Aperiodic Tilings and Entropy*

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Abstract. In this paper we present a construction of Kari-Culik aperiodic tile set, the smallest known until now. Our construction is self-contained and organized to allow reasoning on properties of the resulting sets of tilings. With the help of this construction, we prove that this tileset has positive entropy. We also explain why this result was not expected.

1 Introduction

In this paper we focus on aperiodic tilesets. These tilesets can be used to tile the plane but none of the obtained tilings has a period. The role of aperiodic tilesets is crucial in different fields such as logics (see for instance [1]) or for the study of quasi-periodic structures such as quasi-crystals. Furthermore these aperiodic tilesets are a classical tool to prove undecidability problems for planar structures or dynamical systems. We work with the formalization that Wang proposed in [2].

A classical question about a tileset is to measure its entropy. Roughly speaking the entropy of a tileset is positive if "points of freedom are dense". One can easily make it positive for any aperiodic tileset by a cartesian product with a free bit. The number of tiles is then multiplicated by a factor two but the resulting tiling has positive entropy. It is easy to observe that for classical self similar tilesets such as Berger [3], Robinson [4] the entropy is zero. The main question we address is the entropy of the smallest known aperiodic tileset: it was conjectured that its entropy is zero but we prove it is positive. This entropy zero conjecture comes from other works on this tileset and some algorithmic remarks developed in Sect. 4.

Our paper is organized as follows: first we explain exactly the same tileset as Kari and Culik in [5,6]. Our explanation makes it easier to analyze (repeating the proof of aperiodicity). Then in Sect. 3 we formulate a substitutive property that guarantee positive entropy of Kari-Culik tileset. The rest of the section is devoted to the proof. The last section is focused on more refined approaches to the entropy of a tileset.

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2 Presentation of the Tileset

2.1 Source of Aperiodicity

Let us start with an observation. Consider a bi-infinite sequence x_n of positive real numbers, such that either $x_{n+1} = 2x_n$ or $x_{n+1} = x_n/3$ for every n.

Every such sequences are aperiodic. Indeed, for all n and all $k > 0$, we have $x_{n+k} = x_n \times 2^i/3^j$ for some $i, j > 0$. If we had $x_{n+k} = x_n$, then we would have $1 = 2^i/3^j$ for $i, j > 0$, which is a contradiction.

Moreover, there exist some such sequences x_n which lie in the interval $[1/3; 2]$. Starting from some x_0 in this interval, we can always take $x_{n+1} = 2x_n$ if $x_n < 1$, and $x_{n+1} = x_n/3$ otherwise. The same argument works in the opposite direction.

2.2 Aperiodic Sequences and Tilings

A tile is an unit square with colored sides. Consider the (geometric) plane with a unit grid; a tiling is an assignation of a tile to each square of the grid, in such manner that matching borders have the same color. Thus, in a tiling, we have a bi-infinite sequence of colors along any horizontal or vertical line of the grid.

We are going to focus on the horizontal lines of our tilings. If we use three colors (say, 0, 1 and 2) for the top and bottom sides of the tiles, we will get bi-infinite sequences over the alphabet $\{0, 1, 2\}$. Such sequences might have an **average**, i.e. a limit of averages over finite parts as the length of the parts increases (In our tiling we prove that unique average always exists see Prop. 2).

Our goal is to construct a set of tiles with the following two properties:

1. for every tiling, if the averages of all horizontal lines exist, they form a sequence x_n with the property defined in Sect. 2.1.
2. for every such sequence x_n, we can find a tiling where averages exist and are equal to x_n.

This tile set will be aperiodic. If it had a periodic tiling, it would also have a bi-periodic tiling. In a bi-periodic tiling, all horizontal lines have an average (due to horizontal periodicity), and form a periodic sequence (due to vertical periodicity), which is impossible. The existence of tilings is a consequence of the second claim.

