Chapter 8
Didactic Engineering as a Research Methodology: From Fundamental Situations to Study and Research Paths

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8.1 Didactic Engineering as a Research Methodology

The notion of didactic engineering (DE) has been at the core of the project of a science of didactics founded by Guy Brousseau in the 1970s along with the theory of didactic situations (TDS). In a recent paper presenting the origin of DE, Brousseau (2013) explains its necessity and locates it in the interface between research and teaching:

> Didactic engineering was a necessary and 'concrete' domain between a poorly invested activity, teaching mathematics, and an absent science, Didactics. The latter was supposed to, on the one hand, newly define both of them and, on the other hand, find its contingency in their confrontation and complementarity. 'Do not content yourself only with evidence', 'systematically reproduce', 'analyze in order to save experiences', 'only accept exogenous concepts under their testing in didactic engineering'—those have been the guiding principles [of Didactics]. (Brousseau, 2013, p. 4, our translation)

In the entry *didactic engineering* of the new *Encyclopaedia in Mathematics Education*, Michèle Artigue tries to clarify this intermediate role between the reality of classrooms and the science of didactics:

> The idea of didactical engineering (DE) […] contributed to firmly establish the place of design in mathematics education research. Foundational texts regarding DE such as (Chevallard, 1982) make clear that the ambition of didactic research of understanding

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> and improving the functioning of didactic systems where the teaching and learning of mathematics takes place cannot be achieved without considering these systems in their concrete functioning, paying the necessary attention to the different constraints and forces acting on them. Controlled realizations in classrooms should thus be given a prominent role in research methodologies for identifying, producing and re-producing didactic phenomena, for testing didactic constructions. (Artigue, 2014, p. 159)

It is important to keep in mind that, in the theory of didactic situations (TDS), didactic engineering was part of a collective project, led by Brousseau, to build an empirical science of didactic phenomena where the issue of the empirical validation of results was to be carefully taken into account. This is how he remembers those beginnings, in the same text quoted above:

> My contribution was to design, project and start creating a proper science, which has to be responsible for the original theoretical concepts needed by engineering and for submitting them to the exigencies of any mature science, enriched by its scientific peer to peer relationships with other educational approaches. (*ibid*., p. 4, our translation)

In this context and as Artigue (2008, p. 4) explains, didactic design was called to fulfill two different needs: to take into account the complexity of classrooms, at a time when research mainly relied on laboratory experiments and questionnaires; and to articulate the relationships between research and teaching innovation. She also highlights five main characteristics of DE as a theory-based intervention: the central role given to the notion of *situation* in both the modeling of mathematical knowledge and the organization of its teaching; the crucial attention paid to the epistemology of knowledge and the need to rebuild any mathematical content as the answer to an issue raised within a social situation; the importance given to the characteristics of the empirical *milieu* of the situation and of the students' interaction with this *milieu*; the three different functionalities assigned to mathematical knowledge, *action—formulation—validation*; and the vision of the teacher's role as the organizer of the relationships between the adidactic[1] and the didactic dimensions of situations (*devolution, institutionalization*).

As we shall see, DE appears as a research methodology to be closely related to the TDS, although it exceeds this initial framework:

> As a research methodology, DE emerged with this ambition, relying on the conceptual tools provided by the Theory of Didactical Situations (TDS), and conversely contributing to its consolidation and evolution (Brousseau, 1997). It quickly became a well-defined and privileged methodology in the French didactic community, accompanying the development of research from elementary school up to university level […] (Artigue, 1990, 1992). From the nineties, DE migrated outside its original habitat, being extended to the design of teacher preparation, and professional development sessions, used by didacticians from other disciplines […] and also by researchers in mathematics education in different countries. (Artigue, 2014, pp. 159–160)

What are then the main characteristics of DE that are preserved in the evolution of TDS and the approaches sharing its main epistemological principles, such as the

[1] In an adidactic situation, students interact with a milieu only considering the logic of the problem approached, without taking into account the teachers' didactic intentions.

Anthropological Theory of the Didactic (ATD) we are considering here? We are answering this question using the four-phase structure of DE as a research methodology proposed by Michèle Artigue (2008). It will help us distinguish the theoretical assumptions underlying all DE works and emphasize its internal role in didactics research as a *phenomenotechnique*, that is, as a tool to produce didactic phenomena.

At the starting point of a DE process, we are locating a concrete content or issue to be taught and learned and usually a didactic problem related to it. The first phase, called *preliminary analysis*, mainly includes an epistemological questioning of the mathematical content at stake and of the necessity to introduce it at school, and a study of the conditions and constraints offered by the institutions where the teaching and learning process is to take place. This is an essential first step where research hypotheses are formulated and the content to be taught and learned is questioned, usually considering different kinds of hypothetical didactic phenomena involved. It is also in this phase where previous research results can be reinvested.

The second phase concerns the *design and a priori analysis*. This phase corresponds to the statement of how the content at stake is considered or modeled within didactics research. A mathematical and a didactic level may be distinguished here, to first "define" or "characterize" the content (mathematical analysis), and then to propose how to make it emerge from problematic questions within a sequence of concrete situations (didactic analysis). In the theoretical frames here considered, these analyses are carried out in terms of mathematical and didactic situations (TDS) or mathematical and didactic praxeologies (ATD).

The third phase includes the implementation of the previously designed didactic process, its observation, and data collection. At this experimental level, an *"in vivo" analysis* is usually developed, when interpreting in real time (or straight after) what is taking place in the classroom. Finally, the a posteriori *analysis* culminates the DE process. It is organized in terms of the contrast, validation, and development of the research hypotheses and design proposals of the previous phases, usually often leading to the formulation of new problems, related to both fundamental research and teaching development (Fig. 8.1).

