. J. Symb. Log. **18**(1), 19–29 (1953)
73. Henkin, L.: A generalization of the concept of ω-consistency. J. Symb. Log. **19**(3), 183–196 (1954)
74. Henkin, L.: Review: G. Kreisel, “On a problem of Henkin’s” [157]. J. Symb. Log. **19**(3), 219–220 (1954)
75. Henkin, L.: Review: K. Menger, Calculus. A modern approach [192]. J. Symb. Log. **19**(3), 227–229 (1954)
76. Henkin, L.: Review: K. Menger, “The ideas of variable and function” [193]. J. Symb. Log. **19**(3), 227 (1954)
77. Henkin, L.: The nominalistic interpretation of mathematical language. Bull. Soc. Math. Belg. **7**, 131–136 (1955)
78. Henkin, L.: The representation theorem for cylindric algebras. In: Skolem, Th., Hasenjaeger, G., Kreisel, G., Robinson, A., Wang, H., Henkin, L., Łoś, J. (eds.) Mathematical Interpretation of Formal Systems, pp. 85–97. North-Holland, Amsterdam (1955)
79. Henkin, L.: Review: A. Robinson, “Les rapports entre le calcul déductif et l’interprétation sémantique d’un système axiomatique” [215]. J. Symb. Log. **20**(2), 185–186 (1955)
80. Henkin, L.: Review: A. Robinson, “On axiomatic systems which possess finite models” [212]. J. Symb. Log. **20**(2), 186 (1955)
81. Henkin, L.: Review: E.W. Beth, “Sur le parallelisme logico-mathematique” [8]. J. Symb. Log. **20**(2), 184–185 (1955)
82. Henkin, L.: Review: G. Asser, “Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe” [2]. J. Symb. Log. **20**(3), 282 (1955)
83. Henkin, L.: Review: H. Rasiowa and R. Sikorski, “On existential theorems in non-classical functional calculi” [209]. J. Symb. Log. **20**(1), 80 (1955)
84. Henkin, L.: Review: H. Rasoiowa and R. Sikorski, “Algebraic treatment of the notion of satisfiability” [208]. J. Symb. Log. **20**(1), 78–80 (1955)
85. Henkin, L.: Review: L. Rieger, “On countable generalised σ-algebras, with a new proof of Godel’s completeness theorem” [211]. J. Symb. Log. **20**(3), 281–282 (1955)
86. Henkin, L.: Review: P. Bernays et al., “Discussion générale” [6]. J. Symb. Log. **20**(2), 186 (1955)
87. Henkin, L.: La Structure Algébrique des Théories Mathématiques. Collection de Logique Mathematique, Series A, vol. XI. Gauthier-Villars, Paris (1956)
88. Henkin, L.: Review: A. Mostowski et al., “The present state of investigations on the foundations of mathematics” [200]. J. Symb. Log. **21**(4), 372–373 (1956)
89. Henkin, L.: Review: J. Łoś, “The algebraic treatment of the methodology of elementary deductive systems” [167]. J. Symb. Log. **21**(2), 193–194 (1956)

90. Henkin, L.: Review: J. Reichbach, "Completeness of the functional calculus of first-order" [210]. J. Symb. Log. **21**(2), 194 (1956)
91. Henkin, L.: Cylindrical algebras of dimension 2. Preliminary report. Bull. Am. Math. Soc. **63**(1), 26 (1957)
92. Henkin, L.: A generalization of the concept of ω-completeness. J. Symb. Log. **22**(1), 1–14 (1957)
93. Henkin, L.: Review: A. Church, "Introduction to mathematical logic, Vol. I" [24]. Bull. Am. Math. Soc. **63**(5), 320–323 (1957)
94. Henkin, L.: Review: K.J. Hintikka, "An application of logic to algebra" [147]. J. Symb. Log. **22**(2), 216–217 (1957)
95. Henkin, L.: The foundations of mathematics. In: Klibansky, R. (ed.) Philosophy in the Mid-Century, vol. 1, pp. 116–129 (1958). Firenze La Nuova Italia Editrice
96. Henkin, L.: Review: A. Heyting, "Logique et intuitionnisme" [144]. J. Symb. Log. **23**(1), 33 (1958)
97. Henkin, L.: Review: E.W. Beth, "Réflexions sur l'organisation et la méthode de l'enseignement mathématique" [9]. J. Symb. Log. **23**(1), 34 (1958)
98. Henkin, L.: Review: A. Mostowski, "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques" [199]. J. Symb. Log. **24**(3), 234–235 (1959)
99. Henkin, L.: Review: L. Nolin, "Sur l'algebre des predicats" [202]. J. Symb. Log. **24**(3), 235 (1959)
100. Henkin, L.: On mathematical induction. Am. Math. Mon. **67**(4), 323–338 (1960)
101. Henkin, L.: Review: H. Putnam, "Three-valued logic" [203], P. Feyerabend, "Reichenbach's interpretation of quantum-mechanics" [33], and I. Levi "Putnam's three truth values" [163]. J. Symb. Log. **25**(3), 289–291 (1960)
102. Henkin, L.: Review: J. Rohstein, "Communication, organization and science" [216]. J. Symb. Log. **25**(3), 256 (1960)
103. Henkin, L.: Cylindric algebras. Lectures presented at the 1961 seminar of the Canadian Mathematical Congress. Université de Montréal (1961). Notes made with the collaboration and (occasional elaboration) of Marc Venne
104. Henkin, L.: Some remarks on infinitely long formulas. In: Infinitistic Methods; Proceedings of the Symposium on Foundations of Mathematics (Warzawa 1959), pp. 167–183. Pergamon, Elmsford (1961)
105. Henkin, L.: Are logic and mathematics identical? Science **138**, 788–794 (1962)
106. Henkin, L.: Nominalistic analysis of mathematical language. In: Nagel, E. (ed.) Logic, Methodology, and Philosophy of Science, pp. 187–193. Stanford University Press, Stanford (1962)
107. Henkin, L.: An extension of the Craig–Lyndon interpolation theorem. J. Symb. Log. **28**(3), 201–216 (1963)
108. Henkin, L.: Mathematics and logic. In: Rosenthall, E.M. (ed.) Proceedings of the Fifth Canadian Mathematical Congress, University of Montreal 1961, pp. 184–200. University of Toronto Press, Toronto (1963)
109. Henkin, L.: New directions in secondary school mathematics. In: Ritchie, R.W. (ed.) New Directions in Mathematics, pp. 1–6. Prentice Hall, New York (1963)
110. Henkin, L.: Review: A. Monteiro, "Matrices de Morgan caractéristiques pour le calcul propositionnel classique" [196]. J. Symb. Log. **28**(2), 174–175 (1963)
111. Henkin, L.: A theory of propositional types. Fundam. Math. **52**(1), 323–344 (1963). Errata to the paper in Fundam. Math. **53**(1), 119 (1964)
112. Henkin, L.: Review: E.W. Beth, Formal methods. An introduction to symbolic logic and to the study of effective operations in arithmetic and logic [11]. J. Symb. Log. **30**(2), 235–236 (1965)
113. Henkin, L.: The axiomatic method in mathematics courses at the secondary level. In: The Role of Axiomatics and Problem Solving in Mathematics. pp. 49–56 (1966). Conference Board of the Mathematical Sciences
114. Henkin, L.: Logical Systems Containing Only a Finite Number of Symbols (1967). Université de Montreal
115. Henkin, L.: Review: M.H. Stone, "Free Boolean rings and algebras" [223]. J. Symb. Log. **32**(3), 415 (1967)
116. Henkin, L.: Relativization with respect to formulas and its use in proofs independence. Compos. Math. **20**, 88–106 (1968)
117. Henkin, L.: Review: M. Krasner, "Les algèbres cylindriques" [156]. J. Symb. Log. **36**(2), 337 (1971)

118. Henkin, L.: Review: M.L' Abbé, "Structures algébriques suggérées par la logique mathématique" [158]. J. Symb. Log. **36**(2), 337 (1971)
119. Henkin, L.: Linguistic aspects of mathematical education. In: Learning and the Nature of Mathematics, pp. 211–218 (1972). Science Research Associates
120. Henkin, L. (ed.): Proceedings of the Tarski Symposium. Am. Math. Soc., Providence (1974)
121. Henkin, L.: Identity as a logical primitive. Philosophia **5**(1–2), 31–45 (1975)
122. Henkin, L.: The logic of equality. Am. Math. Mon. **84**, 597–612 (1977)
123. Henkin, L.: The roles of action and of thought in mathematics education—one mathematician's passage. In: Fisher, N.D., Keynes, H.B., Wagreich, P.D. (eds.) Changing the Culture: Mathematics Education in the Research Community. CBMS Issues in Mathematics Education, vol. 5 (1995)
124. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**(2), 127–158 (1996)
125. Henkin, L., Addison, J.W., Tarski, A. (eds.): The Theory of Models. North-Holland, Amsterdam (1965)
126. Henkin, L., Avital, S.: On equations that hold identically in the system of real numbers. Int. J. Math. Educ. Sci. Technol. **12**, 223–234 (1981)
127. Henkin, L., Davis, R.B.: Aspects of mathematics learning that should be the subject of testing. In: Testing, Teaching, and Learning. Report of a Conference on Research in Testing, pp. 60–82 (1979). National Institute od Education
128. Henkin, L., Davis, R.B.: Inadequately tested aspects of mathematics learning. In: Testing, Teaching, and Learning. Report of a Conference on Research in Testing, pp. 83–97 (1979). National Institute od Education
129. Henkin, L., Hadar, N.: Children's conditional reasoning, Part II: Towards a reliable test of conditional reasoning ability. Educ. Stud. Math. **9**, 97–114 (1978)
130. Henkin, L., Leonard, W.: A Euclidean construction? Math. Mag. **51**(5), 294–298 (1978)
131. Henkin, L., Monk, J.D.: Cylindric algebras and related structures. In: Proceedings of the Tarski Symposium, pp. 105–121. Am. Math. Soc., Providence (1974)
132. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras, Part I. North-Holland, Amsterdam (1971)
133. Henkin, L., Monk, J.D., Tarski, A.: Cylindric set algebras and related structures. In: Cylindric Set Algebras, pp. 1–129. Springer, Berlin (1981)
134. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras, Part II. North-Holland, Berlin (1985)
135. Henkin, L., Monk, J.D., Tarski, A.: Representable cylindric algebras. Ann. Pure Appl. Log. **31**, 23–60 (1986)
136. Henkin, L., Mostowski, A.: Review: A.I. Mal'cév, "On a general method for obtaining local theorems in group theory" [181] and "On representations of models" [182]. J. Symb. Log. **24**(1), 55–57 (1959)
137. Henkin, L., Resek, D.: Relativization of cylindric algebras. Fundam. Math. **82**, 363–383 (1974/1975)
138. Henkin, L., Smith, W.N., Varineau, V.J., Walsh, M.J.: Retracing Elementary Mathematics. Macmillan, London (1962)
139. Henkin, L., Suppes, P., Tarski, A. (eds.): The Axiomatic Method. North-Holland, Amsterdam (1959)
140. Henkin, L., Tarski, A.: Cylindrical algebras. In: Summaries of Talks Presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University, vol. 3, pp. 332–340. Am. Math. Soc., Providence (1957). Also in: Givant, S.R. and McKenzie, R.N. (eds.): Alfred Tarski. Collected papers, vol. 4, pp. 647–655. Birkhäuser (1986)
141. Henkin, L., Tarski, A.: Cylindric algebras. In: Dilworth, R.P. (ed.) Lattice Theory. Proceedings of the Symposium on Pure Mathematics, vol. 2, pp. 83–113. Am. Math. Soc., Providence (1961). Also in: Givant, S.R. and McKenzie, R.N. (eds.): Alfred Tarski. Collected papers, vol. 4, pp. 45–75. Birkhäuser (1986)
142. Herbrand, J.: Recherches sur la théorie de la démonstration. Number 33. Travaux de la Socièté des Sciences et des Lettres de Varsovie, Classe III sciences mathématiques et physiques (1930)
143. Heyting, A.: On weakened quantification. J. Symb. Log. **11**, 119–121 (1946)
144. Heyting, A.: Logique et intuitionnisme. In: Applications Scientifiques de la Logique Mathématique, Actes du 2e Colloque International de Logique Mathématique, Paris 25–30 Août 1952, Number 5 in A, pp. 75–82. Institut Henri Poincaré, Gauthier-Villars, Paris (1954)

145. Heyting, A.: Die formalen Regeln der intuitionistichen Logik. Sitzungsberichte der preussischen Akademie der Wissenschaften, pp. 42–56. Springer, Berlin (1930)
146. Hilbert, D., Bernays, P.: Grundlagen der Mathematik, Vol. I. Springer, Berlin (1934)
147. Hintikka, K.J.: An application of logic to algebra. Math. Scand. **2**, 243–246 (1954)
148. Horsten, L.: Philosophy of mathematics. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. The Metaphysics Research Lab. Center for the Study of Language and Information. Stanford University, Summer 2012 Edition (2012). http://plato.stanford.edu/archives/sum2012/entries/philosophy-mathematics/
149. Jaśkowski, S.: Sur certains groupes formés de classes d'ensembles et leur applications aux définitions des nombres. Stud. Soc. Sci. Torun., Sect. A **1**(3), 23–35 (1948)
150. Jaśkowski, S.: Sur les variables propositionelles dépendantes. Stud. Soc. Sci. Torun., Sect. A **1**(2), 17–21 (1948)
151. Kalman, J.A.: Lattices with involution. Trans. Am. Math. Soc. **87**, 485–491 (1958)
152. Kalmár, L.: Quelques formes générales du théorème de Gödel. C. R. Hebd. Séances Acad. Sci. **229**, 1047–1049 (1949)
153. Kalmár, L.: Une forme du théorème de Gödel sous des hypothèses minimales. C. R. Hebd. Séances Acad. Sci. **229**, 963–965 (1949)
154. Kleene, S.C.: Recursive predicates and quantifiers. Trans. Am. Math. Soc. **53**, 41–73 (1943)
155. Krasner, M.: Une généralisation de la notion de corps. J. Math. Pures Appl. **17**(3–4), 367–385 (1938)
156. Krasner, M.: Les algèbres cylindriques. Bull. Soc. Math. Fr. **86**, 315–319 (1958)
157. Kreisel, G.: On a problem of Henkin's. Indag. Math. **15**, 405–406 (1953)
158. L'Abbé, M.: Structures algébraiques suggérées par la logique mathématique. Bull. Soc. Math. Fr. **86**, 299–314 (1958)
159. Leblanc, H.: The semiotic function of predicates. J. Philos. **46**, 838–844 (1949)
160. Leblanc, H.: On definitions. Philos. Sci. **17**, 302–309 (1950)
161. Lesniewski, S.: Grundzuge eines neuens Systems der Grundlagen der Mathematik. Fundam. Math. **14**, 1–81 (1929)
162. Lesniewski, S.: Einleitende Bemerkungen zur Fortsetzung meiner Mitteilung u.d.T. 'Grundzuge eines neuens systems der Grundlagen der Mathematik'. Collect. Log. **1**, 1–60 (1939). Translated into English as: Introductory remarks to the continuation of my article 'Grundzuge eines neuens systems der Grundlagen der Mathematik' in [184, pp. 116–169]
163. Levi, I.: Putnam's three truth values. Philos. Stud. (Minneapolis) **10**, 65–69 (1959)
164. Lewis, C.I., Langford, C.H.: Symbolic Logic. The Century Philosophy Series, 2nd edn. Century, New York (1932). Dover (1959)
165. Loomis, L.H.: On the representation of σ-complete Boolean algebras. Bull. Am. Math. Soc. **53**, 757–760 (1947)
166. Łoś, J.: On the categoricity in power of elementary deductive systems and some related problems. Colloq. Math. **3**, 58–62 (1954)
167. Łoś, J.: The algebraic treatment of the methodology of elementary deductive systems. Stud. Log. **2**, 151–212 (1955). Errata on p. 329
168. Lyndon, R.C.: The representation of relational algebras. Ann. Math., Ser. 2 **51**, 707–729 (1950)
169. Löb, M.H.: Solution of a problem of Leon Henkin. J. Symb. Log. **20**(2), 115–118 (1955)
170. Löwenheim, L.: Über Möglichkeiten im Relativkalkül. Math. Ann. **76**, 447–470 (1915)
171. MacColl, H.: Symbolical reasoning. Mind **5**, 45–65 (1880)
172. MacColl, H.: Symbolic reasoning (II). Mind, n.s. **6**, 493–510 (1897)
173. MacColl, H.: Symbolic reasoning (III). Mind, n.s. **9**, 75–84 (1900)
174. MacColl, H.: Symbolic reasoning (IV). Mind, n.s. **11**, 352–368 (1902)
175. MacColl, H.: Symbolic reasoning (V). Mind, n.s. **12**, 355–364 (1903)
176. MacColl, H.: Symbolic reasoning (VI). Mind, n.s. **14**, 74–81 (1905)
177. MacColl, H.: Symbolic reasoning (VII). Mind, n.s. **14**, 390–397 (1905)
178. MacColl, H.: Symbolic reasoning (VIII). Mind, n.s. **15**, 504–518 (1906)
179. MacNeille, H.M.: Partially ordered sets. Trans. Am. Math. Soc. **42**, 416–460 (1937)
180. Mal'cév, A.: Untersuchungen aus dem Gebiete der mathematischen Logik (Investigations in the realm of mathematical logic). Mat. Sb. **1**, 323–332 (1936) (in Russian). English translation in [183], pp. 1–14

181. Mal'cév, A.: Ob odnom obščém métodé polučéniá lokal'nyh téorém téorii grupp (On a general method for obtaining local theorems in group theory). Ivanovskij Gosudarstvénnyj Pédagogičéskij Institut, Učényé Zapiski. Fiz.-Mat. Nauki **1**, 3–9 (1941) (in Russian). English translation in [183], pp. 15–21
182. Mal'cév, A.: O prédstavléniáh modéléj (On representations of models). Dokl. Akad. Nauk SSSR **108**, 27–29 (1956) (in Russian). English translation in [183], pp. 22–26
183. Mal'cév, A.I.: The Metamathematics of Algebraic Systems. Collected Papers: 1936–1967. North-Holland, Amsterdam (1971)
184. McCall, S. (ed.): Polish Logic 1920–1939 (1967). Oxford
185. McKinsey, J.C.C.: A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. J. Symb. Log. **6**, 117–134 (1941)
186. McKinsey, J.C.C.: On the number of complete extensions of the Lewis systems of sentential calculus. J. Symb. Log. **9**, 42–45 (1944)
187. McKinsey, J.C.C., Tarski, A.: The algebra of topology. Ann. Math., Ser. 2 **45**, 141–191 (1944)
188. McKinsey, J.C.C., Tarski, A.: On closed elements in closure algebras. Ann. Math., Ser. 2 **47**, 122–162 (1946)
189. McKinsey, J.C.C., Tarski, A.: Some theorems about the sentential calculi of Lewis and Heyting. J. Symb. Log. **13**(1), 1–15 (1948)
190. Menger, K.: Algebra of analysis. In: Notre Dame Mathematical Lectures, No. 3. Lithoprinted. University of Notre Dame, Notre Dame (1944)
191. Menger, K.: Are variables necessary in calculus? Am. Math. Mon. **56**, 609–620 (1949)
192. Menger, K.: Calculus. A Modern Approach. The Bookstore, Illinois Institute of Technology, Chicago (1953)
193. Menger, K.: The ideas of variable and function. In: Proceedings of the National Academy of Sciences of the United States of America, vol. 39, pp. 956–961 (1953)
194. Monk, J.D.: In memoriam: Leon Henkin 1921–2006. Bull. Symb. Log. **15**(3), 326–331 (2009)
195. Montague, R., Henkin, L.: On the definition of 'formal deduction'. J. Symb. Log. **21**(2), 129–136 (1956)
196. Monteiro, A.: Matrices deMorgan caractéristiques pour le calcul propositionnel classique. An. Acad. Bras. Ciênc. **32**, 1–7 (1960)
197. Mostowski, A.: On absolute properties of relations. J. Symb. Log. **12**, 33–42 (1947)
198. Mostowski, A.: Proofs of non-deducibility in intuitionistic functional calculus. J. Symb. Log. **13**, 204–207 (1948)
199. Mostowski, A.: Quelques observations sur l'usage des méthodes non finitistes dans la métamathématiques. In: Le Raisonnement en Mathématiques et en Sciences Expérimentales, vol. 70, pp. 19–29 (1958). Colloques internationaux du Centre National de la Recherche Scientifique, Éditions du Centre National de la Recherche Scientifique
200. Mostowski, A., Grzegorczyk, A., Jaśkowski, S., Łoś, J., Mazur, S., Rasiowa, H., Sikorski, R.: The Present State of Investigations on the Foundations of Mathematics. Rozprawy Matematyczne, vol. 9 (1955). Państwowe Wydawnictwó Naukowe
201. Newman, M.H.A., Turing, A.M.: A formal theorem in Church's theory of types. J. Symb. Log. **7**(1), 28–33 (1942)
202. Nolin, L.: Sur l' algèbre des prédicats. In: Le Raisonnement en Mathématiques et en Sciences Expérimentales, vol. 70, pp. 33–37 (1958). Colloques internationaux du Centre National de la Recherche Scientifique, Éditions du Centre National de la Recherche Scientifique
203. Putnam, H.: Three-valued logic. Philos. Stud. (Minneapolis) **8**, 73–80 (1957)
204. Quine, W.V.: On universals. J. Symb. Log. **12**, 74–84 (1947)
205. Rasiowa, H.: Algebraic treatment of the functional calculi of Heyting and Lewis. Fundam. Math. **38**, 99–126 (1952)
206. Rasiowa, H., Sikorski, R.: A proof of the completeness theorem of Gödel. Fundam. Math. **37**, 193–200 (1951)
207. Rasiowa, H., Sikorski, R.: A proof of the Skolem–Lowenheim theorem. Fundam. Math. **38**, 230–232 (1951)
208. Rasiowa, H., Sikorski, R.: Algebraic treatment of the notion of satisfiability. Fundam. Math. **40**, 62–95 (1953)
209. Rasiowa, H., Sikorski, R.: On existential theorems in non-classical functional calculi. Fundam. Math. **41**(1), 21–28 (1954)

210. Reichbach, J.: Completeness of the functional calculus of first order. Stud. Log. **2**, 245–250 (1955). Errata on p. 329
211. Rieger, L.: On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem. Czechoslov. Math. J. **1**, 29–40 (1951)
212. Robinson, A.: On axiomatic systems which posses finite models. Methodos **3**, 140–149 (1951)
213. Robinson, A.: On the Metamathematics of Algebra. North-Holland, Amsterdam (1951)
214. Robinson, A.: On the application of symbolic logic to algebra. In: Providence American Mathematical Society, Editor, Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., 30 August–6 September 1950, vol. I, pp. 686–694 (1952)
215. Robinson, A.: Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique. In: Colloques Internationaux du Centre Nacional de la Recherche Scientifique, Editor, les Méthodes Formelles en Axiomatique Paris Décembre 1950, vol. 36, pp. 35–52 (1953). Discussion included (pp. 51–52)
216. Rothstein, J.: Communication, Organization and Science (1958). The Faslcon's Wing Press
217. Russell, B.: The Principles of Mathematics. Cambridge University Press, Cambridge (1903)
218. Scholz, H.: Zur Erhellungdes Verstehens. In: Wenke, H. (ed.) Geistige Gestalten und Probleme, Eduard Spranger und 60. Geburstag (27 Juni 1942), pp. 291–310. Quelle and Meyer, Leipzig (1942)
219. Scholz, H., Kreisel, G., Henkin, L.: Problems. J. Symb. Log. **17**(2), 160 (1952)
220. Skolem, T.: Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen. Number 4. Skrifter utgit av Videnskapsselskapet i Kristiania, I. Matematisk-naturvidenskabelig klasse (1920)
221. Skolem, T.: Über einige Grundlagenfragen der Mathematik. Number 4. Skrifter utgitt av Det Norske Viden-skaps-Akademi i Oslo, I. Matematisk-naturvidenskapelig klasse (1929)
222. Skolem, T.: Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. Fundam. Math. **23**, 150–161 (1934)
223. Stone, M.H.: Free Boolean rings and algebras. An. Acad. Bras. Ciênc. **26**, 9–17 (1954)
224. Suppes, P., Henkin, L., Joja, A., Moisil, Gr.C. (eds.): Logic, Methodology and Philosophy of Science IV. North-Holland, Elsevier, Amsterdam (1973)
225. Tang, T.-C.: Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication. Bull. Am. Math. Soc. **44**, 737–744 (1938)
226. Tarski, A.: O wyrazie pierwotnym logistyki. Prz. Filoz. **26**, 68–89 (1923). A somewhat modified version was published in French in two parts under separate titles: "Sur le terme primitif de la Logistique", Fundam. Math. **4**, 196–200 (1923), and "Sur les truth-functions au sens de MM, Russell et Whitehead", ibid. **6**, 69–74 (1924). English translation "On the primitive term of logistic" in [236, pp. 1–23] is based partly on the Polish and partly on the French original
227. Tarski, A.: Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I. Monatshefte Math. Phys. **37**, 361–404 (1930). English translation "Fundamental concepts of the methodology of the deductive sciences" in Tarski, A.: Logic, Semantic, Metamathematics, Oxford (1956), pp. 60–109
228. Tarski, A.: Grundzüge des Systemenkallkül, Erster Teil. Fundam. Math. **25**, 503–526 (1935). English translation "Foundations of the calculus of systems" in Tarski, A.: Logic, Semantic, Metamathematics, Oxford (1956), pp. 342–364
229. Tarski, A.: Der Wahrheitsbegriff in den formalisierten Sprachen. Stud. Philos. **1**, 261–405 (1936). German translation by L. Blaustein, with added Nachwort
230. Tarski, A.: Grundzüge des Systemenkallkül, Zweiter Teil. Fundam. Math. **26**, 283–301 (1936). English translation "Foundations of the calculus of systems" in Tarski, A.: Logic, Semantic, Metamathematics, Oxford (1956), pp. 364–383
231. Tarski, A.: Der Aussagenkalkül und Topologie. Fundam. Math. **31**(1), 103–134 (1938)
232. Tarski, A.: A Decision Method for Elementary Algebra and Geometry. The Rand Corporation, Santa Monica (1948)
233. Tarski, A.: A Decision Method for Elementary Algebra and Geometry. University of California Presses Universitaires de France, Paris (1951). Revised edition of [232] prepared for publication with the assistance of J.C.C. McKinsey
234. Tarski, A.: Some notions and methods on the borderline of algebra and metamathematics. In: Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, USA, 30 August–6 September 1950, vol. I, pp. 705–720. Am. Math. Soc., Providence (1952)

235. Tarski, A.: Contributions to the theory of models. In: Proceedings Konilnklijke Nederlandse Akademie van Wetenschappen, Series A, pp. 57 & 58, 572–588, 56–64 (1954–1955). Also Indag. Math. **16**, 572–588 (1954), and **17**, 56–64 (1955)
236. Tarski, A.: Logic, Semantic, Metamathematics (1956). Oxford
237. Vaught, R.L.: Applications of the Löwenheim–Skolem–Tarski theorem to problems of completeness and decidability. Indag. Math. **16**, 467–472 (1954)

C. Martínez Vidal (✉)
Department of Logic and Moral Philosophy, Faculty of Philosophy, University of Santiago de Compostela, Praza de Mazarelos, 15782 Santiago de Compostela, Spain
e-mail: mconcepcion.martinez@usc.es

J.P. Úbeda Rives
Department of Logic and Philosophy of Science, Faculty of Philosophy and Education Sciences, University of Valencia, Av. Blasco Ibáñez, 30, 46010 Valencia, Spain
e-mail: Jose.P.Ubeda@uv.es

Henkin's Theorem in Textbooks

Enrique Alonso

Abstract Our aim in this paper is to examine the incorporation and acceptance of Henkin's completeness proof in some textbooks on classical logic. The first conclusion of this paper is that the inclusion of Henkin's completeness proof into the standards of Logic was neither quick nor easy. Surprising as it may seem today, most of the textbooks published in the 1950s did not include a section for this proof, nor presented it in any way. A question we should try to answer is at what moment does Henkin's proof of completeness for first order logic begin to be considered as a part of the standards of elementary logic. This point brings us to a discussion on the way in which the specific gains of Henkin's proof have been assessed in literature. The possibility of using Henkin's methods in a wide variety of formal systems made completeness a general property belonging to foundations of logic, leaving the realm of model theory for quantification languages where it was previously located.

Keywords History of logic · Case studies in history of mathematics · Completeness in elementary logic

1 Logic Textbooks

The study of the textbooks of a certain discipline is one of the best ways to understand what is at stake at every moment in a field, in this case, the field of contemporary logic. Our aim in this paper is to examine the incorporation and acceptance of Henkin's completeness proof in some textbooks on classical logic. We have carried out a critical analysis of some 34 works[1] that can be considered classical by logicians from almost every country or tradition. All of these textbooks come from the Anglo-Saxon tradition which dominated the development of logic in the twentieth century. We have thus explicitly ignored works, many of which are of great value, belonging to local traditions in communities that could be considered as peripheral.[2] All the works considered in this paper were published before the beginning of the 1980s. We consider that contemporary logic reaches maturity in the late 1970s. As a result, we believe that subsequent textbooks only add minor updates on previously existing material and approaches. In Table 1, we show the distribution of the material over the decades.

[1]The list is featured prominently in Sect. 8.

[2]For details on the dissemination of logic in Spain, see [2].

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_11

Table 1 Works analyzed

Period	Number of works analyzed
1950–1960	9
1960–1970	18
1970–1980	7

Table 2 Textbooks by title

Terms	Number of works including the term in the title
Mathematical	17
Symbolic	8
Formal	2
Others	9

The choice of the works analyzed here has been made based on an expert assessment of their influence at the time. We do not claim the list to be neither objective nor exhaustive. The evaluation of the influence of a work depends too much on the context of each reader so that any claim of objectivity could be considered pretentious. We have made our selection by focusing on works whose title makes reference to some of the several ways to name contemporary logic. Terms such as mathematical logic, the symbolic logic, or formal logic are common. The distribution found in our list is shown in Table 2.

As can be seen from the list, the term "mathematical" is most frequently used. Nevertheless, many of the textbooks including this term in their title do not cover the kind of content we could expect today. In some cases, we only find what could be taken as a promising introduction to the topic.

Another factor that must be considered is the origin of many of these manuals. In many cases, the textbooks originate from previous works published in other languages. We can identify at least five translations from German, but we also have translations from French, Polish, and Russian original works. This usually is the result of the invitations handed out to various European teachers to visit American institutions to offer courses on logic. The notes used often gave rise to a manual that was then published. In other cases, the text came from the translation of a work or notes used in these researchers' universities of origin.

Finally, we must emphasize the interest in the field of logic shown by certain scientific publishing companies during the late 1960s. Reidel, Springer, North-Holland, and Dover are amongst the most relevant publishing houses in this period of consolidation of the discipline. In the previous period, we observe a greater dispersion of publishers. Sometimes, we find general publishers giving a growing discipline an opportunity, but it is more common to find university companies publishing the research of its scholars.

2 Church and Kleene Textbooks

The first conclusion of this paper is that the inclusion of Henkin's completeness proof into the standards of logic was neither quick nor easy. The first mention of this result appears in [8] and [22]. From this we might deduce that it was accepted without hesitation and included immediately in the manuals of the time. However, this was certainly not the case.

As is well known, Church had been Henkin's mentor at Princeton since the arrival of the latter in 1941. In fact, Henkin chose to study at Princeton because of Church's presence at that university.[3]

During the course on logic that Church included in his doctoral program and that spanned over two semesters, Gödel's completeness theorem was analyzed. The course looked at the theorem's reductive character; that is, the use of techniques—normal forms and quantifier elimination—to reduce general problems to simpler and better known situations. The same approach is adopted in Sect. 44 of [8], where a detailed proof of Gödel's original result is exposed. This work by Church was one of the first textbooks to include detailed comments in the English language of this complex proof. Paradoxically, the first official translation of Gödel's original paper on completeness published in German in the Monatshefte für Mathematik und Physik[4] in 1930 appears some years later in the classical work by Van Heijenoort From Frege to Gödel in 1967.[5] It is therefore quite possible that Gödel's proof was known by a number of English logicians through Church's work and not directly through Gödel's. This may partly explain Church's reluctance to adopt Henkin's proof in his manual. The first time Henkin's proof is mentioned in Church's textbook is in Sect. 49, entitled Historical Notes. In this section the content of the preceding section is explained from a historical point of view. Church says that:

> The first proof of completeness in the latter sense is that of Gödel, which is reproduced in Sect. 44. Another proof of completeness of the functional calculus of first order is due to Leon Henkin and is reproduced in Sect. 45 (see further Sect. 54).[6]

If we analyze Sect. 45 in detail, we can see the use of the general technique ideated by Henkin in his proof, but no explicit reference to the author is made. The first explicit mention to Henkin appears in the historic section, Sect. 49, which comes obviously some pages later. It is also relevant that Sect. 45 is entitled *Löwenheim's theorem and Skolem's generalization* and not, as it could be expected, something making mention to Henkin. The reason is that Church employs Henkin's theorem[7]—without mentioning the author—to show that it is possible to obtain both the Gödel completeness theorem and Löwenheim–Skolem theorem as mere corollaries.[8] The section explicitly including the name of Henkin in its title is devoted to the completeness proof for second-order logic, the result that constitutes the core of his doctoral thesis.

It is true however that in a footnote Church clarifies that:

> The method which is used in this section to prove a weak completeness theorem for the functional calculus of second order is due to Leon Henkin (in his dissertation, Princeton University, 1947). It is essentially the same as the method used in Sect. 45 (also due to Henkin, cf. footnote 465) to prove Gödel's completeness theorem for the functional calculus of first order.[9]

---

[3]See [16, p. 132].

[4]See [12].

[5]The translation was elaborated on that occasion by Professor Stefan Bauer-Mengelberg (see [35, p. 593]) and was reviewed and approved by Gödel himself.

[6]See [8, p. 291].

[7]Sometimes the term *Henkin completeness theorem* is used to make reference to any formulation of the type "every consistent set of formulas of first-order logic is satisfiable, that is, has some model." This time, Church specifies that this consistent set is satisfiable in a model of cardinal enumerable.

[8]See [8, pp. 244–245].

[9]See [8, p. 307, nt. 510].

If we gather these data about the presence of Henkin's completeness theorem for first-order logic in the work of Church, then what is actually observed is the attempt to locate its commentary in those places that really represent significative improvements with respect to Gödel's proof. This is really what could be expected from someone who knew the work of Henkin in its true value.

Thus, it is not so strange that Church reserves the term *Henkin's theorem* to refer to the result for second-order logic. Church knew very well that the completeness theorem for second order was obtained previously and that the techniques developed to obtain it were then used by Henkin to extend the result to first-order logic. Moreover, he is aware that the undoubted advantage of the method developed by Henkin over Gödel's proof relies on the ability to handle models of any cardinal. In this context, the connection of Henkin's proof with the Löwenheim–Skolem theorem seems obvious.

Kleene's work, although published before the Church's one, had surely less diffusion, inter alia because Church's work had circulated as lecture notes for a long time previously. Kleene is quite explicit with respect to the value of Henkin's proof when he asserts that:

> Henkin 1949 gave a proof employing a minimum of knowledge of the deductive properties of the predicate calculus. We give a proof which is intermediate in this respect between Hilbert–Bernays' and Henkin's.[10]

However, the proof supplied in his book is basically a Gödel-like proof.

3 The Completeness Proof in Textbooks

Although these early works invite us to imagine that Henkin's proof was included in the standards of contemporary logic in a relatively seamless manner, the truth is that this process took many years. In fact, the completeness theorem itself experienced some difficulties to be finally considered a common topic in elementary textbooks. Surprising as it may seem today, most of the textbooks published in the 1950s did not include a section for this proof, nor presented it in any way. This also occurs in the following decade, although during this period, there are already many works that include the proof in some way. It is in the decade from 1970–1980 when the problem of the completeness for first-order logic acquires enough importance as to deserve an entire section including a detailed proof, the corresponding corollaries, and a commentary of the general implications of the theorem. In fact, none of the textbooks analyzed in this period skipped a chapter or section on completeness. Table 3 shows the specific figures.

Although it is surely inappropriate to talk in the 1950s of the survival of the logicist paradigm, it is true that many of the manuals from that period had a certain logicist flavor.

Table 3 Textbooks including chapters or sections on "completeness"

Ages	Including *completeness*	Not including *completeness*
1950–1960	2	6
1960–1970	8	11
1970–1980	9	–

[10]See [22, p. 389].

This fact is related to the lack of contents from the model theory. Nevertheless, this may be simply because the model theory had not yet had the appropriate development and dissemination. The concept of logic transmitted by works like *Introduction to Symbolic Logic and its Applications* by Carnap, *The Elements of Mathematical Logic* by Rosenbloom, *Logic for Mathematicians* by Rosser, or *Symbolic Logic* by Lewis and Langford is that logic is mainly concerned with the logical calculi. Logic is a tool to derive conclusions from given premises. In a more general sense, logic is mainly oriented toward formalizing mathematical theories and analyzing the result of this formalization. So it is not surprising at all that the definition of completeness offered in these manuals is that of completeness of a theory and not completeness of a logical system.[11] The instrumental value of logic as a tool to analyze ordinary language is also present in many of these textbooks. Reichenbach [30] and Suppes [33] promote an interpretation of logic that would eventually go on to be very popular in university faculties of humanities and philosophy.

In general, it can be said that in this period, there is little interest in the study of the metatheoretical properties of formalisms. On the rare occasions that metatheoretical contents are included and commented on, they are centered more on the decidability or independence of certain axiomatizations of elementary logic. The emphasis is usually placed on the rigor of the methods employed, the ability of formal languages to formalize well-known informal axioms, or on the possibility to derive conclusions from a very small number of axioms. The line of study originally opened up by Church and Kleene years ago was not the one followed by most academics in the 1950s. The mainstream in logic turned their attention toward some of the initial intuitions posed by Russell in the Principia or Hilbert and Bernays in the Grundgesetze.

In the 1960s, the situation changes, bringing about an equilibrium between the manuals that do include comments on the completeness theorem for first-order logic in their indexes and those that do not. In those cases in which the completeness theorem does not appear in the subject index, we do find some kind of comment even if this does not lead to a detailed proof. A distinctive feature in the 1960s is that a discussion of the completeness theorem is not often found in an appropriate place. Sometimes it is included as part of the discussion on foundations, and others within a section devoted to formal semantics. In general, however, there is no clear idea of the correct location for this kind of content.

The manuals that discuss the topic often refer to other works for a detailed demonstration; usually, Church [8] and Hilbert and Ackermann [18] (the English translation of the second German edition). References to Henkin's proof are rare in this context, and in general Gödel's proof is mentioned as the first to establish this result for elementary logic.

Nevertheless, we still can find some manuals that do not make any mention to the completeness of first-order logic; Lee [23] and Hackstaff [13] are good examples. These textbooks exhibit a clear debt to the original ideas of logicism that somehow do not fully disappear.

4 Henkin's Theorem as a Part of the Received View

A question we should try to answer is at what moment does Henkin's proof of completeness for first-order logic begin to be considered as a part of the standards of elementary

[11] See [6, p. 173].

logic. The answer is not an easy one to reach in any case. Rather than in a particular work or a certain milestone, what we find is a series of handbooks that give form to what would become the standards of formal logic in the next decade. In this sense, the *Introduction to Mathematical Logic* by Mendelson, which was first published in 1964,[12] deserves special mention. Apart from a detailed discussion of Henkin's proof, this study is of interest because of where it chooses to deal with the completeness theorem: it appears as a separate section under a heading entitled Quantification Theory. We do not intend to argue at this point where the right place is to show a proof of the completeness for FOL. What is of interest to us is that completeness is understood from now on as part of the general study of quantification theories. It is no longer a strange content that is difficult to locate, but a proper part of the study of the most common formalisms. Also remarkable is the effort made to disclose not the original version but the modification of it made by Hasenjaeger[13] in 1953. This latter adaptation is now considered canonical.

A few years later, in 1971, at least two other books appear that follow the line of research embarked upon by Mendelson. These studies are *Mathematical Logic and Formalized Theories* by R. Rogers and *Metalogic. An Introduction to the Metatheory Standard First Order Logic* by Hunter. Both preserve the location of the completeness for FOL given in the textbook by Mendelson. In the first, an explicit reference to the Mendelson's work as the correct point of reference is made.

In all these cases (in Mendelson's work and then in Rogers and Hunter), some classical results in model theory are commented on with relation to the completeness theorem. The Löwenheim–Skolem and compactness theorems are obvious results in this context, but other topics such as categoricity or elementary equivalence begin to be included alongside more typical contents.[14] What becomes evident from these comments is that completeness itself was seen for a long time as a proper content of model theory rather than as a basic metatheoretical property of formal systems. The complexity of the existing proofs and the notions involved probably contributed toward that view. Consider, for example, that the use of nonconstructive resources was regarded as a kind of border between pure logic and set theory.[15]

In short, it appears that it is during the final years of the 1960s and early 1970s that completeness begins to take the leap from model theory into the standards of metatheory of logic. We refer to these standards as the received view in the study of elementary logic. This shift means abandoning the notion that completeness is a subdiscipline of logic, model theory, and seeing it as a basic property in the study of any formalism.

5 The Gödel's Damn

A common phenomenon in textbooks literature is the survival of certain denominations, regardless of its content. Gödel's name was and is still associated with the completeness theorem of first-order logic even though the proof is not the original proof given by

[12] In 1979, a second edition appears.

[13] See [26, p. 67]. See also [14]. Later we will discuss this change in relation to alternative proofs.

[14] See, for example, [26].

[15] See Gödel's apologize in [11, p. 63].

Gödel. We frequently refer to Gödel's completeness theorem in this context, partly as a way of preserving relevant historical facts. And there is nothing wrong with this. It is true that Gödel was the first to prove this fundamental result, and we had to wait a long time until the method devised by Henkin furnished an essentially different proof. But is that enough to warrant keeping Gödel's name in the completeness theorem of FOL? Rather than offering a verdict, I will discuss what is and what might be inappropriate and what is perfectly admissible. I do not believe that there is any problem in using Gödel's name to refer to the completeness theorem for FOL when this is formulated in a completely general manner. More than a way to grant an undisputed priority, we believe it is a good way to preserve the true history of the discipline. What may be less suitable would be, for example, associating Gödel with formulations that are more akin to those of Henkin. It would be strange to speak of Gödel's theorem in order to refer to the fact that every consistent set of formulas has a model. This is simply not the original formulation offered by Gödel, and it introduces notions that were not around when Gödel proved his completeness theorem.

Much worse would be to go ahead and prove Gödel's completeness theorem by adopting the Henkin's technique without making this clear. To our surprise, we have identified at least two works in which this phenomenon occurs: *Introduction to Mathematical Logic* by H. Hermes and *An Outline of Mathematical Logic* by Grzegorzyk, which both offer this misleading interpretation. In neither case can we ascribe this to ignorance, which makes it even more interesting. I believe it responds to a common process in the construction of the canon of a discipline. To obtain a fundamental result, we are supposed to use the best available techniques at every moment. It is up to those interested in history to fix when and how results were obtained, but the only important concern for textbooks is to fulfill pedagogical and disciplinary targets in the best possible way. The dates on which these works were published, in the mid-1970s, could however facilitate an undesirable confusion, which fortunately does not seem to have spread amongst the scholarly community.

Some textbooks emphasize the difference between the statement of Gödel's completeness theorem and what is described as Henkin's theorem. A possible solution to stress the differences between both statements is that is adopted in *A Course in Mathematical Logic* by Bell and Machover [3]. In this work, Gödel's theorem is described as a weak completeness theorem, whilst Henkin's theorem is considered a full and strong completeness result. It is true that in Gödel 1929 and 1930, [11, 12], his completeness theorem is formulated as *every valid formula of the restricted functional calculus is provable*, and it is postulated that the method used does not lead to a strong completeness result, understood without restrictions on the cardinal of the set of premises.[16] This observation partially justifies the decision taken by Bell and Machover but at the cost of a certain degree of artificiality. In the same work,[17] the term Henkin's theorem is openly used to refer to strong completeness or to some of the formulations commonly associated with his property, that is, all consistent set of formulas of a language L of first order is satisfiable in a model of cardinal less than or equal to that of L. This approach is similar to that adopted by Schoenfield in his *Mathematical Logic* and is the most common solution in cases where some distinction is made between the proper contents of Gödel's completeness theorem and Henkin's one.

[16] See [3, p. 117].

[17] See [3, p. 121].

6 The Real Value of Henkin's Completeness Theorem

This point brings us to a discussion on the way in which the specific gains of Henkin's proof have been assessed in literature. On some occasions, Henkin's proof has been introduced as a kind of simplification of Gödel's original proof. Hermes' manual responds to this approach when he says that the proof that we shall consider is not Gödel's original proof, but a simpler proof due to Henkin.[18] I am of the opinion that almost all those familiar with introductory textbooks and courses in modern logic can recognize certain success for this way of putting things. This is obviously not the most appropriate way of setting the real value of Henkin's theorem, simply because it can never be considered a mere simplification of Gödel's proof. It represents an entirely different way of facing the problem of completeness in formal languages, one completely different from Gödel's. In any case, I do not think that this statement and others like it are down to ignorance or a twisted interpretation of the contents of each of these results. It is more plausible to see it just as a way of commenting on Gödel's proof in a more verbose way, justifying its replacement by another more accessible one to the nonspecialist. There would therefore be a pedagogical intention behind this, rather than a focus on historical fact. Nevertheless, this movement can result in a false impression with respect to the relationship between both proofs.

However, Henkin's theorem is most often introduced as an alternative result, based on concepts and techniques quite different from Gödel's. Mates, for example, perfectly summarizes this point of view when he argues that *Henkin's proof is formulated in terms of certain notions that possess interest even independently of their use in the present connection*.[19] Besides its use of concepts from model theory, Henkin's proof is usually valued for its relative independence from the limitations of language and the specific presentation of the system used to characterize derivability. Bell and Machover insist precisely on assigning the name of Henkin's theorem to the version that allows us to extend the result to languages of any cardinality. But perhaps it is Enderton who exposes this point of view more clearly when he says that Henkin's proof *unlike Gödel's original proof generalizes easily to languages of any cardinality*.[20] Kleene [22] notices the relative independence of the method followed by Henkin with respect to the characterization of derivability: *Henkin* [15] *gave a proof employing a minimum of knowledge of the deductive properties of the predicate calculus*.[21]

I believe that one of the great contributions of the method followed by Henkin consists of its ability to be extended to formal systems far from the realm of first-order logic and quantification languages. For a long time, the community of logicians seemed to consider completeness as a property of quantification formal systems and not as a general property of any formalism. I think that the dependence of Gödel's proof of specific resources of quantificational vocabularies contributed decisively to set this impression. The easy extension of Henkin's technique to some nonclassical logics and especially modal logics based on Kripke semantics was certainly a key factor in understanding completeness as a

[18] See [31, p. 61].

[19] See [25, p. 136].

[20] See [10, p. 139].

[21] See [22, p. 389].

fundamental property of all kinds of formalisms. However, this should be considered as a mere hypothesis that would require an independent study that goes beyond our goals at this time.

Another fact that requires some commentary is the tendency to see Henkin's proof as one amongst others stated during a very particular moment in time. Mendelson,[22] for example, states that:

> The proof given here is due to Henkin [15], as simplified by Hasenjaeger [14]. The result was originally proved by Gödel [12]. Other proofs have been published by Rasiowa-Sikorski [28, 29] and Beth [4], using (Boolean) algebraic and topological methods, respectively. Still other proofs may be found in Hintikka [19, 20] and in Beth [5].[23]

This commentary seems to respond to the idea that the method followed by Henkin was a direct consequence of a wide and extensive comprehension of the meaning of the completeness of formal systems by the scientific community at the time. It is true that Henkin never had to dispute the primacy of his proof over those by Rasiowa–Sikorski or Beth. But comments like those of Mendelson's conveyed the impression that the understanding of the nature itself of logical completeness and the techniques needed to prove this property in more diverse contexts was around during the late 1940s and early 1950s.

Perhaps as a result of this feeling, or simply an understanding of the flexibility of the strategy followed by Henkin, a wide variety of proofs appeared based on minor—or not so minor—changes. The so-called Hasenjaeger's simplified proof might initially had belonged to this category of alternative proofs to be later considered as the canonical discussion of Henkin's proof. In 1949's paper, Henkin did not devise Lindenbaum's construction over the extended language, including in just one one step all the new constants needed as witnesses to avoid ω-inconsistent first-order theories. Instead, Henkin considered a series of successive extensions, complicating the process and taking elegance away from the whole proof. Hasenjaeger[24] seems to be the first to change this aspect, including all the new constants in one single step. Nevertheless, Church [8, p. 311, nt. 513], indicates that it was Henkin himself who, in 1950, instilled in Church the desire to extend the language in that single step.

Smullyan represents an entirely different case. In his *First Order Logic*, he openly presents a methodology opposed to what he called the Henkin–Hasenjaeger completeness proof.[25] As is well known, the technique used by Smullyan is based on the analytic tableaus developed by Beth.[26] Although at the time this was a very interesting and visual way to approach the issue of completeness, the fact is that this new development never acquired the relevance Smullyan demanded. Perhaps less pretentious, but in the same line of thinking, we find the development Schütte offers in his 1962's *Lecture Notes in Mathematical Logic*. In his notes, Schütte develops an original proof that is not always easy to follow, in which he exhibits his personal point of view with respect to logic. It is commonplace in this period to take advance of some topic (completeness in this case) to offer

[22] See also [24, p. 192].

[23] See [26, p. 67].

[24] See [14].

[25] See [32, p. viii].

[26] *Ibid*.

strategies reflecting the philosophical views of the author. Smullyan or Schütte are mere examples of this way in order to understand the contents and targets of textbooks.

A final point that deserves our attention is the commentary of nonconstructive steps in the completeness proof for first-order logic. Most of the manuals analyzed pay little attention to this point. Only those works sensitive to the history of logic or those interested in the differences between finitary and nonfinitary proofs devote some lines to this matter. Church [8] and Kleene [22], probably share both components and expressly mention this fundamental element of completeness proof although in relation to the original proof by Gödel.[27] Curry [9] also raises the question when pointing toward a potential conflict between Church's [7] undecidability theorem for first-order logic and the actual construction of models for any consistent set of formulas. If such thing were possible in a finitary way for any set whatsoever, then we could invalidate any nonprovable formula by a finite construction, thus making the logic itself decidable. Curiously, Curry does not offer more direct evidence of the use of nonconstructive resources in the proof:

> This is necessarily a nonconstructive result, because Church proved that HK^* is recursively undecidable; and if there were a constructive method of deciding whether or not a proposition A is valid in every model, then it would furnish a constructive decision method.[28]

Mendelson is much clearer when he directly refers to the actual proof by Henkin:

> Notice that M is not necessarily effectively constructible. The interpretation of the predicate letters depends upon the concept of provability and this, as was noted at the end of Lemma 2.11, may not be effectively decidable.[29]

In this case, M is the model resulting from Lindenbaum's construction. It is clear that, in general, there is some reluctance in the manuals to indicate clearly the point at which the Henkin's proof includes a nonconstructive step. It is true that Gödel's proof contains a direct appeal to a nonconstructive use of the principle of excluded middle,[30] a step that is not present in Henkin's proof. Lindenbaum's construction, adopted by Henkin, to extend a consistent set to a maximally consistent one is not in itself constructive or not. If the derivability for FOL were a decidable relation between premises and conclusion, the method could show a way to obtain a model from the initial given set. It is not easy to understand the relative absence of more explicit comments on this fact in the textbooks here analyzed. This may respond, as we have noted in other areas of our study, to the intent not to confuse the reader with complex considerations in a proof that is not always easy to follow in detail.

7 The Contribution of Henkin's Theorem to the *Received View*

One of the conclusions to be drawn from this study is that the long way completeness had to travel to gain the role now played in foundations of logic. To refer to the standards and basic skills of contemporary logic, I will choose, following a well-stated tradition, the

[27] See [8, p. 235] and [22, p. 394].

[28] See [9, p. 354].

[29] See [26, p. 69].

[30] See [11, p. 63].

term *the received view of logic*. For the received view, logic consists of the study of the relation of consequence, that is, a relationship established between sets of formulas, the premises, single formulas, and the conclusions. This abstract relationship can be defined and analyzed in several ways. It can be understood under the standards of proof theory, and then we talk of *formal derivability*, or from the point of view of the model theory, in which case we deal with the relation of semantic consequence. Formal derivability and semantic consequence define relationships that are not conceived as isolated. Obviously, we expect that some equivalence result could be stated, but it is not, as all we know, a direct consequence of definitions. Semantic and formal derivability relations are defined in such different terms that nothing guarantees equivalence. This is one of the most appreciated milestones of the received view, and nobody can feel any kind of surprise with these words. Thomason [34], for example, says that:

> The chapters dealing with this topic [completeness] discuss equivalences between syntactic and semantic concepts and proceed to establish these equivalences as metatheorems. The metatheorems are proved using the techniques developed by L. Henkin.[31]

And Hermes and Monk insist on the point stating that:

> One of the substantial achievements of modern logic has been to show that the notion of consequence can be replaced by a provably equivalent notion of derivability which is defined by means of a calculus.[32]

> Then we prove the completeness theorem, which shows the equivalence between the proof-theoretic notion and the corresponding semantic notion.[33]

This way of dealing with the study of derivability and semantic consequence relationships is a standard in introductory courses to logic in virtually any region or context. Despite its virtues, it also has obvious problems that have been identified for a long time. The first has to do with what we call the issue of syntax vs semantics priority.[34] Why do we talk about the soundness or completeness of a calculus and not of the models? Although we speak of equivalence between syntactic and semantic notions, the fact is that one of the terms of this equivalence is more valuable than the other: there is no unsound calculus, although we accept incomplete logics with respect to the class of intended models for its language. I think that this specific point of the received view was decisively put forward thanks to the effect that Henkin's proof had on the assessment of completeness. The possibility of using Henkin's methods in a wide variety of formal systems made completeness a general property belonging to foundations of logic, leaving the realm of model theory for quantification languages where it was previously located. An example of this tendency is given by Hunter [21], who offers an independent proof of completeness for classical propositional logic using the method of Henkin. This strange decision seems to be based on pedagogical criteria and is just a way of introducing the basic notions of Henkin's proof whilst avoiding the difficulties arising from quantificational languages. Nevertheless, it conveys the impression that completeness makes sense in any possible context, even in those in which we deal with decidable formal systems. This might be

[31] See [34, p. viii].

[32] See [17, Preface].

[33] See [27, p. 194].

[34] See [1, p. 80].

an incorrect and misleading way to assess the relative importance of such properties as decidability and completeness. A way that contrasts with the original idea Gödel had on the importance of completeness as a metatheoretical property of logical systems. Suppes fixes the point when he says that:

> In one sense the existence of a decision procedure for truth-functional arguments trivializes the subject. Fortunately or unfortunately, no such trivialization of the logic of quantification is possible.[35]

In general, the value of completeness only can be appreciated in the absence of a decidability result for a given formal systems. In that case, a completeness proof guarantees at least that we have a way to recursively enumerate the logical truths of this logic enumerating them as theorems of an appropriate calculus. Bell and Machover take time to discuss this matter, stating that:

> Another method of detecting logical validity is provided by the first-order predicate calculus. By the completeness theorem, $\models \alpha$ iff $\vdash \alpha$. So, if we want to know whether α is logically valid, we can search systematically for a first-order proof of α. If such a proof exists, we shall sooner or later discover it.[36]

In any case, this is nothing new, but a way to interpret the value of completeness that goes back to Gödel himself[37] and one that was also picked up by Kleene in his comments to Gödel's completeness theorem included in Sect. 75 of *Introduction to Metamathematics*.[38] From this point of view, completeness only has value to ensure some control over the class of logical truths, a control fully guaranteed in a decidable logical system.

This is not, however, a conclusion that could be ascribed to Henkin himself, but rather a direct consequence of the success of a proof that was able to remove the rank and importance of a handful of metatheoretical properties and rearrange them in a new and promising way.

8 Appendix: References by Period of Time

8.1 1950–1960

- Carnap, R. 1954. *Introduction to Symbolic Logic and its Applications*, New York: Dover Pub.
- Church, A. 1956. *Introduction to Mathematical Logic*, Princeton: Princeton University Press.
- Kleene, S.C. 1952. *Introduction to Metamathematics*, New York: D. Van Nostrand.
- Lewis, C.I. and Langford, C.H. 1959. *Symbolic Logic*, New York: Dover Publications.
- Quine, W.V. 1951. *Mathematical Logic*, Harvard Univ. Press. 2nd Edition.
- Reichenbach, H. 1947. *Elements of Symbolic Logic*, New York: The Free Press.
- Rosenblomm, P. 1950. *The Elements of Mathematical Logic*, New York: Dover Publications.

[35] See [33, p. 69].

[36] See [3, p. 123].

[37] See [12, p. 589].

[38] See [22, p. 423].

- Rosser, J.B. 1953. *Logic for Mathematicians*, New York: McGraw-Hill.
- Suppes, P. 1957. *Introduction to Logic*, New York: D. van Nostrand Company.

8.2 1960–1970

- Ambrose, A. and Lazerowitz, M. 1962. *Fundamentals of Symbolic Logic*, New York: Holt, Rinehart and Winston. 2nd Edition.
- Beth, E.W. 1962. *Formal Methods. An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic*, Dordrecht: Reidel.
- Curry, H.B. 1963. *Foundations of Mathematical Logic*, New York: McGraw-Hill.
- Fraenkel, A.A. 1963. *Set Theory and Logic*, Reading: Addison-Wesley Publishing Company.
- Hackstaff, L.H. 1966. *Systems of Formal Logic*, Dordrecht: Reidel.
- Kreisel, G. and Krivine, J.L. 1967. *Elements of Mathematical Logic: Model Theory*, Amsterdam: North-Holland.
- Lee, H.N. 1962. *Symbolic Logic*, London: Routledge.
- Lorenzen, P. 1965. *Formal Logic*, Dordrecht: Reidel.
- Manicas, K. 1968. *Workbook for Essentials of Logic*, New York: American Book Company.
- Margaris, A. 1967. *First Order Mathematical Logic*, New York: Dover Publications.
- Mates, B. 1965. *Elementary Logic*, New York: Oxford University Press.
- Mendelson, E. 1972. *Introduction to Mathematical Logic*, New York: D. van Nostrand Company. 2nd Edition.
- Nidditch, P.H. 1962. *The Development of Mathematical Logic*, London: Routledge.
- Prior, N. 1962. *Formal Logic*, Oxford: Clarendon Press. 2nd Edition.
- Rescher, N. 1968. *Topics in Philosophical Logic*, Dordrecht: Reidel.
- Schoenfield, J.R. 1967. *Mathematical Logic*, Reading: Addison-Wesley.
- Schütte, K. 1962. *Lecture Notes in Mathematical Logic*, Pennsylvania State University.
- Smullyan, R. 1968. *First-Order Logic*, New York: Springer-Verlag.

8.3 1970–1980

- Bell, J.L. and Machover, M. 1977. *A Course in Mathematical Logic*, Amsterdam: North-Holland.
- Enderton, H.B. 1972. *A Mathematical Introduction to Logic*, New York: Academic Press.
- Grzegorczyk, A. 1974. *An Outline of Mathematical Logic*, Dordrecht: Reidel.
- Hermes, H. 1973. *Introduction to Mathematical Logic*, Berlin: Springer-Verlag.
- Hunter, G. *Metalogic. An Introduction to the Metatheory of Standard First Order Logic*, Berkeley: University of California Press.
- Manin, Y.I. 1974. *A Course in Mathematical Logic*, New York: Springer-Verlag.
- Monk, J.D. 1976. *Mathematical Logic*, New York: Springer-Verlag.
- Thomason, R.H. 1970. *Symbolic Logic*, London: The McMillan Company.
- Rogers, R. 1971. *Mathematical Logic and Formalized Theories*, Amsterdam: North-Holland.

References

1. Alonso, E.: La Completitud como propiedad redundante. Notae Philosophicae Scientiae Formalis **2**(2), 74–86 (2013)
2. Alonso, E., Marraud, H.: La lógica que aprendimos. Theoria **18**(3) 327–349 (2003)
3. Bell, J.L., Machover, M.: A Course in Mathematical Logic. North-Holland, Amsterdam (1977)
4. Beth, E.: A topological proof of the theorem of Löwenheim Skolem–Gödel. Indag. Math. **13**, 436–444 (1951)
5. Beth, E.: The Foundations of Mathematics. North-Holland, Amsterdam (1959)
6. Carnap, R.: Introduction to Symbolic Logic and its Applications. Dover, New York (1954)
7. Church, A.: An unsolvable problem for elementary number theory. Am. J. Math. **59**, 345–363 (1936)
8. Church, A.: In: Introduction to Mathematical Logic. Princeton University Press, Princeton (1956)
9. Curry, H.B.: Foundations of Mathematical Logic. McGraw-Hill, New York (1963)
10. Enderton, H.B.: A Mathematical Introduction to Logic. Academic Press, New York (1972)
11. Gödel, K.: On the completeness of the calculus of logic. In: Feferman, S., et al. (eds.) Collected Works, vol. I, pp. 61–101. Oxford University Press, New York (1986)
12. Gödel, K.: Die Vollstandigkeit der Axiome des logischen Funktionenkalkuls. Monatshefte fur Mathematik und Physik **37**, 349–360 (1930)
13. Hackstaff, L.H.: Systems of Formal Logic. Reidel, Dordrecht (1966)
14. Hasenjaeger, G.: Eine Bemerkung zu Henkin's Beweis für die Vollständigkeit des Prädikatenkalküls Ersten Stufe. Journal of Symbolic Logic **18**, 42–48 (1953)
15. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**, 159–166 (1949)
16. Henkin, L.: The discovery of my completeness proofs. The Bulletin of Symbolic Logic **2**(2), 127–158 (1996)
17. Hermes, H.: Introduction to Mathematical Logic. Springer, Berlin (1973)
18. Hilbert, D., Ackermann, W.: Principles of Mathematical Logic. Chelsea, New York (1950)
19. Hintikka, J.: Form and content in quantification theory. Acta Philos. Fenn. **8**, 11–55 (1955)
20. Hintikka, J.: Notes on the quantification theory. Comment. Phys.-Math. Soc. Sci. Fenn. **17**, 1–13 (1955)
21. Hunter, G.: Metalogic. An Introduction to the Metatheory of Standard First Order Logic. University of California Press, Berkeley (1971)
22. Kleene, S.C.: Introduction to Metamathematics. Van Nostrand, New York (1952)
23. Lee, H.N.: Symbolic Logic. Routledge, London (1962)
24. Margaris, A.: First Order Mathematical Logic. Dover, New York (1967)
25. Mates, B.: Elementary Logic. Oxford University Press, New York (1965)
26. Mendelson, E.: Introduction to Mathematical Logic, 2nd edn. Van Nostrand, New York (1972)
27. Monk, J.D.: Mathematical Logic. Springer, New York (1976)
28. Rasiowa, H., Sikorski, R.: A proof of the completeness theorem of Godel. Fundam. Math. **37**, 193–200 (1951)
29. Rasiowa, H., Sikorski, R.: A proof of the Skolem–Lowenheim theorem. Fundam. Math. **38**, 230–232 (1952)
30. Reichenbach, H.: Elements of Symbolic Logic. Free Press, New York (1947)
31. Rogers, R.: Mathematical Logic and Formalized Theories. North-Holland, Amsterdam (1971)
32. Smullyan, R.: First-Order Logic. Springer, New York (1968)
33. Suppes, P.: Introduction to Logic. Van Nostrand, New York (1957)
34. Thomason, R.H.: Symbolic Logic. McMillan Company, London (1970)
35. Van Heijenoort, J.: From Frege to Gödel a Source Book in Mathematical Logic, pp. 1879–1931. Harvard University Press, Cambridge (1967)

E. Alonso (✉)
Department of Logic and Philosophy of Science, Faculty of Philosophy, Avda. Tomás y Valiente s/n, Autonomous University of Madrid, Campus Canto Blanco, 28049 Madrid, Spain
e-mail: enrique.alonso@uam.es

Henkin on Completeness

María Manzano

Abstract *The Completeness of Formal Systems* is the title of the thesis that Henkin presented at Princeton in 1947 under the supervision of Alonzo Church. A few years after the defense of his thesis, Henkin published two papers in the *Journal of Symbolic Logic*: the first, on completeness for first-order logic (Henkin in J. Symb. Log. 14(3):159–166, 1949), and the second one, devoted to completeness in type theory (Henkin in J. Symb. Log. 15(2):81–91, 1950). In 1963, Henkin published a completeness proof for propositional type theory (Henkin in J. Symb. Log. 28(3):201–216, 1963), where he devised yet another method not directly based on his completeness proof for the whole theory of types.

In this paper, these tree proofs are analyzed, trying to understand not just the result itself but also the process of discovery, using the information provided by Henkin in Bull. Symb. Log. 2(2):127–158, 1996.

In the third section, we present two completeness proofs that Henkin used to teach us in class. It is surprising that the first-order proof of completeness that Henkin explained in class was not his own but was developed by using Herbrand's theorem and the completeness of propositional logic. In 1963, Henkin published *An extension of the Craig–Lyndon interpolation theorem*, where one can find a different completeness proof for first-order logic; this is the other completeness proof Henkin told us about.

We conclude this paper, by introducing two expository papers on this subject. Henkin was an extraordinary insightful professor, and in 1967, he published two works that are very relevant for the subject addressed here: *Truth and provability* (Henkin in Philosophy of Science Today, pp. 14–22, 1967) and *Completeness* (Henkin in Philosophy of Science Today, pp. 23–35, 1967).

Keywords Henkin · Truth · Provability · Completeness · Type theory · First-order logic · Propositional type theory · Equality · Interpolation · Craig · Herbrand

1 Introduction

Henkin published two papers in the *Journal of Symbolic Logic*: the first, *The completeness of the first order functional calculus* [6] in 1949, and the second, *Completeness in type theory* [7] in 1950. *A theory of propositional types* [11] was published in *Fundamenta Mathematicae* in 1963.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_12

In this paper, we analyze these three proofs, trying to understand not just the result itself but also the process of discovery, using the information provided by Henkin in *The discovery of my completeness proofs* [16], published in 1996 in *Bulletin of Symbolic Logic*.[1]

We begin our work by pointing out some of Henkin's stated influences, especially three of them: (1) Gödel's completeness theorem, as well as his article on the consistency of the axiom of choice (where he builds a constructible universe), (2) Russell's theory of types and his expository explanation of the axiom of choice, and (3) Church's formulation of the theory of types and the important role played by both the lambda operator and the description operators in foundational issues.

The next section is devoted to Henkin's three completeness theorems, mentioned above, published in 1949, 1950 and 1963. In the first subsection, we focus on the unexpected discovery process and the role played by the lambda operator and the particular formulation of the axiom of choice. The model he builds is interpreted from a *nominalistic* point of view, following Henkin's papers [8] and [10]. In the second subsection, we see how the method used in the completeness proof for type theory is modified to obtain a completeness result for first-order logic. Moreover, we compare his proof with what today is referred to as *Henkin's method*. In the last subsection, devoted to completeness in propositional type theory, we try to answer several questions, among them: *why was Henkin interested in such a theory?*, *why a new method of proof? Can completeness for propositional type theory be derived from the already known completeness proof for type theory or for first-order logic also developed by Henkin?* We show that in this proof, the nominalistic position is more revealing than ever.

In the next section we address two completeness proofs that Henkin taught in class. The story behind this is that of María Manzano, who during the academic year of 1977–1978 attended his class of *metamathematics* for doctorate students at Berkeley. It is surprising that the first-order proof of completeness that Henkin explained in class was not his own but was developed by using Herbrand's theorem and the completeness of propositional logic. In 1963, Henkin published the paper entitled *An extension of the Craig–Lyndon interpolation theorem* [11], where one can find a different proof of completeness for first order logic; this was the other completeness proof Henkin taught in class.

We conclude this paper introducing two expository papers on this subject. Henkin was an extraordinary insightful professor as regards the clarity of his expositions, and he devoted some effort to writing informative papers. In particular, in 1967, he published two papers that are very relevant for the subject broached here: *Truth and provability* [13] and *Completeness* [14].

2 Henkin's Declared Influences

When Henkin was a student of Alonzo Church, the weak completeness theorem of first-order logic had the formulation given by Gödel [4] and the method of proof was based on a reduction to the propositional case.

[1] The paper was dedicated to his maestro Alonzo Church on the occasion of his 91 birthday; it was to be a book chapter, but the book was never published.

> In the presentation of Gödel's completeness proof, emphasis was given to its reductive character: the provability of a logically valid formula is reduced first to the provability of its Skolem normal form, an then to the provability of some tautology in a specific set of propositional formulas.
> (Henkin [16, p. 132])

In Gödel, we also find a strong completeness result, which is obtained by using weak completeness and compactness.

> THEOREM IX. *Every denumerably infinite set of formulae of the restricted functional calculus either is satisfiable* (that is, all formulae of the system are simultaneously satisfiable) *or possesses a finite subsystem whose logical product is refutable.*
> IX follows immediately from:
> THEOREM X. *For a denumerably infinite system of formulae to be satisfied it is necessary and sufficient that every finite subsystem be satisfiable.*
> (Gödel [4, p. 119])

A very similar version of this theorem IX is directly proved by Henkin in his doctoral thesis in 1947, both for type theory and for first-order logic. The method of proof is the main contribution of Henkin's thesis, and it relies on the effective building of a model satisfying the formulas in the consistent set. In the completeness proof for type theory, the clue lay in the lambda operator's ability to define a constructible hierarchy, combined with the description operator's ability to provide *formal beings*. To this effect, the reading of Gödel's monograph on the consistency of the axiom of choice and the generalized continuum hypothesis inspired Henkin.

> I admired the metamathematical treatment whereby the comprehension schema of set formation is obtained from finitely many axioms, and the sophisticated handling of inner-model constructions by means of the notion of the 'absoluteness' of various set-theoretical notions. I was intrigued by the creation of a universal choice function in the realm of constructible sets.
> (Henkin [16, p. 131])

Gödel's formulation of the theorem on the consistency of the axiom of choice has this form.

> THEOREM Let T be a system of axioms for set theory obtained from v. Neumann's system S^* by leaving out the axiom of choice (Ax. III3*); then, if T is consistent, it remains so, if the following propositions 1–4 are adjoined simultaneously as new axioms:
> 1. The axiom of choice (i.e., v. Neumann's Ax. III3*)
> 2. The generalized Continuum-Hypothesis (i.e., the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds for any ordinal α)
> [...]
> A corresponding theorem holds, if T denotes the system of Prin. Math. or Fraenkel's system of axioms for set theory, leaving out in both cases the axiom of choice but including the axiom of infinity.
> (Gödel [5, p. 556])

The model Gödel provides in this proof consists of *'all "mathematically constructible" sets, where the term "constructible" is to be understood in the semiintuitionistic sense which excludes impredicative procedures. This means "constructible" sets are defined to be those sets which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders'*.[2]

Gödel also explains that the proposition *"Every set is constructible"*—formulated as A—can be proved to be consistent with the axioms in T because is true in a model consisting of the constructible sets. Not only that, but when added to T as an axiom *'seems*

[2]See [5, p. 556].

to give a natural completion of the axioms of set theory, in so far it determines the vague notion of an arbitrary infinite set in a definite way'.[3] For Gödel, it is very important that the consistency of A prevails even when inaccessible numbers are admitted. *'Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning of this phrase'.*[4]

Among the influences of his thesis director, we believe that the lambda abstractor was philosopher's stone not just for Alonzo Church[5] but also for Henkin, as we will explain later on.

Church created the lambda calculus using functions as the basic concept, and he made a clear distinction between the value of a function for a given argument, $F(x)$, and the function itself, $\lambda x F(x)$. Functional abstraction allows the naming of functions in the language. Definability was then a hot topic, but we should also keep in mind that Church's thesis, identifying effective calculable functions with those definable by the λ operator had already been formulated, and Henkin was one of Church's students.

Henkin was immensely attracted by Church's language for the theory of types,[6] in particular, by the lambda operator of functional abstraction acting on types, λ.

> The class of *type symbols* [...] is the least class of symbols which contains the symbols ι and 0 and is closed under the operation of forming the symbol $(\alpha\beta)$ from the symbol α and β. [...] 0 being the type of propositions, ι the type of individuals, and $(\alpha\beta)$ the type of functions of one variable for which the range of independent variable comprises the type β and the range of the depended variable is contained in the type α.
> [...]
> Certain formulas are distinguished as being *well-formed* and as having a certain *type*, in accordance with the following rules: (1) a formula consisting of a single proper symbol is a well-formed formula and has the type indicated by the subscript; (2) if x_β is a variable with subscript β and M_α is a well formed formula of type α, then $(\lambda x_\beta M_\alpha)$ is a well-formed formula of type $\alpha\beta$; (3) if $F_{\alpha\beta}$ and A_β are well-formed formulas of types $\alpha\beta$ and β, respectively, then $(F_{\alpha\beta} A_\beta)$ is a well-formed formula of type α.
> (Church [2, pp. 56–57])

Another interesting operator of Church's language was $\iota_{a(0a)}$; these symbols were introduced to play the role of selection operators whose interpretations should be choice functions. The formula $(\iota_a B_0)$ is meant to be an abbreviation for $(\iota_{a(0a)}(\lambda x_a B_0))$ and it functions like the English word "the". This operator provides a very succinct formulation of both the axiom of descriptions

$$9^a.\ f_{0a}x_a \supset .\big[(y_a)(f_{0a}y_a \supset x_a = y_a)\big] \supset f_{0a}(\iota_{a(0a)} f_{0a})$$

and the axiom of choice

$$11^a.\ f_{0a}x_a \supset f_{0a}(\iota_{a(0a)} f_{0a})$$

that Henkin enjoyed very much. Not only choice and descriptions axioms are included in the calculus, but also infinity and extensionality

$$10^{a\beta}.\ (x_\beta)[f_{a\beta}x_\beta = g_{a\beta}x_\beta] \supset f_{a\beta} = f_{a\beta}$$

[3] See [5, p. 557].

[4] See [5, p. 557].

[5] This idea is developed with some details in [19] and [21].

[6] As presented in [2].

These axioms are added to the proper logical axioms in order to obtain classical mathematical theories: for elementary number theory, it is necessary to add the axioms of description and infinity; to obtain real number theory (analysis), also extensionality and choice are appended.[7] We may wonder *'what are they doing in this formal deductive system?* [...] *The answer is that Church wished to show how a logistic system can be applied to provide a foundation of mathematics, or at least Peano arithmetic and real analysis'*.[8] In fact, a large section of Church's paper is devoted to proving that Peano's postulates of arithmetic are theorems of this calculus. Eleven axioms and some rules are introduced for this calculus; among its rules, the λ-conversion ones play the relevant role:

> II. To replace any part $((\lambda x_\beta M_\alpha)N_\beta)$ of a formula by the result of substituting N_β for x_β throughout M_α, provided that the bound variables of M_α are distinct both from x_β and from the free variables of N_β.
> III. Where A_α is the result of substituting N_β for x_β throughout M_α, to replace any part A_α of a formula by $((\lambda x_\beta M_\alpha)N_\beta)$, provided that the bound variables of M_α are distinct both from x_β and from the free variables of N_β.
> (Church [2, p. 60])

At that time, type theory was a strong candidate for being a formal foundation for logic and mathematics since Russell had eliminated the main paradoxes by identifying the source of contradictions and then provided a language where the vicious circle is avoided:

> In all the above contradictions (which are merely a selection from an indefinite number) there is a common characteristic, which we may describe as self-reference or reflexiveness. [...] Thus all our contradictions have in common the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself.
> (Russell [28, pp. 154–155])[9]

Henkin declares in [16] that in 1938, as a second year student in Columbia University, in Russell's book he found an exciting formulation of the axiom of choice:

> [...] let me to browse in Bertrand Russell's Principles of Mathematics, [...] It was in that book that I first read about the principle of choice. I was enormously impressed by Russell's example of a shoe store with infinitely many pairs of shoes and socks: How easy is to specify one shoe from each pair in the shop [...] and how seemingly impossible is to specify one sock from each pair!
> (Henkin [16, pp. 128–129])

3 Henkin's Completeness Papers

As we have already mentioned, a few years after the defense of his thesis, Henkin published two papers in the *Journal of Symbolic Logic*: the first, on completeness for first-order logic [6] and the second devoted to completeness in type theory [7]. In the 1950 paper, completeness is formulated as *'Theorem 1: If Λ is any consistent set of cwffs, there is a general model (in which each domain D_α is denumerable) with respect to which Λ is satisfiable'*.[10] In the 1949 paper, it has the following form (for a calculus without the

[7] As you can see in [2, p. 61].

[8] See [16, p. 146].

[9] These pages refer to the reprint [28].

[10] See [7, p. 85].

equality symbol): THEOREM *'If Λ is a set of formulas of S_0 in which no member has any occurrence of a free individual variable, and if Λ is consistent, then Λ is simultaneously satisfiable in a domain of individuals having the same cardinal number as the set of primitive symbols of S_0'*.[11]

Having such a similar statement with Gödel's formulation, we might wonder what is new in Henkin's completeness theorems. The obvious answer is that the method itself is completely new, and the main difference is that Henkin built the model instead of reducing the problem to the completeness of propositional logic via Skolem normal forms.

In the introduction to his 1949 paper, Henkin said:

> [...] the new method of proof which is the subject of this paper possesses two advantages. In the first place an important property of formal systems which is associated with completeness can now be generalized to systems containing a non-denumerable infinity of primitive symbols. While this is not of especial interest when formal systems are considered as *logics*—i.e., as means of analyzing the structures of languages—it leads to interesting applications in the field of abstract algebra. In the second place the proof suggests a new approach to the problem of completeness for functional calculi of higher order.
> (Henkin [6, p. 159])

It is interesting to recall that the publication order is the reverse of the discovery of the proofs. The completeness for first-order logic was accomplished when he realized that he could modify the proof obtained for type theory in an appropriate way. We consider this to be of great significance because the effort of abstraction needed for the first proof (that of type theory) provided a broad perspective that allowed him to leave apart some prejudices and to make the decisive changes needed to reach his second proof.

3.1 Completeness in the Theory of Types

The theorem of completeness establishes the correspondence between deductive calculus and semantics. Gödel had solved it positively for first-order logic and negatively for any logical system able to contain arithmetic. The lambda calculus for the theory of types, as presented in [2], with the usual semantics over a standard hierarchy of types, was able to express arithmetic and hence could only be incomplete. Henkin showed that if the formulas were interpreted in a less rigid way, accepting other hierarchies of types that did not necessarily have to contain all the functions but at least did contain the definable ones, then it is easily seen that all consequences of a set of hypotheses are provable in the calculus. The valid formulas with this new semantics, called general semantics, are reduced to coincide with those generated by the calculus rules.

Henkin's proof that every consistent set of formulas has a model is performed by a constructive building of the model. Surprisingly, the model uses the expressions themselves as objects; in particular their elements are equivalence classes of closed expressions, the equivalence relationship being that of formal derivability of equality. We shall approach this construction while trying to figure out what Henkin had in mind and why he ended up defining the general models. On the one hand, we know that Henkin defended a Nominalistic position in several of his writings ([8] and [10]) well in harmony with his general

[11] See [6, p. 162].

models, even though we are not sure whether he had maintained this Nominalistic position early in the 1940s or whether instead it was his vision in the 1950s, as—so to speak—an inspired afterthought. What is certain is that he was interested in the elements of the hierarchy of types that possesses a name in type theory, as Henkin himself explained to us in [16]. On the other hand, the axiom of choice played a crucial role in the discovery.

Steps Towards the Discovery

We are very lucky because Henkin wrote a very interesting paper [16] telling us of his discovery process. We will like to concentrate on the hierarchy of types and on its inner hierarchy of nameable types; the second hierarchy was the source of his discovery and, apparently, it maintains a relationship with the former similar to the relationship of Gödel's constructible hierarchy with Zermelo's hierarchy of sets. We shall pinpoint two flashes that illuminated Henkin in the process of manipulating types inside the hierarchy.

Hierarchy of Types The types are structured in a hierarchy that has the following as basic types: (1) $\mathcal{D}_\iota$ is a nonempty set; that of individuals of the hierarchy, (2) $\mathcal{D}_0$ is the domain of truth values (since we are in binary logic, these values are reduced to T and F). The other domains are constructed from the basic types as follows: if $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ have already been constructed, we define $\mathcal{D}_{(\alpha\beta)}$ as the domain formed by all the functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$.

To talk about this hierarchy, Church's formal language of [2] is introduced. Henkin said: *'I decided to try to see just which objects of the hierarchy of types did have names in $\mathcal{T}$'*.[12] That is, he intended to mentally represent the functions in the hierarchy of types that can be named by lambda expressions. To visualize this, he considered a hierarchy of types with a universe given by the set of natural numbers and a language with two constants: one as the name for the zero and another for the successor function. In the universe of subsets of the universe of individuals, there will clearly be both objects with and without a name because with a countable infinite universe of individuals the set of subsets is uncountable but the sets with a name are countable.

Hierarchy of Nameable Types These peculiar elements of the hierarchy can be named by closed expressions. For each type α, the set $\mathcal{D}^n_\alpha$ contains the nameable elements,

$$\mathcal{D}^n_\alpha = \left\{ f \in \mathcal{D}_\alpha : \text{ there is a } F_\alpha \in \mathit{cwff} \text{ s.t. } \Im(F_\alpha) = f \right\}$$

(where $\Im$ stands for the interpretation of the formal language in the hierarchy of types).