2.3 Tilings and Automata

Our tileset should guarantee that some relation holds between any two consecutive lines of a tiling (namely, "$x = 2y$ or $x = y/3$"). Thus, let us consider a stripe (a horizontal line of tiles), as displayed on Fig. 1. We call the sequence of bottom numbers a_n, top numbers b_n, and the matching left and right numbers q_n. Such a stripe can be viewed as a run of a non-deterministic automaton, where q_n are the traversed states, and (a_n, b_n) are the input.

More precisely, each tile (q', a, q, b) correspond to a transition $q \xrightarrow{(a,b)} q'$, where (a, b) is the input. This is illustrated by Fig. 2. Since a tileset only have a

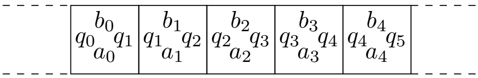


Fig. 1. Example of a horizontal tiled line

finite number of colors and tiles, the running automaton must be a finite-state automaton reading pairs of letters. Note that our automaton has no initial state; it runs infinitely in both directions.

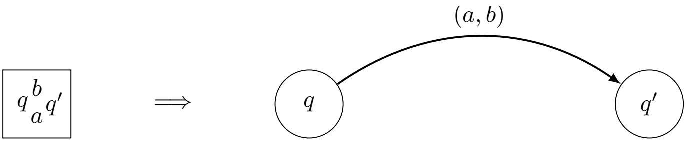


Fig. 2. Translation of a tile to a transition

We can see that there exists a bi-infinite run of the automaton on the sequence

$$\ldots(a_{-2}, b_{-2}), (a_{-1}, b_{-1}), (a_0, b_0), (a_1, b_1), (a_2, b_2)\ldots$$

if and only if there exists a tile horizontal stripe that carries the sequence $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2\ldots$ on the bottom and $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2\ldots$ on the top.

If we try to extract a set of tiles from several automata, and take the union of the results, we will get a tileset which performs a run of one of the automata on each line. We have to ensure that the set of states of the several automata are disjoint, which guarantees that automata will never be mixed within a single line.

2.4 Construction of Actual Automata

Let us construct a finite-state automaton which reads sequences of couples (a_n, b_n) and checks if $|\sum_i b_i - 2\sum_i a_i|$ is bounded; and another one which checks if $|3\sum_i b_i - \sum_i a_i|$ is bounded. The sequences a_n and b_n are on alphabets of two integers, for instance a_n is on $\{0, 1\}$ and b_n on $\{1, 2\}$.

These automata are constructed in the following way: fix a set of states Q, and have all transitions $q \xrightarrow{(a,b)} q'$ to satisfy the following relation:

$$q' = q + 2a - b \text{ (automaton for } b = 2a)$$
$$q' = q + a/3 - b \text{ (automaton for } b = a/3)$$

Thus, the automata will compute the cumulative sum of a_n and the cumulative sum of $2b_n$ (resp. $b_n/3$), and hold the difference into its current state. Since the number of states is finite, the difference must be bounded. As a consequence, if a couple of sequences (a_n, b_n) is accepted by the first (resp. second) automaton and a_n have an average, then b_n have an average which is twice (resp. one third of) a_n's one.

It only remains to set the alphabet for the a_n and b_n sequences. These alphabets are directly connected to the allowed range for averages of a_n and b_n. For instance, if a_n is on alphabet $\{1, 2\}$, its average can be any real number between 1 and 2. Likewise, we have to set an alphabet for b_n. As an additional restriction, we can make automata in such manner that they reject some finite patterns, like 000. Sequences on alphabet $\{0, 1\}$ without any pattern "000" cannot have an average lesser than 1/3. Using this fact, we can restrict allowed ranges for averages in a more precise way.

We can describe our piecewise linear function with two automata on alphabet $\{0; 1; 2\}$ (one for each linear piece).

