# Chapter 10
# Reliability Properties of Systems with Three Dependent Components

**Mehmet Yılmaz, Nihan Potas and Birol Topçu**

**Abstract** This study aims to investigate the reliability properties of the systems constituted of three identical components that are dependent of each other. Besides this, it is studied to compare between the lifetimes of such systems in the sense of hazard rate ordering. Illustrative examples are given by considering some trivariate exponential distributions that are commonly used in reliability literature for the joint distribution of the component lifetimes.

**Keywords** Hazard rate ordering · Systems with dependent components · Hazard rate function · Exponential lifetimes · Three-component systems

## 10.1 Introduction

In a reliability theory, when constructing a system model, a common assumption is to consider a system consisting of the independent components. But in many practical systems, the component's functioning affects the others. Therefore, it is necessary to consider the lifetimes of the components to be dependent on each other. For example, a student must take a three-step test to prove his/her competency. His effort in the first stage may have been influenced by environmental conditions and stress, which will probably effect his success in the other stages. Though the independence assumption is appropriate in certain systems, many common reliability modeling practices are

M. Yılmaz (✉)
Faculty of Science, Department of Statistics, Ankara University, 06100 Ankara, Turkey
e-mail: yilmazm@science.ankara.edu.tr

N. Potas
Department of Biostatistics, University of North Carolina at Chapel Hill,
Gilling School of Global Public Health, Chapel Hill, NC, USA
e-mail: potas@unc.edu

B. Topçu
Namık Kemal University, Tekirdağ, Turkey
e-mail: topcubirol@gmail.com



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inappropriate. If the component lifetime is influenced by environmental exposure, then the components within a system may be dependent on each other, or this dependence may be a result of functional dependence. The study of the system reliability without independent component lifetimes is one of the more difficult problems in the reliability engineering. Though the assumption of independence can often be used to obtain joint distribution of the component lifetimes, sometimes such an assumption is questionable. So, the joint attitude of such component lifetimes needs to be modelled by convenient multivariate distribution to include this dependence structure. Multivariate exponential distributions with exponential margins are useful in reliability modeling. Because these derivations are based on shock models, and the residual lifetime of the each component is independent of its age.

In this paper, we compare the hazard rate functions of systems. Each of them consists of three dependent components whose lifetimes have trivariate exponential distributions. Furthermore, the paper gives a reliability relationship among five systems when the component lifetimes are exchangeable. An ordering relation such as hazard rate ordering between the system lifetimes assuming exponential marginals has been studied by many authors. One may refer to those works in literature such as Navarro and Shaked (2006), Navarro et al. (2006a, b), Navarro and Lai (2007), Joo and Mi (2010).

This paper is organized as follows. In Sect. 10.2, basic definitions and the reliability properties of the system with three dependent components are given. We discuss an ordering relation for possible five coherent systems. In Sect. 10.3, we illustrate the discussion by considering four trivariate exponential distributions. For the joint distribution of the component lifetimes, we assume Farlie-Gumbel-Morgenstern, Marshall-Olkin, Gumbel Type I, Gumbel-Hougaard and Sarhan-Balakrishnan type trivariate exponential distributions which are widely used in reliability literature.

## 10.2 Basic Tools and Main Result

We provide a concise introduction to some basic tools and notions that will be relevant for achieving the main result. Let $X$ and $Y$ be both random variables with survival functions $\overline{F}_X$ and $\overline{F}_Y$, and density functions $f_X$ and $f_Y$. The hazard rate function of $X$ is defined by $h_X(t) = f_X(t)/\overline{F}_X(t)$ for all $t$ such that $\overline{F}_X(t) > 0$. The hazard rate order is defined as $X \leq_{hr} Y \Leftrightarrow h_Y(t) \leq h_X(t)$ for all $t \geq 0$, or $\overline{F}_X(t)/\overline{F}_Y(t)$ is increasing in $t$. For more on the definition of the hazard rate order, readers may refer to Shaked and Shanthikumar (1994), and Lai and Xie (2006).