It needs to be highlighted that, even if the a priori analysis precedes the in vivo and a posteriori analyses, there is always a constant interaction between the outcomes of the different phases: results from the a posteriori analysis may not only suggest introducing changes in the design of the teaching process, but also developing the characterization of the content at stake (preliminary analysis). It may also contribute to the *science of didactics* with the results obtained and the open problems raised, leading to new theoretical or methodological developments. In this sense, DE is not a development practice where previously established research results are transformed into teaching proposals. It is a way to empirically contrast assumptions about the possibilities of the diffusion of mathematical knowledge and the phenomena hindering it. As Brousseau said:

> My ambition has been to turn didactic engineering not into a socio-professional cover, but a scientific activity based on a coherent and 'proper' body of scientific knowledge. (2013, p. 6, our translation)

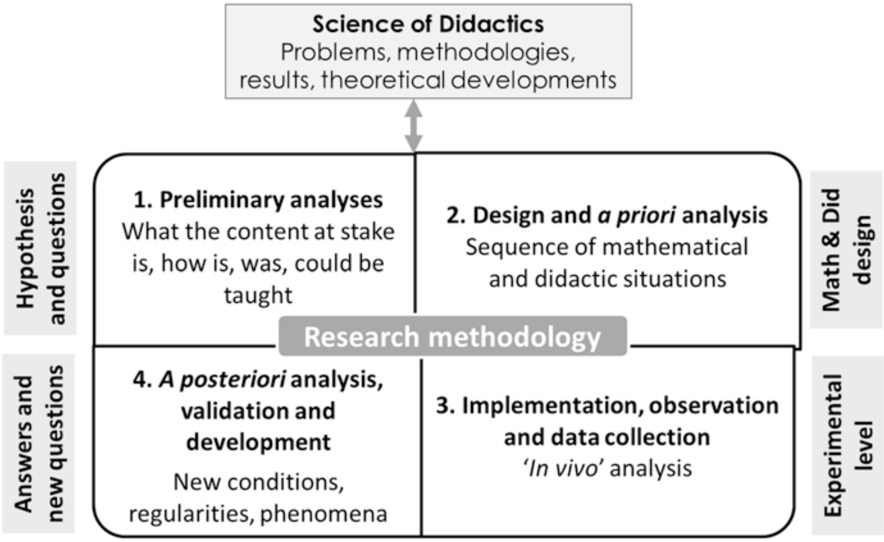

Fig. 8.1 Phases of the DE research methodology within TDS

DE is one among many other empirical methods elaborated and used by didactic research. We will not refer to it for instance when what is investigated is not directly a teaching and learning process but results from evidence coming from other sources gathered through naturalistic observation of institutions and their outcomes (including classes, historical documentation, etc.) or through direct intervention via interviews, questionnaires, etc.

8.2 Didactic Engineering Within the Theory of Didactic Situations

After this brief introduction to the notion of DE, we are presenting two examples of research based on DE processes, one from the TDS about the measure of quantities at primary school level, and another one from the ATD and the teaching of modeling processes at university level. In spite of the initial difference between both investigations, strong commonalities are being stressed, relying on what we propose to conceive as the mainstream of DE in Didactics.

8.2.1 An Example: Measuring Quantities at Primary School

We are using the case of the measurement of quantities at primary school to illustrate the four phases of the DE methodology within the TDS and, especially, their interactions. This example corresponds to a crucial issue in elementary mathematics education and has been the object of many investigations in the TDS that have not been widely disseminated in the international community. We will describe it in a brief and necessarily simplified way. More details can be found in Bessot and Eberhard (1983), Brousseau and Brousseau (1987, 1991–1992), Brousseau (2002), Douady and Perrin-Glorian (1989), Perrin-Glorian (2002, 2012), and Sierra (2006).

8.2.1.1 Preliminary Analysis

The starting point of the research is not the teacher's problem, *How to introduce the measurement of quantities in primary school?* Rather, it is the insertion of this problem into a broader questioning including epistemological as well as social issues, such as: *Why is it necessary to teach the measurement of quantities at primary school? What mathematical entities and practices are related to it? What social activities? How is it related to other mathematical notions, such as numbers, ratios, relationships, areas, volumes?*

To answer these questions, one should take into account the processes of didactic transposition (Chevallard, 1982) and the analysis of the activities that have been, are, and could be taught at school, an analysis that usually leads to the identification of *didactic phenomena*. For instance, it can be shown (Brousseau, 1997; Chambris, 2010; Perrin-Glorian, 2002, 2012) that, with the introduction of New Maths into the French curriculum in the 1970s, magnitudes and quantities disappeared from school mathematics, where they supported the construction of numbers. Only some basic practical measures and the metric system remained. Curricula have changed a lot since then, but the synthesis between quantities and sets to support the construction of numbers has still not been solved. Some indicators of this phenomenon are the fact that the choice of the unit of measure (gauge) is never raised, the blurry role played by units in modeling strategies and calculations, and the frequent situation that mathematical work is dominated by "abstract" numbers instead of "concrete" ones, that is, those directly representing physical quantities. Many years ago, Hans Freudenthal described this absence in the following terms:

> To count people and eggs there are natural units. To measure quantities, one needs gauges; the result of the measuring procedure is a number, which measures the quantity. There is a variety of gauges, because there is a variety of magnitudes; length, area, volume, height, mass, work, current intensity, air pressure, and monetary value are notions that become magnitudes by measuring procedures. Sometimes it is not clear why some magnitudes need different gauges. […] A few of these gauges are learned in arithmetic instruction, and as far as he needs it, the physicist develops a rational measure system. In between a large domain is no man's land. This is the fault of the mathematician. (1973, pp. 197–198)

During the same period, Hassler Whitney (1968) developed a mathematical theory of physical quantities to justify calculations, not between numbers but between quantities (such as $6\text{ m} \div 2\text{ s} = 3\text{ m/s}$ or $5.25€/\text{m} \times 0.8\text{ m} = 4.20€$), thus trying to build a bridge between engineering or science practices and mathematical ones. However, his proposals remained in the "scholarly mathematics" and have not permeated the prevailing school mathematical culture where calculations are very often done with abstract numbers and where units appear (if at all) only at the end at the process.

8.2.1.2 A Priori Analysis: Design of Mathematical and Didactic Situations

In order to face the complex problem quickly outlined previously, research in didactics needs to elaborate its own vision about measure and quantities or, more precisely, a *reference epistemological model* (Bosch & Gascón, 2006). In the TDS, reference epistemological models are formulated in terms of fundamental situations defined as games of action, communication, and validation, in interaction with an experimental *milieu*. The situation proposed by Brousseau (2002) defines the measure and quantities in terms of three intertwined *universes* and different situations between them. The first universe is the world of concrete measurable objects and their material comparison (putting objects side by side, on the two plates of a weighing scale, into a liquid, etc.). The second one is the universe of quantities (lengths, weights, areas, volumes, prices, etc.) as equivalence classes of objects considering analogical measures, where objects do not need to be manipulated but can be compared through some intermediate measures (gauges). The third is the universe of units, numbers, and change of units, obtained after defining a single privileged gauge for each magnitude (Fig. 8.2). We can thus obtain a general definition of measure in terms of triplets, including two universes and a situation to link them: something to measure (objects); a way to put objects into correspondence (adding specific conditions to get a measure application); and a positive numerical structure to express the measure (also with specific conditions depending on the number of units considered and other requirements). Usually, in school culture, only the first and third universes are considered, and only the third acquires a mathematical status.