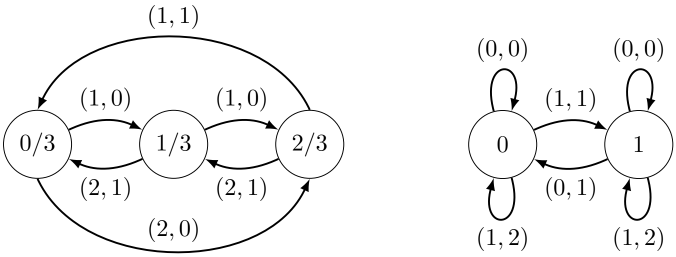


Fig. 3. Automata $M_{1/3}$ and M_2

These automata have 6 transitions each, yielding an almost aperiodic set of 12 tiles. Indeed one slight change must be made to avoid the tiling with only the "0" tile. This change will be presented further in this paper.

2.5 Existence of a Tiling

Remark that if we want to tile the whole plane, our automata cannot be fed with any sequences, even if those sequences have averages. Indeed, when the M_2 automaton reads with the sequence "$a_n = 0011$", the output sequence b_n contains both a 0 and a 2. However, automata $M_{1/3}$ only accepts sequences over $\{1, 2\}$ for a_n, and M_2 only accepts $\{0, 1\}$. Thus, if a stripe has 0011 on its bottom line, then it has both a 0 and a 2 on its top line. The next stripe cannot be a run of M_2 nor $M_{1/3}$, and the tiling does not go to infinity.

As a consequence, we need to show that there exists sequences of average x, for each positive real x, which are accepted by our automata. In order to achieve this, we will use Sturmian sequences. Define:

$$B_x(k) = \lfloor x(k+1) \rfloor - \lfloor xk \rfloor$$

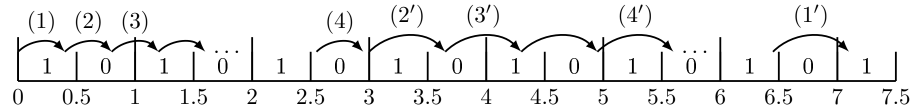
B_x is the **Sturmian sequence** of slope x, and $B_x(k)$ is its k^{th} letter. This sequence is bi-infinite over alphabet $\{\lfloor x \rfloor, \lceil x \rceil\}$. Since the sum over k of $B_x(k)$

is telescopic, it is easy to calculate the average of this sequence and check it is actually x.

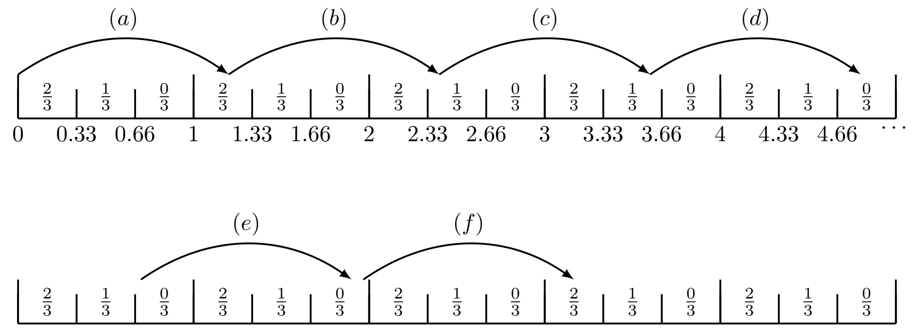
Let us think a bit about Sturmian sequences. Fix a real number x. Imagine you are on an infinite, measured line, and you are making jumps of length x along the line. Whenever you make a jump, write down the number of integers you jumped over: this is the Sturmian sequence of slope x.

We can get the Sturmian sequence of $2x$ by making jumps of length x, and counting the number of multiples of 0.5 we jumped over. There are twice more multiples of 0.5 than integers; thus, in the long run, we actually get the Sturmian sequence of $2x$. This idea is illustrated on Fig. 4a.

If we want to get the Sturmian sequence of $x/3$ from the sequence of x, we have to consider multiples of 3. They actually occur three times less often than integers. This is illustrated on Fig. 4b.



(a) Multiplication by 2: eight types of transitions, but (2) = (2′) and (3) = (3′), yielding six distinct types



(b) Division by 3: six types of transitions

Fig. 4. Multiplications of Sturmian sequences

Note that, in Fig. 4a, jumping over a non-integer multiple of 0.5 (the "small obstacles") increments the difference between "twice number of integers jumped" and "number of multiples of 0.5 jumped" by 1. By contrast, jumping over an integer (big obstacles) decrements this difference by 1. Since we want it as close to 0 as possible, two states are enough (before and after the small obstacle).

