# Chapter 2
# Frameworks and Principles for Task Design

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> *The natural sciences are concerned with how things are.*
> *Design, on the other hand, is concerned with how things might be.*
>
> (Herbert. A. Simon, 1969)

## 2.1 Introduction

The opening citation, drawn from *The Sciences of the Artificial* by H.A. Simon (1969), on design being concerned with "how things might be," evokes the idea that the field of mathematics education has been involved in design ever since its beginnings—going back to the time of Euclid and perhaps even Pythagoras, for whom the term *mathema* meant *subject of instruction*. However, as Wittmann (1995) remarked, in a paper titled *Mathematics Education as a Design Science*, the design of teaching units was never a focus of the educational research community until the mid-1970s. Artigue (2009), too, has argued that "didactical design has always played an important role in the field of mathematics education, but it has not always been a major theme of theoretical interest

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in the community" (p. 7). According to Cobb, Confrey, diSessa, Lehrer, and Schauble (2003), design experiments are conducted to develop theories, not merely to tune empirically *what works*: "a design theory explains why designs work and suggests how they may be adapted to new circumstances" (p. 9). This movement in design within mathematics education from thoughtful tinkering, growing out of intuition and classroom development, to theoretically based research has not only been interlaced with the emergence of an international research community in mathematics education but also been accompanied by additional influences from the discipline of mathematics and, of no less importance, the work of psychologists (Kilpatrick, 1992).

The objective of this chapter is to give an overview of the current state of the art related to frameworks and principles for task design so as to provide a better understanding of the design process and the various interfaces between teaching, researching, and designing. In so doing, it aims at developing new insights and identifying areas related to task design that are in need of further study. The chapter consists of three main sections. The first main section (Sect. 2.2) begins with a historical overview of the emergence of the mathematics education research community, followed by developments within psychology at large that came to bear on design research in mathematics education. This is followed by a discussion of the ways in which mathematics education researchers took up some of these developments and adapted them to fit with a focus on mathematical content, thereby producing their own design frames. After this historical overview, the section offers a conceptualization of current frameworks for task design in mathematics education and describes the characteristics of the design principles/tools/heuristics offered by these frames. The second main section (Sect. 2.3) presents a set of cases that illustrate the relations between frameworks for task design and the nature of the tasks that are designed within a given framework. Because theoretical frameworks and principles do not account for all aspects of the process of task design, the third main section (Sect. 2.4) addresses additional factors that influence task design and the diversity of design approaches across various professional communities in mathematics education. The chapter concludes with a discussion of the progress made in the area of task design within mathematics education over the past several decades and includes some overall recommendations with respect to frameworks and principles for task design and for future design-related research.

## 2.2 Emergence and Development of Frameworks and Principles for Task Design

### *2.2.1 Brief History of the Emergence of Design-Related Work from the 1960s to the 1990s*

When Wittmann (1995) remarked that the design of mathematical teaching units was never a focus of research until the mid-1970s, he was referring indirectly to the fact that it was only at that time that mathematics education research coalesced as a separate field of study. From the early 1900s, psychologists in various countries had been conducting empirical research on how mathematics is learned, while mathematicians and mathematics educators were more interested in focusing on the mathematical content to be taught and learned (Kilpatrick, 1992). Nevertheless, the post-Sputnik wave of mathematics education reform in the late 1950s and 1960s introduced many new task types using research methods and insights from prior psychological research. However, there was no community as such that could be called a mathematics education research community. That changed in the late 1960s and 1970s. In 1969, the first International Congress on Mathematical Education (ICME) took place in Lyon. A round table at that congress set the stage for the formation in 1976 of what was to quickly become the largest association of mathematics education researchers in the world, the International Group for the Psychology of Mathematics Education (PME). The emergence of this community was accompanied by the creation of several research journals, including *Educational Studies in Mathematics* in 1968, *Zentralblatt für Didaktik der Mathematik* in 1969, and *Journal for Research in Mathematics Education* in 1970. In several countries, research institutes were formed, such as the Shell Centres for Mathematical Education at Chelsea College and at the University of Nottingham in 1968, the Instituts de Recherche pour l'Enseignement des Mathématiques in France in 1969, the Netherlands Institute for the Development of Mathematical Education (IOWO) at Utrecht University in 1971, and the Institut für Didaktik der Mathematik in Bielefeld in 1973. With its annual meetings, its journals, and the fertilization made possible by cross-national collaborations, an international community of mathematics education researchers had taken shape. The late 1960s and 1970s thus signaled a huge surge and interest in research in mathematics education.

#### 2.2.1.1 Influences from Psychology at Large

This surge in research in mathematics education had to rely almost exclusively in its early days on psychology as a source of theory (Johnson, 1980). Piaget's (1971) cognitively oriented, genetic epistemology is but one example of the psychological frames adopted by the emerging mathematics education research community in its studies on the learning of mathematics. However, other forces were beginning to be felt during these years—forces related to design that were being conceptualized and developed by psychologists with an interest in education.