To be operative, this definition needs to be specified in terms of sequences of games or adidactic situations passing through phases of action (solving a problem through empirical interaction with a *milieu*), communication (explaining the answer so that another person can follow and even reproduce the solution), and validation (justifying the solution without referring to the contingency of the *milieu*). Depending on the educational level and institution considered, the types of situations may obviously vary. Their design is part of both the delimitation of the reference epistemological model (mathematical situations) and their concrete realization under specific conditions (didactic situations).

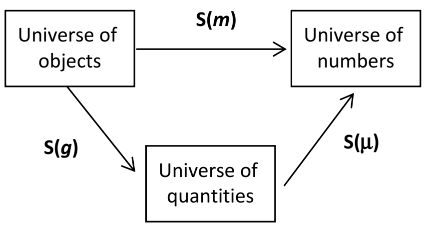

Fig. 8.2 Universes of measure (Brousseau, 2002)

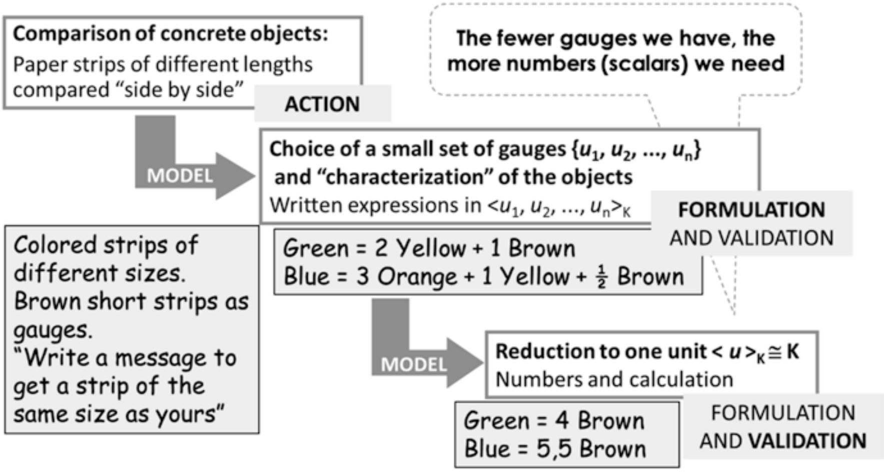

Fig. 8.3 Measuring situations: a priori analysis

Let us take a short example from Brousseau and Brousseau (1987) which is part of a larger DE design: a situation for grade 4 of primary school where it is proposed to introduce the measure of "length" through the following situation of communication. The *milieu* is composed of similar strips of different lengths and colors, with repetitions: short brown strips of the same length, medium red strips of different lengths, and long blue strips of different lengths. In this *milieu*, two strips can be compared by putting them side by side (action). The communication game, played by teams, consists in, given a blue strip, asking another team to bring as many smaller strips as necessary to build a new strip of the same size (Fig. 8.3). The aim of the activity is to raise the need for gauges to simplify the comparison of objects (common units for the messages) and, since there is no simple relationship between long and short strips, to move to the choice of a single unit and its fractions to simplify the messages without decreasing the precision of the measure.

8.2.1.3 Implementation, Observation, and Data Collection

In the first part of the sequence related to the strips communication game (where small brown strips are called "*u*"), it can be seen how the initial messages may fail and students learn how to make more precise messages to get a strip of the same size as theirs. The types of messages produced are "2 *u* plus 3 quarters of a *u*", "5 strips and fold the small *u* strip in 2", "3 times *u*, half, half of the half, half of the half of the half", "2 strips and another one with a small part missing", etc. (Fig. 8.4).

Fig. 8.4 Measuring situations: experimentation (students' productions)

The width of the strips was also used, thus including a new gauge that was not forecasted in the a priori analysis: "5 brown strips plus the width of a brown strip and half the width" (Brousseau & Brousseau, 1987, pp. 3–6).

During the very first experimentation of the sequence, an interesting problem appeared that partially discredited the a priori analysis. Because the students were familiar with the ruler and with measuring lengths in cm, some of them did not feel the need to choose a small strip as a gauge and started writing their messages using cm: "2 brown strips, one small strip and not a whole one, 3 or 4 cm have to be eliminated". It was then very difficult, and artificial, to move the students back to the single brown strip as unit, keeping cm aside.

Due to this unexpected event, after the first two sessions of working with lengths, it was decided to change lengths for weights, less familiar to the students, and avoid the use of metric units. Strips were replaced by small objects (pencil cases, small glasses, exercise books, etc.); different sizes of nails and small plates were introduced as gauges, and the comparison was made with a two-plate scale. This shows how the experimentation and in vivo analysis can make the design and a priori analysis evolve. We can consider that the reference epistemological model was also enriched through the experimentation, showing new conditions for the construction of the process of measuring quantities, as for instance the difficulties for the second universe (quantities) to exist without being directly absorbed by the third one (numbers), and also the relationships between the set of scalars needed in this second universe and the number of generators (gauges) used (Sierra, 2006).

8.2.1.4 A Posteriori Analysis: Results, New Phenomena, New Research Questions

The situations about comparing lengths and weights are part of a long sequence of 30 activities which form the DE work described in (Brousseau & Brousseau, 1987) to introduce the measuring of quantities in grade 4 of primary school. It contains the following activities:

- Measurement of lengths: communication game; studying the messages
- Measurement of weights: communication game; messages; work on the writings; comparing expressions; conversions; adding weights; comparing sums and total weight; transformation in basis 60
- Measurement of time: time and duration; calculation with numbers in basis 60
- Legal units of weight: presentation; conversions
- Finding the weight of an empty recipient: first part; second part (challenges)
- Measurement of lengths: adding lengths; decimal measures
- Writing decimal measures: length measures; decimal length and weight measures; comparison of decimal measures; order in decimal measures
- Operations with decimal measures: addition; multiplication by an integer; subtraction

In a later work, Brousseau and Brousseau (1991–1992) present some crucial issues derived from this research and describe some of the related phenomena. As we have seen before, there is, for instance, the fact that familiar *milieus* (such as lengths) are not always didactically productive, even if they may initially facilitate the devolution of a situation. A similar didactic phenomenon occurs with the teaching of rational or negative numbers, when the fractional or directional measures that are used to introduce them become a didactic obstacle when defining their multiplication or division.

