Chapter 7
E-Textbooks for Mathematical Guided Inquiry: Design of Tasks and Task Sequences

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7.1 Textbook Culture: Traditions and Challenges

A textbook is a message from the professional community to students about *what* they should learn. It also represents the ideas of the author about *how* the content should be taught and learned. It plays a central role in school pedagogy and classroom norms, and its authoritative image has been the dominant aspect of the common classroom culture, often identified as *textbook culture*. Love and Pimm's (1996) "text on texts" is an important resource for understanding the features that textbooks are usually assumed to have: textbooks are closed in the sense that both text and images have been created in the past; they include problems for exercising but not aimed at questioning the content; they are linear and follow the "linear textual flow of reading" (p. 381), and usually they consist of cycles of expositions, examples, and exercises.

Traditional textbook culture assumes that teachers make sure their students learn the content of the book, usually in the specified order, because the book acts as a model for standards and for the way standards are assessed. The textbook is supposed to provide guidance and present opportunities for students to learn, making the objectives and ideas of the curriculum more readily apparent. For teachers, it also provides guidance in bringing their teaching in line with the expectations of the external authority, which may be the school, the syllabus, or some central assessment. In this function, the textbook serves as syllabus and timekeeper, and its author is considered to be the authorized entity charged with delivering content and pedagogy. Note the direct etymological link between *author* and *authority* (further described by Young, 2007), underscoring the authoritarian position of the textbook as written by a recognized expert author or group of authors. As an *author(iz)ed*

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object, the textbook is considered to be a solid personal resource for the learner. Learning with a textbook in this manner is often referred to as *learning by the book*, that is, a passive type of learning. *Teaching by the book* refers to teaching that treats the textbook as an authority that should be fully accepted. Both are known components of textbook culture, which varies only slightly across different contexts worldwide. Herbel-Eisenmann's analysis of the sources of textbook authority (2009) distinguishes between the textbook as an objective representation of knowledge, where the authority is an intrinsic property of the text, and the participatory relations between textbook and teaching. Teachers use various practices to confer authority onto the text and simultaneously onto themselves.

Developments across nations attempt to change the mathematics teaching standards and to shift the teacher's authority toward *guided inquiry* teaching. Although mathematics classrooms have increasingly adopted various formats of constructivism, reviews of post-reform math textbooks do not find that the practice of using textbooks has changed. Post-reform studies that examined the teachers' adoption of new curricula found that, although teachers attempt to reform their pedagogy to emphasize guided inquiry, the student-teacher-book relations have not changed and the textbook remains a central formal authoritative resource for the teacher and the learner. Learning mathematics with understanding has become a shared objective, but teachers seldom have opportunity and time to develop mathematical tasks, textbooks, or teaching sequences in which richness of tasks and learning for understanding are emphasized (Pepin, 2012). Analyzing newly developed textbooks, Nathan, Long, and Alitalia (2007) found that many new books have remained similar to the traditional ones.

Reyes, Reyes, Tarr, and Chavez (2006) studied for 3 years how newly developed textbooks supported by the National Science Foundation in the USA affected math teaching and learning in a US middle school. According to the authors, half the teachers declared that "My math book is my bible," and the other half were influenced more by the state-determined curriculum and assessment materials. The teachers covered 60–70 % of lessons of the specially designed-to-reform textbook, just as they did when using non-reform textbooks. McNaught (2009) presented similar findings in a study of the Core-Plus curriculum project (used as an example of an integrated content textbook), showing that around 60 % of the content of the textbook was taught, but not necessarily from the textbook, and about one third of the teaching was based on other supplementary materials.

Studying Swedish teachers as they guided students solving tasks from the textbook, Johansson (2007) showed that perception of the textbook as a resource of ultimate correctness has affected the teachers' decision not to question a textbook solution. In sum, (a) the textbook remains the single printed and bound object that acts as authoritative pedagogic guideline for what should be learned and for how it should be taught and assessed; (b) new textbooks that attempt to adopt a somewhat less authoritative tone create a didactic challenge and are not likely to be fully adopted by experienced teachers; and generally speaking, (c) although the textbook is assumed to provide devices for actively involving the students in examples and exercises, the studies focus on engagements with textbooks that are reserved mostly for the teachers.

The studies on interactions with textbooks present a somewhat more complex picture of the relations between mathematics, the text, teachers, and students than traditionally assumed. Teaching does not depend on a single textbook: approximately 30 % is accomplished using other resources, mainly previous textbooks that the teachers feel are more likely to achieve their goals. Moreover, different nations have different cultures, traditions, and expectations from teaching, all reflected in the textbook. The comparison between Norwegian and French textbooks by Pepin, Gueudet, and Trouche (2013) is illustrative. Despite the authority of textbooks, an increasing number of studies report on teachers looking for new models and views on teaching aimed at understanding the use of textbooks. Remillard (2012) examined how teachers are positioning themselves with the textbook and found that standard-based resources present a challenge to many teachers and require orientation, and that many teachers use them in ways not intended by the designers. Rezat (2012) analyzed six phases describing how teachers mediate textbook use. Patterns that Rezat identified in students' uses of textbooks are strongly related to the choice of order of expositions, order of pages, and of tasks used by the teacher. It suggests that the power of indirect teacher mediation of the textbook is underestimated.