In 1965, Robert Gagné published *The Conditions of Learning*. Based on models from behaviorist psychology, Gagné's (1965) nine conditions of learning were viewed as principles for instructional design—*instructional design* being defined in Wikipedia as the "practice of creating instructional experiences which make the acquisition of knowledge and skill more efficient, effective, and appealing." Gagné classified cognitive learning into the three areas of verbal information, cognitive strategies, and intellectual skills but tended to emphasize the learning and automating of procedures.

In parallel with the instructional design approach being developed by Gagné and others, a new field was emerging, that of cognitive science, often referred to as the *information processing* movement. As Anderson (1995/2000) remarked in his book, *Learning and Memory*, "at the height of the behaviourist era, around 1950, learning was perceived as the key issue in psychology; … [but] learning was pushed somewhat from center stage by the cognitive movement in the 1960s" (p. vii). Advances in design considerations were stimulated by the theorizing of the cognitive scientist and Nobel laureate, H.A. Simon (1969), in *The Sciences of the Artificial*. He advocated the notion that the design process involved generating alternatives and then testing these alternatives against a range of requirements and constraints. Some of H.A. Simon's design ideas were taken up by the educational psychologist Robert Glaser (1976) in his *Components of a Psychology of Instruction: Toward a Science of Design.*

Glaser distinguished, in line with Bruner, between the descriptive nature of theories of learning and what he referred to as the prescriptive nature of theories of instruction. In integrating design considerations into instructional research, he argued:

> Regardless of the descriptive theory with which one works, four components of a prescriptive theory for the design of instructional environments appear to be essential: (a) analysis of the competence, the state of knowledge and skill, to be achieved; (b) description of the initial state with which learning begins; (c) conditions that can be implemented to bring about change from the initial state of the learner to the state described as the competence; and (d) assessment procedures for determining the immediate and long-range outcomes of the conditions that are put into effect to implement change from the initial state of competence to further development. (Glaser, 1976, p. 8)

Glaser emphasized that the structure of the subject-matter discipline may not be the most useful for facilitating the learning of less expert individuals. While reiterating H.A. Simon's notion that the design process involves the generation of alternatives, he did not designate specific principles on which the "generation of alternatives" might be based. Presumably, these would be related to various theories of learning, especially as design was considered to involve the application of a descriptive theory of learning to the generation of a prescriptive theory of instruction—but, according to Glaser, not necessarily foregrounding subject-matter considerations. Thus, mathematics education researchers would need to develop during the years to come their own scientific approaches to designing environments for the learning of mathematics and to generating frameworks for task design in particular.

#### 2.2.1.2 Early Design Initiatives of the Mathematics Education Research Community

During the 1970s, the focus within the mathematics education research community was squarely on the learning of mathematics and the development of models of that learning. For example, the paper that Hans Freudenthal presented at PME3, held in Warwick, UK, in 1979 (one of the 24 research reports presented at PME that year) dealt with the development of reflective thinking (Freudenthal, 1979); Alan Bishop's, with visual abilities and mathematics learning (Bishop, 1979); and

Richard Skemp's, with goals of learning and qualities of understanding (Skemp, 1979). Nevertheless, two of the 1979 PME papers did touch upon issues related to tasks: one by Claude Janvier and the other by Alan Bell.

Janvier argued that, with the discovery learning movement, emphasis had been put on the "notion of appropriate learning environments and on the idea of rich situations likely to bring about discoveries or to encapsulate rich abstract ideas" (1979, p. 135). In his paper, he made use of one of the tasks (the racing car graph) devised for his doctoral research at the University of Nottingham in order to study various issues involved with the use of situations. At the conclusion of his paper, he remarked that his results were in line with Freudenthal's phenomenological approach, which promoted the use of large-scale situations involving weeks of work and stressing the child's point of view more than that of mathematical structures. In an earlier work, *Weeding and Sowing*, Freudenthal (1978) had introduced the approach of didactical phenomenology, which begins with a thorough mathematical analysis of the topic from which are generated hypothesized learning levels—an approach that he referred to as "developmental research" (see Gravemeijer & Cobb, 2006, 2013) and which was further elaborated by Streefland (1990) and Gravemeijer (1998). Freudenthal's (1979) PME3 text, which reflected his ongoing work, sowed the seeds for a mathematical-psychological approach to task design—an approach that was to develop during the late 1980s and 1990s into the instructional theory specific to mathematics education known as Realistic Mathematics Education.

The PME3 paper presented by Alan Bell focused on the learning that develops from different teaching approaches with various curriculum units that had been designed for the South Nottinghamshire project. The teaching methods that were explored included "embodiment, guided discovery approaches, and cognitive conflict" (Bell, 1979, p. 5). In Bell's research, design considerations were thus seen more through the lens of particular teaching methods than as approaches to the design of tasks per se.