Let $T_i$ be the lifetime of the $i$th system $(i = 1, 2, \ldots, 5)$ and $X_i$, $(i = 1, 2, 3)$ stand for the lifetimes of the components. Then five systems are demonstrated respectively as $\min\{X_1, X_2, X_3\} \equiv T_1$, $\min\{X_1, \max\{X_2, X_3\}\} \equiv T_2$, $\max\{X_1, \min\{X_2, X_3\}\} \equiv T_3$, $\max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3\}\} \equiv T_4$, and $\max\{X_1, X_2, X_3\} \equiv T_5$. Detailed discussions can be found in Barlow and Proschan (1975). Throughout the paper, $S_{(\cdot)}$ denotes the reliability function of the system, and $\overline{F}(x_1, x_2, x_3)$ denotes the joint survival function of the component lifetimes. Then the survival functions of the five systems respectively are:

$$\begin{aligned}
S_1(t) &= \overline{F}(t,t,t) = \Pr(X_1 > t, X_2 > t, X_3 > t) \\
S_2(t) &= 2\overline{F}(t,t,0) - \overline{F}(t,t,t) \\
S_3(t) &= \overline{F}(t,0,0) + \overline{F}(t,t,0) - \overline{F}(t,t,t) \\
S_4(t) &= 3\overline{F}(t,t,0) - 2\overline{F}(t,t,t) \\
S_5(t) &= 3\overline{F}(t,0,0) - 3\overline{F}(t,t,0) + \overline{F}(t,t,t).
\end{aligned}$$

We assume that component lifetimes are identical then we reveal the following two new equations as follows:

$$S_2(t) = \frac{S_1(t) + 2S_4(t)}{3} \tag{10.1}$$

and

$$S_3(t) = \frac{S_5(t) + 2S_4(t)}{3}. \tag{10.2}$$

Furthermore, following relation can be written for both identical and non-identical cases

$$S_1(t) + S_4(t) + S_5(t) = \overline{F}(t,0,0) + \overline{F}(0,t,0) + \overline{F}(0,0,t). \tag{10.3}$$

From (10.1) and (10.2), we respectively write below equalities for the hazard rates of $T_2$ and $T_3$

$$h_2(t) = \alpha_{14}(t)h_1(t) + (1 - \alpha_{14}(t))h_4(t) \tag{10.4}$$

and

$$h_3(t) = \alpha_{45}(t)h_4(t) + (1 - \alpha_{45}(t))h_5(t) \tag{10.5}$$

where, $\alpha_{14}(t) = \frac{S_1(t)}{S_1(t)+2S_4(t)}$ and $\alpha_{45}(t) = \frac{2S_4(t)}{S_5(t)+2S_4(t)}$.

Furthermore, let us consider the systems with two components which are series and parallel systems. Their hazard rates can be represented by a convex combination of the hazard rates of three-component system. Accordingly, $T_{ss}$ and $T_{ps}$ denote respective lifetimes of series and parallel systems. Then corresponding hazard rates is denoted by $h_{ss}(t)$ and $h_{ps}(t)$. From now on, we can write mentioned convex combinations as follows

$$h_{ss}(t) = \alpha_{12}(t)h_1(t) + (1 - \alpha_{12}(t))h_2(t) \tag{10.6}$$

and

$$h_{ps}(t) = \alpha_{35}(t)h_3(t) + (1 - \alpha_{35}(t))h_5(t) \tag{10.7}$$

where $\alpha_{12}(t) = \frac{S_1(t)}{S_1(t)+S_2(t)}$ and $\alpha_{35}(t) = \frac{S_3(t)}{S_3(t)+S_5(t)}$.

It can be seen from the Eq. (10.4) that $h_2(t)$ is represented as a convex combination of $h_1(t)$ and $h_4(t)$. Here, it is expected and reasonable that $h_1(t)$ is greater than $h_4(t)$ for all $t$. However, a counter-example was given for the coherent system by Navarro and Shaked (2006) (see Example 2.2). Now, suppose that $h_1(t) > h_4(t)$ for some $t$. In this case, $h_2(t)$ lies between $h_1(t)$ and $h_4(t)$. Conversely for some $t$, $h_1(t) < h_4(t)$ then $h_2(t)$ still lies between them. That is, if there exist an ordering amongst $T_1$, $T_2$ and $T_3$ then these lifetimes must be ordered as $T_1 \leq_{hr} T_2 \leq_{hr} T_4$ or $T_4 \leq_{hr} T_2 \leq_{hr} T_1$. Similar interpretation can be made for $T_3$ from the Eq. (10.5). On the other hand, according to the Eqs. (10.6) and (10.7), if one can reveal two usual orderings between $h_{ss}(t)$ and $h_1(t)$ and between $h_{ps}(t)$ and $h_5(t)$ then five lifetimes can be ordered respectively. It is easily seen that