Technology has been an important resource in several textbooks attempting reform. But such technological resources as the Open Education Resources, which accompany the newer books, are reported to be considered enrichment, while the textbook remains the core external authority. Textbook publishers are addressing a wide range of expected changes in the affordances of the digital object, including the material aspects of weight and cost, the quality and attractiveness of the material, the richness of the modes of presentation, and the opportunities for personalization. As textbooks rapidly change from print to digital formats, it is assumed that the ways in which they will be used will also change. Publishers allow teachers to personalize digital textbooks for their courses, emphasizing flexibility and inexpensive dynamic changes. Schoolteachers can personalize the textbook by selecting from existing chapters and content and even individualizing the book for the student.

Korea, considered to be one of the leading countries in math and science achievement, became one of the foremost innovators in the area of e-textbooks, especially in school math and sciences. Korea holds an integrative view in which textbooks remain the central learning resource, surrounded by other types of facilitating media. Paper textbooks are being digitized, integrating live resources such as hyperlinks, multimedia, dictionaries, references, and other data sources into a learning management system connecting students, assessment, etc. Other educational systems are adopting a similar view of the new textbook (Taizan, Bhang, Kurokami, & Kwon, 2012). The Israeli education system requires that each textbook appear in at least one of three formats: a digitized textbook, a digitized textbook that is enriched with external links and multimodal materials, or a textbook that is especially designed to work in a digital environment and includes online tools for authoring, learning, and management. An assumption of the integrative view is that the digital textbook can also function as printed text and assume the traditional format of the textbook. Thus, the change of the *object* prompted changes in the publishing process (editions, price, attractiveness) and in the ways of integrating digital add-ons with the instructional materials. The different formats of multimodal interactive textbooks in mathematics have yet to be thoroughly studied.

The previous literature review addresses the studies of printed textbooks, but the findings regarding mediation and patterns of use by students and teachers point to challenges that should be considered when using non-sequential and multimodal interactive texts. An important issue is that of authority; Gueudet and Trouche (2012) argued that the notion of author and authorship is often less transparent in online sources than in printed material, as does Usiskin (2013) when discussing the disappearance of transparency in e-Textbooks. Another is that of order or sequence. It is commonly believed that digital books enable changes of sequence and flexibility, but at the same time such flexibility may cause lack of clarity. In this situation, the main challenge is to rethink the sets of concepts and images used to guide us in thinking about the structure of traditional printed textbooks and to consider the consequences of interactivity, multimodality, and personalization on the design and structure of use—primarily the teacher.

7.2 The Design of an Interactive Unit

Yerushalmy, Katriel, and Shternberg (2002) designed an interactive digital textbook characterized by the extensive roles assigned to visual semiotic means to interactivity between the reader and the visual mathematical objects and processes, and by a conceptual order of digital pages that could be rearranged to serve a variety of instructional paths (for more details, see Yerushalmy, 2013). The design of the *VisualMath* textbook is situated in the larger view field of mathematical guided inquiry and school algebra, taking the function (on real number field) to be its mathematical and pedagogical root. We have long been seeking less formal control structures that attempt to respond to or to signal subjective schemes and views that teachers and students bring to their engagement with the text. Two fundamental principles directed the design:

- A unit, a pedagogical structure for a collection of algebra tasks, should be regarded as a gallery in an interactive museum. The design principles and views of the units were borrowed from a museum setting and were consistent with the distinction Kress and van Leeuwen (1996) made to describe linear and nonlinear texts. They described nonlinear texts as an "exhibition in a large room which visitors can traverse any way they like… It will not be random that a particular major sculpture is placed in the center of the room, or that a particular major painting has been hung on the wall opposite the entrance, to be noticed first by all visitors entering the room" (p. 223). Just as a curated exhibit, a unit should present opportunities for readers to focus on a concept through multi-sensual experience, making the objectives of the collection apparent. Each unit attempts to be a coherent collection of tasks, which, although they can be used in any flexible order according to the decision of the teacher or reader, is constructed to deliver the mathematical lesson by (a) offering a balanced collection of multimodal mathematical activities that include inquiry tasks, problems, and exercises (in general, all the tasks require non-routine thinking and reasoning by interacting and experiencing the mathematics) and (b) addressing various modes of learning and teaching, as described by Chazan and Schnepp (2002) and Lobato, Clarke, and Ellis (2005). Although the tasks are written for the student, it is assumed that each task can be used in different modes, including independent study by students, group collaboration, and teacher-conducted whole-class discussion demonstrating a process of inquiry that supports systematizing and institutionalizing experimental results: (c) a design that meets the institutional demands, including tasks that support transforming inductive results into consolidated mathematics (Barzel, Leuders, Predinger, & Hußmann, 2013 categorize textbook tasks along these terms). Each unit addresses three types of tasks (Kieran, 2004): *transformational activities* that are mostly rule-based manipulations, *generational activities* focused on representing and interpreting situations from outside or within the mathematics, and *integrative activities* that require the use of manipulations and generational actions but also go beyond these, to meta-processes such as generalizations, predictions, etc.