Going up in the hierarchy, it is easy to see that the nameable domain $\mathcal{D}^n_{(\alpha\beta)}$ of type $(\alpha\beta)$ is such that each $f \in \mathcal{D}^n_{(\alpha\beta)}$ allows a map from $\mathcal{D}^n_\beta$ to $\mathcal{D}^n_\alpha$ to be defined. Even though each function in $\mathcal{D}^n_{(\alpha\beta)}$ has $\mathcal{D}_\alpha$ rather than $\mathcal{D}^n_\alpha$ as its range—that is, $f : \mathcal{D}_\beta \longrightarrow \mathcal{D}_\alpha$—we can see that the value of nameable elements in the domain $\mathcal{D}^n_\beta$ is also a nameable element in the domain $\mathcal{D}^n_\alpha$. The reason is as follows: for any $f \in \mathcal{D}^n_{(\alpha\beta)}$, we know that there is an $F_{\alpha\beta}$ such that $\Im(F_{\alpha\beta}) = f$ and this function provides any $g \in \mathcal{D}^n_\beta$ a value $f(g)$ in $\mathcal{D}^n_\alpha$

[12] See [16, p. 146].

(since g has the form of $\Im(G_\beta) = g$ and so $f(g) = \Im(F_{\alpha\beta})(\Im(G_\beta)) = \Im(F_{\alpha\beta}G_\beta) \in \mathcal{D}^n_\alpha$). Therefore, we can replace each $f \in \mathcal{D}_{(\alpha\beta)}$ by the restriction of f to $\mathcal{D}^n_\beta$:

$$\mathcal{D}^{n*}_{(\alpha\beta)} = \left\{ f^* : f \in \mathcal{D}^n_{(\alpha\beta)} \text{ and } f^* = f \upharpoonright \mathcal{D}^n_\beta \right\}.$$

Henkin wanted to know if this restricted class itself formed a hierarchy: *'There was, however, a problem with this idea: What if the hierarchy contracted under the proposed reduction of the domains of functions? In other words, could there be distinct functions f and g in some $D^n_{(\alpha\beta)}$ such that $f^* = g^*$?'*[13]

The answer to this question is *no*, simply because when f and g are in $\mathcal{D}^n_{(\alpha\beta)}$, then $\Im(F_{\alpha\beta}) = f$ and $\Im(G_{\alpha\beta}) = g$ for some cwff. Assume that $f \neq g$. Let us take $X_{0\beta}$ as the lambda expression $\lambda x_\beta \sim (F_{\alpha\beta}x_\beta = G_{\alpha\beta}x_\beta)$ representing the set of elements that give different values under the functions involved. By using the axiom of choice we see that if $f \neq g$, then $\Im(X_{0\beta}) \neq \emptyset$, and $\Im(\iota_{\beta(0\beta)}X_{0\beta})$ is an element $y \in \mathcal{D}_\beta$ such that $f(y) \neq g(y)$. Therefore, $f^* \neq g^*$.

The new hierarchy $\mathcal{D}^{n*}_\alpha$ is isomorphic to the previous one $\mathcal{D}^n_\alpha$. Moreover, the hierarchy obeys the rules of lambda conversion because for a function f of type $(\alpha\beta)$ named by a lambda expression, $\Im(\lambda x_\beta N_\alpha) = f$, the value for a given argument $\Im(M_\beta) = m$ is

$$\begin{aligned} f(m) &= \Im(\lambda x_\beta N_\alpha)\big(\Im(M_\beta)\big) \\ &= \Im\big((\lambda x_\beta N_\alpha)M_\beta\big) \\ &\quad \text{and, by lambda conversion,} \\ &= \Im\left(N_\alpha \frac{M_\beta}{x_\beta}\right). \end{aligned}$$

Flash Number One He then realized that he had crossed the bridge that separates the world of semantical models from the world of syntactic deductive systems.

> As I struggled to see the action of functions more clearly in this way, I was struck by the realization that I have used λ-conversion, one of the formal rules of inference in Church's deductive system for the language of the Theory $\mathcal{T}$. All my efforts had been directed toward *interpretations* of the formal language, and now my attention was suddenly drawn to the fact that these were related to the formal deductive system for that language.
> (Henkin [16, p. 150])

Specifically, to identify objects named by both M_α and N_α, he made use of a criterion based on the calculus, namely, the fact that $\vdash (M_\alpha = N_\alpha)$.

> In particular, I saw that using the symbol $\vdash$ for formal provability (or derivability) as usual, we can define for each type symbol α, a domain $\mathcal{D}'_\alpha$ satisfying the following conditions: (i) Each cwff (closed wff, without free variables) M_α denotes an element M'_α of $\mathcal{D}'_\alpha$ and each element of $\mathcal{D}'_\alpha$ is denoted by some cwff M_α; (ii) for any cwff $F_{\alpha\beta}$, $F'_{\alpha\beta}$ is a function mapping $\mathcal{D}'_\beta$ into $\mathcal{D}'_\alpha$; and (iii) for any cwffs M_α and N_α, $M'_\alpha = N'_\alpha$ if, and only if, $\vdash (M_\alpha = N_\alpha)$.
> (Henkin [16, pp. 150–151])

In other words, Henkin saw a way of defining a hierarchy of names modulo equivalent classes in the deductive calculus. The definition of the universes $\mathcal{D}'_\alpha$ was based on

[13]See [16, p. 149].

recursion on types, and the building of $\mathcal{D}'_{\alpha b}$ from $\mathcal{D}'_{\alpha}$ and $\mathcal{D}'_{b}$ required the axiom of choice working in parallel with the constants $\iota_{\alpha(0\alpha)}$ mentioned above. Fortunately, Henkin's previous understanding of this operator was of great help. The construction seemed to work smoothly, with the only exception of the universe of truth values, $\mathcal{D}'_0$. *'In particular, if* M^0 *is a Gödel sentence such that neither* $\vdash M^0$ *nor* $\vdash\sim M^0$*, then* $(0_1 = 0_1)'$, $(\sim 0_1 = 0_1)'$, *and* $M^{0\prime}$ *are three distinct elements of* D'_0*'*.[14]

Flash Number Two He then realized that to reduce the universe of objects named by propositions (the truth values) to only two, the set of axioms had to be expanded until it constituted a maximal consistent set.

> As soon as I observed this, it occurred to me that if we were to add further cwffs of type 0 to the list of formal axioms, this would have the effect of reducing the number of elements in $\mathcal{D}'_0$ and that ultimately, by taking a maximal consistent set of axioms, the number of elements in $\mathcal{D}'_0$ would be two. [. . .] Immediately I realized that my discovery provided a kind of completeness proof for a system very much like the system PM of type theory which Gödel had proved incomplete.
> (Henkin [16, p. 151])

The Proof

Hierarchy of Equivalent Classes of Names On this occasion, equivalent classes of closed formulas (cwffs) rather than proper objects are used to build the hierarchy. A maximal consistent set Γ is needed and the new equivalence relation is defined as: Two cwffs M_α and N_α of type α will be called *equivalent* if $\Gamma \vdash M_\alpha = N_\alpha$. On p. 86 of *Completeness in the theory of types* [7], Henkin says:

> We now define by induction on α a frame of domains $\{D_\alpha\}$, and simultaneously a one-one mapping Φ of equivalent classes onto the domains D_α such that $\Phi([A_\alpha])$ is in D_α.
> D_0 is the set of two truth values and $\Phi([A_0])$ is T or F according as A_0 or $\sim A_0$ is in Γ
> [. . .]
> D_ι is simply the set of equivalence classes $[A_\iota]$ of all cffs of type ι. And $\Phi([A_\iota])$ is $[A_\iota]$
> [. . .]
> Now suppose that D_α and D_β have been defined, as well as the value of Φ for all equivalence classes of formulas of type α and β and that every element of D_α, or D_β, is the value of Φ for some $[A_\alpha]$ or $[B_\beta]$ respectively. Define $\Phi([A_{\alpha\beta}])$ to be the function whose value, for the element $\Phi([B_\beta])$ of D_β is $\Phi([A_{\alpha\beta} B_\beta])$.

He has to show that Φ is a function on equivalent classes and does not depend on the particular representative chosen, and also that the function is one-to-one. In the inductive step, to prove that $\Phi([A_{\alpha\beta}]) = \Phi([B_{\alpha\beta}])$ implies $[A_{\alpha\beta}] = [B_{\alpha\beta}]$, he uses choice and extensionality in a similar way as he did when building the hierarchy of nameable types; in particular, he uses the following theorem:

$$\begin{aligned}&\vdash A_{\alpha\beta}\big(\iota_{(\beta(0\beta))}\big(\lambda x_\beta\big(\sim (A_{\alpha\beta}x_\beta = B_{\alpha\beta}x_\beta)\big)\big)\big)\\ &\quad = B_{\alpha\beta}\big(\iota_{(\beta(0\beta))}\big(\lambda x_\beta\big(\sim (A_{\alpha\beta}x_\beta = B_{\alpha\beta}x_\beta)\big)\big)\big) \supset .A_{\alpha\beta} = B_{\alpha\beta}\end{aligned}$$

[14]See [16, p. 151]. In this paper, the type of individuals is 1, that is why he writes $(0_1 = 0_1)'$ instead of $(0_\iota = 0_\iota)'$.

We might wonder what the elements of D_ι are, since we do not have individual constants. Of course, the selection operator $\iota_{\beta(0\beta)}$ acting on expressions of the appropriate type, say $X_{0\beta}$, produces elements of any type β.

Using this construction, Henkin was able to achieve his completeness result:

Theorem 1 (Henkin [7, p. 85]) *'If Λ is any consistent set of cffs* (*sentences*), *there is a general model* (*in which each domain $\mathcal{D}_\alpha$ of $\mathcal{M}$ is denumerable*) *with respect to which Λ is satisfiable'.*

To prove this theorem, the set Λ is extended to a maximal consistent set Γ, which serves both as an oracle and as building blocks for the model, the following lemma being the relevant step.

Lemma 2 (Henkin [7, p. 87]) *'For every ϕ and B_β, we have $V_\phi(B_\beta) = \Phi([B_\beta^\phi])$'.*

The Definition of General Model As we have seen, Henkin in his proof uses a hierarchy with countable domains whose elements are equivalent classes of names, *is it legitimate?*, *what about the definition of general models?* He introduces first the definition of a broader class of structures:[15] *'By a frame, we mean a family of domains, one for each type symbol, as follows: D_ι is an arbitrary set of individuals, D_0 is the set of two truth values, T and F, and $D_{\alpha\beta}$ is some class of functions defined over D_β with values in D_α'.*

Inside this class, the general models are placed: *'A frame such that for every assignment ϕ and wff A_α of type α, the value $V_\phi(A_\alpha)$ given by the rules* (i), (ii), and (iii) *is an element of D_α is called a general model'*. Thus, general models are frames characterized by being able to provide interpretations for any expression. However, *is that a proper definition?* Henkin is aware of the fact that *'Since this definition is impredicative, it is not immediately clear that any non-standard models exist. However, they do exist* [...]*'* but he does not seem to mind too much because immediately afterwards he goes on to prove Theorem 1 above, where such a model is constructed. Moreover, right at the beginning of the 1950 paper, in a footnote, Henkin declares that in the second-order case, the universes must be closed under certain operations and gives as examples that of complementation and projection, but he does not give an algebraic definition.[16]

Nominalism The models he builds are in accordance with a Nominalistic position, as Henkin himself affirms in *On nominalism*, published in 1953: *'In fact, such an interpretation is implicit in a recent paper discussing the problem of the completeness of the higher-order functional calculi'*.[17] In 1955, Henkin gave a lecture at the Société Belge de Logique et de Philosophie des Sciences, where he synthesized this position as follows:

> [...] the thesis of Nominalists that there is only one kind of objects—physical objects—and that no kind of abstract object can reasonably be asserted to exist. The problem posed by this thesis is that of reinterpreting familiar language, especially mathematical language, which under the ordinary interpretation has reference to many kinds of abstract entities.
> (Henkin [10, p. 137])

[15] The quotes in the following paragraph all belong to: Henkin [7, p. 85].

[16] Such a definition can be given; for type theory I gave one in [17] and it is also included in the book [18].

[17] See [9, p. 22].

He proposes *'a single realm of individuals'*, but he is aware of the problem posed by the so-called *universals*, '*which have been variously interpreted as denoting* Platonic ideas, *universal attributes or* properties, *and more recently simply* classes*'*.[18] The traditional Nominalistic view is that these words are not names of anything but are used to specify something about physical objects. In first-order logic, such a Nominalist position seems to pose no problem to a Tarskinian interpretation of logical language. In second-order logic, *'it is natural to reinterpret class* variables *as symbols for which predicates can be substituted'*.[19] $\forall X\varphi$ is interpreted as being true for every replacement of X in φ by a predicate; this predicates are provided by auxiliary languages. Since there are many possible auxiliary languages, the interpretation is not unique. He also explains that there are auxiliary languages that provide acceptable interpretations in the sense of being in accordance with the rules of inference. *'Furthermore we can show that the sentences which are formally provable are precisely those sentences which are true under all the proposed interpretations'*.[20] One of the problem such a nominalistic position must face is the interpretation of the formula S

$$\neg\exists F\forall G\exists x\forall y(Fxy \leftrightarrow Gy)$$

whose classical interpretation is that no one-to-one and onto function maps the totality of individuals of a domain with the class of all sets of these individuals. Cantor proved that the formula S is a theorem of classical mathematics; accordingly, it is also a theorem of a second order logic[21] and therefore should be true. For Quine and Goodman, this argument seems to show up the difficulty involved in finding a Nominalist interpretation of the theory of models of standard mathematics when this is limited to countable magnitudes. Regarding this, Henkin's position is the one expected today if we are using nonclassical interpretation:

> [...] the difficulty is only illusory. For it is only under the classical interpretation that the sentence S expresses the proposition that sets are more numerous than individuals. It may well happen that under a nominalistic reinterpretation of the language the sentence S continues to be true but comes to mean something else.
> (Henkin [10, p. 139])

Currently, it is clear that we have to choose between expressive power or complete calculus; in the latter case, the old ghost of Skolem's paradox has returned and we obtain nonstandard models of arithmetic, as Henkin explains at the end of his 1950 paper: *'The Peano axioms are generally thought to characterize the number-sequence fully in the sense that they form a categorical axiom set any two models for which are isomorphic. As Skolem points out, however, this condition obtains only if "set" is interpreted with its standard meaning'*.[22,23]

[18] See [10, p. 138].

[19] See [10, p. 139].

[20] See [10, p. 140].

[21] Recall that at that time logicians include in the logic a bunch of axioms that allow the formulation of natural numbers and even real numbers.

[22] See [7, p. 89].

[23] In *The little mermaid* [20], we ended the paper, devoted to second order logic, saying:

3.2 The Completeness of the First-Order Functional Calculus

It seems natural to think that Henkin's completeness theorem for first-order logic was proved before the completeness for type theory since often we obtain a result for a weaker logic and then try to extend it to a stronger logic. Moreover, if we look at the publication data of this theorem in the JSL, that is precisely the order. Additionally, Henkin himself declares in his 1949 paper: *'In the second place the proof suggests a new approach to the problem of completeness for functional calculi of higher order'*.[24]

Surprisingly, in his 1996 paper he states that he obtained the proof of completeness of first-order logic by readapting the argument found for the theory of types, not the other way around. Henkin declares that after proving completeness for type theory, he wished to extend the previous method and applied it to prove completeness for first-order logic. It was clear that to do so he had to get rid off the axiom of choice; in particular, Church's elegant formulation using the selector operator. As we have already explained, this axiom plays a relevant role in the construction of the hierarchy.

> But when I wrote down details of the proof [...], I saw that the axiom of choice is needed there in a more general way [...] to show that whenever we have a wff M such that $\vdash (\exists x_b)M_0$, then we also have $\vdash (\lambda x_b M_0)(\iota_{b(0b)}(\lambda x_b M_0))$. The fact that this condition holds is a direct consequence of having Axiom Schema 11^b [...], that schema is trivially equivalent to $(\exists x_b f_{0b} x_b) \supset f_{0b}(\iota_{b(0b)} f_{0b})$. It did not take me very long to notice that, in fact, the form of the wff following $(\lambda x_b M_0)$ played no role in the completeness proof; it is only necessary to have some cwff N_b such that $\vdash (\lambda x_b M_0)N_b$ holds if $\vdash (\exists x_b)M_0$ holds.
> (Henkin [16, p. 152])

That is why he extends the consistent set Λ not just to a maximal consistent set, but to one containing witnesses.

> It is easy to see that Γ_ω possesses the following properties:
>
> (i) Γ_ω is a maximal consistent set of cwffs of S_ω.
> (ii) If a formula of the form $(\exists x)A$ is in Γ_ω, then Γ_ω also contains a formula A' obtained from the wff A by substituting some constant u_{ij} for each free occurrence of the variable x.
>
> (Henkin [6, p. 163])

The model is built using the set Γ_ω as an oracle. The universe of the model is the set of constants, and the relation symbols are interpreted as n-ary relations on this universe, according to what our oracle declares.

> In fact we take as our domain I simply the set of individual constants of S_ω and we assign to each such constant (considered as a symbol in an interpreted system) itself (considered as an individual) as denotation.
> [...]
> Every propositional symbol, A, of S_0 is a cwff of S_ω; we assign to it the value T or F according as $\Gamma_\omega \vdash A$ or not. Let G be any functional symbol of degree n. We assign to it the class of those n-tuples $\langle a_1, \ldots, a_n\rangle$ of individual constants such that $\Gamma_\omega \vdash G(a_1, \ldots, a_n)$.
> (Henkin [6, p. 163])

> It is clear that you can have both: expressive power plus good logical properties. You cannot be a mermaid and have an immortal soul.
> [...]
> And the little mermaid got two beautiful legs (with a lot of pain, as you might know). But even in stories everything has a price; you know, the poor little mermaid lost her voice.

[24] In [6, p. 159].

If we compare this proof with the one we use today, then the main difference is that in Henkin's original proof the extension of the language to one with enough witnesses is not done at once—that is, in an infinite succession of steps—but in an infinite succession of infinite steps.

In this paper, he also extends the result for a language with a set of primitive symbols of any cardinality. He ends the paper introducing a language with equality and proving a completeness theorem for it, using an equivalence relation on terms to build the universe.

Leaving aside the difference already mentioned, in the form of extending Γ to Γ_ω, let us analyze in some detail what the differences are between his proof in [6] and the standard one we use nowadays, following what we usually identify as *Henkin's strategy*. We accept that the important issue is to be able to show that each consistent set of formulas has a model, and hence, that syntactic consistency and semantical satisfiability are equivalent (soundness assumed). For a countable language, the proof is performed in two steps:

1. Every consistent set of formulas is extended to a maximal consistent set with witnesses.
2. Once we have the maximal consistent set with witnesses, we use it as a guide to build the precise model the formulas of this set are describing. This is possible because a maximally consistent set is a very detailed description of a structure.

Completeness theorem is proved in its strong sense, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$ for any Γ, φ such that $\Gamma \cup \{\varphi\} \subseteq \mathsf{Sent}(L)$. One prove completeness and its corollaries following the path:

Lindenbaum's lemma + Henkin's lemma } → Henkin's theorem ↗ Compactness, ↘ Löwenheim–Skolem

Henkin's theorem
↓
Strong completeness
↓
Weak completeness

These theorems are understood as follows:

- *Lindenbaum lemma:* If $\Gamma \subseteq \mathsf{Sent}(L)$ is consistent, then there exists Γ^* such that $\Gamma \subseteq \Gamma^* \subseteq \mathsf{Sent}(L^*)$, Γ^* is maximally consistent and contains witnesses.
- *Henkin's lemma:* If Γ^* is a maximally consistent set of sentences and contains witnesses, then Γ^* has a countable model.
- *Henkin's theorem:* If $\Gamma \subseteq \mathsf{Sent}(L)$ is consistent, then Γ has a model whose domain is countable.
- *Strong completeness:* If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$ for any $\Gamma \cup \{\varphi\} \subseteq \mathsf{Sent}(L)$.
- *Weak completeness:* If $\models \varphi$, then $\vdash \varphi$ for any $\varphi \in \mathsf{Sent}(L)$.
- *Compactness theorem:* Γ has a model iff every finite subset of it has a model for any $\Gamma \subseteq \mathsf{Sent}(L)$.
- *Löwenheim–Skolem:* If Γ has a model, then it has a countable model for any $\Gamma \subseteq \mathsf{Sent}(L)$.

Of course, for a noncountable language the theorems are modified accordingly. Also, the schema can be modified to be able to prove a similar result for open formulas, not just for sentences.

If we have a closer look at Henkin's proof in [6], we see that, there, only one result is labeled "theorem"; there is also one lemma and two corollaries. His *theorem* is what we have called here *Henkin's theorem*, and the formulation is the one given here in Sect. 3, p. 154; he never mentions neither proves what we call *strong completeness*, certainly a too easy corollary to be mentioned in the *JSL*. It seems that in his thesis, he used this denomination *strong completeness* for what we are calling here *Henkin's theorem*. In the following argument, we can see why he adopted such a decision:

> Gödel *used* completeness to prove the statement given in our Theorem I, which I have called *strong* completeness. This nomenclature is justified because it is trivial to restate Theorem I in the following form: If S is any set of $\mathcal{L}$-sentences and r is any logical consequence of S (i.e., r is satisfied in every $\mathcal{L}$-structure that satisfies all sentences of S), then r is formally derivable from S (using the formal axioms and rules of inference of $\mathcal{L}$). When Theorem I is formulated in this way, Corollary I becomes a *special case* of Theorem I (the case where S is empty), so if the corollary expresses completeness, we can say that the theorem expresses strong completeness.
> (Henkin [16, p. 135])

What we are calling here *Henkin's lemma* has in his paper this form: *'For each A cwff of* S_ω *the associated value is T or F according as* $\Gamma_\omega \vdash A$ *or not'*.[25] The content of the two corollaries corresponds to what we are calling *weak completeness* and *Löwenheim–Skolem* and Henkin's denomination for them was *completeness* and *Skolem–Löwenheim*. There is no mention of compactness theorem, even thought Henkin told us in [16] that it was part of his thesis:

> The remaining two corollaries of Theorem I are as follows. [...] Corollary III: A set S of wffs of $\mathcal{L}$ is simultaneously satisfied in some $\mathcal{L}$-structure $\mathcal{M}$ if, and only if, each finite subset S_1 of S is satisfied in some $\mathcal{L}$-structure $\mathcal{M}_1$. This result is now called the *compactness property* of first-order logic, and has become one of the principal tools of model theory. The compactness property was not part of Gödel's dissertation [3], but was added in the version written for publication [4].
> (Henkin [16, p. 135])[26]

3.3 A Theory of Propositional Types

In 1963, Henkin published *A theory of propositional types* [12], where he presented a completeness proof for this theory. In this paper, he devised yet another method not directly based on his completeness proof for the whole theory of types.

Before we start giving a brief account of this proof, let us pose a few introductory questions. First are the questions about the nature of propositional type theory and its formal language, *What types are there?*, *Which language is used to deal with them?* The second is *why was Henkin interested in such a theory?* The third is about this specific completeness theorem, *why a new method of proof? Couldn't completeness for propositional type theory be derived from the already known completeness proof for type theory or for first-order logic also developed by Henkin?*

At the beginning of the paper, Henkin answers our last question:

[25] In [6, p. 163].

[26] Gödel papers are [3] and [4].

> The completeness of a theory of types in terms of non-standard models was proved in [7], but this result does not seem to imply our present completeness theorem. It is true that by adding suitably to the earlier proof the present result can be obtained, but such a proof would not have the constructive character possessed by the usual completeness proofs for propositional logic, and we have preferred therefore to indicate another method of proof which seems more appropriate for a theory of types each of which is finite.
> (Henkin [12, p. 324])

He also states what his motivation was:

> Our interest was drawn to a theory of propositional types by the problem of constructing non-standard models of a full theory of types. Since many problems of ordinary predicate logic can be reduced to questions about propositional logic (as in Herbrand's theorem, for example), our hope has been that insight into the totality of models for a full theory of types could be obtained from a study of all models of the much simpler propositional type theory.
> (Henkin [12, pp. 324–325])

Henkin was certainly also interested in developing a logic with lambda and equality as the sole primitives.

Henkin announces another paper we were unable to find: *'We reserve for a future paper, however, a discussion of the models of our present system other than the standard model PT of propositional types'*.[27]

To answer the first of our questions, we present the hierarchy of propositional types as well as the language and its semantics.

Hierarchy of Propositional Types According to Henkin's definition, $\mathfrak{PT}$ is the least class of sets containing $\mathcal{D}_0$ as an element, which is closed under passage from $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ to $\mathcal{D}_{\alpha\beta}$. Here $\mathcal{D}_0$ is the two truth values set, $\mathcal{D}_0 = \{T, F\}$, whereas $\mathcal{D}_{\alpha\beta}$ is the set of all functions that map $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$. To give some examples, $\mathcal{D}_{00}$ is the type of functions from $\mathcal{D}_0$ to $\mathcal{D}_0$; one such a function is negation, the other three are the identity function, the constant function with value F, and the constant function with value V. The binary connectives are in $\mathcal{D}_{(00)0}$.

Equational Proposition Type Theory To build the theory of propositional types, Henkin introduces a formal language with variables for each propositional type, the lambda abstractor, λ, and a collection of equality constants, $\mathcal{Q}_{(0\alpha)\alpha}$, one for each type α. To be more specific, expressions of this theory are either: (1) variables of any type X_α, (2) the constants $\mathcal{Q}_{(0\alpha)\alpha}$, (3) $A_{\alpha\beta}B_\beta$, or (4) $\lambda X_\beta B_\alpha$.

Interpretations of these on the hierarchy $\mathfrak{PT}$ are defined recursively using assignments g that give values to variables of all types. In particular, under a given interpretation $\mathfrak{I} = \langle \mathfrak{PT}, g\rangle$, we have: (1) $\mathfrak{I}(X_\alpha) = g(X_\alpha)$, (2) $\mathfrak{I}(\mathcal{Q}_{(0\alpha)\alpha})$ is the identity on type α, (3) $\mathfrak{I}(A_{\alpha\beta}B_\beta)$ is the value of the function $\mathfrak{I}(A_{\alpha\beta})$ for the argument $\mathfrak{I}(B_\beta)$ and (4) $\mathfrak{I}(\lambda X_\beta B_\alpha)$ is the function of $\mathcal{D}_{\alpha\beta}$ whose value for any $\mathbf{x} \in \mathcal{D}_\beta$ is the element $\mathfrak{I}^{\mathbf{x}}_{X_\beta}(B_\alpha)$ of $\mathcal{D}_\alpha$.[28]

In this language, Henkin was able to define all connectives and quantifiers, that is, using only the biconditional $\mathcal{Q}_{(00)0}$ and λ, the remaining connectives and quantifiers $\forall X_\alpha$—for each propositional variable of any propositional type α—are presented as defined operators.

[27] In [12, p. 325].

[28] Here $\mathfrak{I}^{\mathbf{x}}_{X_\beta} = \langle \mathfrak{PT}, g^{\mathbf{x}}_{X_\beta}\rangle$ where $g^{\mathbf{x}}_{X_\beta}$ is an X_β-variant of g.

As we shall see, this language allows not only the aforementioned definition of all logical constants, but is also able to provide a name for each object in the hierarchy. With these names, Henkin offers an interesting completeness theorem, as we shall see in the next section.

Identity as a logical primitive is the title of a paper published in 1975 by Henkin [15]. At the start he declares: *'By the relation of identity we mean that binary relation which holds between any object and itself, and which fails to hold between any two distinct objects'*.[29] Owing to the central role this notion plays in logic, you can be interested either in how to define it using other logical concepts or in the opposite scheme. In the first case, one investigates what kind of logic is required. In the second one, one is interested in the definition of the other logical concepts (connectives and quantifiers) in terms of the identity relation, using also abstraction. In his expository paper, the following question is posed and affirmatively answered: *Can we define with only equality and abstraction the remaining logical symbols?*

Henkin explains that the idea of reducing the other concepts to identity is an old one, which was tackled with some success in 1923 by Tarski [26], who solved the case for connectors; three years later, Ramsey [25] raised the whole subject; it was Quine [24] who introduced quantifiers in 1937. It was finally answered in 1963 by Henkin [12], where he developed a system of propositional type theory (followed by Andrews' improvement [1]). Henkin's 1975 paper is included in a volume of *Philosophia. Philosophical Quarterly of Israel*, completely devoted to identity.

Let us introduce the basic definitions: connectives, quantifiers, and descriptor.

Definition 3 (Defined Operators) Truth and falsity, negation, conjunction and quantifiers are defined operators.

1. $T^n ::= ((\lambda X_0 X_0) \equiv (\lambda X_0 X_0))$ is a sentence of type 0
2. $F^n ::= ((\lambda X_0 X_0) \equiv (\lambda X_0 T^n))$ is a sentence of type 0
3. $\neg^n ::= (\lambda X_0 (F^n \equiv X_0))$ of type (00)
4. $\wedge^n ::= \lambda X_0 (\lambda Y_0 (\lambda f_{00} (f_{00} X_0 \equiv Y_0)) \equiv (\lambda f_{00} (f_{00} T^n)))$ of type (00)0
5. $\forall X_\alpha A_0 := ((\lambda X_\alpha A_0) \equiv (\lambda X_\alpha T^n))$ is a sentence of type 0.

Description Operator In order to treat the description operator properly, one fixes one element for each type; this element would serve as the denotation of improper descriptions. The setting is done by induction on types: for type 0, we just take $\mathbf{a}_0 = F$; for type $(\alpha\beta)$, we take the constant function $\mathbf{f}_{\alpha\beta}$ with value $\mathbf{a}_\alpha$ for every element of $\mathcal{D}_\beta$, where $\mathbf{a}_\alpha$ is the element in $\mathcal{D}_\alpha$ already chosen. Thus, $\mathbf{f}_{\alpha\beta}\mathbf{x} = \mathbf{a}_\alpha$ for each $\mathbf{x} \in \mathcal{D}_\beta$.

Now, using these elements, an election function $\mathbf{t}^{(\alpha)}$ can be defined for each type,

> For any arbitrary type α let $\mathbf{t}^{(\alpha)}$ be the function of $\mathcal{D}_{\alpha(0\alpha)}$ such that, for any $\mathbf{f} \in \mathcal{D}_{0\alpha}$, $(\mathbf{t}^{(\alpha)}\mathbf{f})$ is the unique element $\mathbf{x} \in \mathcal{D}_\alpha$ for which $(\mathbf{fx}) = T$, in case there is such a unique element $\mathbf{x}$, or else $(\mathbf{t}^{(\alpha)}\mathbf{f}) = \mathbf{a}_\alpha$ if there is no $\mathbf{x}$, or if there are more than one $\mathbf{x}$, such that $(\mathbf{fx}) = T$. We shall show inductively that for each α there is a closed formula $\iota_{\alpha(0\alpha)}$ such that $(\iota_{\alpha(0\alpha)})^d = \mathbf{t}^{(\alpha)}$. Then for any formula A_0 and variable X_α, we shall set $(\jmath X_\alpha A_0) = (\iota_{\alpha(0\alpha)}(\lambda X_\alpha A_0))$.
> (Henkin [12, p. 328])

[29] In [15, p. 31].

Completeness of Propositional Type Theory

Now we would like to explain the method Henkin developed in his beautiful proof. The main idea is to use the theory just introduced to give a name to every object in $\mathfrak{PT}$; since the theory of propositional types only uses λ and $\equiv$,[30] the names of all types in the hierarchy are obtained using only lambda and equality. Henkin crosses the bridge between objects of $\mathfrak{PT}$ and formulas of the language in both directions: in one direction, for any $\mathbf{x} \in \mathcal{D}_\alpha$, he introduces a closed expression of the formal language, termed $\mathbf{x}^n$, which acts as a name of it; in the other direction, any closed expression A_α of type α denotes an object $(A_\alpha)^d$ of the domain $\mathcal{D}_\alpha$. As we shall see later, the very important result is that every object $\mathbf{x}$ in $\mathfrak{PT}$ receives as its name a closed expression $\mathbf{x}^n$ of the theory whose denotation is $\mathbf{x}$—namely, $(\mathbf{x}^n)^d = \mathbf{x}$.

Nameability Theorem Names and denotations do match: *'In particular, we shall associate, which each element* $\mathbf{x}$ *of an arbitrary type* D_α*, a closed formula* $\mathbf{x}^n$ *of type* α *such that* $(\mathbf{x}^n)^d = \mathbf{x}$'.[31]

This theorem is proved by induction on the construction on the hierarchy. Names for the basic object T and F of type 0 are given in Definition 3. For type $(\alpha\beta)$, assuming that the theorem is proven for types α and β, we set a name for every function $\mathbf{f}$ that maps every element of the finite type $\mathcal{D}_\alpha$, say $\mathcal{D}_\alpha = \{\mathbf{y}_1, \ldots, \mathbf{y}_q\}$, to the corresponding $\mathbf{f}(\mathbf{y}_i)$ in $\mathcal{D}_\beta$. To this effect, the names of the objects in $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ (whose existence is assumed by induction hypothesis) and the descriptor operator are used. To introduce $\mathbf{f}^n$, we need to formalize the following: when variable X_α is just the name of object $\mathbf{y}_i$—that is, $X_\alpha \equiv \mathbf{y}_i^n$—the function $\mathbf{f}$ matches it to the unique Z_β naming $\mathbf{f}(\mathbf{y}_i)$—that is, $Z_\beta \equiv (\mathbf{f}(\mathbf{y}_i))^n$. In particular,

$$\mathbf{f}^n := \lambda X_\alpha . \jmath Z_\beta . \big[\big(X_\alpha \equiv (\mathbf{y}_1)^n\big) \wedge \big(Z_\beta \equiv \mathbf{f}(\mathbf{y}_1)^n\big)\big] \vee \cdots \vee \big[\big(X_\alpha \equiv (\mathbf{y}_q)^n\big) \wedge \big(Z_\beta \equiv \mathbf{f}(\mathbf{y}_q)^n\big)\big].$$

To be able to prove the nameability theorem the finiteness of the domains is a must as well as the description operator introduced above.

Completeness For the theory of propositional types, Henkin offers a calculus based on λ and equality rules.[32] This calculus is complete. The important result from where the completeness theorem easily follows has the amazing form:

Lemma 4 *For any formula* A_α *and assignment* g,

$$\vdash A_\alpha \frac{(g(X_{\beta_1}))^n \cdots (g(X_{\beta_m}))^n}{X_{\beta_1} \cdots X_{\beta_m}} \equiv \big(\Im(A_\alpha)\big)^n,$$

where $freeVar(A_\alpha) = \{X_{\beta_1} \ldots X_{\beta_m}\}$*, and* $\Im$ *is the interpretation using* $\mathfrak{PT}$ *and* g*—namely,* $\Im = \langle \mathfrak{PT}, g \rangle$.

[30] We would use the symbol $\equiv$ instead of $\mathcal{Q}_{(0\alpha)\alpha}$ for any α.

[31] See [12, p. 326].

[32] This calculus was improved by Andrews [1]. Please read the beautiful paper in this book where Andrews himself tell us the whole personal business involved.

In Henkin's words: LEMMA. *Let A_α be any formula and φ an assignment. Let $A_\alpha^{(\varphi)}$ be the formula obtained from A_α by substituting, for each free occurrence of any variable X_β in A_α, the formula $(\varphi X_\beta)^n$. Then $\vdash A_\alpha^{(\varphi)} \equiv (V(A_\alpha, \varphi))^n$.*[33]

The lemma is proved by induction on the length of A_α.

The obvious question we ask is *how completeness theorem can be derived from this lemma.* That is, *how can we prove that $\models A_0$ implies $\vdash A_0$ for any formula of type 0?*

Proposition 5 *Lemma 4 implies completeness.*

Proof If A_0 is closed, then $\models A_0$ implies $\Im(A_0) = T$ for any assignment g. Thus, the lemma gives $\vdash A_0 \equiv (\Im(A_0))^n$, which turns to be $\vdash A_0 \equiv T^n$, where T^n is the name of the truth value true.

But using the calculus, in particular, Axiom 2—of the form, $(A_0 \equiv T^n) \equiv A_0$—and the rule of replacement R we obtain the desired result, $\vdash A_0$.

In the event of A_0 being a valid formula but not a sentence, we pass from A_0 to the sentence $\forall X_{\gamma_1} \dots X_{\gamma_r} A_0$—where $freeVar(A_0) = \{X_{\gamma_1} \dots X_{\gamma_r}\}$. We know that $\models \forall X_{\gamma_1} \dots X_{\gamma_r} A_0$, and using the previous argument, $\vdash \forall X_{\gamma_1} \dots X_{\gamma_r} A_0$. Applying the rules of the calculus, we obtain $\vdash A_0$. □

4 Completeness Proofs in Henkin's Course

The story behind this is that of María Manzano, who during the academic year of 1977–1978 attended his class of *metamathematics* for doctorate students at Berkeley. Before each class, Henkin would give us a text of some 4–5 pages that summarized what was to be addressed in the class.

4.1 Herbrand's Theorem Yields Completeness

It is surprising that the first-order completeness proof that Henkin explained in class was not his own but was developed by using Herbrand's theorem and the completeness of propositional logic. In what follows, I will summarize the proof, but I will also try to keep close to the spirit of Henkin's purple notes.

Theorem 6 (Herbrand's Theorem) *Herbrand's Theorem provides an effective way to associate with any first-order sentence A, a set (infinite) of sentences of propositional logic Ψ such that: $\vdash A$ in FOL iff there is some $H \in \Psi$ such that $\vdash_{PL} H$ in PL ($\vdash_{PL}$ means that we just use sentential axioms and detachment).*

The above result can be regarded as a special case of the following:

[33] See [12, p. 341].

Theorem 7 *Let L be a first-order language: We can extend L to L' by adjoining a set $\mathcal{C}$ of individual constants, and we can effectively endow a set Δ of sentences of L' with the following property: For any set of sentences $\Gamma \cup \{A\} \subseteq \mathsf{Sent}(L)$,*

$$\Gamma \vdash A \quad \textit{iff} \quad \Gamma \cup \Delta \vdash_{PL} A.$$

Proof In the first place, we build a set Δ where

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$$

where Δ_1 consists of the sentences $\exists x_i B \to B(c_{i,B})$ (for each $\exists x_i B \in \mathsf{Sent}(L')$). Δ_2 consists of various formal axioms for quantifiers (from first-order logic), and Δ_3 consists of various formal axioms for the equality symbol (if there is one in the language L, otherwise it is $\emptyset$).

In the spirit of Herbrand's theorem, an effective method of transforming any given derivation of A from $\Gamma \cup \Delta$ in *PL* into a formal derivation of A from Γ in *FOL* was given, which solves half of the theorem.

$$\Gamma \cup \Delta \vdash_{PL} A \quad \text{implies} \quad \Gamma \vdash A.$$

As for the other half,

$$\Gamma \vdash A \quad \text{implies} \quad \Gamma \cup \Delta \vdash_{PL} A,$$

let us now assume that we do not have a proof of A from $\Gamma \cup \Delta$ in *PL*, $\Gamma \cup \Delta \nvdash_{LP} A$. If we use the completeness of propositional logic, then $\Gamma \cup \Delta \not\models_{PL} A$, and we conclude that there is an assignment g for atoms of L'_0 that extends to an interpretation $\Im$ such that $\Im(\Gamma \cup \Delta) = T$ but $\Im(A) = F$.

In order to prove the theorem, from this interpretation $\Im$ we obtain a first-order structure $\mathcal{A}$ such that $\models_{\mathcal{A}} \Gamma$ but $\not\models_{\mathcal{A}} A$ and hence $\Gamma \not\models A$.

By soundness of first-order logic, $\Gamma \nvdash A$. □

Predicate Logic: Reduction to Sentential Logic Using the previous theorem, we effectively reduce the completeness problem for first-order logic to that of sentential logic. To this effect, Henkin gave an argument to support the following statement.

Proposition 8 *Theorem 7 and completeness of PL implies completeness of FOL.*

Let us have a look of Henkin's notes:

Note that a proof of the kind described above, provides a completeness proof for 1st order logic. For the theorem shows that whenever *not* $\Gamma \vdash A$ in L then *not* $\Gamma \cup \Delta \vdash_s A$, and the proof of the theorem then shows that *not* $\Gamma \models A$ (by furnishing $\mathcal{A}$ which satisfies Γ but not A); contrapositively, whenever $\Gamma \models A$ then $\Gamma \vdash A$.
Since we use the completeness of sentential logic in our proof, we effectively *reduce the completeness problem for 1st order logic to that of sentential logic*.
(Henkin, Math 225B Notes. 1/27/78)

He finished his class with two remarks:

Remark 3. The structure $\mathcal{A}$ described in the preceding remarks will have as its domain the set of new constants adjoined to L in forming L' (or possibly equivalence classes of such constants under

a suitable equiv. rel'n). Hence we obtain as a corollary the *downward Skolem–Löwenheim Thm*: If an infinite L-structure $\mathcal{B}$ satisfies Γ_0, then a struct. $\mathcal{A}$ of cardinality $=\text{card}(L_0)$ satisfies Γ_0.
Remark 4. We shall show that the theory defined by $\Gamma \cup \Delta$ in L' is a conservative extension of the theory defined by Γ in L: Whenever $A \in L_0$ and $\Gamma \cup \Delta \vdash A$ in L', then $\Gamma \vdash A$ in L.
(Henkin, Math 225B Notes. 1/27/78)

Another completeness proof he also developed in class was his result based on Craig's interpolation theorem.

4.2 An Extension of the Craig–Lyndon Interpolation Theorem

In 1963, Henkin published the paper *An extension of the Craig–Lyndon interpolation theorem* [11], where we can find a different proof of completeness for first order logic. Craig had shown the following theorem:

Theorem 9 *If A and C are any formulas of predicate logic such that $A \vdash C$, then there is a formula B such that* (i) *$A \vdash B$ and $B \vdash C$, and* (ii) *each predicate symbol occurring in B occurs both in A and in C.*

Henkin recalls that due to the fact that the relations $\vdash$ and $\models$ coincide in extension (by soundness and strong completeness theorems), the above theorem is also valid if we replace the syntactic notion of derivability by the semantical notion of consequence. However, his idea was to obtain completeness from a slightly modified version of Craig's theorem.

Notice, however, that if we alter Craig's theorem by replacing the symbol "$\vdash$" with "$\models$" in the hypothesis, but leaving "$\vdash$" unchanged in condition (i) of the conclusion, then the resulting proposition yields the completeness theorem as an immediate corollary.

The main theorem to be proved is:

Theorem 10 *Let Γ and Δ any sets of nnfs* (*negation normal formula*) *such that $\Gamma \models \Delta$. There is a nnf B such that* (i) *$\Gamma \vdash B$ and $B \vdash \Delta$, and* (ii) *any predicate symbol with a positive or negative occurrence in B has an occurrence of the same sign in some formula of Γ and in some formula of Δ.*

The strong completeness theorem is implied by the previous one.

The proof of the theorem is done by contraposition, and to arrive at the conclusion that $\Gamma \not\models \Delta$, Henkin inductively builds two sets of sentences and defines a model based on them using the technique he himself developed in his classical completeness proof [6].

5 Henkin's Expository Papers on Completeness

Henkin was an extraordinary insightful professor in the clarity of his expositions, and he devoted some effort to writing expository papers. In particular, in 1967, he published two very relevant ones for the subject we are investigating here: *Truth and provability* and *Completeness*, published in *Philosophy of Science Today* [23].

5.1 Truth and Provability

In less than 10 pages, Henkin gives a very intuitive introduction to the concept of truth and its counterpart, that of provability, in the same spirit of Tarski's expository paper *Truth and proof* [27]. The latter was published in *Scientific American* two years after Henkin's contribution. This not so surprising since Henkin had by then been working in Berkeley with Tarski for about 15 years, and the theory of truth was Tarski's contribution.

The main topics Henkin was able to introduce (or at least to touch upon) were the very relevant ones, including: the *use/mention* distinction, the desire of *languages with infinite sentences* and the need of a *recursive definition of truth*, the *language/metalanguage* distinction, the need to avoid reflexive paradoxes, the concept of *denotation* for terms, and the interpretation of *quantified formulas*. He also explains what an *axiomatic theory* is and how it works in harmony with a *deductive calculus*; properties as *decidability* and *completeness/incompleteness of a theory* are mentioned at the end. We admire the way these concepts are introduced, with such élan, and the chain Henkin establishes: how each concept is needed to support the next.

Henkin begins by restricting the scope of the suggestive word "true": *'we shall limit ourselves to a much narrower concept of truth, namely, as an attribute of sentences: What does it mean to say that a sentence is true?'*[34] He then goes on to introduce Tarski's conception; in first, he offers a very basic sentence as an example and a proposed specification of its truth conditions that allows him to pinpoint how relevant it is to distinguish use and mention (*the name of an object and the object itself*); in the second place, he states that for a language with just a finite number of sentences, the definition could work. *'But the most decisive point against it is our unwillingness to admit that there are only a finite number of sentences'*.[35]

The need of a recursive definition is clearly motivated as the only way of dealing with an infinite set of sentences. He goes on to mention two major difficulties the definition of *true sentence* must face; the first is the ambiguity and lack of precision of natural languages, and the second is based on the liar's paradox. In natural language, one can formulate sentences that make assertions about themselves and this autoreflexive ability is a source of paradoxes. Without explicitly using these words, Henkin identifies the problems that are associated with the lack of distinction between *language* and *metalanguage*. The need for an artificial language is then justified, and Henkin goes on to say that Tarski was able to give a *'mathematically precise definition'* of the concept of *'true sentence'*, such a definition having a recursive character and obeying general rules like *the law of the excluded middle*. In this way, semantics acquires the citizenship it have been deprived off before. Henkin also explains that *'sentences are built up not only from shorter sentences but from components coming from several grammatical categories. For this reason a recursive definition of truth must deal simultaneously with other semantical notions, such as denotation'*.[36] Finally, he substantiates Tarski's treatment of expressions containing variables and his *'notion of a sequence of objects (in our case integers) satisfying a formula'*. He finished this short presentation of the solutions offered by this mathematical definition

[34] See [13, p. 14].

[35] See [13, p. 15].

[36] See [13, p. 18].

of true sentence (which includes denotation of terms and the definition of satisfiability for quantified formulas) saying: *'Tarski's treatment of expressions containing variables is often considered the key idea of his definition of truth'*.[37]

He then emphasizes the fact that *'the* conditions *under which S is true, does* not *furnish the information as to whether S is in* fact *true'*.[38] The truth of empirical sentences is tested by direct verification (something often hard) but for mathematical sentences the situation is *completely impractical*. In this way he had created the climax to introduce the notion of a calculus: *'Fortunately, we have another method to establish the truth of a sentence, S, quite different from direct verification. Namely, we may infer the truth of S from a knowledge of the truth of certain* other *sentences, say T, U and V'*[39] and to introduce its basic ingredients; namely, *deductions*, *hypothesis*, *conclusion* and *laws of logic*. *'These laws* (...) *never lead from true sentences to a false one'*.[40]

He then brings in the notion of an axiomatic theory, *'We sometimes attempt to organize our knowledge in a certain domain, say D, by seeking to infer all the true sentences dealing with D from one fixed set of hypothesis'*[41] emphasizing the fact that even though axiomatic theories had been known since Euclid, the laws of logic had not received the requisite interest until the nineteenth century with Boole. *'In this way the logicians created a fully formalized axiomatic theory, called a* formal deductive theory, *by means of which we could formulate and study the laws of logic with mathematical precision'*.[42]

The mechanical character of proofs are praised *'for if we did not have such a mechanical means of testing proofs, we would be entitled to ask for a proof that any alleged proof was indeed a proof!'*[43] He poses the important distinction between having a calculus and having a decision procedure for theoremhood.

Finally, he addresses the concept of the *completeness of a theory*. First, he mentions that *'The artificial languages devised by mathematical logicians as a basis for their formal deductive theories were precisely those languages to which Tarski had turned in developing his definition of truth'*,[44] and notes how important is to be able to prove that *each provable sentence is true*. After this he says: *'What is not at all clear, in general, is the converse question: Is each true sentence a theorem? In other words, is there a proof for every true sentence?'*[45] and adds that this is the problem of completeness.

The chapter ends with the incompleteness result, *'The unexpected discovery, however, was that in the case of languages dealing with certain domains, it is impossible to obtain a complete deductive system!'*[46] which he explains in a very simple way by going back to the method of avoiding liar's paradox in formal languages *'it must be impossible to*

[37]See [13, p. 19].

[38]See [13, p. 19].

[39]See [13, p. 19].

[40]See [13, p. 19].

[41]See [13, p. 19].

[42]See [13, p. 19].

[43]See [13, p. 19].

[44]See [13, p. 21].

[45]See [13, p. 21].

[46]In [13, p. 22].

express the concept of true sentence within that language itself'[47] and seeing that for *provable sentence* the situation differs:

> No matter how we select axioms and rules of inference to obtain a formal deductive theory for this language, the resulting notion of *provable sentence* can be expressed *in the language itself*. [...] It follows that no matter which formal deductive theory we select for such a language, the resulting notion of *provable sentence* will differ from that of *true sentence*—since the former can be expressed in the language itself while the latter cannot. Thus, all of these theories are incomplete.
> (Henkin [13, p. 22])

5.2 Completeness

In this short expository paper Henkin explores the complex landscape of the notions of completeness. He introduces the notion of logical completeness—both weak and strong—as an extension of the notion already introduced of *completeness of an axiomatic theory*.[48] This presentation differs notably from the standard way these notions are introduced today; usually, the completeness of the logic precedes the notion of completeness of a theory and, often, to avoid misunderstandings we separate both concepts as much as possible, as if relating them were some sort of terrible mistake or even anathema. Gödel's incompleteness theorem[49] is presented, as well as its negative impact on the search for a complete calculus for higher-order logic. The paper ends with the introduction of his own completeness result for higher-order logic with general semantics. The utilitarian way Henkin uses to justify his general models as a way of sorting the provable sentences from the unprovable ones in the class of valid sentences (in standard models) is very peculiar.

As we have seen in the preceding section, completeness of a theory was described as:

> The question of whether, conversely, every true sentence is provable is the problem of completeness. We note, therefore, that the question of completeness always presupposes a given *language*, a given *interpretation* of the language by means of which its sentences convey information about some domain, and a given *axiomatic theory* formulated within the language.
> (Henkin [14, pp. 23–24])

No doubt, he is talking about completeness of a theory since, a couple of pages later (after defining what a model is) he adds: *'As we have indicated above, our notion of completeness for the theory J is relative to a given model'*.[50]

The two first pages of the paper are devoted to praising the importance of the power of abstraction provided by axiomatic theories in the realm of logic. Let us quote Henkin describing the *important transformation* that the concept of an axiomatic theory has undergone in a century: First, when the community of logicians became aware that the same theory may have different interpretations or models:

---

[47] See [13, p. 22].

[48] In [22], we analyze the evolution of the completeness theorem from Gödel to Henkin in some detail.

[49] The previously mentioned anathema is even stronger when Gödel's incompleteness result is mentioned.

[50] In [14, p. 25].

> This transformation came about through the realization that a given system of symbols and sentences can be subjected to more than one interpretation, so that a single language can be employed to refer simultaneously to many different domains. Although this possibility was implicit at least as early as Descartes's discovery of analytic geometry, its significance for axiomatic mathematics was not appreciated until the invention of non-Euclidean geometry by Bolyai, Lobachevsky and Gauss in the last century.
> (Henkin [14, p. 24])

Secondly, when they realized that theorems in an axiomatic theory automatically become true in any model for these axioms:

> The realization that sentences proved in an axiomatic theory give information simultaneously for a great many domains has had a revolutionary effect on both pure and applied mathematics. As regards applications, it meant that by moving to a more abstract level one could achieve a great economy of effort, handling problems from diverse domains by means of a single investigation.
> (Henkin [14, p. 25])

We believe that it took a considerable degree of abstraction that includes, on the one hand, the realization that the nature of the objects that constitute the universe of a structure is irrelevant and, on the other, that what matters are the relationships that hold these objects together. Henkin did not mention this, but says that the new role played by the formal language reverses the investigation in the area of pure mathematics: *'Instead of starting with a fixed domain and inquiring which sentences are true about it, one starts with fixed sentences and seeks to analyze the totality of domains in which these are true'*.[51] Therefore, the crossing of the bridge between language and structures is not only in the right-left direction—as when we define $Th(\mathcal{A})$—but also in the opposite—when we define $Mod(\Gamma)$.

He then announces that he is going to introduce *'an extension of the completeness concept'*. We believe that this presentation of *completeness of a logic* as an extension of *completeness of a theory* is very relevant and historically well-founded, even thought not many logicians are aware of this fact today. The first concept Henkin brings in to develop this idea is that of *model* in terms that are very familiar to us today: *'Let us use the term "model" (for a given language L) to mean a domain of objects together with an interpretation whereby the symbols of L are made to refer to this domain. Such a model determines each sentence of L as true or false'*.[52]

The first step in that extension of the notion of completeness is when we take a theory J and investigate not just if this theory is complete for a single model $\mathcal{M}$—namely, if $\models_{\mathcal{M}} \varphi$ implies $J \vdash \varphi$—but also whether the theory is complete for a class $\mathfrak{C}$ of models: *'we say that a sentence of L is* valid in $\mathfrak{C}$ *if it is true of every model in* $\mathfrak{C}$*. And we say that J is* complete for the class $\mathfrak{C}$ *if every sentence of L which is valid in* $\mathfrak{C}$ *can be proved in J'*.[53] The final leap between the notions of *completeness of a theory* and the *completeness of a logical calculus* can be seen in the following quote:

> In case the class $\mathfrak{C}$ happens to contain a single model $\mathcal{M}$, this notion of completeness reduces to the earlier one. At the other extreme, the class $\mathfrak{C}$ may contain *all* models for the language L. A theory complete for this class is said to be *logically complete*.
> (Henkin [14, p. 26])

---

[51] See [14, p. 25].

[52] See [14, p. 25].

[53] See [14, p. 26].

Henkin also tells us a bit about the history of this new notion of completeness: *'The first explicit formulation and solution of a completeness problem is due to Emil Post'*[54] adding that decidability was a related issue in PL: *'As a by product of his work, Post obtained a decision procedure for the class of theorems of sentential logic—that is, a completely automatic method which can be applied to any sentence of the system and which indicates, after a finite number of steps, whether or not the sentence is provable'*.[55] and highlighting Gödel's result on first-order logic: *'Kurt Gödel, who was able to establish similar completeness theorems for deductive theories based upon first-order predicate logic'*.[56]

Then, first-order logic is presented in some detail and some examples of particular structures are given. At this point, not only weak completeness but also the strong completeness result for first order logic is presented:

> Gödel completeness theorem applies to a wide class of axiomatic theories which are based on first-order languages. For he showed that if one takes an *arbitrary* set of sentences as new axioms, in addition to the logically valid axioms of the original deductive theory, and if one then considers the class $\mathfrak{C}$ of all those models for which each of the new axioms is true, then every formula which is valid in $\mathfrak{C}$ (that is, true for each model of $\mathfrak{C}$) will be provable in the enriched deductive system. (Henkin [14, p. 28])

This new concept of completeness can also be understood as yet another extension of the concept of the *completeness of a theory* in a class $\mathfrak{C}$ of models, when this class is precisely the models of a set of sentences, $Mod(\Gamma)$—namely, $\models_{Mod(\Gamma)} \varphi$ implies $\Gamma \vdash \varphi$. Henkin does not state this, probably because the connection is too obvious. However, he does mention other relevant issues related to completeness: the negative result of the decidability of validity in the first-order case *'Gödel's proof of logical completeness* [...] *did not lead to a decision procedure for the class of logically valid first-order sentences'*[57] unlike the positive one when completeness of a theory in a single model is concerned: '[...] *the proof furnished a decision procedure for the class of provable sentences'*.[58]

The theorem of compactness and some of its mathematical applications are mentioned, as well as some of the many positive results on the completeness of particular mathematical theories obtained in the 1919–1930 period.

Henkin places Gödel's incompleteness result in this context, saying that it *'came as a shock to the mathematical world'*. He explains how the result originally dealt with the theory of a single model—namely, the first-order theory of the model of natural language $\mathcal{N}$—but was extended to a higher-order theory G, where arithmetic can be axiomatized up to isomorphism level. *'Because these axioms exclude models that differ mathematically from $\mathcal{N}$, it was generally felt that the theory G must be complete for $\mathcal{N}$. Yet Gödel showed that it was not'*.[59] Henkin devotes some paragraphs to explaining Gödel's method by highlighting: *arithmetization*, the ability to formulate the autoreflexive statement Q saying of itself that is not provable in G, and the capability to prove that *'the notion prov-*

---

[54] See [14, p. 26].

[55] See [14, p. 26].

[56] In [14, p. 27].

[57] See [14, p. 28].

[58] See [14, p. 29].

[59] See [14, p. 31].

able in G can be expressed in the language G itself'. Finally, Henkin explains how this result can be expanded to any higher-order calculus: *'Gödel was able to obtain a very general incompleteness theorem.* [...] *there cannot be a complete theory for the logically valid sentences of a higher order language'*.[60]

The paper ends by highlighting two very relevant problems raised for Gödel's incompleteness result that helps Henkin to introduce his own completeness proof for high-order logic with general semantics. He mentions two problems; let us focus on the second one:

> Second, when we have at hand a particular formal deductive system J, which is known to be incomplete for a certain class $\mathfrak{C}$ of models [...] some sentences which are valid for $\mathfrak{C}$ are provable in J while others are not. Accordingly, we may seek general criteria for distinguishing these two kinds of sentences.
> (Henkin [14, p. 32])

When the logic concerned is first-order, the strong completeness result tells us that whenever P is not a theorem of J, there is a model $\mathcal{A}$ of J which happens to be a countermodel of P—since $J \nvdash P$ implies $\not\models_{\mathcal{A}} P$. But *'if the language of J is of higher order, the situation is generally different'*.[61] What we find more surprising is that he presents his general models as a way of *'sorting the provable from the unprovable'*.[62] In particular, for the above-mentioned theory G, it can be shown that there are generalized models *'satisfying the axioms of the theory G whose structure is very different from that of $\mathcal{N}$. Furthermore, it can be shown that every sentence unprovable in G must be false for one of these models* [...] *G is complete for the class of all those generalized models which satisfy all of its axioms'*.[63]

The important outcome being that this result is not just a peculiarity of G, but *'such a completeness theorem can be established not only for G, but for* arbitrary *theories of higher order'*. Let us finish quoting his last paragraph:

> The quest for general criteria by which to identify complete theories has led to several fruitful new metamathematical concepts. And in seeking a means of characterizing the class of provable sentences of an incomplete theory, we have been led to discover new mathematical structures and new ways of interpreting the language of mathematics.
> (*Henkin* [14, pp. 34–35])

References

1. Andrews, P.: A reduction of the axioms for the theory of propositional types. Fundam. Math. **52**, 345–350 (1963)
2. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**(2), 56–68 (1940)
3. Gödel, K.: On the completeness of the calculus of logic. In: Kurt Gödel Collected Works Vol. I. Publications 1929–1936, pp. 61–101. Oxford University Press, Oxford (1929/1986)
4. Gödel, K.: The completeness of the axioms of the functional calculus of logic. In: Kurt Gödel Collected Works Vol. I. Publications 1929–1936, pp. 103–123. Oxford University Press, Oxford (1930/1986)

[60] See [14, pp. 31–32].

[61] In [14, p. 33].

[62] See [14, p. 34].

[63] See [14, p. 34].

5. Gödel, K.: The consistency of the axiom of choice and of the generalized continuum-hypothesis. In: Proc. Natl. Acad. Sci., pp. 556–557 (1938)
6. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**(3), 159–166 (1949)
7. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**(2), 81–91 (1950)
8. Henkin, L.: Some notes on nominalism. J. Symb. Log. **18**(1), 19–29 (1953)
9. Henkin, L.: Banishing the rule of substitution for functional variables. J. Symb. Log. **18**(3), 201–208 (1953)
10. Henkin, L.: The nominalistic interpretation of mathematical language. Bull. Soc. Math. Belg. **7**, 137–141 (1955)
11. Henkin, L.: An extension of the Craig–Lyndon interpolation theorem. J. Symb. Log. **28**(3), 201–216 (1963)
12. Henkin, L.: A theory of propositional types. Fundam. Math. **52**, 323–344 (1963). Errata to the paper "A theory of propositional types". Fundam. Math. **53**(1), 119 (1964)
13. Henkin, L.: Truth and provability. In: Morgenbesser, S. (ed.) Philosophy of Science Today, pp. 14–22. Basic Books, New York (1967)
14. Henkin, L.: Completeness. In: Morgenbesser, S. (ed.) Philosophy of Science Today, pp. 23–35. Basic Books, New York (1967)
15. Henkin, L.: Identity as a logical primitive. Philosophia, Philos. Q. Isr. **5**(1–2), 31–45 (1975)
16. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**(2), 127–158 (1996)
17. Manzano, M.: Los sistemas generales. In: Quintanilla, M.A. (ed.) Estudios de Lógica y de Filosofía de la Ciencia, pp. 65–100. Ediciones de la Universidad de Salamanca, Salamanca (1982)
18. Manzano, M.: Extensions of First Order Logic. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge (1996)
19. Manzano, M.: Life, work and some miracles of Alonzo Church. J. Hist. Philos. Logic **18**(4), 211–232 (1997)
20. Manzano, M., Kurucz, A., Sain, I.: The little mermaid. In: Martínez, C., et al. (eds.) Truth in Perspective, pp. 83–111. Ashgate Publishing Limited, Aldershot (1998)
21. Manzano, M.: Lambda abstractor was philosopher's stone for Alonzo Church. In: Cachro, J., et al. (eds.) Proceedings of the 11th International Congress of Logic, Methology and Philosophy of Science, pp. 435–436. Faculty of Philosophy, Jagiellonian University, Cracow (1999)
22. Manzano, M., Alonso, E.: Completeness from Gödel to Henkin. J. Hist. Philos. Logic **35**(1), 50–75 (2014)
23. Morgenbesser, S. (ed.): Philosophy of Science Today. Basic Books, New York (1967)
24. Quine, W.: Logic based on inclusion and abstraction. J. Symb. Log. **2**(4), 145–152 (1937)
25. Ramsey, F.P.: The foundations of mathematics. Proc. Lond. Math. Soc. **25**(2), 338–384 (1926)
26. Tarski, A.: Sur le terme primitif de la logistique. Fundam. Math. **4**, 196–200 (1923)
27. Tarski, A.: Truth and proof Sci. Am. **June**, 63–70 (1969). Also see pp. 75–77
28. Russell, B.: Mathematical logic as based in the theory of types. In: van Heijenoort, J. (ed.) From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931, pp. 154–155. Harvard University Press, Cambridge (1908/1967)

M. Manzano (✉)
Universidad de Salamanca, Salamanca, Spain
e-mail: mara@usal.es

Part III
Extensions and Perspectives in Henkin's Work

The Countable Henkin Principle

Robert Goldblatt

Abstract This is a revised and extended version of an article that encapsulates a key aspect of the "Henkin method" in a general result about the existence of finitely consistent theories satisfying prescribed closure conditions. This principle can be used to give streamlined proofs of completeness for logical systems, in which inductive Henkin-style constructions are replaced by a demonstration that a certain theory "respects" some class of inference rules. The countable version of the principle has a special role and is applied here to omitting-types theorems, and to strong completeness proofs for first-order logic, omega-logic, countable fragments of languages with infinite conjunctions, and a propositional logic with probabilistic modalities. The paper concludes with a topological approach to the countable principle, using the Baire Category Theorem.

Keywords Deducibility · Inference · Finitely consistent · Maximally consistent · Countable Henkin Principle · Lindenbaum's Lemma · Completeness · Omitting types · Probabilistic modality · Archimedean inference · Baire Category Theorem

The *Henkin method* is the technique for constructing maximally consistent theories satisfying prescribed closure conditions that was introduced by Leon Henkin in his 1947 doctoral dissertation. The method involves building up the desired theory by induction along an enumeration of some relevant class of formulas, with choices being made at each inductive step to include certain formulas, in such a way that when the induction is finished, the theory has the properties desired. The character of this procedure is neatly captured in a phrase of Sacks [20, p. 30], who attributes its importance to the fact that it "takes into account the ultimate consequences of decisions made at intermediate stages of the construction".

Famously, Henkin used his method to give the first new proof of completeness of first-order logic since the original 1929 proof of Gödel and to prove completeness of a theory of types with respect to "general" models [12, 13]. He obtained the type theory result first and thought it would be of greater interest, as he explained in the remarkable article [16], in which he tells the story of his "accidental" discovery of these completeness proofs while trying to solve a different problem. In fact logicians have paid more attention to the first-order construction, which was eventually adapted to propositional and first-order versions of modal, temporal, intuitionistic and substructural logics. There are now numerous kinds of logical formalism whose model-theoretical analysis owes something of its origin to Henkin's pioneering ideas.

In the present article the Henkin method is used to derive a general principle about the existence of maximal theories closed under abstract "inference rules". This principle

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_13

may then be used to give alternative proofs of completeness theorems, proofs in which the Henkin method is replaced by a demonstration that a certain theory "respects" a certain class of inference rules. This alternative approach is illustrated by a re-working of the completeness and omitting-types theorems for first-order logic and certain fragments of $\mathsf{L}_{\infty\omega}$. Some of these involve a countability constraint, and that case of the approach has a particularly direct form that we call the Countable Henkin Principle. It is used to obtain general extension results about the existence of "rich" theories that are closed under countably many inference rules. We apply those extension results to give a proof of strong completeness of a propositional logic with probabilistic modalities, motivated by a recent topological analysis in [17].

The proof of the Countable Henkin Principle itself helps to clarify why some completeness proofs work only under a countability constraint. The essential point is that after building a denumerably long increasing sequence of *consistent* theories, one must take the union of them to proceed to a further stage. But if the proof theory is non-finitary, the union may not preserve consistency (but only the property of being finitely consistent), and the construction cannot be iterated into the transfinite.

This paper is a revised and extended version of the article [6]. That had a previous revision [8], which added applications to completeness for the Barcan formula in quantified modal logic, and for propositional mono-modal logic with infinitary inference rules. Here those modal applications are replaced by new Sects. 4–7, which include the formulation of the Rich Extension theorems, their alternative topological proof using the Baire Category Theorem, and the discussion of probabilistic modal logic. Also, the Sect. 3.2 on the Omitting-Types Theorem has been rewritten from a more general standpoint, and a new Sect. 3.3 added, which relates this to Henkin's generalizations of ω-consistency [14] and ω-completeness [15].

1 The Abstract Henkin Principle

Consider a formal language that includes a distinguished formula $\bot$ and a unary connective $\neg$. Think of $\bot$ as a constant false sentence, and $\neg$ as negation. Let Φ be any class of formulas of this language such that $\bot \in \Phi$ and Φ is closed under application of the connective $\neg$.

Let $\vdash$ be a subset of $2^{\Phi} \times \Phi$, that is, a binary relation from the powerset of Φ to Φ. For $\Gamma \subseteq \Phi$ and $\varphi \in \Phi$, write $\Gamma \vdash \varphi$ if (Γ, φ) belongs to $\vdash$ and $\Gamma \nvdash \varphi$ otherwise. The relation $\vdash$ is called a *deducibility relation on* Φ if it satisfies

D1 If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;
D2 If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
D3 If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\} \vdash \bot$, then $\Gamma \vdash \bot$;
D4 $\Gamma \cup \{\neg\varphi\} \vdash \bot$ iff $\Gamma \vdash \varphi$.

A subset Γ of Φ is called $\vdash$*-consistent* if $\Gamma \nvdash \bot$ and *finitely* $\vdash$*-consistent* if each finite subset of Γ is $\vdash$-consistent in this sense. Γ is *maximally* $\vdash$*-consistent* if it is $\vdash$-consistent but has no $\vdash$-consistent proper extension in Φ. Replacing "$\vdash$-consistent" by "finitely $\vdash$-consistent" in this last definition yields the notion of Γ being *maximally finitely* $\vdash$*-consistent*.

Now from D1 it follows that any $\vdash$-consistent set is finitely $\vdash$-consistent. The relation $\vdash$ is called *finitary* if, conversely, it satisfies

D5 Every finitely $\vdash$-consistent set is $\vdash$-consistent, that is, if $\Gamma \vdash \bot$, then for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vdash \bot$.

If $\mathscr{C}$ is a collection of finitely $\vdash$-consistent subsets of Φ that is linearly ordered by set inclusion, that is, $\Gamma \subseteq \Delta$ or $\Delta \subseteq \Gamma$ for all $\Gamma, \Delta \in \mathscr{C}$, then the union $\bigcup \mathscr{C}$ of $\mathscr{C}$ is finitely $\vdash$-consistent. This follows immediately from the fact that any finite subset of $\bigcup \mathscr{C}$ is a subset of some $\Gamma \in \mathscr{C}$. Thus, if

$$P = \{\Delta \subseteq \Phi : \Gamma \subseteq \Delta \ \& \ \Delta \text{ is finitely} \vdash\text{-consistent}\},$$

then under the partial ordering of set inclusion P fulfills the hypothesis of Zorn's Lemma. From the latter we deduce the following:

Lindenbaum's Lemma *Every finitely $\vdash$-consistent subset of Φ has a maximally finitely $\vdash$-consistent extension in Φ.*

(Note that this result uses no properties of $\vdash$ other than the definitions of the concepts referred to in the statement of the lemma.)

An ordered pair (Π, χ) with $\Pi \subseteq \Phi$ and $\chi \in \Phi$ will be called an *inference* in Φ. As motivation, the reader may care to think of Π as a set of "premises" and χ as a "conclusion", but the notion of inference is quite abstract and applies to any such pair. A set Γ will be said to *respect* the inference (Π, χ) when

$$(\Gamma \vdash \varphi, \text{ all } \varphi \in \Pi) \quad \text{implies} \quad \Gamma \vdash \chi.$$

Γ is *closed under* (Π, χ) if

$$\Pi \subseteq \Gamma \quad \text{implies} \quad \chi \in \Gamma.$$

Γ respects (is closed under) a *set* $\mathscr{I}$ of inferences if it respects (is closed under) each member of $\mathscr{I}$.

The cardinality of a set X will be denoted card X. If κ is a cardinal number, then X is *κ-finite* if card $X < \kappa$. A *κ-finite extension* of X is a set of the form $X \cup Y$ with Y κ-finite. In other words, a κ-finite extension of X is a set obtained by adding *fewer than* κ elements to X.

Theorem 1 *Let $\vdash$ be a finitary deducibility relation on Φ. If $\mathscr{I}$ is a set of inferences in Φ of cardinality κ, and Γ is a $\vdash$-consistent subset of Φ such that*

$$\textit{every } \kappa\textit{-finite extension of } \Gamma \textit{ respects } \mathscr{I},$$

then Γ has a maximally $\vdash$-consistent extension in Φ that is closed under $\mathscr{I}$.

This theorem will be established by first separating out that part of its content that does not involve Lindenbaum's Lemma. To do this requires a further concept: a set $\Gamma \subseteq \Phi$ will be said to *decide* (Π, χ) if

$$\text{either } \chi \in \Gamma, \quad \text{or} \quad \text{for some } \varphi \in \Pi, \quad \neg\varphi \in \Gamma.$$

Γ decides a set of inferences if it decides each member of the set.

The following result holds for any deducibility relation.

Lemma 1

(1) *If Γ decides (Π, χ) and $\Gamma \subseteq \Delta$, then Δ decides (Π, χ).*
(2) *If Γ decides (Π, χ), then Γ respects (Π, χ).*
(3) *If Γ is finitely $\vdash$-consistent, and Γ decides (Π, χ), then Γ is closed under (Π, χ).*
(4) *If Γ is $\vdash$-consistent, and Γ respects (Π, χ), then for some $\psi \in \Phi$, $\Gamma \cup \{\psi\}$ is $\vdash$-consistent and decides (Π, χ).*

Proof (1) Immediate.

(2) Suppose $\Gamma \vdash \varphi$ for all $\varphi \in \Pi$. Then if $\Gamma \nvdash \chi$, by D2 $\chi \notin \Gamma$, so if Γ decides (Π, χ), then $\neg\psi \in \Gamma$ for some $\psi \in \Pi$. But by assumption $\Gamma \vdash \psi$, and so by D4 $\Gamma \cup \{\neg\psi\} \vdash \bot$, that is, $\Gamma \vdash \bot$. But then by D1, $\Gamma \cup \{\neg\chi\} \vdash \bot$, and so by D4 again, $\Gamma \vdash \chi$. Hence, $\Gamma \vdash \chi$.

(3) Suppose Γ decides (Π, χ), and $\Pi \subseteq \Gamma$. Then if $\chi \notin \Gamma$, then $\neg\psi \in \Gamma$ for some $\psi \in \Pi \subseteq \Gamma$. Now by D2, $\{\psi\} \vdash \psi$, and so by D4, $\{\psi, \neg\psi\} \vdash \bot$. But $\{\psi, \neg\psi\} \subseteq \Gamma$, so then Γ is not finitely $\vdash$-consistent.

(4) If $\Gamma \cup \{\neg\varphi\}$ is $\vdash$-consistent for some $\varphi \in \Pi$, then the result follows with $\psi = \neg\varphi$. Otherwise, for all $\varphi \in \Pi$, $\Gamma \cup \{\neg\varphi\} \vdash \bot$, and so by D4, $\Gamma \vdash \varphi$. But Γ respects (Π, χ), hence $\Gamma \vdash \chi$. Since $\Gamma \nvdash \bot$, D3 then implies that $\Gamma \cup \{\chi\} \nvdash \bot$, so the result follows with $\psi = \chi$. □

Abstract Henkin Principle *Let $\vdash$ be a finitary deducibility relation on Φ. If $\mathscr{I}$ is a set of inferences in Φ of cardinality κ, and Γ is a $\vdash$-consistent subset of Φ such that*

$$(*) \quad \textit{every } \kappa\textit{-finite extension of } \Gamma \textit{ respects } \mathscr{I},$$

then Γ has a $\vdash$-consistent extension Δ that decides $\mathscr{I}$.

Note that by applying Lindenbaum's Lemma to the $\vdash$-consistent extension Δ given by the conclusion of this result, an extension of Γ is obtained that is maximally $\vdash$-consistent (since $\vdash$ is finitary), decides $\mathscr{I}$ by Lemma 1(1), and hence is closed under $\mathscr{I}$ by Lemma 1(3). This argument proves Theorem 1.

To prove the Abstract Henkin Principle, let $\{(\Pi_\alpha, \chi_\alpha) : \alpha < \kappa\}$ be an indexing of the members of $\mathscr{I}$ by the ordinals less than κ. A sequence $\{\Delta_\alpha : \alpha < \kappa\}$ of extensions of Γ is then defined such that

(i) Δ_α is $\vdash$-consistent;
(ii) $\Delta_\gamma \subseteq \Delta_\alpha$ whenever $\gamma < \alpha$;
(iii) $\mathrm{card}(\Delta_\alpha - \Gamma) \leq \alpha$, hence Δ_α is a κ-finite extension of Γ;

and such that $\Delta_{\alpha+1}$ decides $(\Pi_\alpha, \chi_\alpha)$. The definition proceeds by transfinite induction on α.

Case 1: If $\alpha = 0$, put $\Delta_\alpha = \Gamma$, so that Δ_α is $\vdash$-consistent by assumption, and $\mathrm{card}(\Delta_\alpha - \Gamma) = 0 = \alpha$.

Case 2: Suppose $\alpha = \beta + 1$, and assume inductively that Δ_β has been defined such that (i)–(iii) hold with β in place of α. Then since Δ_β is a κ-finite extension of Γ, hypothesis $(*)$ on Γ implies that Δ_β respects (Π_β, χ_β). Hence, by Lemma 1(4), there is a $\psi \in \Phi$ such that $\Delta_\beta \cup \{\psi\}$ is $\vdash$-consistent and decides (Π_β, χ_β). Put $\Delta_\alpha = \Delta_\beta \cup \{\psi\}$, so that (i) holds for α. Since $\Delta_\beta \subseteq \Delta_\alpha$, and $\gamma < \alpha$ iff $\gamma \leq \beta$, (ii) follows readily. For (iii), since $\Delta_\alpha - \Gamma \subseteq (\Delta_\beta - \Gamma) \cup \{\psi\}$, $\mathrm{card}(\Delta_\alpha - \Gamma) \leq \mathrm{card}(\Delta_\beta - \Gamma) + 1 \leq \beta + 1 = \alpha$.

Case 3: Suppose α is a limit ordinal and that for all $\beta < \alpha$, Δ_β has been defined to satisfy (i)–(iii). Put

$$\Delta_\alpha = \bigcup_{\beta<\alpha} \Delta_\beta.$$

Then (ii) is immediate for α. For (i), observe that Δ_α is the union of a chain of $\vdash$-consistent, hence finitely $\vdash$-consistent, sets Δ_β, and so Δ_α is finitely $\vdash$-consistent as in the proof of Lindenbaum's Lemma. But $\vdash$ is finitary, so Δ_α is then $\vdash$-consistent. For (iii), observe that

$$(\Delta_\alpha - \Gamma) = \bigcup_{\beta<\alpha} (\Delta_\beta - \Gamma)$$

and note that by the inductive hypothesis, if $\beta < \alpha$, then $\operatorname{card}(\Delta_\beta - \Gamma) \leq \beta < \alpha$. Thus, $(\Delta_\alpha - \Gamma)$ is the union of a collection of at most $\operatorname{card}\alpha$ sets, each of which has at most $\operatorname{card}\alpha$ members. Hence, $\operatorname{card}(\Delta_\alpha - \Gamma) \leq \operatorname{card}\alpha \leq \alpha$.

This completes the definition of Δ_α for all $\alpha < \kappa$. Now put $\Delta = \bigcup_{\alpha<\kappa} \Delta_\alpha$. Then by the argument of Case 3, Δ is a $\vdash$-consistent extension of Γ. Moreover, for each $\beta < \alpha$, $\Delta_{\beta+1}$ decides (Π_β, χ_β) by Case 2, and so Δ decides (Π_β, χ_β) by Lemma 1(1).

2 The Countable Case

In the proof of the Abstract Henkin Principle, the assumption that $\vdash$ is finitary is used only in Case 3, and in the final formation of Δ, to show that the union of an increasing sequence of $\vdash$-consistent sets is $\vdash$-consistent. But if κ is countable, then Case 3 does not arise. Case 2 is iterated countably many times, and then Δ is constructed as the union of the Δ_α. Then if $\vdash$ is not finitary, Δ may not be $\vdash$-consistent. However, it will at least be *finitely* $\vdash$-consistent, and this gives the following result.

Countable Henkin Principle *Let $\vdash$ be any deducibility relation on Φ. If $\mathscr{I}$ is a countable set of inferences in Φ, and Γ is a $\vdash$-consistent subset of Φ such that*

$$(*) \quad \Gamma \cup \Sigma \text{ respects } \mathscr{I} \quad \text{for all finite } \Sigma \subseteq \Phi,$$

then Γ has a finitely $\vdash$-consistent extension that decides $\mathscr{I}$.

The extension of Γ deciding $\mathscr{I}$ in this result can be taken to be *maximally finitely* $\vdash$-consistent, by applying Lindenbaum's Lemma and Lemma 1(1). It is also closed under $\mathscr{I}$ by Lemma 1(3).

The Countable Henkin Principle will be used below to give proofs of a number of results, including an omitting-types theorem for countable first-order languages, and (strong) completeness theorems for countable fragments of $\mathsf{L}_{\infty\omega}$ and a probabilistic modal logic. The analysis given here provides one way of "putting one's finger" on the role of countability restrictions in such applications.

If the ambient formal language has a conjunction connective, allowing the formation of the conjunction $\bigwedge \Sigma$ of any *finite* subset Σ of Φ, then a natural constraint on $\vdash$ would be to require that for all $\Gamma \subseteq \Phi$ and all $\varphi \in \Phi$,

$$\Gamma \cup \Sigma \vdash \varphi \quad \text{iff} \quad \Gamma \cup \left\{\bigwedge \Sigma\right\} \vdash \varphi.$$

A deducibility relation satisfying this condition will be called *conjunctive*. Thus, for a conjunctive deducibility relation, hypothesis $(*)$ in the Countable Henkin Principle can be weakened to

$$\Gamma \cup \{\psi\} \text{ respects } \mathscr{I} \quad \text{for all } \psi \in \Phi.$$

3 Classical Applications

3.1 Completeness for First-Order Logic

Let L be a set of relation, function, and individual-constant symbols, and Γ a set of sentences in the first-order language of L that is consistent under the standard deducibility relation of first-order logic.

The *Completeness Theorem* asserts that Γ has a model. To prove this, a new language $\mathsf{K} = \mathsf{L} \cup \mathsf{C}$ is formed by adding to L a set C of new individual constants of cardinality κ, where κ is the maximum of $\operatorname{card} \mathsf{L}$ and $\aleph_0$. The usual construction of a model for Γ involves two phases.

Phase 1: Γ is extended by the "Henkin method" to a maximally consistent set Γ^* of K-sentences such that for each K-formula $\varphi(x)$ with at most one variable (x) free,

(a) if $\exists x \varphi \in \Gamma^*$, then $\varphi(\mathsf{c}) \in \Gamma^*$ for some $\mathsf{c} \in \mathsf{C}$.

Phase 2: A model $\mathfrak{A}^*$ is defined based on the quotient set $\mathsf{C}/\!\sim$, where $\sim$ is the equivalence relation

$$\mathsf{c} \sim \mathsf{d} \quad \text{iff} \quad (\mathsf{c} = \mathsf{d}) \in \Gamma^*.$$

For each K-formula $\psi(x_1, \ldots, x_n)$, this model satisfies

(b) $\mathfrak{A}^* \models \psi[\mathsf{c}_1/\!\sim, \ldots, \mathsf{c}_n/\!\sim]$ iff $\psi(\mathsf{c}_1, \ldots, \mathsf{c}_n) \in \Gamma^*$.

In particular, $\mathfrak{A}^* \models \sigma$ iff $\sigma \in \Gamma^*$, where σ is any K-sentence, so since $\Gamma \subseteq \Gamma^*$, $\mathfrak{A}^* \models \Gamma$.