There is also another example related to a very interesting experience with one of the reported activities, *the weight of the receptacle*. In spite of the errors of measure of the full and half-full receptacle, the students postulate and confirm an affine relationship between the volume of water and the total weight of the receptacle, which enables them to deduce the weight of the empty receptacle. It thus shows a complex relationship between the students' reasoning in a validation situation and the empirical *milieu* used, because taking into account the errors of measure appears as a mathematical necessity. And it submerges students at the core of scientific activity: "[Due to the measure errors], children became aware that when a theory or a method is made to forecast or obtain a result, the fact that its application happens once or twice is not enough for it to be accepted as true or valid. It has to 'work' in all cases, something which can only be established through reasoning" (*ibid.*, p. 80).

8.2.2 Didactic Engineering in the Science of Didactics

8.2.2.1 The COREM as a *Didactron*

Even if DE can be understood as a general research methodology closely related to the constitution of Didactics as a scientific domain, its existence cannot be separated from the COREM, *Centre d'Observations et de Recherches pour l'Enseignement des Mathématiques*. It was created in 1973 by Guy Brousseau as a research laboratory of the University of Bordeaux 1 and was integrated in the elementary school Jules Michelet in Talence (Bordeaux, France).[2] Till its closure in 1999, the COREM functioned as what Brousseau amusingly called a *didactic accelerator* or *didactron*.

In the COREM, new teaching proposals based on the TDS were regularly experienced by researchers, in close cooperation with the teachers of the school, who participated in the design, a priori analysis, teaching, observation, and a posteriori analysis of the lessons. Furthermore, all didactic engineering components, from the conception of situations to their setting up, managing, and observation, were the concern of all the staff, teachers, and researchers (Brousseau, 2013, p. 7). In fact, Michelet School was (and still is) a normal public elementary French school with 4 classes of preschool level and 10 classes of primary level (2 groups per grade), with pupils from the neighborhood and the same curricular and administrative requirements as any other French school. The teachers at the school were also normal ones, in the sense that no specific educational training was required, even if they were asked to participate in research activities. The peculiarity is that they worked in teams of 3 teachers per 2 classes, devoting one third of their time to the COREM, where they attended seminars and meetings with researchers, made observations, and had teaching preparation sessions with the other teachers of the team. According to Greslard and Salin, "The complexity of the [COREM] functioning is due to the fact that the creators of the project wanted to avoid the educational vocation of the school being altered by the investigations, and that these could later be carried out in the best possible methodological conditions" (1999, p. 30, our translation). It also supposed a detailed regulation of the interactions between researchers, teachers, and the classes observed.

Usually, in the development of a didactic engineering process, researchers presented a teaching proposal partially including the a priori analysis (goals expected, problems addressed, strategies forecast) to the team of teachers. Then they jointly elaborated the details of the sequence of lessons up to the preparation of a *didactic card* (*fiche didactique*) for each lesson. Researchers prepared the observation and decided on the kind of data to be gathered. During the lessons, observers had to try to be as invisible as possible, and teachers were supposed to forget that they were observed, taking their own decisions about the teaching of the lessons. Immediately after each lesson, a short meeting took place for the teacher, researchers, and other possible observers to share impressions, starting with the teacher's

[2]A detailed presentation can be found at http://faculty.washington.edu/warfield/guy-brousseau.com/index.html and http://guy-brousseau.com/le-corem/

report of the experience. The observation of “ordinary lessons” which had not been organized through a didactic engineering process also took place at the COREM on a regular basis, although with a less structured procedure.

Throughout the existence of the COREM, all materials related to the teaching of mathematics were gathered, including class preparation activities (both the normal ones and those specially designed for research), students' productions, video recording of the lessons, reports from teachers and researchers after the lessons, annual planning of the courses, research seminars, etc. There was also a classroom especially designed for observations, with an extra surrounding area for observers, a windowed cubicle for the observers in an outer room, and technicians doing video and audio registration. Since 2010, the COREM archives have been made available by the Centre of Resources in Didactics of Mathematics Guy Brousseau (CRDM-GB) of the Spanish university Jaume 1 of Castelló (Valencia) (http://www.imac.uji.es/CRDM). Video recordings can also be accessed at visa.inrp.fr/visa. The list of didactic engineering realizations observed is very long, including teaching proposals about the main mathematical content from preschool to grade 5:

Reasoning and logic (preschool, grade 1): Designation, equality, lists, belonging to a list; Classing, sets, propositions, no-and-or, equivalence, equality; Comparisons, physical quantities ordering; number, length, mass, price, capacity; Order, <, >, next, previous; $P(E)$; Implicit theorems, demonstration (race to 20); Theorems, proofs (bigger number)

Quantities and measure (preschool, grades 1, 4, 5): Natural quantities (cardinal, lengths, masses, prices); Capacities; Sums, products, extractions, partitions; Volumes, capacities; Rational and decimal quantities (commensuration, unit partition); Events measure (statistics)

Discrete quantities and arithmetic: Operations on natural numbers (addition, multiplication, subtraction, division); Functions

Rational quantities, arithmetic, and algebra: Rational and decimal numbers, definition, writing, operations; Order (density); Linear applications, enlargements; Numerical dilations, ordering, composition; Structure of rational numbers

Space and geometry: Topology, figures; Fundamental situation of geometry; Congruencies; Dilations

Statistics and probability: Random walk; Confidence interval; Compose probability (two successive events); Approximation to the Law of Large Numbers. (Brousseau, 2013, p. 12)