- Although the organization of each unit resembles a traditional set of textbook tasks—expositions, exercises, and problems—the principle that guided the design of tasks and characterization of activities is rooted in the interactivity of tools and diagrams. Each type of interactive element has its semiotic and pedagogical meaning. Tools are artifacts designed to carry a specific element of mathematics: function-based algebra. An interactive diagram (ID) is a relatively small and simple software application (applet) built around a *pre-constructed example* that serves as a basis for change. Both tools and diagrams were designed to support conjecturing and argumentation by providing various degrees of control to the user. Interactive elements are designed to support the systematic generation of examples in linked multiple representations, to accommodate various entry points, and to provide nonjudgmental mirror feedback that should be interpreted subjectively. Indeed, the challenge in constructing a task around an ID is to design opportunities for action (Yerushalmy, 2005).

This section outlines and exemplifies the design rationale and structure of a unit by analyzing the design of expositions, tools, problems, exercises, and essay tasks included in the *Transformations of Linear Functions* unit (http://visualmath.haifa.ac.il/en/linear_functions/transformation_of_graphs) (in Yerushalmy et al., 2002; Yerushalmy, Shternberg, & Katriel, 2014).

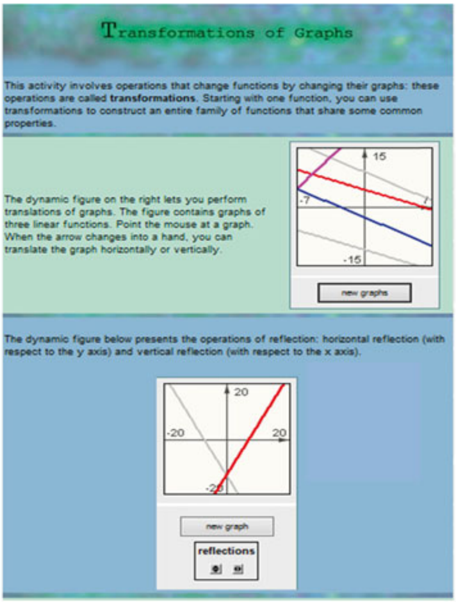
7.2.1 Interactive Exposition

Freisen's (2013) analysis of the structures that characterize the advent of modern textbooks describes the expository nature of textbooks as the main obstacle to considering newer modes of textbooks along the history as an environment supporting the personal construction of knowledge. Following Kuhn (1962), Freisen questioned the role of the exposition in traditional textbooks and the attempt to present the universal truths and to expect students to later recite them back to the teacher. In more recent textbooks, the terminology of the *scripted* exposition has changed to knowledge that is *prompted* by the textbook. Our assumption was that learners need to construct their own examples and to construct the concept image and definition by creating their own example space. The widespread use of examples in mathematics textbooks serves as a means of communication and mediation between learners and ideas. If examples are well selected, the variations between examples are the means by which students can distinguish between essential and redundant features (Bills et al., 2006; Goldenberg & Mason, 2008; Watson & Shipman, 2008). Our challenge was to design expositions that can be worked on and personalized by providing interactive illustrations to be controlled by the reader. Its design was centered on the *illustrating diagram* (see Fig. 7.1).

Exposition: Illustrating Diagrams

Translations and Reflections of Linear Functions

The figure shows how different students can attempt different examples with the exposition.



The role of the *illustrating interactive diagram* that opens the Transformations unit is to capture the central properties and quality of the translation of a linear function on a Cartesian system. It is designed to support the development of an initial working definition to be reconstructed while working on the unit. It is based on a known representation: a line in a Cartesian system and an assumed common, embodied understanding of dragging. The ID appearing in the exposition page are limited in types of representations (graphs only) and in tools used to manipulate the given examples (generation of examples and the translation and reflection tools). Thus, students cannot create counterexamples or turn an example into a non-example. The role of the interactive exposition that affords limited user control is to allow subjective development of awareness of the relevance of the unit features.

Fig. 7.1 Description of the role of illustrating interactive diagram (http://visualmath.haifa.ac.il/)

The exposition was designed to start the example generation, usually by offering a single graphic representation and relatively simple actions, such as viewing an animated example or dragging an interactive shape. The reader was expected to use these to create examples that represent new ideas with known means of control individually, in cooperative groups or in whole-class discussions.

7.2.2 Toolbox and Unit Tools

Tools are artifacts designed to carry specific elements of mathematics. Artifacts become purposeful tools in response to subjective needs and personal actions. The assumption in the design of the *VisualMath* e-textbook was that tools such as the microscope, the calculator, or any software become a way of thinking and knowing and have an epistemological role as they change the traditional assumptions of what we mean by knowing mathematics. A toolbox of ten tools for doing mathematics is part of the e-textbook. The unit tools (or activity tools) are special cases of the tools in the Toolbox, designed to explore specific concepts mainly by limiting the generated types of example spaces through supporting a smaller range of actions. Although they make exploration and inquiry possible, they are designed to call for action in a specific manner that supports the construction of the principal idea of the unit. Toolbox and unit tools are part of the learning environment, and they are always available (see Fig. 7.2).