The Abstract Henkin Principle of this article may be used to give a succinct development of Phase 1. For this, let Φ be the set of all first-order sentences of K, and $\vdash$ the standard (finitary) first-order deducibility relation on Φ. Then Γ is $\vdash$-consistent. The key property of $\vdash$ that will be used is

(c) if $\Delta \vdash \varphi(\mathsf{c})$, and the constant c does not occur in Δ or $\varphi(x)$, then $\Delta \vdash \forall x \varphi(x)$.

Now the closure condition (a) on Γ^* in Phase 1 is equivalent to

$$\text{if } \varphi(\mathsf{c}) \in \Gamma^* \quad \text{for all } \mathsf{c} \in \mathsf{C}, \quad \text{then} \quad \forall x \varphi(x) \in \Gamma^*,$$

that is, to the closure of Γ^* under the inference

$$\varphi_{\mathsf{C}} = \big(\{\varphi(\mathsf{c}) : \mathsf{c} \in \mathsf{C}\}\ \forall x \varphi(x)\big).$$

Let $\mathscr{I}$ be the set of inferences φ_{C} for all first-order K-formulas φ with one free variable. The number of such formulas is κ since card $\mathsf{K} = \kappa$. Hence, card $\mathscr{I} = \kappa$. Thus, to prove the existence of Γ^*, it suffices to show that if Δ is a κ-finite subset of Φ, then

$$\Gamma \cup \Delta \text{ respects } \mathscr{I}.$$

But if card $\Delta < \kappa$, then for any φ, card$(\Delta \cup \{\varphi\}) < \kappa$ since κ is infinite. Hence, fewer that κ members of C appear in $\Delta \cup \{\varphi\}$. But *none* of these constants appear in Γ. Thus, if

$$\Gamma \cup \Delta \vdash \varphi(\mathsf{c}) \quad \text{for all } \mathsf{c} \in \mathsf{C},$$

then

$$\Gamma \cup \Delta \vdash \varphi(\mathsf{c}) \quad \text{for some } \mathsf{c} \text{ not occurring in } \Gamma \cup \Delta \cup \{\varphi\},$$

and so by (c),

$$\Gamma \cup \Delta \vdash \forall x \varphi(x).$$

3.2 Omitting Types

Let L be a *countable* language, and F_n the set of all first-order L-formulas all of whose free variables are among $x_1, \ldots, x_n$. A consistent subset of F_n will be called an *n-type*. An L-structure $\mathfrak{A}$ realises an n-type Σ if there are individuals $a_1, \ldots, a_n$ in $\mathfrak{A}$ such that

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \quad \text{for all } \varphi \in \Sigma.$$

$\mathfrak{A}$ *omits* Σ if it does not realise Σ. If Γ is a consistent set of L-sentences, then an n-type Σ is *isolated over* Γ if there is some formula $\psi \in F_n$ that is consistent with Γ (i.e. $\Gamma \cup \{\psi\}$ is consistent) and has

$$\Gamma \vdash \psi \to \varphi \quad \text{for all } \varphi \in \Sigma.$$

Such a ψ is said to *isolate* Σ *over* Γ. This means that $\Gamma \cup \{\psi\} \nvdash \bot$, whereas $\Gamma \cup \{\psi\} \vdash \varphi$ for all $\varphi \in \Sigma$, that is, $\Gamma \cup \{\psi\}$ fails to respect the rule $(\Sigma, \bot)$. Thus, Σ is *not* isolated over Γ precisely when $\Gamma \cup \{\psi\}$ does respect the rule $(\Sigma, \bot)$ for all $\psi \in F_n$.

The basic omitting-types theorem asserts that if a type Σ is not isolated over Γ, then Γ has a (countable) model that omits Σ. This can be proven by a refinement of the proof of the completeness theorem sketched in Sect. 3.1, and the required model is the structure $\mathfrak{A}^*$ given there.

To simplify the exposition, let Σ be a 1-type. Since each individual of $\mathfrak{A}^*$ is of the form $\mathsf{c}/{\sim}$ for some $\mathsf{c} \in \mathsf{C}$, to ensure that $\mathfrak{A}^*$ does not realise Σ, it suffices, by clause (b) of the description of $\mathfrak{A}^*$, to show that for each $\mathsf{c} \in \mathsf{C}$, there is some formula $\varphi(x_1) \in \Sigma$ such that $\varphi(\mathsf{c}) \notin \Gamma^*$. Since $\bot \notin \Gamma^*$, this amounts to requiring, for each $\mathsf{c} \in \mathsf{C}$, that Γ^* be closed under the inference

$$\Sigma_{\mathsf{c}} = \big(\{\varphi(\mathsf{c}) : \varphi \in \Sigma\}, \bot\big).$$

Lemma 2 *For any* K*-sentence* σ, $\Gamma \cup \{\sigma\}$ *respects* Σ_{c}.

Proof σ may contain members of C other than c. To simplify the notation again, let σ contain just one C-constant d other than c.

Suppose that $\Gamma \cup \{\sigma(\mathsf{c},\mathsf{d})\} \vdash \varphi(\mathsf{c})$, and hence $\Gamma \vdash \sigma(\mathsf{c},\mathsf{d}) \to \varphi(\mathsf{c})$ for all $\varphi(x_1) \in \Sigma$. Then as c and d do not occur in Γ, it follows by standard properties of first-order deducibility that for all $\varphi(x_1) \in \Sigma$,

$$\Gamma \vdash \exists x_2 \sigma(x_1, x_2) \to \varphi(x_1).$$

But the formula $\exists x_2 \sigma(x_1, x_2)$ belongs to F_1, and so since Σ is not isolated over Γ, it follows that $\exists x_2 \sigma(x_1, x_2)$ is not consistent with Γ. Hence, $\Gamma \vdash \neg\exists x_2 \sigma(x_1, x_2)$, and so $\Gamma \vdash \forall x_1 \forall x_2 \neg\sigma(x_1, x_2)$. This implies $\Gamma \vdash \neg\sigma(\mathsf{c},\mathsf{d})$, and so $\Gamma \cup \{\sigma(\mathsf{c},\mathsf{d})\} \vdash \bot$. □

Now since C is countable, there are countably many rules of the form Σ_{c}. Since the standard deducibility relation of first-order logic is conjunctive, the lemma just proved applies to the Countable Henkin Principle and yields, with Lindenbaum's Lemma, a maximally $\vdash$-consistent extension Γ^* of Γ that is closed under Σ_{c} for all $\mathsf{c} \in \mathsf{C}$. But K is countable since L is countable, and so there are countably many inferences of the form φ_{C} for φ a K-formula with at most one free variable. Hence, if the latter inferences are added to the Σ_{c}, there are still only countably many inferences involved altogether, and so Γ^* can be taken to be closed under each φ_{C} as before.

In fact, the whole argument can begin with a countable number of types, not just one. Each type will contribute a countable number of inferences of the form Σ_{c}, and so, since a countable union of countable sets is countable, this will still involve only countably many inferences altogether. Thus, with no extra work, other than these observations about the sizes of sets of inferences, it may be concluded that any countable collection of non-isolated types is simultaneously omitted by some model of Γ.

3.3 C-*Completeness and* C-*Consistency*

Let L be a countable language that includes a set C of individual constants. A C-*model* is any L-structure in which each individual is the interpretation of some constant from C. Thus an L-structure is a C-model iff it omits the 1-type

$$\Delta_{\mathsf{C}} = \big\{\neg(x = \mathsf{c}) : \mathsf{c} \in \mathsf{C}\big\}.$$

A set Γ of L-sentences is C-*complete* if it respects the inference

$$\varphi_{\mathsf{C}} = \big(\big\{\varphi(\mathsf{c}) : \mathsf{c} \in \mathsf{C}\big\},\ \forall x \varphi(x)\big)$$

for all L-formulas $\varphi(x)$, where we continue to take $\vdash$ to be the standard first-order deducibility relation. Now $\Gamma \vdash \forall x \varphi$ iff $\Gamma \vdash \varphi$ since x is not free in Γ, so C-completeness is equivalent to having Γ respect all the inferences $(\{\varphi(\mathsf{c}) : \mathsf{c} \in \mathsf{C}\}, \varphi)$.

The notion of C-completeness was introduced by Henkin in [15], where he showed that

(i) *if Γ is C-complete, then for any sentence σ that is consistent with Γ, there exists a C-model of $\Gamma \cup \{\sigma\}$.*

As a preliminary lemma, he proved that if a set of sentences Γ is C-complete, then so is $\Gamma \cup \{\sigma\}$ for any sentence σ. This implies that (i) is reducible to the assertion

(ii) *every consistent and* C*-complete set of sentences has a* C*-model.*

Now (ii) can be inferred from the Omitting-Types Theorem, provided that we know that any consistent and C-complete set does not have the type Δ_{C} isolated over it. To prove that, let Γ be consistent and C-complete. Suppose further that there is a formula $\psi(x)$ such that $\Gamma \vdash \psi \to \neg(x = \mathsf{c})$ for all $\mathsf{c} \in \mathsf{C}$. Then $\Gamma \vdash \forall x(\psi \to \neg(x = \mathsf{c}))$, hence $\Gamma \vdash \psi(\mathsf{c}) \to \neg(\mathsf{c} = \mathsf{c})$, and so $\Gamma \vdash \neg\psi(\mathsf{c})$ for all $\mathsf{c} \in \mathsf{C}$. Since Γ is C-complete, this yields $\Gamma \vdash \neg\psi$, so ψ is not consistent with Γ and thus does not isolate Δ_C over Γ. The upshot is that Δ_C is not isolated over Γ, so there is a model of Γ that omits Δ_{C} and hence is a C-model.

Henkin's proof of (i) used his general completeness method and applied an earlier result from his paper [14], which states that any *strongly* C*-consistent* set of sentences has a C-model. Here strong C-consistency of Γ can be defined to mean that there is a function assigning to each formula $\varphi(x)$ with only x free a constant c_φ from C such that every sentence of the form

$$\big(\varphi_1(\mathsf{c}_{\varphi_1}) \to \forall x\varphi_1\big) \wedge \cdots \wedge \big(\varphi_n(\mathsf{c}_{\varphi_n}) \to \forall x\varphi_n\big)$$

is consistent with Γ. This implies that the set Δ of all sentences of the form $\varphi(\mathsf{c}_\varphi) \to \forall x\varphi$ is consistent with Γ, and so $\Gamma \cup \Delta$ has a maximally consistent extension Γ^*. By the nature of Δ, this Γ^* is closed under the rules φ_{C} for all $\varphi(x)$, and the structure $\mathfrak{A}^*$ defined from Γ^* as in Sect. 3.1 is a C-model of Γ.

Steven Orey [18] independently proved essentially the same result that a strongly C-consistent set has a C-model, formulating this in the context that C is a collection of constants denoting the natural numbers, in which case a C-model is called an *ω-model*, and the result provides a completeness theorem for *ω-logic*, which adds to the standard first-order axiomatization the general rule that

$$\text{if } \Gamma \vdash \varphi(\mathsf{c}) \quad \text{for all } \mathsf{c}, \quad \text{then} \quad \Gamma \vdash \forall x\varphi$$

(see [3, Proposition 2.2.13]). Both Henkin and Orey gave examples to show that strong C-consistency of Γ is strictly stronger in general than C-*consistency*, which itself means that there is no formula $\varphi(x)$ such that $\Gamma \vdash \varphi(\mathsf{c})$ for all $\mathsf{c} \in \mathsf{C}$ and $\Gamma \vdash \neg\forall x\varphi(x)$. For ω-logic, this is the notion of ω-consistency introduced by Gödel in proving his incompleteness theorems.

3.4 Completeness for Infinitary Conjunction

The infinitary logic $\mathsf{L}_{\infty\omega}$ generated by a language L has a proper class of individual variables, and a proper class of formulas obtained by allowing, in addition to $\neg\varphi$ and $\forall v\varphi$, formation of the conjunction $\bigwedge \Psi$ of any set Ψ of formulas (disjunction being definable by $\bigwedge$ and $\neg$ as usual).

The deducibility relation for infinitary logic has, in addition to the defining properties of deducibility for first-order logic, the axiom schema

$$\bigwedge \Psi \to \varphi \quad \text{if } \varphi \in \Psi,$$

and the rule of deduction that if

$$\Gamma \vdash \psi \to \varphi \quad \text{for all } \varphi \in \Psi,$$

then

$$\Gamma \vdash \psi \to \bigwedge \Psi.$$

This deducibility relation is not finitary because a set of the form $\Psi \cup \{\neg \bigwedge \Psi\}$ will not be $\vdash$-consistent, but all of its finite subsets could be.

Each formula involved in the following discussion will be assumed to have only a finite number of free variables. This restriction is justified by the fact that it includes all subformulas of infinitary *sentences*.

A *fragment* of $\mathsf{L}_{\infty\omega}$ is a set L_A of $\mathsf{L}_{\infty\omega}$-formulas that includes all first-order L-formulas and is closed under $\neg$, $\forall$, *finite* conjunctions, subformulas, and substitution for variables of terms each of whose variables appears in L_A (cf. [2, p. 84]).

A "weak" completeness theorem [2, Sect. III.4] asserts that if L_A is a *countable* fragment of $\mathsf{L}_{\infty\omega}$ and Γ is a set of L_A-sentences that is consistent, then Γ has a model. To prove this, let C be a denumerable set of new constants, $\mathsf{K} = \mathsf{L} \cup \mathsf{C}$, and K_A the set of all formulas obtained from formulas $\varphi \in \mathsf{L}_A$ by replacing *finitely* many free variables by constants $\mathsf{c} \in \mathsf{C}$. Then K_A is countable and is the smallest fragment of $\mathsf{K}_{\infty\omega}$ that contains L_A. A crucial point to note is that each member of K_A contains only finitely many constants from C.

Now let Φ be the (countable) set of sentences in K_A, and $\vdash$ the restriction of the $\mathsf{K}_{\infty\omega}$-deducibility relation to Φ. To obtain a Γ-model, Γ is to be extended to a subset Γ^* of Φ for which the definition of the model $\mathfrak{A}^*$ can be carried through as for first-order logic and for which the condition

(b) $\mathfrak{A}^* \models \psi[\mathsf{c}_1/\sim, \dots, \mathsf{c}_n/\sim]$ iff $\psi(\mathsf{c}_1, \dots \mathsf{c}_n) \in \Gamma^*$

can be established for each formula $\psi(x_1, \dots, x_n)$ *that belongs to* K_A. Then $\mathfrak{A}^*$ will be a Γ-model since $\Gamma \subseteq \Gamma^*$.

In order for (b) to hold for all K_A-formulas, it is sufficient (and necessary) that the following hold.

(i) Γ^* is maximally *finitely* $\vdash$-consistent: this is sufficient to ensure that $\mathfrak{A}^*$ is well defined; $\neg\varphi \in \Gamma^*$ iff $\varphi \notin \Gamma^*$; $\varphi \to \psi \in \Gamma^*$ iff $\varphi \in \Gamma^*$ implies $\psi \in \Gamma^*$; if $\bigwedge \Psi \in \Gamma^*$, then $\varphi \in \Gamma^*$ for all $\varphi \in \Gamma^*$; and if $\forall x\varphi \in \Gamma^*$, then $\varphi(\mathsf{c}) \in \Gamma^*$ for all $\mathsf{c} \in \mathsf{C}$.
(ii) Γ^* is closed under the inference φ_{C} for each K_A-formula φ with at most one free variable.
(iii) If $\bigwedge \Psi \in \mathsf{K}_A$, and $\Psi \subseteq \Gamma^*$, then $\bigwedge \Psi \in \Gamma^*$, that is, if $\bigwedge \Psi \in \mathsf{K}_A$, then Γ^* is closed under the inference $(\Psi, \bigwedge \Psi)$.

Since K_A is countable, there are countably many inferences involved in fulfilling (ii) and (iii). Hence, by the Countable Henkin Principle and Lindenbaum's Lemma, it suffices to show that for all $\sigma \in \Phi$, $\Gamma \cup \{\sigma\}$ respects each such inference. The proof that $\Gamma \cup \{\sigma\}$ respects φ_{C} is just as for first-order logic since, as noted above, σ has only finitely many constants from C, whereas Γ has no such constants.

For an inference of the form $(\Psi, \bigwedge \Psi)$, observe that if

$$\Gamma \cup \{\sigma\} \vdash \varphi \quad \text{for all } \varphi \in \Psi,$$

then

$$\Gamma \vdash \sigma \to \varphi \quad \text{for all } \varphi \in \Psi,$$

so

$$\Gamma \vdash \sigma \to \bigwedge \Psi,$$

and hence

$$\Gamma \cup \{\sigma\} \vdash \bigwedge \Psi.$$

It is left as an exercise for the reader to formulate and derive an omitting-types theorem for countable fragments of $\mathsf{L}_{\infty\omega}$.

4 Richer Extension Theorems

In the application just discussed, a special role was played by maximally *finitely* consistent sets that are closed under given inference rules. We now study conditions under which *every* consistent set can be extended to one of these special sets.

Let $\mathscr{I}$ be a subset of $2^{\Phi} \times \Phi$, that is, a set of inferences. A set Δ of formulas will be called $(\mathscr{I}, \vdash)$-*rich* if

- Δ is maximally finitely $\vdash$-consistent, and
- Δ is closed under $\mathscr{I}$, that is, if $(\Pi, \chi) \in \mathscr{I}$ and $\Pi \subseteq \Delta$, then $\chi \in \Delta$.

So we have already established the following via the Countable Henkin Principle and Lindenbaum's Lemma:

Theorem 2 (Rich Extension I) *Let $\vdash$ be a deducibility relation on Φ, and $\mathscr{I}$ a countable set of inferences in Φ. If Γ is a $\vdash$-consistent subset of Φ such that $\Gamma \cup \Sigma$ respects $\mathscr{I}$ for all finite $\Sigma \subseteq \Phi$, then Γ has an $(\mathscr{I}, \vdash)$-rich extension.*

The relation $\vdash$ will itself be said to *respect* an inference (Π, χ) if every set of formulas respects (Π, χ) under $\vdash$, that is, if the condition

$$(\Gamma \vdash \varphi, \ \text{all } \varphi \in \Pi) \quad \text{implies} \quad \Gamma \vdash \chi$$

holds for every $\Gamma \subseteq \Phi$. Also, $\vdash$ respects $\mathscr{I}$ if it respects each member of $\mathscr{I}$. If this is so, then in particular $\Gamma \cup \Sigma$ will respect $\mathscr{I}$ for all $\vdash$-consistent Γ and all finite $\Sigma \subseteq \Phi$. Thus, Theorem 2 gives the following:

Theorem 3 (Rich Extension II) *Let $\vdash$ be a deducibility relation that respects all members of a countable set $\mathscr{I}$ of inferences. Then every $\vdash$-consistent set of formulas has an $(\mathscr{I}, \vdash)$-rich extension.*

To refine this result further, we invoke the following property:

Cut Rule: If $\Gamma \vdash \varphi$ for all $\varphi \in \Pi$, and $\Pi \vdash \chi$, then $\Gamma \vdash \chi$.

An inference (Π, χ) will be called $\vdash$-*deducible* if $\Pi \vdash \chi$. The Cut Rule states that if (Π, χ) is $\vdash$-deducible, then $\vdash$ respects it. (The converse is always true: since $\Pi \vdash \varphi$ for all $\varphi \in \Pi$, if $\vdash$ respects (Π, χ), then $\Pi \vdash \chi$ follows.) Hence, Theorem 3 gives the following:

Theorem 4 (Rich Extension III) *Let $\vdash$ be a deducibility relation that satisfies the Cut Rule, and let $\mathscr{I}$ be a countable set of $\vdash$-deducible inferences. Then every $\vdash$-consistent set of formulas has an $(\mathscr{I}, \vdash)$-rich extension.*

5 Classical Deducibility

D1–D4 provided minimal assumptions from which to derive the Henkin Principle and the Rich Extension theorems. But in what follows additional properties of $\vdash$ involving an implication connective $\to$ will be needed. So from now on we assume that Φ is the class of formulas of some language that includes all the classical truth-functional connectives. We can take $\bot$ and $\to$ as primitive, and the other connectives $\top$, $\neg$, $\wedge$, $\vee$ as defined from them in the usual way.

Consider the following further properties of a relation $\vdash$ from 2^{Φ} to Φ:

Assumption Rule: If $\varphi \in \Gamma$ or φ is a tautology, then $\Gamma \vdash \varphi$.
Detachment Rule: $\{\varphi, \varphi \to \psi\} \vdash \psi$.
Tautological Rule: If φ is a tautological consequence of Γ, then $\Gamma \vdash \varphi$.
Deduction Rule: $\Gamma \cup \{\varphi\} \vdash \psi$ implies $\Gamma \vdash \varphi \to \psi$.
Implication Rule: $\Gamma \vdash \psi$ implies $(\varphi \to \Gamma) \vdash \varphi \to \psi$, where

$$(\varphi \to \Gamma) = \{\varphi \to \chi : \chi \in \Gamma\}.$$

Lemma 3 *If $\vdash$ satisfies the Assumption, Detachment and Cut rules, then it satisfies the Tautological Rule.*

Proof From the stated rules it follows that the set $\{\varphi : \Gamma \vdash \varphi\}$ contains all members of Γ and all tautologies, and is closed under Detachment. But by standard theory, any such set contains every tautological consequence of Γ. □

Lemma 4 *If $\vdash$ satisfies the Assumption, Detachment, Cut, and Deduction rules, then it satisfies the Implication Rule and the converse of the Deduction Rule, that is,*

$$\Gamma \vdash \varphi \to \psi \quad \textit{implies} \quad \Gamma \cup \{\varphi\} \vdash \psi.$$

Proof Suppose $\Gamma \vdash \psi$. Since the Tautological Rule gives

$$(\varphi \to \Gamma) \cup \{\varphi\} \vdash \chi \quad \text{for all } \chi \in \Gamma,$$

it then follows by the Cut Rule that $(\varphi \to \Gamma) \cup \{\varphi\} \vdash \psi$. Hence, by the Deduction Rule, $(\varphi \to \Gamma) \vdash \varphi \to \psi$, establishing the Implication Rule.

Next, suppose $\Gamma \vdash \varphi \to \psi$. Now by the Assumption Rule, $\Gamma \cup \{\varphi\} \vdash \chi$ for all $\chi \in \Gamma$, so this yields $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$ by the Cut Rule. But also $\Gamma \cup \{\varphi\} \vdash \varphi$, so the Detachment and Cut rules then give $\Gamma \cup \{\varphi\} \vdash \psi$. □

A relation $\vdash$ satisfying the Assumption, Detachment, Cut, and Deduction rules will be called a *classical deducibility relation*. Conditions D1–D4 of Sect. 1 can be derived from these rules, as the reader may verify. When $\emptyset \vdash \varphi$, where $\emptyset$ is the empty set of formulas, we may write $\vdash \varphi$ and say that φ is $\vdash$-*deducible*, Observe that in order for φ to have $\Gamma \vdash \varphi$ for all Γ (e.g. when φ is a tautology), it is enough by D1 to have $\vdash \varphi$.

We note some properties of a maximally finitely $\vdash$-consistent set Δ that hold when $\vdash$ is classical. Membership of Δ reflects the properties of the truth-functions, and Δ contains all $\vdash$-deducible formulas and is closed under the Detachment Rule, that is,

$$\begin{aligned}
&\bot \notin \Delta, \\
&\neg A \in \Delta \quad \text{iff} \quad A \notin \Delta, \\
&A \to B \in \Delta \quad \text{iff} \quad A \notin \Delta \text{ or } B \in \Delta, \\
&\vdash A \quad \text{implies} \quad A \in \Delta,
\end{aligned}$$

etc. Moreover, a finitely $\vdash$-consistent set Γ is maximally finitely $\vdash$-consistent iff it is *negation complete* in the sense that for all $\varphi \in \Phi$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Such facts can be shown by well-known arguments (or see [6, 8, 9] for details).

6 Probabilistic Modal Logic

The theory of rich sets of formulas will now be applied to a system of propositional logic, originating with Aumann [1], that can express assertions of the type "the probability of φ is at least r". This assertion will be written symbolically as $[r]\varphi$. Here r can be any *rational* number in the real unit interval [0, 1]. The set of all such rationals will be denoted $\mathbb{Q}^{01}$, and the letters r, s, t, u are reserved to name them. We write $\mathbb{R}$ as usual for the set of real numbers, and put $\mathbb{R}^{\geqslant 0} = \{x \in \mathbb{R} : x \geqslant 0\}$.

The symbol $[r]$ itself is a unary modal connective, reminiscent of the "box" modality □. But those familiar with modal logic may care to note that $[r]$ is not in general a *normal* modality: the schemes

$$[r]\varphi \wedge [r]\psi \to [r](\varphi \wedge \psi),$$

$$[r](\varphi \to \psi) \to \big([r]\varphi \to [r]\psi\big)$$

are not valid in the semantics defined below, unless $r = 0$ or 1. On the other hand,

$$[1](\varphi \to \psi) \to \big([r]\varphi \to [r]\psi\big)$$

is always valid, and if a formula φ is valid, then so is $[r]\varphi$. The modalities [1] and [0] are both normal, with [0] being of the "Verum" type, that is, $[0]\varphi$ is valid for every formula φ.

An *algebra* $\mathscr{A}$ on a non-empty set X is a non-empty collection of subsets of X that are closed under complements and binary unions. $\mathscr{A}$ is a *σ-algebra* if it is also closed under countable unions. Then $(X, \mathscr{A})$ is called a *measurable space*, and the members of $\mathscr{A}$ are its *measurable sets*. A *measurable function* $f : (X, \mathscr{A}) \to (X', \mathscr{A}')$ between measurable spaces is given by a set function $f : X \to X'$ that pulls measurable sets back to measurable sets, that is, $f^{-1}(Y) \in \mathscr{A}$ for all $Y \in \mathscr{A}'$. For this, it suffices that $f^{-1}(Y) \in \mathscr{A}$ for all sets Y in some generating subset of $\mathscr{A}'$.

A function $\mu : \mathscr{A} \to \mathbb{R}^{\geqslant 0}$ on an algebra $\mathscr{A}$ is *finitely additive* if $\mu(Y_1 \cup Y_2) = \mu(Y_1) + \mu(Y_2)$ whenever Y_1 and Y_2 are disjoint members of $\mathscr{A}$. μ is *countably additive* if $\mu(\bigcup_n Y_n) = \sum_0^\infty \mu(Y_n)$ whenever $\{Y_n \mid n < \omega\}$ is a sequence of pairwise disjoint members of $\mathscr{A}$ whose union $\bigcup_n Y_n$ belongs to $\mathscr{A}$. A *measure* is a countably additive function with $\mu(\emptyset) = 0$. It is a *probability* measure if also $\mu(X) = 1$.

We need the following standard facts about an algebra $\mathscr{A}$:

- Any measure μ on $\mathscr{A}$ is *continuous from above*, meaning that if $Y_0 \supseteq Y_1 \supseteq \cdots$ is a non-increasing sequence of members of $\mathscr{A}$ whose intersection belongs to $\mathscr{A}$, then $\mu(\bigcap_{n<\omega} Y_n) = \lim_{n\to\infty} \mu(Y_n)$.
- A finitely additive $\mu : \mathscr{A} \to \mathbb{R}^{\geqslant 0}$ is a measure if it is *continuous at* $\emptyset$, that is, $\lim_{n\to\infty} \mu(Y_n) = 0$ for any non-increasing sequence $\{Y_n \mid n < \omega\} \subseteq \mathscr{A}$ with $\bigcap_n Y_n = \emptyset$.

The set of all probability measures on a measurable space $(X, \mathscr{A})$ will be denoted $\mathfrak{P}(X, \mathscr{A})$. It becomes a measurable space itself under the σ-algebra generated by the sets $\{\mu \in \mathfrak{P}(\mathscr{A}) : \mu(Y) \geqslant r\}$ for all $Y \in \mathscr{A}$ and $r \in Q^{01}$.

Now let Φ be the set of formulas generated from a countably infinite set of propositional variables by the classical truth-functional connectives and the modalities $[r]$ for all $r \in Q^{01}$. A formula $[r_0]\cdots[r_{n-1}]\varphi$ generated by iterating modalities will be written more briefly as $[r_0 \cdots r_{n-1}]\varphi$. In the case $r = 0$, this is just the formula φ. Notice that Φ is countable since there are countably many variables and Q^{01} is countable.

A *model* $\mathscr{M}$ for this language is given by a measurable space $(X, \mathscr{A})$ together with a measurable function f from $(X, \mathscr{A})$ to the space $\mathfrak{P}(X, \mathscr{A})$ of probability measures, and a *valuation* that assigns a measurable set $[\![p]\!]^{\mathscr{M}} \in \mathscr{A}$ to each propositional variable p. We typically write μ_x for the measure $f(x)$ assigned to $x \in X$ by f. The valuation of variables is extended inductively to all formulas, using the standard Boolean set operations to interpret the truth-functional connectives by

$$[\![\bot]\!]^{\mathscr{M}} = \emptyset, \qquad [\![\varphi \to \psi]\!]^{\mathscr{M}} = \left(X - [\![\varphi]\!]^{\mathscr{M}}\right) \cup [\![\psi]\!]^{\mathscr{M}},$$

and for the modalities putting

$$[\![[r]\varphi]\!]^{\mathscr{M}} = \left\{x \in X : \mu_x[\![\varphi]\!]^{\mathscr{M}} \geqslant r\right\} = f^{-1}\left\{\mu \in \mathfrak{P}(\mathscr{A}) : \mu[\![\varphi]\!]^{\mathscr{M}} \geqslant r\right\}.$$

The measurability of f ensures that if $[\![\varphi]\!]^{\mathscr{M}}$ is measurable, then so is $[\![[r]\varphi]\!]^{\mathscr{M}}$. Hence, inductively every formula φ is interpreted as a measurable set $[\![\varphi]\!]^{\mathscr{M}}$, thought of as the set of points in X that satisfy φ. By writing $\mathscr{M}, x \models \varphi$ to mean that $x \in [\![\varphi]\!]^{\mathscr{M}}$, the following properties of satisfaction are obtained:

$$\mathscr{M}, x \not\models \bot,$$

$$\mathscr{M}, x \models \varphi \to \psi \quad \text{iff} \quad \mathscr{M}, x \not\models \varphi \quad \text{or} \quad \mathscr{M}, x \models \psi,$$

$$\mathscr{M}, x \models [r]\varphi \quad \text{iff} \quad \mu_x [\![\varphi]\!]^{\mathscr{M}} \geqslant r.$$

Satisfaction of a set Γ of formulas is defined by putting $\mathscr{M}, x \models \Gamma$ iff for all $\varphi \in \Gamma$, $\mathscr{M}, x \models \varphi$. A semantic consequence relation is then defined by putting $\Gamma \models \varphi$ iff $\mathscr{M}, x \models \Gamma$ implies $\mathscr{M}, x \models \varphi$ for all points x of all models $\mathscr{M}$. In the case that $\Gamma = \emptyset$, the relation $\emptyset \models \varphi$ means that φ is *valid*, that is, satisfied at every point of every model.

A set Γ is *unsatisfiable* if $\mathscr{M}, x \not\models \Gamma$ for all $\mathscr{M}$ and x. This is equivalent to having $\Gamma \models \bot$. The relation $\models$ is not finitary: the set

$$\Gamma_s = \{[r]p : r < s\} \cup \{\neg[s]p\}$$

has each of its finite subsets satisfiable but is not itself satisfiable because of the Archimedean property that the real number $\mu_x [\![p]\!]^{\mathscr{M}}$ cannot be less than s but closer to s than any rational $r < s$. Thus, $\Gamma_s \models \bot$, but no finite $\Gamma' \subseteq \Gamma_s$ has $\Gamma' \models \bot$.

Our objective is to axiomatize the relation $\models$ by constructing an abstract deducibility relation $\vdash$ and showing that $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$. For this purpose, define an *Archimedean inference* to be a pair of the form

$$\rho = \left(\{[r_0 \cdots r_{n-1} r]\varphi : r < s\}, [r_0 \cdots r_{n-1} s]\varphi\right) \tag{1}$$

with $s > 0$. Such an inference is sound for the semantic consequence relation. In the case $n = 0$, the relation

$$\{[r]\varphi : r < s\} \models [s]\varphi$$

holds just because of the Archimedean property described above, which is independent of the properties of a measure. But when $n > 0$, the proof that

$$\mathscr{M}, x \models \{[r_0 \cdots r_{n-1} r]\varphi : r < s\} \quad \text{implies} \quad \mathscr{M}, x \models [r_0 \cdots r_{n-1} s]\varphi$$

depends on the measure μ_x being continuous from above [17].

Notice that the inference (1) is really only of interest when $s > 0$ because any formula of the form $[0]\varphi$ is valid, hence so is any of the form $[r_0 \cdots r_{n-1} 0]\varphi$.

Let Ax be the set of all formulas that are instances of the following axioms:

A1 $[1](\varphi \to \psi) \to ([r]\varphi \to [r]\psi)$.
A2 $[0]\bot$.
A3 $[r]\neg\varphi \to \neg[s]\varphi$ if $r + s > 1$.
A4 $[r](\varphi \wedge \psi) \wedge [s](\varphi \wedge \neg\psi) \to [r + s]\varphi$ if $r + s \leqslant 1$.
A5 $\neg[r](\varphi) \wedge \neg[s](\psi) \to \neg[r + s](\varphi \vee \psi)$ if $r + s \leqslant 1$.

These axioms are all valid, and the proof of validity requires only the finite additivity of a probability measure.

Let $\mathscr{R}$ be the set of all Archimedean inferences (1). Then $\mathscr{R}$ is countable because each such inference ρ is determined by its conclusion formula $[r_0 \cdots r_{n-1} s]\varphi$ and there are only countably many formulas. We will abbreviate this conclusion of ρ to $\varphi_\rho(s)$, suppressing the parameters $r_0, \ldots, r_{n-1}$, and write each premise correspondingly as $\varphi_\rho(r)$. So ρ takes the form $(\{\varphi_\rho(r) : r < s\}, \varphi_\rho(s))$. Notice that the form of (1) ensures that $\mathscr{R}$ is closed under application of the modalities, in the sense that for any $u \in Q^{01}$, the pair

$$[u]\rho = \big(\big\{[u]\varphi_\rho(r) : r < s\big\}, [u]\varphi_\rho(s)\big) \tag{2}$$

is also an Archimedean inference.

Now let $\vdash$ be the *smallest* deducibility relation that is classical (i.e. satisfies the Assumption, Detachment, Cut, and Deduction rules) and has the following additional properties:

Axiom Deducibility: $\varphi \in \mathsf{Ax}$ implies $\vdash \varphi$.
Almost Sure Rule: $\vdash \varphi$ implies $\vdash [1]\varphi$.
Archimedean Rule: $\{\varphi_\rho(r) : r < s\} \vdash \varphi_\rho(s)$, for all $\rho \in \mathscr{R}$.

Lemma 5

(1) *Monotone Rule*: $\vdash \varphi \to \psi$ *implies* $\vdash [r]\varphi \to [r]\psi$.
(2) $\vdash [r]\top$.
(3) $\vdash [0]\varphi$.
(4) $\vdash \neg[r]\bot$, *if* $r > 0$.
(5) $\vdash [s]\varphi \to [r]\varphi$, *if* $r < s$.
(6) $\vdash [r]\varphi \to \neg[s]\psi$, *if* $r + s > 1$ *and* $\vdash \neg(\varphi \wedge \psi)$.
(7) $[r]\varphi \wedge [s]\psi \to [r+s](\varphi \vee \psi)$, *if* $r + s \leqslant 1$ *and* $\vdash \neg(\varphi \wedge \psi)$.
(8) $\vdash \varphi$ *implies* $\vdash [r]\varphi$.
(9) $\vdash [r_0 \cdots r_{n-1}0]\varphi$.

Proof (Briefly) For (1), from $\varphi \to \psi$ deduce $[1](\varphi \to \psi)$ by the Almost Sure Rule, then apply axiom A1 and the Detachment Rule. (2) comes directly by the Almost Sure Rule. For (3), use the tautology $\bot \to \varphi$ and apply the Monotone Rule (1) and axiom A2.

(4)–(7) are shown as in Lemmas 4.6 and 4.7 of [10]. (8) follows by the Almost Sure Rule and (5). (9) follows from (3) by repetition of (8). □

Theorem 5 (Soundness) *If* $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$.

Proof The rules defining $\vdash$ are all satisfied when $\vdash$ is replaced by $\models$. Since $\vdash$ is specified to be the *smallest* relation satisfying these rules, the result follows. □

To prove the converse of this result (strong completeness), a canonical model will be constructed whose points are the $(\mathscr{R}, \vdash)$-rich sets of formulas. This requires some proof-theoretic preliminaries concerning the question of when a maximally finitely $\vdash$-consistent set Δ is closed under the rules from $\mathscr{R}$. For Δ to be closed under a given rule ρ, with conclusion $\varphi_\rho(s)$, it is enough that Δ decides ρ, that is, either $\varphi_\rho(s) \in \Delta$ or $\neg\varphi_\rho(r) \in \Delta$ for some $r < s$. By properties of maximally finitely consistent sets, this is equivalent to asking that for some $r < s$, $(\neg\varphi_\rho(r) \vee \varphi_\rho(s)) \in \Delta$, or equivalently, $(\varphi_\rho(r) \wedge \neg\varphi_\rho(s)) \notin \Delta$. That provides some motivation for the next result, which uses the full strength of the Archimedean Rule.

Lemma 6 *Let Δ be an $(\mathscr{R},\vdash)$-rich set of formulas. For any rule $\rho \in \mathscr{R}$ with conclusion $\varphi_\rho(s)$, and any positive $t \in Q^{01}$, there exists an $r < s$ with*

$$[t]\big(\varphi_\rho(r) \wedge \neg\varphi_\rho(s)\big) \notin \Delta.$$

Proof Let $L = \{u \in Q^{01} : [u]\varphi_\rho(s) \in \Delta\}$. Then $0 \in L$ since the formula $[0]\varphi_\rho(s)$ is deducible by Lemma 5(3) and so belongs to Δ. Thus, L is non-empty and must have a supremum $l \leqslant 1$. Then in general,

$$[u]\varphi_\rho(s) \in \Delta \quad \text{iff} \quad u \leqslant l. \tag{3}$$

For if $u \leqslant l$ and $r < u$, there must be a $v \in L$ with $r < v$. But $[v]\varphi_\rho(s) \to [r]\varphi_\rho(s)$ is deducible (Lemma 5(5)) and so belongs to Δ, and this leads to $[r]\varphi_\rho(s) \in \Delta$. Hence, we have $\{[r]\varphi_\rho(s) : r < u\} \subseteq \Delta$, which by $\mathscr{R}$-closure of Δ implies $[u]\varphi_\rho(s) \in \Delta$, giving (3).

Now take the case $l = 1$. Then $[1]\varphi_\rho(s) \in \Delta$. But $[t]\neg\varphi_\rho(s) \to \neg[1]\varphi_\rho(s)$ is in Δ by axiom A3, so this implies that $[t]\neg\varphi_\rho(s) \notin \Delta$. Thus, we can pick *any* $r < s$ and conclude that $[t](\varphi_\rho(r) \wedge \neg\varphi_\rho(s)) \notin \Delta$ with the help of the Monotone Rule.

Alternatively, $l < 1$. In that case, choose a $u \in Q^{01}$ with $l < u < l + t$, and then a $v \in Q^{01}$ with $u - t \leqslant v \leqslant l$. Since $l < u$, $[u]\varphi_\rho(s) \notin \Delta$. But Δ is closed under the inference $[u]\rho$ (2), so there is some $r < s$ with $[u]\varphi_\rho(r) \notin \Delta$. To complete the lemma, we need to show that it is *false* that

$$[t]\big(\varphi_\rho(r) \wedge \neg\varphi_\rho(s)\big) \in \Delta. \tag{4}$$

Now $\varphi_\rho(s) \to \varphi_\rho(r)$ is deducible, by Lemma 5(5) and the Monotone Rule. Hence, $[v]\varphi_\rho(s) \to [v](\varphi_\rho(r) \wedge \varphi_\rho(s))$ is deducible by the Tautological and Monotone rules. But $[v]\varphi_\rho(s) \in \Delta$ as $v \leqslant l$ (3), so this implies

$$[v]\big(\varphi_\rho(r) \wedge \varphi_\rho(s)\big) \in \Delta. \tag{5}$$

Assume, for contradiction, that (4) is true. Then we must have $v + t \leqslant 1$ since otherwise by Lemma 5(6), $[v](\varphi_\rho(r) \wedge \varphi_\rho(s)) \to \neg[t](\varphi_\rho(r) \wedge \neg\varphi_\rho(s))$ is deducible, which is incompatible with (5) and (4) both holding. But now if $v + t \leqslant 1$, axiom A4 combines with (5) and (4) to give $[v + t]\varphi_\rho(r) \in \Delta$. Since we already have $[u]\varphi_\rho(r) \notin \Delta$, this contradicts the fact that $u \leqslant v + t$.

Altogether, we have shown that (4) cannot be true, completing the proof. □

Next, let $\{\rho_1, \ldots, \rho_n, \ldots\}$ be an enumeration of $\mathscr{R}$ with

$$\rho_n = \big(\{\varphi_n(r) : r < s_n\}, \varphi_n(s_n)\big).$$

Fix an $(\mathscr{R},\vdash)$-rich set Δ and a positive $t \in Q^{01}$. For each $n \geqslant 1$, define $t_n = t/2^n$. Then by Lemma 6 there exists an $r_n < s_n$ such that

$$[t_n]\psi_n \notin \Delta, \tag{6}$$

where $\psi_n = (\varphi_n(r_n) \wedge \neg\varphi_n(s_n))$.

Lemma 7 *For all $n \geqslant 1$, $[t](\psi_1 \vee \cdots \vee \psi_n) \notin \Delta$.*

Proof Observe that $t_1 + \cdots + t_n = \sum_{i \leqslant n} t/2^i < t \leqslant 1$.

Now for all $i \leqslant n$, $[t_i]\psi_i \notin \Delta$ by (6), hence $\neg[t_i]\psi_i \in \Delta$. So from axiom A5 we get $\neg[t_1 + \cdots + t_n](\bigvee_{i \leqslant n} \psi_i) \in \Delta$, hence $[t_1 + \cdots + t_n](\bigvee_{i \leqslant n} \psi_i) \notin \Delta$. The desired result now follows as $\vdash [t](\bigvee_{i \leqslant n} \psi_i) \to [t_1 + \cdots + t_n](\bigvee_{i \leqslant n} \psi_i)$ by Lemma 5(5). □

For any set Γ of formulas, let $\bigwedge_\omega \Gamma$ be the set of conjunctions of all finite subsets of Γ, that is, $\bigwedge_\omega \Gamma = \{\bigwedge \Sigma : \Sigma \subseteq \Gamma$ and Σ is finite$\}$, and for any $r \in Q^{01}$, let

$$[r]\bigwedge_\omega \Gamma = \Big\{[r]\chi : \chi \in \bigwedge_\omega \Gamma\Big\}.$$

Lemma 8 *For any positive $t \in Q^{01}$, if $[t]\bigwedge_\omega \Gamma$ can be extended to a $(\mathscr{R}, \vdash)$-rich set, then so can Γ.*

Proof Assume that $[t]\bigwedge_\omega \Gamma \subseteq \Delta$ for some $(\mathscr{R}, \vdash)$-rich Δ. Let

$$\Theta_0 = \Gamma \cup \{\neg\psi_n : n \geqslant 1\},$$

where the ψ_n are obtained from Δ as above and satisfy Lemma 7.

Now if Θ_0 is finitely $\vdash$-consistent, it must have a maximally finitely $\vdash$-consistent extension Θ. Then for each $n \geqslant 1$, the formula $\psi_n = (\varphi_n(r_n) \wedge \neg\varphi_n(s_n))$ is not in Θ because $\neg\psi_n \in \Theta$. This, as explained above, ensures that $\neg\varphi_n(r_n) \in \Theta$ or $\varphi_n(s_n) \in \Theta$, so Θ decides ρ_n and therefore is closed under ρ_n. The upshot is that Θ is a maximally finitely $\vdash$-consistent extension of Γ that is $\mathscr{R}$-closed and hence is $(\mathscr{R}, \vdash)$-rich, completing the proof.

So it remains to show that Θ_0 is finitely $\vdash$-consistent. But if it were not, there would be some finite $\Sigma \subseteq \Gamma$ and some n such that

$$\Sigma \cup \{\neg\psi_1, \ldots, \neg\psi_n\} \vdash \bot.$$

That would lead to $\Sigma \vdash \bigvee_{i \leqslant n} \psi_i$ and hence to $\vdash \bigwedge \Sigma \to \bigvee_{i \leqslant n} \psi_i$. By the Monotone Rule it would follow that $\vdash [t]\bigwedge \Sigma \to [t]\bigvee_{i \leqslant n} \psi_i$. But $[t]\bigwedge \Sigma$ belongs to Δ by assumption, so this would imply that $[t]\bigvee_{i \leqslant n} \psi_i \in \Delta$, contradicting Lemma 7. □

We are now ready to construct the canonical model, to be denoted $\mathscr{M}_c$. Let X_c be the set of all $(\mathscr{R}, \vdash)$-rich sets of formulas. For each formula φ, let $|\varphi| = \{\Delta \in X_c : \varphi \in \Delta\}$, and put $\mathscr{A}_c = \{|\varphi| : \varphi$ is a formula$\}$. Then $\mathscr{A}_c$ is an algebra since $X - |\varphi| = |\neg\varphi|$ and $|\varphi| \cup |\psi| = |\varphi \vee \psi|$.

For $\Delta \in X_c$, define $\mu_\Delta : \mathscr{A}_c \to [0, 1]$ by putting

$$\mu_\Delta|\varphi| = \sup\big\{u \in Q^{01} : [u]\varphi \in \Delta\big\}.$$

Thus, the number l defined at the beginning of the proof of Lemma 6 is $\mu_\Delta|\varphi_\rho(s)|$. In general, we have

$$\mu_\Delta|\varphi| \geqslant r \quad \text{iff} \quad [r]\varphi \in \Delta. \tag{7}$$

For if $\mu_\Delta|\varphi| \geqslant r$, then for any $s < r$, by definition of $\mu_\Delta|\varphi|$ as a supremum there is a $u > s$ with $[u]\varphi \in \Delta$. Hence, $[s]\varphi \in \Delta$ by Lemma 5(5). Thus, $\{[s]\varphi : s < r\} \subseteq \Delta$, implying $[r]\varphi \in \Delta$ because Δ is $\mathscr{R}$-closed. The converse holds by definition.

μ_Δ can be shown to be a well-defined finitely additive function as in [10, Theorem 5.4]. By results (3) and (4) of Lemma 5, $\mu_\Delta \emptyset = \mu_\Delta |\bot| = 0$, and by result (2), $\mu_\Delta X_c = \mu_\Delta |\top| = 1$. Moreover, μ_Δ is continuous from above at $\emptyset$. To see why, suppose that $|\varphi_0| \supseteq |\varphi_1| \supseteq \cdots$ is a nested sequence of members of $\mathscr{A}_c$ whose intersection is empty. Then we want to show that $\lim_{n\to\infty} \mu_\Delta |\varphi_n| = 0$. Now by finite additivity, the number sequence $\{\mu_\Delta |\varphi_n| : n < \omega\}$ is non-increasing. If this sequence did not converge to 0, there would exist a positive rational t with $\mu_\Delta |\varphi_n| \geqslant t$ for all n. Then if $\Gamma = \{\varphi_n : n < \omega\}$, then any $\psi \in \bigwedge_\omega \Gamma$ has $|\psi| = |\varphi_m|$ for some m since the $|\varphi_n|$ are nested, hence $\mu_\Delta(|\psi|) \geqslant t$, so $[t]\psi \in \Delta$ by (7). This shows that $[t](\bigwedge_\omega \Gamma)$ is a subset of Δ. Our Lemma 8 then implies that there is a $\Theta \in X_c$ with $\Gamma \subseteq \Theta$. But then $\Theta \in \bigcap_{n<\omega} |\varphi_n|$, contradicting the assumption that this intersection is empty. Hence, $\{\mu_\Delta |\varphi_n| : n < \omega\}$ does converge to 0.

Thus, indeed μ_Δ is continuous from above at $\emptyset$ and so is countably additive and a probability measure. Standard measure theory then tells us that it extends uniquely to a probability measure on $\sigma(\mathscr{A}_c)$, the σ-algebra on X_c generated by $\mathscr{A}_c$. We call this extension μ_Δ as well. The map $\Delta \mapsto \mu_\Delta$ from $(X_c, \sigma(\mathscr{A}_c))$ to its space of probability measures $\mathfrak{P}(X_c, \sigma(\mathscr{A}_c))$ is then a measurable map because each measurable set of the form $\{\mu : \mu|\varphi| \geqslant r\}$ in the space of measures pulls back along this map to the measurable set $|[r]A| \in \mathscr{A}_c$, by (7).

The model $\mathscr{M}_c$ is now defined on the measurable space $(X_c, \sigma(\mathscr{A}_c))$ by putting $[\![p]\!]^{\mathscr{M}_c} = |p|$ for all propositional variables p.

Lemma 9 (Truth Lemma) *Every formula* φ *has* $[\![\varphi]\!]^{\mathscr{M}_c} = |\varphi|$, *that is, for all* $\Delta \in X_c$,

$$\mathscr{M}_c, \Delta \models \varphi \quad \textit{iff} \quad \varphi \in \Delta.$$

Proof By induction on the formation of φ, with the base case holding by the definition of $[\![p]\!]^{\mathscr{M}_c}$. The cases of the truth-functional connectives are standard, and the case of a formula $[r]\varphi$, assuming the result for φ, is given by (7). □

Corollary 1

(1) *Every* $(\mathscr{R}, \vdash)$*-rich set* Δ *is* $\vdash$*-deductively closed, that is,* $\Delta \vdash \varphi$ *implies* $\varphi \in \Delta$, *and is maximally* $\vdash$*-consistent.*
(2) *A set of formulas is* $\vdash$*-consistent iff it has an* $(\mathscr{R}, \vdash)$*-rich extension.*

Proof (1) By the Truth Lemma, every member of Δ is satisfied at the point Δ in $\mathscr{M}_c$, that is, $\mathscr{M}_c, \Delta \models \Delta$. Thus, if $\Delta \vdash \varphi$, then $\Delta \models \varphi$ by soundness, hence $\mathscr{M}_c, \Delta \models \varphi$, and so the Truth Lemma gives $\varphi \in \Delta$.

It follows that Δ is $\vdash$-consistent since if $\Delta \vdash \bot$, then $\bot \in \Delta$, which would contradict the finite $\vdash$-consistency of Δ. But Δ has no proper $\vdash$-consistent extensions since by the definition of "rich" it has no proper finitely $\vdash$-consistent extensions.

(2) From left to right holds by Rich Extension Theorem 4. Conversely, if Γ has an $(\mathscr{R}, \vdash)$-rich extension Δ, then since $\bot \notin \Delta$, we have $\Delta \nvdash \bot$ by (1), hence $\Gamma \nvdash \bot$ by D1, so Γ is $\vdash$-consistent. □

Remark 1 A natural question here is whether this $\vdash$-deductive closure of a rich set can be shown proof-theoretically, rather than by a model-theoretic detour through the canonical model. A strategy for this is to define the notion of a "theory" as being a set with suitable closure properties, including in this case closure under the Detachment Rule and under $\mathscr{R}$, and then to define a relation $\Gamma \vdash^+ \varphi$ to mean that φ belongs to every theory that includes Γ. In other words, the set $\{\varphi : \Gamma \vdash^+ \varphi\}$ is the intersection of all theories that include Γ, and will typically be a theory as well. Thus, if Γ itself is a theory, it will be $\vdash^+$-deductively closed *by definition*. Moreover, rich sets will be theories, hence a rich set is $\vdash^+$-deductively closed. One then shows that $\vdash^+$ satisfies all the rules that define $\vdash$, from which it follows that in general $\Gamma \vdash \varphi$ implies $\Gamma \vdash^+ \varphi$ because $\vdash$ is specified to be the *smallest* relation satisfying these rules. Now if Γ is rich and $\Gamma \vdash \varphi$, we get $\Gamma \vdash^+ \varphi$, and hence $\varphi \in \Gamma$ because a rich set is $\vdash^+$-deductively closed.

This kind of analysis is applied to modal logic in [9], to dynamic/algorithmic logic in [5], and to coalgebraic logic in [4, 10].

We turn now to our main goal for probabilistic modal logic:

Theorem 6 (Strong Completeness)

(1) *Every* $\vdash$*-consistent set of formulas is satisfiable.*
(2) *If* $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$*.*

Proof (1) and (2) are equivalent. We prove (1) and infer (2).

For (1), if Γ is $\vdash$-consistent, then by the Rich Extension Theorem 4 there is a $\Delta \in X_c$ with $\Gamma \subseteq \Delta$. By the Truth Lemma 9, $\mathscr{M}_c, \Delta \models \Delta$, so Γ is satisfied at Δ in $\mathscr{M}_c$. For (2), if $\Gamma \nvdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is $\vdash$-consistent by D4, so is satisfiable by (1), implying $\Gamma \not\models \varphi$. □

Finally on this topic, here are some remarks on the background to the strong completeness result. Heifetz and Mongin [11] gave a finitary axiomatization of the set $\{\varphi : \emptyset \models \varphi\}$ of valid formulas, using a set of axioms similar to our A1–A5, and a special rule that derives assertions about the probabilities of two finite sets of formulas that are equivalent in a suitable sense. Zhou [21, 22] replaced this rule by the infinitary inference scheme

$$\text{from } \vdash \varphi \to [r]\psi \quad \text{for all } r < s, \quad \text{infer} \quad \vdash \varphi \to [s]\psi$$

and showed that the resulting system was equivalent to that of [11]. He also gave an example in [23] to show that the canonical measure associated with a maximally consistent set for this system need not be countably additive.

The present author showed in [10] that the countable additivity of canonical measures follows from the rule

$$\Gamma \vdash \bot \quad \text{implies} \quad [t]\bigwedge_{\omega} \Gamma \vdash \bot \tag{8}$$

for positive t and *countable* Γ. This is itself derivable from the rule

$$\Gamma \vdash \varphi \quad \text{implies} \quad [t]\bigwedge_{\omega} \Gamma \vdash [t]\varphi \tag{9}$$

by taking $\varphi = \bot$ and using $\vdash \neg[t]\bot$ (Lemma 5(4)). For countable Γ, (9) is sound for the present semantics (i.e. is true when $\vdash$ is replaced by $\models$), as can be shown using continuity of a measure from above [10, Theorem 4.8]. The reference to countability of Γ was required because some of the logics discussed in [10] have uncountably many formulas.

In that situation a "rich extension" theorem is not always available, and the analysis of μ_Δ given in [10] depended on $\vdash$ having the *Lindenbaum property* that every $\vdash$-consistent set has a maximally $\vdash$-consistent extension. The semantic consequence relation $\models$ has this property since $\models$-consistency just means satisfiability, and if Γ is satisfied at x in $\mathscr{M}$, then $\{\varphi : \mathscr{M}, x \models \varphi\}$ is a maximally $\models$-consistent set extending Γ. It was shown that $\models$ is characterizable proof-theoretically as the *smallest* deducibility relation having the Lindenbaum property [10, Theorem 5.17].

The striking insight that the set $\mathscr{R}$ of Archimedean inferences suffices to show that μ_Δ is countably additive is due to the authors of [17]. Our Lemmas 6–8 are motivated by the Stone duality constructions of Sect. VI of [17] and are intended to give a proof-theoretic manifestation of that insight. Note the equivalence of Lemma 8 and the rule (8) by Corollary 1(2). The countability of the language and of $\mathscr{R}$ ensures that the Henkin method can be applied to show that sufficiently many $\mathscr{R}$-rich extensions exist to prove Strong Completeness.

7 The Baire Category Connection

The year after Henkin's new first-order completeness proof appeared in print, Helena Rasiowa and Roman Sikorski published another proof based on Boolean-algebraic and topological ideas [19]. This introduced what became known as the *Rasiowa–Sikorski Lemma*, stating that any non-unit element of a Boolean algebra belongs to a prime ideal that preserves countably many prescribed joins. They proved this by using the fact that the Stone space of a Boolean algebra satisfies the Baire Category Theorem: the intersection of countably many open dense sets is dense.

There are intimate relationships between the Rasiowa–Sikorski Lemma, the Baire Category Theorem for certain kinds of spaces, and the Countable Henkin Principle. These are explored in [7]. Here we will illustrate the connection by giving a rapid version of a topological proof of our Rich Extension Theorem 2. As will be evident, this is rather more elaborate than the direct deducibility-theoretic proof.

Let X_m be the set of maximally finitely $\vdash$-consistent sets of subsets of Φ, where $\vdash$ is a classical deducibility relation. For each $\varphi \in \Phi$, let $|\varphi|_m = \{\Delta \in X_m : \varphi \in \Delta\}$. The collection $\{|\varphi|_m : \varphi \in \Phi\}$ is a basis for a compact Hausdorff topology on X_m, in which each basic open $|\varphi|_m$ is also closed, because $X_m - |\varphi|_m = |\neg\varphi|_m$.

Now let Γ be $\vdash$-consistent. Define $\Gamma^\vdash = \{\varphi : \Gamma \vdash \varphi\}$, and restrict X_m to the set $X_\Gamma = \{\Delta \in X_m : \Gamma^\vdash \subseteq \Delta\}$. Then $X_\Gamma = \bigcap\{|\varphi|_m : \varphi \in \Gamma^\vdash\}$, so X_Γ is a closed subset of X_m and hence becomes a compact Hausdorff space itself under the subspace topology. A basis for this subspace topology is $\{|\varphi|_\Gamma : \varphi \in \Phi\}$, where $|\varphi|_\Gamma = |\varphi|_m \cap X_\Gamma = \{\Delta \in X_\Gamma : \varphi \in \Delta\}$.

Note that $\Gamma \subseteq \Gamma^\vdash$ by D2. Also, $\Gamma^\vdash$ is closed under $\vdash$-deducibility since if $\Gamma^\vdash \vdash \chi$, then $\Gamma \vdash \chi$ by the Cut Rule. Consequently, $\Gamma^\vdash$ is $\vdash$-consistent because Γ is. Therefore, by Lindenbaum's Lemma, X_Γ is non-empty. Moreover, using the Deduction Rule and its converse, we get that in general

$$\Gamma \cup \{\psi\} \vdash \varphi \quad \text{iff} \quad \Gamma^+ \cup \{\psi\} \vdash \varphi \quad \text{iff} \quad (\psi \to \varphi) \in \Gamma^+. \tag{10}$$

Given an inference $\rho = (\Pi, \chi)$, define $U_\rho = \{\Delta \in X_\Gamma : \Delta \text{ is closed under } \rho\}$. Since members of X_Γ are negation complete, a given $\Delta \in X_\Gamma$ is closed under ρ iff it either contains χ or contains $\neg\varphi$ for some $\varphi \in \Pi$. Thus,

$$U_\rho = |\chi|_\Gamma \cup \bigcup \{|\neg\varphi|_\Gamma : \varphi \in \Pi\},$$

which shows that U_ρ is open, being a union of (basic) open sets.

Lemma 10 *If $\Gamma \cup \{\psi\}$ respects ρ for all $\psi \in \Phi$, then the open set U_ρ is dense in the space X_Γ.*

Proof Let $\Gamma \cup \{\psi\}$ respect ρ for all $\psi \in \Phi$. To prove density of U_ρ, it suffices to show that any non-empty basic open set $|\psi|_\Gamma$ intersects U_ρ. So assume that $|\psi|_\Gamma \neq \emptyset$.

First, take the case that for some $\varphi \in \Pi$, the set $\Gamma^\vdash \cup \{\psi, \neg\varphi\}$ is $\vdash$-consistent. Then this set extends by Lindenbaum's Lemma to a $\Delta \in X_\Gamma$ that belongs to $|\psi|_\Gamma \cap |\neg\varphi|_\Gamma \subseteq |\psi|_\Gamma \cap U_\rho$, giving the desired result that $|\psi|_\Gamma \cap U_\rho \neq \emptyset$.

The alternative case is that for all $\varphi \in \Pi$, we have $\Gamma^\vdash \cup \{\psi, \neg\varphi\} \vdash \bot$; hence, $\Gamma^\vdash \cup \{\psi\} \vdash \varphi$ by D4, so $\Gamma \cup \{\psi\} \vdash \varphi$ by (10). But $\Gamma \cup \{\psi\}$ respects ρ, so then $\Gamma \cup \{\psi\} \vdash \chi$. Hence, $(\psi \to \chi) \in \Gamma^\vdash$ by (10).

By assumption there exists some $\Delta \in |\psi|_\Gamma$. Since $\Gamma^\vdash \subseteq \Delta$, we get $(\psi \to \chi) \in \Delta$ and $\psi \in \Delta$, giving $\chi \in \Delta$ and thus $\Delta \in |\psi|_\Gamma \cap |\chi|_\Gamma \subseteq |\psi|_\Gamma \cap U_\rho$. So again $|\psi|_\Gamma \cap U_\rho \neq \emptyset$ as required. □

Now to prove the Rich Extension theorem, let $\mathscr{I}$ be a countable set of inferences such that $\Gamma \cup \{\psi\}$ respects $\mathscr{I}$ for all $\psi \in \Phi$. Define $U_{\mathscr{I}} = \bigcap\{U_\rho : \rho \in \mathscr{I}\}$. Then U_ρ is the set of all $(\mathscr{I}, \vdash)$-rich extensions of $\Gamma^\vdash$. By the lemma just proved, $U_{\mathscr{I}}$ is the intersection of countably many open dense subsets of the space X_Γ. Hence, $U_{\mathscr{I}}$ is dense in X_Γ since the Baire Category Theorem holds for any compact Hausdorff space. Since $X_\Gamma \neq \emptyset$, it follows that $U_{\mathscr{I}} \neq \emptyset$. Any member of $U_{\mathscr{I}}$ is an $(\mathscr{I}, \vdash)$-rich extension of Γ fulfilling Theorem 2.

References

1. Aumann, R.J.: Interactive epistemology II: Probability. Int. J. Game Theory **28**, 301–314 (1999)
2. Barwise, J.: Admissible Sets and Structures. Springer, Berlin (1975)
3. Chang, C.C., Keisler, H.J.: Model Theory, 2nd edn. North-Holland, Amsterdam (1977)
4. Friggens, D., Goldblatt, R.: A modal proof theory for final polynomial coalgebras. Theor. Comput. Sci. **360**, 1–22 (2006)
5. Goldblatt, R.: Axiomatising the Logic of Computer Programming. Lecture Notes in Computer Science, vol. 130. Springer, Berlin (1982)
6. Goldblatt, R.: An abstract setting for Henkin proofs. Topoi **3**, 37–41 (1984)
7. Goldblatt, R.: On the role of the Baire category theorem and dependent choice in the foundations of logic. J. Symb. Log. **50**, 412–422 (1985)
8. Goldblatt, R.: An abstract setting for Henkin proofs. In: Mathematics of Modality, CSLI Lecture Notes, vol. 43, pp. 191–212. CSLI Publications, Stanford (1993). Expanded version of [6]
9. Goldblatt, R.: A framework for infinitary modal logic. In: Mathematics of Modality, CSLI Lecture Notes, vol. 43, pp. 213–229. CSLI Publications, Stanford (1993)
10. Goldblatt, R.: Deduction systems for coalgebras over measurable spaces. J. Log. Comput. **20**(5), 1069–1100 (2010). doi:10.1093/logcom/exq029. Cited 12 December 2008
11. Heifetz, A., Mongin, Ph.: Probability logic for type spaces. Games Econ. Behav. **35**, 31–53 (2001)

12. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**, 159–166 (1949)
13. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**, 81–91 (1950)
14. Henkin, L.: A generalisation of the concept of ω-consistency. J. Symb. Log. **19**, 183–196 (1954)
15. Henkin, L.: A generalisation of the concept of ω-completeness. J. Symb. Log. **22**, 1–14 (1957)
16. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**(2), 127–158 (1996)
17. Kozen, D., Larsen, K.G., Mardare, R., Panangaden, P.: Stone duality for Markov processes. In: Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2013), pp. 321–330 (2013). Available via doi:10.1109/LICS.2013.38
18. Orey, S.: ω-consistency and related properties. J. Symb. Log. **21**, 246–252 (1956)
19. Rasiowa, H., Sikorski, R.: A proof of the completeness theorem of Gödel. Fundam. Math. **37**, 193–200 (1950)
20. Sacks, G.E.: Saturated Model Theory. Benjamin, Reading (1972)
21. Zhou, Ch.: A Complete Deductive System for Probability Logic with Application to Harsanyi Type Spaces. Ph.D. thesis, India University (2007)
22. Zhou, Ch.: A complete deductive system for probability logic. J. Log. Comput. **19**(6), 1427–1454 (2009)
23. Zhou, Ch.: Intuitive probability logic. In: Ogihara, M., Tarui, J. (eds.) TAMC 2011: Theory and Applications of Models of Computation—8th Annual Conference. Lecture Notes in Computer Science, vol. 6648, pp. 240–251. Springer, Berlin (2011)

R. Goldblatt (✉)
School of Mathematics, Statistics and Operations Research, Victoria University of Wellington,
PO Box 600, Wellington 6140, New Zealand
e-mail: rob.goldblatt@msor.vuw.ac.nz

Reflections on a Theorem of Henkin

William Gunther and Richard Statman

Abstract The $\lambda\delta$-calculus is the λ-calculus augmented with a discriminator which distinguishes terms. We consider the simply typed $\lambda\delta$-calculus over one atomic type variable augmented additionally with an existential quantifier and a description operator, all of lowest type. First we provide a proof of a folklore result which states that a function in the full type structure of $[n]$ is $\lambda\delta$-definable from the description operator and existential quantifier if and only if it is symmetric, that is, fixed under the group action of the symmetric group of n elements. This proof uses only elementary facts from algebra and a way to reduce arbitrary functions to functions of lowest type via a theorem of Henkin. Then we prove a necessary and sufficient condition for a function on $[n]$ to be $\lambda\delta$-definable without the description operator or existential quantifier, which requires a stronger notion of symmetry.

Keywords Lambda calculus · Lambda delta calculus · Types · Typed lambda calculus · Simply typed lambda calculus · Type theory · Classical type theory · First-order logic · Henkin semantics · Typed lambda calculus semantics · Delta discriminator · Description operator

1 Introduction

Let $\mathscr{M}^n$ be the full type structure over a ground domain of size n. It is folklore that a member of $\mathscr{M}^n$ is symmetric if and only if it is definable in type theory. The origin of this theorem is murky. It is safe to say that it was not known to Newton. Robin Gandy told the senior author that he knew it in the 1940s. It is not unlikely it was known to Church before this. It occurred to the senior author as a student in the 1970s after reading Andrews [1] and Lauchli [5]. There are not many proofs in the literature. A proof appears in Van Benthem [8] in the 1990s, but it is incomplete. A proof follows immediately from an observation due to Leon Henkin [4].

In this note, we intend to do two things. First, we shall generalize the folklore theorem to Church's $\lambda\delta$-calculus [3] for the $\mathscr{M}^n$, and also to the "profinite" model which is the "limit" of the $\mathscr{M}^n$. Second, we shall provide a straightforward proof of the folklore theorem itself using only simple facts about the symmetric group and its action on equivalence relations.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_14

2 Preliminaries

We first do a review of the lambda calculus, simple types, Henkin-style semantics, and extending simple typed lambda calculus to type theory.

2.1 Lambda Calculus

Untyped Lambda Calculus

The untyped lambda calculus serves as the underlying language of our more disciplined systems. We give a quick reminder.

Definition 1 Fix some countable set of variables $V = \{x_1, x_2, \ldots\}$. We define the set of λ-terms, which we denote by Λ inductively:

Variables $V \subseteq \Lambda$
Abstraction If $x \in V$ and $M \in \Lambda$, then $(\lambda x \centerdot M) \in \Lambda$
Application If $M, N \in \Lambda$, then $(MN) \in \Lambda$.

We will always identify terms that are the same up to a renaming of bound variables (α equivalence). Moreover, we define a notion of convertibility $=_{\beta\eta}$ formed by the reduction rules

$$(\lambda x \centerdot M)N \to_\beta M[x := N]$$
$$(\lambda x \centerdot Mx) \to_\eta M \quad \text{variable } x \text{ does not appear free in } M$$

Using this notion of reduction, a term M is in $\beta\eta$ *normal form* if there is no term N such that $M \to_{\beta\eta} N$.

We write $\Lambda^{\emptyset}$ for the set of all closed terms, that is, terms with no free variables.

Remark 1 (Notational Conventions for Terms) We will follow all ordinary conventions when writing terms. When parentheses are omitted, we will associate terms to the left, so that MNP is parsed as $(MN)P$. The symbol $\centerdot$ which is "Church's dot notation" will be used for binders to remind the reader that we are binding in the largest scope possible; so in $\lambda x \centerdot MN$ the x is bound by the λ in both M and N. We write λ-terms with uppercase Latin letters, like F, G, M, N.

What follows are a few useful closed terms that we will use throughout the paper:

$$S := \lambda xyz \centerdot xz(yz)$$
$$K = \textbf{True} := \lambda xy \centerdot x$$
$$K^* = \textbf{False} := \lambda xy \centerdot y$$
$$\textbf{And} := \lambda mn \centerdot \lambda xy \centerdot m(nxy)y$$
$$\textbf{Not} := \lambda m \centerdot \lambda xy \centerdot myx$$

Simple Types

Here, we will put a typing discipline on Λ.

Definition 2 On Λ we define the simple typing system in the style of Church, which we notate $\lambda_\to$. We will have only one type constant, which we denote 0. The set of types, T, is inductively defined as the smallest set containing 0 and closed under arrows, that is, if α and β are types, then $\alpha \to \beta$ is a type. We will use lower case Greek letters at the beginning of the alphabet for types, such as α, β, γ.

For the terms of the system, we mimic what we did the untyped case, but with restrictions. We defined the set $\Lambda_\to(\alpha)$, the set of typed terms of type α, by induction:

Variables If $x \in V$ and α is a type, then $x^\alpha \in \Lambda_\to(\alpha)$.
Abstraction If $M \in \Lambda_\to(\beta)$ and $x \in V$, then

$$\left(\lambda x^\alpha . M\right) \in \Lambda_\to(\alpha \to \beta)$$

Application If $M \in \Lambda_\to(\alpha \to \beta)$ and $N \in \Lambda_\to(\alpha)$ then

$$MN \in \Lambda_\to(\beta)$$

We write $\Lambda^{\emptyset}_\to(\alpha)$ for the set of closed terms of type α. We write $\Lambda_\to$ for $\bigcup_\alpha \Lambda_\to(\alpha)$. Similarly for $\Lambda^{\emptyset}_\to$. Instead of writing $M \in \Lambda_\to(\alpha)$, we will usually write $M : \alpha$, which we read "M has type α" or "M is in α."

Example 1 The closed terms we had previously defined are all typeable. For example, one can see both **True** and **False** have types $0 \to (0 \to 0)$.

Definition 3 (Long Normal Form) We define the long ($\beta\eta$) normal form of a term $M : \alpha$ by induction on the type α. If $M : 0$, then M is in long normal form if and only if it is of the form $xM_1 \dots M_m$ where each M_i is in long normal form. If $M : \alpha \to \beta$, then M is in long normal form if M is of the form $\lambda f^\alpha . N$, where N is in long normal form. Every term has a unique long normal form, which one can obtain by η expansions.

Example 2 The long normal form of the term $\lambda x^0 . f^{0\to(0\to0)} x$ is

$$\lambda x^0 \lambda y^0 . \left(f^{0\to(0\to0)} x\right) y$$

Remark 2 (Notational Conventions for Types) We will associate types the right to facilitate Currying; therefore $\alpha \to \beta \to \gamma$ is parsed as $\alpha \to (\beta \to \gamma)$. We will tend to not decorate variables with types when it is otherwise deducible from context what type the variable is.

We will use the notation $\alpha^n \to \beta$ to stand for $(\alpha \to (\alpha \to \cdots (\alpha \to \beta) \cdots))$. That is, a term of this type expects n-many inputs of type α and returns an output of type β. Also to improve readability, we will write the type for booleans $\alpha \to \alpha \to \alpha$ as $Bool_\alpha$.

2.2 Semantics

In this section, we will define a set-theoretic framework for which we can interpret typed lambda terms. We first give some more general definitions because we will later introduce different semantics.

Definition 4 Suppose we have an indexed family of sets $\mathscr{M}(\alpha)$ for each type α. Let $\cdot_{\alpha,\beta}$ be a map from $\mathscr{M}(\alpha \to \beta) \times \mathscr{M}(\alpha) \to \mathscr{M}(\beta)$. We say that this is a *typed applicative structure* if it is extensional; that is, for every $f, g \in \mathscr{M}(\alpha \to \beta)$, if we know for every $n \in \mathscr{M}(\alpha)$ $f \cdot_{\alpha,\beta} n = g \cdot_{\alpha,\beta} n$, then $f = g$.

Definition 5 Fix a natural number n. We define a model $\mathscr{M}^n$ as follows, by induction on the type α:

$$\mathscr{M}^n(0) := \{1, \ldots, n\}$$
$$\mathscr{M}^n(\alpha \to \beta) := \big\{ f \mid f : \mathscr{M}^n(\alpha) \to \mathscr{M}^n(\beta) \big\}$$

That is, we interpret the type 0 to be the set $[n]$, the first n natural numbers, and the type $\alpha \to \beta$ is the function space of objects of type α to objects of type β. Note that this is clearly a typed applicative structure where $\cdot$ is just function application. This particular typed applicative structure is called the *full type structure over* $[n]$.

We will write these set-theoretic functions as lowercase Latin letters like f, g, h.

Now that we have a framework which to interpret types, we can make an evaluation of the terms into this framework.

Definition 6 Fix a natural number n and a function φ mapping typed variables x^α to members of $\mathscr{M}^n(\alpha)$. Then we define the evaluation of term with respect to φ by induction on the term:

$$[\![x]\!]^n_\varphi = \varphi(x)$$
$$[\![\lambda x \centerdot M]\!]^n_\varphi = \lambda\!\!\lambda f \centerdot [\![M]\!]^n_{\varphi[x:=f]}$$
$$[\![MN]\!]^n_\varphi = [\![M]\!]^n_\varphi\big([\![N]\!]^n_\varphi\big)$$

2.3 Type Theory

To begin a path from simply typed lambda calculus to a type theory, we need an equality symbol, which we shall call δ. Following the example of Andrews [1], such an addition for all types would lead us to the study of higher order logic. For our purposes, we will just be dealing with first order (classical) logic, and our equality symbol is only for ground type 0.

To add equality, we augment our language with a new constant symbol δ. For the typing rules, we just say that δ is a term of type $0 \to 0 \to \mathrm{Bool}_0$, defined by the following axiom:

$$(x = y \implies \delta xyuv = u) \wedge \big(\neg(x = y) \implies \delta xyuv = v\big)$$

In Statman [7] it was proven that under $\beta\eta$ conversion, the equational consequences of this axiom are exactly the same as from these rules:

$$
\begin{array}{rll}
\delta MMUV = U & & \text{(Reflexivity)} \\
\delta MNUU = U & & \text{(Identity)} \\
\delta XYUV = \delta YXVU & & \text{(Symmetry)} \\
\delta XYXY = Y & & \text{(Hypothesis)} \\
P(\delta MN) = \delta MN(P\mathbf{True})(P\mathbf{False}) & & \text{(Monotonicity)} \\
\delta MN(\delta MNUV)W = \delta MNUW & & \text{(Stutter)} \\
\delta MNU(\delta MNWV) = \delta MNUV & & \text{(Stammer)}
\end{array}
$$

In the previous section on the untyped lambda calculus, we define closed terms representing the booleans of **True** and **False**, as well as **And** and **Not**. For first-order type theory we must add a first-order quantifier, $\exists : (0 \to \mathrm{Bool}_0) \to \mathrm{Bool}_0$ with the rule

$$\exists M = \begin{cases} \mathbf{True} & \text{if } Mn = \mathbf{True} \text{ for some } n : 0 \\ \mathbf{False} & \text{otherwise} \end{cases}$$

and a description operator ι of type $(0 \to \mathrm{Bool}_0) \to 0 \to 0$ with the rule

$$\iota Mm = \begin{cases} n & \text{if } Mn = \mathbf{True} \text{ and } n \text{ is unique such} \\ m & \text{otherwise} \end{cases}$$

Also, we can extend our semantics to handle the terms involving $\delta, \exists$, and ι in the obvious way; for example, for the equality operator, $[\![\delta]\!]^n_\varphi$ is the characteristic function of equality. The following shows that the semantics provided by $\mathscr{M}(n)$ is sound and complete for $\beta\eta\delta$.

Theorem 1 (Soundness and Completeness) *Let M and N be terms of type α.*

1. (*Soundness*) *If $M =_{\beta\eta\delta} N$, then for every $n \in \mathbb{N}$ and every φ, we have $[\![M]\!]^n_\varphi = [\![N]\!]^n_\varphi$*
2. (*Completeness*) *If $M \neq_{\beta\eta\delta} N$, then there are some $n \in \mathbb{N}$ and φ such that $[\![M]\!]^n_\varphi \neq [\![N]\!]^n_\varphi$*

Proof Proof in Statman [7]. □

3 Henkin's Theorem

We would like to be able to say that every finite function in the above semantics can be represented in some way in the $\lambda\delta$-calculus.

Theorem 2 (Henkin's Theorem) *Fix an assignment of variables to types φ, a natural number n, and a type α.*

- *There is a* $\delta_\alpha : 0^n \to \alpha \to \alpha \to \text{Bool}_\alpha$ *such that for every* $f, g, h, j \in \mathscr{M}^n(\alpha)$,

$$\left([\![\delta_\alpha]\!]^n_\varphi 1 \cdots n\right) fghj = \begin{cases} h & \textit{if } f = g \\ j & \textit{if } f \neq g \end{cases}$$

- *If* $f \in \mathscr{M}^n(\alpha)$, *then there is* $F : 0^n \to \alpha$ *such that*

$$\left([\![F]\!]^n_\varphi 1 \cdots n\right) = f$$

Proof Go by induction on the type α.

If $\alpha = 0$, then define

$$\delta_0 = \lambda x_1 \ldots x_n \bullet \delta$$

If $f \in \mathscr{M}^n(0)$, then $f \in [n]$, so $f = i$ for some $1 \leq i \leq n$. Then we can just F be the ith projection:

$$\lambda x_1 \ldots x_n \bullet x_i$$

Suppose $\alpha = \beta \to \gamma$. We have δ_β and δ_γ, closed terms of types β and γ, respectively, that have the desired property. Enumerate all elements of $\mathscr{M}^n(\beta)$, $\{m_1, m_2, \ldots, m_k\}$. By induction hypothesis we know that these are representable; that is, there are closed terms $M_1, \ldots, M_k$ such that $[\![M_i]\!]^n_\varphi 1 \ldots n = m_i$ for every i. So define

$$\begin{aligned}\delta_{\beta\to\gamma} = \lambda \bar{x} F G \bullet \\ & \delta_\gamma \bar{x}\big(F(M_1\bar{x})\big)\big(G(M_1\bar{x})\big) \wedge \delta_\gamma \bar{x}\big(F(M_2\bar{x})\big)\big(G(M_2\bar{x})\big) \\ & \qquad \wedge \cdots \wedge \delta_\gamma \bar{x}\big(F(M_k\bar{x})\big)\big(G(M_k\bar{x})\big)\end{aligned}$$

where $\bar{x}$ is shorthand for $x_1 \ldots x_n$, and $M \wedge N$ is $\mathbf{And}(M)(N)$. One can easily verify this as desired.

Let f be a function in $\mathscr{M}^n(\beta \to \gamma)$. Note that $f(m_i) \in \mathscr{M}^n(\gamma)$ by the definition of the semantics. Therefore, set $p_i = f(m_i)$. By induction hypothesis, there are closed terms P_i for every $1 \leq i \leq k$ such that $([\![P_i]\!]^n_\varphi 1 \ldots n) = p_i$. We define F by doing cases on what our input is:

$$\begin{aligned} F = \lambda \bar{x} m \bullet & \text{If } \delta_\beta \bar{x}(m)(M_1\bar{x}) \text{ then } (P_1\bar{x}) \text{ else} \\ & \quad \text{If } \delta_\beta \bar{x}(m)(M_2\bar{x}) \text{ then } (P_2\bar{x}) \text{ else} \\ & \quad\quad \ldots \\ & \quad\quad\quad \text{If } \delta_\beta \bar{x}(m)(M_{k-1}\bar{x}) \text{ then } (P_{k-1}\bar{x}) \text{ else } (P_k\bar{x}) \end{aligned}$$

where "If M then N else P" is "shorthand" for $(MN)P$. Similarly, this is easily shown to be as claimed. □

The following is an easy corollary to the completeness result stated above and Henkin's theorem.

Corollary 1 *Take $M, N : \alpha_1 \to \alpha_2 \to \cdots \to \alpha_k \to 0$ with all free variables having type 0. If $M \neq_{\beta\eta\delta} N$, then there are some n and closed terms $F_i : 0^n \to \alpha_i$ such that*

$$M(F_1\bar{x})\cdots(F_k\bar{x}) \neq_{\beta\eta\delta} N(F_1\bar{x})\cdots(F_k\bar{x})$$

where $\bar{x}$ includes all variables free in both M and N.

Proof By soundness, for some φ and n, we have $[\![M]\!]^n_\varphi \neq [\![N]\!]^n_\varphi$. Take $\bar{x}$ to be a sequence of length n of free variables of type 0, containing all the free variables of M and N. Then clearly $[\![\lambda\bar{x}.M]\!]^n_\varphi \neq [\![\lambda\bar{x}.N]\!]^n_\varphi$.

These are set-theoretic functions, therefore there are $f_1 \in \mathscr{M}^n(\alpha_1), \ldots, f_k \in \mathscr{M}^n(\alpha_k)$ such that $([\![\lambda\bar{x}.M]\!]^n_\varphi 1\ldots n)f_1\ldots f_k \neq ([\![\lambda\bar{x}.N]\!]^n_\varphi 1\ldots n)f_1\ldots f_k$. By Henkin, all these f_i have closed terms of type $0^n \to \alpha_i$ representing them; denote those closed terms F_i.