$$T_1 \leq_{hr} T_2 \leq_{hr} T_4 \leq_{hr} T_3 \leq_{hr} T_5 \tag{10.8}$$

when each five system consist of independent components. We find out the conditions when the components are dependent the five systems reveal the same ordering above. Navarro and Shaked (2006) investigated some conditions of the hazard rate order of the order statistics. Their ideas give us to consider ordering amongst for all systems with three components. To show the ordering relation amongst the five lifetimes we propose the following Lemma.

**Lemma 1** *$T_1 \leq_{hr} T_2 \leq_{hr} T_4 \leq_{hr} T_3 \leq_{hr} T_5$ holds if one of the conditions below is satisfied*:

(i) *both ratios $\frac{S_1(t)}{S_4(t)}$ and $\frac{S_4(t)}{S_5(t)}$ are decreasing function of $t$,*

*or*

(ii) *for all $t > 0$, $h_{ss}(t) \leq h_1(t)$ and $h_{ps}(t) \geq h_5(t)$.*

*Proof*

(i) Suppose that the ratio $\frac{S_1(t)}{S_4(t)}$ decreases in $t$. Then from (ii), two facts $\frac{S_2(t)}{S_4(t)} \downarrow t$ and $\frac{S_2(t)}{S_1(t)} \uparrow t$ are obtained. According to the definition of the hazard rate ordering, we have $T_2 \leq_{hr} T_4$ and $T_1 \leq_{hr} T_2$. According to the transitivity property of the hazard rate ordering, then we get

$$T_1 \leq_{hr} T_2 \leq_{hr} T_4. \tag{10.9}$$

Secondly, assume that the ratio $\frac{S_4(t)}{S_5(t)}$ decreases in $t$. Then from (10.3), two facts $\frac{S_3(t)}{S_5(t)} \downarrow t$ and $\frac{S_3(t)}{S_4(t)} \uparrow t$ are obtained. Hence, we write $T_3 \leq_{hr} T_5$ and $T_4 \leq_{hr} T_3$ and both relations imply

$$T_4 \leq_{hr} T_3 \leq_{hr} T_5. \tag{10.10}$$

By combining (10.9) with latter relation, we get the desired result.

(ii) From the Eq. (10.6), we have

$$h_{ss}(t) - h_1(t) = (1 - \alpha_{12}(t))(h_2(t) - h_1(t)) \tag{10.11}$$

This result is combined with Eq. (10.4), above equality is enhanced as

$$h_{ss}(t) - h_1(t) = (1 - \alpha_{12}(t))(1 - \alpha_{14}(t))(h_4(t) - h_1(t)). \tag{10.12}$$

By the assumption $h_{ss}(t) \leq h_1(t)$, according to Eqs. (10.11) and (10.12) we obtain two relations such as $h_2(t) \leq h_1(t)$, and $h_4(t) \leq h_1(t)$. From the Eq. 10.4, by considering $h_4(t) \leq h_1(t)$ then $h_4(t) \leq h_2(t) \leq h_1(t)$. Analogously we have two other relations from the Eqs. 10.7 and 10.5 that is, we have $h_5(t) \leq h_3(t)$, and $h_5(t) \leq h_4(t)$. Finally we write $h_5(t) \leq h_4(t) \leq h_1(t)$. By considering 5 again, we have

$$h_5(t) \leq h_3(t) \leq h_4(t) \leq h_1(t) \tag{10.13}$$

then we get the desired result. □

We investigate the results of (i) and (ii) of the Lemma 1 by considering some trivariate exponential distributions for the joint distribution of the component lifetimes.