8.2.2.2 An Experimental Epistemology

In the research program set up by the TDS, the experimental work carried out by DE processes is crucial, as it represents a way to empirically test epistemological and didactic proposals formulated in terms of sequences of adidactic and didactic situations. In a sense, the TDS appeared as a reaction to the New Mathematics reform of the 1960s and 1970s that Brousseau considered as “a utopia totally ignoring all the difficulties and laws of the dissemination of knowledge and practices in a society […], which believed and died in the illusion of transparency of didactic facts" (2004, p. 23). This explains the importance given to the empirical contrast of teaching proposals before their dissemination, as well as the necessity to base them on a consistent and explicit framework of theoretical assumptions. That was the very precise role of DE:

> *Didactic engineering* became, de facto, a part of Didactics of mathematics where precise, observable and reproducible teaching devices, specific to different forms of knowledge of determined mathematical entities, were conceived and also empirically, experimentally and theoretically studied […] (Brousseau, 2013, p. 6, our translation)

In this context, the notion of *situation* does not only appear as a way to describe teaching activities, but was also first used as a model to conceive mathematical activities specific to each mathematical content to be taught. It contains the requirement to characterize each piece of mathematical knowledge by a set of conditions making it progressively appear as the answer to a given problematic question. The adidactic situations are thus a way to show the functionality of mathematical knowledge in the institutional environment of the students who have to learn it.

This ambitious project requires a double rupture: researchers need to allow themselves to question mathematics as it is usually conceived and presented by mathematics scholars and by school institutions, elaborating their own alternative reconstructions of mathematical knowledge and activities (the *reference epistemological models*). They also need to have the same attitude towards other disciplines (psychology, pedagogy, sociology, etc.) concerning the effects of their proposals on mathematical practices and knowledge. This is why it is important that the results obtained are empirically based, protecting researchers from adopting unfounded ideologies or implicit institutional viewpoints on both educational facts and mathematical knowledge.

8.3 Didactic Engineering Within the Anthropological Theory of the Didactic

8.3.1 From Situations to Study and Research Paths

The TDS conception of DE is located in what Chevallard (2012) calls the *paradigm of questioning the world*: mathematical contents, just like the content of any other subject matter, should not be taught as if their value and importance were taken for granted. On the contrary, they need to be constructed and appear for the students as true answers to real questions. The search for a fundamental situation to represent, model, and rebuild any given piece of knowledge is in fact a way for didactics research to assume its own responsibility in the search for the possible *raisons d'être* of mathematical contents within the students' reach, and for the rationale of their teaching at school. For instance, the epistemological question, *what is the* *measure of quantities and how can it be constructed through a sequence of situations?*, includes the primary question: *what is the measure of quantities for, and why is it important to learn it?*

The Anthropological Theory of the Didactic, as it has been developed by Yves Chevallard (1992, 2006, 2007, 2012), shares the TDS essential epistemological questioning, the search for a rationale for any piece of knowledge to be taught, and the central place given to problematic issues in learning and teaching processes. The modeling in terms of fundamental situations is replaced by two main theoretical tools: the notion of *praxeology* used to describe any kind of human activity (Chevallard, 1999) and the *Herbartian schema*, named after the J. F. Herbart (1776–1841), into account the way praxeologies are built, taught, learned, or disseminated as the answer to a given problematic question (Chevallard, 2011).

If the starting point of the teaching and learning process is a given praxeology P a group of students X should learn under the supervision of a group of teachers Y, then the didactic process involving X, Y, and P can be described in terms of *study and research activities* structured in six didactic moments or dimensions closely linked to the structure of praxeologies (Barbé, Bosch, Espinoza, & Gascón, 2005; Bosch & Gascón, 2010; Chevallard, 1999). However, this is not the only possible pattern to represent teaching and learning. A didactic process does not necessarily begin with the delimitation of a given piece of knowledge to be taught, but can also be motivated by the need to consider a problematic question Q_0 a group of students X wants (or has) to answer with the help of a group of teachers Y. What then appears is a sequence of linked study and research activities called *study and research paths* (*SRP*), which can be formalized using the general "Herbartian schema" as follows:

$$\left[S\left(X;Y;Q_0\right)\rightarrow\left\{A^{\lozenge}{}_i,O_j,Q_k,A_k\right\}\right]\rightarrow A^{\blacktriangledown}$$

The starting point of an SRP should be a "lively" question of genuine interest for the community of study, what we call a *generating question* referred to as Q_0. The question has to be taken seriously, not as a mere opportunity to cover some fixed a priori mathematical content. Elaborating answers to Q_0 must become the main purpose of the study and an end in itself.

The study of Q_0 evolves and opens many other *derived questions* Q_k that appear as the starting point of new SRP or new branches of the initial one. One needs to constantly ask whether these derived questions are relevant in the sense of being capable of leading *temporary answers* A_k that can be helpful in elaborating a *final answer* $A^{\blacktriangledown}$ for Q_0. As a result, the study of Q_0 and its derived questions Q_k leads to successive temporary answers A_k tracing out the *possible routes* to be followed in the effective experimentation of the SRP. The work of producing $A^{\blacktriangledown}$ can thus be described as a *tree of questions* Q_k *and temporary answers* A_k related to each other through a modeling process.

The implementation of the SRP usually requires resorting to external preestablished answers $A_i^{\lozenge}$ to the derived questions Q_k, as well as some other objects O_j used to test the available answers, elaborate new ones, and formulate new questions. The preestablished answers $A_i^\lozenge$ are accessible through different means of communication and diffusion called the *media* (in the sense of "mass media"). However, knowledge provided by the *media* corresponds to constructions that have usually been elaborated to answer other questions than those specifically approached. Thus, it has to be "deconstructed" and "reconstructed" according to the new needs. This is the main role of the *experimental milieu*, *M*, containing empirical objects O_j as well as other old, well-established answers $A_i^\lozenge$. *Milieu M* evolves throughout the study process and becomes one of the main guarantees of a successful outcome. It is usual that, during the SRP, emerges the need to make a given $A^\lozenge$ available to *X* because it is required or seems necessary to produce $A^\blacktriangledown$, The specific branch of the SRP starting in this case is called a *study and research activity* (SRA) focused on $A^\lozenge$. In this sense, SRP together with SRA provides a general modeling tool to describe any kind of teaching and learning process, from those based on the direct transmission of knowledge to those centered on inquiry activities.