Therefore, by completeness, the result follows. □

Consider for each type α the set $\Lambda^\delta_\to(\alpha)$, that is, the set of terms of $\lambda^\delta_\to$ of type α. Define the set $\mathscr{T}(\alpha)$ by

$$\mathscr{T}(\alpha) := \Lambda^\delta_\to(\alpha)/=_{\beta\eta\delta}$$

That is, all properly typed terms of type α, modulo $\beta\eta\delta$ conversion. As a consequence to the above theorem, we have that $\mathscr{T}$ is an typed applicative structure. For each natural number n, we consider the set of n free variables $X := \{x_1, x_2, \ldots, x_n\}$ of type 0. We can then take the set of all terms M in $\mathscr{T}$ that are λ-definable from this set (and δ).

This is not necessarily a typed applicative structure. For we may have two terms M_1 and M_2 that are not extensionally equal, but are with respect to all terms λ-definable from $X \cup \{\delta\}$. That is, none of the witnesses that M_1 and M_2 are different are λ-definable from $X \cup \{\delta\}$. Therefore, we consider only the equivalence classes formed by equality under δ restricted only to the ground set X. So, if we have M_1 and M_2 as above, we collapse them. We call the resulting model *Gandy hull* of $X \cup \{\delta\}$ in $\mathscr{T}$. This is a typed applicative structure, which we will denote by $\mathscr{T}^n$. For more information on the Gandy hull construction, see [2].

One can see that there is a natural relationship between $\mathscr{T}^n$ and $\mathscr{M}^n$. There is a natural homomorphism from $\mathscr{T}^n$ to $\mathscr{M}^n$, which is completely determined by a mapping of X to $[n]$. Further, we can look at some infinite models. For instance, we can define $\mathscr{M}^\omega$ to be the full type structure over the natural numbers; so $\mathscr{M}^\omega(0) = \{1, 2, 3, \ldots\}$. We can then take the Gandy hull of $\{1, 2, 3, \ldots\} \cup \delta$ in this model and get a model $\mathscr{M}$.

This model could be obtained another way. Fix a bijection from free variables of type 0 and ω. Then one can build a corresponding homomorphism from $\mathscr{T}$ to $\mathscr{M}^\omega$. The image is exactly $\mathscr{M}$. These models are discussed further in Statman [6].

4 Folklore Theorem

Definition 7 The *symmetric group on n elements*, which we denote as S_n, is the subset of $\mathscr{M}^n(0 \to 0)$ that are bijections. These form a group with the operation of composition. We call members of the group permutations. We shall use lower case Greek letters in the middle of the alphabet to stand for permutations, for example, π, ρ, σ, τ.

Of course, members of S_n act canonically of type 0 by application. But, we can lift this action to higher types. Consider $\pi \in S_n$. We define $\pi_\alpha \in \mathscr{M}^n(\alpha \to \alpha)$ by induction on α. If $\alpha = 0$, then we just take $\pi_0(n) = \pi(n)$. If $\alpha = \beta \to \gamma$, then we define

$$\pi_\alpha(f) = \pi_\gamma \circ f \circ \pi_\beta^{-1}$$

Therefore, we have an action of S_n on our entire model $\mathscr{M}^n$, where π acts on $f : \alpha$ by $\pi_\alpha(f)$. For this action, we will write $\pi \cdot f$.

If $f \in \mathscr{M}^n$, then we denote the stabilizer of f under this action $\mathrm{St}(f)$; that is, $\mathrm{St}(f)$ is the set of all permutations that fix f.

$$\mathrm{St}(f) := \{\pi \in S_n \mid \pi \cdot f = f\}$$

We call an $f \in \mathscr{M}^n$ *symmetric* if $\mathrm{St}(f) = S_n$, that is, f is fixed under the action of all permutations.

Remark 3 We can say that S_n acts on $\mathscr{T}^n$ as well. Any permutation of the free variables elicits an automorphism on the entire set $\mathscr{T}^n$. The converse, however, is false; there are automorphisms of $\mathscr{T}^n$ that do not come from permutations of the variables. Therefore, when we call $F \in \mathscr{T}^n$ symmetric, we mean preserved under all automorphisms, not just the "inner" automorphisms arising from permutations of variables.

Theorem 3 (Folklore Theorem) *$f \in \mathscr{M}^n$ is symmetric if and only if it is λ-definable from $\delta, \iota, \exists$.*

Proof The right-to-left direction is straightforward. For δ, ι, and $\exists$ are all symmetric, as are combinators S and K. S and K form a basis for all λ terms, and λ-definable objects are closed under application. Thus, we have that all λ-definable objects are indeed symmetric.

The left-to-right direction will constitute the rest of this section of the paper.

Definition 8 For each function $f : 0^n \to \alpha$, we associate a functional $f^+ : (0 \to 0) \to \alpha$ such that

$$f^+\pi = f(\pi 1)(\pi 2)\ldots(\pi n)$$

A function f is said to be *regular* if for every $g \in \mathscr{M}^n(0 \to 0)$ where $g \notin S_n$, we have $f^+g = g(1)$.

For the present moment, we will restrict our attention only on functions $f : 0^n \to 0$.

Remark 4 Note that the action $\pi \cdot f$ in this case is the following:

$$\pi \cdot f = \mathbb{\lambda}\bar{x}\,.\,\pi\big(f\big(\pi^{-1}x_1\big)\ldots\big(\pi^{-1}x_n\big)\big)$$

This and that $\mathrm{St}(f)$ is a subgroup implies that $\pi \in \mathrm{St}(f)$ if and only if

$$\mathbb{\lambda}\bar{x}\,.\,\pi^{-1}\big(f(\pi x_1)\ldots(\pi x_n)\big) = f$$

Definition 9 Fix an $f : 0^n \to 0$ regular. We define a relation $\sim_f$ on S_n by

$$\pi \sim_f \sigma \iff \pi^{-1}\big(f\big(\pi(1)\big)\dots\big(\pi(n)\big)\big) = \sigma^{-1}\big(f\big(\sigma(1)\big)\dots\big(\sigma(n)\big)\big)$$

We restrict this relation to be a right congruence by taking its *right congruence hull*, which we denote $\sim_f^*$ and define by

$$\pi \sim_f^* \sigma \iff \forall \rho \in S_n \centerdot \pi\rho^{-1} \sim_f \sigma\rho^{-1}$$

Lemma 1 *For* $f : 0^n \to 0$ *regular and* $\pi \in S_n$, *the following are equivalent*:

1. $\pi \in St(f)$.
2. *For all* $\rho \in S_n$, *we have* $\pi\rho \sim_f \rho$.
3. $\pi \sim_f^* \mathrm{id}$.

Proof ((1) ⇒ (2)). Take $\pi \in \mathrm{St}(f)$. By remark we have

$$\lambda\bar{x} \centerdot \pi^{-1}\big(f(\pi x_1)\dots(\pi x_n)\big) = f$$

Fix $\rho \in S_n$. Apply $\rho(1), \rho(2), \dots, \rho(n)$ to the left:

$$\pi^{-1}\big(f\big(\pi\rho(1)\big)\dots\big(\pi\rho(n)\big)\big) = f\big(\rho(1)\big)\dots\big(\rho(n)\big)$$

which gives us

$$\rho^{-1}\big(\pi^{-1}\big(f\big(\pi\rho(1)\big)\dots\big(\pi\rho(n)\big)\big)\big) = \rho^{-1}\big(f\big(\rho(1)\big)\dots\big(\rho(n)\big)\big)$$

which implies that $\pi\rho \sim_f \rho$.

((2) ⇒ (3)). Take $\rho \in S_n$. We want to show that $\pi\rho^{-1} \sim_f \mathrm{id}\,\rho^{-1}$. The right-hand side is of course just ρ^{-1}; therefore, this follows immediately from (2).

((3) ⇒ (1)). We want to show that

$$\lambda\bar{x} \centerdot \pi^{-1}\big(f(\pi x_1)\dots(\pi x_n)\big) = f$$

By extensionality, it suffices to show that the above holds after an arbitrary application. Moreover, let us fix an arbitrary $g : 0 \to 0$ (not necessarily in S_n). The application of $g(1)$ to the left of both sides, followed by $g(2)$, etc., up to $g(n)$, is an arbitrary application as g is arbitrary; thus, it suffices to show

$$\pi^{-1}\big(f\big(\pi g(1)\big)\dots\big(\pi g(n)\big)\big) = f\big(g(1)\big)\dots\big(g(n)\big)$$

If $g \notin S_n$, then by regularly of f, both sides are exactly $g(1)$. Otherwise, call $\rho := g$ is a member of S_n. By (3) (using the right congruence property on ρ^{-1}), we have that $\pi\rho \sim_f \rho$. This means that

$$\rho^{-1}\pi^{-1}\big(f\pi\big(\rho(1)\big)\dots\pi\big(\rho(n)\big)\big) = \rho^{-1}\big(f\big(\rho(1)\dots\rho(n)\big)\big)$$

which is exactly what we wanted. □

From the above one sees that $\sim_f^*$ partitions S_n into equivalence classes, where St(f) is the class that contains id. All other classes can be written as unions of right cosets of the stabilizer, by property (2).

Let $\mathscr{B}$ be the set of equivalence classes. For each $B \in \mathscr{B}$, let $\chi_B : 0^n \to \text{Bool}_0$ denote its characteristic function. That is,

$$\chi_B^+(\pi) = \begin{cases} \textbf{True} & \text{if } \pi \in B \\ \textbf{False} & \text{otherwise} \end{cases}$$

By the definition of the equivalence relation, if π and σ are in a block B, then $\pi^{-1}(f^+\pi) = \sigma^{-1}(f^+\sigma) =: i$. So when f is given inputs corresponding to a permutation in B, f is just the ith projection function. Thus, to define f, we need only know which block the given input it in. Therefore, f itself is $\lambda\delta$-definable from the set $\{\chi_B \mid B \in \mathscr{B}\}$ via the function

$$\begin{aligned} F = \lambda x_1 \ldots x_n \centerdot \text{If } \chi_{B_1} x_1 \ldots x_n \text{ then } (x_{i_1}) \text{ else} \\ \text{If } \chi_{B_2} x_1 \ldots x_n \text{ then } (x_{i_2}) \text{ else} \\ \ldots \\ \text{If } \chi_{B_j} x_1 \ldots x_n \text{ then } (x_{i_j}) \text{ else } (x_1) \end{aligned}$$

where $\{B_1, \ldots B_j\} = \mathscr{B}$, and i_k is the coordinate that f projects on block B_k.

Lemma 2 *If f is regular, symmetric of type $0^n \to 0$, then f is $\lambda\delta$-definable.*

Proof Since f is symmetric, by (1), there is only one block of the equivalence class formed by $\sim_f^*$. Since f is $\lambda\delta$-definable from the set of blocks, we have that f is $\lambda\delta$-definable outright. □

Now, consider arbitrary symmetric $f : \alpha_1 \to \alpha_2 \to \cdots \to \alpha_k \to 0$. It suffices, given the above, to show that f is definable from regular, symmetric functions of type $0^n \to 0$. Consider the set of lists

$$\mathscr{L} = \Lambda_{\to}^{\delta}(\alpha_1) \times \cdots \times \Lambda_{\to}^{\delta}(\alpha_k) = \big\{\langle f_1, f_2, \ldots, f_k\rangle \mid f_i : \alpha_i\big\}$$

For each list $L = \langle f_1, \ldots, f_k\rangle$ in $\mathscr{L}$, we define a function $c_L : 0^n \to 0$, which we call the *Lth coordinate function* defined by

$$c_L = \lambda\!\!\lambda x_1 \ldots x_n \centerdot \begin{cases} f(F_1 x_1 \ldots x_n) \ldots (F_k x_1 \ldots x_n) & \text{if } x_1, \ldots, x_k \text{ distinct} \\ x_1 & \text{otherwise} \end{cases}$$

where the $F_i : 0^n \to \alpha_i$ are the terms from Henkin's theorem corresponding to f_i. Each coordinate function is regular (by the cases defining it) and also symmetric (as f is). So, each c_L is $\lambda\delta$-definable. Thus, we need only show that f is definable from its coordinate functions; f is not definable outright, but we need to use ι and $\exists$. We begin by the remark that the function **alldiff** : $0^n \to \text{Bool}_0$, which returns **True** if all the first n inputs are different, and **False** otherwise is $\lambda\delta$-definable.

$$\textbf{alldiff} := \lambda x_1 \cdots x_n \centerdot \delta(x_1)(x_2)(\textbf{False})\big(\cdots \delta(x_{n-1})(x_n)(\textbf{False})(\textbf{True}) \cdots\big)$$

Now, we can define f:

$$f = \lambda x_1 \dots x_k . \iota \Bigg(\lambda z . \exists y_1 \dots y_n . (\mathbf{alldiff}\, y_1 \dots y_n) \wedge$$

$$\bigvee_{\substack{L \in \mathscr{L} \\ L = \langle F_1, \dots, F_k \rangle}} \big(\delta(x_1)(F_1 y_1 \dots y_n) \wedge \dots \wedge \delta(x_k)(F_k y_1 \dots y_n)$$

$$\wedge\, (c_L y_1 \dots y_n x_1 \dots x_k)(z) \big) \Bigg)$$

Therefore, we have that f is definable if each of the c_L is definable; the c_L are regular functions of type 0^n, which are therefore definable if they are symmetric. It is easy to see that if f is symmetric, then so are its coordinate functions. Therefore, if f is symmetric, then we can substitute the $\lambda\delta$-definition of c_L into each of the c_L above and get a $\lambda\delta$-definition of f (using $\exists$ and ι). □

Moreover, we can prove the following strengthening:

Theorem 4 *Fix a function $f \in \mathscr{M}^n$ and $A \subseteq \mathscr{M}^n$. Then if $(\bigcap_{g \in A} St(g)) \subseteq St(f)$, then f is $\lambda\delta$-definable from functions in A along with $\iota, \exists$.*

Proof It is easy to see that $\mathrm{St}(f) = \bigcap \mathrm{St}(c_L)$, where the c_L are the coordinate functions of f; for since f and its coordinate functions are definable from each other, any permutation that fixes f must fix its coordinate functions, and any that fixes all its coordinate functions fixes the function.

Therefore, it suffices that we prove the theorem only for $f : 0^n \to 0$ and, similarly, assume all $g \in A$ be of type $0^n \to 0$. We suppose that $(\bigcap_{g \in A} \mathrm{St}(g)) \subseteq \mathrm{St}(f)$. By Lemma 1, since $\mathrm{St}(g)$ is exactly the block of the equivalence relation $\sim_g^*$ containing id, it follows that the set of left cosets of $\bigcap_{g \in A} \mathrm{St}(g)$ is a finer partition of S_n than the set of left cosets of $\mathrm{St}(f)$, which are exactly the blocks of $\sim_f^*$.

Therefore, on any left coset of $\bigcap_{g \in A} \mathrm{St}(g)$ we have that f behaves like a projection operator since the coset is entirely contained in a block of $\sim_f^*$, which in turn is contained in a block of $\sim_f$. Thus, for any permutations π, we can identify the left coset of $\bigcap_{g \in A} \mathrm{St}(g)$ that π is in. f acts uniformly on that block as a projection function, so we can make a definition similar to the above definition of f by its blocks in $\sim_f$. □

5 Definability and Symmetry

Let us return our attention to the term model $\mathscr{T}$, where members are terms with possible free variables among $x_1, x_2, \dots$. We first state the following result of Lauchli [5].

Proposition 1 (Lauchli) *There is a closed term $F \in \Lambda_\delta^\emptyset(\alpha)$ if and only if there is an $F \in \mathscr{T}(\alpha)$ symmetric (recall: for terms in $\mathscr{T}(\alpha)$, symmetric means fixed under all automorphisms).*

Proof In Lauchli [5], it is stated and proved in terms of intuitionist logic: $\vdash_I \alpha$ if and only if there is an "invariant" function of type α. □

Theorem 5 *Any $F \in \mathscr{T}$ is λδ-definable if and only if it is symmetric.*

Proof It is easy to see that every element of $\mathscr{T}$ that is λδ-definable is symmetric since it is $\beta\eta\delta$ equal to a closed term, which is fixed under all automorphisms. We will just prove the converse.

Let $F \in \mathscr{T}$ by symmetric; consider F to be of type $\alpha_1 \to \cdots \to \alpha_k \to 0$. Write F as $Gx_1 \ldots x_n$, where G is closed and free variables of F are among $x_1, \ldots, x_n$. By the proposition above, we can get a closed term $H : \alpha_1 \to \cdots \to \alpha_k \to 0$, which has the following form in long normal form:

$$\lambda y_1 \ldots y_k \centerdot H'$$

where H' has type 0 and free variables only among $y_1, \ldots y_k$. Consider

$$G \underbrace{H' \ldots H'}_{n \text{ times}}$$

This is a term of type $\alpha_1 \to \cdots \to \alpha_k \to 0$, which has free variables only among $y_1, \ldots y_k$. Thus, the term

$$M := \lambda y_1 \ldots y_k \centerdot G \underbrace{H' \ldots H'}_{n \text{ many}} y_1 \ldots y_k : \alpha_1 \to \cdots \to \alpha_k \to 0$$

is closed.

Recall that F is symmetric. Therefore,

$$F = Gx_1 \ldots x_n =_{\beta\eta\delta} Gy_1 \ldots y_n$$

for any variables $y_i : 0$. Thus, by a substitution, we have that $F =_{\beta\eta\delta} GY_1 \ldots Y_n$ for any $Y_i : 0$. Therefore, $M =_{\beta\eta\delta} F$ and is closed, thus is a λδ-definition of F. □

Corollary 2 *Let $h : \mathscr{T} \to \mathscr{M}$ be defined as $x_i \mapsto i$. This is called the* canonical homomorphism. *A function $f \in \mathscr{M}$ is λδ-definable if and only if there is $F \in h^{-1}(f)$ symmetric.*

Proof Once again the forward direction is straightforward. For the backward direction, we just apply the last theorem. By the last theorem, if $F \in h^{-1}(f)$ is symmetric, then it is λδ-definable by some closed term G. Since $h(G) = f$ and G is closed, G is also a good λδ-definition for f. □

Definition 10 We call a homomorphism $h : \mathscr{T}^n \to \mathscr{M}^m$ *canonical* if $x_i \mapsto i$ for all $1 \leq i \leq m$.

We say that an $F \in \mathscr{T}^n$ is *supersymmetric* if for every homomorphism $\varphi : \mathscr{T}^n \to \mathscr{T}^n$, $\varphi(F)$ is symmetric.

Theorem 6 *$f \in \mathscr{M}^m$ is λδ-definable if and only if there is some $n > m$ and $F \in \mathscr{T}^n$ supersymmetric such that for all canonical homomorphisms $h : \mathscr{T}^n \to \mathscr{M}^m$, we have $h(F) = f$.*

Proof The left to right direction is trivial since f being $\lambda\delta$-definable gives us a closed term that will satisfy all the requirements.

For the other direction, fix $f \in \mathscr{M}^m$ of type $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_k \to 0$. Suppose that $n > m$ and $F \in \mathscr{T}^n$ is supersymmetric where for all homomorphisms $h : \mathscr{T}^n \to \mathscr{M}^m$, have $h(F) = f$. Write $F = F'x_1 \dots x_j$ where F' is a closed term.

The idea is as follows: we will do induction on the number of free variables on F, j. We will construct a new term M that has $j-1$ free variables and still has the property that it is supersymmetric and is sent to f under all canonical homomorphisms. At the end of our construction, we will have eliminated all free variables and will have constructed a closed term M that is sent to f under all canonical homomorphisms. But, since M will be closed, M is a $\lambda\delta$-definition for f.

To start the induction, if $j = 1$, then $F = F'x_1$. As $n > m \geq 1$, we know $n > 1$, so that $x_n \neq x_1$. F is supersymmetric, and therefore it is symmetric, so under the automorphism sending x_1 to x_n, we know $F'x_1 = F'x_n$. As $n > m$, we have freedom with our canonical homomorphism to send x_n anywhere; in particular, for any $1 \leq s \leq m$, we can define canonical homomorphism h where $h(x_n) = s$. Therefore, $f = F's$ for all s. Therefore, we may replace x_1 in F by anything of type 0, and it would still be sent to f through any canonical homomorphism.

By Lauchli [5], there is a closed term G of type $\alpha_1 \to \cdots \alpha_k \to 0$. We can write F as $\lambda z_1 \dots z_k \centerdot F'x_1 z_1 \dots z_k$ by doing η expansions. Then, replacing x_1 to form the term $\lambda z_1 \dots z_k \centerdot F'(Gz_1 \dots z_k) z_1 \dots z_k$, we have a closed $\lambda\delta$ term equal to f.

If $j > 1$, then we wish to eliminate the variable x_j. If $j > m$, then we already have freedom to send x_j to any number in a canonical homomorphism h. Therefore, for every $1 \leq s \leq m$, by picking a canonical homomorphism that sends x_j to s we have

$$f = h\big(F'x_1 \dots x_j\big) = F'1 \dots n\big(h(n+1)\big) \dots \big(h(j-1)\big)s$$

Since s is unrestricted, we can replace x_j with anything of type 0, and the above is still preserved. In particular, doing an η expansion of F gives us $F = \lambda z_1 \dots z_k \centerdot F'x_1 \dots x_j z_1 \dots z_k$, and then replacing x_j, we get:

$$f = h\underbrace{\big(\lambda z_1 \dots z_k \centerdot F'x_1 \dots x_{j-1}\big(F'x_1 \dots x_1 z_1 \dots z_k\big) z_1 \dots z_k\big)}_{M}$$

M has only $j-1$ free variables. It remains to show that M is supersymmetric. This, however, is not hard to see. Under the map $x_i \mapsto x_1$ for all $1 \leq i \leq j$, we have that, since F is supersymmetric, $F'x_1 \dots x_1$ is symmetric and therefore preserved under all automorphisms. Therefore, for any homomorphism $\varphi : \mathscr{T}^n \to \mathscr{T}^n$, we will have $\varphi(M)$ symmetric since $\varphi(F)$ was symmetric and M is just F with a free variable replaced by a symmetric term.

If $1 < j \leq m < n$, we have by the symmetry of F by the automorphism switching x_j and x_n that

$$F = F'x_1 \dots x_{j-1} x_n$$

Now, we have the freedom to send x_n anywhere under any canonical homomorphism, and thus we can repeat what we did above to eliminate x_n. □

References

1. Andrews, P.B.: General models, descriptions, and choice in type theory. J. Symb. Log. **37**(2), 385–394 (1972)
2. Barengregt, H., Dekkers, W., Statman, R.: Lambda Calculus with Types. Cambridge University Press, Cambridge (2011)
3. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**(2), 56–68 (1940)
4. Henkin, L.: A theory of propositional types. Fundam. Math. **52**(3), 323–344 (1963)
5. Läuchli, H.: An abstract notion of realizability for which intuitionistic predicate calculus is complete. Stud. Logic Found. Math. **60**, 227–234 (1970)
6. Statman, R.: Completeness, invariance and lambda-definability. J. Symb. Log. **47**(1), 17–26 (1982)
7. Statman, R.: Church's lambda delta calculus. In: Logic for Programming and Automated Reasoning, pp. 293–307. Springer, Berlin (2000)
8. van Benthem, J., Doets, K.: Higher-order logic. In: Gabbay, D.M., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn. vol. 1, pp. 189–244. Springer, Berlin (2001)

W. Gunther (✉) · R. Statman
Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
e-mail: wgunther@andrew.cmu.edu

R. Statman
e-mail: rs31@andrew.cmu.edu

Henkin's Completeness Proof and Glivenko's Theorem

Franco Parlamento

Abstract We observe that Henkin's argument for the completeness theorem yields also a classical semantic proof of Glivenko's theorem and leads in a straightforward way to the weakest intermediate logic for which that theorem still holds. Some refinements of the completeness theorem can also be obtained.

Keywords Consistency · Completeness · Intermediate logics

1 Introduction

Henkin's argument for the completeness theorem consists of the proof that a consistent set of sentences in first-order logic is satisfiable, from which, by a classical reduction ad absurdum, one concludes that if F is a semantic consequence of a set of sentences Γ, then there must be a derivation in classical first-order logic of F from Γ. Given the non-constructive character of Henkin's proof, it is perhaps somewhat surprising that it offers a way to obtain results concerning the relationship between intuitionistic logic and classical logic, such as Glivenko's theorem for propositional logic, and a characterization of the weakest extension of first-order intuitionistic logic for which that theorem holds with respect to the full first-order language. It seems also remarkable that the proof under consideration is the one that Henkin pointed out to Smullyan and can be found in Smullyan's celebrated book *First Order Logic* [6, p. 96], with the comment that "It is hard to imagine a more direct completeness proof!" The crucial observation is that the proof directly entails the following facts: (1) for a set of propositional formulae to be satisfiable, it suffices its consistency with respect to intuitionistic logic; (2) for a set of first-order sentences to be satisfiable, it suffices its consistency with respect to the extension of intuitionistic logic obtained by allowing the inference of $\neg\neg\forall x F$ from $\forall x \neg\neg F$; (3) the latter strengthening is not required if only $\forall$-free sentences are involved; (4) if, furthermore, they are also $\rightarrow$-free, even the *ex-falso* rule can be dispensed with. The latter two observations lead to obtain directly refined forms of the completeness theorem.

2 Henkin's Proof and Glivenko's Theorem

In the following, we will adopt the natural deduction system with both $\neg$ and $\bot$ taken as primitive and denote by **N** the system that has only the introduction and elimination

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_15

rules for each logical constant, namely the system for intuitionistic logic, and by **NC** the system for classical logic, obtained by adding to **N** the classical reduction ad absurdum rule $\bot_c$. For Γ a set of formulae, $\mathscr{L}(\Gamma)$ denotes the set of all formulae built over the symbols that occur in Γ, including $\bot$ and $\neg$, even if they do not occur in Γ. Given a set of formulae Σ and a formula F, $\Sigma \rhd F$ denotes the derivability relation of **N**; more precisely, $\Sigma \rhd F$ means that there is a deduction $\mathscr{D}$ in **N** such that all the formulae that occur in $\mathscr{D}$ belong to $\mathscr{L}(\Gamma)$, the active assumption of $\mathscr{D}$ belongs to Σ, and the conclusion of $\mathscr{D}$ is F. Similarly, $\Sigma \rhd_c F$ denotes the derivability relation of **NC**. $\mathbf{N}_p$ and $\mathbf{NC}_p$ denote the systems that are obtained from **N** and **NC** by excluding the rules concerning quantifiers and the corresponding derivability relations $\Sigma \rhd_p F$ and $\Sigma \rhd_{cp} F$. A *purely propositional* formula is a formula that contains only propositional parameters, $\bot$, and propositional connectives. Since, assuming that every set can be well ordered, it does not require any additional ado, we formulate Henkin's argument with respect to an arbitrary set of formulae, rather than to the countable ones as in [6].

Proposition 1 *If Γ is a set of purely propositional formulae, and $\Gamma \not\rhd_p \bot$, then Γ is satisfiable in classical propositional logic.*

Proof Let $<$ be a well-ordering of $\mathscr{L}(\Gamma)$ whose order type is a cardinal. By $<$-recursion, for every $F \in \mathscr{L}(\Gamma)$, we define a set of formulae Γ_F of $\mathscr{L}(\Gamma)$ as follows:

$$\Gamma_F = \begin{cases} \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} & \text{if } \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} \not\rhd_p \bot \\ \Gamma \cup \bigcup_{G<F} \Gamma_G & \text{otherwise} \end{cases}$$

Clearly, Γ_F is defined for every F in $\mathscr{L}(\Gamma)$, and if $G \le F$, then $\Gamma_G \subseteq \Gamma_F$. Since the property $\Sigma \not\rhd_p \bot$ is preserved under unions of increasing (under inclusion) chains of sets, by $<$-induction it follows that for every $F \in \mathscr{L}(\Gamma)$, $\Gamma_F \not\rhd_p \bot$. Hence, letting $\bar{\Gamma} = \bigcup_{F \in \mathscr{F}} \Gamma_F$, for the same reason, we have that $\bar{\Gamma}$ is consistent with respect to $\rhd_p$, namely:

(i) $\bar{\Gamma} \not\rhd_p \bot$

Since $\Sigma \not\rhd_p \bot$ is preserved under subsets, $\bar{\Gamma}$ is maximal with respect to the property $\Sigma \not\rhd_p \bot$, namely, letting $\bar{\Gamma}, F$ denote $\bar{\Gamma} \cup \{F\}$, we have:

(ii) if $\bar{\Gamma}, F \not\rhd_p \bot$, then $F \in \bar{\Gamma}$

For, if $\bar{\Gamma}, F \not\rhd_p \bot$, then, by the definition of Γ_F, we have that $F \in \Gamma_F$ and hence that $F \in \bar{\Gamma}$ by the definition of $\bar{\Gamma}$. As a consequence, we also have:

(iii) if $\bar{\Gamma} \rhd_p H$, then $H \in \bar{\Gamma}$

In fact from $\bar{\Gamma} \rhd_p H$ it follows that $\bar{\Gamma}, H \not\rhd_p \bot$, otherwise by the cut rule, namely the possibility of composing deductions, we would get $\bar{\Gamma} \rhd_p \bot$, against (iii). Hence, by (ii), $H \in \bar{\Gamma}$. Notice that so far only the structural properties of weakening and cut of $\rhd_p$, satisfied also by classical as well as by minimal propositional logic, have been used. The logical rules are needed to verify the following:

(1) $\neg G \in \bar{\Gamma}$ if and only if $G \notin \bar{\Gamma}$
(2) $G \wedge H \in \bar{\Gamma}$ if and only if $G \in \bar{\Gamma}$ and $H \in \bar{\Gamma}$
(3) $G \vee H \in \bar{\Gamma}$ if and only if $G \in \bar{\Gamma}$ or $H \in \bar{\Gamma}$

(4) $G \to H \in \bar{\Gamma}$ if and only if $G \notin \bar{\Gamma}$ or $H \in \bar{\Gamma}$

More precisely: (1) is a consequence of (i) and the $\neg$-elimination rule in the *only if* direction and of (ii), the $\neg$-introduction rule, and (iii) in the other. (2) is a consequence of the $\wedge$-elimination rules and (iii) in the *only if* direction and of the $\wedge$-introduction rules and (iii) in the other. As for (3), assuming that $G \vee H \in \bar{\Gamma}$, we cannot have both $G \notin \bar{\Gamma}$ and $H \notin \bar{\Gamma}$ since, otherwise, by (1), $\neg G \in \bar{\Gamma}$ and $\neg H \in \bar{\Gamma}$, and by the $\neg$ and $\vee$-elimination rules we would have $\bar{\Gamma} \rhd_p \bot$ against (i). Conversely, if $G \in \bar{\Gamma}$ or $H \in \bar{\Gamma}$, then, by the appropriate $\vee$-introduction rule and (iii), we have that $G \vee H \in \bar{\Gamma}$. As for (4), if $G \to H \in \bar{\Gamma}$ and $G \in \bar{\Gamma}$, then by the $\to$-elimination rule and (iii) it follows that $H \in \bar{\Gamma}$, so that either $G \notin \bar{\Gamma}$ or $H \in \bar{\Gamma}$. Conversely, if $G \notin \bar{\Gamma}$, then by (1), $\neg G \in \bar{\Gamma}$. Since, by the $\neg$-elimination, *ex-falso*, and the $\to$-introduction rules, $G \to H$ is deducible from $\neg G$, by (ii) we have that $G \to H \in \bar{\Gamma}$. The same conclusion is reached if $H \in \bar{\Gamma}$ since, by the $\to$-introduction rule, $G \to H$ follows from H. Thus, in both cases, $G \to H \in \bar{\Gamma}$. On the ground of (1)–(4), it is immediate to verify, by induction on the height of formulae, that the assignment v that gives the value $\mathbf{t}$ to a propositional parameter P if and only $P \in \bar{\Gamma}$ assigns the value $\mathbf{t}$ to a (purely propositional) formula F if and only if $F \in \bar{\Gamma}$. In particular, v satisfies all the formulae in Γ, showing that Γ is indeed satisfiable. □

Corollary 1 (Glivenko's Theorem) *If a purely propositional formula F is deducible in $\mathbf{NC}_p$, then $\neg\neg F$ is deducible in $\mathbf{N}_p$.*

Proof If F is deducible in $\mathbf{NC}_p$, then $\neg F$ cannot be satisfied in classical propositional semantics. By the previous proposition $\neg F \rhd_p \bot$; hence, $\rhd_p \neg\neg F$ by the $\neg$-introduction rule. □

From the proof of Proposition 1 it is apparent that the availability in $\mathbf{N}_p$ of the *ex-falso* rule is needed only to establish condition (4) concerning $\to$. Therefore, we have also a proof that if a set of propositional formulae is $\to$-free, namely it does not contain occurrences of $\to$ and is consistent with respect to minimal propositional logic, then it is satisfiable in classical propositional semantics, so that if a purely propositional formula is $\to$-free and is deducible in classical propositional logic, then its double negation is deducible in minimal propositional logic.

3 Extension to First-Order Logic

In extending to the first-order language, the previous proof we will strictly follow the one suggested by Henkin and reported in [6, p. 96]. Let us recall that a canonical interpretation of $\mathscr{L}(\Gamma)$ is an interpretation whose domain is the set of all closed terms of $\mathscr{L}(\Gamma)$ and in which function and constant symbols are given their canonical interpretation. Thus, a canonical interpretation is completely determined by which atomic sentences of $\mathscr{L}(\Gamma)$ are considered to be true. For C a set of constants, let $\mathscr{L}(\Gamma, C)$ be defined as $\mathscr{L}(\Gamma)$ except that also constants in C are allowed.

Proposition 2 *If Γ is a set of first-order sentences and $\Gamma \not\rhd_c \bot$, then Γ is satisfiable in a canonical interpretation of $\mathscr{L}(\Gamma, C)$ for any set C of constants not occurring in Γ of cardinality not less than $|\mathscr{L}(\Gamma)|$.*

Proof Let C be any set of constants not occurring in Γ of cardinality not less than $|\mathscr{L}(\Gamma)|$, $\mathscr{F}$ be the set of all sentences in $\mathscr{L}(\Gamma, C)$, and $<$ be a well-ordering of $\mathscr{F}$ whose order type is a cardinal. Let Γ_F be defined by $<$-recursion as follows:

$$\Gamma_F = \begin{cases} \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} & \text{if } \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} \not\rhd_c \bot \\ & \text{and } F \text{ is neither of the form } \exists x H \\ & \text{nor of the form } \neg\forall x H \\ \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} \cup \{H\{x/c\}\} & \text{if } \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} \not\rhd_c \bot \\ & \text{and } F \text{ is } \exists x H \\ \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} \cup \{\neg H\{x/c\}\} & \text{if } \Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\} \not\rhd_c \bot \\ & \text{and } F \text{ is } \neg\forall x H \\ \Gamma \cup \bigcup_{G<F} \Gamma_G & \text{otherwise} \end{cases}$$

where c is one of the constants in C not occurring in $\Gamma \cup \bigcup_{G<F} \Gamma_G \cup \{F\}$.

Γ_F is defined for every F since in the second and third cases of the definition of Γ_F, which we name $C_\exists$ and $C_{\neg\forall}$, there are always constants in C that do not belong to $\Gamma \cup \bigcup_{G<F} \Gamma_G$ because of cardinality reasons. The rest of the proof proceeds as for Proposition 1 except that we have to:

(a) check that also in cases $C_\exists$ and $C_{\neg\forall}$, the underivability of $\bot$ is preserved,
(b) complete the proof that the canonical interpretation of $\mathscr{L}(\Gamma, C)$ in which an atomic sentence is considered to be true if and only if it belongs to $\bar{\Gamma}$ satisfies a sentence F if and only if $F \in \bar{\Gamma}$, by taking care also of existential and universal sentences.

As for (a), the underivability of $\bot$ is not lost in the $C_\exists$ case since the $\exists$-elimination rule ensures that if c does not occur in $\Sigma, \exists x H$, then

($C'_\exists$) if $\Sigma, \exists x H, H\{x/c\} \rhd_c \bot$ then $\Sigma, \exists x H \rhd_c \bot$.

For, given a deduction $\mathscr{D}$ of $\bot$ from $\Sigma, \exists x H, H\{x/c\}$, it suffices to replace c by a variable y not occurring in $\mathscr{D}$, to obtain a deduction $\mathscr{D}\{c/y\}$ of $\bot$ from $\Sigma, \exists x H, H\{x/y\}$, from which an application of the $\exists$-elimination rule yields a deduction of $\bot$ from $\Sigma, \exists x H$. Dually, the $\forall$-introduction rule is needed in dealing with the $C_{\neg\forall}$ case, but **N** is not strong enough. **NC** certainly suffices since, assuming that c does not occur in $\Sigma, \neg\forall x H$, the following holds:

($C'_{\neg\forall}$) if $\Sigma, \neg\forall x H, \neg H\{x/c\} \rhd_c \bot$ then $\Sigma, \neg\forall x H \rhd_c \bot$.

In fact, if $\mathscr{D}$ is a deduction of $\bot$ from $\Sigma, \neg\forall x H, \neg H\{x/c\}$ and y is a variable not occurring in $\mathscr{D}$, then the following is a deduction of $\bot$ from $\Sigma, \neg\forall x H$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma, \neg\forall x H, [\neg H\{x/y\}] \\ \mathscr{D}\{c/y\} \\ \bot\end{matrix}}{H\{x/y\}}\bot_c}{\forall x H} \qquad \neg\forall x H}{\bot}$$

As for (b), it suffices to check that:

($\exists'$) $\exists x H \in \bar{\Gamma}$ if and only if for some closed term t in $\mathscr{L}(\Gamma, C)$, $H\{x/t\} \in \bar{\Gamma}$;
($\forall'$) $\forall x H \in \bar{\Gamma}$ if and only if for all closed term t in $\mathscr{L}(\Gamma, C)$, $H\{x/t\} \in \bar{\Gamma}$.

By (iii) and (1) in the proof of Proposition 1, the $\exists$-introduction rule and the $\forall$-elimination rule ensure the *if* direction of ($\exists'$) and the *only if* direction of ($\forall'$), respectively. The *only if* direction of $\exists'$ is a direct consequence of $C_\exists$. Finally, for the *if* direction of $\forall'$, we note that if $\forall x H \notin \bar{\Gamma}$, by (1) in the proof of Proposition 1, $\neg\forall x H \in \bar{\Gamma}$, and hence, by $C_{\neg\forall}$, $\neg H\{x/c\} \in \bar{\Gamma}$ for some constant c, contradicting, by (i) in the proof of Proposition 1, the assumption that for every closed term t, $H\{x/t\} \in \bar{\Gamma}$. □

Since $\bot_c$ is needed only to ensure that ($C'_{\neg\forall}$) holds, taking into account the remark concerning minimal propositional logic at the end of the previous section, the following corollary is immediate.

Corollary 2

(a) *If Γ is a set of $\forall$-free sentences and $\Gamma \not\rhd \bot$ in $\mathscr{L}(\Gamma)$, then Γ is satisfiable in a canonical interpretation of $\mathscr{L}(\Gamma, C)$ for any set C of constants not occurring in Γ of cardinality not less than $|\mathscr{L}(\Gamma)|$.*
(b) *If a $\forall$-free sentences is deducible in first-order logic, then its double negation is deducible in intuitionistic first-order logic.*
(c) *If Γ is a set of $\{\forall, \rightarrow\}$-free sentences, consistent with respect to minimal logic, then Γ is satisfiable in a canonical interpretation of $\mathscr{L}(\Gamma, C)$ for any set C of constants not occurring in Γ of cardinality not less than $|\mathscr{L}(\Gamma)|$.*
(d) *If a $\{\forall, \rightarrow\}$-free sentence is deducible in classical first-order logic, then its double negation is deducible in first-order minimal logic.*

While $\bot_c$ suffices to establish $C'_{\neg\forall}$, it is more than what is needed for that purpose. In fact, to establish $C'_{\neg\forall}$, we could use the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma, \neg\forall x H, [\neg H\{x/y\}]\\ \mathscr{D}\{x/y\}\\ \bot\end{array}}{\neg\neg H\{x/y\}}}{\forall x \neg\neg H}}{\neg\neg\forall x H} \qquad \neg\forall x H}{\bot}$$

based on the rule that allows the inference of $\neg\neg\forall x H$ from $\forall x\neg\neg H$. Thus, if, in accordance with [2] and [1], we denote with **MH** the system that is obtained by adding to **N** such a rule, to be called the double negation shift (DNS) rule, and let $\rhd_{mh}$ stand for the corresponding derivability relation, we have the following:

Corollary 3 *If Γ is a set of sentences such that $\Gamma \not\rhd_{mh} \bot$, then, for any set C of constants not occurring in $\bar{\Gamma}$ of cardinality not less than $|\mathscr{L}(\Gamma)|$, Γ is satisfiable in a canonical interpretation over $\mathscr{L}(\Gamma, C)$.*

Corollary 4 (Glivenko's Theorem for MH) *If a sentence F is deducible in* **NC**, *then* $\neg\neg F$ *is deducible in MH*.

Furthermore, we have the following corollary which refines the completeness theorem.

Corollary 5 (Completeness Theorem) *If for a set C of constants not occurring in Γ, F, of cardinality not less than $|\mathscr{L}(\Gamma)|$, there is no canonical interpretation of $\mathscr{L}(\Gamma, F, C)$ in which Γ is satisfied while F is not; a fortiori, if $\Gamma \models F$, then $\Gamma \rhd_c F$. Furthermore, if the use of the DNS rule is allowed or Γ, F is $\forall$-free, then there is a deduction of F from Γ that contains only one final application of $\bot_c$, and if Γ is both $\forall$ and $\to$-free, such a derivation, except for its final application of the $\bot_c$ rule, can be taken to belong to minimal logic.*

That **MH** is indeed an optimal answer to the question of which logical strength is needed to ensure that the consistency of a set of sentences implies the existence of a classical structure in which it is satisfied can be seen as follows. Obviously, the *DNS* rule is derivable from the corresponding *DNS* principle, namely the schema $\forall x \neg\neg F \to \neg\neg\forall x F$. If **L** is strong enough to ensure that consistency implies existence, in the sense just explained, then Glivenko's theorem holds for **L** and double negations of formulae that are classically valid are derivable in **L**. In particular, the double negations of the formulae in the *DNS* principle are derivable in **L**. As it is easily seen, the formulae in the *DNS* principle are equivalent in **N** to their double negations. Therefore, provided that **L** includes intuitionistic logic, the *DNS* principle and, hence, also the *DNS* rule are derivable in **L**, so that **L** must be at least as strong as **MH**.

Remark Other ways of introducing **MH** are considered in [2, 4, 7], and [5].

Remark The replacement of the constant c with a new variable in the proof of Proposition 2 can be avoided, provided that we adopt first-order languages endowed with individual parameters together with truth-value semantics and the $\exists$-elimination and $\forall$-introduction rules are extended by allowing individual parameters to occur in place of their proper variables (see [3] for a brief outline). In that case, Proposition 2 extends to formulae that do not contain free variables but may contain individual parameters, and for the proof, the set C of new constants is replaced by a set of new individual parameters.

Acknowledgements Work supported by funds PRIN-MIUR of Italy.

References

1. Van Dalen, D.: Intuitionistic logic. In: Handbook of Philosophical Logic, vol. III, pp. 225–339. Reidel, Dordrecht (1986)
2. Gabbay, D.M.: Applications of trees to intermediate logics. J. Symb. Log. **37**(1), 135–138 (1972)
3. Parlamento, F.: Truth-value semantics and functional extensions for classical logic of partial terms based on equality. Notre Dame J. Form. Log. **55**(3), 383–395 (2014)
4. Rauszer, C.: Formalizations of intermediate logics, Part I. In: Methods in Mathematical Logic—Proceedings of the Sixth Latin American Symposium, Caracas, 1983, Lecture Notes in Mathematics, vol. 1130, pp. 360–384. Springer, Berlin (1985)

5. Seldin, J.P.: On the proof theory of the intermediate logic MH. J. Symb. Log. **51**(3), 626–647 (1986)
6. Smullyan, R.: First-Order Logic. Springer, Berlin (1968)
7. Umezava, T.: On logics intermediate between intuitionistic and classical logic. J. Symb. Log. **24**(1), 141–153 (1959)

F. Parlamento (✉)
Department of Mathematics and Computer Science, University of Udine, via Delle Scienze 206, 33100 Udine, Italy
e-mail: franco.parlamento@uniud.it

From Classical to Fuzzy Type Theory

Vilém Novák

Abstract Higher-order logic—the type theory (TT)—is a powerful formal theory that has various kinds of applications, for example, in linguistic semantics, computer science, foundations of mathematics and elsewhere. It was proved to be incomplete with respect to standard models. In fifties and sixties of the last century, L. Henkin proved that there is an axiomatic system of TT that is complete if we relax the concept of model to the, so called, generalized one. The difference is that domains of functions in generalized models need not contain all possible functions but only subsets of them. Henkin then proved that a formula of type o (truth value) of a special theory T of the theory of types is provable iff it is true in all general models of T. Mathematical fuzzy logic is a special many-valued logic whose goal is to provide tools for capturing the vagueness phenomenon via degrees. It went through intensive development and many formal systems of both propositional as well as first-order fuzzy logic were proved to be complete. This endeavor was crowned in 2005 when also higher-order fuzzy logic (called the Fuzzy Type Theory, FTT) was developed and its completeness with respect to general models was proved. The proof is based on the ideas of the Henkin's completeness proof for TT. This paper addresses several complete formal systems of the fuzzy type theory. The systems differ from each other by a chosen algebra of truth values. Namely, we focus on three systems: the Core FTT based on a special algebra of truth values for fuzzy type theory—the EQ-algebra, then IMTL-FTT based on IMTL$_\Delta$-algebra of truth values and finally the Ł-FTT based on MV$_\Delta$-algebra of truth values.

Keywords Fuzzy type theory · EQ-algebra · Residuated lattice · IMTL-algebra · MV-algebra · Higher-order fuzzy logic · Mathematical fuzzy logic · Δ-operation

1 Introduction

Leon Henkin was a great logician of twentieth century who significantly contributed to many areas of mathematical logic including the higher-order one. Recall that B. Russell in [36] introduced the type theory as a general formal logic using which he wanted to prevent paradoxes occurring in set theory. For a long time, however, higher-order logic was taken as the system that is, unlike the predicate first-order logic, incomplete. The

The research was supported by the European Regional Development Fund in the IT4 Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070).

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_16

problem was in formulation of semantics of the former, namely, only standard models containing all the possible elements for each type were taken into account. Henkin came in the 1950s with ingenious idea to consider also general models, that is, sets of elements of some (higher) types in these models may lack some elements (see [18]). As the basic formal system, he chose elegant λ-calculus in the formulation of Church [3]. In 1963, he simplified his theory and published second highly influential paper on this topic [19].

In parallel with these works, the development of many-valued logic was continued. Recall that this logic was founded by J. Łukasiewicz in the 1920s (see [21, 22]). A significant step forward was the discovery of MV-algebras in [2]. Many-valued logic, however, was not taken seriously as the real logic. The change came after introduction of the concept of a fuzzy set by Zadeh [38]. The first deep analysis of the potential of logic based on the latter concept was given by Goguen [14]. Since then, we speak about *fuzzy logic*. On the basis of Goguen's paper, Pavelka [35] introduced a propositional version of fuzzy logic as a well-established constituent of mathematical logic. His logic was further extended by the author of this paper to first-order one and concluded in the book [34] as the so-called *fuzzy logic with evaluated syntax*.

In the 1990s, Hájek [16] initiated development of mathematical fuzzy logic (MFL) as an extended formal system with many-valued semantics (we speak about *fuzzy logic with traditional syntax*). He was followed by many authors (see, e.g., [7, 8, 15, 33]), so that MFL became a highly sophisticated branch of mathematical logic. As a consequence of this approach, there now exist many formal systems of fuzzy logic that differ in the chosen structure of truth values. The latter then determines special properties of the given system of MFL.

Let us remark that one of the main arguments in favor of MFL consists in its ability to provide a working model of various manifestations of the vagueness phenomenon. The latter is present in all kinds of human reasoning and is a distinguished feature of the semantics of natural language.

The fact that fuzzy logic touched the vagueness phenomenon led to a large number of its successful applications. Quite often, they rely on a simple model of the meaning of some words of natural language (see, e.g., [25]). From linguistic point of view, however, these applications are more or less naive. Moreover, since fuzzy logic has been developed only up to the first order, these applications are rather limited. In the accepted linguistic theory, an important role is played by intensional logic (see [9, 13, 37]), which is based on the simple theory of types. This raises the question whether the latter can be generalized also to fuzzy one. The positive answer is the *fuzzy type theory* (FTT). This theory was formulated by V. Novák first in [26] and then fully elaborated in [27] in the style developed for classical type theory by L. Henkin. Recall that a specific feature of the type theory is the crucial role of equality among objects of the respective types.[1] In fuzzy logic, the equality is generalized to a many-valued (fuzzy) one, that is, there can exist not precisely equal couples of objects.

In this chapter, we give an overview of the present state of FTT. Note that the basic (but not sufficient) step in the development of FTT is to replace the first axiom of type theory, stating that there are just two truth values by a sequence of axioms characterizing a chosen algebra of truth values. FTT is proved to be complete in the sense of the original Henkin's proof of the completeness of type theory with respect to generalized models.

[1] In case of truth values, we speak about equivalence instead of equality.

Two interesting problems occurred when developing FTT. The first one concerns the classical axiom

$$(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha} x_\alpha \equiv f_{\beta\alpha} y_\alpha).$$

Interpretation of the equality formula $\equiv$ in FTT is a fuzzy equality that is a binary fuzzy relation. A truth value of $m \equiv m'$ laying between 0 and 1 means that the elements m, m' are not fully equal; they can be only "similar" to each other in the given degree. The axiom above, however, states that equality of arguments leads to equality of function values for *any* function $f_{\beta\alpha}$. It turns out that this requirement cannot be fulfilled in the many-valued case, unless the equality is just the classical one (i.e., crisp). The only way to overcome this problem is to introduce in the language a special connective Δ whose interpretation in an algebra of truth values is a function that puts the truth values smaller than 1 to 0. This trick enables us to preserve elegance of the classical type theory.

Another problem is methodological. As will be seen below, if we consider a residuated lattice[2] as the structure of truth values, then one of the basic operations in it is the *residuation* operation $\rightarrow$ that naturally interprets implication in fuzzy logic. But then equivalence is interpreted by a derived operation of biresiduation defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$$

(for all a, b). We thus encounter a methodological discrepancy for FTT: the *basic connective* is interpreted by a *derived algebraic operation*. This initiated the attempt to introduce a special algebra for the fuzzy type theory in which fuzzy equality (equivalence) is basic while implication is derived. Such an algebra is called an EQ-algebra and is studied in [29, 31].

The chapter consists of five sections. In Section 2, we give overview of the considered algebras of truth values. Section 3 contains a description of the formal systems of FTT. In Section 4, we mention some important properties of FTT. Finally, in Section 5, we discuss the concept of FTT and outline some possibilities for its further development.

In this paper, we will freely use the following standard concepts: A fuzzy set is a function $A : M \rightarrow E$ where M is a universe and E a support of an algebra of truth values. The *kernel* of A is the set $Ker(A) = \{m \in M \mid A(m) = 1\}$. A fuzzy set A is *normal* if $Ker(A) \neq \emptyset$ and *subnormal* otherwise. The set of all normal fuzzy sets on M is denoted by $\mathscr{N}\mathscr{F}(M)$. An n-ary fuzzy relation R on M for some $n \geq 0$ is a fuzzy set $R : M^n \rightarrow E$.

2 Truth Values in FTT

The first axiom of the classical type theory states that there are just two truth values. In fuzzy type theory, this axiom is replaced by a sequence of axioms characterizing structure of the truth values that is an algebra with convenient properties (see [5, 11]). In this section, we will overview some of the main ones.

[2] A residuated lattice is accepted as the fundamental algebra of truth values considered in most systems of mathematical fuzzy logic.

2.1 Residuated Lattices

According to the analysis done in mathematical fuzzy logic, the algebra of truth values should be a residuated lattice with additional properties. More precisely, this should be an integral, residuated, prelinear, lattice-ordered monoid

$$\mathscr{E} = \langle E, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle, \tag{1}$$

where $\langle E, \vee, \wedge, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1. The $\otimes$ is a multiplication operation such that $\langle E, \otimes, 1\rangle$ is a commutative monoid, and $\rightarrow$ is a residuation operation. Both these operations fulfill the *adjunction property*

$$a \otimes b \leq c \quad \Longleftrightarrow \quad a \leq b \rightarrow c, \quad a, b, c \in E. \tag{2}$$

The residuation operation is a natural interpretation of logical implication.

2.2 MTL and IMTL-Algebras

MTL-algebras are residuated lattices that are *prelinear*, that is, they fulfill also the equality

$$(a \rightarrow b) \vee (b \rightarrow a) = 1, \quad a, b \in E. \tag{3}$$

Negation is the function $\neg a = a \rightarrow 0$. It is involutive if

$$\neg\neg a = a, \quad a \in E. \tag{4}$$

The latter is also called the *law of double negation*. An MTL-algebra in which (4) holds is called an IMTL-algebra.

We can define also the following additional operations in residuated lattices:

$$\begin{aligned}
a \oplus b &= \neg(\neg a \otimes \neg b), \quad &&(\textit{strong sum})\\
a^n &= \underbrace{a \otimes \cdots \otimes a}_{n\text{-times}}, \quad &&(n\textit{-fold strong power})\\
na &= \underbrace{a \oplus \cdots \oplus a}_{n\text{-times}}, \quad &&(n\textit{-fold strong sum})\\
a \leftrightarrow b &= (a \rightarrow b) \wedge (b \rightarrow a) \quad &&(\textit{biresiduation})
\end{aligned}$$

for all $a, b \in L$.

2.3 MV-Algebras

This is the class of the most important algebras of truth values. The concept of an MV-algebra was introduced by Chang [2].

An MV-algebra is an algebra

$$\mathscr{E} = \langle L, \oplus, \otimes, \neg, 0, 1\rangle \tag{5}$$

in which the following identities are valid:

$$a \oplus b = b \oplus a, \qquad a \otimes b = b \otimes a, \tag{6}$$

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c, \qquad a \otimes (b \otimes c) = (a \otimes b) \otimes c, \tag{7}$$

$$a \oplus 0 = a, \qquad a \otimes 1 = a, \tag{8}$$

$$a \oplus 1 = 1, \qquad a \otimes 0 = 0, \tag{9}$$

$$a \oplus \neg a = 1, \qquad a \otimes \neg a = 0, \tag{10}$$

$$\neg(a \oplus b) = \neg a \otimes \neg b, \qquad \neg(a \otimes b) = \neg a \oplus \neg b, \tag{11}$$

$$a = \neg\neg a, \qquad \neg 0 = 1, \tag{12}$$

$$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a. \tag{13}$$

The lattice operations in MV-algebras can be introduced by

$$a \vee b = \neg(\neg a \oplus b) \oplus b = (a \otimes \neg b) \oplus b, \tag{14}$$

$$a \wedge b = \neg(\neg a \vee \neg b) = (a \oplus \neg b) \otimes b. \tag{15}$$

Moreover, if we put $a \to b = \neg a \oplus b$, then it can be proved that $\to$ is a residuation w.r.t. $\otimes$ and that the prelinearity (3) also holds. Hence, we can prove that each MV-algebra is an MTL-algebra. Important property is the following:

Theorem 16.1 *A residuated lattice $\mathscr{E}$ is an MV-algebra iff*

$$(a \to b) \to b = a \vee b \tag{16}$$

for all $a, b \in L$.

For many properties of all the above-introduced algebras, see [4, 16, 34].

Any residuated lattice with $E = [0, 1]$ is called *standard*. It is known that multiplication in the standard MTL-algebra is a so-called left-continuous t-norm (see [20]) that is an operation generalizing the boolean conjunction operation (logical "AND"). It can be easily verified that each boolean algebra is a special case of residuated lattice and it is at the same time an MTL-algebra and an MV-algebra.

A natural interpretation of logical equivalence in fuzzy logic is the biresiduation operation $\leftrightarrow$. This is a binary operation that is *separated*, that is, it has the property $a \leftrightarrow b = 1$ iff $a = b$. Moreover, it is *reflexive* ($a \leftrightarrow a = 1$), *symmetric*, and *transitive* in the following sense:

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \le a \leftrightarrow c$$

for all $a, b, c \in L$.

In fuzzy type theory, we moreover need a unary operation of $\Delta : E \to E$ whose role is to "extract" the boolean substructure from the given algebra of truth values. It is algebraically defined using the following axioms:

$$\Delta 1 = 1, \tag{17}$$

$$\Delta(a \vee b) \leq \Delta a \vee \Delta b, \tag{18}$$

$$\Delta a \leq a, \tag{19}$$

$$\Delta a \leq \Delta\Delta a, \tag{20}$$

$$\Delta(a \to b) \leq \Delta a \to \Delta b, \tag{21}$$

$$\Delta a \vee \neg\Delta a = 1. \tag{22}$$

If $\mathscr{E}_\Delta$ is linearly ordered, then Δ can be defined as

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

To stress the presence of the Δ operation, we often speak about an MTL$_\Delta$-algebra or an MV$_\Delta$-algebra.

2.4 Fundamental Examples

There are three fundamental examples of the standard MTL-algebra of truth values (i.e., it is defined on the interval $E = [0, 1]$) that play essential role in mathematical fuzzy logic:

Standard Gödel algebra

$$a \otimes b = a \wedge b$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

$$\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a \leftrightarrow b = \begin{cases} 1 & \text{if } a = b \\ a \wedge b & \text{otherwise} \end{cases}$$

Standard product algebra

$$a \otimes b = a \cdot b$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{otherwise} \end{cases}$$

$$\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a \leftrightarrow b = \begin{cases} 1 & \text{if } a = b \\ \frac{a \wedge b}{a \vee b} & \text{otherwise} \end{cases}$$

Standard Łukasiewicz MV-algebra

$$a \otimes b = 0 \vee (a + b - 1),$$

$$a \to b = 1 \wedge (1 - a + b),$$

$$\neg a = 1 - a,$$

$$a \leftrightarrow b = 1 - |a - b|,$$

where $a, b \in [0, 1]$. The standard Łukasiewicz MV-algebra is the only standard residuated lattice (up to isomorphism) whose residuation operation is continuous.

Note that the multiplication operation (i.e., a t-norm) in all three examples is continuous but only the standard Łukasiewicz MV-algebra has also the continuous residuation. The negation is involutive also only in the latter case. An example of a left-continuous t-norm whose residuation generates an involutive negation is the *nilpotent minimum* defined by

$$a \otimes b = \begin{cases} a \wedge b & \text{if } a + b > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Its residuum is defined by

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise,} \end{cases} \tag{25}$$

where $\neg a = 1 - a$. Thus, the standard algebra with the multiplication (24) and residuation (25) is an IMTL-algebra.

2.5 EQ-Algebras

As mentioned in Introduction, the methodological problem raising from the algebras considered above is that the basic connective in FTT is fuzzy equivalence and so implication is derived, whereas in a residuated lattice, the biresiduation operation (see Section 2.2) as interpretation of equivalence is derived from residuation. Therefore, we face an exciting task to find some other typical algebra of truth values in which the fuzzy equivalence is basic while the implication is derived. Such an algebra does exists and is called an EQ-algebra (see [31]):

$$\mathscr{E} = \langle E, \wedge, \otimes, \sim, 0, 1, \rangle, \tag{26}$$

where for all $a, b, c, d \in E$:

(E1) $\langle E, \wedge, 1\rangle$ is a commutative idempotent monoid (i.e., $\wedge$-semilattice). We put $a \leq b$ iff $a \wedge b = a$, as usual. Then 1 is the top and 0 the bottom element.
(E2) $\langle E, \otimes, 1\rangle$ is a monoid, $\otimes$ is isotone w.r.t. $\leq$.
(E3) $a \sim a = 1$ (reflexivity).
(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution).
(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence).
(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity).
(E7) $a \otimes b \leq a \sim b$ (boundedness).
(E8) $a \sim 1 = a$ (goodness).

The operation $\wedge$ is called meet (infimum), $\otimes$ is called multiplication, and $\sim$ is a fuzzy equality. Note that the multiplication is, in general, noncommutative.

Axiom (E3) expresses reflexivity, (E4) is the substitution axiom, (E5) is the congruence axiom, (E6) is a monotonicity axiom, and (E7) is the axiom of boundedness. Axiom (E8) may not be included. If yes, then we call the EQ-algebra *good*.

We define:

$$a \to b = (a \wedge b) \sim a, \tag{27}$$

$$a \vee b = \big((a \to b) \to b\big) \wedge \big((b \to a) \to a\big). \tag{28}$$

It should be emphasized that every residuated lattice is a good EQ-algebra with fuzzy equality being the biresiduation (i.e., $a \sim b = (a \to b) \wedge (b \to a)$).

An example of a nonresiduated noncommutative EQ-algebra on [0, 1] is the EQ-algebra with multiplication defined as follows:

$$a \otimes b = \begin{cases} 0, & 2a + b \leq 1, \\ \min\{a, b\}, & 2a + b > 1, \end{cases} \qquad a \bar{\otimes} b = \begin{cases} 0, & a + 2b \leq 1, \\ \min\{a, b\}, & a + 2b > 1. \end{cases}$$

Both $\otimes$ and $\bar{\otimes}$ are isotone monoidal operations on [0, 1], but they are not commutative. Furthermore, put

$$a \sim b = \begin{cases} 1, & a = b, \\ \max(\frac{1}{2} - a, b), & a > b, \\ \max(\frac{1}{2} - b, a), & a < b. \end{cases}$$

Then $\mathscr{E} = \langle [0, 1], \wedge, \otimes, \sim, 1 \rangle$ and $\mathscr{E}' = \langle [0, 1], \wedge, \bar{\otimes}, \sim, 1 \rangle$ are noncommutative EQ-algebras that are not residuated.

The *delta operation* $\Delta : E \to E$ in an EQ-algebra E is defined by the following axioms:

(i) $\Delta 1 = 1$,
(ii) $\Delta a \leq a$,
(iii) $\Delta a \leq \Delta \Delta a$,
(iv) $\Delta(a \sim b) \leq \Delta a \sim \Delta b$,
(v) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$.
(vi) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,
(vii) $\Delta a \vee \neg \Delta a = 1$

($a, b \in E$). If $\mathscr{E}$ is linearly ordered, then the Δ operation can be defined as in (23).

2.6 *Fuzzy Equality Between Arbitrary Objects and Weakly Extensional Functions*

In FTT, we must introduce also a *fuzzy equality* on an arbitrary set M.

Definition 16.1 Let $\mathscr{E}$ be an EQ-algebra with support E, and M a set. A fuzzy equality $\stackrel{\circ}{=}$ on M is a binary fuzzy relation on M, that is, a function

$$\stackrel{\circ}{=} : M \times M \to E$$

such that the following holds for all $m, m', m'' \in M$:[3]

(i) $[m \mathrel{\dot{=}} m] = 1$ (reflexivity)
(ii) $[m \mathrel{\dot{=}} m'] = [m' \mathrel{\dot{=}} m]$ (symmetry)
(iii) $[m \mathrel{\dot{=}} m'] \otimes [m \mathrel{\dot{=}} m'] \leq [m \mathrel{\dot{=}} m']$ ($\otimes$-transitivity).

We say that $\mathrel{\dot{=}}$ is *separated* if

$$\left[m \mathrel{\dot{=}} m'\right] = 1 \quad \text{iff} \quad m = m'$$

for all $m, m' \in M$.

As a special case, objects of the set M can be functions. Then the fuzzy equality can be induced in the sense of the following lemma.

Lemma 16.1 ([27]) *Let M, N be sets, and $\mathrel{\dot{=}}_N$ be a fuzzy equality on the set N. Then the function $\mathrel{\dot{=}}_{NM}$: $N^M \times N^M \to E$ defined for every $f, g \in N^M$ by*

$$[f \mathrel{\dot{=}}_{NM} g] = \bigwedge_{m \in M} \left[f(m) \mathrel{\dot{=}}_N g(m)\right] \tag{29}$$

is a fuzzy equality. If $\mathrel{\dot{=}}_N$ is separated, then $\mathrel{\dot{=}}_{NM}$ is also separated.

Finally, we need the following concept. Let $f : M \to N$ be a function, and $\mathrel{\dot{=}}_M, \mathrel{\dot{=}}_N$ be fuzzy equalities defined on M, N, respectively. We say that f is *weakly extensional* if for all $m, m' \in M$,

$$\left[m \mathrel{\dot{=}}_M m'\right] = 1 \quad \text{implies that} \quad \left[f(m) \mathrel{\dot{=}}_N f\left(m'\right)\right] = 1. \tag{30}$$

3 Fuzzy Type Theory

As mentioned, there are more fuzzy type theories that differ in the chosen algebra of truth values. The latter determines the choice of logical axioms and also necessary definitions of special formulas. Theoretical investigations show that there are many formal systems of propositional fuzzy logics. For example, in [6], 69 various fuzzy logic systems are analyzed that are based on residuated lattices. When considering EQ-algebras, this number can still be extended because EQ-algebras are more general than residuated lattices. The EQ-logics based on nonresiduated EQ-algebras have already been formed (see [10]). Moreover, each of these fuzzy logic systems is extended to a first-order one. Therefore, we may expect that each fuzzy logic can be extended also to a higher-order one.

On the other hand, however, we doubt that this has any practical sense. Therefore, we will confine only to few distinguished fuzzy type theories. Namely, we will consider in this paper three of them: the *Core FTT* based on general non-commutative EQ_Δ-algebra of truth values; this can be extended (by adding further axioms) to IMTL-FTT (IMTL *fuzzy type theory*) whose algebra of truth values is an IMTL_Δ-algebra; and finally, Ł-FTT (*Łukasiewicz fuzzy type theory*) whose algebra of truth values is an MV_Δ-algebra. All the considered algebras must be linearly ordered.

[3] As usual, we write $m \mathrel{\dot{=}} m'$ instead of $\mathrel{\dot{=}}(m, m')$. The symbol $[m \mathrel{\dot{=}} m']$ denotes an (arbitrary) truth value of $m \mathrel{\dot{=}} m'$.

3.1 Syntax

The basic syntactical objects of fuzzy type theories are classical (see [1]), namely the concepts of type and formula. The atomic types are ε (elements) and o (truth degrees). The set of types is the smallest set *Types* satisfying:

(i) $\varepsilon, o \in Types$,
(ii) if $\alpha, \beta \in Types$, then $(\alpha\beta) \in Types$.

Recall that higher-order types represent functions.

A *language* J of FTT consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ where $\alpha \in Types$, auxiliary symbol λ, and brackets. A set of formulas[4] over the language J is the smallest set such that for each $\alpha, \beta \in Types$, the following are specified:

(i) If $x_\alpha \in J$ is a variable, $\alpha \in Types$, then x_α is a formula of type α.
(ii) If $c_\alpha \in J$ is a constant, $\alpha \in Types$, then c_α is a formula of type α.
(iii) If $B_{\beta\alpha}$ is a formula of type $\beta\alpha$ and A_α a formula of type α, then $(B_{\beta\alpha} A_\alpha)$ is a formula of type β.
(iv) If A_β is a formula of type β and $x_\alpha \in J$ a variable of type α, then $\lambda x_\alpha A_\beta$ is a formula of type $\beta\alpha$.

The set of formulas of type α, $\alpha \in Types$, is denoted by $Form_\alpha$. If $A \in Form_\alpha$ is a formula of type $\alpha \in Types$, then we will often write A_α. This means that if $\alpha \neq \beta$, then A_α and A_β are *different* formulas. A set of all formulas of the language J is $Form = \bigcup_{\alpha \in Types} Form_\alpha$.

Common Constants and Definitions

Specific constants always present in the language of FTT are the following.

(i) $\mathbf{E}_{(o\alpha)\alpha}$, $\alpha \in Types$ (fuzzy equality),
(ii) $\mathbf{D}_{oo}$ (delta connective),
(iii) $\iota_{\varepsilon(o\varepsilon)}, \iota_{o(oo)}$ (description operators).

The fundamental connective in FTT is the *fuzzy equality* defined as follows:

(i) $\equiv_{(o\alpha)\alpha} := \lambda x_\alpha \lambda y_\alpha (\mathbf{E}_{(o\alpha)\alpha} y_\alpha) x_\alpha$, $\alpha \in \{o, \varepsilon\}$,
(ii) $\equiv_{(o(\beta\alpha))(\beta\alpha)} := \lambda f_{\beta\alpha} \lambda g_{\beta\alpha} (\forall x_\alpha)(f_{\beta\alpha} x_\alpha \equiv g_{\beta\alpha} x_\alpha)$.

We will write fuzzy equality in the form $(A_\alpha \equiv B_\alpha)$ and omit the type at "$\equiv$". Note that $(A_\alpha \equiv B_\alpha)$ is a formula of type o. If $\alpha = o$, then "$\equiv$" is the logical (fuzzy) equivalence.

Further basic formulas in FTT are the following:

(a) Representation of truth and falsity:

$$\top := (\lambda x_o x_o \equiv \lambda x_o x_o), \qquad \bot := (\lambda x_o x_o \equiv \lambda x_o \top).$$

[4] In the present literature on classical type theory, especially that focused on computer science, the formulas are often called *lambda-terms*. We will keep the original logical term introduced by A. Church and L. Henkin to emphasize that we are developing the *logic*.

(b) Delta:

$$\Delta := \lambda x_o \mathbf{D}_{oo} x_o.$$

(c) Negation:

$$\neg := \lambda x_o(\bot \equiv x_o).$$

(d) Implication:

$$\Rightarrow := \lambda x_o \lambda y_o \big((x_o \wedge y_o) \equiv x_o\big).$$

(e) General quantifier:

$$(\forall x_\alpha) A_o := (\lambda x_\alpha A_o \equiv \lambda x_\alpha \top).$$

Further details can be found in [27, 29].

Fundamental Axioms

The following axioms are common for all kinds of FTT:

(Fund1) $\Delta(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha} x_\alpha \equiv f_{\beta\alpha} y_\alpha)$,
(Fund2) $(\forall x_\alpha)(f_{\beta\alpha} x_\alpha \equiv g_{\beta\alpha} x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$,
(Fund3) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha} x_\alpha \equiv g_{\beta\alpha} x_\alpha)$,
(Fund4) $(\lambda x_\alpha B_\beta) A_\alpha \equiv C_\beta$,
where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α,
(Fund5) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha,\ \alpha = o, \varepsilon$,
(Fund6) $(x_o \equiv \top) \equiv x_o$.

Notice that axioms (Fund1)–(Fund5) are practically the same as axioms (2)–(5) from [1]. Axiom (Fund1) differs from the classical one by the connective Δ. The reason lays in the assumption that all the considered equalities are fuzzy. Without using Δ, however, this axiom would be too strong and enforce degeneration of the fuzzy equality just to the classical one. The Δ connective overcomes this inconvenience as follows: each fuzzy equality is separated, and so, this axiom, in fact, says that all functions behave classically—classically equal arguments require equality of the functional values. Since interpretation of Δ is a function $\Delta(a)$ assigning zero to all $a < 1$, interpretation of (Fund1) gives always truth value 1. Consequently, this axiom imposes no restriction in cases where (interpretation of) x_o is equal to y_o in some general degree only.

Axioms (Fund2) and (Fund3) characterize the considered functions. For formal reasons, we had to split one axiom of classical type theory into two ones. Axiom (Fund4) is an important axiom of lambda conversion. Axiom (Fund5) characterizes the description operator in the same way as in classical TT. Finally, axiom (Fund6) is a general characterization of the algebra of truth values that must be *separated*, that is, equality of truth values in the degree 1 is the same as classical equality.

Inference Rules and Provability in FTT

The following inference rules are common to all fuzzy type theories.

(R) *Let* $A_\alpha \equiv A'_\alpha \in Form_o$ *and* $B_o \in Form_o$ *be formulas. Then we infer from them a formula* B'_o *that comes from* B_o *by replacing one occurrence of* A_α *by* A'_α*, provided that the occurrence of* A_α *in* B_o *is not an occurrence of a variable immediately preceded by* λ.

(N) *Let* $A_o \in Form_o$ *be a formula. Then from* A_o *infer* ΔA_o.

The rule (R) is in unchanged form taken from [18] (or [1]). The rule (N) is the *necessitation rule* first introduced in [16].

The concept of *provability* and *proof* are defined in the same way as in classical logic. A *theory* T over FTT is a set of formulas of type o, that is, $T \subseteq Form_o$. As usual, T is determined by a *set of its special axioms*, which is again a subset of $Form_o$. If T is a theory, then its language will be written as $J(T)$.

Semantics

Interpretation of formulas in FTT is realized in a *frame* that is a tuple

$$\mathscr{M} = \left\langle (M_\alpha, \stackrel{\circ}{=}_\alpha)_{\alpha \in Types}, \mathscr{E}_\Delta, I_o, I_\varepsilon \right\rangle \tag{31}$$

such that the following holds:

(i) The $\mathscr{E}_\Delta$ is an algebra of truth degrees. We put $M_o = E$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathscr{E}_\Delta$.

(ii) $=_\alpha$: $M_\alpha \times M_\alpha \to L$ is a fuzzy equality on M_α, that is, $\stackrel{\circ}{=}_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$.

(iii) $I_o : \mathscr{N}\mathscr{F}(M_o) \to M_o$, $I_\varepsilon : \mathscr{N}\mathscr{F}(M_\varepsilon) \to M_\varepsilon$ are functions interpreting the basic description operators. We assume that $I_\alpha(A) \in Ker(A)$ for every normal fuzzy set $A \in \mathscr{N}\mathscr{F}(M_\alpha)$, $\alpha \in \{o, \varepsilon\}$. If the given fuzzy set is subnormal, then the operations I_o, I_ε are not defined.

Interpretation of a formula A_α, in general, is assignment of an element from the corresponding set M_α to it. It is defined recurrently starting with an assignment p of elements from M_α to variables (of the same type). Given the assignment p, we write interpretation of A_α in $\mathscr{M}$ as $\mathscr{M}_p(A_\alpha) \in M_\alpha$. If the assignment is not important, then we omit the subscript p.

Let x_α be a variable, and $p, p' \in \mathrm{Asg}(\mathscr{M})$ be two assignments such that $p'(x_\alpha) \neq p(x_\alpha)$ and $p'(y_\gamma) = p(y_\gamma)$ for all $y_\gamma \neq x_\alpha$ (i.e., p' differs from p only in the variable x_α). In this case, we will write $p' = p \backslash x_\alpha$. Now we define:

(i) If x_α is a variable, then $\mathscr{M}_p(x_\alpha) = p(x_\alpha)$.

(ii) If c_α is a constant, then $\mathscr{M}_p(c_\alpha)$ is a specific element from M_α. If $\alpha \neq o, \varepsilon$, then $p(c_\alpha)$ is a weakly extensional function. As a special case:

  (a) $\mathscr{M}_p(\mathbf{E}_{(o\alpha)\alpha}) : M_\alpha \times M_\alpha \to L$ is a fuzzy equality $=_\alpha$.
  In more detail: if $\alpha = \varepsilon$, then $\mathscr{M}_p(\mathbf{E}_{(o\varepsilon)\varepsilon})$ is a given fuzzy equality $=_\varepsilon$; if $\alpha = o$, then $\mathscr{M}_p(\mathbf{E}_{(oo)o})$ is either the operation $\sim$ of fuzzy equality in the EQ-algebra

$\mathscr{E}_\Delta$ or it is the biresiduation $\leftrightarrow$. Otherwise, $\mathscr{M}_p(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality $=_\alpha$ defined by (29).

(b) $\mathscr{M}_p(\mathbf{D}_{oo}) : L \to L$ is the delta operation

$$\mathscr{M}_p(\mathbf{D}_{oo})(a) = \Delta(a)$$

for all $a \in L$.

(iii) An interpretation of a formula $B_{\beta\alpha}A_\alpha$ of type β is

$$\mathscr{M}_p(B_{\beta\alpha}A_\alpha) = \mathscr{M}_p(B_{\beta\alpha})\big(\mathscr{M}_p(A_\alpha)\big).$$

(iv) An interpretation of a formula $(\lambda x_\alpha\, A_\beta)$ of type $\beta\alpha$ is a function

$$\mathscr{M}_p(\lambda x_\alpha\, A_\beta) = F : M_\alpha \to M_\beta,$$

which is weakly extensional w.r.t. "$=_\alpha$" and "$=_\beta$", and which assigns to each $m_\alpha \in M_\alpha$ the element $F(m_\alpha) = \mathscr{M}_{p'}(A_\beta)$ determined by an assignment p' such that $p' = p\backslash x_\alpha$ and $p'(x_\alpha) = m_\alpha$.

A model of a theory T is a general frame $\mathscr{M}$ for which $\mathscr{M}_p(A_o) = 1$ for all axioms A_o of T. Note that the definition of a general model corresponds to the definition of a safe model introduced in [16]. A formula A_o is true in the theory T, $T \models A_o$, if it is true in the degree 1 in all its models.

3.2 *Special Fuzzy Type Theories*

In this subsection, we will present three basic kinds of fuzzy type theories that differ from each other by the algebra of truth values. From the syntactical point of view, this means that the fuzzy type theories differ in definitions of some special formulas and in additional logical axioms.

Core FTT

This is the basic fuzzy type theory whose truth values form a noncommutative good EQ-algebra. The other type theories can be obtained from the Core FTT by adding further logical axioms and adding or modifying some of the definitions.

The language J is extended by the following constants: $\mathbf{C}_{(oo)o}$ (conjunction) and $\mathbf{S}_{(oo)o}$ (strong conjunction). Then, the following additional connectives and existential quantifier are defined:

(i) *Conjunction:*

$$\wedge := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o} y_o) x_o.$$

(ii) *Strong conjunction (fusion):*

$$\& := \lambda x_o \lambda y_o (\mathbf{S}_{(oo)o} y_o) x_o.$$

(iii) *Disjunction:*

$$\vee := \lambda x_o \lambda y_o \big((x_o \Rightarrow y_o) \Rightarrow y_o\big) \wedge \big((y_o \Rightarrow x_o) \Rightarrow x_o\big).$$

(iv) *Existential quantifier:* Let $A_o \in Form_o$, x_α be a variable of type α, and let y_o do not occur in A_o. Then we put:

$$(\exists x_\alpha) A_o := (\forall y_o)\big((\forall x_\alpha)\Delta(A_o \Rightarrow y_o) \Rightarrow y_o\big) \tag{32}$$

(y_o does not occur in A_o).

Logical axioms of the Core FTT are (Fund1)–(Fund6) plus the following:

Axiom of fuzzy equality

C-EQ1 $(x_\varepsilon \equiv y_\varepsilon)\&(y_\varepsilon \equiv z_\varepsilon) \Rightarrow (x_\varepsilon \equiv z_\varepsilon)$.

Axioms of truth values

Let $\bigcirc \in \{\&, \wedge\}$.

C-Tv111 $(x_o \wedge y_o) \equiv (y_o \wedge x_o)$,
C-Tv212 $(x_o \bigcirc y_o) \bigcirc z_o \equiv x_o \bigcirc (y_o \bigcirc z_o)$,
C-Tv313 $(x_o \bigcirc \top) \equiv x_o$,
C-Tv414 $(\top \& x_o) \equiv x_o$,
C-Tv515 $(x_o \wedge x_o) \equiv x_o$,
C-Tv616 $((x_o \wedge y_o) \equiv z_o)\&(t_o \equiv x_o) \Rightarrow (z_o \equiv (t_o \wedge y_o))$,
C-Tv717 $(x_o \equiv y_o)\&(z_o \equiv t_o) \Rightarrow (x_o \equiv z_o) \equiv (y_o \equiv t_o)$,
C-Tv818 $(x_o \Rightarrow (y_o \wedge z_o)) \Rightarrow (x_o \Rightarrow y_o)$,
C-Tv919 $(x_o \Rightarrow y_o) \Rightarrow ((x_o \wedge z_o) \Rightarrow y_o)$,
(FT-tval10a) $\Delta(x_o \Rightarrow y_o) \Rightarrow (x_o \& z_o \Rightarrow y_o \& z_o)$,
(FT-tval10b) $\Delta(x_o \Rightarrow y_o) \Rightarrow (z_o \& x_o \Rightarrow z_o \& y_o)$,
C-Tv10110 $((x_o \Rightarrow y_o) \Rightarrow z_o) \Rightarrow ((y_o \Rightarrow x_o) \Rightarrow z_o) \Rightarrow z_o$.

Axioms of delta

(C-Delt11) $(g_{oo}(\Delta x_o) \wedge g_{oo}(\neg \Delta x_o)) \equiv (\forall y_o) g_{oo}(\Delta y_o)$,
(C-Delt22) $\Delta(x_o \wedge y_o) \equiv \Delta x_o \wedge \Delta y_o$,
(C-Delt33) $\Delta(x_o \vee y_o) \Rightarrow \Delta x_o \vee \Delta y_o$,
(C-Delt44) $\Delta x_o \vee \neg \Delta x_o$.

Axioms of quantifiers

(C-qu1nt1) $\Delta(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$, x_α is not free in A_o,
(C-qu2nt2) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha) A_o \Rightarrow B_o)$, x_α is not free in B_o,
(C-qu3nt3) $(\forall x_\alpha)(A_o \vee B_o) \Rightarrow ((\forall x_\alpha) A_o \vee B_o)$, x_α is not free in B_o.

Semantics of the Core FTT

The structure $\mathscr{E}_\Delta$ of truth values is a linearly ordered good EQ_Δ-algebra (26) whose multiplication operation $\otimes$ is noncommutative.

Let $\mathscr{M}$ be a general frame (31). Then we define:

(i) $\mathscr{M}(\mathbf{C}) := \wedge$.
(ii) $\mathscr{M}(\mathbf{S}) := \otimes$.