This works the same for Fig. 4b. Jumping over "small obstacles" increments the difference between "one third of the obstacles jumped" and "number of big obstacles jumped". Jumping over "big obstacles" decrements this difference by 2. As a conclusion, only 3 states are needed.

One can finally check that all possible types of jumps are displayed on Fig. 4, and that each of them corresponds to a transition of our automata (Fig. 3). For instance, type (1) corresponds to $1 \xrightarrow{(0,0)} 1$, and (2) corresponds to $1 \xrightarrow{(0,1)} 0$. More generally, in M_2, a_n corresponds to the number of jumped obstacles (of any size) by an arrow, and b_n corresponds to the number of jumped big obstacles. In $M_{1/3}$, it is permuted: a_n is the number of jumped big obstacles, and M_2 the number of jumped obstacles.

As a conclusion, (B_x, B_{2x}) is always accepted by M_2 and (B_{3x}, B_x) is always accepted by $M_{1/3}$. Thus one can take any sequence x_n from Sect. 2.1, write the Sturmian sequence of x_n on line n of the tiling, and get valid runs for automata. Thus one get valid tilings.

2.6 Aperiodicity

This construction ensures that each tiling corresponds to a specific sequence which is identically null or aperiodic. Then we just have to avoid this null sequence. Culik presented one way to achieve that in [6]. The idea is to forbid three consecutive uses of the M_2 automaton. This can be done by adding only one tile. Consider a new color $0'$ which value is 0 in the average, such that above a 0 there can be either a 1 or a $0'$, and above a $0'$ there always is a 1. Thus there cannot be three consecutive *not* $-$ 1 in a row, ensuring that the all zero configuration is forbidden. All tilings with this tileset are aperiodic. This tileset is displayed on Fig. 6.

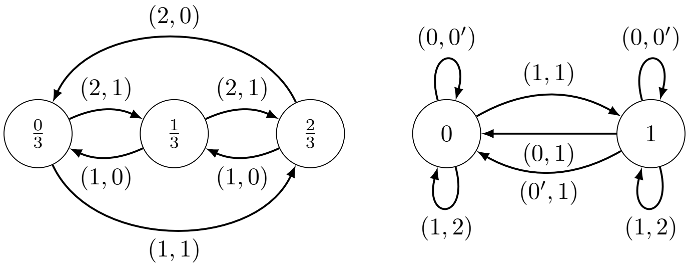


Fig. 5. Kari + Culik automata

3 Positive Entropy

3.1 Introduction

Let S be a palette and $C_{\mathcal{S}}(n)$ be the number of different patterns of size $n \times n$ which appear in a tiling. Then the *entropy* of $\mathcal{S}$ is defined by $H(\mathcal{S}) = \lim_n \frac{\log C_{\mathcal{S}}(n)}{n^2}$ (the limit always exists).

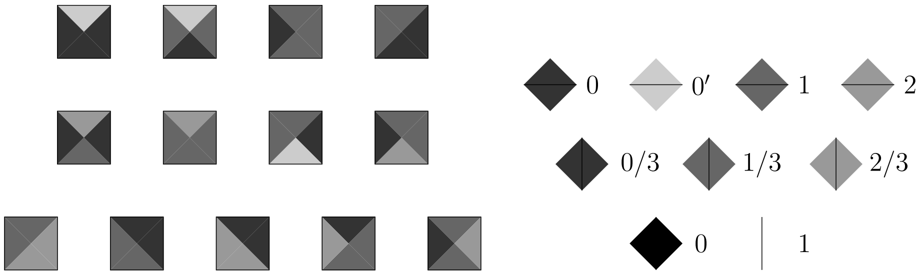


Fig. 6. Kari + Culik Tileset and colors meaning

The question we address is whether the Kari-Culik tileset has positive entropy. A usual method for proving such a fact is to exhibit a *substitutive pair*, i.e., a couple of different patterns with the same borders.