## 10.3 Illustrations

We consider FGM type trivariate exponential distribution to illustrate Lemma 1, (ii). Marshall-Olkin, Gumbel Type I, Gumbel-Hougaard and Sarhan-Balakrishnan type trivariate exponential distributions which are widely used in reliability literature are considered to illustrate Lemma 1, (i).

### *10.3.1 Farlie Gumbel Morgenstern Distribution with Identical Marginals*

Suppose component lifetimes are identically distributed as exponential with $\lambda$ parameter. The survival distribution is given by at time $t$

$$S(t,t,t)=\exp(-3\lambda t)\left[1+3\theta(1-\exp(-\lambda t))^2-\theta(1-\exp(-\lambda t))^3\right]$$

where $\theta\in\left(0,\frac{1}{2}\right)$. For more detailed discussion on this distribution can be found in Regina and Johnson (1976), Johnson and Kotz (1977) and Kotz et al. (2000).

First we compare $h_1(t)$ with $h_{ss}(t)$. To do this, we will check the monotonicity of the ratio $S_1(t)/S_{ss}(t)$. Accordingly, by letting $1-\exp(-\lambda t)=u$ then the monotonicity of this ratio is equivalent to monotonicity of $\rho_\theta(u)$ which is given by

$$\rho_\theta(u)=\frac{(1-u)^3[1+3\theta u^2-\theta u^3]}{(1-u)^2[1+\theta u^2]}.$$

We interest the sign of the first derivative of $\rho_\theta(u)$ to determine monotonic property of $\rho_\theta(u)$. Consequently, after some rearrangement the first derivative can be written as follows

$$\frac{d\rho_\theta(u)}{du}=-2(1-u)\left[1-\frac{1}{(1+\theta u^2)^2}\right]-\left(1-\frac{1}{(1+\theta u^2)}\right)+\left[\frac{4\theta u(1-u)}{(1+\theta u^2)^2}-1\right].$$

It is seen that first two summands are negative signed. Besides, for the last summand by using the fact that $4\theta u(1-u)\leq 1$, we have an upper bound for $d\rho_\theta(u)/du$ which is given by

$$\frac{d\rho_\theta(u)}{du}\leq-\left(1-\frac{1}{(1+\theta u^2)^2}\right).$$

Hence it can be seen that $\rho_\theta(u)$ decreases in $u$. Therefore equivalently the ratio $S_1(t)/S_{ss}(t)$ decreases in $t$. As a result, $h_1(t)\geq h_{ss}(t)$. Now we will compare $h_5(t)$ and $h_{ps}(t)$ in a similar way. Monotonicity of $S_5(t)/S_{ps}(t)$ is equivalent to monotonicity of the $\varphi_\theta(u)$ which is given by

$$\varphi_\theta(u)=\frac{1-u^3\left[1+3\theta(1-u)^2+\theta(1-u)^3\right]}{1-u^2\left[1+\theta(1-u)^2\right]}.$$

The first derivative of this ratio can be arranged as

$$\begin{aligned}\frac{d\varphi_\theta(u)}{du}=&\,u\frac{2\theta^2uA(u)^2(2-u)+2\theta[1-4A(u)]+2[1-3\theta A(u)]}{\left(1-u^2\left[1+\theta(1-u)^2\right]\right)^2}\\&+\frac{u[1-4\theta A(u)(1-u)]+\theta u^3}{\left(1-u^2\left[1+\theta(1-u)^2\right]\right)^2}\end{aligned}$$

where $A(u) = u(1-u)$. Since each sums in the numerator is nonnegative the sign of $d\varphi_\theta(u)/du$ is nonnegative. Therefore $S_5(t)/S_{ps}(t)$ is nondecreasing function of $t$ that is $h_{ps}(t) \le h_5(t)$. Hence assumptions of Lemma 1 (ii) are fulfilled we say that when the lifetimes have a joint distribution as a special case of FGM the possible five coherent system lifetimes preserve the ordering as well as in the independence case.