(iii) $\mathscr{M}(\mathbf{E}_{(oo)o}) :=\sim$.
(iv) $\mathscr{M}(\mathbf{E}_{(o\varepsilon)\varepsilon})$ is a fuzzy equality due to Definition 16.1 given explicitly.
(v) Let $f, f' \in M_{\beta\alpha}$, where $\alpha, \beta \in Types$. Then

$$\mathscr{M}\big(f \doteq_{\beta\alpha} f'\big) = \bigwedge_{m \in M_\alpha} \big(\mathscr{M}_{\beta\alpha}(f)(m) \doteq_\beta \mathscr{M}_{\beta\alpha}\big(f'\big)(m)\big).$$

IMTL-FTT

The algebra of truth values $\mathscr{E}_\Delta$ of this fuzzy type theory is an IMTL$_\Delta$-algebra, that is, (commutative) MTL-algebra with the law of double negation.

The language J is extended by the constant $\mathbf{C}_{(oo)o}$ (conjunction) and the following additional connectives and existential quantifier are defined:

(i) *Conjunction:*

$$\wedge := \lambda x_o \lambda y_o (\mathbf{C}_{(oo)o} y_o) x_o.$$

(ii) *Strong conjunction (fusion):*

$$\& := \lambda x_o \lambda y_o \neg\big((x_o \wedge \neg y_o) \equiv x_o\big).$$

(iii) *Disjunction:*

$$\vee := \lambda x_o \lambda y_o \big((x_o \Rightarrow y_o) \Rightarrow y_o\big) \wedge \big((y_o \Rightarrow x_o) \Rightarrow x_o\big).$$

(iv) *Strong disjunction:*

$$\nabla := \lambda x_o \lambda y_o . \big(\neg(\neg x_o \& \neg y_o)\big).$$

(v) *Existential quantifier:* Let $A_o \in Form_o$, and x_α be a variable of type α. Then we put:

$$(\exists x_\alpha) A_o := \neg(\forall x_\alpha)\neg A_o. \tag{33}$$

All the connectives will be written in infix form as usual.

As a special case, if $A \in Form_o$, then we put

$$A^n := \underbrace{A \& \cdots \& A}_{n\text{-times}}, \quad (n\text{-}fold\ strong\ conjunction)$$

$$nA := \underbrace{A \nabla \cdots \nabla A}_{n\text{-times}} \quad (n\text{-}fold\ strong\ disjunction).$$

Note that $\bot, \top, (\forall x_\alpha)A_o, (\exists x_\alpha)A_o \in Form_o$, $\neg \in Form_{oo}$ and $\Rightarrow, \vee, \&, \nabla \in Form_{(oo)o}$.
Logical axioms of IMTL-FTT are (Fund1)–(Fund6) plus the following:

Axiom of fuzzy equality

(I-EQ1) $(x_\varepsilon \equiv y_\varepsilon) \Rightarrow ((y_\varepsilon \equiv z_\varepsilon) \Rightarrow (x_\varepsilon \equiv z_\varepsilon))$.

Axioms of truth values

(I-Tv111) $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \wedge (y_o \Rightarrow x_o))$,

(I-Tv2l2) $(x_o \Rightarrow y_o) \Rightarrow ((y_o \Rightarrow z_o) \Rightarrow (x_o \Rightarrow z_o))$,
(I-Tv3l3) $(x_o \Rightarrow (y_o \Rightarrow z_o)) \equiv (y_o \Rightarrow (x_o \Rightarrow z_o))$,
(I-Tv4l4) $((x_o \Rightarrow y_o) \Rightarrow z_o) \Rightarrow (((y_o \Rightarrow x_o) \Rightarrow z_o) \Rightarrow z_o)$,
(I-Tv5l5) $(\neg y_o \Rightarrow \neg x_o) \equiv (x_o \Rightarrow y_o)$,
(I-Tv6l6) $x_o \wedge y_o \equiv y_o \wedge x_o$,
(I-Tv7l7) $x_o \wedge y_o \Rightarrow x_o$,
(I-Tv8l8) $(x_o \wedge y_o) \wedge z_o \equiv x_o \wedge (y_o \wedge z_o)$.

Axioms of delta

(I-Delt11) $(g_{oo}(\Delta x_o) \wedge g_{oo}(\neg \Delta x_o)) \equiv (\forall y_o) g_{oo}(\Delta y_o)$,
(I-Delt22) $\Delta(x_o \wedge y_o) \equiv \Delta x_o \wedge \Delta y_o$,
(I-Delt33) $\Delta(x_o \vee y_o) \Rightarrow \Delta x_o \vee \Delta y_o$.

Axiom of quantifiers

(I-Qu1nt1) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$, where x_α is not free in A_o.

Semantics of the IMTL-FTT

The structure of truth values $\mathscr{E}_\Delta$ is a linearly ordered IMTL$_\Delta$-algebra.

Let $\mathscr{M}$ be a general frame (31). Then we define:

(i) $\mathscr{M}(\mathbf{C}) := \wedge$.
(ii) $\mathscr{M}(\mathbf{E}_{(oo)o}) := \leftrightarrow$.
(iii) $\mathbf{E}_{(o\varepsilon)\varepsilon}$ is a fuzzy equality due to Definition 16.1 given explicitly.
(iv) Let $f, f' \in M_{\beta\alpha}$, where $\alpha, \beta \in Types$. Then

$$\mathscr{M}\big(f \mathrel{\mathring{=}}_{\beta\alpha} f'\big) = \bigwedge_{m \in M_\alpha} \big(\mathscr{M}_{\beta\alpha}(f)(m) \mathrel{\mathring{=}}_\beta \mathscr{M}_{\beta\alpha}\big(f'\big)(m)\big).$$

Łukasiewicz-FTT (Ł-FTT)

The algebra of truth values $\mathscr{E}_\Delta$ of this FTT is a linearly ordered MV$_\Delta$-algebra that can be taken as a special case of IMTL$_\Delta$-algebra. Therefore, the language J of the Ł-FTT is the same as the language of the IMTL-FTT. There is a difference in the definition of *disjunction:*

$$\vee := \lambda x_o \lambda y_o \big((x_o \Rightarrow y_o) \Rightarrow y_o\big).$$

The definitions of the other connectives and quantifiers are the same as for IMTL-FTT.

Logical axioms of Ł-FTT are (Fund1)–(Fund6) plus the following:

Axiom of fuzzy equality

(Ł-EQ1) $(x_\varepsilon \equiv y_\varepsilon) \Rightarrow ((y_\varepsilon \equiv z_\varepsilon) \Rightarrow (x_\varepsilon \equiv z_\varepsilon))$.

Axioms of truth values

Ł-Tv1l1 $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \wedge (y_o \Rightarrow x_o))$,
Ł-Tv2l2 $x_o \Rightarrow (y_o \Rightarrow x_o)$,
Ł-Tv3l3 $(x_o \Rightarrow y_o) \Rightarrow ((y_o \Rightarrow z_o) \Rightarrow (x_o \Rightarrow z_o))$,
Ł-Tv4l4 $(\neg y_o \Rightarrow \neg x_o) \equiv (x_o \Rightarrow y_o)$,

Ł-Tv5l5 $(x_o \vee y_o) \equiv (y_o \vee x_o)$,
Ł-Tv6l6 $(x_o \wedge y_o) \equiv x_o \& (x_o \Rightarrow y_o)$.

Axioms of delta

(Ł-Delt11) $(g_{oo}(\Delta x_o) \wedge g_{oo}(\neg \Delta x_o)) \equiv (\forall y_o) g_{oo}(\Delta y_o)$,
(Ł-Delt22) $\Delta(x_o \wedge y_o) \equiv \Delta x_o \wedge \Delta y_o$,
(Ł-Delt33) $\Delta(x_o \vee y_o) \Rightarrow \Delta x_o \vee \Delta y_o$.

Axiom of quantifiers

(Ł-Qu1nt1) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$ where x_α is not free in A_o.

Semantics

The algebra $\mathscr{E}_\Delta$ of truth values of Ł-FTT is an MV$_\Delta$-algebra that is a special case of an IMTL$_\Delta$-algebra. Therefore, the definition of frame is the same as in the case of IMTL-FTT.

4 Properties of Fuzzy Type Theories

It should be noted that both IMTL-FTT and Ł-FTT can be obtained from the Core-FTT by adding new axioms.

4.1 Extensions of Core-FTT

From Core-FTT to IMTL-FTT

The following axioms must be added:

(CI1) $((x_o \& y_o) \Rightarrow z_o) \equiv (x_o \Rightarrow (y_o \Rightarrow z_o))$.
(CI2) $\neg\neg x_o \equiv x_o$.
(CI3) $(x_o \& y_o) \equiv \neg(x_o \Rightarrow \neg y_o)$.
(CI4) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$, x_α is not free in A_o.
(CI5) $(\exists x_\alpha) A_o \equiv \neg(\forall x_\alpha) \neg A_o$.

Axiom (CI1) enforces residuation, Axiom (CI2) is the law of double negation, Axiom (CI3) defines strong conjunction from implication, and, finally, axiom (CI4) is a stronger version of Axiom (C-quant1). Finally, axiom (CI5) precisely relates the existential quantifier with the general one.

From Core-FTT to Ł-FTT

We must add Axioms (CI1)–(CI5) plus the following:

(CŁ) $(x_o \vee y_o) \equiv (x_o \Rightarrow y_o) \Rightarrow y_o$.

Note that due to this axiom, each IMTL-algebra of truth values in the frame must be the MV-one (cf. Theorem 16.1).

4.2 Few Basic Properties

Theorem 16.2 (Soundness) *The fuzzy type theory is sound in the following sense: If* $\vdash A_o$, *then* $\mathscr{M}_p(A_o) = 1$ *for every assignment* p *and every general model* $\mathscr{M}$.

The following version of the deduction theorem is important in the proof of the generalized completeness of FTT. Note the presence of the connective Δ. The theorem does not hold in FTT without it.

Theorem 16.3 (Deduction Theorem) *Let* T *be a theory of FTT,* $A_o \in Form_o$ *a closed formula. Then*

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \Delta A_o \Rightarrow B_o$$

for every formula $B_o \in Form_o$.

Theorem 16.4 *Let* T *be a theory of FTT.*

(a) *If* $T \cup \{A_o\} \vdash C_o$ *and* $T \cup \{B_o\} \vdash C_o$, *then* $T \cup \{A_o \vee B_o\} \vdash C_o$. (*Proof by cases*)
(b) $T \cup \{A_o \Rightarrow B_o\} \vdash C_o$ *and* $T \cup \{B_o \Rightarrow A_o\} \vdash C_o$ *then* $T \vdash C_o$. (*Prelinearity property*)

This theorem provides interesting technical tool for proving various properties of special theories. Let us remark that these properties are specific for many systems of fuzzy logic.

The following theorem is specific for FTT and characterizes equality between elements of all types.

Theorem 16.5 (Equality Theorem)

$$\vdash \Delta(x_\beta \equiv y_\beta) \Rightarrow \big(\Delta(f_{\alpha\beta} \equiv g_{\alpha\beta}) \Rightarrow (f_{\alpha\beta} x_\beta \equiv g_{\alpha\beta} y_\beta)\big). \tag{34}$$

4.3 Canonical Model of FTT

In this subsection, we will describe construction of the canonical (general) model of a consistent theory of FTT. The construction follows the method developed for classical type theory by L. Henkin. The method uses syntactical material for the construction. For each type $\alpha \in Types$, we construct the corresponding set M_α. For elementary types, this construction is straightforward, but for complex types $\beta\alpha$, we have to construct sets $M_{\beta\alpha}$ as sets of weakly extensional functions.

First, we will introduce several important concepts that are analogous to the classical ones.

Definition 16.2 Let T be a theory. We say that:

(i) T is *contradictory* if $T \vdash \bot$. Otherwise, it is *consistent*.
(ii) T is *maximal consistent* if each its extension T', $T' \supset T$, is inconsistent.

(iii) T is *linear*[5] if for every two formulas A_o, B_o,

$$T \vdash A_o \Rightarrow B_o \quad \text{or} \quad T \vdash B_o \Rightarrow A_o.$$

(iv) T is *extensionally complete* if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \nvdash A_{\beta\alpha} \equiv B_{\beta\alpha}$, it follows that there is a closed formula C_α such that $T \nvdash A_{\beta\alpha} C_\alpha \equiv B_{\beta\alpha} C_\alpha$.

The following somewhat tricky theorem makes it possible to prove the generalized completeness of FTT.

Theorem 16.6

(a) *Every consistent theory T can be extended to a maximally consistent linear theory.*
(b) *Every consistent theory T can be extended to an extensionally complete consistent theory $\overline{T}$.*

Construction of the Good EQ-Algebra of Truth Values

First, we construct the set M_o of truth values and its appropriate algebraic structure. Let T be a theory and define equivalence on the set of closed formulas from $Form_o$ by

$$A_o \approx B_o \quad \text{iff} \quad T \vdash A_o \equiv B_o. \tag{35}$$

It can be verified that $\approx$ is the equivalence. The equivalence class of a formula A_o is denoted by $|A_o|$, and we put $M_o = Form_o|\approx$.

Now we define the following operations on the set M_o:

$$|A_o| \wedge_T |B_o| = |A_o \wedge B_o|, \tag{36}$$
$$|A_o| \otimes_T |B_o| = |A_o \& B_o|, \tag{37}$$
$$|A_o| \sim_T |B_o| = |A_o \equiv B_o|, \tag{38}$$
$$\Delta_T\big(|A_o|\big) = |\Delta A_o|, \tag{39}$$
$$|A_o| \vee_T |B_o| = |A_o \vee B_o|, \tag{40}$$
$$1_T = |\top|, \qquad 0_T = |\bot|. \tag{41}$$

Note that we immediately obtain that $|A_o| \rightarrow_T |B_o| = |(A_o \wedge B_o) \equiv A_o| = |(A_o| \wedge_T |B_o|) \sim_T |A_o|$.

By the straightforward syntactical proof we obtain the following theorem.

Theorem 16.7 *Let T be a linear extensionally complete theory. Then the algebra*

$$\mathscr{E}_T = \langle M_o, \wedge_T, \otimes_T, \sim_T, \Delta_T, 1_T, 0_T \rangle \tag{42}$$

is a linearly ordered algebra of truth values that is either a good EQ_Δ *algebra for the Core-FTT, or an* IMTL_Δ*-algebra for IMTL-FTT, or an* MV_Δ*-algebra for Ł-FTT.*

Let us remark that because of linearity of $\mathscr{E}_T$ (this is the consequence of the linearity of T), the operation Δ_T coincides with (23).

---

[5]Such a theory in [16] and also in [27] is called complete.

4.4 Construction of the Canonical Frame

In this section, we will follow the basic ideas used by Henkin in his proof of the generalized completeness theorem for classical type theory (see [18, 19]; cf. also [1]).

Let T be a linear consistent extensionally complete theory. We will extend the equivalence (35) to closed formulas of all types as follows:

$$A_\alpha \sim B_\alpha \quad \text{iff} \quad T \vdash A_\alpha \equiv B_\alpha \tag{43}$$

(in the same way as in (35), we can verify that this is an equivalence). The equivalence class of a formula A_α of type α is denoted by $|A_\alpha|$.

We now need to define all the domains of the canonical frame that, in general, consist of weakly extensional functions. Therefore, we define a special function $\mathscr{V}$ whose domain and range are formulas or their equivalence classes. Then we define the sets of the canonical frame by

$$M_\alpha = \left\{\mathscr{V}(A_\alpha) \,\middle|\, A_\alpha \in \mathit{Form}_\alpha\right\}, \quad \alpha \in \mathit{Types}. \tag{44}$$

The construction proceeds inductively:

(CI1) If $\alpha = o$, then $\mathscr{V}(A_o) = |A_o|$, that is, $M_o = \mathit{Form}_o|\approx$. Furthermore, we put $\stackrel{\circ}{=}_o := \sim_T$, where $\sim_T$ is defined in (38).
(CI2) If $\alpha = \varepsilon$, then $\mathscr{V}(A_\varepsilon) = |A_\varepsilon|$, thats is, $M_\varepsilon = \mathit{Form}_\varepsilon|\approx$.
(CI3) If $\alpha = \gamma\beta$, then we put $\mathscr{V}(A_{\gamma\beta}) \subseteq M_\beta \times M_\gamma$, which is the relation consisting of couples

$$\left\langle \mathscr{V}(B_\beta), \mathscr{V}(A_{\gamma\beta} B_\beta)\right\rangle$$

for all closed $B_\beta \in \mathit{Form}_\beta$ and $A_{\gamma\beta} \in \mathit{Form}_{\gamma\beta}$.

The fuzzy equality in each set M_α, $\alpha \neq o$, is defined by

$$\stackrel{\circ}{=}_\alpha \left(\mathscr{V}(A_\alpha), \mathscr{V}(B_\alpha)\right) := |A_\alpha \equiv B_\alpha|. \tag{45}$$

It follows from (36)–(41) and this description that the operations from $\mathscr{E}_T$ are included in $M_{oo} \cup M_{o(oo)}$.

For complex types, we must also prove that definition (45) conforms with (29). This can be done on the basis of Axioms (Fund2) and (Fund2) and also using the fact that the theory T is extensionally complete. The latter also enables to prove that all the constructed functions are weakly extensional.

After constructing the canonical frame, we must define interpretation of formulas in it. Let p be an assignment of elements to variables, that is, $p(x_\alpha) = \mathscr{V}(A_\alpha) \in M_\alpha$ for all $\alpha \in \mathit{Types}$. Then we put:

(CI1) If x_α is a variable, then $\mathscr{M}_p^T(x_\alpha) = p(x_\alpha)$.
(CI2) If c_α, $\alpha \neq o$ is a constant, then $\mathscr{M}_p^T(c_\alpha) = \mathscr{V}(c_\alpha) \in M_\alpha$. Furthermore,

(a) $\mathscr{M}_p^T(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality (45), $\alpha \in \mathit{Types}$.
(b) Interpretation of the conjunction $\mathbf{C}_{(oo)o}$, strong conjunction $\mathbf{S}_{(oo)o}$, and $\mathbf{D}_{(oo)o}$ are the operations (36), (37), and (39), respectively.

(CI3) Interpretation of the formula $\lambda x_\alpha\, A_\beta$ of type $\beta\alpha$ is the function

$$\mathscr{M}_p^T(\lambda x_\alpha\, A_\beta) : \mathscr{V}(B_\alpha) \mapsto \mathscr{V}\big((\lambda x_\alpha\, A_\beta) B_\alpha\big) \tag{46}$$

for each assignment $p' = p \backslash x_\alpha$, where $p'(x_\alpha) = \mathscr{V}(B_\alpha)$.

(CI4) Interpretation of the description operator is the function

$$\mathscr{M}_p^T(\iota_{\alpha(o\alpha)}) : \mathscr{V}(A_{o\alpha}) \mapsto \mathscr{V}(\iota_{\alpha(o\alpha)} A_{o\alpha}), \quad \alpha = o, \varepsilon. \tag{47}$$

The construction of the canonical frame enables us to prove the generalized completeness theorem for fuzzy type theory.

Theorem 16.8 ([27, 29])

(a) *A theory T is consistent iff it has a general model $\mathscr{M}$.*
(b) *For every theory T and a formula A_o,*

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

5 Discussion

The main goal of this chapter is to demonstrate that the ideas of L. Henkin enabled also to develop a many-valued (fuzzy) type theory with complete syntax with respect to generalized (fuzzy) models. Similarly as in the case of propositional and first-order fuzzy logic, there are more kinds of FTT depending on the chosen algebra of truth values. In this chapter, we presented three kinds of FTT, namely Core-FTT based on EQ_Δ-algebras, IMTL-FTT based on IMTL_Δ-algebras, and Ł-FTT based on MV-algebras. Of course, there are more convenient algebras of truth values, and so, more kinds of fuzzy type theory can be introduced. Though not yet proved, we argue that each first-order fuzzy logic can be extended to a higher-order one, that is, to the corresponding FTT.

Let us remark that FTT generalizes the classical type theory and is obtained by extension of its list of axioms. Thus, for example, both IMTL-FTT and Ł-FTT reduce[6] to classical simple type theory by adding the axiom

$$x_o \vee \neg x_o.$$

This axiom introduces the law of excluded middle with respect to the ordinary disjunction (interpreted by the lattice join operation). Note that the law of excluded middle holds in fuzzy logic with respect to the strong disjunction ∇, which is, in general, different from $\vee$. Of course, in classical logic, both disjunctions coincide.

The reader may appreciate depth and originality of Henkin's ideas. Though we needed some special tricks in the construction of the canonical model, we demonstrated that the basic Henkin construction can be applied in FTT, too.

[6]To be precise, the resulting type theory is isomorphic with the classical simple theory of types.

Let us spare a thought about usefulness of FTT and various kinds of it. We argue that FTT is a powerful formal tool using which we can solve various problems raised in modeling of semantics of natural language and concepts (cf. [9]) caused by the presence of vagueness. A typical manifestation of the latter is the sorites (heap) paradox. Using means of Ł-FTT, we gave in [28] a simple and elegant solution of this paradox and demonstrated that it is by no means paradoxical.

Another interesting problem is modeling of noncommutativity of conjunction. This occurs quite often in natural language and commonsense reasoning. Therefore, if we want to model the latter, then the formal logic must also capture noncommutativity. Note that fuzzy logic including FTT has two different conjunctions. The first one is the ordinary conjunction $\wedge$ interpreted by the lattice meet $\wedge$. This is always commutative. The second one is the strong conjunction & interpreted by the multiplication operation $\otimes$. This conjunction should be used for safe connection of two conjuncts with various truth values if we do not know whether the meanings of both conjuncts are in some sense related or not. In fuzzy logic based on residuated lattices, however, noncommutativity of the strong conjunction enforces two implications (see [17]). This looks unnatural since we have no interpretation for the second implication. In Core-FTT, on the other hand, the strong conjunction is noncommutative, and still only one implication is sufficient.

Fuzzy type theory, namely the Ł-FTT, is used also as a metatheory for development of the so-called *fuzzy natural logic* (FNL). This is a formal theory whose paradigm is to develop a working mathematical model of the *semantics* of special natural language expressions including their *vagueness* and then to develop a mathematical model of special natural (commonsense) *human reasoning schemes* that are language independent. The first steps in FNL have already been done (cf. [30, 32]).

An open problem is modification of the syntax of FTT so that the general model can also contain partial functions. In classical TT, this was solved Farmer [12]. Note that this direction of research is motivated especially by the desire to model the meaning of concepts (see [23, 24]). One of additional tasks is also the study of the complexity of sets of tautologies of all FTTs. Let us also remark that our theory suggests interesting philosophical questions concerning the role of (fuzzy) equality/equivalence in (fuzzy) logic with respect to implication.

References

1. Andrews, P.: An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof. Kluwer Academic, Dordrecht (2002)
2. Chang, C.C.: Algebraic analysis of many valued logics. Trans. AMS **93**, 74–80 (1958)
3. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**, 56–68 (1940)
4. Cignoli, R.L.O., D'Ottaviano, I.M.L., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning. Kluwer Academic, Dordrecht (2000)
5. Cintula, P., Esteva, F., Gispert, J., Godo, L., Noguera, C.: Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies. Ann. Pure Appl. Log. **160**, 53–81 (2009)
6. Cintula, P., Hájek, P., Horčík, R.: Formal systems of fuzzy logic and their fragments. Ann. Pure Appl. Log. **150**, 40–65 (2007)
7. Cintula, P., Hájek, P., Noguera, C. (eds.): Handbook of Mathematical Fuzzy Logic, vol. 1. College Publications, London (2011)

8. Cintula, P., Hájek, P., Noguera, C. (eds.): Handbook of Mathematical Fuzzy Logic, vol. 2. College Publications, London (2011)
9. Duží, D., Jespersen, B., Materna, P.: Procedural Semantics for Hyperintensional Logic. Springer, Dordrecht (2010)
10. Dyba, M., Novák, V.: EQ-logics: non-commutative fuzzy logics based on fuzzy equality. Fuzzy Sets Syst. **172**, 13–32 (2011)
11. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets Syst. **124**, 271–288 (2001)
12. Farmer, W.M.: A partial functions version of Church's simple theory of types. J. Symb. Log. **55**, 1269–1291 (1990)
13. Gallin, D. (ed.): Intensional and Higher-Order Modal Logic (with Applications to Montague Semantics). North-Holland, Amsterdam (1975)
14. Goguen, J.A.: The logic of inexact concepts. Synthese **19**, 325–373 (1968–1969)
15. Gottwald, S.: A Treatise on Many-Valued Logics. Research Studies Press Ltd., Baldock, Herfordshire (2001)
16. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer Academic, Dordrecht (1998)
17. Hájek, P.: Fuzzy logics with noncommutative conjunctions. J. Log. Comput. **13**, 469–479 (2003)
18. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**, 81–91 (1950)
19. Henkin, L.: A theory of propositional types. Fundam. Math. **52**, 323–344 (1963)
20. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer Academic, Dordrecht (2000)
21. Łukasiewicz, L.: O logice trojwartoscowej. (On three-valued logic). Ruch Filoz. **5**, 170–171 (1920) (in Polish)
22. Łukasiewicz, L., Borkowski, L.: Selected Works, Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam (1970)
23. Materna, P.: Concepts and Objects. Acta Philosophica Fennica, vol. 63, Helsinki (1998)
24. Materna, P.: Conceptual Systems. Logos Verlag, Berlin (2004)
25. Novák, V.: Fuzzy Sets and Their Applications. Adam Hilger, Bristol (1989)
26. Novák, V.: Towards fuzzy type theory. In: Proc. Int. Seminar on Multiple-Valued Logic ISMVL 2003, pp. 65–70. Meiji Univ. Tokyo, Tokyo (2003)
27. Novák, V.: On fuzzy type theory. Fuzzy Sets Syst. **149**, 235–273 (2005)
28. Novák, V.: A comprehensive theory of trichotomous evaluative linguistic expressions. Fuzzy Sets Syst. **159**(22), 2939–2969 (2008)
29. Novák, V.: EQ-algebra-based fuzzy type theory and its extensions. Log. J. IGPL **19**, 512–542 (2011)
30. Novák, V.: Reasoning about mathematical fuzzy logic and its future. Fuzzy Sets Syst. **192**, 25–44 (2012)
31. Novák, V., de Baets, B.: EQ-algebras. Fuzzy Sets Syst. **160**, 2956–2978 (2009)
32. Novák, V., Pavliska, V., Perfilieva, I., Štěpnička, M.: F-transform and fuzzy natural logic in time series analysis. In: Proc. Int. Conference EUSFLAT-LFA'2013, Milano, Italy (2013)
33. Novák, V., Perfilieva, I. (eds.): Discovering the World with Fuzzy Logic. Studies in Fuzziness and Soft Computing, vol. 57. Springer, Heidelberg (2000)
34. Novák, V., Perfilieva, I., Močkoř, J.: Mathematical Principles of Fuzzy Logic. Kluwer Academic, Boston (1999)
35. Pavelka, J.: On fuzzy logic I, II, III. Z. Math. Log. Grundl. Math. **25**, 45–52 (1979). Also see pp. 119–134, 447–464
36. Russell, B.: Mathematical logic as based on the theory of types. Am. J. Math. **30**, 222–262 (1908)
37. Tichý, P.: The Foundations of Frege's Logic. De Gruyter, Berlin (1988)
38. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)

V. Novák (✉)
Institute for Research and Applications of Fuzzy Modelling, National Supercomputing Centre IT4 Innovations, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic
e-mail: Vilem.Novak@osu.cz

The Henkin Sentence

Volker Halbach and Albert Visser

Abstract In this paper we discuss Henkin's question concerning a formula that has been described as expressing its own provability. We analyze Henkin's formulation of the question and the early responses by Kreisel and Löb and sketch how this discussion led to the development of provability logic. We argue that, in addition to that, the question has philosophical aspects that are still interesting.

Keywords Self-reference · Fixed Points · Second Incompleteness Theorem · Provability Logic

1 Henkin's Question

In the problem section of the Journal of Symbolic Logic, Vol. 17, No. 2 (June 1952), on p. 160, Leon Henkin posed the following problem

> *A problem concerning provability*. If Σ is any standard formal system adequate for recursive number theory, a formula (having a certain integer q as its Gödel number) can be constructed which expresses the proposition that the formula with Gödel number q is provable in Σ. Is this formula provable or independent in Σ? (Received February 28, 1952.)

There seems to be a certain *naiveté* about the question. There are many formal systems adequate for recursive number theory. It is not just that we can vary the axioms, but we can vary the proof system; we can even vary the setup of the syntax. Moreover, the construction of Henkin's sentence involves arithmetization, which can be implemented in any number of ways. We can choose different Gödel numberings, but even with a given Gödel numbering, we can encode the syntax in different ways as linear strings of symbols, trees, or still something else. Once all these decisions have been made, one needs to single out a formula for expressing provability. Once such a formula is fixed, one can think of many constructions for obtaining a sentence with code q that says that the sentence with Gödel number q is provable. Different choices at any stage will produce a different sentence. Why should we expect that all these formulae share the same status with respect to provability or independence?

Of course, one could insist that we simply fix one specific set of choices for all these "parameters," a set of choices that we intuitively recognize as being correct, straightforward or "natural." But nobody seriously thinks that there is just one admissible way of arithmetizing syntax, of picking a provability predicate, and so on. When contemplating arithmetization, we always understand that a chosen implementation is just one of the many ways, but that our results should be robust with respect to particular choices as long

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_17

as they are "reasonable" or "natural" because we do not make use of "accidental" features of our choices. This is by no means trivial. But in the case of Henkin's problem and similar problems, it was not to be expected in 1952 that a solution would be robust.[1]

Henkin's way of posing the question differs significantly from those found in the more recent literature. In the modern literature, Henkin's question or Henkin's problem is usually described as the question *whether or not the sentence expressing its own provability is provable* [12, p. 148] or even as the question *whether the sentence expressing its own provability* [...] *is true or false, and provable or not* [37]. It is far from clear whether these abbreviated forms are adequate renderings of Henkin's original question or whether they capture Henkin's intention. Today *Henkin's Problem* is used more like a proper name for a family of logical questions and less as a description of the question asked by Henkin in 1952. Here we would like to take a closer look at what Henkin's did ask—and what he did not ask.

In contrast to many modern accounts, Henkin did not make use of the notion of self-reference in the formulation of his question. He did not describe the sentence as one that *says of itself that it is provable* or the like. The usual catchphrases like *self-reference* are strangely absent from Henkin's question and also from the immediate replies and discussions, even though Gödel had already described his own sentence as a sentence stating its own unprovability [19, p. 175].

In the proof of Gödel's First Incompleteness Theorem and many other results, the notion of self-reference is not needed; The Gödel sentence γ only needs to be a fixed point of nonprovability, that is, it must satisfy $\Sigma \vdash \gamma \leftrightarrow \neg\mathsf{Bew}(\ulcorner\gamma\urcorner)$; whether γ says something about itself and what it says is irrelevant for the proof. But Henkin also did not ask whether a fixed point of the provability predicate, that is, a sentence η with $\Sigma \vdash \eta \leftrightarrow \mathsf{Bew}(\ulcorner\eta\urcorner)$ is provable or not. Any provable formula such as $0 = 0$ clearly is a fixed point of any formula that may be called a provability predicate; and these trivial fixed points are clearly not what Henkin was after.

Henkin also did not ask whether the sentence obtained by applying a certain canonical diagonal construction to the provability predicate is provable or not. As we shall see in our discussion of Kreisel's answer, this is not equivalent to Henkin's requirement that the formula with Gödel number q should "express[es] the proposition that the formula with Gödel number q is provable," as this may be achievable without applying the standard Gödel diagonal construction to a given provability predicate.

Moreover, Henkin did not ask whether his sentence is *refutable*. He probably noted that if Σ refutes this sentence, then Σ ipso facto proves its nonprovability. This, in its turn, implies that Σ proves the consistency of Σ, thus contradicting the Second Incompleteness Theorem—if we assume that Σ is consistent. To make such reasoning valid, the provability predicate involved must have some of the properties usually ascribed to it.

Henkin employed intensional language in the question: the formula in question is supposed to *express* the provability of a formula with a certain Gödel number. Is this use

[1] One may compare this with the truth-teller sentence that states its own Σ_1-truth. The answer to the question whether this sentence is provable, refutable, or independent depends on assumptions on the coding, the diagonalization method, and so on [27]. So Henkin's question for Σ_1-truth instead of provability only admits an answer that is far less robust than Löb's answer to Henkin's original question, which is extremely robust. Among the "Henkin-like" problems, the robustness of the answer to Henkin's original problem may be more the exception than the rule.

of intensionality to be viewed as merely a *façon de parler* or does it carry some serious weight? As we will see, Henkin's review of Kreisel's paper contains some evidence that Henkin did indeed take that business of *expressing something* seriously.

One should view Henkin's question as including the challenge to give a definite mathematical extensional meaning to Henkin's intensional description of his sentence. Kreisel and others took up the challenge by breaking it into two problems, once a formal system Σ and a coding are fixed:[2] First one needs to provide conditions that must be satisfied by a formula to express provability. Then, in the second step, from that formula a sentence with code q that ascribes to q the property of being provable must be constructed.

As we will see, both Kreisel and Löb developed criteria, albeit entirely different ones, that must be satisfied by a formula to qualify as a provability predicate. Kreisel argued that the answer to Henkin's question depends on which provability is used and that different provability predicates can be employed to obtain provable and even refutable Henkin sentences. Hence, the burden on his conditions for expressing provability was heavier. The provability predicates used in his examples need to be recognized as correct arithmetizations of provability. As we shall see, neither Henkin nor Löb agreed that his examples were good examples of arithmetizations of provability.

Löb's answer was positive: Henkin's sentence is provable; and his answer was definitive: all sentences of this kind are provable. Consequently, Löb needed only necessary conditions on formulae for expressing provability. The set of formulae satisfying his conditions only needs to *include* all predicates that we would recognize as good arithmetizations of provability. It is perfectly all right if the conditions admit cases that we would not recognize as good arithmetizations (as in fact they do). The same applies to the second step. Löb's result holds for all diagonal sentences, that is, sentences η satisfying $\Sigma \vdash \eta \leftrightarrow \mathsf{Bew}(\ulcorner \eta \urcorner)$ for the chosen provability predicate Bew; and all formulae with code q that express that the formula with q via the predicate Bew will be diagonal sentences. That there are also other diagonal sentences merely shows that Löb's result is stronger than needed to answer Henkin's question.

Henkin published his question in 1952. This is 21 years after the appearance of Gödel's paper *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I* [19] and 13 years after the first publication of part II of *Grundlagen der Mathematik* by Hilbert and Bernays [23], which contained a careful analysis of the proof of the Second Incompleteness Theorem. It seems to us that all ingredients for Henkin's question were already present in Gödel's 1931 paper. Even if you want to insist that a careful analysis of the proof of Gödel's second theorem is needed as the background for the question, then these were certainly available in the 1939 book by Hilbert and Bernays. So, why was the question not asked earlier? A first point is that one should never underestimate the size of the search space even for elementary questions. After all, the other question (received 19 March 1952) asked by Henkin in the same list of questions is whether the Ordering Theorem is equivalent to the Axiom of Choice for classes of finite sets. This other question also only involves concepts that were around for quite some time. Secondly, the former question clearly was off the beaten track. It was not part of existing research lines. It is an almost whimsical question posed in a playful mood. The question gives the reader a feeling of contingency. It could as well never have been posed. Seeing all the exciting developments that followed it, we may be glad that it actually was asked.

[2] In what follows, we somewhat neglect the problems involving the choice of the formal system Σ and a coding of syntax. See [27] for some additional remarks.

2 Kreisel's Solution

In his 1953 paper [31], Georg Kreisel summarized his reply to Henkin as follows:

> We shall show below that the answer to Henkin's question depends on which formula is used to 'express' the notion of *provability in Σ*.

Thus, Kreisel's reply precisely concerns the point where we said that Henkin's question had a certain *naiveté*. Note the scare quotes around *express*, which suggest that Kreisel thought that the business of expressing should be taken *cum grano salis*. Kreisel gives the following condition for *expressing provability*:

> A formula $\mathfrak{P}(a)$ is said to express provability in Σ if it satisfies the following condition: for numerals a, $\mathfrak{P}(a)$ can be proved in Σ if and only if the formula with number a can be proved in Σ.

We can generalize Kreisel's condition for provability to a more general condition for expressing a property.

Kreisel's Condition A formula $\varphi(x)$ is said to express a property P in Σ if and only if, for all numbers n, we have $\Sigma \vdash \varphi(\underline{n})$ iff n has property P.

In metamathematics, Kreisel's Condition became the formal notion of weak representability:[3] A formula $\varphi(x)$ is said to *weakly represent* a set S of numbers if and only if $\Sigma \vdash \varphi(\underline{n})$ iff $n \in S$.

The main problem with Kreisel's Condition is that it is counterintuitive on two scores. First, according to it, for example, over Peano Arithmetic (PA), many predicates express provability that intuitively do not. Examples of such predicates are Rosser provability, Feferman provability, and the two notions of Kreisel–Henkin provability discussed below. Secondly, consider some predicate Bew that we recognize as a good arithmetization of provability in, say, $\Theta := \mathsf{PA} + \mathsf{incon}(\mathsf{PA})$. Then, by Kreisel's condition, Bew does *not* express provability-in-Θ. However, by the Friedman–Goldfarb–Harrington Theorem, we may manufacture, by Rosser trickery, a predicate that *does* express provability in Θ by the light of the Kreisel Condition.[4] With some more effort, we may even build such a predicate satisfying the Löb Conditions too.

We note that for Kreisel's *it depends* answer, it is needed that all predicates admitted by his Condition are indeed recognized as expressing provability. It does no harm when some predicates not recognized by the condition do express provability. Thus, Kreisel's condition must be a sufficient condition. Conversely, for Löb, with his positive answer, one wants that whatever is recognized as expressing provability should be admitted by Löb's condition. It does no harm when certain predicates admitted by Löb's condition are recognized as not representing provability. For instance, over PA, the predicate $x = x$ satisfies the Löb conditions. Nobody would think that this predicate expresses provability. But, also, nobody would think this is a problem for Löb's answer to Henkin's question. Thus, Löb's condition needs to be necessary.

---

[3] Feferman [17] introduced and used the term "numerate" for "weakly represent."

[4] See, for instance, [51] for a discussion.

Kreisel constructed two sentences that are both supposed to satisfy Henkin's Condition; one of them is provable, the other refutable. Let Basic be the Tarski–Mostowski–Robinson theory R extended by the recursion equations for all primitive recursive functions.

Kreisel's Observation Let Σ be a consistent theory that extends Basic.[5] Then the following hold:

a. There is a formula $\mathsf{Bew}_I(x)$ and a term t_1 such that the following three conditions are satisfied:

   i. Bew_I weakly represents provability in Σ.
   ii. $\Sigma \vdash t_1 = \ulcorner \mathsf{Bew}_I(t_1) \urcorner$.
   iii. $\Sigma \vdash \mathsf{Bew}_I(t_1)$.

b. Similarly, there is a provability predicate $\mathsf{Bew}_{II}(x)$ and a term t_2 such that

   i. Bew_{II} weakly represents provability in Σ.
   ii. $\Sigma \vdash t_2 = \ulcorner \mathsf{Bew}_{II}(t_2) \urcorner$.
   iii. $\Sigma \vdash \neg \mathsf{Bew}_{II}(t_2)$.

The examples employed by Kreisel in the proof are of some interest. In particular, the example for $\mathsf{Bew}_I(t_1)$ foreshadows Kreisel's [32] proof of Löb's theorem, as was pointed out by [44]. Henkin suggested simpler examples that are mentioned by [31] in footnotes. We will use Henkin's examples and refer the reader to Smoryński's paper for an exposition of Kreisel's original examples.

Proof We start with a proof for the second part (b). Fix some predicate $\mathsf{Bew}(x)$ that weakly represents Σ-provability in Σ. In case Σ is Σ_1-sound, a standard arithmetization of provability will do. In the unsound case, one uses the theorem that any recursively enumerable set is weakly representable in a consistent recursively enumerable extension of the Tarski–Mostowski–Robinson theory R. This is a direct consequence of the Friedman–Goldfarb–Harrington Theorem.[6] Using the canonical diagonal construction (or any other method), one obtains a term t_2 satisfying the condition

$$\Sigma \vdash t_2 = \ulcorner t_2 \neq t_2 \wedge \mathsf{Bew}(t_2) \urcorner \tag{1}$$

and defines $\mathsf{Bew}_{II}(x)$ as

$$x \neq t_2 \wedge \mathsf{Bew}(x).$$

Condition b(ii), that is, $\Sigma \vdash t_2 = \ulcorner \mathsf{Bew}_{II}(t_2) \urcorner$, is then obviously satisfied by the choice (1) of t_2. Since Σ refutes $t_2 \neq t_2 \wedge \mathsf{Bew}(t_2)$, item b(iii) is satisfied as well.

It remains to verify b(i), which is the claim that $\mathsf{Bew}_{II}(x)$ weakly represents Σ-provability. In other words, we must establish the following equivalence for all formulae φ:

$$\Sigma \vdash \varphi \quad \text{iff} \quad \Sigma \vdash \mathsf{Bew}_{II}\big(\ulcorner \varphi \urcorner\big). \tag{2}$$

[5] Kreisel asked that the theory be Σ_1-sound, but that demand is superfluous.

[6] See, for instance, [51] for a discussion.

If φ is different from $t_2 \neq t_2 \wedge \mathsf{Bew}(t_2)$, then this is obvious from the definition of $\mathsf{Bew}_{II}(x)$, using the fact that Bew weakly represents provability in Σ. In the other case, the left-hand side of the equivalence is refutable, and so is the right-hand side by (1). This concludes the proof of part (b) of Kreisel's Observation.

We turn to case (a). If we assume that our theory is Σ_1-sound and sufficiently strong (e.g., if it extends the arithmetical version of Buss' theory S^1_2), then the canonical provability predicate can be used as $\mathsf{Bew}_I(x)$, and t_1 can be obtained in any way, including the usual Gödel diagonal construction. Claim a(iii) follows then by Löb's theorem. (See [35] or, e.g., [12].)

Since Löb's Theorem was not known, Henkin and Kreisel had to use a different construction.[7] Henkin suggested the following construction. He picked a term t_1 such that

$$\Sigma \vdash t_1 = \ulcorner t_1 = t_1 \vee \mathsf{Bew}(t_1) \urcorner$$

and defines $\mathsf{Bew}_I(x)$ as

$$x = t_1 \vee \mathsf{Bew}(x). \qquad \square$$

Clearly, the provability predicates Bew_I and Bew_{II} are somewhat peculiar. Although they satisfy Kreisel's Condition, hardly anyone considers them to be proper provability predicates. As we shall see soon, Henkin was the first to reject them and claim that the sentences $\mathsf{Bew}_I(t_1)$ and $\mathsf{Bew}_{II}(t_2)$ do not fit the description in his question.

However, the alleged Henkin sentences $\mathsf{Bew}_I(t_1)$ and $\mathsf{Bew}_{II}(t_2)$ exhibit another peculiarity that is neither discussed by Kreisel nor by Henkin:[8] They are not obtained by applying the usual diagonal construction to the respective provability predicates Bew_I and Bew_{II}. Rather Kreisel finessed the predicates Bew_{II} in such a way that simply substituting the term t_2 for the free variable in Bew_{II} produces a formula with Gödel number q such that the value of t_2 is q. So one can reasonably claim that $\mathsf{Bew}_{II}(t_2)$ is "a formula (having a certain integer q as its Gödel number) [...] which expresses the proposition that the formula with Gödel number q is provable in Σ" [24] if Bew_{II} is taken to express provability. Similar remarks apply to $\mathsf{Bew}_I(t_1)$ of course. So, with the possible exception of the choice of the provability predicates, Kreisel provided a correct answer to Henkin's question.

However, one may wonder whether Kreisel answered the questions that are currently called *Henkin's Problem*. In other words, is $\mathsf{Bew}_{II}(t_2)$ self-referential and does it state its own provability? In particular, does $\mathsf{Bew}_{II}(t_2)$ ascribe to itself the property expressed by $\mathsf{Bew}_{II}(x)$—whether it is a good provability predicate or not? Usually, when one considers a sentence that says about itself that it has the property expressed by a formula $\psi(x)$, one often intends to talk about the sentence that is obtained from $\psi(x)$ by the usual diagonal construction or a variant thereof. What exactly the usual diagonal construction and its variant are may be unclear, but $\mathsf{Bew}_{II}(t_2)$ has not been obtained by anything that resembles such a method.

This sheds a light on the usual reformulation of Henkin's problem: It is often stated as a problem about a formula that states its own provability or that says about itself that it

[7] Note also that the Kreisel–Henkin construction works in some very weak cases where it is not clear that we have Löb's theorem.

[8] It was first noted by Craig Smoryński in [42].

is provable. Of course, one may speculate that Henkin intended to ask his question about this formula and Kreisel tried to address the question understood in this sense. It also seems that later authors understood Henkin's question as being about sentences that state their own provability. But the equivalence to the original formulation is not obvious.

At any rate, if the usual diagonal construction involving the substitution function is applied to Bew_{II}, one obtains a *provable* sentence. This follows easily from Löb's theorem, which of course was not known at the time Kreisel published his paper. If that sentence is seen as the only sentence saying about itself that it has the property expressed by Bew_{II}, then Kreisel fails to provide a counterexample to the claim that the sentence stating its own provability is provable—irrespective of whether Bew_{II} is a provability predicate or not. So Kreisel was somewhat imprecise in summarizing his result: He had shown 'that the answer to Henkin's question depends on which formula is used to "express" the notion of *provability in* Σ'—but also on how the formula with code q is obtained that ascribes provability to q via this provability predicate.

However, after all, it can be shown that, if Kreisel's Condition is adopted, it *only* depends which provability predicate is chosen whether the Henkin sentence is provable or not. We can even use the standard diagonal construction to obtain a refutable Henkin sentence from the given provability predicate. For in [27] it has been shown that there still another provability that yields a refutable Henkin sentence if the standard diagonal construction is applied to it. Such a provability predicate can be obtained by tinkering with the Kreisel–Henkin construction.

3 Henkin's Review

In 1954 Leon Henkin responds to Kreisel's paper in a review [25] in the Journal of Symbolic Logic. Henkin's main critical point is the following.

> A clear explication of the concept of *that which is expressed by a formula* must be based on an axiomatic treatment of this notion (perhaps along the lines of Church XVII 133). However, it seems fair to say that in one sense, at least, neither formula $P_1(a)$ nor $P_2(a)$ expresses the propositional function *a is provable*; but the former, for example, expresses the proposition *a is provable or is equal to q*, which is a different proposition even though it has the same extension. The direct way to express *a is provable* is, of course, by the formula $(\exists x)B(x, a)$. But the methods of the present paper give no indication as to whether the formula $(\exists x)B(x, q)$ whose Gödel number is denoted by q is provable.

The reference Church XVII 133 is to a review in JSL by Rulon Wells [53] of Church's paper *A formulation of the logic of sense and denotation* [15]. Regrettably, the desired axiomatic explication of *that which is expressed by a formula* never materialized. The remark about *expressing* underscores the fact that Henkin took the philosophical problem of intensionality quite seriously—no scare quotes for him. The subsequent remarks about P_1 and P_2 show that Henkin rejected Kreisel's Condition as a sufficient condition for *expressing provability*.

Finally, Henkin insisted that Kreisel did not solve the problem for the intended predicate $(\exists x)B(x, q)$, where $B(x, y)$ is, as Henkin put it in [25], the "standard formula such that $B(m, n)$ or its negation is provable according as m does or does not denote the number of a formal proof of a formula whose Gödel number is denoted by n." It is not completely clear what the standard formula is, given that Henkin did not fix a formal system Σ;

but for systems like PA, the standard formulas can be thought of as those found in the literature.

Henkin's insistence on a less contrived provability predicate is at least consistent with the fact that he asked in his original 1952 question whether his sentence is provable or independent. As remarked above, he may have reasoned that the refutability of the Henkin sentence would imply consistency contradicting Gödel's Second Incompleteness Theorem. But this applies only if the consistency statement and the Henkin sentence are formulated with a well-behaved provability predicate. The Second Incompleteness Theorem fails for the Rosser provability predicate, for instance. For Kreisel's predicate Bew_{II}, the second incompleteness theorem holds; after all, it agrees with the standard one on all sentences except for $\neg\mathsf{Bew}_{II}(t_2)$. Kreisel's provability predicate Bew_{II}, however, does not satisfy Löb's second derivability condition LC2 below. Of course, Henkin had published his question and the review of Kreisel's reply before Löb's derivability conditions were formulated, so Henkin could resort only to intensional properties of the provability predicates and what they do or do not "express."

At any rate concentration on the "standard" provability predicate led to the breakthrough and the commonly accepted answer to Henkin's question.

4 Löb's Paper

We are again one year later. In his celebrated JSL paper [35], Martin Löb starts by echoing Henkin's review:

> One approach to this problem is discussed by Kreisel in [4]. However, he still leaves open the question whether the formula $(\exists x)\mathfrak{B}(x, a)$, with Gödel number a, is provable or not. Here $\mathfrak{B}(x, y)$ is the number-theoretic predicate which expresses the proposition that x is the number of a formal proof of the formula with Gödel-number y.

So we see that Löb adhered to Henkin's intensional phrasing of the question.

Let us write $\vdash$ for $\Sigma \vdash$ and $\Box\varphi$ for $\mathsf{Bew}(\underline{\ulcorner\varphi\urcorner})$. Then, we can state Löb's conditions like this:[9]

LC1 $\vdash \varphi \;\Rightarrow\; \vdash \Box\varphi$.
LC2 $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
LC3 $\vdash \Box\varphi \to \Box\Box\varphi$.

Löb derives what is known as Löb's Rule from his conditions. We have: if $\vdash \Box\varphi \to \varphi$, then $\vdash \varphi$.

The reasoning is as follows. Suppose (a) $\vdash \Box\varphi \to \varphi$. By Gödel's Fixed Point Lemma, we can find a sentence λ such that (b) $\vdash \lambda \leftrightarrow (\Box\lambda \to \varphi)$. Now reason in Σ. Suppose (c) $\Box\lambda$. Then, by LC3, (d) $\Box\Box\lambda$. By (b), (c), LC1, and LC2, we find: (e) $\Box(\Box\lambda \to \varphi)$. Combining (d) and (e) using LC2, we may conclude $\Box\varphi$, and, hence, by (a): φ. Thus, by canceling assumption (c), we have found (f) $\Box\lambda \to \varphi$. By (b) and (f), we have (g) λ. We have derived λ without assumptions, hence, by LC1, (h) $\Box\lambda$. Combining this with (f), we may conclude φ.

[9] Actually, Löb mentions more conditions in his paper. However, upon analysis, we only need the ones given here.

Löb's solution of Henkin's Problem now follows immediately. Suppose $\vdash \eta \leftrightarrow \Box\eta$. Then, a fortiori, $\vdash \Box\eta \to \eta$, and, hence, by Löb's Rule, $\vdash \eta$.

In footnote 2, Löb states:

> In a previous version of this note the method of proof was applied specifically to Henkin's problem. The present more general formulation of our result was suggested by the referee.

Albert Visser asked George Kreisel who he thought was the referee of Löb's paper. Kreisel answered that, of course, this must have been Henkin. Later Albert Visser asked Henkin whether he was the referee, and Henkin confirmed that this was indeed the case.

Löb's Principle is the formalized form of Löb's Rule: $\vdash \Box(\Box\varphi \to \varphi) \to \Box\varphi$. We can derive Löb's Principle by formalizing the reasoning leading to Löb's Rule. However, we can also derive Löb's Principle from Löb's Rule. Reason in Σ. We suppose that (i) $\Box(\Box(\Box\varphi \to \varphi) \to \Box\varphi)$ and (ii) $\Box(\Box\varphi \to \varphi)$. From (ii) and $\mathsf{LC3}$, we have (iii) $\Box\Box(\Box\varphi \to \varphi)$. Combining (i) and (iii) using $\mathsf{LC2}$, we find (iv) $\Box\Box\varphi$. From (ii) and (iv), using $\mathsf{LC2}$, we get: (v) $\Box\varphi$. By canceling assumptions (ii) and (i), we find:

$$\text{(vi)} \quad \Box\big(\Box(\Box\varphi \to \varphi) \to \Box\varphi\big) \to \big(\Box(\Box\varphi \to \varphi) \to \Box\varphi\big).$$

Then, using Löb's Rule, we find: $\Box(\Box\varphi \to \varphi) \to \Box\varphi$, as desired.

Conversely, both Löb's Rule and $\mathsf{LC3}$ follow from Löb's Principle using only $\mathsf{LC1}$ and $\mathsf{LC2}$.

To derive Löb's Rule from Löb's Principle, suppose that (A) $\vdash \Box\varphi \to \varphi$. Then, by $\mathsf{LC1}$, we find (B) $\vdash \Box(\Box\varphi \to \varphi)$. Löb's Principle then gives us (C) $\vdash \Box\varphi$. By (A) and (C), we have $\vdash \varphi$.

The derivation of $\mathsf{LC3}$ from Löb's Principle is due to Dick de Jongh. It works as follows. Reason in Σ. Suppose (α) $\Box\varphi$. Then, we find (β) $\Box(\Box(\varphi \wedge \Box\varphi) \to (\varphi \wedge \Box\varphi))$, by $\mathsf{LC1}$ and $\mathsf{LC2}$. Hence, by Löb's Principle, (γ) $\Box(\varphi \wedge \Box\varphi)$. We may conclude by $\mathsf{LC1}$ and $\mathsf{LC2}$ that $\Box\Box\varphi$.

What do Löb's answer to Henkin's question and Gödel's Second Incompleteness Theorem have in common? Löb showed that any fixed point of $\mathsf{Bew}(x)$ is provably equivalent to $0 = 0$. Gödel showed that any fixed point of $\neg\,\mathsf{Bew}(x)$ is provably equivalent to $\mathsf{con}(\Sigma)$. Thus, both fixed point equations have, modulo provable equivalence, a unique solution with a self-reference free formulation. As we will see in the next section, the proper framework to formulate, study, and generalize this insight is *Provability Logic*.

5 Provability Logic

The first to note the possibility of reading formal provability in a theory as a modal operator was Kurt Gödel in his paper [20]. The main result of the paper is the translation of intuitionistic propositional logic IPC in the modal system $\mathsf{S4}$. At the end of the paper, Gödel remarks:

> Es ist zu bemerken, daß für den Begriff "beweisbar in einem bestimmten formalen System S" die aus $\mathfrak{S}$ beweisbaren Formeln nicht alle gelten. Es gilt z.B. für ihn $B(Bp \to p)$ niemals, d.h. für kein System S, das die Arithmetik enthalt. Denn andernfalls wäre beispielsweise $B(0 \neq 0) \to 0 \neq 0$ und daher auch $\sim B(0 \neq 0)$ in S beweisbar, d.h. die Widerspruchsfreiheit von S wäre in S beweisbar.

Here $\mathfrak{S}$ is S4. In the English translation of the Collected Works, Gödel's text becomes:

> It is to be noted that for the notion "provable in a certain formal system S" not all of the formulas provable in $\mathfrak{S}$ hold. For example, $B(Bp \to p)$ never holds for that notion, that is, it holds for no system S that contains arithmetic. For, otherwise, for example, $B(0 \neq 0) \to 0 \neq 0$ and therefore also $\sim B(0 \neq 0)$ would be provable in S, that is, the consistency of S would be provable in S.

In this paper, we will just look at that part of provability logic that is directly connected to Henkin's question: the study of fixed points. For a treatment of Gödel's remarks that is closely connected to Provability Logic, see [4].

Löb's logic GL is a modal propositional logic that has, in addition to the axioms and rules of propositional logic, the following principles:[10]

L1 $\vdash \varphi \;\Rightarrow\; \vdash \Box\varphi$.
L2 $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
L3 $\vdash \Box\varphi \to \Box\Box\varphi$.
L4 $\vdash \Box(\Box\varphi \to \varphi) \to \Box\varphi$.

The principles of GL correspond to the Löb Conditions plus Löb's Theorem. Thus, these principles are schematically valid for arithmetical interpretations where the $\Box$ is interpreted by Bew. We note that by the remarks of Sect. 4 the axiom L3 is superfluous. We can prove that the following strengthened version of Löb's Rule is admissible over GL. Let $\boxdot\chi$ stand for $\chi \wedge \Box\chi$. We have:

$$\text{SLR} \quad \text{if } \Box\psi_0, \ldots, \Box\psi_{n-1}, \boxdot\chi_0, \ldots, \boxdot\chi_{k-1}, \Box\varphi \vdash \varphi, \qquad \text{then}$$
$$\Box\psi_0, \ldots, \Box\psi_{n-1}, \boxdot\chi_0, \ldots, \boxdot\chi_{k-1} \vdash \varphi$$

We have seen in Sect. 4 that both Gödel's fixed point equation and Henkin's fixed point equation have unique self-reference free solutions. Provability Logic gives us the proper context both to formulate and to generalize these results.

Let us say that φ is *modalized* in p if all occurrences of p in φ are in the scope of $\Box$. A first observation is that, if φ is modalized in p, then φ has a unique fixed point (modulo provable equivalence) w.r.t. p. The uniqueness of fixed points was proved independently by Dick de Jongh (unpublished), Giovanni Sambin [39] and Claudio Bernardi in 1974 [10].

Let φp be modalized in p, and let q be a fresh propositional variable. Then we have:

$$\mathsf{GL} \vdash \big(\boxdot(p \leftrightarrow \psi p) \wedge \boxdot(q \leftrightarrow \psi q)\big) \to (p \leftrightarrow q).$$

To see this, reason in GL. Suppose (a) $\boxdot(p \leftrightarrow \psi p)$, (b) $\boxdot(q \leftrightarrow \psi q)$, and (c) $\Box(p \leftrightarrow q)$. Since φp is modalized in p and φq is modalized in q, we find from (c): (d) $\varphi p \leftrightarrow \varphi q$.[11] Ergo, by (a) and (b), (e) $p \leftrightarrow q$. By SLR, we may conclude $p \leftrightarrow q$ without assumption (c).

We turn to the existence of explicit fixed points in the modal language. Suppose φp is modalized in p. Then there is a formula χ, where the free variables of χ are included in the free variables of φ minus p, such that $\mathsf{GL} \vdash \chi \leftrightarrow \varphi\chi$.

[10] It would be more appropriate to call this logic simply L. Unfortunately, L also suggests *language*, so the designation GL was preferred.

[11] The substitution principle used here can be proved by induction of φ.

As is proper for great results, the theorem has many proofs. The existence of explicit fixed points was first proved by Dick de Jongh in 1974 (unpublished). Dick provided both a semantical and a syntactical proof. In 1976 another proof was given by Giovanni Sambin [39]. Also in 1976, George Boolos found a proof of explicit definability using characteristic formulas. In 1978, Craig Smoryński [41] proved explicit definability via Beth's Theorem.[12] There is an improved version of Sambin's approach by Giovanni Sambin and Silvio Valentini [46] in 1982 and an improved version of Boolos' proof by Zachary Gleit and Warren Goldfarb [18] in 1990. Finally, in 1990 there is a proof by Lisa Reidhaar-Olson [38] that is close to the proof of Sambin–Valentini. In 2009, Luca Alberucci and Alessandro Facchini [3] provide a proof using the modal μ-calculus.

We give the proof that is due to Craig Smoryński. In the proof we will assume the interpolation theorem for GL that can be proved both by semantic methods and by proof-theoretical methods. We assume that φp is modalized in p and that q does not occur in φp. We note that the uniqueness theorem gives us:

$$\begin{aligned}\Box(p \leftrightarrow \varphi p) \wedge \Box(q \leftrightarrow \varphi q) &\vdash_{\mathsf{GL}} \Box\boxdot(p \leftrightarrow \varphi p) \wedge \Box\boxdot(q \leftrightarrow \varphi q)\\ &\vdash_{\mathsf{GL}} \Box(p \leftrightarrow q)\\ &\vdash_{\mathsf{GL}} \varphi p \leftrightarrow \varphi q\end{aligned}$$

It follows that

$$\Box(p \leftrightarrow \varphi p) \wedge \varphi p \vdash_{\mathsf{GL}} \Box(q \leftrightarrow \varphi q) \to \varphi q.$$

Let χ be an interpolant between $\Box(p \leftrightarrow \varphi p) \wedge \varphi p$ and $\Box(q \leftrightarrow \varphi q) \to \varphi q$. By substituting χ for p and q we obtain:

$$\Box(\chi \leftrightarrow \varphi\chi) \wedge \varphi\chi \vdash_{\mathsf{GL}} \chi \vdash_{\mathsf{GL}} \Box(\chi \leftrightarrow \varphi\chi) \to \varphi\chi.$$

We may conclude that $\mathsf{GL} \vdash \Box(\chi \leftrightarrow \varphi\chi) \to (\chi \leftrightarrow \varphi\chi)$ and, hence, $\mathsf{GL} \vdash \chi \leftrightarrow \varphi\chi$.

The story of the fixed points is not finished here. First, the uniqueness and explicit definability results extend to interpretability logic as was shown in [16]. A Smoryński-style proof of this result is provided in [2]. See also [26]. Secondly, the fixed points of provability logic connect it to that other great modal logic of fixed points, the modal μ-calculus. See [48, 52], and [3]. See also Giacomo Lenzi's survey [33] of results concerning the μ-calculus.

The world looks different when the Löb conditions fail. The uniqueness/non-uniqueness of the Rosser fixed points was studied by David Guaspari and Robert Solovay [22]. Their answer is a laconic *it depends*. Interestingly, Kreisel never found their work convincing. His experience with his own *it depends* led him to hope for a missing natural condition ....

What happens in case of the Feferman provability predicate was studied by Albert Visser [50], by Craig Smoryński [43], and by Volodya Shavrukov [40]. Visser shows that there are infinitely many pair-wise nonequivalent Henkin fixed points for Feferman provability. Smoryński shows that under rather natural assumptions the Gödel fixed point

[12] In her paper [36], Larisa Maksimova shows that, conversely, Beth's theorem follows from the existence of explicit fixed points. See also [26].

for the Feferman predicate is unique. Shavrukov gives a beautiful modal derivation of the same result.

Of course, there is much more to provability logic than the treatment of fixed points. For example, there is Solovay's celebrated arithmetical completeness theorem ([45]). However, this further story is outside of the scope of this paper. We just provide some pointers to the further literature.

The history of the mathematical modality *provability in a certain formal system* has been described in the paper [13]. This paper is warmly recommended to the reader. The systematic content of provability logic can be found in the textbooks [12] and [42]. It is worth looking at both books since they offer a somewhat different perspective. For more recent treatments, containing also new material, see also [1, 14, 30, 34, 47].

We have seen that from Henkin's question and Löb's work, the field of Provability Logic emerged. Provability Logic, apart from being a beautiful subject, has some application outside of its own domain.

- Michael Beeson employed fp-realizability, a form of realizability based on provability to prove the independence of the Myhill–Shepherdson theorem and of the Kreisel–Lacombe–Schoenfield theorem from Heyting arithmetic. See [5] and [49]. Beeson's result uses Löb's principle.
- Research on the Provability Logic of Heyting Arithmetic inspired an axiomatization of the admissible rules of the intuitionistic propositional calculus. See [28].
- S.N. Goryachev connects in his work [21] Provability Logic and reflection principles. His results are used by Lev Beklemishev for the analysis of reflection principles. See [6] and [8].
- Japaridze's polymodal logic [29] is used by Lev Beklemishev to study ordinal notations. See [7, 9, 11].

6 Concluding Remarks

The question of what it means for a formula to *express* a property like provability has fallen from grace since the success of Löb's work. First, the question seems rather hopeless, and second, Löb showed that, at least for some important results, we can successfully get by without an answer to the question. Does this mean that the question has for once and for all been laid to rest? We do not think so. First, even if, perhaps, the question is mathematically less important, then it is still relevant philosophically. If we say, for example, that Peano arithmetic does not prove its own consistency, is this merely a *façon de parler*—to be paraphrased away by a more mathematical pronouncement, or does it really mean what it seems to mean?

Can we tell a better story about arithmetical *ventriloquism*? Such a story should at least take into account that content ascriptions like *the formula* Bew *expresses provability in* Σ are heavily contextual. For instance, the ascription only makes sense against the background of some Gödel numbering. Perhaps, we need, as Henkin thought, to have a theory of content as a *prolegomemon*. But, maybe, the task is rather to describe on the basis of our everyday understanding of how formulas express properties, to explain how formulas that are obviously about something else (like numbers), still manage to express properties of sentences against the background of conventional choices relating, for example, numbers and formulas.

We think that there is reason to have hope for progress. In a sense, we have all the needed information concerning what is going on. After all, the good cases where we think that we really construct a formula Bew expressing provability are open for detailed inspection. *All there is, is here.* Also we have lots of deviant examples where we could have doubts like the nonstandard Gödel numberings with in-built self-reference. What is lacking is an articulated analysis bringing to the fore what is good and what is bad.

As we have seen, Henkin's playful question led to the development of Provability Logic. Moreover, it touches immediately upon philosophical questions concerning intensionality in mathematics. Voltaire said "Il est encore plus facile de juger de l'esprit d'un homme par ses questions que par ses réponses" (it is easier to judge a man by his questions than by his answers). Clearly, Leon Henkin is doing very well on Voltaire's criterion.

Acknowledgements We thank Volodya Shavrukov for his comments on the penultimate version.

References

1. Artemov, S.N., Beklemishev, L.D.: Provability logic. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn. vol. 13, pp. 229–403. Springer, Dordrecht (2004)
2. Areces, C., de Jongh, D., Hoogland, E.: The interpolation theorem for IL and ILP. In: Proceedings of AiML98. Advances in Modal Logic, Uppsala, Sweden (1998)
3. Alberucci, L., Facchini, A.: On modal μ-calculus and Gödel–Löb logic. Stud. Log. **91**(2), 145–169 (2009)
4. Artemov, S.: Logic of proofs. Ann. Pure Appl. Log. **67**(1), 29–59 (1994)
5. Beeson, M.: The nonderivability in intuitionistic formal systems of theorems on the continuity of effective operations. J. Symb. Log. **40**, 321–346 (1975)
6. Beklemishev, L.D.: Proof-theoretic analysis by iterated reflection. Archive **42**, 515–552 (2003). doi:10.1007/s00153-002-0158-7
7. Beklemishev, L.D.: Provability algebras and proof-theoretic ordinals. Ann. Pure Appl. Log. **128**, 103–124 (2004)
8. Beklemishev, L.D.: Reflection principles and provability algebras in formal arithmetic. Russ. Math. Surv. **60**(2), 197–268 (2005)
9. Beklemishev, L.D.: The Worm principle. Logic Colloquium'02. Lect. Notes Log. **27**, 75–95 (2006)
10. Bernardi, C.: The uniqueness of the fixed-point in every diagonalizable algebra. Stud. Log. **35**(4), 335–343 (1976)
11. Beklemishev, L.D., Joosten, J.J., Vervoort, M.: A finitary treatment of the closed fragment of Japaridze's provability logic. J. Log. Comput. **15**(4), 447–463 (2005)
12. Boolos, G.: The Logic of Provability. Cambridge University Press, Cambridge (1993)
13. Boolos, G., Sambin, G.: Provability: the emergence of a mathematical modality. Stud. Log. **50**, 1–23 (1991)
14. Beklemishev, L.D., Visser, A.: Problems in the logic of provability. In: Gabbay, D.M., Concharov, S.S., Zakharyashev, M. (eds.) Mathematical Problems from Applied Logic I. Logics for the XXIst Century. International Mathematical Series, vol. 4, pp. 77–136. Springer, New York (2006)
15. Church, A.: A formulation of the logic of sense and denotation. In: Structure, Method, and Meaning, pp. 3–24 (1951)
16. de Jongh, D.H.J., Visser, A.: Explicit fixed points in interpretability logic. Stud. Log. **50**, 39–50 (1991)
17. Feferman, S.: Arithmetization of metamathematics in a general setting. Fundam. Math. **49**, 35–92 (1960)
18. Gleit, Z., Goldfarb, W.: Characters and fixed points in provability logic. Notre Dame J. Form. Log. **31**, 26–36 (1990)

19. Gödel, K.: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. Monatshefte Math. **38**, 173–198 (1931)
20. Gödel, K.: Ein Interpretation des intuitionistischen Aussagenkalküls. In: Ergebnisse eines mathematischen Kolloquiums, vol. 4, pp. 39–40 (1933). Reprinted as: An interpretation of the intuitionistic propositional calculus. In: Feferman, S. (ed.), Gödel Collected Works I, pp. 300–303, Oxford (1986)
21. Goryachev, S.: On interpretability of some extensions of arithmetic. Mat. Zametki **40**, 561–572 (1986) (in Russian). English translation in Math. Notes **40**
22. Guaspari, D., Solovay, R.M.: Rosser sentences. Ann. Math. Log. **16**, 81–99 (1979)
23. Hilbert, D., Bernays, P.: Grundlagen der Mathematik II, Springer, Berlin (1939). 2nd edn. in (1970)
24. Henkin, L.: A problem concerning provability. J. Symb. Log. **17**, 160 (1952)
25. Henkin, L.: Review of G. Kreisel: On a problem of Henkin's. J. Symb. Log. **19**(3), 219–220 (1954)
26. Hoogland, E.: Definability and Interpolation: Model-Theoretic Investigations. Institute for Logic, Language and Computation, Amsterdam (2001)
27. Halbach, V., Visser, A.: Self-reference in Arithmetic. Logic Group Preprint Series, vol. 316. Faculty of Humanities, Philosophy, Utrecht (2013)
28. Iemhoff, R.: On the admissible rules of intuitionistic propositional logic. J. Symb. Log. **66**(1), 281–294 (2001)
29. Japaridze, G.: The polymodal logic of provability. In: Intensional Logics and Logical Structure of Theories: Material from the Fourth Soviet–Finnish Symposium on Logic, Telavi, pp. 16–48 (1985)
30. Japaridze, G., de Jongh, D.: The logic of provability. In: Handbook of Proof Theory, pp. 475–546. North-Holland, Amsterdam (1998)
31. Kreisel, G.: On a problem of Henkin's. Indag. Math. **15**, 405–406 (1953)
32. Kreisel, G., Takeuti, G.: Formally self-referential propositions for cut free classical analysis and related systems. Diss. Math. **118**, 1–50 (1974)
33. Lenzi, G.: Recent results on the modal μ-calculus: a survey. Rend. Ist. Mat. Univ. Trieste **42**, 235–255 (2010)
34. Lindström, P.: Provability logic—a short introduction. Theoria **62**(1–2), 19–61 (1996)
35. Löb, M.H.: Solution of a problem of Leon Henkin. J. Symb. Log. **20**, 115–118 (1955)
36. Maksimova, L.: Definability theorems in normal extensions of the probability logic. Stud. Log. **48**(4), 495–507 (1989)
37. Raatikainen, P.: Gödel's incompleteness theorems. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy (2013). Available via http://plato.stanford.edu/archives/win2013/entries/goedel-incompleteness/
38. Reidhaar-Olson, L.: A new proof of the fixed-point theorem of provability logic. Notre Dame J. Form. Log. **31**(1), 37–43 (1990)
39. Sambin, G.: An effective fixed-point theorem in intuitionistic diagonalizable algebras. Stud. Log. **35**, 345–361 (1976)
40. Shavrukov, V.Yu.: A smart child of Peano's. Notre Dame J. Form. Log. **35**, 161–185 (1994)
41. Smoryński, C.: Beth's theorem and self-referential sentences. In: Macintyre, A., Pacholski, L., Paris, J. (eds.) Logic Colloquium '77. Studies in Logic. North-Holland, Amsterdam (1978)
42. Smoryński, C.: Self-reference and Modal Logic. Springer, New York (1985)
43. Smoryński, C.: Arithmetic analogues of McAloon's unique Rosser sentences. Arch. Math. Log. **28**, 1–21 (1989)
44. Smoryński, C.: The development of self-reference: Löb's theorem. In: Drucker, T. (ed.) Perspectives on the History of Mathematical Logic, pp. 110–133. Springer, Berlin (1991)
45. Solovay, R.M.: Provability interpretations of modal logic. Isr. J. Math. **25**, 287–304 (1976)
46. Sambin, G., Valentini, S.: The modal logic of provability. The sequential approach. J. Philos. Log. **11**(3), 311–342 (1982)
47. Švejdar, V.: On provability logic. Nord. J. Philos. Log. **4**(2), 95–116 (2000)
48. Van Benthem, J.: Modal frame correspondences and fixed-points. Stud. Log. **83**(1–3), 133–155 (2006)
49. Visser, A.: On the completeness principle. Ann. Math. Log. **22**, 263–295 (1982)
50. Visser, A.: Peano's smart children: a provability logical study of systems with built-in consistency. Notre Dame J. Form. Log. **30**(2), 161–196 (1989)
51. Visser, A.: Faith & Falsity: a study of faithful interpretations and false Σ^0_1-sentences. Ann. Pure Appl. Log. **131**(1–3), 103–131 (2005)

52. Visser, A.: Löb's logic meets the μ-calculus. In: Middeldorp, A., van Oostrom, V., van Raamsdonk, F., de Vrijer, R. (eds.) Processes, Terms and Cycles, Steps on the Road to Infinity. Essays Dedicated to Jan Willem Klop on the Occasion of His 60th Birthday. LNCS, vol. 3838, pp. 14–25. Springer, Berlin (2005)
53. Wells, R.: Review of: A formulation of the logic of sense and denotation, by Alonzo Church. J. Symb. Log. **17**(2), 133–134 (1952)

V. Halbach
New College, OX1 3BN Oxford, England, UK
e-mail: volker.halbach@new.ox.ac.uk

A. Visser (✉)
Philosophy, Faculty of Humanities, Utrecht University, Janskerkhof 13, 3512BL Utrecht, The Netherlands
e-mail: a.visser@uu.nl

April the 19th

María Manzano

Abstract This paper is about my book (Manzano, Extensions of first-order logic, 1996), published by Cambridge University Press in 1996. The main purpose of it being to pinpoint Henkin's influence concerning the translation technique proposed in the book.

Several extensions of first order logic are introduced in Extensions, while trying to pursue the thesis that most reasonably logical systems can be naturally translated into many-sorted first order logic. I did credit most of the ideas involved in my translation to Henkin's papers (Completeness in the theory of types, 1950, and Banishing the rule of substitution for functional variables, 1953).

Keywords Henkin · Translations · Correspondence theory · Many-sorted logic · Extensions of first-order logic · Type theory · General models · Comprehension schema · Rule of substitution

1 April 19, 1996

On April the 19th, 1996, I was invited by *The Group in Logic and the Methodology of Science* (University of California, Berkeley) to give a talk at the *Berkeley Logic Colloquium*—which was followed by the so called *Logic Tea*. A very special date indeed, as we were to celebrate Henkin's 75 birthday. Unfortunately, I was unable to attend since while I was in Stanford walking to the meeting point where some colleagues were to pick me and my 10-year son Ulises up, we were both run over on a pedestrian crossing. The accident was not as bad as it could have been—our left arms broken—but we were still unable to congratulate Leon Henkin on occasion of his 75th birthday.

A couple of weeks later, I did give the talk, its main purpose being to underscore Henkin's influence on my book, *Extensions of First Order Logic*, published by Cambridge University Press that year.

In 1994 I sent Henkin the manuscript of that book and he replied in a beautiful letter in which he suggested that I should give the talk during my forthcoming visit to the States:

> And now, when I begin really to look in detail at what have you put into the book, I am very excited to see the presentation of some of my early ideas that you have incorporated beautifully in the work [...] not only the completeness of second-order logic in the sense of non-standard models, but my little paper on induction models from 1960, which has always been my favorite among my expository papers!
> Most amazing to me is that this book will appear in the Cambridge Tracts on Theoretical Computer Science [...] for there was no such field as computer science when I wrote these papers!
> [...]

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_18

> I am very grateful that you've written these works, Mara, and very grateful that you have sent me copies. If you wish to mention how our ideas came together when you came to work in Berkeley, "my cup (of joy) will overflow".
> *Henkin, August 12, 1994. Personal letter to María Manzano.*

After my talk, Henkin seemed to be pleased with what I said and he insisted that I had taken his ideas further than he ever dreamed; needless to say that I was happy and honoured, even though I was able to recognize his kind disposition as the main basis of his public comments. A few months later, he sent me a printed copy of his *The Discovery of my Completeness Proofs* [9] with this autograph: *'To Mara Manzano, who made great use of the ideas that I wrote about, bringing them beyond where I imagined they might go.'*

2 Two of Henkin's Papers

2.1 Completeness in the Theory of Types [6]

Henkin's completeness theorems are the main content of his doctoral thesis, *The completeness of formal systems*,[1] defended in 1947 under the direction of Alonzo Church. He published two papers in the *Journal of Symbolic Logic*: the first, on completeness for first-order logic [5], in 1949, and the second devoted to completeness in type theory [6], in 1950. It is interesting to recall that the publication order is the reverse of the actual discovery of the proofs.

At the beginning of *Completeness in the Theory of Types*, the author makes reference to Gödel's results, both on completeness and incompleteness. Regarding completeness for first-order logic, we read: *'each formula of the calculus is a formal theorem which becomes a true sentence under every one of a certain intended class of interpretations of the formal system'*,[2] and regarding incompleteness for second-order logic, Henkin says: '[...] *no matter what (recursive) set of axioms are chosen, the system will contain a formula which is valid but not a formal theorem'*.[3]

Of course, the latter result is obtained allowing only standard models: '[...] *the individual variables are interpreted as ranging over an (arbitrary) domain of elements while the functional variables of degree n range over all sets of ordered n-tuples of individuals'*.[4] It is in this context that he announces that the incompleteness problem is solvable if *'a wider class of models'* (non-standard models) is allowed:

> If we redefine the notion of valid formula to mean one which expresses a true proposition with respect to every one of *theses* models, we can then prove that the usual axiom system for the second-order calculus is complete: a formula is valid if and only if it is a formal theorem.
> (Henkin [6, pp. 81–82])

---

[1] In [9] Henkin said that the dissertation has never been published.

[2] In [6, p. 81].

[3] In [6, p. 81].

[4] In [6, p. 81].

In the aforementioned paper Henkin introduces three different interpretations for type theory: (1) *standard models*, (2) *frames* and (3) *general models*. The later matches the calculus, since a completeness result for Church's calculus [1] is proved in [6]. Let us quote him:

> By a *standard model*, we mean a family of domains, one for each type-symbol, as follows: D_ι is an arbitrary set of elements called *individuals*, D_o is the set consisting of the two truth values, T and F, and $D_{\alpha\beta}$ is the set of all functions defined over D_β with values in D_α.
> [. . .]
> By a *frame*, we mean a family of domains, one for each type symbol, as follows: D_ι is an arbitrary set of *individuals*, D_o is the set of two truth values, T and F, and $D_{\alpha\beta}$ is some class of functions defined over D_β with values in D_α.
> [. . .]
> A frame such that for every assignment ϕ and wff A_α of type α, the value $V_\phi(A_\alpha)$ given by the rules (i), (ii), and (iii) is an element of D_α is called a *general model*.
> (Henkin [6, pp. 83–85])

The last interpretation yields the well-known Henkin's completeness result for type theory:

> Now we define a valid formula in the general sense as a formula A_o such that $V_\phi(A_o)$ is T for every assignment ϕ with respect to any general model. We shall prove a completeness theorem for the formal system by showing that A_o is valid in the general sense if and only if $\vdash A_o$.
> (Henkin [6, p. 85])

To achieve such a result he proves: *'If Λ is any consistent set of cwffs, there is a general model (in which each domain D_α is denumerable) with respect to which Λ is satisfiable*.'[5]

2.2 *Banishing the Rule of Substitution for Functional Variables* [7]

Henkin published this paper in 1953, and its main result was the introduction of substitution-free calculi for classical logics to replace Church's. At that time it was common to use axiomatic calculi with a finite number of axioms instead of scheme axioms, and the substitution rule was necessary even for the propositional calculus. When first-order or second-order calculi were introduced, new and even more difficult substitution rules were added; a look at Church's book *Introduction to Mathematical Logic* [2] suffices to glimpse what I am saying. Nevertheless, in that book Church proposed other calculi for propositional and first-order logics, where the objectionable rules are reduced to a minimum. The difficult substitution rules appear in Church's second-order calculus.

From this point of departure, and knowing Henkin's devotion to pedagogy, it is easy to identify Henkin's main purpose as a pedagogical one:

> Thus the beginning student who is introduced to the calculi through such a formulation is forced to cope from the outset with details which have proved treacherous even to the initiate. For this reason it is desirable to seek for alternative formulations of the functional calculi in which this type of substitution is not mentioned.
> (Henkin [7, p. 201])

As Henkin points out, for first-order logic Church defines the system F^1 where the substitution rule is only for quantifier elimination:

[5] In [6, p. 85].

[...] F^1 takes as axioms all instances of the following two schemata:
(i) $(\mathfrak{a})(\mathfrak{A} \supset \mathfrak{B}) \supset (\mathfrak{A} \supset (\mathfrak{a})\mathfrak{B})$, where $\mathfrak{a}$ is any individual variable not free in $\mathfrak{A}$
(ii) $(\mathfrak{a})\mathfrak{A} \supset S'^{\mathfrak{a}}_{\mathfrak{b}}\mathfrak{A}|$, where $\mathfrak{b}$ is any individual variable such that no free occurrence of $\mathfrak{a}$ in $\mathfrak{A}$ is in a part of $\mathfrak{A}$ of the form $(\mathfrak{b})\mathfrak{C}$.
(Henkin [7, p. 202])

Henkin mentions that the natural extension of this calculus to the second-order case, which he calls F^*—namely, extending the quantifier rules to deal with the new predicate variables—is not a complete one: '*Unfortunately the system F^* is woefully incomplete, for even a simple formula as*

$$(\exists x)(\exists G)G(x)$$

easily seen to be valid under the standard interpretation of the second-order calculus, is not a formal theorem of F^*.'[6]

No doubt he has in mind not just the classical higher-order interpretation but also the general interpretation introduced by himself in his pioneering paper [6].

