From Algebra to Logic: There and Back Again
The Story of a Hierarchy*

(Invited Paper)

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Abstract. This is a survey about a collection of results about a (double) hierarchy of classes of regular languages, which occurs in a natural fashion in a number of contexts. One of these occurrences is given by an alternated sequence of deterministic and co-deterministic closure operations, starting with the piecewise testable languages. Since these closure operations preserve varieties of languages, this defines a hierarchy of varieties, and through Eilenberg's variety theorem, a hierarchy of pseudo-varieties (classes of finite monoids that are defined by pesudo-identities). The point of this excursion through algebra is that it provides reasonably simple decision algorithms for the membership problem in the corresponding varieties of languages. Another interesting point is that the hierarchy of pseudo-varieties bears a formal resemblance with another hierarchy, the hierarchy of varieties of idempotent monoids, which was much studied in the 1970s and 1980s and is by now well understood. This resemblance provides keys to a combinatorial characterization of the different levels of our hierarchies, which turn out to be closely related with the so-called rankers, a specification mechanism which was introduced to investigate the two-variable fragment of the first-order theory of the linear order. And indeed the union of the varieties of languages which we consider coincides with the languages that can be defined in that fragment. Moreover, the quantifier alternation hierarchy within that logical fragment is exactly captured by our hierarchy of languages, thus establishing the decidability of the alternation hierarchy.

There are other combinatorial and algebraic approaches of the same logical hierarchy, and one recently introduced by Krebs and Straubing also establishes decidability. Yet the algebraic operations involved are seemingly very different, an intriguing problem...

Formal language theory historically arose from the definition of models of computation (automata, grammars, etc) and relied for its first step on combinatorial reasoning, especially combinatorics on words. Very quickly however, algebra and

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logic were identified as powerful tools for the classification of rational languages, e.g. with the definition of the syntactic monoid of a language and Büchi's theorem on monadic second-order logic. It did not take much time after that to observe that, conversely, formal language theory is itself a tool for algebra and logic.

The results which we will present are an illustration of this back-and-forth movement between languages, algebra and logic. They deal with a hierarchy of classes of rational languages which arises in different contexts and turned out to solve a problem in logic, namely the decidability of the quantifier alternation hierarchy within the two-variable fragment of first-order logic $\mathsf{FO}^2[<]$.

The full picture uses a collection of results in logic, combinatorics on words and algebra which were obtained independently of the quantifier alternation hierarchy by various authors over several decades.

Let $\mathcal{R}_0$ be the class of piecewise testable languages, which is natural from a combinatorial and automata-theoretic point of view, and corresponds to the first level of the quantifier alternation hierarchy within $\mathsf{FO}^2[<]$ (and within $\mathsf{FO}[<]$) as well). This class is rather simple and reasonably well understood, see [1, 10]. We first consider the hierarchies of classes of languages obtained from $\mathcal{R}_0$ by alternatingly closing it under deterministic and co-deterministic closure: we let $\mathcal{L}_0 = \mathcal{R}_0$, $\mathcal{R}_{k+1}$ (resp. $\mathcal{L}_{k+1}$) be the deterministic (resp. co-deterministic) closure of $\mathcal{L}_k$ (resp. $\mathcal{R}_k$).

Results from the 1970s and 1980s [9, 12] show that the classes $\mathcal{R}_k$ and $\mathcal{L}_k$ are varieties (whether a language L belongs to one of these classes depends only on its syntactic monoid) and describe the corresponding varieties of finite monoids $\mathbf{R}_k$ and $\mathbf{L}_k$. Results from the 1960s [5] (see also [6, 11, 17]) shows that their membership problems are decidable and they form an infinite hierarchy.

A first view of the structure of the lattice formed by these varieties can be obtained by using purely algebraic results from the 1970s on a seemingly different hierarchy, that of varieties of idempotent monoids [2]. The theory of the latter varieties is particularly well understood, and one can exhibit for each of them structurally elegant identities and solutions of the word problem (of the corresponding relatively free object) [3].

To completely elucidate the structure of the lattice generated by the $\mathcal{R}_k$ and $\mathcal{L}_k$, Kufleitner and Weil introduced the notion of condensed rankers [8]. These are a rather natural extension of the algorithm to solve the word problem in the relatively free idempotent monoids and have natural connections with deterministic and codeterministic products. But they are also – and foremost – a variant of the rankers introduced by Weiss and Immerman [18] (following the turtle programs of Schwentick, Thérien and Vollmer [13]) to characterize the levels of the quantifier alternation hierarchy of $\mathsf{FO}^2[<]$. As a result one can show that the k-th level of this hierarchy coincides with the intersection $\mathcal{R}_{k+1} \cap \mathcal{L}_{k+1}$, thus proving the decidability of each level of the hierarchy [7].