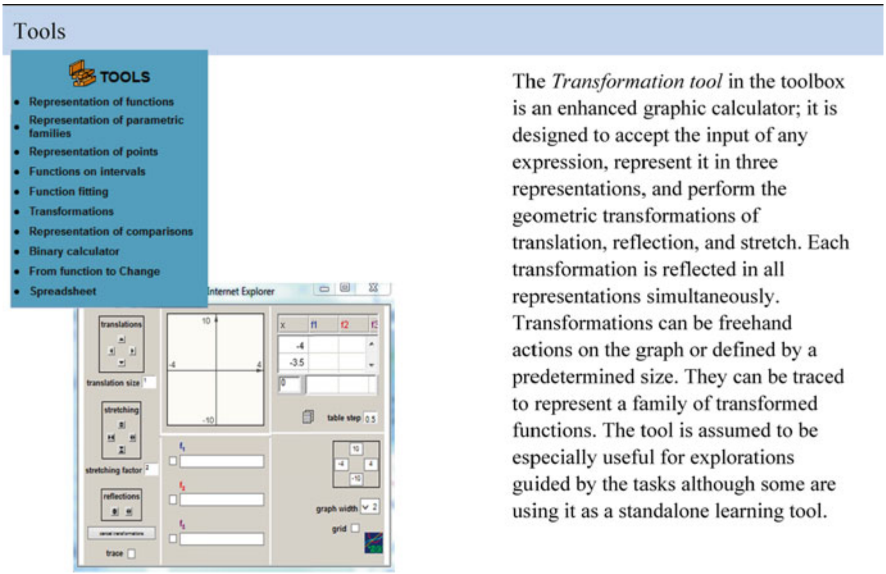
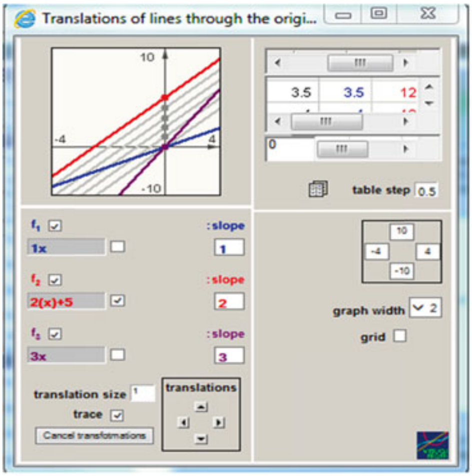


Fig. 7.2 Descriptions of toolbox and unit tools (http://visualmath.haifa.ac.il/)

Unit Tools



The *Translations of lines through the origin* is a transformation tool designed to focus on an object $f(x) = mx$ and an operation (translation). Symbolic input is not allowed. New examples can be generated only by changing the original given functions using translations, or by free input of the size of the slope. Thus, by limiting the example space, it demonstrates how terms in linear expressions reflect or correspond to parallel lines. The tool is on one hand general, as one can present any linear function and line; on the other hand it is limited to translations and starts with a specific generic example, creating boundaries for the story to be learned.

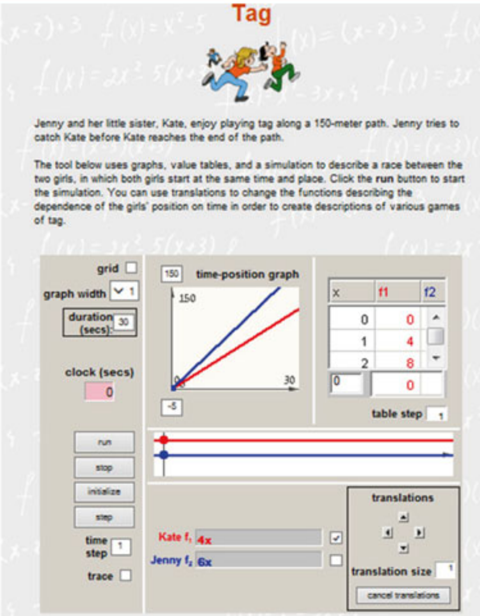
Fig. 7.2 (continued)

7.2.3 *Tasks or Problems and Exercises*

In the pedagogical endeavor for which the *VisualMath* book is designed, each activity was intended to grant opportunities for students to explore in order to formulate ideas. The traditional order of direct teaching of procedures followed by drilling, practice, and word problems (applications of the taught algorithms) was not a relevant consideration. The objectives of the tasks were to use explorations out of which conjectures grow, are discussed, are explained, and are informally and formally proved or rejected. The tasks require making sense of problems and, as the new core standards state, require the students to "persevere in solving them," spending longer on analyzing givens, constraints, relationships, and goals. Although problem solving can always be helped by use of appropriate tools, it should be carried out strategically, constructing viable arguments and critiquing the reasoning of others. The problems are designed to provide an opportunity to obtain peculiar, conflicting, or unexpected examples. The primary means of designing problems are interactive diagrams (IDs). Especially useful are the *guiding diagrams* (GIDs), designed to be the principal delivery channel of the message of the activity. Similar to the narrator's voice in Goldenberg (1999), GIDs are designed to call for action in a specific manner that supports the construction of the principal ideas of the task.