To solve the incompleteness situation, he defines two equivalent calculi, F^2 and F^{**}, and mentions the completeness result of his 1950 paper [6]:

[...] we construct a system F^2 which is like F^* except that we include the additional axiom schema
(iii) $(\mathfrak{a})\mathfrak{A} \supset \overline{S}^{\mathfrak{c}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)}_{\mathfrak{B}}\mathfrak{A}|$
[...] F^2 contains all of the theorems of the usual second order functional calculus, and indeed can be shown to be complete (in a certain sense).
(Henkin [7, p. 202])

The problem with calculus F^2 is that it includes "the troublesome notion of substitution for functional variables". That is why Henkin proposed the calculus F^{**} whose advantage is that it takes comprehension schema as an axiom:

F^{**} is obtained from F^* by adding the axiom schema
(iv) $(\exists\mathfrak{c})(\mathfrak{a}_1)\ldots(\mathfrak{a}_n)(\mathfrak{c}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n) \equiv \mathfrak{B})$,
where $\mathfrak{B}$ is any wff, $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ are any distinct individual variables, and $\mathfrak{c}$ is any n-adic functional variable not occurring freely in $\mathfrak{B}$.
(Henkin [7, p. 203])

For the most part of this paper, Henkin concentrates in proving the equivalence between F^2 and F^{**}. However, the relevant part for the translation technique concerning my own work is mentioned at the end of the paper:

[...] there is another advantage which we would claim for this system, namely, that it calls attention to the subsystem F^*.
[...]
By a *model* let us here understand a sequence $(K_0, K_1, \ldots, K_n)$ such that K_0 is an arbitrary (non-empty) set, K_1 is an arbitrary (non-empty) class of subsets of K_0, and for $n > 1$ K_n is an arbitrary (non-empty) class of n-ary relations on K_0.
[...]
The formal theorems of the system F^* consist precisely of those formulas which are completely valid with respect to every model, as can be shown by the method of [6].
[...]
This observation suggests in a natural way consideration of certain subsystems of F^{**} containing F^*, which can be defined by weakening axiom schema (iv).
(Henkin [7, pp. 206–207])

[6] See [7, p. 202].

As I shall explain in what follows, F^* could be taken as a many-sorted first-order calculus and the calculi between F^* and the second-order F^{**} could be obtained in most cases by restricting comprehension to formulas that are translations of a variety of logics into the many-sorted framework.

3 Translations

Currently, the proliferation of logical systems used in mathematics, computer science, philosophy, and linguistics makes their relationships between them and their possible translations into one another pressing issues. Translations between logics have been formulated as an ambitious paradigm whose tools would serve for handling the *existing* multiplicity of logics.

1. From a *proof-theoretical* point of view, the style of comparing logics will rest upon morphisms between calculi. The "labelled deductive systems" of Gabbay emerge.[7]
2. From a *model-theoretical perspective*, one will presumably compare logics by defining morphisms between the structures those logics are attempting to describe, as in the correspondence theory of van Benthem.[8]
3. From a *suprastructural* point of view, we define morphisms between categories. Among the most abstract approaches to logic, we should highlight the "general logics" of Meseguer.[9]

It is worth noting that in the three cases we ended up with classical logic. The paradigm of logical translation assumed in my book belongs to the model-theoretical tradition mentioned in item 2 and the target logic is many-sorted logic (*MSL*).

Why MSL? you might ask.

In itself, *MSL* is a natural logic for reasoning about more than one type of objects, very flexible. Many-sorted or heterogeneous logic is not only natural for reasoning about more than one type of objects, but it is also efficient, since its proof theory is well developed (it has a sound and complete calculus). Moreover, *MSL* can be used as a unifier framework in which to place other logics.

3.1 Extensions of First Order Logic [10]

In that book I consider various extensions of first-order logic which have applications in philosophy, computer science, mathematics, linguistics, and artificial intelligence. I give a detailed and elaborated treatment of the following useful systems: Second-order logic (*SOL*), type theory (*TT*), modal logic (*PML*, *FOML–S*5), dynamic logic (*PDL*), and many-sorted logic (*MSL*).

[7] See [3] and [4].

[8] See [13] and [14].

[9] See [12] and [11].

The second goal of the book is to work out the thesis that most reasonable logical systems can be naturally translated into many-sorted first-order logic. This thesis is maintained throughout the book, but only appears openly and explicitly in the last chapter. At that point, all the logics addressed in the book are placed in direct correspondence with *MSL* because this logic offers a unifying framework in which to place other logics.

From my point of view, the appeal of the approach is that it is so intuitive and easy that only common sense is needed to understand the construction. Furthermore, as the basic ingredients change, the recipe can be adapted to cook different dishes. It is difficult to trace the development of this approach because almost all non-classical logics have found their standard counterparts at birth. Nevertheless, I like to credit most of the ideas of this translation impulse to the aforementioned papers of Henkin.

However, I do not wish to be misleading. In his 1950 paper we do not find translations of formulas or the overt appearance of a many-sorted calculus. Regarding higher-order logic, something close to a many-sorted calculus was introduced in the paper of 1953, but many-sorted logic was still not mentioned and neither were translations between logical systems. As I have already stated, in this paper Henkin proposes the comprehension axiom as a way to avoid the troublesome rule of substitution of the previous higher-order calculus. Leaving aside the differences between formal languages, *MSL* calculus is *SOL* calculus without comprehension. Moreover, this new calculus allows us to isolate calculi between the *MSL* calculus and the higher-order ones, by weakening comprehension. This idea is used both in modal logic and in dynamic logic.

Lessons from *Completeness in the Theory of Types* When one has a logic, one often has a set of formulas and two ways of selecting formulas from this set; the rules of the calculus, selecting the logical theorems, and the semantics, choosing the validities.

Are these sets the same? When all the logical theorems are valid we say that the calculus is sound, and when the reverse inclusion is true, we say that it is complete. Unfortunately, for higher-order logic, the latter is not true, since there are valid formulas that cannot be derived in any possible calculus. We learn from Gödel's result that the set of valid formulas is too big, indeed unapproachable. Therefore, with the standard semantics, the set of valid formulas of *SOL* will never be generated by a finite deductive calculus. The standard semantics is being determined by structures $\mathcal{A} = \langle A, \langle A_n \rangle_{n \in N}, \ldots \rangle$ where $A_1 = \wp A$, $A_2 = \wp(A \times A)$, etc. Here, the concept of a subset is taken from the metatheory of sets, in the same sense as we usually take the concept of identity in first-order logic from metatheory. With this I mean that we do not include it in the structure, but that it is taken as a logical or primitive concept.

However, even knowing that no deductive calculus is able to select as theorems all valid formulas in standard semantics (i.e., there are no complete calculi for *SOL*), we can define sound ones. The idea is very simple: The set of valid formulas is so big because the concept of a standard structure is far too restrictive. However, if we leave open, definable by each structure, the concept of a subset, the situation changes radically. If we also accept as structures for *SOL* the non-standard ones (i.e., those where $A_n \subseteq \wp A^n$, but not necessarily equals), the set of valid formulas reduces. Nevertheless, if the selection of what is put into A_n is left completely arbitrary, the comprehension axiom might fail; that is, some sets defined by formulas of the language may have been left out. To make the calculus *SOL* sound and complete, we need general Henkin structures where the domains are closed under definability. As we have already seen, this is exactly what Henkin did in his paper of 1950 when he defined *frames* and *general models*. The classes of standard structures, general structures and frames are ordered by set inclusion

$$\mathfrak{SS} \subseteq \mathfrak{GS} \subseteq \mathfrak{F}$$

and accordingly the sets of validities in each class obey the reverse order

$$\models_{\mathfrak{F}} \subseteq \models_{\mathfrak{GS}} \subseteq \models_{\mathfrak{SS}}$$

In fact, the inclusion is strict:

$$\models_{\mathfrak{F}} \subset \models_{\mathfrak{GS}} \subset \models_{\mathfrak{SS}}$$

since there are sentences true in all general structures that fail in some frames as well as sentences true in standard structures that fail in some general structures. The Comprehension axiom

$$\exists X \forall u (Xu \leftrightarrow \varphi)$$

is an example of a sentence not necessarily true in frames but valid in general structures. There are also sentences true in standard structures that fail in some general structures. As an example, let us take the sentence

$$\pi_1 \wedge \pi_2 \wedge \pi_3 \rightarrow \gamma$$

where π_1, π_2 and π_3 are the second-order categorical Peano axioms for arithmetic, and γ is the well-known Gödel formula asserting of itself that it is not a theorem of Peano arithmetic.

From Henkin's paper we learn that a modification of the semantics can adapt validities (in the new semantics) to logical theorems.[10] To obtain the completeness theorem, non-standard semantics is required; that is, validities in general structures and logical theorems in second-order calculus are the same:

$$\models_{\mathfrak{GS}} = \vdash_{SOL}$$

Even though the move to general structures is a very convenient one, we believe that it is not an ad hoc construction lacking mathematical or philosophical inspiration. In fact, these structures could be connected with the philosophical position of Nominalism on the one hand and with the constructible universe on the other hand.[11]

Lessons from *Banishing the Rule of Substitution for Functional Variables* As we have already mentioned, in this paper Henkin introduces two calculi F^* and F^{**}. Both calculi are incomplete with standard semantics: namely, the set $\vdash_{F^*}$ of theorems of F^*

[10] In [10] this idea is used in connection with dynamic logic where non-standard models are also introduced. In DL the non standard character came from the fact that the loop operator is not taken as the least reflexive and transitive closure of a given relation, but as the least in the structure.

[11] See [8, p. 22].

is a proper subset of the set $\models_{\mathfrak{S}\mathfrak{S}}$ of validities in standard structures, and so is the set of $\vdash_{F^{**}}$ of theorems of F^{**}:

$$\vdash_{F^*} \subset \models_{\mathfrak{S}\mathfrak{S}} \quad \text{and} \quad \vdash_{F^{**}} \subset \models_{\mathfrak{S}\mathfrak{S}}$$

We have already seen that the second-order calculus is complete with the general semantics. Henkin mentioned that we can obtain a completeness result for the F^* calculus with the *frame semantics*: '*The formal theorems of the system F^* consist precisely of those formulas which are completely valid with respect to every model, as can be shown by the method of* [6]'.[12] In fact,

$$\models_{\mathfrak{F}} = \vdash_{F^*}$$

Henkin does not mention it, but we could prove completeness for F^{**} with general semantics simply by using completeness of F^*. The advantage here is that the property of being a general structure can be axiomatized using comprehension. Thus,

$$\models_{\mathfrak{G}\mathfrak{S}} \varphi \quad \Longleftrightarrow \quad \Delta \models_{\mathfrak{F}} \varphi$$

(where Δ contains all comprehension axioms as well as extensionality axiom and the disjoint universes requirement).[13]

There is another idea, appearing explicitly in the 1953 paper, which is also useful. The idea is the following: If we weaken comprehension (for instance, for first-order formulas, or for translations of dynamic or modal formulas, or any other recursive set), then we obtain a calculus between F^* and F^{**}. And it is easy to find a semantics for the logic thus defined. Of course, this class of structures is placed between $\mathfrak{F}$ and $\mathfrak{G}\mathfrak{S}$. The new logic, call it *XL*, will also be complete. The reason is that this class of models is again axiomatizable. In the book, this idea is exploited both in propositional modal logic and propositional dynamic logic. There, instead of using the calculus F^* with the frame semantics, we move directly to many-sorted logic.

Using this philosophy, in the book, I introduce a many-sorted language *MSL*, with a higher-order appearance, which will be used as a target language into which to translate a variety of languages. In the first place, the logic we wish to translate into *MSL* is *SOL* (supplemented with the general semantics of Henkin) and well-known calculi are defined. The two second-order calculi with a "pedigree" are called *MSL* and *SOL*. *MSL* is a simple extension of a calculus for *FOL* obtained by extending the quantification rules to cover all the new quantified formulas. *SOL* is obtained by adding the comprehension schema to *MSL*.

4 Reducing Other Logics to Many-Sorted Logic

It is argued in [10] that *MSL* would be the target logic in translation issues, owing to its efficient proof theory, flexibility, naturalness, and versatility to adapt to reasoning about more than one type of object. As we shall show in what follows, it is indeed the perfect

[12] In [7, p. 207].

[13] See [10, p. 285].

candidate to act as a unifying framework in which we may situate and compare most of the many logics available to us.

Now I am going to explain the general idea[14] while putting the example of *PML* (propositional modal logic).[15]

4.1 Representation Theorems

The general plan is as follows.

The signature of the logic being studied (call it *XL*) is transformed into a many-sorted signature; the expressions of the logic *XL* are translated into *MSL*, and the structures of the logic *XL* are converted into many-sorted structures. Thus, we need to define a recursive function, *Trans*, to do the translation and a direct conversion of structures, $Conv_1$.

A possible approach is to add to the many-sorted structures new universes containing all the categories of mathematical objects we wish to talk about (and we are able to talk to) in the logic *XL*. Thus, we can put into $Conv_1(\mathcal{A})$ universes containing the sets and relations defined in the original structure, $\mathcal{A}$, by the constructs of *XL*. As a consequence, we seem to be shifting to *SOL* instead of *MSL*. However, we already know how to avoid using *SOL* explicitly by using *MSL* instead; that is, we can have many-sorted universes, but we have to add membership relations and extensionality.

With the direct conversion of structures we wish to obtain as a result the equivalence of the validity in the original structures for *XL*, let us call them simply *Str*(*XL*), with validity of a certain class of many-sorted formulas (the translations) in the class $\mathfrak{S}^*$ of converted structures (where $\mathfrak{S}^* = Conv_1 Str(XL)$).

Our first goal is to state and prove the following proposition.

Proposition 1 *For every sentence* φ *of the XL logic*,

$$\models_{Str(XL)} \varphi \textit{ in } XL \quad \textit{iff} \quad \models_{\mathfrak{S}^*} Trans(\varphi) \textit{ in } MSL.$$

Of course, the first question to ask is whether or not $\mathfrak{S}^*$ can be replaced by the models of a set Δ of many-sorted formulas. Thus, the key to both definitions, *Trans* and $Conv_1$, is to simplify the proof of the semantic equivalence, and in this respect the relevance of the results obtained depends strongly on the possibility of axiomatizing $\mathfrak{S}^*$.

Our first goal is to prove a representation theorem:

Theorem 2 (Representation Theorem) *There is a recursive set of* L^**-sentences* Δ, *with* $\mathcal{S}^* \subseteq Mod(\Delta)$, *such that*

$$\models_{Str(XL)} \varphi \textit{ in } XL \quad \textit{iff} \quad \Delta \models_{Str(MXL^*)} \forall Trans(\varphi) \textit{ in } MSL$$

for every sentence φ *of the XL logic.*

[14]For a detailed exposition of this technique, see [10, pp. 263–276].

[15]See [10, pp. 312–332].

From this theorem, the enumerability for the logic *XL* follows. Thus, we learn that a calculus for *XL* is a natural demand, but we also learn that in *MSL* we can simulate such a calculus and that we can then use the theorem prover of *MSL*, if any.

The Representation Theorem in Propositional Modal Logic

As an example, let us see the case where we reduce *PML* into a particular many-sorted language MSL^*. This many-sorted language contains: $\perp$, $\neg$, $\rightarrow$, the unary relation symbols P_0, P_1, $P_2, \ldots$ the binary relation symbol S, a membership sign and two equalities: for individuals and for sets. The variables are also of two kinds, individuals and sets. We use the well-known standard translations, where we write formulas stating in first-order logic the semantic truth conditions for the modal formulas. Namely,

$$Trans(\perp)[u] := u \neq u$$

$$Trans(\neg\varphi)[u] := \neg Trans(\varphi)[u]$$

$$Trans(\varphi \rightarrow \psi)[u] := Trans(\varphi)[u] \rightarrow Trans(\psi)[u]$$

$$Trans(\Box\varphi)[u] =: \forall v\big(Suv \rightarrow Trans(\varphi)[v]\big)$$

It is easy to see that the translation of both axiom K and $Df\Diamond$ expresses evident properties of quantification, and so these formulas can be proved in *MSL* as logic theorems. On the other hand, the translation of axioms T and 4 are formulas needing certain hypothesis to be proved. *Which ones, we wonder?*

Let us see the translation of $\Box\varphi \rightarrow \varphi$

$$Trans(T)[u] := \forall v\big(Suv \rightarrow Trans(\varphi)[v]\big) \rightarrow Trans(\varphi)[u]$$

and of $\Box\varphi \rightarrow \Box\Box\varphi$

$$Trans(4)[u] := \forall v\big(Suv \rightarrow Trans(\varphi)[v]\big) \rightarrow \forall v\big(Suv \rightarrow \forall w\big(Svw \rightarrow Trans(\varphi)[w]\big)\big)$$

The many-sorted structures we shall use are obtained from the modal structures by adding a universe containing sets of states or worlds. The intuition behind this construction is that we wish to have all the sets that can be defined by modal formulas represented explicitly.

Given a modal structure $\mathcal{A}$

$$\mathcal{A} = \big\langle \mathbf{W}, \mathbf{R}, \big\langle P^{\mathcal{A}} \big\rangle_{P \in ATOM.PROP} \big\rangle$$

we build a frame by adding a universe of sets, $\mathbf{W}'$. We say that $\mathcal{AF}$ is a frame built on $\mathcal{A}$ iff

$$\mathcal{AF} = \big\langle \mathbf{W}, \mathbf{W}', \mathbf{R}, \big\langle P^{\mathcal{A}} \big\rangle_{P \in ATOM.PROP} \big\rangle$$

where:

1. $\mathbf{W}' \subseteq \wp\mathbf{W}$
2. $P^{\mathcal{A}} \in \mathbf{W}'$, for all $P \in ATOM.PROP$.

A very special class of frames is that of general structures built on modal structures. To build a general structure, we include in $\mathbf{W}'$ the set *DEF* of algebraically defined subsets of $\mathbf{W}$ (that is, $DEF \subseteq \mathbf{W}'$) plus all the singletons, and all the sets of points accessible from a given one. *DEF* contains the empty set, $\emptyset$, the whole $\mathbf{W}$, and the interpretations of all the atomic formulas. *DEF* is closed under Boolean operations

$$\forall T (T, S \in DEF \Rightarrow T \cup S \in DEF \& \mathbf{W} - T \in DEF)$$

and under the operation

$$\forall T \big(T \in DEF \Rightarrow Dom\big(R \cap (\mathbf{W} \times T)\big) \in DEF\big)$$

Out of *DEF*, but in $\mathbf{W}'$, we have the singletons of all elements of $\mathbf{W}$. $\mathbf{W}'$ also obeys this rule

$$\forall w \big(w \in \mathbf{W} \Rightarrow Rec\big(R \cap \big(\{w\} \times \mathbf{W}\big)\big) \in \mathbf{W}'\big)$$

Let us use $\mathfrak{F}$ to refer to the class of all frames built on modal structures and $\mathfrak{G}$ to refer to the class of all general structures built on modal structures defined as above.

We can prove that $\mathfrak{G}$ is contained in the class of frames built on modal structures whose universe $\mathbf{W}'$ contains the sets defined by modal formulas in its own modal structures. Also, it can be proved that a modal formula φ defines in a modal structure $\mathcal{A}$ the same set its translation defines in any general structure built on $\mathcal{A}$, $\mathcal{AG}$:

$$\mathcal{AG} = \big\langle \mathbf{W}, \mathbf{W}', \mathbf{R}, \langle P^{\mathcal{A}} \rangle_{P \in ATOM.PROP} \big\rangle$$

That is,

$$\mathcal{AG}\big(\lambda u Trans(\varphi)[u]\big) = \mathcal{A}(\varphi)$$

Finally, we prove that validity in *PML* is equivalent to validity of the universal closure of the translation in the class $\mathfrak{G}$. That is,

$$\models \varphi \text{ in } PML \quad \Longleftrightarrow \quad \models_{\mathfrak{G}} \forall u Trans(\varphi)[u]$$

Now the question is, *can we give axioms for* $\mathfrak{G}$?

The answer is yes, since we define the *MODO* theory having as axioms comprehension sentences for translations of modal formulas, for atomic formulas with equality, and for atoms using the binary predicate S, representing the accessibility relation, and the membership relation ε:

$$\forall \exists X \forall u (\varepsilon u X \leftrightarrow \varphi) \quad \text{where } \varphi \in Trans(PML) \cup I \cup \Sigma$$

I contains all the formulas of the form $u = v$, and Σ contains all the formulas of the form, Suv. We also add extensionality because we want to simulate second-order logic in many-sorted logic.

Now we can prove the following:

$$Mod(MODO) = \mathfrak{G}$$

Using these results, we obtain a representation theorem for the minimal modal logic, K:

$$\models \varphi \text{ in } PML \quad \Longleftrightarrow \quad MODO \models_{\mathfrak{F}} \forall u Trans(\varphi)[u] \text{ in } MSL$$

Remark 3 The representation theorem proves enumerability for the logic *PML*.

Now that we have a representation theorem for the logic K, we look forward a representation theorem for the logic $S4$. Let us use $MODO(S4)$ to refer to the many-sorted theory obtained by adding to *MODO* the 'second-order' abstract conditions for the modal formulas T and 4—call them $MS(T)$ and $MS(4)$.

$$MS(T) := \forall uY\big(\forall v(Suv \to \varepsilon_1 vY) \to \varepsilon_1 uY\big)$$
$$MS(4) := \forall uY\big(\forall v(Suv \to \varepsilon_1 vY) \to \forall v\big(Suv \to \forall w(Svw \to \varepsilon_1 wY)\big)\big)$$

Let $\mathfrak{D}$ denote the class of reflexive and transitive Kripke structures. The representation theorem for $S4$ has the form

$$\models_{\mathfrak{D}} \varphi \text{ in } PML \quad \Longleftrightarrow \quad MODO(S4) \models_{\mathfrak{F}} \forall u Trans(\varphi)[u] \text{ in } MSL$$

This result is obtained easily because, in the many-sorted calculus, we can prove that the usual first-order formulations of reflexivity and transitivity are equivalent to $MS(T)$ and $MS(4)$ modulo the *MODO* theory.

Remark 4 This situation is better than the one encountered in *PML* with the semantics of models. There, a reflexive Kripke structure is a model of T, a transitive one is a model of 4, etc. But there are irreflexive models of T and intransitive ones of 4. Thus, if we think that the modal axioms are trying to define their accessibility relation, this could be considered as a kind of failure. Instead, we can recourse to the so called semantics of frames for *PML*. I propose to go just to this environment (the general structures as defined here), which offers the characterization of the semantic properties of the accessibility relation, but without losing a calculus.

4.2 The Main Theorem

Now we leave the example for a moment and ask ourselves what else can be done. *Is the representation theorem our top (maximum) goal?*

When the *XL* logic under scrutiny has a concept of logical consequence, we may try to prove *the main theorem*; that is, that a consequence in *XL* is equivalent to the consequence of its translation in *MSL*, *modulo* the theory Δ.

Theorem 5 (Main Theorem) *There is a recursive set* $\Delta \subseteq Sent(L^*)$ *with* $\mathcal{S}^* \subseteq Mod(\Delta)$, *such that*

$$\Pi \models_{Str(XL)} \varphi \quad \textit{iff} \quad Trans(\Pi) \cup \Delta \models_{Str(MXL^*)} Trans(\varphi)$$

for all $\Pi \cup \{\varphi\} \subseteq Sent(XL)$.

To prove the main theorem, a reverse conversion of structures should be defined; our goal in defining it is to prove that starting from a model of Δ, say, $\mathcal{B}$, and formula φ,

$$\models_{Conv_2(\mathcal{B})} \varphi \quad \text{iff} \quad \models_{\mathcal{B}} \forall Trans(\varphi)$$

Remark 6 From theorem 5 it is possible to prove *Compactness* and *Löwenheim–Skolem* for *XL*. Therefore, the logic under investigation could have a strong complete calculus.

Example 7 (The Main Theorem in PML) In *PML* the reverse conversion is only the inverse of the direct conversion (we need only to erase the universe $\mathbf{W}'$), and we obtain easily the main theorem. By using it, we obtain the theorems of compactness and Löwenheim-Skolem for the logics K and $S4$ for free. The proof of the main theorem also uses the following crucial fact

$$MODO \vdash MS(T) \leftrightarrow Reflexivity$$

$$MODO \vdash MS(4) \leftrightarrow Transitivity$$

4.3 Deductive Correspondence

When the *XL* logic also has a deductive calculus, we can try to use the machinery of correspondence to prove, if possible, soundness and completeness for *XL*.

Since we already have the main theorem and completeness and soundness in *MSL*, we would like to complete this picture by proving the double arrow in the bottom line.

$$\begin{array}{ccc} Trans(\Pi) \cup \Delta \models_{Str(MSL)} Trans(\varphi) & \iff & \Pi \models_{Str(XL)} \varphi \\ & & \Updownarrow \\ Trans(\Pi) \cup \Delta \vdash_{MSL} Trans(\varphi) & \iff & \Pi \vdash_{XL} \varphi \end{array}$$

The left arrow is usually very easy because you can add the required axioms to Δ. In some cases it is better to use the canonical model construction.

Example 8 (Testing the Modal Calculus K and S4) In the case of the logic K, the left arrow is obtained easily because the translations of both K and $Df\Diamond$ express obvious properties of quantification. The rule of necessitation is also preserved under translations. As far as the logic $S4$ is concerned, using comprehension and $MS(T)$ as well as $MS(4)$, we obtain the desired translation of any given occurrence of T or 4 as theorems of $Trans(\Pi) \cup \Delta$.

In modal logic, given a logic, say B, there is the canonical model, $\mathcal{B}_B$, whose universe $\mathbf{W}$ contains all maximal B-consistent sets and whose accessibility relation is defined as

$$\big\{\langle x, y\rangle : \{\varphi : \Box\varphi \in x\} \in y\big\}$$

From this model, by direct conversion, we obtain the canonical general structure,

$$\mathcal{B}_B\mathfrak{G}.$$

For the logic K, this structure is not only a model of *MODO*, but we can also prove that the translation of a modal formula φ is true at a point t of the canonical general structure if and only if, in that point, the formula is in t

$$\models_{\mathcal{B}_B\mathfrak{G}[t]} Trans(\varphi)[u] \iff \varphi \in t$$

We can now prove

$$\models_{\mathcal{B}_B \mathfrak{S}} \forall u Trans(\varphi)[u] \quad \Longrightarrow \quad \vdash_K \varphi$$

Finally, we have

$$MODO \vdash_{MSL} \forall u Trans(\varphi)[u] \quad \Longleftrightarrow \quad \vdash_K \varphi$$

Using these results we finally arrive at the completeness and soundness of the minimal modal logic, K, of *PML*. We have avoided making a direct proof; we just dragged these properties from *MSL*.

For the logic $S4$ a similar result is also obtained:

$$MODO(S4) \vdash_{MSL} \forall u Trans(\varphi)[u] \quad \Longleftrightarrow \quad \vdash_{S4} \varphi$$

Thus, a soundness and completeness result is now easily accomplished.

Remark 9 If somebody ask us what have been achieved, the answer is that we now have a bunch of logics in our net.

References

1. Church, A.: A formulation of the simple theory of types. J. Symb. Log. **5**(2), 56–68 (1940)
2. Church, A.: Introduction to Mathematical Logic, vol. I. Princeton University Press, Princeton (1956)
3. Gabbay, D.M. (ed.) What Is a Logical System? In: What Is a Logical System? pp. 181–215. Oxford University Press, Oxford (1994)
4. Gabbay, D.M.: Labelled Deductive Systems, vol. 1. Oxford University Press, Oxford (1996)
5. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**(3), 159–166 (1949)
6. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**(2), 81–91 (1950)
7. Henkin, L.: Banishing the rule of substitution for functional variables. J. Symb. Log. **18**(3), 201–208 (1953)
8. Henkin, L.: Some notes on nominalism. J. Symb. Log. **18**(1), 19–29 (1953)
9. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**(2), 127–158 (1996)
10. Manzano, M.: Extensions of First Order Logic. Cambridge University Press, Cambridge (1996)
11. Martí-Oliet, N., Meseguer, J.: General logics and logical frameworks. In: What Is a Logical System? Oxford University Press, Oxford (1994)
12. Meseguer, J.: General logics. In: Ebbinghaus, et al. (eds.) Logic Colloquium'87, pp. 275–329. North-Holland, Amsterdam (1989)
13. Van Benthem, J.: Modal Logic and Classical Logic. Bibliopolis, Naples (1983)
14. Van Benthem, J.: Correspondence theory. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn., vol. 3, pp. 325–408. Kluwer Academic, Dordrecht (2001). 1st edn. 1986

M. Manzano (✉)
Department of Philosophy, University of Salamanca, Salamanca, Spain
e-mail: mara@usal.es

Henkin and Hybrid Logic

Patrick Blackburn, Antonia Huertas, María Manzano, and Klaus Frovin Jørgensen

Abstract Leon Henkin was not a modal logician, but there is a branch of modal logic that has been deeply influenced by his work. That branch is hybrid logic, a family of logics that extend orthodox modal logic with special propositional symbols (called nominals) that name worlds. This paper explains why Henkin's techniques are so important in hybrid logic. We do so by proving a completeness result for a hybrid type theory called HTT, probably the strongest hybrid logic that has yet been explored. Our completeness result builds on earlier work with a system called BHTT, or basic hybrid type theory, and draws heavily on Henkin's work. We prove our Lindenbaum Lemma using a Henkin-inspired strategy, witnessing $\Diamond$-prefixed expressions with nominals. Our use of general interpretations and the construction of the type hierarchy is (almost) pure Henkin. Finally, the generality of our completeness result is due to the first-order perspective, which lies at the heart of both Henkin's best known work and hybrid logic.

Keywords Hybrid logic · Modal logic · Higher-order logic · Rigidity · Henkin constants · Henkin models · Bounded fragment

1 Introduction

Leon Henkin was not a modal logician, but there is a branch of modal logic that has been deeply influenced by his work. That branch is hybrid logic, a family of logics that extends orthodox modal logic with special propositional symbols (called nominals) that name worlds. This paper explains why Henkin's influence on hybrid logic runs so deep, and we do so by proving a completeness result for a hybrid type theory that we call HTT. But before diving into higher-order logic, let us informally introduce the two central ideas of basic hybrid logic.

The first idea is to add special propositional symbols called nominals to an orthodox modal language and to insist that in every model, nominals are true at precisely one world. Nominals name worlds by being true there and there only. Consider the following example:

$$\neg\Diamond i.$$

Here i is a nominal (nominals are conventionally written i, j, and k). Suppose we are working in a model in which the nominal i names the world w. Then this expression will be true at w if and only if w is not accessible from itself via R, the usual modal accessibility relation between worlds.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_19

The second idea is to add modalities of the form $@_i$, where i can be any nominal. The intended semantics is written into the notation: a formula of the form $@_i\varphi$ is true at a world w if and only if φ is true at the world w' named by the nominal i. Note that a formula of the form $@_i\varphi$ is either true at all worlds or false at all worlds: if φ is true at the world named by i, then $@_i\varphi$ is true at all worlds w; otherwise, it is (everywhere) false. To put it another way, $@_i$ is a *rigidifying* operator. In any model, the world w where we evaluate $@_i\varphi$ is irrelevant: this expression always returns the truth value that φ has at the world named i.

The basic propositional hybrid language has been intensively investigated. It is decidable, indeed PSPACE-complete, just like orthodox propositional modal logic (see [2]), and it extends the expressive power of orthodox propositional modal logic (see [3] for a useful survey and [9] for a detailed account). Most relevantly for this paper, its completeness theory is well understood; simple and general results covering many important classes of models are known ([4], Chap. 7.3, and [7] are good starting points, and [9] is close to definitive). These results lift to first-order hybrid logic in full generality, and in [1] they were lifted to higher-order logic with the completeness proof for BHTT, basic hybrid type theory.

The BHTT system is an almost direct combination of Henkin-style type theory with basic propositional hybrid logic. But the "almost" is important since in a higher-order setting it is natural to interpret @ as a rigidifier that can be applied not merely to formulas, but to expressions of *arbitrary* type. That is, suppose α_a is an expression of type a. In BHTT, when we evaluate $@_i\alpha_a$ at a world w, it rigidly returns the value of α_a at the world named i. This interpretation of @ is conceptually and technically appealing, and we use @ as a general rigidifier in this paper too.

HTT, the higher-order logic we work with in this paper, is BHTT enriched with the $\downarrow$ binder. Consider again the expression $\neg\Diamond i$. When working with $\downarrow$, we are free to replace nominals with state variables (typically written s and t) and to use $\downarrow$ to bind the result. So we can form:

$$\downarrow s(\neg\Diamond s).$$

So syntactically, $\downarrow$ binds out nominals.[1] But what is its semantic effect?

The $\downarrow$ binder works locally: it binds state variables to the world of evaluation. That is, when we evaluate an expression of the form $\downarrow s\varphi$ at a world w, the state variable s is bound to w, and all occurrences of s within the scope of $\downarrow s$ are interpreted as names for w. Consider $\downarrow s(\neg\Diamond s)$ again. When we evaluate it at w, s is bound to w, and this means that the s in $\neg\Diamond s$ is to be interpreted as a name for w. Hence, $\neg\Diamond s$ is true at w if and only if w is not accessible from itself via R. So there is an important difference between $\neg\Diamond i$ and $\downarrow s(\neg\Diamond s)$. In any model, the nominal i is a *fixed* name for a world; hence, $\neg\Diamond i$ only tests for irreflexivity when evaluated at the unique world named i. But $\downarrow s(\neg\Diamond s)$ binds the state variable s to the world of evaluation; hence, it is an expression that tests for irreflexivity at *every* world in a model.

Basic propositional hybrid logic enriched with $\downarrow$ has also been intensively investigated. It is undecidable (see [5]), and elegant model-theoretically (see [2] and [9]). Moreover, its completeness theory is well understood, and we will build on what is known in this paper.

[1]Some authors bind nominals directly, forming expressions like $\downarrow i(\neg\Diamond i)$. There is nothing wrong with this, but it seems neater to draw a syntactic distinction between state variables (which are open to binding) and nominals (which are not).

We proceed as follows. In Sect. 2 we define the syntax and semantics of HTT; we illustrate a common pattern of interaction between @ and ↓ and define substitution in detail. In Sect. 3 we axiomatize HTT over the class of all models. We discuss three variant axiomatizations; thinking about their differences will help us see why (and where) Henkin's techniques are important in hybrid logic. In Sect. 4 we prove completeness, building on the earlier proof for BHTT. As we shall see, the Lindenbaum construction for BHTT does not need to be modified; ↓ is handled automatically. In Sect. 5 we lift a general result from propositional hybrid logic to HTT; we show that we can add certain hybrid theories as additional axioms and automatically gain completeness and show that these axioms are equivalent in expressive power to the bounded fragment of first-order logic. In Sect. 6 we conclude by asking where Henkin's influence is most important in hybrid logic. We answer the question by looking more closely at a key proof rule in one of our axiomatizations.

2 Syntax and Semantics of HTT

In this section we introduce the syntax and (standard and general) semantics for HTT. Our definitions are those given for BHTT in [1] extended with the clauses for ↓ and state variables (the nominal-like variables that ↓ binds). After our preliminary work, we discuss substitution and rigidity.

2.1 Syntax

Definition 1 (Syntax of HTT) Types. Let t and e be two fixed objects. We define the set TYPES of types of HTT to be

$$\mathsf{TYPES} ::= t \mid e \mid \langle a, b \rangle \quad \text{with } a, b \in \mathsf{TYPES} \text{ and } a \neq t.$$

Meaningful Expressions The set ME_a of *meaningful expressions of type a* consists of the basic and complex expressions of type a we now define.

Basic Expressions For each type $a \neq t$, we have a denumerably infinite set CON_a of *nonlogical constants* $c_{n,a}$, where n is a natural number. Constants of type t are *truth* and *falsity*, that is, $\mathsf{CON}_t = \{\top, \bot\}$, and we define CON to be $\bigcup_a \mathsf{CON}_a$. For each type $a \neq t$, we have a denumerably infinite set VAR_a of *variables* $v_{n,a}$, where n is a natural number, and we define VAR to be $\bigcup_a \mathsf{VAR}_a$. Finally, for type t, we have a denumerably infinite set NOM of nominals i_n and a denumerably infinite set SVAR of state variables s_n, where n is a natural number. We define HYB to be $\mathsf{NOM} \cup \mathsf{SVAR}$.

Summarizing, for each natural number n, we have:

$$i_n \in \mathsf{ME}_t \mid s_n \in \mathsf{ME}_t \mid c_{n,a} \in \mathsf{ME}_a \mid v_{n,a} \in \mathsf{ME}_a \quad \text{with } a \neq t.$$

Complex Expressions These are generated as follows:

$$(\gamma_{\langle b,a\rangle}\beta_b) \in \mathsf{ME}_a \mid (\lambda u_b \alpha_a) \in \mathsf{ME}_{\langle b,a\rangle} \mid (@_i \alpha_a) \in \mathsf{ME}_a \mid (@_s \alpha_a) \in \mathsf{ME}_a$$

$$\left\{\left(\alpha_a = \alpha'_a\right), (\neg\varphi_t), (\varphi_t \wedge \psi_t), (\forall u_b \varphi_t), (\Box\varphi_t), (\downarrow s \varphi_t)\right\} \subseteq \mathsf{ME}_t,$$

where $\alpha_a, \alpha'_a \in \mathsf{ME}_a$, $\beta_b \in \mathsf{ME}_b$, $\gamma_{\langle b,a\rangle} \in \mathsf{ME}_{\langle b,a\rangle}$, $u_b \in \mathsf{VAR}_b$, $i \in \mathsf{NOM}$, $s \in \mathsf{SVAR}$, and $\varphi_t, \psi_t \in \mathsf{ME}_t$. As this notation illustrates, we sometimes explicitly give the type of a meaningful expression (writing, for example, α_a as we just did) to emphasize that $\alpha \in \mathsf{ME}_a$. The remaining booleans, the quantifier $\exists$ and the diamond $\Diamond$, are defined in the familiar way. We routinely drop outermost brackets and drop others when this will not result in ambiguity. Given a set of expressions Δ, we define $\mathsf{CON}(\Delta)$, $\mathsf{NOM}(\Delta)$, $\mathsf{VAR}(\Delta)$, and $\mathsf{SVAR}(\Delta)$ to be (respectively) the sets of constants, nominals, variables, and state variables occurring in expressions in Δ. We often call expressions of type t formulas.

For those unfamiliar with propositional or first-order hybrid logic, the following point should be stressed: *nominals can occur in two distinct syntactic positions, and state variables can occur in three*. To give some simple examples, the following expressions are meaningful expressions of type t in which the nominal i and the state variable s occur in *formula position*:

$$i \qquad s \qquad i \vee \neg i \qquad \Box(s \to i) \to (\Box s \to \Box i) \qquad \Diamond\Diamond s \to \Diamond s.$$

The following are also meaningful expressions of type t, but here we see the nominal i and the state variable s also occurring in *operator position*, that is, in expressions of the forms $@_i$ and $@_s$, respectively:

$$@_i i \qquad @_i s \qquad @_s(i \vee \neg i) \qquad @_i(s \to i) \to (@_i s \to @_i i) \qquad @_s(\Diamond\Diamond s \to \Diamond s).$$

Finally, here are three examples in which state variables occur in *binding position*, that is, in patterns like $\downarrow s$ and $\downarrow t$:

$$\downarrow s(\neg\Diamond s) \qquad \downarrow s(\Diamond\Diamond s \to \Diamond s) \qquad \downarrow s\Box\Box\downarrow t\, @_s \Diamond t.$$

In these examples, all occurrences of the state variables s and t—whether in formula, operator, or binder position—have been bound by the $\downarrow$ binder. Now, there are two other binders in our language, namely the familiar $\forall$ and λ binders that bind ordinary variables, so before going further, let us be precise about the concepts of freedom and bondage for the three binders of HTT.

Definition 2 (Freedom and Bondage) Given a meaningful expression α, the set of *free variables* occurring in α_a (notation $\mathsf{FREE}(\alpha)$) is given by:

$$\mathsf{FREE}(v) = \{v\} \quad \text{for } v \in \mathsf{VAR}$$

$$\mathsf{FREE}(\tau) = \emptyset \quad \text{for } \tau \in \mathsf{CON} \cup \mathsf{NOM} \cup \mathsf{SVAR}$$

$$\mathsf{FREE}(\alpha = \beta) = \mathsf{FREE}(\alpha\beta) = \mathsf{FREE}(\alpha \wedge \beta) = \mathsf{FREE}(\alpha) \cup \mathsf{FREE}(\beta)$$

$$\mathsf{FREE}(\neg\alpha) = \mathsf{FREE}(\Box\alpha) = \mathsf{FREE}(\alpha)$$

$$\mathsf{FREE}(@_i\alpha) = \mathsf{FREE}(@_s\alpha) = \mathsf{FREE}(\downarrow s\alpha) = \mathsf{FREE}(\alpha)$$

$$\mathsf{FREE}(\forall u\alpha) = \mathsf{FREE}(\lambda u\alpha) = \mathsf{FREE}(\alpha)\backslash\{u\}.$$

Given a meaningful expression α, the set of *free state variables* occurring in α_a (notation $\mathsf{SFREE}(\alpha)$) is defined as follows:

$$\mathsf{SFREE}(s) = \{s\} \quad \text{for } s \in \mathsf{SVAR}$$
$$\mathsf{SFREE}(\tau) = \emptyset \quad \text{for } \tau \in \mathsf{CON} \cup \mathsf{NOM} \cup \mathsf{VAR}$$
$$\mathsf{SFREE}(\alpha = \beta) = \mathsf{SFREE}(\alpha\beta) = \mathsf{SFREE}(\alpha \wedge \beta) = \mathsf{SFREE}(\alpha) \cup \mathsf{SFREE}(\beta)$$
$$\mathsf{SFREE}(\neg\alpha) = \mathsf{SFREE}(\Box\alpha) = \mathsf{SFREE}(\alpha)$$
$$\mathsf{SFREE}(\forall u\alpha) = \mathsf{SFREE}(\lambda u\alpha) = \mathsf{SFREE}(\alpha)$$
$$\mathsf{SFREE}(@_i\alpha) = \mathsf{SFREE}(\alpha)$$
$$\mathsf{SFREE}(@_s\alpha) = \mathsf{SFREE}(\alpha) \cup \{s\}$$
$$\mathsf{SFREE}(\downarrow s\alpha) = \mathsf{SFREE}(\alpha)\backslash\{s\}.$$

If a variable v is free in a meaningful expression α, then it is bound in both $\forall v\alpha$ and $\lambda v\alpha$. Similarly, if a state variable s is free in a meaningful expression α, then it is bound in $\downarrow s\alpha$. A meaningful expression α_t of type t is called a *sentence* if and only if all the variables and state variables it contains are bound, that is, if and only if $\mathsf{FREE}(\alpha_t) \cup \mathsf{SFREE}(\alpha_t) = \emptyset$.

Summing up, variable binding and state variable binding are distinct. Ordinary variables can only be bound by $\forall$ or λ, whereas state variables can only be bound by the $\downarrow$ binder. Moreover, it should be clear (at least syntactically) that state variables are essentially nominals open to $\downarrow$ binding, so we could have stated the syntactic clauses for nominals and state variables more compactly by stipulating that for all $h_n \in \mathsf{HYB}$, $h_n \in \mathsf{ME}_t$, and $@_h\alpha_a \in \mathsf{ME}_a$.

Nominals and $@_i$ operators are the tools characteristic of basic hybrid logic, and the BHTT system defined in [1] is built over them. The $\downarrow$ binder takes us to a richer hybrid logic, and the HTT of this paper differs from BHTT precisely by its addition. Nominals and expressions of the form $@_i\alpha_a$ (where i is any nominal) will play the central role in the completeness result for HTT: models will be built Henkin-style out of equivalence classes of witness nominals, and expressions of the form $@_i\alpha_a$ will specify how information is to be distributed. Indeed, as far as our fundamental completeness result is concerned, state variables and the $\downarrow$ binder play a rather passive role: HTT can be axiomatized by adding a single axiom schema to the axiomatization of BHTT. The $\downarrow$ binder will make its presence felt when we strengthen our fundamental completeness result.

2.2 Semantics

Definition 3 (HTT Models) A *standard structure* (or *standard model*) for HTT is a pair $\mathscr{M} = \langle \mathscr{S}, \mathsf{F}\rangle$ such that

1. $\mathscr{S} = \langle \langle \mathsf{D}_a \rangle_{a \in \mathsf{TYPES}}, W, R\rangle$ is a *standard skeleton*, where:

a. $\langle \mathsf{D}_a \rangle_{a \in \mathsf{TYPES}}$, the *standard type hierarchy*, is defined as follows:

$$\mathsf{D}_t = \{T, F\} \text{ is the set of truth values,}$$

$$\mathsf{D}_e \neq \emptyset \text{ is the set of individuals,}$$

$$\mathsf{D}_{\langle a,b \rangle} = \mathsf{D}_b^{\mathsf{D}_a} \text{ is the set of all functions from } \mathsf{D}_a \text{ into } \mathsf{D}_b$$
$$\text{for } a, b \in \mathsf{TYPES},\ a \neq t.$$

b. $W \neq \emptyset$ is the set of worlds.
c. $R \subseteq W \times W$ is the accessibility relation.
d. The pair $\langle W, R \rangle$ is called a *frame*; when working with a given model $\mathscr{M}$, we sometimes talk about its *underlying frame*.

2. The *denotation function* F assigns to each nonlogical constant a function from W to elements of appropriate type and assigns to each nominal a function from W to the set of truth values. More precisely:

   a. For any constant $c_{n,a}$, we define $\mathsf{F}(c_{n,a}) : W \longrightarrow \mathsf{D}_a$. Moreover, $(\mathsf{F}(\top))(w) = T$ and $(\mathsf{F}(\bot))(w) = F$ for any world $w \in W$.
   b. $\mathsf{F}(i) : W \longrightarrow \{T, F\}$ such that $(\mathsf{F}(i))(v) = T$ for a unique $v \in W$. To simplify notation, we sometimes write $\mathsf{F}(i) = \{v\}$ and say that v is the denotation of i or the world named by i.

Note that we are working with a constant domain semantics: we have a fixed type hierarchy $\langle \mathsf{D}_a \rangle_a \in \mathsf{TYPES}$, and F interprets all constants on this fixed domain. Furthermore, recall that a central idea of propositional hybrid logic is to use propositions as names. Because nominals are true at precisely one world in any model, they can be thought of as naming that world by being true there and there only. Our interpretation of nominals in type theory imports this basic idea to the richer setting: the interpretational constraint ensures that nominals act as world names.

We interpret ordinary variables and state variables via assignments:

Definition 4 An *assignment* g is a function with domain $\mathsf{VAR} \cup \mathsf{SVAR}$ such that for any variable $v_{n,a}$, we have $g(v_{n,a}) \in \mathsf{D}_a$, and for any state variable s, we have $g(s) \in W$.

An assignment g' is a v-variant of g if and only if it coincides with g on all values except, perhaps, on the value assigned to the variable v. We use g_v^θ to denote the v-variant of g whose value for $v \in \mathsf{VAR}_a$ is $\theta \in \mathsf{D}_a$. Similarly, g' is an s-variant of g if and only if it coincides with g on all values except, perhaps, on the value assigned to the state variable s. We use g_s^w to denote the s-variant of g whose value for s is $w \in W$.

There are two things to note about this definition. First, it treats ordinary variables in the manner familiar from Henkin's work. Second, it treats state variables as syntactic entities that name worlds. An ordinary nominal is true at a unique world. An assignment maps a state variables to a unique world. State variables are essentially nominals open to binding by the $\downarrow$ binder.

Definition 5 (HTT Interpretations) A *standard interpretation* is a pair $\langle \mathscr{M}, g \rangle$, where $\mathscr{M}$ is a standard structure for HTT, and g is a variable assignment on $\mathscr{M}$. Given a standard structure $\mathscr{M} = \langle\langle\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}, W, R\rangle, \mathsf{F}\rangle$ and an assignment g, we recursively define, for any meaningful expression α, the standard interpretation of α with respect to the model $\mathscr{M}$ and the assignment g, at the world w, denoted by $[\![\alpha]\!]^{\mathscr{M},w,g}$, as follows:

1. $[\![\tau]\!]^{\mathscr{M},w,g} = (\mathsf{F}(\tau))(w)$ for $\tau \in \mathsf{CON} \cup \mathsf{NOM}$.
2. $[\![v_{n,a}]\!]^{\mathscr{M},w,g} = g(v_{n,a})$ for $v_{n,a} \in \mathsf{VAR}_a$.
3. $[\![s]\!]^{\mathscr{M},w,g} = T$ if $g(s) = w$ and F if $g(s) \neq w$ for $s \in \mathsf{SVAR}$.
4. $[\![\lambda u_b \alpha_a]\!]^{\mathscr{M},w,g} = f$, where, for any $\theta \in \mathsf{D}_b$, $f : \mathsf{D}_b \longrightarrow \mathsf{D}_a$ is the function defined by $f(\theta) = [\![\alpha_a]\!]^{\mathscr{M},w,g^{\theta}_{u_b}}$.
5. $[\![\alpha_{\langle b,a\rangle}\beta_b]\!]^{\mathscr{M},w,g} = [\![\alpha_{\langle b,a\rangle}]\!]^{\mathscr{M},w,g}([\![\beta_b]\!]^{\mathscr{M},w,g})$.
6. $[\![\alpha_a = \beta_a]\!]^{\mathscr{M},w,g} = T$ iff $[\![\alpha]\!]^{\mathscr{M},w,g} = [\![\beta]\!]^{\mathscr{M},w,g}$.
7. $[\![\neg\varphi_t]\!]^{\mathscr{M},w,g} = T$ iff $[\![\varphi_t]\!]^{\mathscr{M},w,g} = F$.
8. $[\![\varphi_t \wedge \psi_t]\!]^{\mathscr{M},w,g} = T$ iff $[\![\varphi_t]\!]^{\mathscr{M},w,g} = T$ and $[\![\psi_t]\!]^{\mathscr{M},w,g} = T$.
9. $[\![\forall x_a \varphi_t]\!]^{\mathscr{M},w,g} = T$ iff for all $\theta \in \mathsf{D}_a$, $[\![\varphi]\!]^{\mathscr{M},w,g^{\theta}_{x_a}} = T$.
10. $[\![\Box\varphi_t]\!]^{\mathscr{M},w,g} = T$ iff for all $v \in W$ such that $\langle w, v\rangle \in R$, $[\![\varphi_t]\!]^{\mathscr{M},v,g} = T$.
11. $[\![@_i \alpha_a]\!]^{\mathscr{M},w,g} = [\![\alpha_a]\!]^{\mathscr{M},v,g}$, where $\{v\} = \mathsf{F}(i)$.
12. $[\![@_s \alpha_a]\!]^{\mathscr{M},w,g} = [\![\alpha_a]\!]^{\mathscr{M},v,g}$, where $v = g(s)$.
13. $[\![\downarrow\! s\varphi_t]\!]^{\mathscr{M},w,g} = T$ iff $[\![\varphi_t]\!]^{\mathscr{M},w,g^{w}_{s}} = T$.

Some remarks on the clauses covering nominal and state variables. Consider Clause 1 when τ is a nominal. This covers occurrences of i in formula position, and in such cases, i should be true at precisely the world it denotes. Because of the constraint on the interpretation of nominals, this is what Clause 1 gives us. Next consider Clause 11, which covers occurrences of i in operator position. We want $@_i\alpha$ to be an expression (of the same type as α) that rigidly yields the value of α at the world named by i. Clause 11 gives us this.

Now for state variables. Clause 3 covers occurrences of s in formula position. State variables are assigned a unique world, the world they name. As they are expression of type t, they should be true at that world and at that world only, which is what Clause 3 insists upon. Clause 12 deals with s in operator position and says that $@_s\alpha_a$ is true at a world w if and only if α_a is true at v, the world named by s. This mirrors the clause for nominals in operator position, and indeed we could collapse Clauses 11 and 12 together by stating that for all $h \in \mathsf{HYB}$, $[\![@_h\alpha_a]\!]^{\mathscr{M},w,g}$ is $[\![\alpha_a]\!]^{\mathscr{M},v,g}$, where v is the world named by h.

But state variables were only introduced to support the $\downarrow$ binder that distinguishes HTT from BHTT, so Clause 13 is the real novelty. This says that $\downarrow\! s\varphi$ binds s to the world of evaluation and that all occurrences of s in φ within its scope are to be interpreted as names for this world. So to speak, $\downarrow\! s$ creates a temporary name s for the world of evaluation. Consider again the expression $\downarrow\! s(\neg\Diamond s)$ we discussed in the introduction. As we said there, $\downarrow$ binds s to the world of evaluation. The semantics just defined guarantees that if we evaluate this expression at any world w in any model, it will be true precisely when w is not accessible (via R) from itself.

Let us look at a second example, which illustrates a common theme: $\downarrow$ binding a state variable occurring in an @ operator under its scope. Let Woman be an expression of type $\langle e, t\rangle$ that picks out the women in each possible world, and let Potus be an expression of type e that picks out the President of the United States in each possible world. Suppose that an American voter murmurs: *The President of the United Sates might be a woman.*

We shall consider three readings of this sentence. The first is that the voter is thinking about a possible world in which Barack Obama is a woman:

$$\downarrow s\Diamond\big(\mathsf{Woman}(@_s\mathsf{Potus})\big).$$

This expression is correctly typed: Potus is of type e, hence so is $@_s\mathsf{Potus}$, hence Woman can be applied to it yielding the type t expression $\mathsf{Woman}(@_s\mathsf{Potus})$. Prefixing this with $\downarrow s\Diamond$ yields the above sentence. This expresses the first reading of the utterance: s is bound to the utterance world, which means that the embedded expression $@_s\mathsf{Potus}$ must be evaluated at the utterance world too, yielding the value Obama. So the voter is musing about what it might be like in a possible world in which Obama is a woman.

On the other hand, maybe our voter simply meant this:

$$\Diamond(\mathsf{Woman\ Potus}).$$

That is, perhaps our voter merely meant that there was some possible world in which the president in that world (whoever she may be) is a woman. Well, perhaps. Indeed, if our voter had lived in (say) 1950, this reading may have been all that the voter considered possible.

But it is also possible (indeed, nowadays more likely) that our voter is musing about the growing number of powerful American woman politicians and means that: *The President of the United Sates might be a woman existing in the utterance world*. We can express this by

$$\downarrow s\Diamond\big((@_s\mathsf{Woman})\ \mathsf{Potus}\big).$$

This is also correctly typed: Woman is of type $\langle e, t\rangle$, hence so is $@_s\mathsf{Woman}$, and so this can be applied to the type e expression Potus. This yields an expression of type t, and, as before, prefixing $\downarrow s\Diamond$ gives us a sentence. But note the difference: the bound state variable s forces Woman to be evaluated at the utterance world, hence the sentence is true if and only if there is an individual in some accessible possible world who is president there and a woman in the utterance world.

This example illustrates an important point: $@$ and $\downarrow$ are a powerful team. The $@$ operator is a rigidifying operator for all types: it universalizes a local value. The $\downarrow$ binder enables us to force evaluation of state variable at the current world: it is a *localizing* binder. And when, as in the Potus example, an occurrence of $\downarrow s$ is used to "store" a value for s which is later "retrieved" by occurrences of $@_s$ under its scope, we are able to shift the evaluation of embedded expressions to the world named s and hence to draw interesting distinctions.

2.3 General Semantics

The standard semantics for higher-order logic is strong: if we define validity as truth in all standard structures, then the set of validities cannot be axiomatized. In 1950 Henkin proposed a weaker notion of interpretation for higher-order logic (see Henkin [13, 14]). As he showed, defining validity with respect to *general interpretations* lowers the logical complexity of validity (since there are more generalized structures than standard ones, it becomes easier to falsify a formula, so there are fewer validities), and this new notion of validity can be axiomatized in a natural way. We follow Henkin's approach and prove our completeness results with respect to general interpretations.

Definition 6 (HTT Skeletons and Structures) A *type hierarchy* is a family $\langle \mathsf{D}_a \rangle_{a \in \mathsf{TYPES}}$ of sets defined recursively as follows:

$$\mathsf{D}_e \neq \emptyset,$$
$$\mathsf{D}_t = \{T, F\},$$
$$\mathsf{D}_{\langle a,b \rangle} \subseteq \mathsf{D}_b^{\mathsf{D}_a} \quad \text{for } a, b \in \mathsf{TYPES},\ a \neq t.$$

A *skeleton* $\mathscr{S} = \langle \langle \mathsf{D}_a \rangle_{a \in \mathsf{TYPES}}, W, R \rangle$ is a triple satisfying all the conditions of a standard skeleton except that $\langle \mathsf{D}_a \rangle_{a \in \mathsf{TYPES}}$ is a (not necessarily standard) type hierarchy. A *structure* (or *model)* is a pair $\mathscr{M} = \langle \mathscr{S}, \mathsf{F} \rangle$ where $\mathscr{S}$ is a skeleton and F is a denotation function.

This definition encapsulates the idea that we do not need all the functions from D_a to D_b to interpret expressions of type $\langle a, b \rangle$. However, we do need enough functions to interpret all the expressions of our language, which motivates the following:

Definition 7 (General Interpretation) A *general interpretation* is a pair $\langle \mathscr{M}, g \rangle$ where $\mathscr{M}$ is a structure, g a variable assignment, and for any meaningful expression in ME_a, its interpretation (as given by Definition 5) is in D_a.

That is, from now on, given a (not necessarily standard) model $\mathscr{M}$, an assignment g, and an expression α, we will interpret α in $\mathscr{M}$ using the clauses given in Definition 5.

We can now define consequence and validity. Note that these definitions really do generalize the standard ones since every standard interpretation is a generalized interpretation (but not conversely).

Definition 8 (Consequence and Validity) Let $\Gamma \cup \{\varphi\} \subseteq \mathsf{ME}_t$, and $\mathscr{M}$ be a structure. We define consequence and validity as follows:

Consequence: We say that φ is a consequence of Γ, written $\Gamma \models \varphi$, if and only if for all general interpretations $\langle \mathscr{M}, g \rangle$ and all $w \in W$, if $[\![\gamma]\!]^{\mathscr{M},w,g} = T$ for all $\gamma \in \Gamma$, then $[\![\varphi]\!]^{\mathscr{M},w,g} = T$.

Validity: We say that φ is valid, written $\models \varphi$, if and only if φ is a consequence of the empty set (that is $\emptyset \models \varphi$).

A useful way of thinking about generalized interpretation is as a mechanism that reduces higher-order logic to first-order logic, or (perhaps better) as a mechanism that picks out the higher-order validities that are essentially first-order from the rich space of standard higher-order validities; for useful discussions, see [16] and [15]. We mention this because we are going to use Henkin's method of constants to prove a completeness result, Theorem 7, that covers a wide range of frame classes. From an orthodox modal perspective, this result is atypically general. But when viewed from the first-order perspective that underpins Henkin's work, its generality is natural since, as we shall see in Sect. 5, the hybrid machinery used in this paper is essentially first-order. Indeed, the pure nominal-free sentential fragment we shall discuss there is essentially hybrid notation for the bounded fragment of the first-order language of frames.

2.4 Substitution and Rigidity

To conclude this section, some syntactic lemmas concerning substitution and rigidity. Our first lemma is Lemma 9 from [1] extended to cover state variables. It is proved by induction on the structure of terms, and we leave it to the reader. The inductive steps for λ and $\forall$ can be found in [1].

Lemma 1 (Agreement for Variables and State Variables) *Let g and h be assignments that agree on the values assigned to the free variables and state variables of α; that is, f and g agree on the values they assign to all the elements of* $\mathsf{FREE}(\alpha_a) \cup \mathsf{SFREE}(\alpha_a)$. *Let $\langle \mathscr{M}, g\rangle$ and $\langle \mathscr{M}, h\rangle$ be general interpretations. Then for any world w, we have that* $[\![\alpha_a]\!]^{\mathscr{M},w,g} = [\![\alpha_a]\!]^{\mathscr{M},w,h}$.

We now define substitution. We first deal with substitution for state variables. This is simpler than ordinary variable substitution since the only expressions substitutable for a free state variable are nominals and state variables, that is, elements h of HYB. Any such h is a basic expression, and since neither nominals nor state variables can be bound by λ or $\forall$, we do not have to worry about accidental binding from these sources. So we need merely specify how any $h \in \mathsf{HYB}$ should be substituted into formula, operator, and binding positions.

Definition 9 (State Variable Substitution) Let $h \in \mathsf{HYB}$. We define, for all $\alpha_a \in \mathsf{ME}_a$, the *substitution of h for a state variable s in α_a*, written $\alpha_a(\frac{h}{s})$, as follows:

1. $\tau\left(\frac{h}{s}\right) := \tau$ for $\tau \in \mathsf{CON} \cup \mathsf{VAR} \cup \mathsf{NOM}$.
2. $s'\left(\frac{h}{s}\right) := \begin{cases} h & \text{if } s' = s, \\ s' & \text{if } s' \neq s. \end{cases}$
3. $(\lambda u_p \beta_b)\left(\frac{h}{s}\right) := \lambda u_p\left(\beta_b\left(\frac{h}{s}\right)\right)$.
4. $(\beta_{\langle b,a\rangle}\delta_b)\left(\frac{h}{s}\right) := \beta_{\langle b,a\rangle}\left(\frac{h}{s}\right)\delta_b\left(\frac{h}{s}\right) \Big| (\beta_b = \delta_b)\left(\frac{h}{s}\right) := \beta_b\left(\frac{h}{s}\right) = \delta_b\left(\frac{h}{s}\right)$.
5. $(\neg\varphi)\left(\frac{h}{s}\right) := \neg\left(\varphi\left(\frac{h}{s}\right)\right) \Big| (\varphi \wedge \psi)\left(\frac{h}{s}\right) := \varphi\left(\frac{h}{s}\right) \wedge \psi\left(\frac{h}{s}\right)$.
6. $(\forall u_p \beta_b)\left(\frac{h}{s}\right) := \forall u_p\left(\beta_b\left(\frac{h}{s}\right)\right)$.

7. $(\Box\psi)\left(\frac{h}{s}\right) := \Box\left(\psi\left(\frac{h}{s}\right)\right)$.

8. $(@_i\beta_b)\left(\frac{h}{s}\right) := @_i\left(\beta_b\left(\frac{h}{s}\right)\right)$.

9. $(@_{s'}\beta_b)\left(\frac{h}{s}\right) := \begin{cases} @_h(\beta_b(\frac{h}{s})) & \text{if } s' = s, \\ @_{s'}(\beta_b(\frac{h}{s})) & \text{if } s' \neq s. \end{cases}$

10. $\left(\downarrow s'\beta_b\right)\left(\frac{h}{s}\right) := \begin{cases} \downarrow s'\beta_b & \text{if } s \notin \mathsf{SFREE}(\downarrow s'\beta_b), \\ \downarrow s'\beta_b & \text{if } s \in \mathsf{SFREE}(\downarrow s'\beta_b) \text{ and } h = s', \\ \downarrow s'(\beta_b(\frac{h}{s})) & \text{if } s \in \mathsf{SFREE}(\downarrow s'\beta_b) \text{ and } h \neq s'. \end{cases}$

With state variable substitution $(\frac{h}{s})$ at our disposal, we can define ordinary variable substitution $(\frac{\gamma_c}{v_c})$, the substitution of a (possibly complex) expression γ_c for a free variable of type c. Since γ_c may be complex, accidental binding is an issue. However, it is a well understood issue. The following definition is (for the most part) standard: it uses the usual type-theoretic definitions that prevent accidental binding of ordinary variables by λ and ∀. Only the ↓ clause requires comment. As we know, we can freely substitute ordinary variables under the scope of the ↓ binder; accidental binding of ordinary variables in γ_c by ↓ is impossible. But γ_c may contain free occurrences of the state variable s, and we must prevent $\downarrow s$ from accidentally binding these. But this is easily done: we need merely make use of state variable substitution $(\frac{h}{s})$ as just defined.

Definition 10 (Variable Substitution) For all $\alpha_a \in \mathsf{ME}_a$, the *substitution of* γ_c *for a variable* v_c *in* α_a, written $\alpha_a(\frac{\gamma_c}{v_c})$, is defined as follows:

1. $\tau\left(\frac{\gamma_c}{v_c}\right) := \tau \quad \text{for } \tau \in \mathsf{CON} \cup \mathsf{NOM} \cup \mathsf{SVAR}.$

2. $v_a\left(\frac{\gamma_c}{v_c}\right) := \begin{cases} \gamma_c & \text{if } v_a = v_c, \\ v_a & \text{if } v_a \neq v_c. \end{cases}$

3. $(\lambda u_p\beta_b)\left(\frac{\gamma_c}{v_c}\right) := \begin{cases} \lambda u_p\beta_b & \text{if } v_c \notin \mathsf{FREE}(\lambda u_p\beta_b), \\ \lambda u_p(\beta_b(\frac{\gamma_c}{v_c})) & \text{if } v_c \in \mathsf{FREE}(\lambda u_p\beta_b),\ u_p \notin \mathsf{FREE}(\gamma_c), \\ (\lambda x_p(\beta_b\frac{x_p}{u_p}))(\frac{\gamma_c}{v_c}) & \text{if } v_c \in \mathsf{FREE}(\lambda u_p\beta_b), u_p \in \mathsf{FREE}(\gamma_c), \\ & x_p \text{ new.} \end{cases}$

4. $\left(\beta_{\langle b,a\rangle}\delta_b\right)\left(\frac{\gamma_c}{v_c}\right) := \beta_{\langle b,a\rangle}\left(\frac{\gamma_c}{v_c}\right)\delta_b\left(\frac{\gamma_c}{v_c}\right) \Bigg| \left(\beta_b = \delta_b\right)\left(\frac{\gamma_c}{v_c}\right) := \beta_b\left(\frac{\gamma_c}{v_c}\right) = \delta_b\left(\frac{\gamma_c}{v_c}\right).$

5. $(\neg\varphi)\left(\frac{\gamma_c}{v_c}\right) := \neg\left(\varphi\left(\frac{\gamma_c}{v_c}\right)\right) \Bigg| (\varphi \wedge \psi)\left(\frac{\gamma_c}{v_c}\right) := \varphi\left(\frac{\gamma_c}{v_c}\right) \wedge \psi\left(\frac{\gamma_c}{v_c}\right).$

6. $(\forall u_p\psi)\left(\frac{\gamma_c}{v_c}\right) := \begin{cases} \forall u_p\psi & \text{if } v_c \notin \mathsf{FREE}(\forall u_p\psi), \\ \forall u_p(\psi(\frac{\gamma_c}{v_c})) & \text{if } v_c \in \mathsf{FREE}(\forall u_p\psi),\ u_p \notin \mathsf{FREE}(\gamma_c), \\ (\forall x_p(\psi\frac{x_p}{u_p}))(\frac{\gamma_c}{v_c}) & \text{if } v_c \in \mathsf{FREE}(\forall u_p\psi),\ u_p \in \mathsf{FREE}(\gamma_c), \\ & x_p \text{ new.} \end{cases}$

7. $(\Box\psi)\left(\frac{\gamma_c}{v_c}\right) := \Box\left(\psi\left(\frac{\gamma_c}{v_c}\right)\right)$.

8. $(@_i\beta_b)\left(\frac{\gamma_c}{v_c}\right) := @_i\left(\beta_b\left(\frac{\gamma_c}{v_c}\right)\right)$.

9. $(@_s\beta_b)\left(\frac{\gamma_c}{v_c}\right) := @_s\left(\beta_b\left(\frac{\gamma_c}{v_c}\right)\right)$.

10. $(\downarrow s\psi)\left(\frac{\gamma_c}{v_c}\right) := \begin{cases} \downarrow s\psi & \text{if } v_c \notin \mathsf{FREE}(\downarrow s\psi), \\ \downarrow s\psi(\frac{\gamma_c}{v_c}) & \text{if } v_c \in \mathsf{FREE}(\downarrow s\psi),\ s \notin \mathsf{SFREE}(\gamma_c), \\ (\downarrow t(\psi\frac{t}{s}))(\frac{\gamma_c}{v_c}) & \text{if } v_c \in \mathsf{FREE}(\downarrow s\psi),\ s \in \mathsf{SFREE}(\gamma_c),\ t \text{ new.} \end{cases}$

We now define *rigid expressions*. These are expressions that have the same value at all worlds. They play an important role in our axiomatization, and equivalence classes of rigid expressions are the Lego bricks of the Henkin-style type hierarchy construction we use in the completeness proof.

The following definition is Definition 11 of [1] extended to cover state variables in both formula and operator position.

Definition 11 (Rigid Expressions) The set RIGIDS, consisting of *rigid meaningful expressions*, is defined as follows:

$$\mathsf{RIGIDS} ::= \bot \mid \top \mid h \mid v_a \mid @_h\theta_a \mid \lambda v_b\alpha_a \mid \gamma_{\langle b,a\rangle}\beta_b \mid \alpha_b = \beta_b \mid \neg\varphi_t \mid \varphi_t \wedge \psi_t \mid \forall v_a\varphi_t,$$

where $h \in \mathsf{HYB}$, $\theta_a \in \mathsf{ME}_a$, and $\alpha_a, \beta_b, \gamma_{\langle b,a\rangle}, \varphi_t, \psi_t \in \mathsf{RIGIDS}$. We say that $\alpha \in \mathsf{RIGIDS}_a$ if α is rigid and of type a, that is, if $\alpha \in \mathsf{RIGIDS} \cap \mathsf{ME}_a$.

Unsurprisingly, $\downarrow$ is conspicuous by its absence since $\downarrow$ does not rigidify. As a simple example, consider the sentence $\downarrow s(\Diamond s)$. In any model containing both reflexive and irreflexive worlds, this sentence is true at all the reflexive worlds and false at the rest.

Lemma 2 *Let* $\langle \mathscr{M}, g\rangle$ *be a general interpretation. Then for any* $\gamma \in \mathsf{RIGIDS}$, *we have that* $[\![\gamma]\!]^{\mathscr{M},w,g} = [\![\gamma]\!]^{\mathscr{M},v,g}$ *for all* $w, v \in W$.

Proof By induction on the construction of rigid expressions. The steps for $\lambda v_b\alpha_a$ and $\forall v_a\varphi$ can be found in the proof of Lemma 12 in [1]. □

Rigid expressions behave straightforwardly with respect to substitution:

Lemma 3 (Rigid Substitution) *Let* $\langle \mathscr{M}, g\rangle$ *be a general interpretation. Then for all worlds* w, *all* $\alpha_a \in \mathsf{ME}_a$, *all* $h \in \mathsf{HYB}$, *and all state variables* s,

$$\left[\!\!\left[\alpha_a\left(\frac{h}{s}\right)\right]\!\!\right]^{\mathscr{M},w,g} = [\![\alpha_a]\!]^{\mathscr{M},w,g^{\overline{h}}_s},$$

where $\overline{h}$ *is an abbreviation for* $[\![h]\!]^{\mathscr{M},w,g}$, *that is, it is the world named by* h.

Furthermore, for all worlds w, all $\alpha_a \in \mathsf{ME}_a$, all meaningful expressions γ_c of type c, and all variables $v_c \in \mathsf{VAR}_c$,

$$\left[\!\!\left[\alpha_a\left(\frac{\gamma_c}{v_c}\right)\right]\!\!\right]^{\mathscr{M},w,g} = [\![\alpha_a]\!]^{\mathscr{M},w,g\frac{\overline{\gamma_c}}{v_c}},$$

where $\overline{\gamma_c}$ is an abbreviation for $[\![\gamma_c]\!]^{\mathscr{M},w,g}$.

Proof By induction on the construction of expressions. Use Lemma 1. □

3 Axiomatizing HTT

We shall now axiomatize HTT by adding a single axiom schema to the BHTT axiomatization of [1]. We call this axiomatization K1. At the end of this section we note two more axiomatizations, K2 and K3.[2] The differences between these systems are unimportant as far as our technical results are concerned, but at the end of the paper we will discuss what K2 tells us about Henkin's influence in hybrid logic.

3.1 Axioms

As axioms we take all HTT instances of propositional tautologies together with all HTT instances of the following schemas; we use h and h' as metavariables over elements of HYB.

1. *Distributivity schemas*:
   a. $\Box$-*distributivity*: $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
   b. @-*distributivity*: $\vdash @_h(\varphi \to \psi) \to (@_h\varphi \to @_h\psi)$.
   c. $\forall$-*distributivity*: $\vdash \forall x_b(\varphi \to \psi) \to (\forall x_b\varphi \to \forall x_b\psi)$.
2. *Quantifier schemas*:
   a. $\forall$-*elimination*: For β_b rigid, $\vdash \forall x_b\varphi \to \varphi(\frac{\beta_b}{x_b})$.
   b. *Vacuous*: $\vdash \varphi \to \forall y_a\varphi$, where y_a does not occur free in φ.
3. *Equality schemas*:
   a. *Reflexivity*: $\vdash \alpha_a = \alpha_a$.
   b. *Substitution*: For α_a, β_a rigid, $\vdash \alpha_a = \beta_a \to (\delta_c(\frac{\alpha_a}{v_a}) = \delta_c(\frac{\beta_a}{v_a}))$.
4. *Functional schemas*:
   a. *Extensionality*: $\vdash \forall v_b(\gamma_{\langle b,a\rangle} v_b = \delta_{\langle b,a\rangle} v_b) \to \gamma_{\langle b,a\rangle} = \delta_{\langle b,a\rangle}$, where v_b does not occur free in $\gamma_{\langle b,a\rangle}$ or $\delta_{\langle b,a\rangle}$.

[2] In modal logic, the basic proof system is usually called K in honor of Saul Kripke. The three axiomatizations considered here are alternative ways of providing a basic proof system for HTT.

b. *β-conversion*: For rigid β_b, $\vdash (\lambda x_b \alpha_a)\beta_b = \alpha_a(\frac{\beta_b}{x_b})$.
c. *η-conversion*: $\vdash (\lambda x_b \gamma_{\langle b,a\rangle} x_b) = \gamma_{\langle b,a\rangle}$, where x_b is not free in $\gamma_{\langle b,a\rangle}$.

5. *Basic hybrid schemas*:

a. *Selfdual*: $\vdash @_h\varphi \leftrightarrow \neg @_h \neg\varphi$.
b. *Intro*: $\vdash h \rightarrow (\varphi \leftrightarrow @_h\varphi)$.
c. *Back*: $\vdash \Diamond @_h\varphi \rightarrow @_h\varphi$.
d. *Ref*: $\vdash @_h h$.
e. *Agree*: $\vdash @_{h'} @_h \alpha_a = @_h \alpha_a$.

6. *Domain schema*:

a. *Hybrid Barcan*: $\vdash \forall x_b @_h\varphi \leftrightarrow @_h \forall x_b \varphi$.

7. *Basic hybrid type theory schemas*:

a. *Equality-at-named-worlds*: $\vdash @_h(\beta_b = \delta_b) = (@_h\beta_b = @_h\delta_b)$.
b. *Rigid function application*: $\vdash @_h(\gamma_{\langle b,a\rangle}\beta_b) = (@_h\gamma_{\langle b,a\rangle})(@_h\beta_b)$.
c. *Rigids are rigid:* If α_a is rigid, then $\vdash @_h\alpha_a = \alpha_a$.

8. *Downarrow schema*:

a. *DA*: $\vdash @_i(\downarrow s\varphi \leftrightarrow \varphi(\frac{i}{s}))$.

The first seven groups of schemas are those used in [1] to axiomatize BHTT. In the present paper, these schemas range over HTT expressions (not merely BHTT expressions), and operators of the form $@_i$ and $@_j$ have been replaced by operators of the form $@_h$ and $@_{h'}$ to reflect the addition of state variables, but otherwise the systems are identical. Moreover, with the exception of the *Basic hybrid type theory schemas*, which were introduced in [1], these schemas are familiar from either higher-order logic or hybrid logic.

Now, to move from BHTT to HTT, we need only add the *DA* schema. Readers familiar with hybrid logic will recognize this as a standard schema used to prove completeness when $\downarrow$ is added to basic (propositional or first-order) hybrid logic. The *DA* schema spells out the locality of $\downarrow$ with admirable precision: when we are working at a world named i, we are free (reading the $\leftrightarrow$ in the left to right direction) to eliminate $\downarrow$ by substituting i for bound occurrences of s, or (reading it in the right to left direction) to use $\downarrow s$ to bind out occurrences of i. Because this schema captures the local semantics of $\downarrow$ so cleanly, the completeness proof for BHTT needs only relatively minor modifications to extend it to HTT.

3.2 Rules of Proof

1. *Modus Ponens*: If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$.
2. *Generalizations*:

a. *Gen*$_\Box$: If $\vdash \varphi$, then $\vdash \Box\varphi$.
b. *Gen*$_@$: If $\vdash \varphi$, then $\vdash @_h\varphi$.
c. *Gen*$_\forall$: If $\vdash \varphi$, then $\vdash \forall x_a\varphi$.

3. *Rigid replacement*: If $\vdash \varphi$, then $\vdash \varphi'$, where φ' is obtained from φ by:

a. Uniformly replacing expressions $h \in \mathsf{HYB}$ by expressions $h' \in \mathsf{HYB}$.
b. Uniformly replacing free variables of type a by rigid expressions of type a.

4. *Name*: If $\vdash @_h\varphi$ and h does not occur in φ, then $\vdash \varphi$.
5. *Bounded Generalization*: If $\vdash @_h\Diamond j \rightarrow @_j\varphi$, $j \neq h$, and j does not occur in φ, then $\vdash @_h\Box\varphi$.

These are standard rules from hybrid and classical logic. The restriction in the rigid replacement rule (that only nominals and state variables can replace nominals and state variables) reflects the fact that nominals and state variables are names for worlds, and replacement must respect this. The additional restriction (that variables can only be freely replaced by rigid terms) reflects the fact that assignment functions interpret variables rigidly, and replacement must respect this too. We shall discuss *Name* and *Bounded Generalization* shortly when we consider alternative axiomatizations.

Definition 12 A *proof* of φ is a finite sequence $\alpha_1, \ldots, \alpha_n$ of expressions such that $\alpha_n := \varphi$ and for every $1 \leq i \leq n-1$, either α_i is an axiom, or α_i is obtained from previous expressions in the sequence using the rules of proof. We write $\vdash \varphi$ whenever we have such a sequence and say that φ is an HTT-*theorem*.

Definition 13 If $\Gamma \cup \{\varphi\}$ is a set of meaningful expressions of type t, a *proof of φ from Γ* is a proof of $\vdash \gamma_1 \wedge \cdots \wedge \gamma_n \rightarrow \varphi$ where $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$. A meaningful expression φ of type t is *provable from a set of expressions Γ*, written $\Gamma \vdash \varphi$, if and only if there is a proof of φ from Γ. We call Γ *inconsistent* if and only if for all formulas φ, $\Gamma \vdash \varphi$. Otherwise, Γ is *consistent*.

Theorem 1 (Soundness) *For all $\varphi \in \mathsf{ME}_t$, we have that $\vdash \varphi$ implies $\models \varphi$.*

Proof Straightforward but tedious. □

That was K1, simply the BHTT axiomatization enriched with the *DA* schema. But more should be said about the *Name* and *Bounded Generalization* rules. Neither is an orthodox modal rule of proof. However, both draw on familiar ideas from classical logic; indeed, both rules can be viewed as (axiomatic simulations of) natural deduction rules. Consider *Name*. This can be read as saying:

> *If φ can be proved to hold at an arbitrary world h not mentioned in φ, then we can (so to speak) discharge the $@_h$ and prove φ.*

And *Bounded Generalization* can be read similarly:

> *If φ can be proved to hold at some arbitrary successor world j of h, which is not mentioned in φ, then we can (so to speak) discharge the $@_h\Diamond j$ assumption and prove that φ occurs at all successors of h.*

Natural deduction systems for hybrid logic have been intensively studied (for a monograph-length treatment, see [8]), and the ideas just sketched are central to many such proof systems.

And this leads us to K2. This is the axiomatization obtained by discarding *Bounded Generalization* and replacing it with the *Paste* rule (note that j must be new):

$$\frac{\vdash @_i\Diamond j \wedge @_j\varphi \rightarrow \theta}{\vdash @_i\Diamond\varphi \rightarrow \theta}.$$

That is, we replace one nonorthodox rule by another (note: we retain *Name*). The *Paste* rule is interesting for two reasons. Firstly, as we shall see when we discuss the Lindenbaum construction, this is the rule that most directly licenses the use of Henkin-style witness nominals. Secondly, as we shall discuss at the end of the paper, *Paste* is essentially a lightly disguised tableau rule. But let us defer further discussion of the second option and turn to the third.

We obtain K3 when we discard both *Name* and *Paste* and replace them with the following axioms and rules:

$Gen_{\downarrow}$: if $\vdash \varphi$ then $\vdash \downarrow s \varphi$.
$Name_{\downarrow}$: $\vdash \downarrow s(s \rightarrow \varphi) \rightarrow \varphi$, where s does not occur in φ.
Bounded Generalization$_{\downarrow}$: $\vdash @_i \Box \downarrow s @_i \Diamond s$.

This approach (due to Balder ten Cate) is elegant. The new rule is orthodox; indeed, it mirrors our other *Gen* rules. And the two new axioms show that when we have ↓ in our language, the proof-theoretic effect of the *Name* and *Bounded Generalization* rules can be captured by axioms.[3]

4 Completeness

We now prove the completeness of our axiomatization(s). This involves only minor modifications of the completeness proof for BHTT. We sketch what is required, highlighting issues involving the ↓ binder.

4.1 The Lindenbaum Construction

As with any Henkin proof, we shall build our model out of the expressions contained in a maximal consistent set of formulas, that is, out of the items in some $\Delta \subseteq \mathsf{ME}_t$. Moreover, we are going to build the maximal consistent sets we require using a Henkin-style strategy. That means that each ∃-formula will be witnessed by a constant of appropriate type, and we will build the functional hierarchy out of equivalence classes of these elements; this part of our proof follows Henkin's recipe almost to the letter. But working Henkin-style in hybrid logic also means that we are going to use our hybrid machinery to imitate Henkin's strategy in the modal part of the language: each ◊-formula will be witnessed by a nominal, and the worlds will be built out of equivalence classes of witness nominals. This motivates the following definition.