Our method is a small variation of the above: we prove that our tileset contains two substitutive pairs and for each sufficiently large square one of the pair items appears. Our substitutive pairs are shown in Fig. 7.

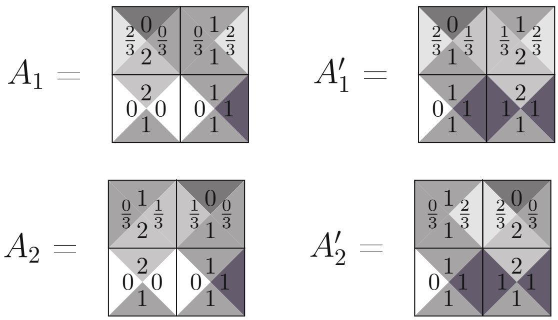


Fig. 7. Substitutive pairs

3.2 Coming Back to the Function

Recall that our function is $f : [\frac{1}{3}; 2] \mapsto [\frac{1}{3}; 2]$ such that

$$f(x) = \begin{cases} 2x & \text{if } x \in [\frac{1}{3}; 1] \\ \frac{1}{3}x & \text{if } x \in [1; 2] \end{cases}$$

Lemma 1. *The orbits of f are dense.*

Proof. It is well-known that irrational rotations on the circle have dense orbits. Thus, we map the interval $[\frac{1}{3}; 2]$ on the unit circle in such manner that the function f corresponds to a rotation of irrational angle.

We consider the following mapping:

$$\phi : [\frac{1}{3}; 2] \to [0; 1]$$
$$\phi(x) = \frac{\log(x) + \log 3}{\log(2) + \log 3} \mod 1$$

We view the interval $[0; 1]$ as a circle by identifying point 0 with point 1.

$$\phi(2x) = \frac{\log(2) + \log(x) + \log 3}{\log(2) + \log 3} \mod 1 = \phi(x) + \frac{\log(2)}{\log(2) + \log 3} \mod 1 \quad (1)$$
$$\phi(\frac{x}{3}) = \frac{\log(x)}{\log(2) + \log 3} \mod 1 = \phi(x) + \frac{\log(2)}{\log(2) + \log 3} \mod 1 \quad (2)$$

Both transitions map to the same irrational rotation of angle $\frac{\log 2}{\log 2 + \log 3}$.

Proposition 1. *Given any interval the maximal number of iterations of f between two occurrences in this interval is bounded.*

Proof. Consider any interval $I =]a; a+\alpha[$ in $[0; 1] \mod 1$. As the orbits of f are dense from starting point a, there exists N such that $f^N(a) \mod 1$ is in $]a; a + \alpha/2[$. Thus $f^N(a) = a + \beta$ with $\beta < \alpha/2$. From any point x in I, either $x + \beta$ or $x - \beta$ is in I. Thus either $f^{N\lfloor 1/\beta \rfloor}(x)$ or $f^{N\lceil 1/\beta \rceil}(x)$ is in I.Hence our required bound on the number of iterations of f is $N\lceil 1/\beta \rceil$.

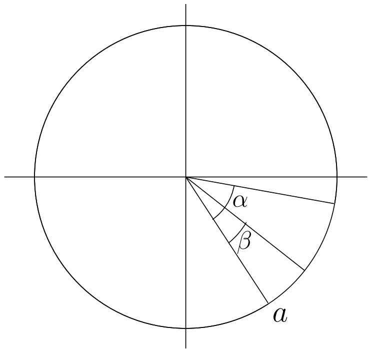


Fig. 8. Crossing intervals

Now let us examine the colors that appear on any horizontal line of the tiling. On this line colors represent 0 and 1 or 1 and 2 (0 and $0'$ being interpreted as the same zero).

Proposition 2. *Any horizontal line in any tiling has an average (in the sense of frequencies of numbers defined above).*

Proof. Our proof is based on the remark that on each line we have either 0 and 1 or 1 and 2 but never 0 and 2.