As in many algebra books, exercises call for writing an expression, equation, or inequality. In function-based algebra, the expressions are functions and thus represent a procedure. Rather than asking to simplify or to solve, most exercises ask for equivalent expressions and encourage more than a single correct answer, or they require the construction of a function or of an equation that meets the specified conditions. The design assumption regarding exercises was that they are performed after students have adopted a correct (if not complete) concept image and concept definition and have practiced with understanding the principal processes related to the concept. The primary means used to design exercises are *elaborating diagrams*, which include a wide range of representations and controls within the representations. Elaborating diagrams leave it up to the learner to solve the task by offering a variety of general-purpose tools. The exercise can be solved without the support of the interactive diagram, using known procedures mentally as a paper exercise. Alternatively, the answer can be reached by trials with multiple representation reflecting feedback. Students can use openly available input, watch the feedback, compare the linked results, and gradually improve their guesses. See examples in Figs. 7.3 and 7.4.

Tasks: Guiding Diagrams

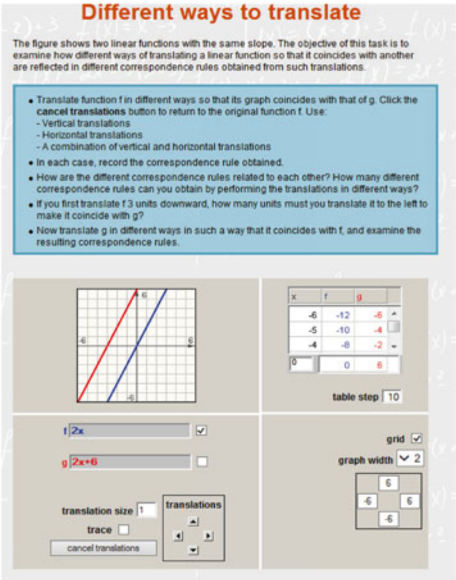


Tag recasts mathematical stories as tag games. It asks students to analyze the given games by describing the motion of pairs and to invent other games of tag.

- Kate started 30 meters ahead of Jenny.
- Kate started 60 seconds ahead of Jenny.
- Jenny's run starts 60 meters before the marked beginning of the path.
- Jenny waited 15 seconds after Kate started to run before starting herself.

Although the game is navigated by the two expressions (and not by direct dragging, the animated motion or the table of values), there is no way to change the expressions by typing in expressions and numerals. The only available tools are translations.

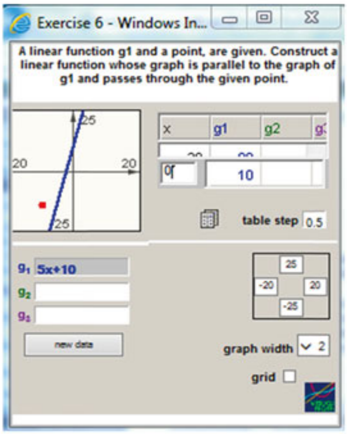
Fig. 7.3 Guiding diagrams and elaborating diagrams (http://visualmath.haifa.ac.il/)



Different ways to translate problem focuses on the expression as a mathematical story. For any two parallel lines, there are many ways of translating one to coincide with the other:

For example, translating $2x + 6$ by 3 to the right is expressed symbolically as $2(x - 3) + 6$, which is equivalent to $2x$.
Translations are geometric manipulations; input or editing of expressions is not allowed. This restriction, which is a matter of principle in the design of guiding diagrams, helps focus the exploration on the connections between the geometric changes of the object and its symbolic algebraic description.

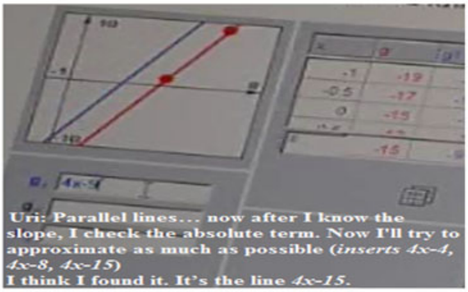
Exercises: Elaborating Diagrams



The exercise (or the infinite number of exercises generated by the restricted randomness of the point and the line – in this case it is always a point that is not on any axis) is a traditional task that requires an answer. It is designed as a graphic calculator tool. Input is free, trials can be attempted, and feedback is given by the linked graph and values. Specific tools, such as the transformation tools, which could have been helpful here, are missing on purpose. It is assumed that students can solve the exercise on paper without the tool, and use the interactive diagram for checking, or that they are exploring the exercise as beginners and need, at least at first, to find their own way of solving the task with an interactive that does not limit such exploration. The library of exercises is presented in an order of increasing difficulty based on the location of the point.

Fig. 7.3 (continued)

<u>*The task*</u>: Write an expression describing the given line graph containing the two points marked on the diagram.



"Their attempts demonstrate diverse working strategies in using the free symbolic input and the three function representations in the diagram. One student (Uri) used the free symbolic input to write an expression and systematically change its parameter values. We interpret his work as building a generic example. The second student (Ilana) used the free input only after the interviewer pointed out this option. She started her solution by entering various non-explicit function expressions and other invalid syntax inputs that were coincidental combinations of variables and coordinates of intersection points. Ilana's process of trial and error reached a turning point when she incidentally plotted and interpreted the constant function graph. Once she was able to link between a constant term and a graph, she was able to generalize the meaning of the b parameter in $f(x) = ax + b$, but she misunderstood the meaning of the parameter a. Dani looked for numeric values on the graph and in the table, which he then used for his paper and pencil computations. When he started using the free symbol input, he realized that he had made a mistake in his work on paper. He went back to calculate the slope and the constant term on paper, and rewrote an expression with the ID to check it. Once the line of the input expression looked parallel with the original graph, Dani's focus changed from finding the Y-intercept to comparing the two lines and achieving a correct constant term by making his graph overlap the given line.