This broadened conception of didactic processes can be used to describe almost any form of teaching and learning strategy, and it prevents researchers from assuming any kind of specific form of school organization as normal or natural. Furthermore, it encourages taking into account a broad set of conditions and constraints affecting the teaching and learning processes that far exceed the limits of the classroom. At the same time, the border between mathematics and didactics (in the sense of teaching and learning) is blurred: doing mathematics includes study, research, and supervision; learning mathematics includes collectively carrying out an activity of study and research; and teaching mathematics corresponds to leading or supervising a research and study activity.

It is in this context that DE experimentations are carried out in the setting of ATD, in very different conditions than those established by the COREM, although they maintain the main methodological gestures exposed at the beginning of the chapter.

8.3.2 An Example: Teaching Modeling at University Level

The second case of DE we are presenting approaches the problem of teaching mathematical modeling at university level. This case illustrates some of the tools used in the framework of the ATD going through the four phases of a DE methodology process (see Fig. 8.5). More details about this particular case can be found in (Barquero, 2009; Barquero, Bosch, & Gascón, 2008). Some other research about the design and integration of SRP at different school levels, and even in teachers' professional development, has been established in the same framework and following similar methodologies (García, Gascón, Ruiz-Higueras, & Bosch, 2006; Hansen & Winslow, 2010; Rodríguez, Bosch, & Gascón, 2008; Ruiz-Munzón, Matheron, Bosch, & Gascón, 2012; Winsløw, Matheron, & Mercier, 2013).

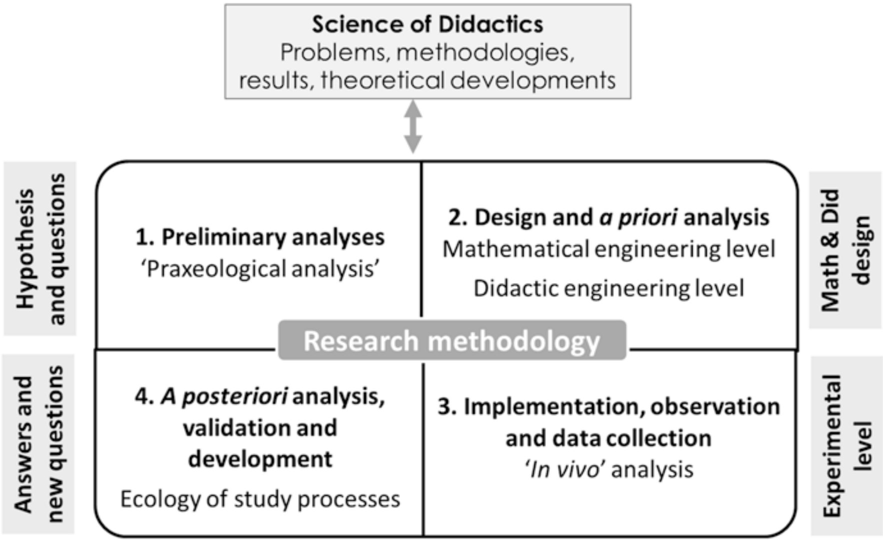

Fig. 8.5 Phases of the DE research methodology within the ATD

8.3.2.1 Preliminary Analysis

The starting point of the research here considered is the integration of mathematical modeling in first-year university courses of Mathematics for Natural Sciences. When analyzing what kind of mathematics is taught at this level, one could think that natural sciences university degrees would offer favorable institutional conditions to teach mathematics as a modeling tool, as mathematical models are becoming more and more essential to the understanding, use, and development of scientific disciplines. However, this seems far away from reality: despite the fact that mathematical models appear in the syllabi of almost all the courses, teaching mathematical models often comes at the end of the process, if there is time left for it. The dominant ideology is that modeling represents a mere application of some preestablished knowledge, leaving little room for the process of proposing, constructing, validating, and questioning mathematical models. We define as *applicationism* this spontaneous epistemology, which appears to be dominant in many university institutions (Winsløw et al., 2013).

According to the ATD and to the epistemological principles considered, if we start from the principle that intra-mathematical modeling is part of mathematical modeling, then many mathematical activities can be reformulated as modeling activities. It is considered that, in a modeling process, both the initial system considered and the models used have a praxeological structure. Mathematical modeling activity then appears as a process of (re)construction and articulation of mathematical praxeologies which become progressively broader and more complex, the main aim of which is to provide answers to problematic questions.

Thus, mathematical modeling cannot only be considered as an aspect or modality of mathematical activity but has to be placed at the core of it. This integration constitutes an essential aspect of our *research problem*, which opens the issue of the design of teaching proposals where mathematical modeling adopts an explicit and crucial role, emerging from initial problematic questions and able to link mathematical content that now appears as tools or models to provide answers to questions. Our working hypothesis is to suggest an SRP as one of the appropriate teaching proposals to move toward the (new) paradigm of *questioning the world* proclaimed by Chevallard, and which explicitly situates mathematical modeling problems at the heart of teaching and learning processes.

Several investigations from different theoretical perspectives have shown that mathematical modeling activities can exist at school under appropriate conditions, at all levels and in almost all curricular content. However, besides the good progress and encouraging results in research for the integration of modeling, many researchers have pointed out the existence of strong limitations hindering the large-scale dissemination of mathematical modeling practices in the classroom. For instance, Burkhardt makes the following harsh statement:

> [W]e know how to teach modelling, have shown how to develop the support necessary to enable typical teachers to handle it, and it is happening in many classrooms around the world. The bad news? 'Many' is compared with one; the proportion of classrooms where modelling happens is close to zero. (2008, p. 2091)

The research problem that has to be addressed is thus the study of the conditions that make the teaching of mathematical modeling possible at school, as well as the constraints that hinder its development as a normalized activity. Of course this problem depends on how mathematical modeling is conceived by both the research community (epistemological model) and the institutions where it is to be disseminated (the usual school and scholar epistemology, which takes here the form of applications). In ATD, it is referred to as the problem of the *ecology* of mathematical modeling in current educational environments. It can be specified with central questions about: what kinds of limitations and constraints exist in our current educational systems that prevent mathematical modeling from being widely incorporated in daily classroom activities? What kind of conditions could help a large-scale integration of mathematical modeling at school?

According to our previous analysis, the problem of the ecology of mathematical modeling becomes the problem of the ecology of SRPs and of their capacity to ensure the development of modeling activities. In the following section, we outline our partial answer to this enormous problem, focusing on the mathematical and didactic design of a particular SRP at university level with respect to the question of how to predict population dynamics.