The average of any segment has a value in $[0;2]$ which is a compact set. Consider two non overlapping growing sequences of subpatterns. Suppose their averages have different limits. Then we can take subpatterns of different averages as large as we want. With enough runs of the automata, one of this subpattern will have average less than 1 and the other, greater than 1, on the same line. But no automaton can read such a line, which makes a contradiction.

Remark that Kari's basic idea is that the averages of the lines obey the function f.

We prove below that a specific family of patterns appears dense in our tiling. The following lemma gives us the horizontal density, and the vertical density is obtained by combination of this lemma and Proposition 1.

Lemma 2. *The family of patterns* $\{01^\alpha 0 | \alpha > 3\}$ *appear with positive density in each line that has average in* $]\frac{4}{5};\frac{9}{10}[$.

The choice of $\frac{9}{10}$ is arbitrary, the result remain true for any value in $]\frac{4}{5};1[$.

Proof. On each line with average greater than 4/5 the pattern 1111 must appear with positive density. On each line with average less than 9/10, 0 appear with a positive density, otherwise we would have a contradiction with Proposition 2.

We now have a family of linear patterns that appear in a dense way in our tiling. Let us prove that each time one of this patterns appear, one element of our substitutive pairs appear.

Let us consider the two lines above this pattern. Above the 0's will always be 1's because reading 11 implies that the output alphabet is $\{1;2\}$, and a 2 is never above a 0 by construction. Above the 1's there will be 2's, expect for one 1. We then distinguish three cases depending on the position of the 1 in the block of 2's : the leftmost, somewhere in the middle or the rightmost. The line above this block of two is the result of a division by 3. This operation is deterministic, there are a priori three possibilities for the phase of the carry on the whole line : 0, 1/3, 2/3.

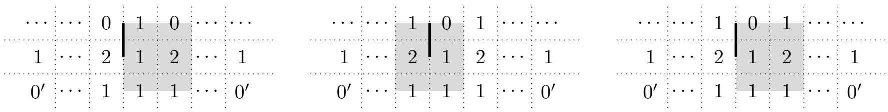
In the middle case (Fig. 9a), the three cases for the phase of the carry are possible. In each of those cases one element of the substitutive pairs appears, either with the bottomleft or the bottomright tile being the apparition of the 1 in the block of 2's.

In the two other cases (leftmost and rightmost) we have two consecutive ones. This prevents the appearance of one of the phases (one can check that block 0011 cannot be continued, drown in red in the pictures).

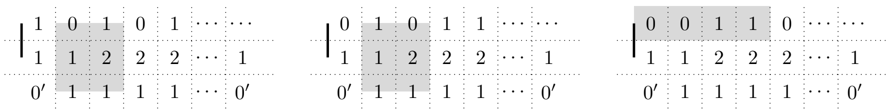
In the leftmost case (Fig. 9b), only two of the three possibilities for the phase of the carry may appear. In both cases, one element of the substitutive pairs appear above the two leftmost 1's of the first line.

In the rightmost case (Fig. 9c), only two of the three phases of the carry are possible. In both cases, one element of the substitutive pairs appear above the two rightmost 1's of the first line.

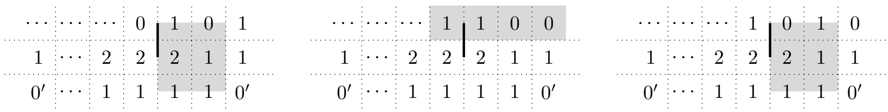
The colored vertical bar correspond to the code of bits presented in Fig. 6.



(a) Middle case



(b) Leftmost case



(c) Rightmost case

Fig. 9. Our case analysis

Theorem 1. *The Kari-Culik tileset have positive entropy.*

Proof. The theorem is a consequence of all the other results of this section : we have presented two substitutive pairs that together appear in a dense way in any tiling of the plane.

3.3 Open Problems

We proved that our two pairs are together dense in any tiling. Is one of those pairs dense alone in a given tiling?

Consider the extended tileset where we forbid one pattern in each of the presented pairs. The obtained tileset is still a palette. Has this tileset a positive entropy? If the answer is positive, is it possible to exclude a finite number of patterns so that the resulting tileset has zero entropy?