The possibility of entering various expressions to be viewed in three linked function representations in the elaborating ID offers a range of working strategies. At the initial stage of the work with expressions using the ID, all three students used the elaborating ID only to check data for the given example. After the initial stage, they proceeded along different paths: systematic search for meanings of parameters in the expression, experimentation with numerical changes in the expression by trial and error, and elaborative work on the expression with paper and pencil, with the ID being used only for checking."

Fig. 7.4 An example of students working on an exercise (adapted from Naftaliev & Yerushalmy, 2011)

7.2.4 Write an Essay Task

Returning to the museum image, the role of the main *integrative task* of writing an *essay* is analogous to choosing the piece of art to be positioned in the center of the gallery. It is placed in such a way as to be visible and considered by all, but it is not necessary to stop by this exhibit for a long time on a first visit. Most units offer a math concept essay. The essay should guide students' exploration or summarize and demonstrate ideas that students already explored with tools, talked about in groups, and tried in other tasks of the unit. The essay can also be regarded by the teacher and students as suggestions for directions to follow (see Fig. 7.5). The suggested directions are the gallery "centerpiece" or the main goals that the unit wants to bring to the fore using its tools, problems, and exercises. In this sense, the suggested outline for the exploration tasks is also a suggestion for the teacher or for any reader interested in what would constitute "knowing the concept" upon completing the unit.

Essay Integrative Task: Use available Tools, Unit Tools, and Interactive Tasks

How are the various transformations expressed in the different representations of a function?
- In the graph
- In the correspondence rule
- In the value table

How do the various transformations affect the properties of a function (such as increase or decrease, points of intersection with the axes, or slope)? Which of the properties change and which do not?

How can transformations help in describing phenomena? When a transformation is applied to a graph that describes a story, what is the meaning of the transformation in the terms of the story?

Fig. 7.5 Example of an essay integrative task (http://visualmath.haifa.ac.il/)

Writing an essay task can serve as a significant tool in supporting the organization of knowledge in a constructivist classroom because it asks to structure divergent results and consolidate the mathematics learned, to generalize results to other settings, and to be skillful. It creates opportunities for students to raise questions that cut across objects, operations, and the terms of the subject. In this sense, essay tasks are considered "expert problems." Expert work consists of solving problems when they arise, searching for the foundations of the task, learning necessary definitions, searching for examples, looking for similar problems, performing meta-level heuristics, and instrumenting artifacts to serve as mathematical and psychological tools. Expert problems in a textbook "are rich tasks, each presented in a form it might arise in mathematics, science or daily life. They require effective use of problem solving strategies, as well as concepts and skills. Performance on these tasks indicates how well a person will be able to do and use the mathematics beyond the mathematics classroom" (Burkhardt & Swan, 2013, p. 440). Standing as the central exhibit of the gallery, the essay can be visited and revisited in various scenarios of exploration: an evolving essay that the student is responsible to complete in the course of the learning period or an individual project that can be developed and expanded to connect with other units (e.g., the exact same essay task appears in the transformations unit of the quadratic function section of the textbook).

7.2.5 Restructuring a Unit Along the Space of Interactive Elements

Our initial attempt in constructing a balanced unit was to support readers, mainly teachers, by keeping the structure of the unit based on more or less traditional components. At the same time, the interactive tools and diagrams became the main consideration of the design and of the pedagogical implementation. The exercise room of the transformation gallery was designed to support practice with meaning, but the simple and open design of the exercises as elaborating IDs caused teachers to often use it as expositions in guided class discussions or as an exploration activity. The unit tools that initially were designed to support specific tasks were used as a learning environment, replacing expositions. And the essay tasks, which were initially designed to support an open-ended project for students or groups to pursue on their own, were assigned as reviews upon completion of the unit, and teachers often used it as a map to guide them in teaching the unit. These pedagogical decisions made it clear that specifying the functions of the gallery rooms along the lines of the structure of textbook units as traditionally described needs to be rethought. We suggest, therefore, changing our perspective and considering designing the interactive units from the vantage point of the semiotics of interactive elements.

We used this semiotic framework to examine the aspects of interactive tasks from the point of view of presentational, orientational, and organizational functions. The *organizational function* refers to the connection between all the components of the task: verbal text, representations, tools, examples, etc. To describe the process of design with IDs, it is useful to look at three types of organization: illustrating, elaborating, and guiding. These types were most apparent in the description of the components of the unit in this section. The example that initially appears in the ID determines the nature of its *presentational function*. Three types of examples are widely used in IDs: specific, random, and generic. Specific examples that present the exact data of the activity of which they are part provide a dynamic illustration of the text without altering the information. In random examples, a specific example is generated within given constraints, presenting different information at various times and for different users. To serve as a generic example, the diagram must be structured to be representative; it must present a situation that could be part of the given task, but its focus is not on the specific data of the activity. The tone in which the text addresses the learner is subject to design decisions having to do with the *orientational function*. The “sketchiness” or “rigorousness” of the diagrams is an important aspect of reader orientation. An example that appears in a diagram can have an accurate appearance and speak in a strict, distant tone, or it can include a more subtle description and adopt a non-authoritative tone. IDs can function both as sketches and as accurate diagrams.