8.3.2.2 A Priori Analysis: Mathematical and Didactic Design

Our generating question Q_0 that leads to the a priori mathematical and didactic design of the SRP, given the size of a population over previous periods of time, focuses on the following questions: *How can we predict the long-term behavior of its size?*

What sort of assumptions about the population, its growth, and its surroundings should be made? How can one create forecasts and test them? In all its implementations, Q_0 was introduced using different populations: first a pheasant population, then a fish population, and finally, a yeast population that was cultivated either in independent containers or mixed.

To provide answers to Q_0 and to the sequence of the derived questions that followed it, the construction of different mathematical models was required. Depending on whether time was considered as a *discrete* or *continuous* magnitude and if population generations were considered *independent* (x_t only depends on x_{t-1}) or *mixed* (x_t depends on $d > 1$ past generations $x_{t-1}, x_{t-2}, \ldots, x_{t-d}$), a four-branch structure of the SRP can be delimited, giving rise to its a priori mathematical design (Fig. 8.6).

Looking into the derived questions opens a sequence of modeling activities that cover most of the content of a first-year course of mathematics for natural science students at university level: sequences and its convergence, one-variable calculus, linear algebra, and ordinary differential equations and their systems. This first mathematical design step is followed by the *didactic* a priori *design* of the SRP. It has inherited the structure defined in the mathematical a priori design and now includes questions about the *mesogenesis* (evolution of the experimental *milieus*), *chronogenesis* (evolution of the new questions and the knowledge introduced through the media), and *topogenesis* (sharing of responsibilities between teacher and students).

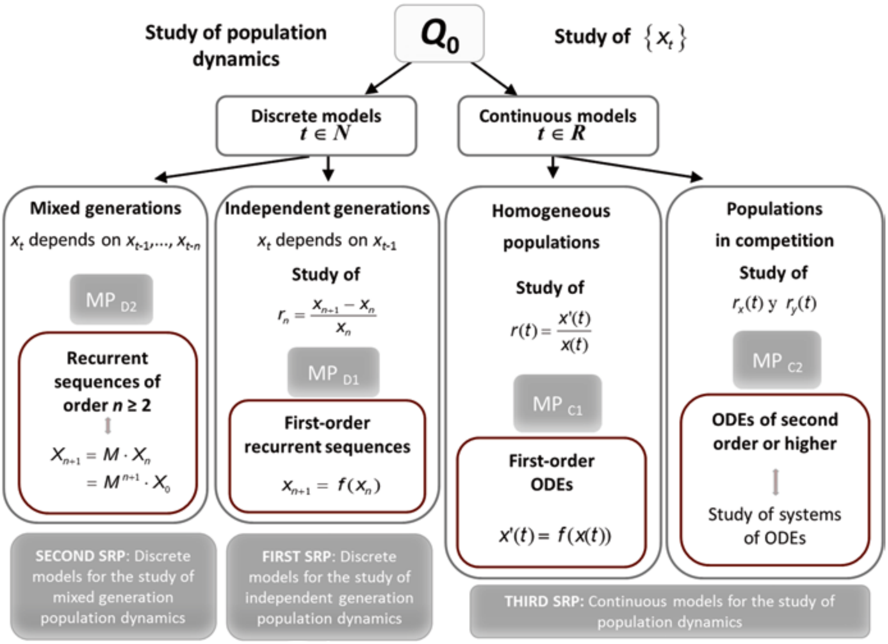

Fig. 8.6 General structure of the SRP branches derived from the study of Q_0 (Barquero, 2009)

Many important decisions are taken at this point to support the necessary change of students' and teachers' common strategies rooted in a dominant university teaching culture. For instance, students were constantly asked to assume new responsibilities so as to formulate new questions and approach them, to provide their own temporary answers to the successive derived questions, to plan the collective work, etc. In turn, the teacher has a new role to play as the *supervisor of the inquiry*, avoiding the temptation of imposing possible answers, inviting the groups of students to defend the successive answers they provide, to help decide on the questions to pursue, etc. Moreover, students should be able to introduce any external work or piece of knowledge they find appropriate in the *milieu*. The whole class will have the task to create the appropriate *milieu* for an internal validation of all those preestablished answers. All those new conditions in the implementation of the SRP required new teaching strategies and new devices: enabling the students to plan the work, elaborate new answers, compare data and models, write reports with temporary answers, validate final answers, defend them, etc.

8.3.2.3 Implementation and In Vivo Analysis

We tested the use of the SRP for five academic years (from 2005/06 to 2009/10) with first-year students of technical engineering degrees at the Autonomous University of Barcelona (Spain), who were attending a 1-year *Mathematical Foundations of Engineering* course. A special educational activity, called the "mathematical modeling workshop," was introduced in the general organization of the course; it was optional for students. The workshop ran in parallel with the lecture and problem practice sessions scheduled in the usual course. In the successive implementation of the SRP, a 2-h weekly session of the workshop took place as follows: students worked in teams of 2 or 3 members and had to develop their own study and propose their own "temporary" answers to the intermediate questions of Q_0.

Throughout the modeling workshop and its successive implementation year after year, the necessity arose to introduce several teaching and learning devices that were nonexistent in our usual university teaching settings. They had to evolve throughout the course and become accepted by the students. On the one hand, at the beginning of each workshop session, the teams were asked to deliver a report summarizing the work carried out in the previous sessions with respect to assumptions considered, main problematic questions dealt with, mathematical models used, "temporary" answers obtained, and new questions opened. In each session, one team was in charge of explaining and defending their report. A discussion followed to state the main progress and to agree on how to continue the study process. Moreover, there was the "secretary of the week", the person in charge of summarizing the work done and the main points of debate during the session. The secretary of the week and the team of the week played a crucial role in the workshop and all their reports were included in the diary of the workshop. At the end of the SRP, each individual student had to write a final report of the entire study: evolution of the main questions studied, work in and with different mathematical models, relationship between them, and so on. On the other hand, students were asked to search for any external information about the mathematical models they were building, and the answers they were providing in the media. Their findings were also explained in the workshop sessions and they discussed how these external mathematical objects could be useful (or not) and how they could be validated and used in relation to the questions they were dealing with.