[3]These three approaches have a common evolutionary history. The earliest was the *Paste* plus *Name* combination found in K2. This was introduced in [7] in the setting of propositional tense logic with ↓ and used to prove analogs of Theorems 5 and 7 of this paper. The *Bounded Generalization* plus *Name* combination used in K1 was designed to provide a natural deduction style counterpart to the tableau style *Paste* rule. Balder ten Cate then showed how this combination could be refined (when ↓ is in the language) with the rules and axioms used in K3. For detailed explorations of the *Bounded Generalization* plus *Name* option and their simplifications with ↓, see [6] and [9]. Another nonorthodox hybrid proof rule (one which is useful even if we do not have @ in our language) is the *COV* rule of [11].

Definition 14 Let Σ be a set of meaningful expressions.

1. Σ is *named* iff one of its elements is a nominal.
2. Σ is $\Diamond$-*saturated* iff for all expressions $@_i\Diamond\varphi \in \Sigma$, there is a witness nominal; that is, a nominal $j \in \mathsf{NOM}$ such that $@_i\Diamond j \in \Sigma$ and $@_j\varphi \in \Sigma$.
3. Σ is $\exists$-*saturated* iff for all expressions $@_i\exists x_a\varphi \in \Sigma$, there is a witness constant, that is, a constant $c_a \in \mathsf{CON}_a$ such that $@_i\varphi(\frac{@_i c_a}{x_a}) \in \Sigma$.

So far so good—but what about the $\downarrow$ binder? If $\exists$ requires witness constants, does not the $\downarrow$ binder, like $\Diamond$, require witness nominals? Don't we also need our maximal consistent sets to be $\downarrow$-saturated? The answer is: yes, we do, but this is given to us automatically, courtesy of the *DA* schema.

Lemma 4 *Let Δ be maximal consistent, and let i be any nominal such that $i \in \Delta$. Then we have that $@_i{\downarrow}s\varphi \in \Delta$ iff $@_i\varphi(\frac{i}{s}) \in \Delta$.*

Proof Recall that *DA* is $\vdash @_i({\downarrow}s\varphi \leftrightarrow \varphi(\frac{i}{s}))$. Hence, if $@_i{\downarrow}s\varphi \in \Delta$, then by the left-to-right direction we have that $\vdash @_i\varphi(\frac{i}{s}) \in \Delta$. Conversely, if $\vdash @_i\varphi(\frac{i}{s}) \in \Delta$, then by the right-to-left direction we have that $@_i{\downarrow}s\varphi \in \Delta$. □

The $\downarrow$ operator, though expressively powerful, is deductively straightforward because of its locality. Unlike $\exists$ and $\Diamond$, which need *new* witness constants and nominals, we can specify in advance the witnesses that $\downarrow$ needs, and *DA* does this. It tells us that, at a world named i, we are free to use i as a $\downarrow$ witness, and furthermore, that we can also use $\downarrow$ to bind out i. Because of this, the Lindenbaum construction for HTT is identical to the construction for BHTT, as we do not need to modify the construction to cope with the $\downarrow$ binder.

Lemma 5 (Lindenbaum) *Let Σ be a consistent set of formulas. Then Σ can be extended to a maximal consistent set Σ^ω that is named, $\Diamond$-saturated, and $\exists$-saturated.*

Proof Let $\{i_n\}_{n\in\omega}$ be an enumeration of a countably infinite set of new nominals, $\{c_{n,a}\}_{n\in\omega}$ an enumeration of a countably infinite set of new constants of type a, and $\{\varphi_n\}_{n\in\omega}$ an enumeration of the formulas of the extended language. We will build $\{\Sigma^n\}_{n\in\omega}$, a family of subsets of ME_t, by induction:

- $\Sigma^0 = \Sigma \cup \{i_0\}$.
- Assume that Σ^n has already been built. To define Σ^{n+1}, we distinguish four cases:
  1. $\Sigma^{n+1} = \Sigma^n$ if $\Sigma^n \cup \{\varphi_n\}$ is inconsistent.
  2. $\Sigma^{n+1} = \Sigma^n \cup \{\varphi_n\}$ if $\Sigma^n \cup \{\varphi_n\}$ is consistent and φ_n is not of the form $@_i\Diamond\psi$ or $@_i\exists x_a\psi$.
  3. $\Sigma^{n+1} = \Sigma^n \cup \{\varphi_n, @_i\Diamond i_m, @_{i_m}\psi\}$if $\Sigma^n \cup \{\varphi_n\}$ is consistent, φ_n has the form $@_i\Diamond\psi$, and i_m is the first nominal not in Σ^n or φ_n.
  4. $\Sigma^{n+1} = \Sigma^n \cup \{\varphi_n, @_i\psi\frac{@_i c_{m,a}}{x_a}\}$ if $\Sigma^n \cup \{\varphi_n\}$ is consistent, φ_n has the form $@_i\exists x_a\psi$, and $c_{m,a}$ is the first constant of type a not in Σ^n or φ_n.

Now, let $\Sigma^{\omega} = \bigcup_{n\in\omega} \Sigma^{n}$. Then Σ^{ω} is named, $\Diamond$-saturated, $\exists$-saturated, and maximal consistent. The proof is the same as that given in [1]. We use *Name* to prove the consistency of Σ^0. We use *Paste* to show the consistency of Case 3; this rule gives us exactly what is required. *Paste* is a primitive rule in K2 and a derived rule in both K1 and K3.

Case 4 introduces a small (but significant) deviation from Henkin's recipe for the quantifiers. We do not simply witness the $\exists$ quantifier with a new constant $c_{m,a}$, rather we use the rigidified constant $@_i c_{m,a}$, where i names the world where the existential formula is to be evaluated. Note that $@_i c_{m,a}$, like $c_{m,a}$, is of type a. The change does not effect consistency, and the proof is essentially the standard one; see [1] for details. □

4.2 Building Hybrid Henkin Structures

We come to the core construction. Recall that a structure $\mathscr{M}$ is a pair of the form $\langle \mathscr{S}, \mathsf{F}\rangle$, where $\mathscr{S} = \langle\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}, W, R\rangle$, and F is a denotation function. So we have two tasks. The first is to define the type hierarchy $\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}$, and the second is to define $\langle W, R\rangle$ and F. Let us deal with the first task.

Definition 15 Let Δ be a named, $\Diamond$-saturated, and $\exists$-saturated maximal consistent set. Then:

- For all $\alpha_a, \beta_a \in \mathsf{RIGIDS}_a$: $\alpha_a \approx_\Delta \beta_a$ iff $\alpha_a = \beta_a \in \Delta$ for every $a \in \mathsf{TYPES}-\{t\}$. The *rigidity equivalence class* of α_a, denoted $[\alpha_a]_\Delta$, is the set $\{\beta_a \mid \alpha_a \approx_\Delta \beta_a\}$.
- For $\varphi, \psi \in \mathsf{ME}_t : \varphi \approx_\Delta \psi$ iff $\varphi = \psi \in \Delta$. The *truth equivalence class* of φ, denoted $[\varphi]_\Delta$, is the set $\{\psi \mid \varphi \approx_\Delta \psi\}$.

When Δ is clear from context, we will usually write $\approx$ instead of $\approx_\Delta$ and $[\alpha]$ instead of $[\alpha]_\Delta$. It is straightforward to check that both rigidity equivalence and truth equivalence are equivalence relations.

This leads to the key result: all the equivalence classes needed when building the type hierarchy can be represented by rigidified constants.

Theorem 2 (Rigid Representatives) *Let Δ be a maximal consistent set that is named, $\Diamond$-saturated, and $\exists$-saturated.*

1. *Let $h \in \mathsf{HYB}$ and $\alpha \in \mathsf{ME}_t$. Then $[\alpha] = [@_h\bot]$ or $[\alpha] = [@_h\top]$.*
2. *Let $h \in \mathsf{HYB}$ and $\alpha_a \in \mathsf{RIGIDS}_a$ such that $a \neq t$. Then there is a constant $c_a \in \mathsf{CON}$ such that $[\alpha_a] = [@_h c_a]$.*

Proof The proof is by induction on type structure. We give the proof for type t expressions since state variables are of type t. The case for type e expressions and the inductive step can be found in [1].

Let $h \in \mathsf{HYB}$ and $\alpha \in \mathsf{ME}_t$. Assume that $\alpha \in \Delta$. But $\alpha \rightarrow (\alpha = \top) \in \Delta$ by propositional logic. Thus, $\alpha = \top \in \Delta$ and $\alpha = @_h\top \in \Delta$ by Axiom 7c and maximal consistency. Hence, $[\alpha] = [@_h\top]$. On the other hand, if we assume that $\alpha \notin \Delta$, both $\neg\alpha$ and $\neg\alpha \rightarrow (\alpha = \bot) \in \Delta$, and similar reasoning lets us conclude that $[\alpha] = [@_h\bot]$. A remark: since for any $h \in \mathsf{HYB}$, we have that $\vdash @_h\bot = \bot$ and $\vdash @_h\top = \top$, by Axiom 7c $[@_h\bot] = [\bot]$

and $[@_h\top] = [\top]$. So the choice of h is irrelevant; there are only two truth equivalence classes. □

Theorem 3 (Hierarchy Theorem) *Given a maximal consistent set Δ that is named, $\Diamond$-saturated, and $\exists$-saturated, there are a family of domains $\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}$ and a function Φ such that*:

1. *Φ is a bijection from* BB (B*uilding* B*locks*) *to $\bigcup_{a\in\mathsf{TYPES}} \mathsf{D}_a$, where*

$$\mathsf{BB} = \bigcup_{a\in\mathsf{TYPES}\setminus\{t\}} \big\{[\alpha_a] \,\big|\, \alpha_a \in \mathsf{RIGIDS}_a\big\} \cup \big\{[\varphi] \,\big|\, \varphi \in \mathsf{ME}_t\big\}.$$

2. $\mathsf{D}_t = \{\Phi([\varphi]) \mid \varphi \in \mathsf{ME}_t\}$ *and* $\mathsf{D}_a = \{\Phi([\alpha_a]) \mid \alpha_a \in \mathsf{RIGIDS}_a\}$ *for* $a \neq t$.

Proof The proof is essentially Henkin's; we refer the reader to [1] for full details. However, we will give the case for type t expressions since we want to be explicit about how state variables are handled. We define D_t to be the two elements set $\mathsf{D}_t = \{[@_j\bot], [@_j\top]\}$, and for every $\varphi \in \mathsf{ME}_t$, we define:

$$\Phi\big([\varphi]\big) = \begin{cases} [@_j\top] & \text{iff} \quad \varphi \in \Delta, \\ [@_j\bot] & \text{iff} \quad \neg\varphi \in \Delta, \end{cases}$$

where the nominal j is arbitrary. It is immediate by the first part of the rigid representatives theorem that Φ is well defined, one-to-one, and onto. Note that the definition of Φ covers state variables since they are of type t. We will need to use the definition of $\Phi([s])$ when we prove Theorem 4. □

Now for the second task, defining $\langle W, R\rangle$ and F. We first define an equivalence relation over elements of HYB.

Definition 16 Let Δ be a maximal consistent set. For $h, h' \in \mathsf{HYB}$, we shall define $h \approx^N h'$ to hold if and only if $@_h h' \in \Delta$. Clearly, if $h \approx^N h'$, then it means that h and h' name the same world, and it is easy to show that $\approx^N$ is an equivalence relation on HYB. For any $h \in \mathsf{HYB}$, we shall define $[h]^N$ to be $\{h' \in \mathsf{HYB} : h \approx^N h'\}$.

Definition 17 (Hybrid Henkin Structures) Let Δ be a maximal consistent set that is named, $\Diamond$-saturated, and $\exists$-saturated. The *hybrid Henkin structure $\mathscr{M} = \langle \mathscr{S}, \mathsf{F}\rangle$ over Δ* is made up of:

1. The skeleton $\mathscr{S} = \langle\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}, W, R\rangle$ defined by:
   a. $\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}$, as given by the hierarchy theorem,
   b. $W = \{[i]^N \mid i \text{ is a nominal}\}$,
   c. $R = \{\langle [i]^N, [j]^N\rangle \mid @_i\Diamond j \in \Delta\}$.
2. F is a function with domain $\mathsf{NOM} \cup \mathsf{CON}$ defined by:
   a. For $c_{n,a} \in \mathsf{CON}$, $\mathsf{F}(c_{n,a})$ is a function from W to D_a such that $\mathsf{F}(c_{n,a})([i]^N) = \Phi([@_i c_{n,a}])$.
   b. For $i \in \mathsf{NOM}$, $\mathsf{F}(i)$ is a function from W to $\mathsf{D}_t = \{[@_i\top], [@_i\bot]\}$ such that $(\mathsf{F}(i))([j]^N) = [@_i\top]$ iff $i \in [j]^N$.

We need to check that hybrid Henkin structures are indeed well-defined structures, but this is straightforward. Further details can be found in [1].

4.3 General Interpretation and Completeness

One last detail remains: defining our variable assignment. The following definition extends the definition in [1] to cover state variables. We remind the reader that Φ is the bijection defined in the proof of the hierarchy theorem and that for any state variable s, $[s]^N$ is the equivalence class of elements of HYB that name the same world as s.

Definition 18 The *hybrid Henkin assignment* on $\mathscr{M}$ is the function g defined as follows. For every $v_a \in \mathsf{VAR}_a$, we have

$$g(v_a) = \Phi\big([v_a]\big),$$

and for every state variable $s \in \mathsf{SVAR}$, we have

$$g(s) = [s]^N.$$

Theorem 4 *Let $\mathscr{M}$ be a hybrid Henkin structure, and g its hybrid Henkin assignment. For all meaningful expressions β_b and for all $i \in \mathsf{NOM}$, we have*

$$[\![\beta_b]\!]^{\mathscr{M},[i]^N,g} = \Phi\big([@_i \beta_b]\big).$$

Proof By induction on the formation of meaningful expressions. A proof covering most steps is given in [1]. Here we give the steps for state variables in formula position and the $\downarrow$ binder.

So suppose that β_b is a state variable s. We want to show, for any nominal i, that $[\![s]\!]^{\mathscr{M},[i]^N,g} = \Phi([@_i s])$. Now, in the hybrid Henkin structure, $T = [@_j \top]$ and $F = [@_j \bot]$, where j is an arbitrary nominal. So by the semantic definition for state variables in formula position, the fact that $g(s) = [s]^N$, and Definition 16 we have that

$$[\![s]\!]^{\mathscr{M},[i]^N,g} = \begin{cases} [@_j \top] & \text{iff} \quad g(s) = [i]^N \quad \text{iff} \quad [s]^N = [i]^N \quad \text{iff} \quad @_i s \in \Delta, \\ [@_j \bot] & \text{iff} \quad g(s) \neq [i]^N \quad \text{iff} \quad [s]^N \neq [i]^N \quad \text{iff} \quad @_i s \notin \Delta. \end{cases}$$

This immediately gives us the equivalence we require since by the definition of Φ given in the proof of the hierarchy theorem and maximal consistency,

$$\Phi\big([@_i s]\big) = \begin{cases} [@_j \top] & \text{iff} \quad @_i s \in \Delta, \\ [@_j \bot] & \text{iff} \quad \neg @_i s \in \Delta \quad \text{iff} \quad @_i s \notin \Delta. \end{cases}$$

This establishes the case for state variables in formula position.

So suppose that β_b is $\downarrow s \varphi$. We now show that $[\![\downarrow s \varphi]\!]^{\mathscr{M},[i]^N,g} = \Phi([@_i \downarrow s \varphi])$. Predictably, this is where we use the DA schema:

$$\begin{aligned}
[\![\downarrow s\varphi]\!]^{\mathscr{M},[i]^N,g} &= [\![\varphi]\!]^{\mathscr{M},[i]^N,g\frac{[i]^N}{s}} && \text{Semantic definition}\\
&= \left[\!\!\left[\varphi\left(\frac{i}{s}\right)\right]\!\!\right]^{\mathscr{M},[i]^N,g} && \text{Lemma 3}\\
&= \Phi\left(\left[@_i\left(\varphi\left(\frac{i}{s}\right)\right)\right]\right) && \text{Inductive hypothesis}\\
&= \Phi\big([@_i\downarrow s\varphi]\big) && \mathsf{DA}.
\end{aligned}$$

The only other novel case is for expressions of the form $@_s\varphi$, and this is straightforward by the induction hypothesis. □

Corollary 1 *A pair $\langle\mathscr{M}, g\rangle$ where $\mathscr{M}$ is a hybrid Henkin structure, and g is its Henkin assignment in a general interpretation.*

Proof The induction underlying the proof of the previous theorem shows that every expression has an interpretation in the appropriate domain of the hierarchy, and this is precisely what we require of general interpretations. □

Theorem 5 (Henkin Theorem) *Every consistent set of meaningful expressions of type t has a general interpretation satisfying it.*

Proof Let Γ be a consistent set of meaningful expressions of type t. By the Lindenbaum Lemma there is a maximal consistent extension Δ of Γ that is named, $\Diamond$-saturated, and $\exists$-saturated. Because Δ is named, there is a nominal k in Δ. By Theorem 4 and Corollary 1 there is a general interpretation $\langle\mathscr{M}, g\rangle$ such that, for all $\beta_t \in \mathsf{ME}_t$, we have that

$$[\![\beta_t]\!]^{\mathscr{M},[k]^N,g} = \Phi\big([@_k\beta_t]\big).$$

Let $\varphi \in \Gamma$. Hence, $@_k\varphi \in \Delta$. But this means that $\Phi([@_k\varphi]) = [@_k\top]$. Hence, $[\![\varphi]\!]^{\mathscr{M},[k]^N,g} = [@_k\top]$. That is, φ is true in this model at the world $[k]^N$. □

Theorem 6 (Completeness) *For all $\Gamma \subseteq \mathsf{ME}_t$ and $\varphi \in \mathsf{ME}_t$, the following holds: $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.*

Proof Standard. □

5 The Bounded Fragment

When logicians talk about completeness for first-order logic, they typically talk about *the* completeness theorem, singular, the result first proved by Gödel in his 1929 Ph.D. thesis and proved in the now standard fashion by Henkin in his celebrated 1949 paper [12]. Modal logicians, on the other hand, typically talk of completeness theorems, plural. Even readers with only passing acquaintance with modal logic may well have heard of modal logics boasting such names as K, T, S4, S5, S4.3, and many many more. Why this difference in perspective?

The reasons are partly historical. Modal logic began as a largely syntactic attempt to pin down such concepts as strict implication, necessity, and possibility, and many systems were considered. Although algebraic methods were sometimes used, it was not until the introduction of possible world semantics in the late 1950s, by pioneers such as Kripke, Hintikka, Kanger, that light was shed on this proliferation of logics. For example, it became clear—indeed, easy to see—that S4 is the set of formulas valid on preordered frames $\langle W, R\rangle$, that is, frames where R is reflexive and transitive. Modal completeness results of the early 1960s typically attempted to show that particular axioms of interest had simple semantic characterizations. There were successful attempts to prove more general completeness results, the best known being Sahlqvist's theorem, but such results tended to be complex. And in the early 1970s, incompleteness results were proved in the (seemingly simple) setting of propositional modal logic. All in all, thinking in terms of *logics* is probably the most natural way of coping with this complex landscape of results and partial results.

In first-order logic, thanks to the strength and simplicity of the completeness theorem, the situation is more straightforward. Instead of thinking in terms of different first-order logics, we tend to think in terms of theories. For example, if we want to work with partial orders (that is, antisymmetric preorders), then we would form the theory consisting of the following sentences:

$$\forall x Rxx \qquad \forall x \forall y (Rxy \wedge Ryx \rightarrow x = y) \qquad \forall x \forall y \forall z (Rxy \wedge Ryz \rightarrow Rxz).$$

The first-order completeness theorem guarantees that the consequences of these sentences are precisely the first-order formulas valid on partial orders. Of course, we can, and often do, talk of "the first-order logic of partial orders." Nonetheless, the simplicity and generality of the completeness theorem makes it natural to think in terms of theories, in a way that is uncommon in modal logic.

As its name suggests, hybrid logic lives somewhere in the middle. We shall now prove a completeness result that covers many useful hybrid theories of frame structure. The results discussed below are all standard in (propositional and first-order) hybrid logic; the point of our discussion is to show how straightforwardly they lift to HTT. Indeed, stronger results are known in the hybrid literature, and many of them lift with similar ease to HTT. But we have selected the result below because it is simple and elegant and showcases the locality of the $\downarrow$ binder, a recurrent theme in this paper.

Let us start with an example. Suppose we are working with HTT but wish to work with partially ordered frames. Then we simply form the following theory:

$$\downarrow s(\Diamond s) \qquad \downarrow s\Box(\Diamond s \rightarrow s) \qquad \downarrow s\Box\Box\downarrow t@_s\Diamond t.$$

These sentences pin down reflexivity, antisymmetry, and transitivity, respectively. Note that the transitivity sentence makes use of the store and retrieve interplay between $\downarrow$ and $@$. This sentence says: starting from a point that we have temporally labelled s, if we label t any point that is modally accessible from s in two steps (the $\Box\Box$), then at s we can modally access t in one step (the $\Diamond t$). It is not difficult to see that this formula will be true at *all* the worlds in an HTT model if and only if the frame of the model is transitive. That is, this hybrid sentence defines transitivity.

Another example. If we want to work with models whose frames are *strict* partial orders (that is, irreflexive, asymmetric and transitive), then we could work with the following sentential theory:

$$\downarrow s(\Diamond\neg s) \qquad \downarrow s(\Box\neg\Diamond s) \qquad \downarrow s\Box\Box\downarrow t\,@_s\Diamond t.$$

Again these expressions define the conditions we are interested in. And this leads to the first question that will occupy us: if we add such sentences as extra axioms, do we *automatically* have completeness? Is the completeness result we have proved for HTT like the completeness theorem for first-order logic, in that it supports thinking in terms of theories rather than logics? It is, but to prove this, we must first be precise about the axioms we intend to use.

As axioms, we allow *pure nominal-free sentences*. The "pure" means that they contain no constants; "nominal free" and "sentences" are self-explanatory. To put it another way, we generate *pure nominal-free expressions* as follows. As basic expressions, we only have state variables, and complex expressions can only have the following form:

$$\neg\varphi \mid \varphi\wedge\psi \mid \Box\varphi \mid @_s\varphi \mid \downarrow s\varphi.$$

This is a small fragment of HTT, containing only expressions of type t. But it is also a well-studied fragment of propositional hybrid logic. Don't be fooled by its apparent simplicity: this fragment is undecidable (see [2]) and capable of defining many important frame classes. The axioms we will work with are the *sentences* of this fragment. The above examples are all sentences of this kind (we define the additional booleans and $\Diamond$ in the usual way).

Most of the information in a model $\mathscr{M} = \langle\langle\langle \mathsf{D}_a\rangle_{a\in\mathsf{TYPES}}, W, R\rangle, \mathsf{F}\rangle$ and an assignment g is irrelevant for pure nominal-free expressions. Indeed, all that is relevant is the frame $\langle W, R\rangle$ and the assignment that g makes to the state variables. Thus, for expressions φ in this fragment, we are simply looking at interpretations at some world $w \in W$ with respect to an assignment g:

$$[\![\varphi]\!]^{\langle W,R\rangle,w,g}.$$

This leads us to *frame validity*. We say that a pure nominal-free expression is valid on a frame $\langle W, R\rangle$ if and only if for all $w \in W$ and all assignments g, we have that $[\![\varphi]\!]^{\langle W,R\rangle,w,g} = T$. We write $[\![\varphi]\!]^{\langle W,R\rangle}$ when φ is valid on $\langle W, R\rangle$.

And now for completeness. We want to use pure nominal-free sentences as additional axioms in HTT. And we want to prove a completeness result that tells us: if we add as extra axioms (for example) the hybrid theory of strict partial orders given above, then for any consistent set of type t expressions, we can always build a verifying model over a *strictly partially ordered* frame. How can we do so? As we have seen, Theorem 5 will indeed give us a model for consistent sets of expressions, and if we have added extra axioms, then the hybrid Henkin structure will make them all true too. But that is not enough: we need to show that the axioms are valid on the underlying frame. But for pure nominal-free sentences, this is easy.

Lemma 6 *Let φ be a pure nominal-free sentence, let $\langle W, R\rangle$ be a frame, and let g be a fixed but arbitrary assignment on $\langle W, R\rangle$. Suppose that for all $w \in W$, we have $[\![\varphi]\!]^{\langle W,R\rangle,w,g} = T$. Then $[\![\varphi]\!]^{\langle W,R\rangle}$, that is, φ is valid on $\langle W, R\rangle$.*

Proof Suppose that for some assignment g and all worlds $w \in W$, we have that $[\![\varphi]\!]^{\langle W,R\rangle,w,g} = T$. But φ is a *sentence*, so the choice of assignment is irrelevant. That is, at any $w \in W$, for any assignment g', we have that $[\![\varphi]\!]^{\langle W,R\rangle,w,g'} = T$. So φ is valid on $\langle W, R\rangle$ as claimed. □

Definition 19 Let φ be a pure nominal-free sentence. Then $\mathsf{Fr}(\varphi)$ is the class of frames $\langle W, R\rangle$ such that $[\![\varphi]\!]^{\langle W,R\rangle}$. That is, $\mathsf{Fr}(\varphi)$ is the class of all frames that validate φ. A sentential theory Th is a set of pure nominal-free sentences; note that all such sets are countable. Then $\mathsf{Fr}(\mathsf{Th})$ is the class of frames $\langle W, R\rangle$ such that for all $\varphi \in \mathsf{Th}$, $\langle W, R\rangle$ belongs to $\mathsf{Fr}(\varphi)$. That is, $\mathsf{Fr}(\mathsf{Th})$ is the class of frames that validate all sentences in the theory Th.

By a *pure nominal-free sentential extension* of our axiomatization(s), we mean the addition as extra axioms of all the pure nominal-free sentences in such a theory Th. A set of type t expressions Γ is Th-consistent if and only it is consistent in this enriched system.

Theorem 7 (Extended Henkin Theorem) *Let* Th *be a set of pure nominal-free sentences. Every* Th*-consistent set* Γ *of meaningful expressions of type* t *has a general interpretation* $\langle \mathscr{M}, g\rangle$ *that satisfies it, such that the frame* $\langle W, R\rangle$ *underlying* $\mathscr{M}$ *belongs to* $\mathsf{Fr}(\mathsf{Th})$.

Proof For the first claim, we proceed as in the proof of Theorem 5. Let Γ be a Th-consistent set of meaningful expressions of type t. We use our Lindenbaum construction to obtain a maximal consistent extension Δ of Γ that is $\Diamond$-saturated, $\exists$-saturated, and named by some nominal k. By Theorem 4 and Corollary 1 there is a general interpretation $\langle \mathscr{M}, g\rangle$ such that, for all $\beta_t \in \mathsf{ME}_t$,

$$[\![\beta_t]\!]^{\mathscr{M},[k]^N,g} = \Phi\big([@_k\beta_t]\big).$$

As we saw, this shows that all expressions in Δ (and hence Γ) are true in this general interpretation at the world $[k]^N$. Note that this includes all the sentences in Th since as Γ is Th-consistent, every sentence in Th will be added to Δ at some stage of the Lindenbaum construction.

But for the second part of the theorem, we need to show that all axioms in Th are valid on the frame underlying $\mathscr{M}$, not merely that they are true. We do so as follows. Let $\varphi \in \mathsf{Th}$. Since φ is an axiom, then for all nominals i, by @-generalization we have that $\vdash @_i\varphi$. So by maximal consistency, all these expressions belong to Δ. But every world in the hybrid Henkin structure is an equivalence class of nominals. So (here we use the displayed equality again) at every world $[i]^N$ we have that $[\![\varphi]\!]^{\mathscr{M},[i]^N,g} = T$. But φ is a pure nominal-free sentence, so by Lemma 6 it is valid on $\langle W, R\rangle$. But φ was an arbitrary element of Th, so all sentences in Th are valid on $\langle W, R\rangle$. Hence, $\langle W, R\rangle$ is in $\mathsf{Fr}(\mathsf{Th})$, which is what we wanted to prove. □

What frame classes are covered by this result? Here is a syntactic answer. At the start of this section we gave the three axioms characteristic of partial orders in a first-order language of frames. It was a simple first-order language: a two-place relation R plus the equality symbol. Here we define the bounded fragment of this first-order frame language. We will use s, t, and so on—that is, the symbols we typically use for state variables—as first-order variables. We generate its *bounded fragment* as follows:

$$Rst \mid s = t \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists t(Rst \wedge \varphi), \quad \text{where } s \neq t.$$

Now, all pure nominal-free expressions translate into the bounded fragment:

$$ST_s(t) = (s = t) \quad \text{for all state variables } t,$$
$$ST_s(\neg\varphi) = \neg ST_s(\varphi),$$
$$ST_s(\varphi \wedge \psi) = ST_s(\varphi) \wedge ST_s(\varphi),$$
$$ST_s(\Diamond\varphi) = \exists t\big(Rst \wedge ST_t(\varphi)\big),$$
$$ST_s(@_t\varphi) = ST_t(\varphi),$$
$$ST_s(\downarrow t\varphi) = \big(ST_s(\varphi)\big)\left(\frac{s}{t}\right).$$

In the translation clause for $\downarrow$ we have used $(\frac{s}{t})$ to indicate the substitution of the (first-order) variable s for free occurrences of the (first-order) variable t. This translation uses the relevant clauses of the *standard translation*, which is widely used in orthodox modal logic and hybrid logic; see [4] or [3] for more detailed accounts. Note that ST_s translates every pure nominal-free *sentence* into a bounded formula with s as its sole free variable. It follows by induction that for all sentences φ, frames $\langle W, R\rangle$, and $w \in W$,

$$[\![\varphi]\!]^{\langle W,R\rangle,w} \quad \text{iff} \quad \langle W, R\rangle \models ST_s(\varphi)[s \leftarrow w].$$

Here $\langle W, R\rangle \models ST_s(\varphi)[s \leftarrow w]$ means first-order satisfaction, with the unique free variable s in $ST_s(\varphi)$ being assigned w as value. Note that we have written $[\![\varphi]\!]^{\langle W,R\rangle,w}$ rather than $[\![\varphi]\!]^{\langle W,R\rangle,w,g}$ because, as φ is a sentence, the choice of variable assignment is irrelevant. Summing up, all pure nominal-free sentences define frame conditions expressible in the bounded fragment with one free variable s. Hence, a pure nominal-free sentence φ is valid on a frame $\langle W, R\rangle$ if and only if $\langle W, R\rangle \models \forall s\varphi$, where $\models$ indicates first-order truth in $\langle W, R\rangle$. So every frame condition expressible by a pure nominal-free sentence can be expressed by a bounded sentence.

And the converse also holds since we can translate the bounded fragment into the pure nominal-free expression as follows:

$$HT(s = t) = @_s t,$$
$$HT(Rst) = @_s\Diamond t,$$
$$HT(\neg\varphi) = \neg HT(\varphi),$$
$$HT(\varphi \wedge \psi) = HT(\varphi) \wedge HT(\psi),$$
$$HT\big(\exists t(Rst \wedge \varphi)\big) = @_s\Diamond{\downarrow}t\big(HT(\varphi)\big).$$

Again, we can show by induction that for every frame $\langle W, R\rangle$, if we translate a bounded first-order formula θ with s as its only free variable, then

$$\langle W, R\rangle \models \theta[s \leftarrow w] \quad \text{iff} \quad [\![{\downarrow}s\,HT(\theta)]\!]^{\langle W,R\rangle,w}.$$

Taken together, these two results show that if we restrict attention to bounded formulas containing at most one free variable, pure nominal-free sentences are a hybrid notation for capturing bounded frame conditions.

The bounded fragment is interesting because it is a *local* fragment of first-order logic: it only lets us quantify over accessible entities. Feferman was the first to consider bounded fragments (see [10]); he did so in set theory, using $\in$ rather than R as his accessibility predicate. The link between bounded fragments and hybrid logic was first explored in [2]; more can be said, here we have only noted the bare essentials. For a start, there are elegant semantic characterizations: roughly speaking, pure nominal-free sentences allow us to talk about frame classes that are both closed under and reflect point generated subframes. Moreover, Theorem 7 can be extended to allow free state variables (or nominals) in the axioms. But for further discussion, we refer the reader to [9], the authoritative source. Here we shall return to Henkin.

We remarked that Henkin's work fits well with hybrid logic because of the first-order perspective that underlies his best known work. In this section we have been explicit about the first-order character of our hybrid technology. As we have previously remarked, Theorem 7 is atypical in its generality, at least when viewed from orthodox modal logic. But there is no puzzle here: it is the first-order character of key hybrid tools that makes such results possible, and it is the use of Henkin models that makes them easy to prove.

6 It's Henkin, All the Way Down

In this paper we proved a general completeness theorem for a hybrid type theory called HTT, probably the strongest hybrid logic that has yet been explored. The result built upon earlier work on a system called BHTT, which only used the basic hybrid tools of nominals and the @ operator. The $\downarrow$ binder is the tool that differentiates HTT from the earlier system, and it is $\downarrow$ that allows us to capture the bounded fragment. The influence of Henkin's work throughout this paper should be obvious. For a start, we used Henkins's methods of constants to allow $\Diamond$-prefixed expressions to be witnessed by nominals. Secondly, our use of general interpretations and the construction of the type hierarchy in the completeness proof are (pretty much) one hundred percent Henkin. Thirdly, the first-order nature of pure nominal-free sentences allowed us to extend our K-style completeness result(s) to cover a wide class of hybrid theories, and the proof of Theorem 7 was straightforward because we were working with a hybrid Henkin structure: as each world is named in such structures, it was easy to verify the validity of the additional axioms.

But to close this paper, we want to claim that the most profound impact of Henkin's work on hybrid logic occurs neither at the level of type theory, nor even at the level of first-order hybrid logic, but at the level of *propositional* hybrid logic, and indeed, right down at the level of *basic* hybrid logic, where the only hybrid tools at our disposal are @ and nominals. To put it another way, a recurring theme in the development of hybrid completeness has been that richer languages hitch a free ride on the underlying basic hybrid logic. This is certainly the case for first-order hybrid logic, and the completeness results for HTT and BHTT repeat this pattern.

Let us look closer. Consider first the role of @. For a start, it drives much of the model construction process. Because it lets us rigidify arbitrary type, it lets us state rigidity restrictions on axioms, prove the rigid representatives theorem, and specify an appropriate world for every formula in a suitably saturated maximal consistent set of sentences (these points are made in more detail in [1], where BHTT was proved complete). Moreover, we have seen that two of our axiomatizations, K1 and K2, pretty much reduce $\downarrow$ to a spectator role as far as axiomatics is concerned. The crucial *DA* schema is a local schema: it tells us how to deal with $\downarrow$ at a named world, but we rely on @ to distribute this schema to all named worlds, and the logic of @ is captured in basic hybrid logic. The K3 axiomatization eliminates the nonorthodox basic rules and shows how $\downarrow$ can play a more active role in the axiomatization. But here we also rely on @ to insist that *DA* holds at all named worlds.

And this mention of nonorthodox rules brings us to the heart of our claim. Such rules are not nonorthodox in any interesting sense. Our earlier remarks on the link between the *Name* and *Bounded Generalization* rules and natural deduction hinted at this, but to close this paper, we turn to the K2 axiomatization, which uses the basic hybrid logic rule *Paste*. The following remarks draw on and elaborate the discussion on pp. 445–447 of [4].

Here is the *Paste* rule (again, j must be new):

$$\frac{\vdash @_i \Diamond j \wedge @_j \varphi \to \theta}{\vdash @_i \Diamond \varphi \to \theta}.$$

As we saw in the proof of our Lindenbaum Lemma, this directly licenses the use of witness nominals. And if we read this rule from bottom-to-top, then we see that it boils down to the following tableau rule:

$$\begin{array}{l} \underline{@_i \Diamond \varphi} \\ @_i \Diamond j,\ j\ \mathit{new} \\ @_j \varphi \end{array}$$

That is, if in the course of a tableau proof we encounter $@_i \Diamond \varphi$, then we are free to decompose this into the near-atomic formula $@_i \Diamond j$ (where j is new) and the simpler $@_j \varphi$. This directly embodies the idea of Henkin-style nominal witnessing and in one form or another is the basis for most tableau systems for hybrid logic. Deductively, at least, what makes hybrid logic better behaved than orthodox modal logic is the possibility of eliminating $\Diamond$ in this way. Tableau rules for the other connectives are easy to define, dealing with $\Diamond$ is the tricky part. This is the rule that opens the door to usable (nonaxiomatic) proof procedures.

But why should we call this rule orthodox? The answer takes us back (yet again) to first-order logic and the world of Henkin. Consider the following table. This uses the standard translation (recall Sect. 5) to show the content of the tableau rule, viewed from the perspective of the first-order frame language (enriched with a first-order constant i for every nominal i):

Hybrid Logic	First-Order Frame Language
$@_i \Diamond \varphi$	$\exists s(Ris \wedge ST_s(\varphi))$ $Rij \wedge ST_j(\varphi)$
$@_i \Diamond j$	Rij
$@_j \varphi$	$ST_j(\varphi)$

The translation of the tableau rule is a fragment of a first-order tableau proof. Moreover, it makes clear that (from a first-order perspective) all that is going on is a skolemisation (that is, a witnessing) of the existential quantifier, followed by a conjunction reduction. First-order orthodoxy reigns.

Ultimately, this is why Henkin's work is so important to hybrid logic. Diamond witnessing is the central idea that has driven the development of hybrid deduction and completeness theory. But in hybrid logic, Henkin's use of witnesses is put to work in a simple, decidable propositional language, namely basic hybrid logic. This gives stability and generality to completeness results for basic hybrid logic and enables them to support completeness in richer hybrid systems (for example, hybrid logics containing the $\downarrow$ binder) and combinations of hybrid logic with first-order logic and type theory. Hybrid logic? It is a suggestive name. But it could without exaggeration be called Henkin-style modal logic.

Acknowledgements The authors are grateful to the Spanish Ministerio de Economía y Competitividad for funding the project *Logica Intensional Hibrida* (*Hybrid Intensional Logic*), FFI2013-47126-P, hosted by the Universidad de Salamanca.

References

1. Areces, C., Blackburn, P., Huertas, A., Manzano, M.: Completeness in hybrid type theory. J. Philos. Log. **43**, 209–238 (2014)
2. Areces, C., Blackburn, P., Marx, M.: Hybrid logics: characterization, interpolation and complexity. J. Symb. Log. **66**(3), 977–1010 (2001)
3. Areces, C., ten Cate, B.: Hybrid logic. In: Handbook of Modal Logic, pp. 821–868. Elsevier, Amsterdam (2007)
4. Blackburn, P., De Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2002)
5. Blackburn, P., Seligman, J.: Hybrid languages. J. Log. Lang. Inf. **4**(3), 251–272 (1995)
6. Blackburn, P., ten Cate, B.: Pure extensions, proof rules, and hybrid axiomatics. Stud. Log. **84**, 277–322 (2006)
7. Blackburn, P., Tzakova, M.: Hybrid languages and temporal logic. Log. J. IGPL **7**(1), 27–54 (1999)
8. Braüner, T.: Hybrid Logic and Its Proof-Theory. Applied Logic Series, vol. 37. Springer, Berlin (2010)
9. ten Cate, B.: Model theory for extended modal languages. Ph.D. thesis, University of Amsterdam. ILLC Dissertation Series DS-2005-01 (2005)
10. Feferman, S.: Persistent and invariant formulas for outer extensions. Compos. Math. **20**, 29–52 (1968)
11. Gargov, G., Goranko, V.: Modal logic with names. J. Philos. Log. **22**, 607–636 (1993)
12. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**(3), 159–166 (1949)
13. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**(2), 81–91 (1950)
14. Henkin, L.: The discovery of my completeness proofs. Bull. Symb. Log. **2**(2), 127–158 (1996)
15. Manzano, M.: Extensions of First Order Logic. Cambridge University Press, Cambridge (1996)
16. Van Benthem, J., Doets, K.: Higher-order logic. In: Handbook of Philosophical Logic, pp. 275–329. Springer, Berlin (1983)

P. Blackburn (✉) · K.F. Jørgensen
Roskilde Universitet, Roskilde, Denmark
e-mail: patrickb@ruc.dk

A. Huertas
Universitat Oberta de Catalunya, Barcelona, Spain

M. Manzano
Universidad de Salamanca, Salamanca, Spain

Changing a Semantics: Opportunism or Courage?

Hajnal Andréka, Johan van Benthem, Nick Bezhanishvili, and István Németi

Abstract The generalized models for higher-order logics introduced by Leon Henkin and their multiple offspring over the years have become a standard tool in many areas of logic. Even so, discussion has persisted about their technical status, and perhaps even their conceptual legitimacy. This paper gives a systematic view of generalized model techniques, discusses what they mean in mathematical and philosophical terms, and presents a few technical themes and results about their role in algebraic representation, calibrating provability, lowering complexity, understanding fixed-point logics, and achieving set-theoretic absoluteness. We also show how thinking about Henkin's approach to semantics of logical systems in this generality can yield new results, dispelling the impression of adhocness.

Keywords Henkin models · Definable predicates · General frames · Absoluteness · General models for recursion and computation

1 General Models for Second-Order Logic

In Henkin [46] general models were introduced for second-order logic and type theory that restrict the ranges of available predicates to a designated family in the model. This enlarged model class for higher-order logics supported perspicuous completeness proofs for natural axiom systems on the pattern of the famous method in Henkin [45] for proving completeness for first-order logic. To fix the historical background for this paper, we review the basic notions here, loosely following the exposition in van Benthem and Doets [17].

The language of second-order logic has the usual vocabulary and syntactic constructions of first-order logic, including quantifiers over individual objects, plus variables for n-ary predicates, and a formation rule for existential and universal second-order quantifiers $\exists X\varphi$, $\forall X\varphi$. *Standard models* $M = (D, I)$ for this language have a domain of objects D and an interpretation map I for constant predicate symbols.[1] Next, *assignments* s send individual variables to objects in D, and predicate variables to real predicates over D, viewed as sets of tuples of objects. Then we have the following standard truth condition:

$$M, s \models \exists X\varphi \quad \text{iff} \quad \text{there is some predicate } P \text{ of suitable arity with } M, s[X := P] \models \varphi.$$

This paper is dedicated to Leon Henkin, a deep logician who has changed the way we all work, while also being an always open, modest, and encouraging colleague and friend.

[1]For simplicity only, we mostly omit individual constants and function terms in what follows.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0_20

Now we take the control of available predicates in our own hands, instead of leaving their supply to set theory. A *general model* is a tuple (M, V) where M is a standard model and V is some nonempty family of predicates on the domain D of M. There may be some further constraints on what needs to be in the family, but we will look into this below.

Now interpretation on standard models is generalized as follows:

$$M, s \models \exists X\varphi \quad \text{iff} \quad \text{for some predicate } P \text{ in } V \text{ of suitable arity: } M, s[X := P] \models \varphi.$$

Henkin [46] proved the following seminal result:

Theorem 1 *Second-order logic is recursively axiomatizable over general models.*

This can be shown by adapting a Henkin-style completeness proof for first-order logic.

However, another insightful road for obtaining Theorem 1 is via translation into *many-sorted first-order logic*.[2] Consider a first-order language with two sorts: "objects" and "predicates," each with their own set of variables. In addition, the domains are connected by a special predicate Exp saying that object x belongs to predicate p. Now we can translate the language of second-order logic into this two-sorted first-order language via a straightforward map τ. The clauses for the logical operators are obvious, with $\exists x$ going into the object domain, and $\exists X$ into the predicate domain. At the atomic level, the translation is as follows:

$$Xy \quad \text{goes to} \quad EyX.$$

In addition, we state one principle that makes predicates in the two-sorted first-order language behave like real set-theoretic predicates in general models:

$$\textit{Extensionality}(EXT) \quad \forall pq: \quad \big(p = q \leftrightarrow \forall x(Exp \leftrightarrow Exq)\big).$$

Proposition 1 *For all second-order formulas φ, φ is valid on all general models iff $\tau(\varphi)$ follows from EXT on all models for the two-sorted first-order language.*

The proof is obvious except for one observation. Interpretations for "predicate variables" p in two-sorted first-order models M can be arbitrary objects in the domain. But Extensionality allows us to identify such objects p one-to-one with the sets $\{x \mid Exp \text{ in } M\}$.

As a result of this embedding, the validities of second-order logic over general models are axiomatizable, and by inspecting details, one can extract an actual axiomatization close to that of two-sorted first-order logic. Moreover, this correspondence can be modulated. We can drop Extensionality if we want even more generalized models, where we give up the last remnant of set theory: namely, that predicates are identified with their set-theoretic extensions. This generalization fits well with intensional views of predicates as concepts or procedures that occur occasionally in logic and more often in philosophy.

But more common in the logical literature is the opposite direction, an even stronger form of set-theoretic influence, where we turn further second-order validities into constraints on general models. A major example is that one often requires the following schema to hold in its two-sorted first-order version:

[2] In what follows in this section, merely for convenience, we shall deal with *monadic second-order logic* only, where second-order quantifiers run only over unary predicates or sets.

Comprehension $\exists X \forall y(Xy \leftrightarrow \varphi)$ for arbitrary second-order formulas φ with X not free in φ.

What this achieves in the above completeness proofs is enforcing the validity of what many people find a very natural version of the logical law of existential instantiation:[3]

$$\varphi(\psi) \to \exists X \varphi(X).$$

One can add many other constraints in this way without affecting completeness. One example is the object dual to Extensionality, $\forall xy(x = y \leftrightarrow \forall p(Exp \leftrightarrow Eyp))$, which expresses a sort of "Individuality" for objects in terms of available predicates.

How far are these first-order versions of second-order logic removed from the logic over standard models? The distance can be measured by means of one standard formula, as established by Montague [72].

Fact 2 *A second-order formula is valid on standard models iff its first-order translation* $\tau\varphi$ *follows from EXT plus the second-order formula* $\forall X \exists p \forall y(Xy \leftrightarrow Eyp)$.

The newly added axiom tells us that all set-theoretic predicates lie represented in the first-order model. Amongst other things, this observation shows that, taking prenex forms, the full complexity of second-order logic resides in its existential validities.

The preceding results turn out to extend to both classical and intensional type theories (Gallin [36]) and other systems. For a modern treatment of the current range of generalized models for higher-order logics and related systems, we refer to Manzano [69, 70].[4]

2 Clearing the Ground: Some Objections and Rebuttals

Before going to concrete technical issues, we set the scene by listing some perspectives on generalized models that can be heard occasionally in the logical community, if not read in the literature.[5] We do this mainly to clear away some rhetoric, before going to the real issues. For instance, the very term "standard model" is already somewhat rhetorical since it prejudges the issue of whether other models might also be natural, by depriving them of a neutral name and making their pursuit "nonstandard" or "deviant."[6]

General models are often considered an ad hoc device with little genuine content. This objection can be elaborated in several ways. One may hold that the natural semantics for second-order logic consists only of those models that have the full power set of the individual domain for their unary predicates, and likewise for higher arities. General models

[3] Comprehension does make the strong philosophical assumption that logical constructions out of existing predicates deliver available predicates, something that might be debated.

[4] There is much more to second-order logic than the general perspective given here. In particular, deep results show that well-chosen fragments of second order logic, over well-chosen special classes of standard models, can have much lower complexity, and have surprising combinatorial content, for instance, in terms of automata theory: cf. Grädel, Thomas, Wilke, eds. [42].

[5] In what follows, for brevity, we write "general models" instead of "generalized models".

[6] Sometimes, a little dose of linguistics suffices to dispel this rhetoric, e.g., by calling 'non-classical' logics 'modern' logics, or 'non-intended' models 'serendipitous' models.

are then a proof-generated device lacking independent motivation and yielding no new insights about second-order logic. And there can be more specific objections as well. For instance, imposing comprehension as a constraint on general models smacks of circularity. In order to define the available predicates, we need to know what they are already, in order to understand the range of the second-order quantifiers in the comprehension principles. But objections can also run in another direction. The general model-based completeness proof for second-order logic tells us nothing new beyond what we already knew for first-order logic. In particular, one learns nothing that is specific to second-order validity.[7]

Now there is something to these objections, but the question is how much. At least, one will always do well to also hear the case for the opposition. Here are a few preliminary considerations in favor of general models, and the proofs involving them.

First of all, it is just a fact that in many natural settings, we do not want the full range of all set-theoretically available sets or predicates since it would trivialize what we want to say. For instance, in natural language, when Russell wrote that "Napoleon had all the properties of a great general," he meant all relevant properties, not some trivial one like being a great general. Likewise, when we grasp Leibniz' principle that objects with the same properties are identical, we do not mean the triviality that object x has the property of being identical to object y, but rather think of significant properties. More generally, while it is true that the intuitive notion of property or predicate comes with a set, its extension, or range of application, it would be unwarranted set-theoretic imperialism to convert this into the statement that every set is a property or predicate.[8]

The same consideration applies to mathematics: even there, set theory is not the norm. In geometry, important shapes correspond to sets of points, but definitely not every set of points is a natural geometrical shape. Hilbert 1899 mainly axiomatizes points, lines, and planes, instead of points and sets of points, and that for excellent mathematical reasons. Likewise, topology looks at open sets only, many theories of spatial objects use only convex sets, and one can mention many similar restrictions. Also, at a more abstract level, category theory does not say that all function spaces between objects must be full, the definition of a category allows us control over the available morphisms between objects, and that is precisely the reason for the elegance and sweep of the category-theoretic framework.

A third line of defense might be a counterattack, turning the table on some of the earlier objections. Intended models for second-order logic provide us with a magical source of predicates that we tend to accept without having enquired into how they got there, and whether they should be there. Moreover, these entities also come with a magical notion of truth and validity that seems to be there without us having to do any honest work in terms of analyzing proof principles that would constrain the sort of entity that we want. From this perspective, it is rather the general models that force us to be explicit about what we want and why. This point can be sharpened up. It is often thought

[7] Contrast this with the sense of achievement, based on innovative proof techniques, when axiomatizing a piece of second-order logic on a class of intended models, such as the monadic second-order logics of trees with successor relations as the "initial segment" relation in Rabin [79].

[8] One example that shows this is the discussion in philosophy of propositions viewed as sets of worlds. Clearly, propositions correspond to sets of worlds where they are true, but it does not follow at all that every set of worlds must be an extension of some proposition. We return to this theme at the end of Sect. 4.

that general models decrease the interest of proofs establishing their properties, but as we shall see later, this is a mistaken impression. There are also natural settings where general models make logical proof analysis and technical results more, rather than less sophisticated.

This introductory discussion is general and inconclusive. Moreover, in the process of getting some ideological issues out of the way, we may even have introduced new ones. For instance, despite appearances, our aim with this paper is not criticizing the use of set-theoretic notions per se. But there is a distinction to be made. The set-theoretic language is a lingua franca for much of logic and mathematics, and it facilitates formulation and communication in a way similar to the role of academic English. What one should be wary of, however, is the often implicit suggestion of a further commitment to very specific claims of set theory qua mathematical theory in our logical modeling.[9] But where the precise border line between set-theoretic language and "set theory" is located seems a delicate matter and one where studying general models may in fact provide a better perspective.

We will now look at more systematic motivations for the use of general models in logic, Henkin-style and eventually also beyond, and assess their utility in more precise technical terms.

3 Logical Perspectives on Controlling Predicates

For a start, we continue with the two themes introduced in the preceding section: controlling the available predicates and the fine-structure of reasoning.

3.1 Proof Theory

Logical analysis of an area of reasoning, even of mathematical proofs, seldom uses all set-theoretically available predicates—but only ones that are definable in some way. And such proofs are crucial: despite the claimed intuitive nature of standard models, we hardly ever model-check assertions in them since these models are too complex and mysterious for that. Instead, we prove that certain second-order formulas hold, say, in the natural numbers, by means of a mathematical argument appealing to accepted general principles. And these proofs will usually employ only very specific predicates, often ones that are definable in some sense, witness a wide literature on formalizing mathematical theories from Bishop [21] to reverse mathematics [83].[10] A survey of predicative proof theories with, for instance, highly restricted comprehension axioms, would go far beyond the confines of this paper, whose main slant is semantic, but we submit that such systems embody a reasoning practice that goes well with Henkin-style models.

---

[9] For this unwarranted commitment, consider the much more radical claim that using academic English carries a commitment to "British" or "American values."

[10] This is more delicate with nonconstructive principles like the axiom of choice, which we forego here.

3.2 Definable Predicates

Restrictions to definable predicates are natural even in Henkin's first-order completeness proof itself that started our considerations. The Hilbert–Bernays completeness theorem says that, in order to obtain counterexamples to nonvalid first-order formulas, models on the natural numbers suffice where the interpreting predicates are Δ^2_0. Even so, not every restriction of predicates to some family of definable sets will be a good choice of a general model class for completeness. For instance, Mostowski [73] showed that first-order logic is not complete when we restrict the interpreting predicates in models to be recursive, or even recursively enumerable. Vaught [90] considered the related problem of complexity for the sentences true in all constructive models.[11] Likewise, if we constrain general models too much, say, to only contain predicates that are first-order definable in the underlying first-order model, we may get harmful complexity. In particular, Lindström [64] proved that, if second-order variables range over first-order definable predicates only, the system is nonaxiomatizable. Thus, whereas a restriction to definable predicates may be a motivation for using general models, there is no general guarantee that this move lowers the complexity of axiomatizability for the logic. Having too many predicates gives us second-order logic, and having too few can also lead to high complexity. We should be in between, and where that lies precisely may differ from case to case.

3.3 New Structure: Dualities

Here is another mathematical consideration in favor of controlling predicates. Doing so drastically may reveal important structure that we would not see otherwise. For instance, consider the frequently rediscovered weak theory of objects and types in "Chu spaces" (Barwise and Seligman [9], Pratt [76], Ganter and Wille [37]). This theory treats objects and types on a par, resulting in a very appealing duality between their behavior,[12] constrained by a notion of structural equivalence capturing the basic categorial notion of adjointness, which allows for model-theoretic analysis (van Benthem [14]). All this elegant structure remains invisible to us in standard models for second-order logic. Similar points hold in terms of algebraic logic, the topic of Sect. 4 below.

3.4 New Theorems: Dependence and Games

It may be thought from the above that working with general models will decrease proof strength, so that, if anything, we lose theorems in this way. But the case of dualities was an instance where we regained new theorems at a higher level that just do not hold for the original version on standard models. Here is another example of this phenomenon, where general models increase mathematical content.

[11] The related result that Peano arithmetic has no nonstandard recursive models is often called "Tennenbaum's theorem," see, e.g., Kaye [60].

[12] For example, in Chu spaces, Extensionality for predicates is exactly Leibniz' principle for objects.

Consider so-called IF logic of branching quantifiers, introduced in Henkin [47] and taken further by Hintikka as a general study of independence in logic (cf. Hintikka and Sandu [54]).[13] Enderton [32] and Walkoe [95] showed that branching quantifiers are the existential functional fragment of second-order logic, having a very high complexity of validity as we have seen in Sect. 1. In line with this, there has been little proof theory of IF logic, leaving the precise nature of reasoning with independent quantification a bit of a mystery. However, one can also start from a natural deduction analysis of dependent and independent quantifiers (Lopez-Escobar [65]) and get an insightful proof system. The natural complete semantics for this proof system turns out to be general models whose available functions satisfy some simple closure conditions. But there is more: these models also represent a crucial move of independent interest.

As is well known, like first-order logic, IF logic has a semantics in terms of evaluation games where truth amounts to existence of a winning strategy for the "verifier" in a game with imperfect information of players about moves by others (Mann, Sandu, and Sevenster [68]). This existential quantifier over strategies ranges over all set-theoretically available strategies, viewed as arrays of Skolem functions. Now the general models correspond to a restriction on *available strategies*, a very intuitive move in modeling games played by real agents, and this can drastically change the structure of the game. In particular, what can happen is that a game that is "determined" in the sense that one of the two players has a winning strategy, now becomes nondetermined. But then a very natural mathematical language extension suggests itself. Nondetermined IF games still have *probabilistic solutions* in mixed strategies by the basic theorems of von Neumann and Nash. And the equilibrium theory of these new games is much richer than that of the original games.[14]

3.5 Conclusion

This concludes our first foray into concrete logical aspects of general models, showing that they raise delicate issues of calibrating proof strength, levels of definability for predicates, and new laws possibly involving attractive language extensions. What we see here is that, with general models, weakness is at the same time wealth. We now turn to a number of major technical perspectives that we will consider in more detail.

4 General Models and Algebraic Representation

Algebraic semantics is one of the oldest ways of modeling logic, going back to the seminal work of Boole, de Morgan, and Schroeder. In this section, we investigate the relation between algebraic models and general models, largely using one case study: the algebraic semantics of modal logic. We identify a number of general issues, revolving around representation and completeness, and toward the end, we show how the interplay of algebraic

[13] See Väänänen [88] for a parallel development of a rich system of "dependence logic."

[14] Hintikka himself suggested a restriction to "definable strategies" for yet different reasons, but such a restriction might be a case of the above over-simplicity inducing high complexity after all.

semantics and general models is very much alive today. Thus, we find a motivation for general models that is very different from the proof-theoretic concerns that were central in earlier sections.[15]

4.1 Algebraic Completeness of Modal Logic

The algebraic semantics of modal logic is given by modal algebras $(A, \Diamond)$, with A a Boolean algebra and $\Diamond$ a unary operation on A satisfying $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. Every modal logic L is complete with respect to the "Lindenbaum–Tarski algebra" of all formulas quotiented by L.[16] The Lindenbaum–Tarski algebra of a logic L validates all and only the theorems of L, and hence every modal logic is complete with respect to a natural corresponding class of modal algebras.[17]

4.2 General Models via Algebraic Representation and Categorial Duality

Modal algebras are related to model-theoretic structures via natural representations. These model theoretic structures, often called "general frames," may be viewed as Henkin models for modal logic generalizing the original Kripke frames. A *Kripke frame* is a pair (X, R) of a set X and a binary relation R. Each Kripke frame gives rise to a modal algebra whose domain is the powerset $\mathscr{P}(X)$ of X with the Boolean operations $\cap$, $\cup$, $()^c$, plus an operation $\Diamond_R$ on $\mathscr{P}(X)$ defined by setting $\Diamond_R(S) = \{x \in X : \exists y \in S \text{ such that } xRy\}$ for each $S \subseteq X$.

However, going in the opposite direction, modal algebras do not normally induce standard frames. However, here is a widely used representation method. Given a modal algebra $(A, \Diamond)$, take the space X_A of all *ultrafilters* on A and define a relation R_A as follows; xR_Ay iff $a \in y$ implies $\Diamond a \in x$ for each object $a \in A$. Moreover, let $\mathfrak{F}(X_A) = \{\varphi(a) : a \in A\}$, where $\varphi(a) = \{x \in X_A : a \in x\}$. The latter is a special family of "good subsets" of the frame (X_A, R_A) satisfying a number of natural closure conditions: it forms a Boolean algebra and is closed under the natural operation for the modality. This is an instance of the following general notion. A *general frame* $(X, R, \mathfrak{F}(X))$ is a triple such that (X, R) is Kripke frame and $\mathfrak{F}(X)$ is a subset of $\mathscr{P}(X)$ closed under $\Diamond_R$, that is, if $S \in \mathfrak{F}(X)$, then $\Diamond_R(S) \in \mathfrak{F}(X)$.[18] General frames are natural models for a modal language since they

[15]In another context, namely in finite-variable fragments of first-order logic, Henkin [48] introduced homomorphic images of the Lindenbaum–Tarski algebras as generalized models, and he explained why and how they can be considered as models. In [49], Henkin calls these generalized models "algebraic models." He proves a completeness theorem and uses these algebraic models to show unprovability of a given formula. So, the present section shows that the ideas originated from Henkin [48, 49] are alive and flourishing.

[16]That is, two formulas φ and ψ are equivalent if $L \vdash \varphi \leftrightarrow \psi$.

[17]More generally, there exists a lattice anti-isomorphism between the lattice of normal modal logics and the lattice of equationally defined classes (varieties) of modal algebras: see [22, 26, 62] for details.

[18]In particular, $(\mathfrak{F}(X), \Diamond_R)$ is a subalgebra of $(\mathscr{P}(X), \Diamond_R)$.

provide denotations for all formulas—and of course, they are general models in Henkin's sense. Moreover, they are the right choice. A basic theorem by Jónsson and Tarski [59] generalizes the Stone representation for Boolean algebras to the modal realm:

Theorem 3 *Each modal algebra is isomorphic to the modal algebra induced by the general frame of its ultrafilter representation.*

But still more can be said. The general frames produced by the Jónsson–Tarski representation satisfy a number of special conditions with a natural topological background. They are *descriptive* in the sense that different points are separated by available sets, nonaccessibility is witnessed in the available predicates, and a natural compactness or "saturation" property holds for the available predicates. Whereas not all Kripke frames are descriptive, the latter property does hold for many other model-theoretic structures.[19] This is interesting as an example where a natural class of general models satisfies third-order closure conditions different from the ones considered in the original Henkin models for second-order logic.

The correspondence between modal algebras and descriptive frames can be extended to a full correlation between two mathematical realms (see any of [22, 26, 62, 80, 93] for details):

Theorem 4 *There exists a categorical duality between* (a) *descriptive frames and definable "p-morphisms"* (*the natural semantic morphisms between descriptive frames that preserve modal theories*) *and* (b) *modal algebras and Boolean-modal homomorphisms.*

Thus, well-chosen classes of general models support rich category-theoretic dualities.

4.3 Completeness and Incompleteness for Standard Relational Models

Whereas all this is true, standard frames do play a central role in a major area of modal logic, its completeness theory. The basic completeness results of modal logic say that, for well-known logics L, theoremhood coincides with validity in the standard frames satisfying the axioms of L. For instance, a formula is a theorem of modal $S4$ iff it is valid in all reflexive-transitive frames—and there are many results of this kind for many modal logics in the literature. By contrast, via the above representation, algebraic completeness would only match theoremhood with validity in the class of general frames for the logic, which is restricted to valuations that take values in the admissible sets. What is going on here?

Consider the Lindenbaum–Tarski algebra for any modal logic L. In general, the Kripke frame underlying the general frame representation of this algebra (often called the "canonical general frame" for the logic) need not be a model for the axioms of L under all valuations. But in many special cases, it is. One general result of this form is the well-known Sahlqvist theorem, which states that every modal logic that is axiomatized by "Sahlqvist

[19] For more details on descriptive frames, we refer to [22, 26, 62, 80].

formulas"—having a special syntactic form whose details need not concern us here—is complete with respect to a first-order definable class of standard frames.

Results like this are often seen as improving the automatic completeness provided by general frames. But even so, they do not detract from the latter's importance. The proof of the Sahlqvist theorem depends essentially on showing that Sahlqvist formulas that hold on the canonical general frame for the logic also hold in its underlying standard frame. Thus, general models can be crucial as a vehicle for *transfer* to standard models.[20]

However, there are limits to the preceding phenomena. Completeness theorems on frame classes are not always obtainable in modal logic. Famous "incompleteness theorems" from the 1970s onward have shown that there are consistent modal logics L and formulas φ such that $L \cup \{\varphi\}$ is consistent, but φ is not satisfiable on any Kripke frame for L. Concrete examples are nontrivial, and several interesting ones may be found in [10, 22, 26]. Incidentally, the consistency is usually proved by exhibiting a general frame where the logic is valid.[21] In fact, frame incompleteness is the norm among modal logics, witness the remarkable classification results in [23], whereas a modern exposition can be found in [26]. What all this says is that, despite appearances, it is completeness for general frames that underlies deduction in modal logic, whereas Kripke-frame completeness is a bonus in special cases.

4.4 Further Representation Methods

Our discussion may have suggested that the algebraic defense of general models depends on one specific representation method for modal logic. But in fact, there is a much broader theory where similar points can be made. For instance, there are more general representations for distributive lattices and Heyting algebras (Priestley [77, 78], Esakia [33], Davey and Priestley [28], [26]), using prime filters instead of ultrafilters to analyze intuitionistic logic and related systems. But new representation methods are appearing continuously. One recent example is the categorical duality between the categories of "de Vries algebras" (complete Boolean algebras with a special relation $\prec$ satisfying natural conditions) and compact Hausdorff spaces: [18, 94]. Significantly, all our earlier modal themes return in this much broader mathematical setting.

Going beyond this, there is also a flourishing representation theory for substructural logics; cf. Dunn [29], Gehrke, Dunn, Palmigiano [30], Gehrke [38], Marra and Spada [71], Galatos and Jipsen [35]. However, in this extended realm, it seems fair to say that

[20]There are also links here to *modal correspondence theory* where we study relational properties expressed by modal axioms on Kripke frames. Correspondence theory assumes a different shape on general frames, though, for instance, many of its classical results still hold when we assume that the available propositions in general frames are closed under first-order definability. We do not pursue this model-theoretic theme in this paper, though it is definitely a case where introducing general models also poses some challenges to existing theory on standard models of a logic.

[21]For a concrete example, consider the tense logic of [86], which has Löb's axiom for the past, making the relation transitive and well founded on Kripke frames, and the McKinsey axiom for the future, which states that above every point, there is a reflexive endpoint. Taken together, these requirements are inconsistent, but they do hold on the general frame consisting of the natural numbers with only the finite and cofinite sets as available propositions.

major open problems remain such as finding "good" representation theorems for residuated lattices. But the very fact that this is considered a serious mathematical open problem illustrates the importance attached to finding Henkin-style model-theoretic structures matching the algebras.

The preceding themes are not exclusively mathematical concerns, they also play inside philosophy. For various conceptual reasons, Humberstone [57] and Holliday [56] have proposed replacing the possible worlds semantics for modal languages by structures of "possibilities" ordered by inclusion. To make this work, one adapts the usual truth definition to clauses for Boolean operations with an intuitionistic flavor and interprets the modality by means of a suitable relation or function among possibilities. This new framework leads to an interesting theory that even improves some features of classical modal logic, and it can be understood in terms of regular open sets in topological spaces; cf. [16]. But of interest to us here is an issue of representation. Each standard possible worlds frame naturally induces a possibilities model defined by extending a relation on a frame into a function on its powerset [56] (this is similar to a coalgebraic perspective of modal logic [93]). But conversely, it is not true that each possibilities frame can be represented as coming from a Kripke frame in this manner. What is the proper comparison then between the two realms? The solution is again a move to general models: what should be compared are possibilities models and *generalized frames* based on a Boolean algebra of regular open sets of a topological space (see [56] and [16] for details of this construction).

4.5 Conclusion

Algebraic models for logical systems are a natural match with general models in Henkin's sense, or at least in Henkin's style, where precise connections are provided by a large and growing body of representation theorems. Of course, this is a special take on the genesis and motivation for general models, which comes with many interesting features of its own. For instance, we have drawn attention to special conditions on the general models produced by representation methods and to the ongoing challenges of representation theory, whereas we have also shown how in this realm, standard models become an interesting special case, which sometimes, but not always, suffices for completeness.

5 General Models, Lowering Complexity, and Core Calculi

Henkin's general models lower the complexity of second-order logic to that of the recursively axiomatizable first-order logic.[22] Behind this move, we can discern a more general theme, that of lowering complexity of given logical systems.[23] And with that theme, there

[22]This is interesting historically since, initially, first-order logic was not the measure of all things. It was proposed only in Hilbert and Ackermann [53] as a well-behaved fragment of higher-order logic, for which Gödel [41] then provided the first completeness result.

[23]In [48], Henkin introduces generalized models for the finite-variable fragments L_n of first-order logic, and he proves a completeness theorem with respect to these generalized models. This is also an example

is no reason to stop at first-order logic, whose notion of validity is recursively enumerable but undecidable. We will see (in Sect. 7) a similar lowering in complexity in connection with models of computation. Continuing in what we see as Henkin's spirit, though going far beyond the basic semantic approach presented in Sect. 1, could there be generalized models for first-order logic that make the system decidable?

This might sound like a purely technical interest, but one can add a deeper consideration. Van Benthem [13] observes how many "standard" logical systems, despite their entrenched status, embody a choice of modeling a phenomenon that combines basic features with details of the particular formal framework chosen. For instance, first-order logic wants to be a core calculus of Boolean connectives and quantification. But in order to do this job, its semantics is couched in terms of set-theoretic notions. Are the latter a harmless medium (the "lingua franca" of our earlier discussion), or do these "wrappings" add extraneous complexity? In particular, is the undecidability of first-order logic a logical core feature of quantification, or a mathematical reflection of the set-theoretic modeling? In order to answer such questions, again, we need a more general modeling for strategic depth and a talent for sniffing out unwarranted second-order or otherwise over-specific features. In this section, we present one line, so-called "general assignment models" for first-order logic, as an instructive exercise in generalized model thinking.[24]

The core semantics of first-order logic works as follows.[25] Models $M = (D, I)$ consist of a non-empty set of objects D plus an interpretation map I assigning predicates over D to predicate symbols in the language. Semantic interpretation involves models M, formulas φ, and assignments s of objects in D to individual variables of the language. A typical and basic example of the format is the truth condition for the existential quantifier:

$$M, s \models \exists x\varphi \quad \text{iff} \quad \text{there is some } d \text{ in } D \text{ with } M, s[x := d] \models \varphi.$$

Whereas the underlying intuitive idea of existential quantification is uncontroversial, this technical clause involves two noteworthy assumptions in terms of supporting machinery. The auxiliary indices of evaluation are taken to be maps from variables to objects, and it is assumed that each such function is available as an assignment. Thus, not in its predicate structure, but in its supporting structure for interpretation, a first-order model comes with a hidden standard second-order object, viz. the full function space D^{VAR}.[26]

for lowering complexity since it is known that L_n cannot have a finitistic Hilbert-style strongly complete proof system for the standard models (see, e.g., [51, Thm. 4.1.3]). Here is a further example of Henkin's positive thinking. Reacting to the negative result just quoted about L_n, he initiated jointly with Monk in [50, Problem 1] the so-called finitization problem, which in turn fruited interesting and illuminating results, for example, by Venema, Stebletsova, Sain, and Simon (for further details, we refer to [84]).

[24]The ideas in this section go back to Németi [74], Venema [91], van Benthem [13], and Andréka, van Benthem, and Németi [6], to which we refer for details. Here, Németi's work, couched in terms of "cylindric relativized set algebras," was inspired by remarks of Henkin about obtaining positive results in algebraic logic, rather than the wave of counterexamples in earlier stages of the theory. We note that investigating relativized cylindric algebras also originates from Leon Henkin; see, for example, Henkin and Resek [52].

[25]As in Sect. 1, we will disregard individual constants and function symbols for convenience.

[26]Later on, we will see what it is more precisely that makes such spaces high-complexity inducing, viz. their geometrical "confluence" or "gluing" properties: see Facts 6 and 7 below.

5.1 A Modal Perspective

This choice is not entirely obvious intuitively, as the first-order language could be interpreted just as well on an abstract universe of states s allowing for binary transitions between indices, yielding a basic clause for modal logic:

$$M, s \models \exists x\varphi \quad \text{iff} \quad \text{there is some } s' \text{ with } s R_x s' \text{ such that } M, s' \models \varphi.$$

We can think of the state space here as some independent computational device that regulates the mechanism of interpretation for our language, or its "access" to the model. Of course, standard models are still around as a special case.

Treated in this modal way, first-order logic retains the essentials of its compositional interpretation, but its core laws do not reflect any set-theoretic specifics of assignment maps. Rather, they form the "minimal modal logic" K that already takes care of a large slice of basic reasoning with quantifiers, including its ubiquitous monotonicity and distribution laws.[27]

The minimal modal logic is decidable and perspicuous, whereas its metatheory closely resembles that of first-order logic (Blackburn, de Rijke, and Venema [22]).

5.2 General Assignment Models

On top of this modal base system, the additional valid principles of first-order logic lie in layers of more specific assumptions on the modal state machinery approaching the full assignment spaces of standard models. Here is one basic level, using the set-theoretic view of assignments as functions from variables to individual objects, but without the existence assumption of having all functions around.

Definition 1 A general assignment model is a structure (M, V) with M a standard first-order model (D, I) and V a set of maps from individual variables to objects in D. Interpretation of first-order formulas in these models is standard except for the following clause:

$$M, s \models \exists x\varphi \quad \text{iff} \quad \text{there is some } d \text{ in } D \text{ such that } s[x := d] \in V \text{ and } M, s[x := d] \models \varphi.$$

By itself, this is just a technical move, but again there is an interesting interpretation. "Gaps" in the space of all assignments encode possible dependencies between variables. Suppose that we have an assignment s and want to change the value of x. Perhaps, the only variant that we have available for s in the special family V also changes its value for y, tying the two variables x, y together. One can then think of the logic of general assignment models as first-order logic freed from its usual independence assumptions.

Theorem 5 *First-order logic over general assignment models is decidable.*

[27] The modal character shows in that the quantifier $\exists x$ is really a labeled modality $\langle x \rangle$ now.

Things start changing with yet further axioms that express existence assumptions on the assignments that must be present. Their content can be brought out by standard modal frame correspondence techniques. We give one illustration, stated, for greater familiarity, in terms of the earlier abstract modal models for first-order logic.

Fact 6 *A modal frame satisfies the axiom $\exists x \forall y Sxy \to \forall y \exists x Sxy$ for all valuations* (*i.e.*, *all interpretations of atomic formulas*) *iff its transition relations R_x, R_y for the variables x, y satisfy the following Church–Rosser confluence property for all states s, t, u: if $s R_x t$ and $s R_y u$, then there is a v with $t R_y v$ and $u R_x v$.*

In general assignment models, the confluence property says that we can make the corresponding changes of values for variables concretely step by step: say, with $t = s[x := d]$, $u = s[y := e]$, and $v = s[x := d][y := e]$. This particular property is significant:

Fact 7 *Adding confluence to general assignment logic makes the logic undecidable.*

The reason is that assignment models satisfying the stated property are rich, or rather, regular, enough to run standard proofs of the undecidability of first-order logic through its ability to encode undecidable tiling problems in a two-dimensional grid [44]. Stated in another way, again we see that the undecidability has to do with mathematical, rather than logical content of the modeling: its ability to express regular geometrical patterns.

5.3 Richer Languages

A striking new phenomenon that occurs with many forms of generalized semantics in logic is that it suggests richer languages to interpret or, at least, more sophisticated versions of the original logical language over standard models.

A concrete case for general assignment models are *polyadic quantifiers* $\exists \mathbf{x} \varphi$ where $\mathbf{x}$ is a finite tuple of variables. In standard first-order logic, this is just short-hand for iterated prefixes of quantifiers $\exists x \exists y \ldots \varphi$. On general assignment models, the natural interpretation for $\exists \mathbf{x} \varphi$ is as existence of some available assignment in the model whose values on the variables in $\mathbf{x}$ can differ, a sort of "simultaneous re-assignment." This is not equivalent to any stepwise iteration of single quantifiers. Nevertheless, the logic allows the following:

Theorem 8 *First-order logic with added polyadic quantifiers is decidable over general assignment models.*

But language extensions can also increase complexity much more drastically. For instance, the general assignment models over a given first-order model M form a natural family under extension of their assignment sets. But this is an interesting structure in its own right, and it makes sense to add a new extension modality interpreted as follows:

$$M, V, s \models \Diamond \varphi \quad \text{iff} \quad \text{there is some } V' \supseteq V \text{ with } M, V', s \models \varphi.$$

Now it is easy to see that standard first-order logic can be embedded in this richer language by interpreting a standard first-order quantifier $\exists x$ as a modal combination $\Diamond\exists x$.

This shows that the new logic with a modality across assignment sets is undecidable. We conjecture that this system is recursively enumerable, although we have not been able to find a straightforward argument to this effect. However, may be our general point is that generalized models do not just lower complexity or provide axiomatizations for fixed logical formalisms. They may also change the whole design of languages and logics.

5.4 From General Models to Fragments

One proof of completeness and decidability for first-order logic over general assignment models mentioned above proceeds by translation into the decidable "guarded fragment" (GF) of first-order logic (Andréka, van Benthem, and Németi [6]). GF allows only quantifications of the following syntactic type:[28]

$\exists y\big(G(x, y) \wedge \varphi(x, y)\big)$, where x, y are tuples of variables, G is an atomic predicate, and x, y are the only free variables occurring in $\varphi(x, y)$.

Theorem 9 *First-order logic over general assignment models can be translated faithfully into the guarded fragment of first-order logic over standard models.*

This is interesting since a full language interpreted over generalized models now gets reduced to a syntactic fragment of that full language, interpreted over standard models. There is also a converse result (van Benthem [15]) tightening the connection: the guarded fragment can be reduced to first-order logic over general assignment models.

Thus, again we see that general assignment models are a laboratory for rethinking what a generalized semantics means. Sometimes, a move to generalized models can also be viewed entirely differently, as one from a full logical language to a sublanguage, where new features of the generalized models show up as syntactic restrictions.

5.5 Conclusion

We have taken the spirit of general models one step further and applied it to another "standard feature" of the semantics of logical systems, its use of assignments sending variables into the domain of objects. This method can be taken further than what we have shown here since it applies to about any logical system.[29] We have found a significant border line between decidable core theories and complexity arising from using set-theoretic objects

---

[28] We omit existing decidable extensions to "loosely guarded," "packed," and, recently, "unary negation" fragments.

[29] For instance, whereas the base system of "dependence logic" in [88] is undecidable and in fact of higher-order complexity, one can again find a standard power set in the background, the set of all sets of first-order assignments, and once this is tamed, the core dependence logic will become decidable again.

such as full powersets. Moreover, as with algebraic representation, we found a number of further general issues that emerge in this special realm, such as possible redesign of logical languages and the interplay between generalizing semantics and passing to fragments.

6 General Models and Absoluteness

In this section, we discuss yet another general perspective on general models, coming from the classical foundations of set theory and logic itself—and motivated, to a certain extent, by the desire to draw a principled border line between the two. As with our earlier topics, we cannot give an extensive technical overview, so we merely present some basics along the way that are needed for the points that we wish to make.[30]

6.1 Absoluteness and Nonabsoluteness

There is a sentence φ_{CH} of second-order logic (SOL) that expresses the continuum hypothesis (CH) in the following sense: for every model M with cardinality at least continuum, we have[31]

$$\mathrm{M} \models \varphi_{\mathsf{CH}} \quad \text{iff} \quad \mathsf{CH} \text{ holds in our metatheory.} \tag{*}$$

Hence, φ_{CH}[32] is logically valid in SOL iff CH holds in the metalevel set theory "floating above our heads." Thus, when contemplating the logical validities of SOL, we have to rely on the set theory we use for modeling/formalizing SOL, once more illustrating the earlier-discussed phenomenon of "wrappings" versus "content." Many authors have agreed that the meaning of a formula φ in a model M should depend on φ and M (and their parts) and not on the entire set theoretic hyperstructure of the whole universe containing φ and M, let alone on deciding highly complex and mysterious assertions like the continuum hypothesis (cf. [63]). And here is where absoluteness kicks in. Meaningfully investigating these kinds of questions and trying to make tangible and precise definitions and statements concerning phenomena such as the above lead to the theory of absolute logics.

A set-theoretic formula φ is called ZF-*absolute* if its truth value does not change in passing from any model **V** that satisfies the axioms of standard Zermelo–Fraenkel set theory ZF to any transitive submodel $\mathbf{V}'$ that satisfies those same axioms. Next, Kripke–Platek set theory KP can be considered as an austere effective fragment of ZF where we omit the axiom of infinity and restrict the replacement and collection schemes to bounded FOL-formulas. KPU is a natural extension of KP where we also allow urelements; see [8, pp. 10–12]. A set-theoretic formula is said to be *absolute* simpliciter if it is KPU-absolute.

---

[30]We thank Jouko Väänänen for sharing many useful corrections and insights concerning absoluteness, only some of which we have been able to do justice to in this brief paper.

[31]Equivalently, CH holds in the "real world," cf. [63, Sect. I.3, pp. 7–8].

[32]More precisely, a suitable variant of it taking into account the cardinality condition on M. For the formula φ_{CH}, see Ebbinghaus et al. [31, pp. 141–142].

A logic $\mathscr{L} = \langle Fm, \models \rangle$ is called *absolute* if the set-theoretic formulas defining Fm and the ternary satisfiability relation $\models$ are both absolute.

Here are a few illustrations for concreteness. Typical absolute concepts are "being an ordinal," "being a finite ordinal," "being the ω (i.e., an ordinal with each element a successor ordinal, but itself not a successor ordinal)," "being the union of two sets ($x = y \cup z$)," etc. However, equally typically, the property "x is the powerset of y" is not ZF-absolute, because whether a collection of elements of y is elevated to the rank of being a set is independent from the $\in$-structure of y.

Intuitively, absoluteness of a logic means that truth or falsity of the predicate $\mathrm{M}, s \models \varphi$ should depend only on the $\in$-structures of φ, M, and s, and not on the "context" that M, φ, and s are living in (an example of such a context would be the powerset of M). Then our (*) above shows that the logic SOL is not ZF-absolute.

Barwise and Feferman [7] initiated the study of absolute logics. The following theorem elucidating the special attention to KPU-absoluteness and making a crucial link to first-order logic is due to Kenneth Manders [67]; see also Akkanen [1], Theorem 9 of Feferman [34], and Theorem 3.1.5 in [87, pp. 620–622]. Below, by $\mathscr{L}_{\omega\omega}$ we understand many-sorted first-order logic (FOL) with equality. Also, logic $\mathscr{L}'$ is said to be stronger than $\mathscr{L}$ if every class C of models defined by a sentence of $\mathscr{L}$ is also definable by a sentence of $\mathscr{L}'$.

Theorem 10 *$\mathscr{L}_{\omega\omega}$ is the strongest logic among those absolute logics whose formulas are hereditarily finite sets and whose structures* M *have domains consisting only of urelements.*

Here we allow urelements in KPU so that we can have a good variety of infinite models built from urelements without forcing the existence of an infinite "pure set."