The elements of this framework were valuable in explaining student learning with various interactive diagrams in different contexts of algebra tasks (Naftaliev & Yerushalmy, 2013). They were also valuable in guiding our design of new instructional resources, and it remains to be explored whether this framework is valuable and productive as a guide for instruction.

7.3 Sequencing in Non-sequential Textbooks

Brown (2009) compares teaching to a jazz player’s use of the notes and argues that rather than designing curriculum materials as one-size-fits-all documents, designers should support different modes of use according to their pedagogical design capacity (p. 31). I support this call and suggest that transparent design of the mathematical idea and in the mapping of the objects and of actions of the mathematical subject matter can be a helpful tool for teachers who design or modify curricular documents and resources. For a collection of semi-ordered materials and multimodal digital “pages,” which to a certain extent stand on their own, to be considered a textbook, the deep structure of the concepts and the interrelations between them must be simple and visible. Two principles guided the design of the *VisualMath* e-textbook. Our first decision was to organize the content along a single view of the algebra, focusing on the algebra of functions: *VisualMath* was designed to use functions as the foundation for mastering algebraic skills with understanding by all students. This is not the common structure of mathematics textbooks that usually represent a progression along various themes and views of the algebra (Rezat & Straser, 2013). The second decision was to organize the materials along a relatively small number of mathematical objects and operations that can mathematically and pedagogically support a variety of progressions and sequences. These two principal design considerations are described in detail in Yerushalmy (2013).

We therefore prepared two lists. One consisted of the mathematical objects involved. In its current form, the *VisualMath* e-textbook accommodates the linear and the quadratic “museum exhibits,” each one appearing as a row on the map: a mathematical object. An additional row represents “any” or generic examples of functions. The other list consisted of operations on the objects and with them. The six operations included do not form an exhaustive list. Rather, they are what Schwartz had called the “interesting middle” (Schwartz, 1995): operations that represent important mathematical concepts and are appropriate and useful to learn as part of function-based school algebra. The operations are represent (a function), modify (reforming the view or structure without changing the function), transform (using operations to transform a function into families of functions), analyze the function and its change, operate with two functions (synthesizing new functions out of two different or identical functions), and compare two functions. The two lists are distinct and were therefore placed in an orthogonal organization in a 2D matrix map, where each cell represents the opportunities for learning resulting from the corresponding operation and object. Each operation with an object can take place in symbolic, graphic, or numeric representations (Fig. 7.6).

Symbolic/ Graphic/ Numeric

Operation / Objects	Represent	Modify	Transform	Analyze (function change)	Operate (with 2)	Compare
Generic						
Linear						
Quadratic						

Fig. 7.6 An organizational map for the *VisualMath* e-textbook (http://visualmath.haifa.ac.il/)

The next example illustrates design intentions aimed at supporting sequencing decisions. Assuming that students are already familiar with the three representations of functions, three conceptual "guided tours" are suggested for teaching the quadratic unit: the *analyzing tour*, the *solving tour*, and the *algebraic structure* (modifying and operating) *tour*. Although the three are complementary, each of the sequences by itself may respond to necessary foundational knowledge of quadratics.

Teachers can plan the course of the *analyzing tour* based on the concept of constant or nonconstant rate of change. This can be achieved by emphasizing the *Analyze* column in the map, which is elaborated in three units of the book (marked in Fig. 7.7): Rate of Change unit in the Linear part, Quadratic Growth unit, and Motion at Changing Speed unit, where motion is modeled by changing speed and constant acceleration.

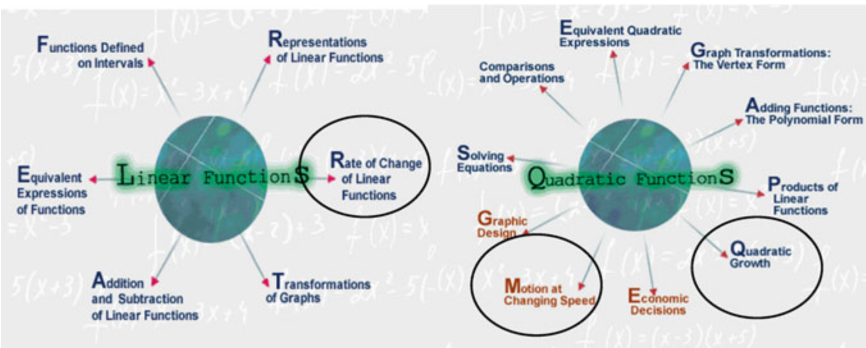


Fig. 7.7 The function analysis sequence based on three units: Rate of Change, Quadratic Growth, and Motion (http://visualmath.haifa.ac.il/)

An alternative focus places *equations and inequalities* in the center. Students have already learned to solve linear equations and probably view them as comparisons of linear functions. They have also experienced the idea of comparison of two functions representing equation or inequality and equivalent comparisons as the conceptual view of the process of solving equation or inequality in function-based algebra. Therefore, another challenging sequence aims at teaching along the two rightmost columns in Fig. 7.6: *Operate with 2* and *Compare* (comprising of 3 units: *Addition and Subtraction of Linear Functions*, *Comparisons and Operations*, and *Solving Equations* units). This sequence, marked in Fig. 7.8, is based on knowledge in the Modify column, but the focus is on the algebraic object of equation and on the concept of equivalence. Many tasks in various other units require solving and can be used to enrich the sequence.