Thanks to the several variations in the successive implementations of the SRP and to its *in vivo* analysis, several aspects could be improved every year. The a priori mathematical and didactic design of the SRP was gradually enriched, that is, after each implementation we had a more detailed description of the derived questions and temporary answers that were likely to appear in each of the SRP branches. Moreover, we obtained more details and got more control of the use and functionality of the different learning and teaching devices that were included throughout the workshop, especially those aspects related to the new conditions of mesogenesis, chronogenesis, and topogenesis:

- What responsibilities did students find more difficult to assume?
- What teaching strategies can help achieve the transfer of passing on more responsibilities to students?
- Do the weekly reports help students to formulate their own assumptions and to pose new questions?
- Does the debate generated at the beginning of each session help students to organize their own work?
- Does the workshop diary encourage the students and the teacher to have a broader perspective of the whole modeling process?

8.3.2.4 A Posteriori Analysis and Ecology

When considering the SRP as a whole, we verified from its first implementation that the sequence of derived questions arising from the generating question Q_0 led the students and the teacher to consider most of the main content of the entire mathematics course (sequences and their convergence, one-variable functions, derivative, ordinary differential equations, matrices, etc.). However, during the workshop, this content appeared in a structure that was completely different from the usual organization proposed in the main course. Instead of the classical "logic of mathematical concepts", the workshop was more guided by the "looking for answers to problematic questions" and "types of models" that progressively appeared. During its five courses of implementation, because the instructors were the same year after year, the first author of the paper and a lecturer who is an expert in applied mathematics, we found it easier to make SRP compatible with the standard formats of teaching (lectures and problem practice sessions). In the end, all these traditional devices were subordinated to the study of questions opened during the workshop. For instance, when questions appeared that needed some theoretical developments, such as "what was the relation between the relative rate of growth and the derivative or how to calculate the n-power of a matrix", we suspended the workshop sessions and spent several lectures and practice sessions on a *study and research activity* centered on the diagonalization of matrices before carrying on with the workshop.

However, it was not easy to preserve and transfer all these good conditions to the new teacher who came to replace us. Although he had all the material and descriptions from the previous SRP and all our assistance, the year after we left the course, the new implementation of the SRP only took 2 weeks. When we asked the teacher why it had taken such a short time, he told us that he only needed three sessions to show the students how to solve the questions and explain all the mathematical models they had to apply… In the end, the traditional ways, focused only on direct transmission and application, seem to have prevailed (Barquero, Bosch, & Gascón, 2013).

Other important constraints that could be identified were mainly related to the difficulties for keeping in mind the generating question of the SRP, given the fact that students were not used to pursuing a question for such a long period of time. SRP requires a strong modification of the usual didactic contract that currently exists at universities, where the teacher provides long lists of different small problems which the students have to solve. On the contrary, some other responsibilities that are usually assigned exclusively to the teacher were easily assumed by the students: searching information about models, discussing different ways of looking for an answer, comparing experimental data and reality, writing and defending reports with partial or final answers, etc. Others, however, were more difficult to share: choosing the relevant mathematical tools, criticizing the scope of the models constructed, posing new questions to continue with the study, planning the work to do, etc.

Last but not least, another strong constraint appeared in all the SRP implementations: the necessity of an *ad hoc mathematical discourse* available to describe the process that had just taken place. The work carried out in the workshop led to a need for new words, concepts, and discourses to talk about what was going on and to formalize it theoretically. The teacher and the students could no longer base their work on previously selected material, such as the one provided by textbooks or by previous lectures. In each case, they had to elaborate their own narrative of the process followed, a collective and original *mathematical text* indispensable to describe the dynamics of the work done and to provide material for the writing of the final answer $A^{\heartsuit}$. This lack of mathematical discourses to express, describe, and formalize the dynamics of mathematical activity brings to light the necessity to develop new mathematical and didactic infrastructures to support self-sufficient modeling activities.

Following Hans Freudenthal's observation in the case of the mathematical work with quantities, we came across other "no man's lands" which appear to be crucial for mathematical modeling to live in our school institutions. The problem is not only "the fault of the mathematician"; it seems to affect the entire educational culture and the conceivable ways of making it evolve.

8.4 Open Questions

As was said in the introduction, this chapter focuses on the notion of DE as it was introduced in the TDS to empirically organize the study of didactic phenomena and new teaching proposals, and its later developments in the ATD with the problem of the ecology of teaching and learning processes. We have left aside other conceptions of DE which are more or less related to them (Margolinas et al., 2011), their contrast with other task-design works, and more general reflections about the role of design and theories in mathematics education (Burkhardt & Schoenfeld, 2003; Design-Based Research Collaborative, 2003; Godino et al., 2013).

In order to encourage the debate and nourish future comparative studies on this issue, we conclude by briefly addressing three main issues that, in our opinion, cannot be left aside in the research work of contrasting and trying to articulate different approaches. First of all, we have seen that, in the research program established by the TDS and developed by the ATD, the transition to the paradigm of questioning the world becomes crucial: mathematical content, as well as any other subject matter, needs to appear as true answers to real questions rather than mere *monuments to visit* (Chevallard, 2012). The necessity to move away from *monumentalism* is not new, but it has not always been considered in the same manner, especially when researchers' epistemological assumptions require a certain distance from assumptions which prevail in teaching and research institutions.

The TDS and the ATD locate the problem of the ecology of design realizations at the heart of DE research, didactics appearing as the scientific study of the conditions for mathematical knowledge (praxeologies) to disseminate in human institutions. Furthermore, the ATD proposes a considerable enlargement of the unit of analysis for research corresponding to the different levels of the scale of didactic codetermination (Chevallard, 2002). In the case of mathematical modeling, some approaches (Burkhardt, 2008; Kaiser & Maaβ, 2007; Lesh & Doerr, 2003; among others) have also highlighted the problem of the large-scale dissemination of new teaching proposals. However, this issue is still far from becoming central in the main stream of research in mathematics education. We need more insight about how other approaches have experienced and proposed to deal with this ecological problem.

It seems clear that the ecological problem needs to engage different partnerships of the educational community: researchers, designers, policy makers, teacher associations, mathematicians, editors, etc. In some approaches, the role of research is clearly distinguished from the teachers' role, even if they are in broad agreement on their tight cooperation. The problem of the roles assigned in mathematics education to the different partners of the education process appears as an unavoidable issue related to the problem of the ecology, especially at a moment when all efforts should be put together, while responsibilities are of course typically different among the partners and institutions involved.

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