### *10.3.2 Marshall-Olkin Trivariate Exponential Distribution*

The survival function of this family is given by

$$\overline{F}(x_1, x_2, x_3) = \exp\left(-\sum_{i=1}^{3}\lambda_i x_i - \sum_{i<j}\lambda_{ij}\max\{x_i, x_j\} - \lambda_{123}\max\{x_1, x_2, x_3\}\right)$$

where $\lambda$'s are corresponding shock parameters which are positive valued. For more details we refer to Marshall and Olkin (1967) and Basu (1997). According to Lemma 1, (i) we will compare $T_4$ and $T_5$. To do this, we check the monotonic property of the ratio $S_4(t)/S_5(t)$ denoted by $r_{45}(t)$. Then,

$$r_{45}(t) = \frac{3\exp(\lambda t) - 2}{1 + 3\exp((2\lambda + \theta)t) - 3\exp(\lambda t)}$$

where $\theta = \lambda_{ij}$ and $\delta = \lambda_{123}$. Hence, we consider the numerator of the first derivative of this ratio; denoting this statement by $g(t)$, then we obtain

$$g(t) = -\lambda - 3(\lambda + \theta)\exp((2\lambda + \theta)t) + 2(2\lambda + \theta)\exp((\lambda + \theta)t)$$

with $g(0) = -\theta$. If $g(t)$ remains negative along the line $(0, \infty)$ then its derivative may be negative signed or its maximum value has a negative sign. Therefore, we check the sign of the derivative of $g(t)$;

$$g'(t) = (\lambda + \theta)(2\lambda + \theta)\exp((\lambda + \theta)t)[2 - 3\exp(\lambda t)] \le 0.$$

As it can be seen that $g(t)$ decreases in $t$. Since $g(0) = -\theta$, $g(t)$ can not be positive valued. This implies $r_{45}(t)$ decreases in $t$. According to Lemma 1, (i) second condition is satisfied. Now, we consider the ratio $S_4(t)/S_1(t)$ denoted by $r_{41}(t)$

$$r_{41}(t) = 3\exp(\lambda t) - 2.$$

We can easily say that $r_{41}(t)$ increases in $t$. This implies $r_{14}(t)$ decreases in $t$. The first condition of Lemma 1, (i) is also satisfied. As a result of Lemma 1, we have the ordering relation for this family.

### 10.3.3 Gumbel Type I Trivariate Exponential Distribution

The survival function of this family is given by

$$\overline{F}(x_1, x_2, x_3) = \exp\left(-\sum_{i=1}^{3} \lambda_i x_i - \sum_{i<j} \lambda_{ij} x_i x_j - \lambda_{123} x_1 x_2 x_3\right)$$

where $\lambda$'s are corresponding shock parameters which are positive valued. Some characterizations and properties of this family can be found in Galambos and Kotz (1978) and Marshall and Olkin (1967). Then we get the ratio $r_{45}(t)$ which is given by

$$r_{45}(t) = \frac{3\exp(\lambda t + 2\theta t^2 + \delta t^3) - 2}{1 + 3\exp(2\lambda t + 3\theta t^2 + \delta t^3) - 3\exp(\lambda t + 2\theta t^2 + \delta t^3)}$$

where $\theta = \lambda_{ij}$ and $\delta = \lambda_{123}$. For the simplicity,

$$A(t) = \lambda t + 2\theta t^2 + \delta t^3$$

and

$$B(t) = \lambda t + \theta t^2$$

are taken. Then the ratio is rewritten as

$$r_{45}(t) = \frac{1 + 3(\exp(A(t) - 1))}{1 + 3e^{A(t)}(\exp(B(t) - 1))}.$$

To show the monotonicity of this ratio we check the sign of the first derivative. We consider the numerator of this derivative. Denoting this statement by $g(t)$, then we have

$$g(t) = 9\exp(A(t))[(\exp(B(t)) - 1)A'(t) - (\exp(A(t)) - 1)\exp(B(t))B'(t)].$$

By using two facts that are $A'(t) \leq \exp(A(t)) - 1$ and $\exp(-B(t)) \geq 1 - B'(t)$ then we can write an upper bound for $g(t)$ which is given by

$$\begin{aligned} g(t) &\leq 9e^{A(t)}\left(e^{A(t)} - 1\right)\left[\left(e^{B(t)} - 1\right) - e^{B(t)}B'(t)\right] \\ &\leq 9e^{A(t)}\left(e^{A(t)} - 1\right)\left[\left(e^{B(t)} - 1\right) + e^{B(t)}\left(e^{-B(t)} - 1\right)\right] \\ &= 0. \end{aligned}$$

This implies $r_{45}(t)$ decreases in $t$. Now, we consider the ratio $r_{41}(t)$

$$r_{41}(t) = 3\exp(\lambda t + 2\theta t^2 + \delta t^3) - 2$$

which increases in $t$. Hence, $r_{14}(t) \downarrow t$. We showed that ordering relation given in Lemma 1 is valid for this family.