The story does not end there: using algebraic methods similar to those described in his book [15], Straubing showed [16] that the k-th level of the quantifier alternation hierarchy of $\mathsf{FO}^2[<]$ is the variety of languages whose syntactic monoid is in the k-th term of the sequence given by $\mathbf{V}_1 = \mathbf{J}$ and $\mathbf{V}_{n+1} = \mathbf{V}_n \Box \mathbf{J}$. Here $\mathbf{J}$ is the class of $\mathcal{J}$-trivial monoids, which characterizes piecewise testable languages by Simon's theorem [14] and $\Box$ denotes the two-sided block product, the bilateral version of the more classical wreath product. Then Straubing and Krebs showed that every one of these classes of finite monoids is decidable [4], thus providing an alternate proof of the decidability of the quantifier alternation hierarchy, but also giving an alternative characterization of the classes $\mathbf{V}_k$: a finite monoid M is in $\mathbf{V}_k$ if and only if it sits in both $\mathbf{R}_{k+1}$ and $\mathbf{L}_{k+1}$.

The coincidence of these two very differently defined hierarchies raises an intriguing question: what connects the block product with the alternate operation of deterministic and co-deterministic closure?...

References

1. Almeida, J.: Finite Semigroups and Universal Algebra. World Scientific, Singapore (1994)
2. Gerhard, J.: The lattice of equational classes of idempotent semigroups. Journal of Algebra 15, 195–224 (1970)
3. Gerhard, J., Petrich, M.: Varieties of bands revisited. Proceedings of the London Mathematical Society 58(3), 323–350 (1989)
4. Krebs, A., Straubing, H.: An effective characterization of the alternation hierarchy in two-variable logic. In: FSTTCS 2012. LIPIcs, vol. 18, pp. 86–98. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl (2012), doi:10.4230/LIPIcs.FSTTCS.2012.86
5. Krohn, K., Rhodes, J., Tilson, B.: Homomorphisms and semilocal theory. In: Arbib, M. (ed.) The Algebraic Theory of Machines, Languages and Semigroups. Academic Press (1965)
6. Kufleitner, M., Weil, P.: On the lattice of sub-pseudovarieties of DA. Semigroup Forum 81, 243–254 (2010)
7. Kufleitner, M., Weil, P.: The FO^2 alternation hierarchy is decidable. In: CSL 2012. LIPIcs, vol. 16, pp. 426–439. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl (2012), doi:10.4230/LIPIcs.CSL.2012.426
8. Kufleitner, M., Weil, P.: On logical hierarchies within FO^2-definable languages. Logical Methods in Computer Science 8(3:11), 1–30 (2012)
9. Pin, J.E.: Propriétés syntactiques du produit non ambigu. In: de Bakker, J.W., van Leeuwen, J. (eds.) ICALP 1980. LNCS, vol. 85, pp. 483–499. Springer, Heidelberg (1980)
10. Pin, J.É.: Varieties of Formal Languages. North Oxford Academic, London (1986)
11. Rhodes, J., Steinberg, B.: The $\mathfrak{q}$-theory of finite semigroups. Springer Monographs in Mathematics. Springer, New York (2009)
12. Schützenberger, M.P.: Sur le produit de concaténation non ambigu. Semigroup Forum 13, 47–75 (1976)
13. Schwentick, T., Thérien, D., Vollmer, H.: Partially-ordered two-way automata: A new characterization of DA. In: Kuich, W., Rozenberg, G., Salomaa, A. (eds.) DLT 2001. LNCS, vol. 2295, pp. 239–250. Springer, Heidelberg (2002)
14. Simon, I.: Piecewise testable events. In: Brakhage, H. (ed.) GI-Fachtagung 1975. LNCS, vol. 33, pp. 214–222. Springer, Heidelberg (1975)
15. Straubing, H.: Finite automata, formal logic, and circuit complexity. Birkhäuser, Boston (1994)

16. Straubing, H.: Algebraic Characterization of the Alternation Hierarchy in $FO^2[<]$ on Finite Words. In: CSL 2011. LIPIcs, vol. 12, pp. 525–537. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl (2011), doi:10.4230/LIPIcs.CSL.2011.525
17. Weil, P.: Some results on the dot-depth hierarchy. Semigroup Forum 46, 352–370 (1993)
18. Weis, P., Immerman, N.: Structure theorem and strict alternation hierarchy for FO^2 on words. Logical Methods in Computer Science 5, 1–23 (2009)