6.2 General Models via Absolute Versions

If an important logic $\mathscr{L}$ turns out to be nonabsolute, then it seems useful to consider and study an absolute version[33] of $\mathscr{L}$ (besides $\mathscr{L}$ itself). Indeed, Henkin's SOL is an absolute version of SOL. Whereas SOL is not ZF-absolute, one can check that the satisfaction relation of Henkin's general models for second-order logic is absolute (and it even remains so if we add a finite number of our favorite axioms valid in standard SOL). One can see this by using Manders' theorem, Theorem 10, and by recourse to the results in Sect. 1, in particular, the close connection between SOL on Henkin models and many-sorted FOL. In the light of Manders' theorem, any absolute version of SOL must have the form of a possibly many-sorted FOL theory.

There is a general method for obtaining absolute versions of given nonabsolute logics $\mathscr{L}$. We may assume that the nonabsoluteness of $\mathscr{L}$ originates from the set-theoretical definition of the ternary satisfiability relation of $\mathscr{L}$ not being absolute. Hence, the set-theoretic formula defining the ternary $\mathrm{M}, s \models \varphi$ contains quantifiers ranging over some objects not in the transitive closures of M, φ, and s. Collect these "dangerous objects" into an extra sort S. Consider the resulting extended models $(\mathrm{M}, S, \in)$. Similarly to the

[33] We use the expression "absolute version" in an intuitive way, not in the technical sense of [8].

example of SOL and its Henkin models above, we arrive at a recursively axiomatizable many-sorted FOL theory T whose "standard" models are the $(M, S, \in)$s. We regard T as an absolute version of the original logic $\mathscr{L}$. By fine-tuning the axioms of T, we can fine-tune the absolute version we want to work with. The models of T that are not standard are the nonstandard models of the absolute version T of $\mathscr{L}$. So, an absolute version of a logic always comes with a notion of a generalized model for $\mathscr{L}$. Indeed, Henkin's models for SOL can be seen as having arisen from devising an absolute version of SOL, and later in Sect. 8 we will see other examples where the notion of a generalized model for computation can be seen as devising an absolute version of a nonabsolute logic. Moreover, in many cases a nonstandard model arising in this way can still be seen as a standard one when viewed from another model of set theory.

If we encounter a logic with high computational complexity, the diagnosis may be nonabsoluteness as the cause for high complexity. This is the case with SOL and with first-order dynamic logic with standard semantics (see below). In all these cases, finding absolute versions leads to natural notions of generalized models and to a lower computational complexity of the absolute version. So, the notions of the theory of absolute logics shed light on generalized models and on lowering complexity.[34]

At this high level of generality, the style of analysis in this section applies very broadly. One of these applications concerns logics of programs, or processes, cf. [4, 5]. Such logics are usually ZF-absolute but not KPU-absolute. They cannot have decidable proof systems, even if we select acceptably small sublanguages in them; see [4, Thm. 1]. The reason for this is that standard semantics assumes programs/processes to run in standard time, that is, along the finite ordinals. We will return to this theme in Sects. 7–8, where we consider generalized semantics for recursion and computation in the setting of fixed-point logics.

6.3 Conclusion

Absoluteness as independence from set theory can be a desideratum motivating forms of generalized semantics and, in fact, a powerful methodology for their design.

7 General Models for Recursion and Computation

Our final topic is an important phenomenon that has often presented challenges to simple-minded semantic views of logical systems, the notion of computation, and in particular, its characteristic nested structures of recursion and inductive reasoning. These logics tend to be of very high complexity since treating computation explicitly often makes the natural numbers definable by a formula of the logic, after which validity starts encoding truth in second-order arithmetic.[35]

[34] Whereas absoluteness is meant to express independence from interpretations of set theory, its very definition depends heavily on set theory—but this may be considered a beneficial case of "catching thieves with thieves."

[35] Likewise, so far, fixed-point logics of recursion, though very natural at first glance, have successfully resisted Lindström-style analysis in abstract model theory.

Intuitively, logics of computation are about processes unfolding over time, and hence it is crucial how we represent time: as a standard set-theoretic object like the natural numbers or as a more flexible process parameter. We will start with this perspective and its generalized semantics. After that, we move to abstract fixed-point logics of computation and generalized semantics that facilitate the analysis of deductive properties and complexity-theoretic behavior. In a final discussion, we show how two perspectives are related.

7.1 Nonstandard Dynamic Logics

Consider logics of programs in the tradition of propositional dynamic logic *PDL*, based on program expressions with the regular operations, where the key role is played by the Kleene star, that is, *transitive closure*. Whereas this propositional system is an elegant decidable modal logic, its natural first-order version with objects and predicates that applies to more realistic programming languages is of highly intractable complexity. What causes this, and how can this be remedied? We survey a few ideas from "nonstandard dynamic logic" *NDL*[36] [5, 27, 39, 40, 43, 75].

First-order dynamic logic cannot have a decidable proof system, even if we select a small sublanguage, as is shown technically in [4]. The underlying reason is that standard semantics assumes processes to run in standard time, that is, along the natural numbers. If we make dependence on time explicit, however, we get a dynamic logic with an explicit time structure or, in other words, with a generalized semantics. In this nonstandard dynamic logic *NDL*, we are not tied to the natural numbers as the only possible time structure, but we can still make as many explicit requirements (as axioms) about time as we wish. As a consequence of this greater flexibility in models, *NDL* has a decidable proof system; see [5].[37]

More concretely, *NDL* is a three-sorted classical first-order logic whose sorts are the time scale T, the data domain D, and a sort I consisting of (not necessarily all) functions from T into D. We think of the elements of I as objects changing in time ("intensions"). Typically, the content of a machine register changes during an execution of a program, and this register is modeled as an element of I. In standard models, T is the set of natural numbers, and I is the set of all functions from T to D. In models of *NDL*, these domains can be much more general, though we can impose basic reasoning principles about computation such as induction axioms for I (over specified kinds of formulas talking about time), comprehension axioms for intensions, and axioms about the time sort such as successor axioms, order axioms, as well as Presburger arithmetic, or even full Peano arithmetic. Also, special axioms about the data structure may be given according to the concrete application at hand.

NDL talks about programs, processes, and actions, and it can express partial and total correctness plus many other important properties such as concurrency, nondeterminism, or fairness.[38]

---

[36] The nonstandard models for this dynamic logic were influenced by Henkin's generalized models for SOL. More positively, one might also call these systems "logics of general computation."

[37] Here the proof system is decidable, not the set of validities: Sect. 8 has a question about this.

[38] *NDL* has been used for characterizing the "information content" of well-known program verification methods, for comparing powers of program verification methods, as well as for generating new ones [5].

7.2 Digression: Extensions to Space–Time

Similar methods have been applied to space–time theories, such as the relativity theory of accelerated observers; see [66, 85]. There are good reasons for formalizing parts of relativity theory in first-order logic FOL [3]. And then another standard structure for time emerges.[39] Physical processes happen in real time, where the world-line of an accelerated observer is an intensional entity whose spatial location changes with time. It is customary to take the relevant time scale here to be the real number line. But this can again be generalized: this time, to create a special relativity theory *AccRel* of accelerated observers where time structure becomes an explicit parameter subject to suitable physically motivated axioms on the temporal order and the world lines of test particles. While this is a "continuous" rather than a "discrete" temporal setting, many of our earlier points apply. For instance, instead of induction on the natural numbers, one will now have axioms of Dedekind continuity expressing that, if a physical property changes from holding to not holding along a world-line, then there is a concrete point of time when the change takes place.[40] Furthermore, one uses comprehension axioms to ensure existence of enough world-lines for physical purposes, for example, for being able to select the inertial world-lines as those that locally maximize time.

We merely mention one striking outcome of this type of analysis with generalized models of time. The famous twin paradox of relativity theory predicts that one of two twins who departs on a journey and undergoes acceleration will be younger upon her return than the other twin who stayed put. From a logical perspective, deriving this result turns out to involve having the real line as a time structure. Using our more general models, it can be shown that the twin paradox cannot be derived by merely imposing temporal axioms; it is also essential to know how physical processes are related to time in terms of induction or continuity axioms about how properties of test particles change along time.

7.3 Modal Fixed-Point Logic on Generalized Models

Let us now move to a much more abstract view of induction and recursion over temporal or ordinal structures as embodied in modal fixed-point logics. In particular, consider the common running example of the modal *mu-calculus*, which has become a powerful mathematical paradigm for the foundational study of sequential computation.[41]

Consider the standard modal language enriched with a least fixed-point operator $\mu x.\varphi$ for all formulas φ, where x occurs under the scope of an even number of negations.[42] By a *general frame* for fixed point logic we mean a general frame $(X, R, \mathfrak{F}(X))$ as in Sect. 4 such that the family $\mathfrak{F}(X)$ is closed under the fixed-point operators. More precisely,

[39] This kind of logical investigations of relativity theory were motivated by Henkin's suggestion (to Andréka and Németi) for "leaving the logic ghetto," that is, for applying logic in other areas of science.

[40] Cf. [12] for extensive model-theoretic discussion of such properties.

[41] The contrast with the earlier *NDL* approach may be understood in terms of moving from "operational" semantics to "denotational" semantics of programs (cf. [75]), but we will not elaborate on this theme here.

[42] For simplicity, we disregard the greatest fixed-point operator, definable as $\nu x.\varphi = \neg\mu x.\neg\varphi$.

here, inductively, each positive formula φ induces a monotone map $F_\varphi : \mathfrak{F}(X) \to \mathfrak{F}(X)$.[43] Next, we take the intersection of all prefixed points of F_φ from $\mathfrak{F}(X)$, that is, we consider $\bigcap\{S \in \mathfrak{F}(X) \mid F_\varphi(S) \subseteq S\}$. If this intersection belongs to $\mathfrak{F}(X)$ for each φ, then we call $(X, R, \mathfrak{F}(X))$ a *general frame for modal fixed-point logic*. This intersection is then the least fixed-point of F_φ, and it is in fact exactly the denotation of the fixed-point formula $\mu x.\varphi$.

This generalized semantics provides a new way to interpret fixed-point operators. Often, say, in spatial logic, we need to restrict attention to some practically realizable subsets of the plane, and fixed-point operators need to be computed with respect to these subsets only. Note also that we lose nothing: if $\mathfrak{F}(X) = \mathscr{P}(X)$, then our generalized truth condition coincides with the standard semantics of the modal mu-calculus.

A major motivation for studying general-frame semantics for fixed-point logic is that, via existing algebraic completeness and representation methods, every axiomatic system in the language of the modal mu-calculus is complete with respect to its general frames. Moreover, the powerful Sahlqvist completeness and correspondence results from Sect. 4 extend from modal logic to axiomatic systems in the modal mu-language for this semantics; cf. [20]. To appreciate this, we note that completeness results for axiomatic systems of modal fixed-point logic with respect to the standard semantics are very rare and require highly complex machinery; see [61, 96] and also [25, 82].

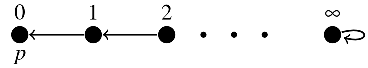
As a further instance of the naturalness of these generalized models, we note a delicate point of algebraic representation theory. Axiomatic systems of the modal "conjugated mu-calculus" axiomatized by Sahlqvist formulas are closed under the well-known *Dedekind–MacNeille completions* in the above general-frame semantics (cf. [19]), whereas no such result holds for the standard semantics (cf. [81]).

Now, following the general line in Sect. 4, we will quickly overview the algebraic semantics for the modal mu-calculus. A modal algebra $(A, \Diamond)$ is a *modal mu-algebra* if for each formula φ positive in x, the meet in A of all the prefixed points of φ exists (see [2] and [20] for details). This meet will be exactly the denotation of the fixed-point formula $\mu x.\varphi$. Similarly to modal logic, the background for our earlier claims of completeness is that axiomatic systems of the modal mu-calculus are complete with respect to modal mu-algebras obtained via the Lindenbaum–Tarski construction; cf. [2, 20]. A modification of the duality between modal algebras and general frames to modal mu-algebras leads to the following notion. A descriptive general frame $(X, R, \mathfrak{F}(X))$ is called a *descriptive mu-frame* if it is a general mu-frame. For each descriptive mu-frame $(X, R, \mathfrak{F}(X))$, the modal algebra $(\mathfrak{F}(X_A), \Diamond_{R_A})$ is a modal mu-algebra. Moreover, we have the following converse: every modal mu-algebra $(A, \Diamond)$ is isomorphic to $(\mathfrak{F}(X_A), \Diamond_{R_A})$ for some descriptive mu-frame $(X, R, \mathfrak{F}(X))$.

The difference between the standard and general-frame semantics for fixed-point operators shows in the following example. Consider the frame $(\mathbb{N} \cup \{\infty\}, R)$ drawn in Fig. 1. We assume that $\mathfrak{F}(\mathbb{N} \cup \{\infty\}) = \{$all finite subsets of $\mathbb{N}$ and cofinite subsets containing the point $\infty\}$. The standard semantics of the formula $\Diamond^* p$ is the set of points that "see points in p with respect to the reflexive and transitive closure of the relation R." Therefore, it is easy to see that the semantics of $\Diamond^* p$ is equal to the set $\mathbb{N}$. Indeed, $\mathbb{N}$ is the least fixed-point of the map $S \mapsto \{0\} \cup \Diamond_R(S)$. However, if we are looking for a least fixed-point of this map in general-frame semantics, then we see that this will be the set $\mathbb{N} \cup \{\infty\}$. We

[43] Here for $S \in \mathfrak{F}(X)$, we have that $F_\varphi(S)$ is the value of φ when the variable x is assigned to S.

Fig. 1 Example



can say that the semantics of the formula $\Diamond^* p$ with respect to the general-frame semantics is the set of all points that "see points in p with respect to the *'admissible' reflexive and transitive closure* of the relation R."[44]

Whereas all this seems to be a smooth extension from the basic modal case, there are also some deficiencies of this generalized semantics. We do not yet have many convincing concrete examples of general mu-frames and especially of descriptive mu-frames. Some simple examples are general frames $(X, R, \mathfrak{F}(X))$, where $\mathfrak{F}(X)$ is a complete lattice. For instance, the lattice of regular open sets $\mathscr{RO}(X)$ in any topological space is a complete lattice, so it allows interpretations of the least and greatest fixed-point operators. However, the meets and joins in such structures $\mathscr{RO}(X)$ are not set-theoretic intersection and union, and analyzing such concrete generalized models for their computational content seems to deserve more scrutiny than it has received so far.

7.4 Digression: A Concrete Fixed-Point Logic for Recursive Programs

Modal fixed-point logics can be related to the earlier systems presented in this section. Our three-sorted system *NDL* essentially described operational semantics for flow-charts, whereas denotational semantics provides more structured meanings for programs than just input/output relations. In particular, [75] extends the Henkin-style explicit-time semantics of *NDL* to a Henkin-style denotational semantics for recursive programs (see also [40, pp. 363–365]). The resulting system NLP_{rec} is a two-sorted classical first-order logic with one sort W for the set of states and another sort S for "admissible" binary relations over W, perhaps subject to the extensionality and comprehension axioms. The semantics for recursive programs p will now be a least admissible fixed-point in such a model.[45] (More details will be discussed in Sect. 8.)

7.5 Comparing Generalized Dynamic Logic and Henkin-Style Second Order Logic

Both the logic *NDL* and fixed-point logic over general models as developed above are strongly connected to Henkin-style SOL (HSOL). We discuss one aspect of this connection here, though a full discussion is beyond the scope of this paper.

In one direction, *NDL* may be viewed as a fragment of Henkin-style SOL:

[44] A similar operation is used in [92] for characterizing in dual terms the notion of subdirectly irreducible modal algebras.

[45] For example, take a recursive program p computing factorial: $p(x) :=$ if $x = 0$ then 1 else $x \cdot p(x-1)$. The state-transition functional F associated with this program is $F(R) := \{\langle 0, 1\rangle\} \cup \{\langle x, x \cdot y\rangle : \langle x-1, y\rangle \in R\}$.

Theorem 11 *NDL admits a natural translation into the language of the Henkin-style second-order logic.*

Proof Idea We would like to interpret *NDL* (hence T, D, I) in HSOL. How can we define the time-structure $(T, +, *)$ of *NDL* from the concepts available in HSOL?[46] Here comes the trick: In our HSOL structure $(D, V, E, E^2, E^3, \ldots)$, we can talk about second-order objects living in V. That is, we can talk about sets, binary and ternary relations. So we reconstruct T as a pair of sets, $H \subseteq D$ and $f \subseteq D \times D$ (i.e., f is a binary relation) with $f : D \to D$ such that (H, f) behaves as the natural numbers with the successor function. For example, we can postulate that there is an object $0 \in H$ that generates the whole of H via f. To see this, it is sufficient to note that for all $H' \subseteq D$, if $0 \in H'$ and H' is closed under f, then $H' \supseteq H$. From the axioms of HSOL we can prove that this definition is correct, that is, that (H, f) is unique up to isomorphism. The latter property is again expressible in HSOL. We skip the rest of the proof, which uses similar ideas. □

A general experience reflected by the literature is that all program properties expressed in HSOL so far were found to be expressible in *NDL*. This motivates the following.

Conjecture 12 *The program properties expressible in the language of Henkin style SOL can also be expressed in NDL.*

We leave a similar detailed comparative discussion of expressive power in generalized frames for the mu-calculus to another occasion.

7.6 Conclusion

We have included logics for computation in this paper since these are often considered a challenge to general model techniques in logic. Indeed, we found two existing approaches to generalizing the semantics of logics with recursive definitions: one replacing standard mathematical structures by a wider class of models for their theory, leading to "generalized counting" on linear orders, the other modifying standard fixed-point semantics in terms of available predicates for approximations. Whereas these generalized approaches to modeling computation seem prima facie different from the earlier ones presented in this paper, in the final analysis presented here, they seem to fall under the general Henkin strategy after all.

[46] Here we need more than monadic HSOL: for example, binary or ternary relations for expressing addition $+$ on T. Therefore, we need to simulate binary and ternary relations in many-sorted HSOL. We simulate binary relations with a ternary "elementhood" relation $E^2 \subseteq D \times D \times V$ such that E^2xyX means that x, y are "objects," X is a binary relation, and Xxy holds (or, in other words, $\langle x, y \rangle \in X$). Similarly, we use a four-place elementhood relation E^3 for simulating ternary relations, etc. Thus, our HSOL model in the general case is of the form $(D, V, E, E^2, E^3, \ldots)$, that is, a structure of the sort considered in Sect. 1.

8 Relating Different Perspectives on Generalized Semantics

We have discussed a number of general perspectives on Henkin's general models and further ways of generalizing semantics by defusing some of the invested set theory. Obvious further questions arise when thinking about connections between the approaches that we have discussed. We will only raise a few of these questions here, and our selection is not very systematic. Even so, the following discussion may help reinforce the interest of thinking about the phenomenon of generalized models in general terms.

8.1 Generalized Fixed-Point Semantics and Absoluteness

We have stated an objection of high complexity against prima facie attractive program logics such as first-order dynamic logic on its standard models. However, another charge would be lack of absoluteness. For a concrete example, consider the fixed-point logic of recursive programs in Pasztor [75].

The meaning of a recursive program p in a model M is defined as the least fixed-point of a functional F associated with p as a unary function over the set W of all possible states for p in M, that is, over $\mathscr{P}(W \times W)$. As an illustration for the idea, consider computing the transitive closure of a relation R as the denotational semantics of a nondeterministic recursive program

$$p(x) = y \quad \text{if either } R(x, y) \quad \text{or} \quad \text{else } \exists z\big[p(x) = z \wedge R(z, y)\big].$$

The functional F associated with this program is

$$F(X) = R \cup (X \circ R), \quad \text{where } \circ \text{ denotes relational composition.}$$

It can be seen that the least fixed-point of the above F is defined by

$$\Phi(x, y) \leftrightarrow \exists \text{ sequence } s \text{ whose domain is a finite ordinal } n+1 \quad \text{such that}$$
$$\big(x = s_0, y = s_n \wedge \forall i < n \, \langle s_i, s_{i+1}\rangle \in R\big).$$

This definition of transitive closure is ZF-absolute. If $\mathbf{V}, \mathbf{V}'$ are ZF-models with $\mathbf{V}'$ a transitive submodel of $\mathbf{V}$, then $\mathbf{V}$ and $\mathbf{V}'$ have the same finite ordinals since ω is ZF-absolute and $\mathbf{V}'$ has to have an ω by the axiom of infinity. However, the definition is not KPU-absolute. Let $\mathbf{V}', \mathbf{V} \models$ KPU where $\mathbf{V}'$ is a transitive submodel of $\mathbf{V}$ from which ω is omitted. This is possible because there is no axiom of infinity in KPU as defined in [8, pp. 10–12]. It may happen that $\mathbf{V}$ has more finite ordinals than $\mathbf{V}'$. Hence, the sequence s in Φ might exist in $\mathbf{V}$ but not in $\mathbf{V}'$. Therefore, for some concrete $x, y \in \mathbf{V}'$, we may have $\Phi(x, y)$ in $\mathbf{V}$ but not in $\mathbf{V}'$. This shows that $\Phi(x, y)$ is not a KPU-absolute formula.

Now Pasztor [75] describes an absolute version of this semantics, which is precisely the logic NLP_{rec} for recursive programs discussed earlier in Sect. 7. In particular, its choice of "admissible transition relations" is entirely analogous to introducing a first-order sort for predicates as we did in Sect. 1. As a result, NLP_{rec} has a complete and decidable inference system. A similar analysis shows how our earlier logic *NDL* can be viewed as the result of making the semantics of first-order dynamic logic absolute in a principled manner.

As a final issue in this connection, recall the earlier-mentioned feature that *NDL* is a recursively axiomatizable but not a decidable theory. Can we make it truly decidable, in line with constructive set theories such as those studied in [24], by merging its semantics with the general assignment models of Sect. 5?[47]

A more general issue arising here is how our earlier motivations for decidable semantics extend to the realm of computation. Our semantics of generalized counting still imports a substantial first-order theory of data structures for the computation to work on and seems an interesting matter for even closer scrutiny. For instance, in line with Sect. 5, we could also define *guarded fragments* of second-order logic whose bounded quantification patterns match restrictions on available assignments of denotations to individual and predicate variables.[48]

8.2 Smoking Out Set Theory in Algebraic Representation

In Sect. 4, we used the case of modal logic to show how general models are completely natural from the viewpoint of algebraic semantics for logical systems—and even the only vehicle that will lead to insightful categorial dualities between algebras and model-theoretic structures.

Even so, second-order logic can hide in unexpected places. For instance, note that the standard Stone representation for Boolean algebra, on which the modal representation is based, *itself employs* a nonabsolute standard set-theoretic object, viz. the *set of all ultrafilters* on a given algebra. This shows in various side effects. One is that the models produced by this method tend to have overly large point sets as their domains. For instance, to represent a countable Boolean algebra, one needs only countable many points to make all necessary distinctions, whereas the Stone space will have uncountably many points.[49]

But a further natural concern may be the heavy dependence of this representation on set-theoretic assumptions like the existence of ultrafilters extending given filters (the "prime ideal theorem"). Indeed, van Benthem [11] raised the question whether this feature can be eliminated and came up with what is essentially a new "possibility semantics" for classical logic where representation takes place in the universe of all consistent sets, by generalizing the semantics of the basic logical operations to the latter setting.[50]

This move suggests a further kind of generalized semantics, where we do not just extend the model class, but where we must also radically generalize our understanding of the meanings of basic expressions in our original language. One way of thinking about this is as turning the tables, using generalized semantics on the very representation method that is supposed to motivate generalized semantics. We leave this as an intriguing unexplored avenue for now.

---

[47] A similar move toward creating decidable theories in physics might follow the lead of Sect. 7 in having only specially designated "space–time trajectories" available in models for physics.

[48] Here guard atoms will now be syntactically "third-order" (though this term loses much of its force in a many-sorted setting), while we must also allow names for designated objects and predicates. We leave technical details of such more radical decidability perspectives for another occasion.

[49] To see that a countable structure suffices, take a countable elementary substructure of the Stone space.

[50] This theme will be resumed in the forthcoming study [16].

8.3 Richer Languages for Generalized Models

We have seen that generalized assignment semantics supports richer first-order languages with natural new operators. This phenomenon is well known from other areas of logic, such as linear logic and other resource-conscious logics, where classical logical operators "split" into several variants, whereas new operators emerge as well. Again, an immediate test question is then whether the same happens in second-order logic, the realm where Henkin started his analysis. Do generalized predicate models support natural extensions of the usual second-order language?

We have not been able to find a good example that stays inside FOL. However, one can add a new modality, as we did for general assignment models in Sect. 5, that says that the matrix formula is true *in some extension of the current predicate range*. Interesting things then become expressible, and the usual language of second-order logic gets embedded, but we are not even sure what sort of higher-order logic this system would really be.

8.4 Trade-Offs Between Generalized Semantics and Language Fragments

We have seen a close analogy between first-order logic with general assignment models and the guarded fragment over standard models, as almost two sides of the same coin. Now well-chosen fragments are of great logical interest, and we noted their importance already in understanding how second-order logic works at deeper semantic and computational levels. What can we say about this in general? Can we reanalyze basic fragments of second-order logic in terms of generalized semantics for the whole language?[51]

8.5 Coda: Nonstandard Truth Conditions

We conclude with one possibly more radical style of generalizing a semantics, by changing truth definitions, a theme that has come up occasionally in our earlier discussions, for example, on constructivizing representation methods.

As a concrete instance, consider the general assignment semantics of Sect. 5 as embodying our theme of "lowering complexity," taking FOL from undecidable to decidable by removing extraneous mathematics from the core definitions of the logical system. Can we lower complexity even further, from decidable to "feasible" logics, whose satisfiability or validity problem would be decidable in polynomial time? No such proposal seems to exist, and it may be of interest to see where the difficulty lies. The complexity of decidability for many logics has to do with the exponential combinatorial explosion associated with *disjunctions*. Enlarging the model class does not seem to solve this, and what may be needed is rather a change in the way in which we interpret disjunction.[52]

[51] As a related issue, could well-known key fragments of *SOL* be captured through their insensitivity in passing between standard models and matching generalized models yet to be found?

[52] One such proposal from the folklore has been to read disjunction as linear combination.

Generalized semantics by tinkering with the interpretations of logical operators can be very powerful. One interesting example of lowering complexity while improving model-theoretic behavior are modal "bisimulation quantifiers" (Hollenberg [55]) that step outside of current models in their interpretation. A bisimulation quantifier is a modality $\langle p\rangle\varphi$ interpreted in pointed models (M, s) as saying that there exists a bisimulation, for the whole language except for the propositional letter p, between (M, s) and some model (N, t) such that φ is true in (N, t). One can think of this as a "tamed form" of second-order quantification over the property p—and its effect is to leave modal logic decidable and even preserves its striking model-theoretic properties such as uniform interpolation.[53]

A similar move would be possible for second-order logic, replacing bisimulation by potential isomorphism, generalizing its semantics in very different ways from the above. However, we will not pursue this more radical form of meaning-changing generalized semantics here.

9 Conclusion

This paper started with an account of Henkin models for second-order logic. To some people, this technique may look like a trick for lazy people, but we have shown how this circle of ideas keeps returning in the field for good reasons. We surveyed some major manifestations of generalized semantics ranging from set theory to algebra and computational logic, some close to Henkin's models, others more remote—and explored the many interesting technical and conceptual issues and results that these bring to light.[54] Our main suggestion has been that it is well-worth thinking about the phenomenon of generalized semantics more generally, both in terms of general insights and concrete technical results when their main features are out side by side. We feel that we have only scratched the surface of what might be a deeper theoretical understanding of what is going on here.[55]

While we have proposed taking a critical look at unwarranted "set-theoretic imperialism" and unquestioning acceptance of set-theoretic structures without a cost-benefit analysis, we have not advocated doing away with set theory altogether since that would mean abandoning a lingua franca that has served the field of logic so well. Likewise, we have not proposed abandoning second-order logic: it is attractive to look at behavior on standard models, and that attraction will not easily go away.[56] Finally, our tolerance for generalized semantics may seem almost all-embracing at times, but we certainly would

[53] Added to propositional dynamic logic, bisimulation quantifiers yield the modal mu-calculus.

[54] Even so, our survey is by no means complete. For example, Gabriel Sandu has suggested that the "substitution account" for the meaning of first-order quantifiers would be another natural candidate for our style of analysis.

[55] To mention just one more delicate point, it is argued in [58] that second-order logic with generalized semantics is model-theoretically stronger than first-order logic because even weak comprehension in the second-order part brings about "strong instability" in the model-theoretic sense. In another vein, [58] compares second-order logic with generalized semantics (plus comprehension) to full second-order rather than to first-order logic. What emerges then is that, by forcing one obtains model-theoretic results for full second-order logic, but with generalized models one can eliminate forcing.

[56] We refer to [89] for a state-of-the-art discussion of set theory side by side with second-order logic.

not advocate trying to make general sense of every auxiliary move by logicians. Some practices in a field are just ad hoc, and nothing is gained by pretending otherwise.

Finally, however, we would like to emphasize the original motivation for writing this paper once more. It is always worthwhile to think about the achievements of Leon Henkin and what they mean in a broader perspective, and we have done so triggered by the various references to his work in this paper. In addition, we hope that the free-thinking and sometimes playful way in which we have done so reflects something of Henkin's open-minded personality that one would do well to emulate.

Acknowledgements The authors wish to thank several colleagues for their helpful comments, including Solomon Feferman, Maria Manzano, Vaughan Pratt, Gabriel Sandu, and, especially, Jouko Väänänen.

The first and fourth authors would like to acknowledge the support of the Hungarian National Grant for Basic Research No. T81188. The third author would like to acknowledge the support of the Rustaveli Foundation of Georgia grant FR/489/5-105/11.

References

1. Akkanen, J.: Absolute logics. Annales Academia Scientiarum Fennica, Series A, I. Mathematica Dissertationes, vol. 100. Suomalainen Tiedeakatemia, Helsinki (1995)
2. Ambler, S., Kwiatkowska, M., Measor, N.: Duality and the completeness of the modal μ-calculus. Theor. Comput. Sci. **151**, 3–27 (1995)
3. Andréka, H., Madarász, J., Németi, I.: Logical axiomatizations of space-time. Samples from the literature. In: Prékopa, A., Molnár, E. (eds.) Non-Euclidean Geometries: János Bolyai Memorial Volume. Mathematics and Its Applications, vol. 581, pp. 155–185. Springer, Berlin (2006)
4. Andréka, H., Németi, I., Sain, I.: Completeness problems in verification of programs and program schemes. In: Mathematical Foundations of Computer Science'79. LNCS, vol. 74, pp. 208–218. Springer, Berlin (1979)
5. Andréka, H., Németi, I., Sain, I.: A complete logic for reasoning about programs via nonstandard model theory. Theor. Comput. Sci. **17**(3), 193–212 (1982). Part II in Number 3, pp. 259–278
6. Andréka, H., van Benthem, J., Németi, I.: Modal logics and bounded fragments of predicate logic. J. Philos. Log. **27**(3), 217–274 (1997)
7. Barwise, J.: Absolute logics and $\mathbf{L}_{\infty\omega}$. Ann. Math. Log. **4**, 309–340 (1972)
8. Barwise, J.: Admissible Sets and Structures. Springer, Berlin (1975)
9. Barwise, J., Seligman, J.: Imperfect information flow. In: Ong, L. (ed.) LICS 1993, LNCS, pp. 252–260. Springer, Berlin (1993)
10. van Benthem, J.: Two simple incomplete modal logics. Theoria **44**(1), 25–37 (1978)
11. van Benthem, J.: Partiality and non-monotonicity in classical logic. Log. Anal. **29**, 225–247 (1986)
12. van Benthem, J.: The Logic of Time. Synthese Library, vol. 156. Reidel, Dordrecht (1991)
13. van Benthem, J.: Exploring Logical Dynamics. Studies in Logic, Language and Information. CSLI Publications/Cambridge University Press, Stanford/Cambridge (1996)
14. van Benthem, J.: Information transfer across chu spaces. Log. J. IGPL **8**(6), 719–731 (2000)
15. van Benthem, J.: Guards, bounds, and generalized semantics. J. Log. Lang. Inf. **14**(3), 263–279 (2005)
16. van Benthem, J., Bezhanishvili, N., Holliday, W.: Modal logic based on possibilities. In preparation
17. van Benthem, J., Doets, K.: Higher-order logic. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. 1, pp. 275–329. Springer, Dordrecht (1983)
18. Bezhanishvili, G.: Stone duality and Gleason covers through de Vries duality. Topol. Appl. **157**(6), 1064–1080 (2010)
19. Bezhanishvili, N., Hodkinson, I.: Preservation of Sahlqvist fixed point equations in completions of relativized fixed point BAOs. Algebra Univers. **68**, 43–56 (2012)
20. Bezhanishvili, N., Hodkinson, I.: Sahlqvist theorem for modal fixed point logic. Theor. Comput. Sci. **424**, 1–19 (2012)

21. Bishop, E.: Foundations of Constructive Analysis. McGraw-Hill, New York (1967)
22. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2001)
23. Blok, W.: On the degree of incompleteness of modal logics. Bull. Sect. Log. **7**(4), 167–175 (1978)
24. Cantone, D., Omodeo, E.G., Policriti, A.: Set Theory for Computing. From Decision Procedures to Logic Programming with Sets. Monographs in Computer Science. Springer, Berlin (2001)
25. ten Cate, B., Fontaine, G.: An easy completeness proof for the modal μ-calculus on finite trees. In: Ong, L. (ed.) FOSSACS 2010. LNCS, vol. 6014, pp. 161–175. Springer, Berlin (2010)
26. Chagrov, A., Zakharyaschev, M.: Modal Logic. Oxford Logic Guides, vol. 35. Clarendon Press, New York (1997)
27. Csirmaz, L., Paris, J.: A property of 2-sorted Peano models and program verification. Z. Math. Log. **30**, 325–334 (1984)
28. Davey, B., Priestley, H.: Introduction to Lattices and Order, 2nd edn. Cambridge University Press, New York (2002)
29. Dunn, M.: Gaggle Theory: An Abstraction of Galois Connections and Residuation, with Applications to Negation, Implication, and Various Logical Operators. Lecture Notes in Computer Science, vol. 478, pp. 31–51. Springer, Berlin (1991)
30. Dunn, M., Gehrke, M., Palmigiano, A.: Canonical extensions and relational completeness of some substructural logics. J. Symb. Log. **70**(3), 713–740 (2005)
31. Ebbinghaus, H.-D., Flum, J., Thomas, W.: Mathematical Logic, 2nd edn. Springer, Berlin (1994)
32. Enderton, H.: Finite partially-ordered quantifiers. Z. Math. Log. **16**, 393–397 (1970)
33. Esakia, L.: Topological Kripke models. Sov. Math. Dokl. **15**, 147–151 (1974)
34. Feferman, S.: Set-theoretical invariance criteria for logicality. Notre Dame J. Form. Log. **51**(1), 3–20 (2010)
35. Galatos, N., Jipsen, P.: Residuated frames with applications to decidability. Trans. AMS **365**, 1219–1249 (2013)
36. Gallin, D.: Intensional Higher-Order Logic. North-Holland, Amsterdam (1975)
37. Ganter, B., Wille, R.: Formal Concept Analysis—Mathematical Foundations. Springer, Berlin (1999)
38. Gehrke, M.: Generalized Kripke frames. Stud. Log. **84**(2), 241–275 (2006)
39. Gergely, T., Úry, L.: First-Order Programming Theories. EATCS Monographs on Theor. Comp. Sci., vol. 24. Springer, Berlin (1991)
40. Gil-Luezas, A., Hortalá-González, T., Rodriguez-Artalejo, M.: Standard versus nonstandard semantics in logics for functional programs. In: Meinke, K., Tucker, J.V. (eds.) Many-Sorted Logics and Their Applications in Computer Science, Proc. Leeds, August 1988, pp. 343–391. Wiley, New York (1993)
41. Gödel, K.: I, Dissertation, pp. 60–101, University of Vienna (1929). Reprinted in Gödel (1986)
42. Grädel, E., Thomas, W., Wilke, T. (eds.): Automata, Logics, and Infinite Games. Springer, Berlin (2002)
43. Hájek, P.: Making dynamic logic first-order. In: Mathematical Foundations of Computer Science'81 (Proc. Strbske Pleso 1981). Lecture Notes in Computer Science, vol. 118, pp. 278–295. Springer, Berlin (1981)
44. Harel, D.: Recurring dominoes: making the highly undecidable highly understandable. Ann. Discrete Math. **24**, 51–72 (1985)
45. Henkin, L.: The completeness of the first-order functional calculus. J. Symb. Log. **14**, 159–166 (1949)
46. Henkin, L.: Completeness in the theory of types. J. Symb. Log. **15**, 81–91 (1950)
47. Henkin, L.: Some remarks on infinitely long formulas. In: Infinitistic Methods. Proceedings of the Symposium on Foundations of Mathematics, Warsaw 2–9 September 1959, pp. 167–183. Panstwowe Wydawnictvo Naukowe, Warsaw (1961)
48. Henkin, L.: Logical systems containing only a finite number of symbols. In: Séminaire de Mathématiques Supérieures, vol. 21, p. 48. Les Presses de l'Université de Montréal, Montréal (1967)
49. Henkin, L.: Internal semantics and algebraic logic. In: Leblanc, H. (ed.) Truth, Syntax and Modality, pp. 111–127. North-Holland, Amsterdam (1973)
50. Henkin, L., Monk, D.: Cylindric algebras and related structures. In: Proceedings of the Tarski Symposium, pp. 105–121. AMS, Providence (1974)

51. Henkin, L., Monk, D., Tarski, A.: Cylindric Algebras. Part II. North-Holland, Amsterdam (1985)
52. Henkin, L., Resek, D.: Relativization of cylindric algebras. Fundam. Math. **82**, 363–383 (1975)
53. Hilbert, D., Ackermann, W.: Grundzügen der theoretischen Logik. Springer, Berlin (1928)
54. Hintikka, J., Sandu, G.: What is the logic of parallel processing? Int. J. Found. Comput. Sci. **6**(1), 27–49 (1995)
55. Hollenberg, M.: Logic and bisimulation. Ph.D. thesis, University of Utrecht (1998)
56. Holliday, W.: Partiality and adjointness in modal logic. In: Gore, R., Kooi, B., Kurucz, A. (eds.) Advances in Modal Logic (AiML 2014), pp. 313–333 (2014). College Publications
57. Humberstone, L.: From worlds to possibilities. J. Philos. Log. **10**, 313–339 (1981)
58. Hyttinen, T., Kangas, K., Väänänen, J.: On second-order characterizability. Log. J. IGPL **21**(5), 767–787 (2013)
59. Jónsson, B., Tarski, A.: Boolean algebras with operators, Part I. Am. J. Math. **73**, 891–939 (1951)
60. Kaye, R.: Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies (Edited by J. Kennedy and R. Kossak). Cambridge University Press, Cambridge (2011)
61. Kozen, D.: Results on the propositional μ-calculus. Theor. Comput. Sci. **27**, 333–353 (1983)
62. Kracht, M.: Tools and Techniques in Modal Logic. Studies in Logic and the Foundations of Mathematics, vol. 142. North-Holland, Amsterdam (1999)
63. Kunen, K.: Set Theory. Studies in Logic and the Foundations of Mathematics, vol. 102. North-Holland, Amsterdam (1980)
64. Lindström, P.: A note on weak second order logic with variables for elementarily definable relations. In: The Proceedings of the Bertrand Russell Memorial Conference (Uldum, 1971), Bertrand Russell Memorial Logic Conf., Leeds, 1973, pp. 221–233
65. Lopez-Escobar, E.: Formalizing a non-linear Henkin quantifier. Fundam. Math. **138**(2), 93–101 (1991)
66. Madarász, J., Németi, I., Székely, G.: Twin paradox and the logical foundation of relativity theory. Found. Phys. **36**, 681–714 (2006)
67. Manders, K.L.: First-order logical systems and set-theoretic definability. Dept. of Philosophy and Mathematics, University of Pittsburgh, pp. 1–37 (1979)
68. Mann, A., Sandu, G., Sevenster, M.: Independence-Friendly Logic: A Game-Theoretic Approach. Cambridge University Press, Cambridge (2011)
69. Manzano, M.: Extensions of First Order Logic. Cambridge University Press, Cambridge (1996)
70. Manzano, M.: Henkin on completeness. In: This Volume (2014)
71. Marra, V., Spada, L.: The dual adjunction between MV-algebras and Tychonoff spaces. Stud. Log. **100**(1–2), 253–278 (2012)
72. Montague, R.: Reduction of higher-order logic. In: Theory of Models (Proc. 1963 Internat. Sympos. Berkeley), pp. 251–264. North-Holland, Amsterdam (1965)
73. Mostowski, A.: A formula with no recursively enumerable model. Fundam. Math. **XLIII**, 125–140 (1955)
74. Németi, I.: Free algebras and decidability in algebraic logic. Doctoral Dissertation for the Hungarian Academy of Sciences (1986). Translation of the relevant chapter is: Decidability of weakened versions of first-order logic. in: Logic Colloquium'92, Stanford, CSLI Publications (1995), pp. 177–241
75. Pasztor, A.: Recursive programs and denotational semantics in absolute logics of programs. Theor. Comput. Sci. **70**, 127–150 (1990)
76. Pratt, V.R.: Chu spaces (1999). Course notes for the school in category theory and applications
77. Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bull. Lond. Math. Soc. **2**, 186–190 (1970)
78. Priestley, H.A.: Ordered topological spaces and the representation of distributive lattices. Proc. Lond. Math. Soc. **24**, 507–530 (1972)
79. Rabin, M.O.: Decidability of second-order theories and automata on infinite trees. Trans. Am. Math. Soc. **141**, 1–35 (1969)
80. Sambin, G., Vaccaro, V.: Topology and duality in modal logic. Ann. Pure Appl. Log. **37**(3), 249–296 (1988)
81. Santocanale, L.: Completions of μ-algebras. Ann. Pure Appl. Log. **154**(1), 27–50 (2008)
82. Santocanale, L., Venema, Y.: Completeness for flat modal fixpoint logics. Ann. Pure Appl. Log. **162**(1), 55–82 (2010)

83. Simpson, S.: Subsystems of Second Order Arithmetic, 2nd edn. Cambridge University Press, Cambridge (2010)
84. Stebletsova, V.: Algebras, relations and geometries, an equational perspective. In: Publications of the Zeno Institute of Philosophy, vol. 32 (2000)
85. Székely, G.: First-order logic investigation of relativity theory with an emphasis on accelerated observers. Ph.D. thesis, Eötvös Loránd University, Faculty of Sciences, Institute of Mathematics, Budapest (2009)
86. Thomason, S.K.: An incompleteness theorem in modal logic. Theoria **40**(1), 30–34 (1974)
87. Väänänen, J.: Set theoretic definability of logics. In: Barwise, J., Feferman, S. (eds.) Model Theoretic Logics, Studies in Logic and Practical Reasoning, pp. 599–643. Springer, Berlin (1985)
88. Väänänen, J.: Dependence Logic: A New Approach to Independence Friendly Logic. Cambridge University Press, Cambridge (2007)
89. Väänänen, J.: Second order logic or set theory? Bull. Symb. Log. **18**(1), 91–121 (2012)
90. Vaught, R.L.: Sentences true in all constructive models. J. Symb. Log. **25**, 39–53 (1960)
91. Venema, Y.: Many-dimensional modal logic. Ph.D. thesis, ILLC, University of Amsterdam (1992)
92. Venema, Y.: A dual characterization of subdirectly irreducible BAOs. Stud. Log. **77**(1), 105–115 (2004)
93. Venema, Y.: Algebras and coalgebras. In: Blackburn, P., van Benthem, J., Wolter, F. (eds.) Handbook of Modal Logic. Studies in Logic and Practical Reasoning, vol. 3, pp. 331–426. Elsevier, Amsterdam (2007)
94. de Vries, H.: Compact spaces and compactifications. An algebraic approach. Ph.D. thesis, University of Amsterdam (1962)
95. Walkoe, W.: Finite partially-ordered quantification. J. Symb. Log. **35**, 535–555 (1970)
96. Walukiewicz, I.: Completeness of Kozen's axiomatisation of the propositional μ-calculus. Inf. Comput. **157**(1–2), 142–182 (2000). LICS 1995 (San Diego, CA)

H. Andréka · I. Németi
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, PF 127, 1364, Hungary

H. Andréka
e-mail: andreka@renyi.hu

I. Németi
e-mail: nemeti.istvan@mta.renyi.hu

J. van Benthem · N. Bezhanishvili (✉)
Institute for Logic, Language and Computation (ILLC), University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands
e-mail: N.Bezhanishvili@uva.nl

J. van Benthem
e-mail: johan.vanbenthem@uva.nl

Appendix
Curriculum Vitae: Leon Henkin

1 Academic Career

The information to elaborate this section was provided by Professor Emeritus of Mathematics John Addison (Department of Mathematics UC Berkeley), in 2009, during a short visit of María Manzano.

1.1 Formation

Leon Henkin went to Lincoln High school, graduated at 16, and went to Columbia College, receiving a B.A. degree in Mathematics and Philosophy in 1941. In 1942 he obtained a M.A. in Mathematics at Princeton, and 5 years later he received a Ph.D. at the same university, under the supervision of Alonzo Church.

1.2 Academic Positions

His beginnings in higher-education lecturing and teaching took place in University of Southern California as Assistant Professor of Mathematics in the period 1949–1953. In 1953 he moved at Berkeley with the same category, obtaining the position of Professor Emeritus in 1991. From 1991 onwards, he was at Mills College, where he collaborated to start the *Summer Math Institute at Mills*, which aim was to prepare mathematically talented undergraduate women students (nationally) for graduate school in mathematics.

Although his academic life was always linked to teaching at Berkeley, Henkin on many occasions traveled abroad with scholarships and grants from various institutions. He had two *Fulbright Research Grants,* one during 1954–1955 in Amsterdam and another in 1979 in Haifa, Israel. During 1960–1961, he was in Dartmouth College, and during 1968–1969 in Souls College (Oxford), in both cases as a *Visiting Professor*. These visits led him to establish a permanent contact with the Netherlands, Belgique, United Kingdom, Israel, Spain, Portugal, Hungary, and France.

M. Manzano et al. (eds.), *The Life and Work of Leon Henkin*, Studies in Universal Logic,
DOI 10.1007/978-3-319-09719-0

1.3 Service to the Department and to the University

Henkin accepted the responsibility as the chairman or acting chairman of the *Department of Mathematics* in four occasions but always for short periods of time. He was chairman of the *Group in Logic and Methodology of Science* on a couple of occasions, a research group with which he felt a strong commitment during his career. It is specially relevant the *Mathematics Opportunity Committee* to which he joined in forming: his membership began in 1970, not leaving it until the early 1980s. There Henkin assumed responsibilities as Chairman on several occasions. From 1981 he began to collaborate with Berkeley *SESAME* program, an interdisciplinary academic unit dedicated to advance the understanding and practice of learning and teaching in science, engineering, and mathematics.

1.4 Collaboration with US Administration Projects

His collaboration with the American administration began in the time of war when serving first in Belmar NJ, 1942–1943, then in the Manhattan District Project in NY, 1943–1945, and finally 1945–1946 in the headquarters of Union Carbide, acquiring some responsibility within the Separation Performance Group.

During 1958–1959, he acted as a consultant for the Air Force Research Group, and during 1960–1963, he was an adviser of the Office of Naval Research.

Along his life, Henkin took part in a large number of committees connected one way or another with federal projects. The interest for minorities and education was in many cases an outstanding reason to accept such responsibilities.

1.5 Editorial Boards

Henkin took part in various editorial committees of relevant periodical publications. Among this work, the one he did in the *Transactions of the American Mathematical Society* deserves special attention, where he held various positions between 1955 and 1967. He also had a long-term collaboration with the *Journal of Symbolic Logic*, which extends from 1953 to 1960, and acted as a referee in numerous publications during the 1960s. A whole chapter of this book, *Leon Henkin the Reviewer*, is devoted to that contribution of Henkin.

1.6 Scientific Associations and Committees

Henkin's dedication to the promotion of scientific activity through various associations was extensive and consistent throughout his career. He devoted careful attention to what could be considered "top level scientific associations," as well as to open doors to women and minorities in his field. It is worth mentioning his role in the *Association for Symbolic Logic*, of which he was President during the period 1962–1964 and vice-president from

1952 to 1954. Additionally, it is noteworthy his participation both in the *National Science Foundation* (NSF), as an adviser and consultant, and in the *National Academy of Sciences*. He was also a member of the *Mathematical Association of America*, where he was part of the *Committee on Educational Media*, in accordance with his interest in the media and the dissemination of science, especially mathematics.

Regarding the committees and institutions to promote equality among minorities to which he belonged, the list is long. Perhaps the most relevant along time is the *Committee on Special Scholarships* to which he adhered in 1963 and was linked till the end of his academic career. He was president in 1969–1970 of the *Berkeley Chapter of the American Association of University Professors*, and from 1970 onwards he was a member of the *Community Teaching Fellowship Program*, a statewide program operating from the University Office of the President. Henkin's interest in promoting equality between women and ethnic minorities is particularly manifest through his collaboration with the *Academic Senate Committee on Special on Status for Women and Ethnic Minorities*.

Finally, his concern with purely didactic aspects of mathematics and logic was also a constant in his career. For instance, since 1980 he served as Chair on the *U.S. Commission on Mathematics Instruction*, until at least 1984. He also participated in the organization of several conferences centered on education and teaching, such as the *International Congress on Mathematics Education* held in 1979–1980 and the *National Conference on Mathematics Education*.

1.7 Invited Conferences

Henkin's local and international recognition arrived relatively early, and hence the number of invited lectures in which he participated throughout his career was relatively large. It is remarkable that those lectures were initially focused on topics such as cylindrical algebras or algebraic logic, later evolving to aspects related to education as well as to logic foundation of mathematics.

The academic year of 1954–1955 was especially prolific in conferences focused on cylindrical algebras given in Europe, especially in various universities in the Netherlands, Belgium, Norway, and Denmark. His address at the *Symposium on Theory of Models* organized in Amsterdam, *"A representation theorem for cylindrical algebras"*, was particularly relevant. In the summer of 1957 he was section leader—and gave lectures—at the *NSF Institute on Logic*, at Cornell University, the first of a cavalcade of large-scale meetings devoted to logic.

His vocation for educational aspects of mathematics and the history of logic emerged in the early 1960s with talks as *"New directions in secondary school mathematics"*, taught at Dartmouth College in 1961–1962, *"Survey of mathematical logic"* and *"Mathematical logic in perspective"*. By that time he also presented his work on propositional type theory, *"A theory of propositional types"*, and on interpolation theorem, *"An interpolation theorem of mathematical logic"*, and he began his first collaborations with broadcast companies with an interview entitled *"Scholarship at the Institute for Advanced Study"*.

In the 1970s he continued presenting his work on algebraic logic and mathematical education. From that period, two invited talks should be highlighted: *"Induction models"* and *"Copernican revolution in mathematics and the idea of the impossible"*. Later, in the 1980s, he developed an intense international activity at many European universities: Belgrade, Kragujevac, Novi-Sad, Madrid, Seville, Barcelona, Lisbon.

1.8 Regular Courses Taught

Henkin never gave up on teaching activity, which he combined with an intense activity as a researcher. One of his first official courses, *"Calculus with Analytic Geometry"*, seems to correspond to the typical responsibilities of a young teacher, but even in this initial period, it can appreciated an attempt to pay attention to his real interests. During 1964–1965, he taught *"Metamathematics"*—two semesters, and the following year *"Mathematical Logic"*—two semesters; he alternated both courses practically throughout his whole teaching career. In the late 1960s, he began to offer *"Concepts of Mathematics for Elementary School Teachers"*, which he taught until the mid 1970s. Other constant along his teaching was a seminar on *"Theory of Cylindric Algebras"* offered during the 1980s. Apart from these items, the rest of his teaching was diverse: *"Abstract Algebra"*, *"Algebra and Trigonometry"*, *"Finite Mathematics"*, *"Foundations of Geometry"*, *"Introduction to the Theory of Sets"* are some of the courses offered by Henkin along his career. Henkin seems to have had no tendency to settle in the area of his main interest accepting, even at the end of his career, subjects that were not strictly related to his research or direct interests. He was always able to switch from a topic into another, showing in every case his potential as a teacher, and his ability to explain complex stuff to every kind of people.

1.9 Doctorate Supervision

The list of Ph.D. students under the supervision of Henkin contains a total nine students, eight of which belong to Berkeley—Hadar Nitsa and Diane Resek, collaborating in this volume are among them—and one to Southern California. It is noteworthy that the last four Ph.D. at Berkeley have to do with aspects of teaching mathematics or the integration of minorities in this type of research.

1. Carol Ruth Karp (1959) *Languages with expressions of infinite length.*
2. Daniel Demaree (1970) *Studies in algebraic logic* (advisor 2. Dana Stewart Scott, unofficial advisor J.D. Monk)
3. Daniel Gallin (1972) (advisor 2. Dana Stewart Scott) *Intensional and higher-order modal logic.*
4. Diane Resek (1974) *Some results on relativized cylindric algebras*.
5. Nitsa Hadar (1975) *Children's conditional reasoning*.
6. David Lawrence Ferguson (1980) *The language of mathematics—how calculus students cope with it.*
7. Hadas Rin (1982) *Linguistic barriers to students' understanding of definitions in a college mathematics course*.
8. Francoise Tourniaire (1984) *Proportional reasoning in grades three, four, and five.*

9. Philip Treisman (1985) *A study of the mathematics performance of black students at the University of California, Berkeley.*

There is an additional list of six other students for whom Henkin was the official chairman or co-chairman of the dissertation committee but for whom the actual chairman was outside the Department. It is relevant to note that in these cases the topics are strictly formal.

1. Charles Malone Howard (1965) (co-chairman with William Craig; unofficial co-advisor J.D. Monk) *An approach to algebraic logic*.
2. Edgar G.K. Lopez-Escobar (1965) (co-chairman with Dana Scott) *Infinitely long formulas with countable quantifier degrees.*
3. Peter Krauss (1966) (co-chairman with Dana Scott) *Probability logic*.
4. Michel Jean (1968) (co-chairman with William Craig) *Pure structures*.
5. Mohamed Amer (1969) (co-chairman with William Porter Hanf) *Boolean algebras of sentences of first-order logic.*
6. Leonard Lipner (1970) (co-chairman with William Craig) *Some aspects of generalized quantifiers.*

There are at least four other students Henkin supervised during their postdoctoral stays: Newton da Costa (Brazil), H.D. Ebbinghaus (Germany), Irineu Bicudo (Brazil), Maria Manzano (Spain). Some of these researchers came from countries without a strong logical tradition, as a result of which Henkin became a reference for the development of logical research in these places; such effect was specially notable in Spain or Brazil.

1.10 Awards

Henkin was awarded the *Chauvenet Prize* in 1964 for his paper *"Are Logic and Mathematics Identical?"—Mathematical Association of America* [32] award to the author of an outstanding expository article on a mathematical topic by a member of the Association.

In 1972, his paper *"Mathematical Foundations for Mathematics"* [52] granted the *Lester R. Ford Award*, which recognizes authors of articles of expository excellence published in *The American Mathematical Monthly* or *Mathematics Magazine*.

In the last years of his career, he was laureate several times in recognition of his work for mathematics education for minorities. He received the *Mathematics Sciences Education Board for Contributions to Making Work outstanding Minorities*, and the *Mathematics Undergraduate Student Association Distinguished Teaching Award*, both in 1990; in the same year, he was the first recipient of the *Gung and Hu Award for Distinguished Service to Mathematics*.

In 1991, Henkin received the *Berkeley Citation*, in recognition to his entire career at that university, and in 1998, he received the *Presidential Award for Excellence in Science, Mathematics, and Engineering*. Finally, the Berkeley campus established a prize inspired by Henkin's life and work, the *Leon Henkin Citation for Distinguished Service*, devoted to recognize the *"exceptional commitment to the educational development of students from groups who are underrepresented in the academy"*.

2 Books, Articles, and Abstracts

1. Henkin, L.: The completeness of the first-order functional calculus. The Journal of Symbolic Logic 14(3), 159–166 (1949)
2. Henkin, L.: Fragments of the propositional calculus. The Journal of Symbolic Logic 14(1), 42–48 (1949)
3. Henkin, L.: Completeness in the theory of types. The Journal of Symbolic Logic 15(2), 81–91 (1950) MR0035006 (11,675e)
4. Henkin, L.: A problem on inverse mapping systems. Proceedings of the American Mathematical Society 1(2), 224–225 (1950)
5. Henkin, L.: An algebraic characterization of quantifiers. Fundamenta Mathematicae 37, 63–74 (1950)
6. Henkin, L.: Banishing the rule of substitution for functional variables. The Journal of Symbolic Logic 18(3), 201–208 (1953)
7. Henkin, L.: On the primitive symbols of Quine's 'Mathematical Logic'. Revue Philosophique de Louvain 51(32), 591–593 (1953)
8. Henkin, L.: Some interconnections between modern algebra and mathematical logic. Transactions of the American Mathematical Society 74(3), 410–427 (1953)
9. Henkin, L.: Some notes on nominalism. The Journal of Symbolic Logic 18(1), 19–29 (1953)
10. Henkin, L.: A generalization of the concept of ω-consistency. The Journal of Symbolic Logic 19(3), 183–196 (1954)
11. Henkin, L.: Boolean representation through propositional calculus. Fundamenta Mathematicae 41(1), 89–96 (1954)
12. Henkin, L.: Γ-completeness. In: Proceedings of the International Congress of Mathematicians, held at Amsterdam, 2–9 September, vol. 2, pp. 403–404. North-Holland, Amsterdam (1954)
13. Henkin, L.: Metamathematical theorems equivalent to the prime ideal theorems for Boolean algebras. Preliminary report. Bulletin of the American Mathematical Society 60(4), 387–388 (1954)
14. Henkin, L.: On a theorem of Vaught. Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings, series A, vol. 58, and Indagationes Mathematicae 17, 326–328 (1955)
15. Henkin, L.: The algebraic structure of mathematical theories. Bulletin de la Société de Mathématique de Belgique 7, 131–136 (1955)
16. Henkin, L.: The nominalistic interpretation of mathematical language. Bulletin de la Société de Mathématique de Belgique 7, 137–141 (1955)
17. Henkin, L.: The representation theorem for cylindrical algebras. In: Skolem, Th., Hasenjaeger, G., Kreisel, G., Robinson, A., Wang, Hao, Henkin, L., Łoś, J. (eds.) Mathematical Interpretation of Formal Systems, pp. 85–97. North-Holland, Amsterdam (1955)
18. Skolem, Th., Hasenjaeger, G., Kreisel, G., Robinson, A., Wang, Hao, Henkin, L., Łoś, J. (eds.): Mathematical Interpretation of Formal Systems. Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam (1955)
19. Henkin, L.: La structure algébrique des théories mathématiques. Collection de Logique Mathématique, Série A, no. XI. Gauthier-Villars, Paris (1956)
20. Henkin, L.: Two concepts from the theory of models. The Journal of Symbolic Logic 21(1), 28–32 (1956)

21. Montague, R., Henkin, L.: On the definition of 'formal deduction'. The Journal of Symbolic Logic 21(2), 129–136 (1956)
22. Henkin, L., Tarski, A.: Cylindrical algebras. In: Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University, vol. 3, pp. 332–340. American Mathematical Society, Providence (1957) [2nd ed., pp. 332–340. Communications Research Division, Institute for Defense Analyses, Princeton (1960)]
23. Henkin, L.: Cylindrical algebras of dimension 2. Preliminary report. Bulletin of the American Mathematical Society 63(1), 26 (1957)
24. Henkin, L.: Sums of squares. In: Summaries of talks presented at the Summer Institute of Symbolic Logic, Cornell University, vol. 2 (1957) [2nd ed., pp. 284–291. Communications Research Division, Institute for Defense Analyses, Princeton (1960)]
25. Henkin, L.: A generalization of the concept of ω-completeness. The Journal of Symbolic Logic 22(1), 1–14 (1957)
26. Henkin, L.: The foundations of mathematics. In: Klibansky, R. (ed.) Philosophy in the Mid-century. A Survey, vol. 1, pp. 116–129. La Nuova Italia, Firenze (1958)
27. Henkin, L., Suppes, P., Tarski, A. (eds.): The axiomatic method, with special reference to geometry and physics. Proceedings of an International Symposium held at the University of California, Berkeley, 26 December 1957–4 January 1958. North-Holland, Amsterdam (1959)
28. Henkin, L.: On mathematical induction. The American Mathematical Monthly 67(4), 323–338 (1960) [Translation: O matematicheskoi induktsii. In: Grindlinger, M.D., Grindlinger, E.I. (translators), with introduction by Essenin-Volpin, A.S. Booklet, 36 pp. Matematicheskaya Bibliotechka, Moscow (1962)]
29. Henkin, L., Tarski, A.: Cylindric algebras. In: Dilworth, R.P. (ed.) Lattice Theory. Proceedings of Symposia in Pure Mathematics, vol. 2, pp. 83–113. American Mathematical Society, Providence (1961) [Reprint in: Givant, S.R., McKenzie, R.N. (eds.) Alfred Tarski. Collected Papers, vol. 4, pp. 647–655. Birkhäuser, Basel (1986)]
30. Henkin, L.: Some remarks on infinitely long formulas. In: Infinitistic Methods. Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 2–9 September 1959, pp. 167–183. Państwowe Wydawnictwo Naukowe, Warszawa and Pergamon Press, Oxford (1961)
31. Henkin, L.: Cylindric Algebras. Lectures presented at the 1961 Seminar of the Canadian Mathematical Congress. Notes made with the collaboration and (occasional elaboration) of Marc Venne. Université de Montréal, Montréal (1961)
32. Henkin, L.: Are logic and mathematics identical? Science 138, 788–794 (1962) [Translation: A lógica e a matemática são idênticas? Boletim da Sociedade Paranaense de Matemática 7(3), 2–15 (1964)]
33. Henkin, L., Smith, W.N., Varineau, V.J., Walsh, M.J.: Retracing Elementary Mathematics. Macmillan, New York (1962)
34. Henkin, L.: Nominalistic analysis of mathematical language. In: Nagel, E., Suppes, P., Tarski, A. (eds.) Logic, Methodology, and Philosophy of Science. Proceedings of the 1960 International Congress, pp. 187–193. Stanford University Press, Stanford (1962)
35. Henkin, L.: Symmetric Euclidean relations. Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings, series A, vol. 65, and Indagationes Mathematicae 24, 549–553 (1962)

36. Henkin, L.: New directions in secondary school mathematics. In: Ritchie, R.W. (ed.) New directions in Mathematics, pp. 1–6. Prentice Hall, New Jersey (1963)
37. Henkin, L.: A class of non-normal models for classical sentential logic. Abstract. The Journal of Symbolic Logic 28(4), 300 (1963)
38. Henkin, L.: A theory of propositional types. Fundamenta Mathematicae 52, 323–344 (1963)
39. Henkin, L.: An extension of the Craig–Lyndon interpolation theorem. The Journal of Symbolic Logic 28(3), 201–216 (1963)
40. Henkin, L.: Mathematics and logic. In: Rosenthall, E.M. (ed.) Proceedings of the Fifth Canadian Mathematical Congress, University of Montreal, 1961, pp. 184–200. University of Toronto Press, Toronto (1963)
41. Henkin, L.: Errata to the paper "A theory of propositional types". Fundamenta Mathematicae 53(1), 119 (1964)
42. Addison, J.W., Henkin, L., Tarski, A. (eds.): The Theory of Models. Proceedings of the 1963 International Symposium at Berkeley. xv+494 pp. North-Holland, Amsterdam (1965)
43. Henkin, L.: Extending Boolean operations. Preliminary report. In: Proceedings of the International Congress of Mathematicians, p. 8. Moskva (1966)
44. Henkin, L.: The axiomatic method in mathematics courses at the secondary level. In: Begle, E.G. (coord.) The Role of Axiomatics and Problem Solving in Mathematics, pp. 49–56. Conference Board of the Mathematical Sciences, Ginn, Boston (1966)
45. Henkin, L.: Logical systems containing only a finite number of symbols. Séminaire Mathématiques Superieures, Université de Montréal, Montréal (1967)
46. Henkin, L.: Systems, Formal, and Models of Formal Systems. In: Edwards, P. (ed.) The Encyclopedia of Philosophy 8, pp. 61–74. The Macmillan Company & The Free Press, New York and Collier Macmillan Limited, London (1967)
47. Henkin, L.: Truth and provability. In: Morgenbesser, S. (ed.) Philosophy of Science Today, pp. 14–22. Basic Books, New York (1967)
48. Henkin, L.: Completeness. In: Morgenbesser, S. (ed.) Philosophy of Science today, pp. 23–35. Basic Books, New York (1967)
49. Henkin, L.: Relativization with respect to formulas and its use in proofs of independence. Dedicated to A. Heyting on the occasion of his 70th birthday. Compositio Mathematica 20, 88–106 (1968)
50. Henkin, L.: Weighted sums of squares. Preliminary report. Notices of the American Mathematical Society 5, 65 (1958)
51. Henkin, L.: Extending Boolean operations. Pacific Journal of Mathematics 32(3), 723–752 (1970)
52. Henkin, L.: Mathematical foundations for mathematics. American Mathematical Monthly 78(5), 463–487 (1971)
53. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras. Part I. Studies in Logic and the Foundations of Mathematics, vol. 64. North-Holland, Amsterdam (1971)
54. Henkin, L.: Linguistic aspects of mathematical education. In: Lamon, W.E. (ed.) Learning and the Nature of Mathematics, pp. 211–218. Science Research Associates, Chicago (1972)
55. Henkin, L.: Internal semantics and algebraic logic. In: Leblanc, H. (ed.) Truth, Syntax and Modality. Proceedings of the Temple University Conference on Alternative Semantics. Studies in Logic and the Foundations of Mathematics, vol. 68, pp. 111–127. North-Holland, Amsterdam (1973)

56. Suppes, P., Henkin, L., Joja, A., Moisil, Gr.C. (eds.): Logic, methodology and philosophy of science IV. Proceedings of the Fourth International Congress for Logic, Methodology and Philosophy of Science, Bucharest, 1971. Studies in Logic and the Foundations of Mathematics, vol. 74. North-Holland, Amsterdam and Państwowe Wydawnictwo Naukowe, Warszawa (1973)
57. Henkin, L., Monk, J.D.: Cylindric algebras and related structures. In: Henkin, L., Addison, J., Chang, C.C., Craig, W., Scott, D., Vaught, R. (eds.) Proceedings of the Tarski Symposium. Held at the University of California, Berkeley, 23–30 June 1971. Proceedings of Symposia in Pure Mathematics, vol. XXV, pp. 105–121. American Mathematical Society, Providence (1974)
58. Henkin, L., Addison, J., Chang, C.C., Craig, W., Scott, D., Vaught, R. (eds.): Proceedings of the Tarski Symposium. Held at the University of California, Berkeley, 23–30 June 1971. Proceedings of Symposia in Pure Mathematics, vol. XXV. American Mathematical Society, Providence (1974)
59. Henkin, L., Resek, D.: Relativization of cylindric algebras. Fundamenta Mathematicae 82, 363–383 (1974/1975)
60. Henkin, L.: Identity as a logical primitive. Philosophia. Philosophical Quarterly of Israel 5(1–2), 31–45 (1975)
61. Henkin, L.: The logic of equality. American Mathematical Monthly 84(8), 597–612 (1977)
62. Henkin, L.: Algebraic aspects of logic: past, present, future. In: Colloque International de Logique, Clermont-Ferrand, 18–25 Juillet 1975, pp. 89–106. Centre National de la Recherche Scientifique, Paris (1977)
63. Henkin, L., Leonard, W.A.: A Euclidean construction? Mathematics Magazine 51(5), 294–298 (1978)
64. Hadar, N., Henkin, L.: Children's conditional reasoning, part II: Towards a reliable test of conditional reasoning ability. Educational Studies in Mathematics 9(1), 97–114 (1978)
65. Davis, R.B., Henkin, L.: Aspects of mathematics learning that should be the subject of testing. In: Tyler, R.W., White, S.H. (chairmen) Testing, Teaching and Learning. Report of a Conference on Research on Testing, 17–26 August 1978, pp. 60–82. The National Institute of Education, Washington (1979)
66. Henkin, L., Davis, R.B.: Inadequately tested aspects of mathematics learning. In: Tyler, R.W., White, S.H. (chairmen) Testing, Teaching and Learning. Report of a Conference on Research on Testing, 17–26 August 1978, pp. 83–97. The National Institute of Education, Washington (1979)
67. Henkin, L.: Linguistic aspects of mathematical education. Preliminary report. Abstracts of papers presented to the American Mathematical Society, p. 332 (1980)
68. Avital, Shm., Henkin, L.: On equations that hold identically in the system of real numbers. International Journal of Mathematical Education in Science and Technology 12(2), 223–234 (1981)
69. Henkin, L., Monk, J.D., Tarski, A.: Cylindric set algebras and related structures. In: Henkin, L., Monk, J.D., Tarski, A., Andréka, H., Németi, I. (authors) Cylindric Set Algebras. Lecture Notes in Mathematics, vol. 883, pp. 1–129. Springer-Verlag, Berlin-Heidelberg (1981)[1]

[1]Normally, the entire book is quoted as written by Henkin, Monk, Tarski, Andréka, and Németi since they all appear in the cover and, even more, sign the "Introduction" as "the authors." However, whereas

70. Henkin, L., Monk, J.D., Tarski, A., Andréka, H., Németi, I.: Cylindric Set Algebras. Lecture Notes in Mathematics, vol. 883. Springer-Verlag, Berlin-Heidelberg (1981)
71. Henkin, L.: Notes for lectures on the theory of cylindric algebras. (Boegrad, Kragujevac Novi Sad), 26 pp.
72. Henkin, L.: Proofs in first-order logics with only finitely many variables. Preliminary report. Abstracts of papers presented to the American Mathematical Society 4, 8–9 (1983)
73. Henkin, L.: The complement of intuitive thinking in mathematics teaching and learning. A response to Eric Wittmann's paper. In: Zweng, M., Green, Th., Kilpatrick, J., Pollak, H., Suydam, M. (eds.) Proceedings of the Fourth International Congress on Mathematical Education, pp. 602–604. Birkhäuser, Boston (1983)
74. Henkin, L., Monk, J.D., Tarski, A.: Cylindric Algebras. Part II. Studies in Logic and the Foundations of Mathematics, vol. 115. North-Holland, Amsterdam (1985)
75. Henkin, L., Monk, J.D., Tarski, A.: Representable cylindric algebras. Annals of Pure and Applied Logic 31, 23–60 (1986)
76. Blackwell, D., Henkin, L.: Mathematics: Report of the Project 2061 Phase I Mathematics Panel. Science for All Americans, vol. 3. American Association for the Advancement of Science, Washington (1989)
77. Henkin, L.: The roles of action and of thought in mathematics education—One mathematician's passage. In: Fisher, N.D., Keynes, H.B., Wagreich, Ph.D. (eds.) Changing the Culture: Mathematics Education in the Research Community. CBMS Issues in Mathematics Education, vol. 5, pp. 3–16. American Mathematical Society in cooperation with Mathematical Association of America, Providence (1995)
78. Henkin, L.: In memoriam: Raphael Mitchell Robinson. The Bulletin of Symbolic Logic 1(3), 340–343 (1995)
79. Henkin, L.: The discovery of my completeness proofs. The Bulletin of Symbolic Logic 2(2), 127–158 (1996)

3 Reviews

1. Henkin, L.: M.H.A. Newman and A.M. Turing. "A formal theorem in Church's theory of types". The Journal of Symbolic Logic 7(3), 122 (1942)
2. Henkin, L.: J.C.C. McKinsey and Alfred Tarski. "Some theorems about the sentential calculi of Lewis and Heyting". The Journal of Symbolic Logic 13(3), 171–172 (1948)
3. Henkin, L.: Jean Cavaillès. "Transfini et continu". The Journal of Symbolic Logic 13(3), 143–144 (1948)
4. Henkin, L.: Alfred Tarski. "A decision method for elementary algebra and geometry". The Journal of Symbolic Logic 14(3), 188 (1949)
5. Henkin, L.: André Chauvin. "Généralisation du théorème de Gödel". The Journal of Symbolic Logic 14(3), 193 (1949)
6. Henkin, L.: André Chauvin. "Structure logiques". The Journal of Symbolic Logic 14(3), 193 (1949)

Henkin, Monk, and Tarski wrote the first part of the book, "Cylindric set algebras and related structures," Andréka and Németi are the authors of the second part.

7. Henkin, L.: Stanisław Jaśkowski. "Sur certains groupes formés de classes d'ensembles et leur application aux définitions des nombres". The Journal of Symbolic Logic 14(1), 66 (1949)
8. Henkin, L.: Stanisław Jaśkowski. "Sur les variables propositionelles dépendantes". The Journal of Symbolic Logic 14(1), 65 (1949)
9. Henkin, L.: László Kalmár. "Une forme du théorème de Gödel sous des hypothèses minimales"; László Kalmár. "Quelques formes générales du théorème de Gödel". The Journal of Symbolic Logic 15(3), 230 (1950)
10. Henkin, L.: Heinrich Scholz. "Zur Erhellung des Verstehens". The Journal of Symbolic Logic 16(1), 53 (1951)
11. Henkin, L.: Hugues Leblanc. "On definitions". The Journal of Symbolic Logic 16(3), 213–214 (1951)
12. Henkin, L.: Hugues Leblanc. "The semiotic function of predicates". The Journal of Symbolic Logic 16(3), 213 (1951)
13. Henkin, L.: James K. Feibleman. "Class-membership and the ontological problem". The Journal of Symbolic Logic 16(3), 213 (1951)
14. Henkin, L.: Abraham Robinson. On the metamathematics of algebra. The Journal of Symbolic Logic 17(3), 205–207 (1952)
15. Henkin, L.: Alfred Tarski. A decision method for elementary algebra and geometry. The Journal of Symbolic Logic 17(3), 207 (1952)
16. Henkin, L.: Kurt Gödel. The consistency of the axiom of choice and the generalized continuum-hypothesis with the axioms of set theory. The Journal of Symbolic Logic 17(3), 207–208 (1952)
17. Henkin, L.: Burton Dreben. "On the completeness of quantification theory". The Journal of Symbolic Logic 18(4), 339 (1953)
18. Henkin, L.: H. Rasiowa. "Algebraic treatment of the functional calculi of Heyting and Lewis". The Journal of Symbolic Logic 18(1), 72–73 (1953)
19. Henkin, L.: G. Kreisel. "On a problem of Henkin's". The Journal of Symbolic Logic 19(3), 219–220 (1954)
20. Henkin, L.: Karl Menger. Calculus. A modern approach. The Journal of Symbolic Logic 19(3), 227–229 (1954)
21. Henkin, L.: Karl Menger. "The ideas of variable and function". The Journal of Symbolic Logic 19(3), 227 (1954)
22. Henkin, L.: Abraham Robinson. "Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique". The Journal of Symbolic Logic 20(2), 185–186 (1955)
23. Henkin, L.: Abraham Robinson. "On axiomatic systems which possess finite models". The Journal of Symbolic Logic 20(2), 186 (1955)
24. Henkin, L.: E.W. Beth. "Sur le parallélisme logico-mathématique". The Journal of Symbolic Logic 20(2), 184–185 (1955)
25. Henkin, L.: Günter Asser. "Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe". The Journal of Symbolic Logic 20(3), 282 (1955)
26. Henkin, L.: H. Rasiowa and R. Sikorski. "Algebraic treatment of the notion of satisfiability". The Journal of Symbolic Logic 20(1), 78–80 (1955)
27. Henkin, L.: H. Rasiowa; R. Sikorski. "On existential theorems in non-classical functional calculi". The Journal of Symbolic Logic 20(1), 80 (1955)

28. Henkin, L.: Ladislav Rieger. "On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem". The Journal of Symbolic Logic 20(3), 281–282 (1955)
29. Henkin, L.: Paul Bernays, E.W. Beth, L.E.J. Brouwer, Jean-Louis Destouches, R. Feys. "Discussion générale"; A. Châtelet. "Allocution d'ouverture"; L.E.J. Brouwer. "Discours final"; The Journal of Symbolic Logic 20(2), 186 (1955)
30. Henkin, L.: Andrzej Mostowski, A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa and R. Sikorski. The present state of investigations on the foundations of mathematics. The Journal of Symbolic Logic 21(4), 372–373 (1956)
31. Henkin, L.: Jerzy Łoś. "The algebraic treatment of the methodology of elementary deductive systems". The Journal of Symbolic Logic 21(2), 193–194 (1956)
32. Henkin, L.: Juliusz Reichbach. "Completeness of the functional calculus of first-order". The Journal of Symbolic Logic 21(2), 194 (1956)
33. Henkin, L.: Introduction to mathematical logic, vol. I, by Alonzo Church. Bulletin of American Mathematical Society 63(5), 320–323 (1957)
34. Henkin, L.: K. Jaakko Hintikka. "An application of logic to algebra". The Journal of Symbolic Logic 22(2), 216–217 (1957)
35. Henkin, L.: A. Heyting. "Logique et intuitionnisme"; G. Kreisel, Abraham Robinson, Ingebrigt Johansson, Georges Kurepa, A. Heyting. "Discussion". The Journal of Symbolic Logic 23(1), 33 (1958)
36. Henkin, L.: E.W. Beth. "Réflexions sur l'organisation et la méthode de l'enseignement mathématique". The Journal of Symbolic Logic 23(1), 34 (1958)
37. Henkin, L.: Andrzej Mostowski. "Quelques observations sur l'usage des méthodes non finitistes dans la méta-mathématiques"; D. Lacombe, A. Mostowski. "Interventions". The Journal of Symbolic Logic 24(3), 234–235 (1959)
38. Henkin, L.: L. Nolin. "Sur l'algébre des prédicats"; A. Mostowski, J. Porte, A. Tarski, J. Riguet. "Interventions". The Journal of Symbolic Logic 24(3), 235 (1959)
39. Henkin, L., Mostowski, A.: A. Mal'cév. "On a general method for obtaining local theorems in group theory"; A. Mal'cév. "On representations of models". The Journal of Symbolic Logic 24(1), 55–57 (1959)
40. Henkin, L.: Hilary Putnam. "Three-valued logic"; Paul Feyerabend. "Reichenbach's interpretation of quantum-mechanics"; Isaac Levi. "Putnam's three truth values". The Journal of Symbolic Logic 25(3), 289–291 (1960)
41. Henkin, L.: Jerome Rothstein. Communication, organization and science. With a foreword by C.A. Muses. The Journal of Symbolic Logic 25(3), 256 (1960)
42. Henkin, L.: António Monteiro. "Matrices de Morgan caractéristiques pour le calcul propositionnel classique". The Journal of Symbolic Logic 28(2), 174–175 (1963)
43. Henkin, L.: Evert W. Beth. Formal methods. An introduction to symbolic logic and to the study of effective operations in arithmetic and logic. The Journal of Symbolic Logic 30(2), 235–236 (1965)
44. Henkin, L.: M.H. Stone. "Free Boolean rings and algebras". The Journal of Symbolic Logic 32(3), 415 (1967)
45. Henkin, L.: Marc Krasner. "Les algèbres cylindriques". The Journal of Symbolic Logic 36(2), 337 (1971)
46. Henkin, L.: Maurice L'Abbé. "Structures algébriques suggérées par la logique mathématique". The Journal of Symbolic Logic 36(2), 337 (1971)

4 Problems and Others

1. Problems in symbolic logic. The Journal of Symbolic Logic 16(2), 160 (1951)
2. Henkin, L.: A problem concerning provability. Journal of Symbolic Logic 17(2), Problems, 160 (1952)
3. Henkin, L.: The ordering theorem. Journal of Symbolic Logic 17(2), Problems, 160 (1952)
4. Henkin, L.: Mathematical induction. Hour-length motion picture. Mathematical Association of America with support from NSF. Filmed by Palmer Films, Inc., at San Francisco studios and Berkeley campus (1960)
5. Henkin, L.: Mathematical induction. Mathematical Association of American Film Manual no. 1, 21 pp. Manual to accompany motion picture "Mathematical induction".
6. Henkin, L.: A letter to reviewer. Philosophia Mathematica 1(2), 118–119 (1964)
7. Henkin, L.: Mr. Simplex Saves the Aspidistra. Mathematical film. Mathematical Association of America (1964)
8. Henkin, L.: Truth and provability. In: Voice of America. Forum Lectures, Philosophy of Science Series. pp. 1–7. Text of talk broadcast by Voice of America (1964)
9. Henkin, L.: Completeness. In: Voice of America. Forum Lectures, Philosophy of Science Series. pp. 1–9. Text of talk broadcast by Voice of America (1964)
10. Henkin, L., Engeler, E.: Mathematics 125A, B: Mathematical logic. Course notes. (1962–1963) [Second edition, 125A only (1967–1968)]
11. Henkin, L., Kessner, A.: Mathematics 15: Mathematics for Elementary School Teachers. Course notes. Produced with support from Educational Development Center (1968–1969)
12. Henkin, L., Monk, J.D., Tarski, A.: Problems. In: Cylindric Algebras. Part II. Studies in Logic and the Foundations of Mathematics, vol. 64, pp. 157–158, 245–246, 463–465. North-Holland, Amsterdam (1971)
13. Henkin, L.: Fractions and rational numbers. In: The Rational Numbers, Chap. 2, pp. 25–54. National Council of Teachers of Mathematics (1972)
14. Henkin, L., Monk, J.D., Tarski, A.: Problems. In: Henkin, L., Monk, J.D., Tarski, A., Andréka, H., Németi, I. (authors) Cylindric Set Algebras. Lecture Notes in Mathematics, vol. 883, pp. 126–128. Springer-Verlag, Berlin-Heidelberg (1981)
15. Henkin, L., Monk, J.D., Tarski, A.: Problems. In: Cylindric Algebras. Part II. Studies in Logic and the Foundations of Mathematics, vol. 115, pp. 179–180, 273. North-Holland, Amsterdam (1985)