### 10.3.4 Gumbel-Hougaard Trivariate Distribution

Distribution function of this family is given by

$$F(x_1, x_2, x_3) = \exp(-([-\ln F(x_1)]^m + [-\ln F(x_2)]^m + [-\ln F(x_1)]^m))^{\frac{1}{m}}$$

and the survival function is

$$\overline{F}(x_1, x_2, x_3) = \exp(-\left(\left[-\ln \overline{F}(x_1)\right]^m + \left[-\ln \overline{F}(x_2)\right]^m + \left[-\ln \overline{F}(x_1)\right]^m\right))^{\frac{1}{m}}$$

where $m > 1$. We refer the reader to Georges et al. (2001) for more details. Then we write the ratio

$$r_{45}(t) = \frac{1 + 2F(t)^b - 3F(t)^a}{1 - F(t)^b}$$

where $b = 3^{\frac{1}{m}}$ and $a = 2^{\frac{1}{m}}$. To show monotonic property of this ratio, we consider the following transformation: Let $1 - F(t)^b = v$ then

$$\varphi(v) = S_4(S_5)^{-1}(v) = 1 + 2(1 - v) - 3(1 - v)^{\frac{a}{b}}$$

Now, taking second derivative of $\varphi(v)$ we have the following result:

$$\varphi''(v) = -3\frac{a}{b}\left(\frac{a}{b} - 1\right)(1 - v)^{\frac{a}{b}} \geq 0$$

Hence, $\varphi(v)$ is a convex function on $(0, 1)$. This implies

$$\frac{S_4(S_5)^{-1}(v)}{v} \uparrow v \equiv \frac{S_4(t)}{S_5(t)} \downarrow t.$$

Now, we consider the ratio $r_{41}(t)$ which is given by

$$r_{41}(t) = 3\overline{F}(t)^{a-b} - 2.$$

Since $a \leq b$ we say that $r_{14}(t) \downarrow t$. For this family, we have the similar ordering relation as discussed for above two families.

### 10.3.5 Sarhan-Balakrishnan Type Trivariate Distribution

This family was introduced by Sarhan and Balakrishnan (2007). Franco and Vivo (2009) have determined the joint survival function of the n-dimensional extension. The survival function of this family is given by

$$\overline{F}(x_1, x_2, x_3) = \exp(-\lambda \max\{x_1, x_2, x_3\}) \prod_{i=1}^{3} \left(1 - (1 - \exp(x_i))^{\theta_i}\right)$$

where $\lambda$ and $\theta$‘s are positive real numbers. In order to show that the monotonic property of the ratio $r_{45}(t)$ we take the equivalence $u = 1 - \exp(-t)$ which simplify the notation. Thus we get the transformation $r_{45}(u)$ as below

$$r_{45}(u) = -2 + 3\frac{1}{\frac{u^{2\theta}}{1+u^{\theta}} + 1}.$$

Since $u^{2\theta}/1 + u^{\theta}$ increases in $u$ this ratio decreases in $u$. Equivalently $S_4(t)/S_5(t)$ decreases in $t$. By similar notation, we consider the ratio $r_{41}(u)$;

$$r_{41}(u) = \frac{3}{(1 - u^{\theta})} - 2$$

where $u = 1 - \exp(-t)$. This implies $r_{14}(t)$ decreases in $t$. We have similar ordering relation for this family.

## 10.4 Conclusion

The ordering relation given as the result of Lemma 1 is also valid in case of the independent component lifetimes. Because of this, in the case of dependent component lifetimes, by considering five trivariate exponential distributions for the joint distribution of component lifetimes, we emphasized that the system consisting of the dependent components satisfies the same ordering relation. In a reliability theory, for the five distribution families which have important role for the system modeling, this ordering properties obtained in this work can be seen as a useful result.

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