Using substitutive pairs we proved that there are tilings which horizontal lines do not represent mechanical words. Is it possible to better characterize the Kari-words : the language of the lines that can appear in a tiling ?

Major Open Problem: Does a sub-shift of finite type A exists such that

– A has positive entropy;
– any sub-shift of finite type B included in A, has positive entropy.

This question addresses spaces of dimension at least 2 and any finite alphabet.

4 Choices, "cylindricity" and Entropy

Positive entropy is a very rough method for understanding the quantity of choice that you meet when you effectively construct a tiling of the plane. Imagine that you walk over the plane in spiral trajectory, placing one matching tile after another. Lets mark in red the cells were you have a "real choice" *i.e.* were you have at least two possibilities that can continue to an infinite tiling of the plane. If the set of red cells is dense then the entropy of the tileset is positive. If you use this approach on selfsimilar tilesets as usually constructed, then the red cells are exponentially rare : as soon as you fix a tile on this cell you impose the next level structure and the size of the such determined areas grows exponentially. From the original construction of Kari [5] it is clear that there are horizontal lines where the density of red points is constant (because of underlying mechanical words). Nevertheless, even if this makes a difference between self similar tilesets and Kari's, this does not prove positive entropy, furthermore it was conjectured that this freedom in representation of mechanical words of same density is strongly coupled.

Another refined version of the entropy appraoch was presented by Thierry Monteil in [7] using the notion of cylindricity. We explain below how this is related with our work.

Consider a vertical cylinder of size n. If you can tile this cylinder with a tileset then two of the horizontal rings are identical, thus one can tile a torus which correspond to a periodic tiling of the plane. If the tileset is aperiodic, for each n there exists a maximal vertical size for a portion of the cylinder to be tilable. The smallest growing function greater than this vertical size is called the *cylindricity* function of the tileset.

Consider any self similar tileset (for instance use the generic approach of Nicolas Ollinger in [8] for generating a Wang tileset from a substitution). If one can tile a cylinder of given size, then we can rewrite all the tiles into blocks and thus obtain a larger cylinder with about the same proportions: depending on the proportion between horizontal and vertical factors. The cylindricity function is greater than x^α with α positive.

Remark that the cylindricity is always greater than a logarithm : consider a tiling of the plane and a vertical segment of size n in an horizontal stripe. The minimal distance for seeing twice the same vertical segment is bounded by an exponential in n (because the tileset is finite).

It would be interesting to study this function for more sophisticated tilesets, for instance the most complicated one [9] or the robust to errors version in [10]. In Kari's tiling if you have a periodic configuration then its image after a few steps will have a period three times larger because when we divide by three, consecutive periods assume different carry phases. Note that the $\times 2$ operation cannot diminish the period. Thus if we have a cylinder of length n and height h, then the period of the first line is at most $(n/3)^{\alpha h}$ where α is a constant (some easy technical adjustments are needed to transform this argument into a complete proof).

From this result it was conjectured in [7] that the logarithm of number of patterns of size $n \times n$ was of order n. This would have produced an entropy zero tiling with strictly more choices than for self similar case. But it is not the case. We proved that the bound given by the cylindricity approach is not tight.

Acknowledgments and Related Results. The authors thank Alexander Shen for his help in stating the above results in a clear way.

Since the pre-publication of our paper in arXiv we received some comments from people working in this area :

- Emmanuel Jeandel used a brute-force program to prove that one of our pairs of entropic patterns always appears starting from a line of ones before a fixed number of steps;
- Nathalie Aubrun and Michael Rao worked on substitutive pairs and discovered the same pairs as ours. Their approach could be promising for solving our open problems but is not yet published;
- Uwe Grimm pointed to us an article by N. Nikola, D. Hexner and D. Levine [11]. Contrarily to what they write in their paper they do not provide a proof of positive entropy with only one of our entropic pairs, the proof is supposed to be in a paper of the same authors without title and labeled as "to be published". If their proof is correct then one of our open problem is solved.

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