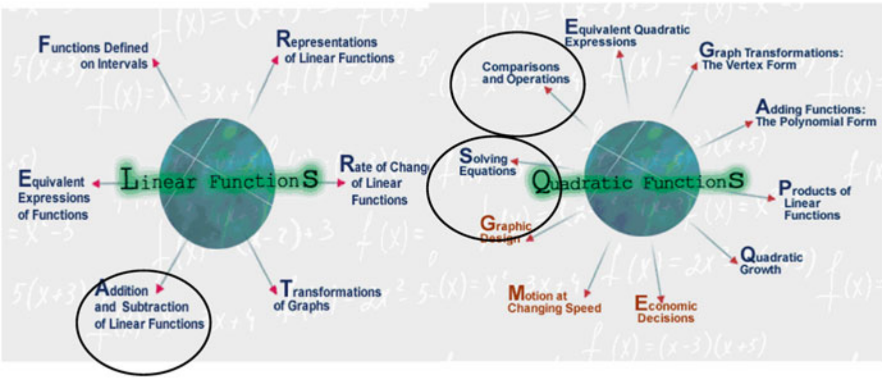


Fig. 7.8 The solving sequence based on three units: *Addition and Subtraction of Linear Functions*, *Comparisons and Operations*, and *Solving Equations* (http://visualmath.haifa.ac.il/)

A third choice of a focus for delving into the algebra of quadratic functions is the *algebraic structure* tour, based on the Operate with 2 and the Modify/Transform columns in Fig. 7.6. The scenario could begin with the investigation of *Products of Linear Functions* unit or with the *Graphic Design* modeling unit that uses the area model to explore the product of functions. The binary product of two linear expressions is one of three different structural forms (product, polynomial, and vertex) appearing in the *Equivalent Quadratic Expression* unit, and the manipulations required to arrive from one to the other are an important part of understanding quadratic expressions. Therefore, this sequence, which is based on five units that are marked in Fig. 7.9, emphasizes manipulations.

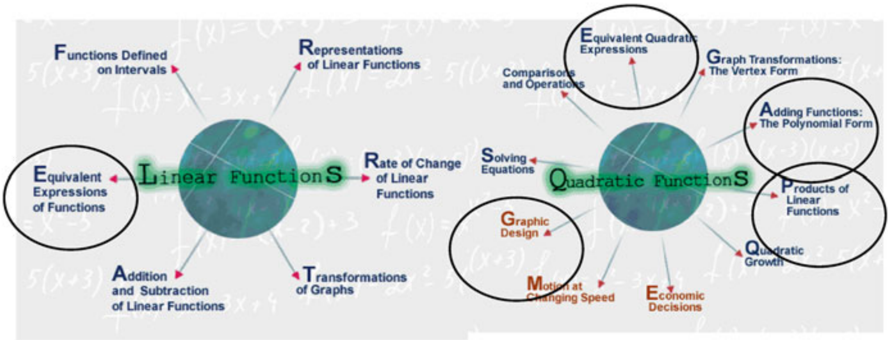


Fig. 7.9 The algebraic structure and manipulation sequence based on five units: *Equivalent Expressions*, *Products of Linear*, *Adding Functions to Polynomials*, *Equivalent Quadratics*, and *Graphic Designs* (http://visualmath.haifa.ac.il/)

7.4 Summary: Concept-Driven Navigation in the Space of Interactive Tasks

We started with a brief review of the traditional roles and images of order and authority of textbooks. We analyzed the difficulties that arise when teachers try to respond to student needs and ways of understanding based on an approach according to which the textbook remains an important resource that determines what should be taught and how. We then proceeded to describe common notions concerning the interactivity of digital textbooks and questioned what the Web and other resources appended to the digitized versions of textbooks can provide to support guided inquiry in the mathematics classroom. Finally, we discussed several design principles of interactive digital textbooks and reviewed examples of central design decisions reflected in the *VisualMath* algebra textbook. The three principles included in the discussion were (a) designing interactive diagrams that provide students with ways to explore within curricular boundaries, (b) suggesting a visual semiotic framework for typifying the conceptual components and terms inherent to the design of interactions within technology-based textbooks, and (c) organizing the textbook into units that respond to the principal objects and operations of the mathematics to be learned and sequenced to achieve personalization. I tried to address the difficulties expressed by many teachers who are deeply committed to changing the way in which mathematics is being taught today but are frustrated in their efforts by the authority of textbooks that dictate their teaching agenda. I argue that the organization of the e-textbook should make it possible for teachers to sequence the curricular material in a way that serves the needs of the classroom and of the students. I therefore suggest that, in the era of digital textbooks, it is necessary that such organizational/design principles should be part of teachers' knowledge.

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