In summary, the work of Hans Freudenthal at IOWO, of Alan Bell at Nottingham, and their colleagues during the 1970s reflected the beginnings of the new community of mathematics education researchers' efforts to grapple with the interaction between curriculum materials and the quality of mathematical teaching and learning—a dimension on which curriculum development efforts over the previous several decades had yielded little information. This embryonic work in task design was characterized mainly by reflection on the nature of mathematics, with aspects drawn from the psychologically based learning theories of the day and supported by personal pedagogical experience, coupled with informal observations of children's, students', or teachers' activity. The main aim seemed a combination of desiring to know more about the nature of learning mathematics and/or improving the teaching of mathematics rather than casting light on the nature of the tasks that might support such teaching or learning.

The 1980s within the mathematics education research community brought some integration of aspects of the design theories of H.A. Simon and others. In his 1984 *ESM* paper (a modified version of his opening address at the 14th annual meeting of German mathematics educators in 1981), titled *Teaching units as the integrating* *core of mathematics education*, Erich Wittmann (1984) argued for tasks displaying the following characteristics: the objectives, the materials, the mathematical problems arising from the context of the unit, and the mostly mathematical, sometimes psychological, background of the unit. He suggested that a teaching unit is not an elaborated plan for a series of lessons but rather it is an idea for a teaching approach that leaves open various ways of realizing the unit. Wittmann viewed the philosophy behind the teaching units as being embedded in Herbert Simon's *Sciences of the Artificial*: teaching units, according to Wittmann, are simply artificial objects constructed by mathematics educators—objects to be investigated within different educational ecologies.

During the years 1985–1988, one of the PME working groups focused on the extent to which its activities had established principles for the design of teaching. In 1988, a collection of papers from this working group was put together by the Shell Centre under the title *The Design of Teaching: Papers from a PME Working Group* and subsequently published in a special issue of *Educational Studies in Mathematics* in 1993. In his editorial for the special issue, Alan Bell wrote:

> Experimental work on the development of understanding in particular mathematical topics is relatively easy to conduct … but studies of the general properties of different *teaching methods* and *materials* are more difficult to set up. … Types of research on teaching which have been found productive, albeit in different ways, are the following: (1) basic psychological studies of aspects of learning …; (2) *developmental activities in which teaching materials are designed on the basis of theory and practical experience and are then taken through several cycles of trial and improvement*, …; and (3) comparative studies in which the same topic is taught to parallel classes by different methods. Examples of each of these types appear in this issue. But the design of teaching is a creative activity, and readers may hope to gain from these articles not only knowledge of some empirically established principles, but also tested ideas for their practical implementation. (Bell, 1993a, pp. 1–2, italics added)

Note in the quote the integration of "teaching methods" and "materials", that is, principles of teaching practice that are in harmony with principles that have been incorporated into the design of the teaching materials—an integration of two types of principles that will be seen to continue to be important in task design within the community over the decades to come. In his introductory article, "Principles for the Design of Teaching," Bell specified the following set of design principles:

> First one chooses a *situation* which embodies, in some *contexts*, the concepts and relations of the conceptual field in which it is desired to work. Within this situation, *tasks* are proposed to the learners which bring into play the concepts and relations. It is necessary that the learner shall know when the task is correctly performed; hence some form of *feedback* is required. When *errors* occur, arising from some *misconception*, it is appropriate to expose the *cognitive conflict* and to help the learner to achieve a resolution. This is one type of intervention which a teacher may make to assist the learning process. (Bell, 1993b, p. 9, italics in the original)

The underlying psychological learning principles supporting this theory of teaching design were said by Bell to include connectedness, structural transfer across contexts, feedback, reflection and review, and intensity.

The late 1970s and 1980s also witnessed Soviet-style teaching experiments, both in individual settings as well as in classrooms (e.g., Kalmykova, 1966; Menchinskaya, 1969), experiments that explored alternate teaching-with-task designs so as to investigate more deeply the learning of various mathematical concepts. The origin of these teaching experiments dated back to the 1920s with the individualized instructional experiments of Vygotsky, who believed that the development of mental abilities was essentially dependent upon instruction.

Another important development during the decade of the 1980s with respect to design was the emergence within France of *didactical engineering* (DE), an exacting theory-based approach to conducting research that had didactical design at its heart (Artigue, 1992). Despite its success as a design-based research practice, certain problems were encountered, according to Artigue (2009), when the rigorous designs were implemented in everyday classroom practice throughout its first decade. It was observed that the original designs went through a certain mutation in practice, leading her to note that "the relationships between theory and practice as regards didactical design are not under theoretical control" (Artigue, 2009, p. 12). This awareness pointed to one of the inherent limitations in theorizing about task design in isolation from considerations regarding instructional practice.

#### 2.2.1.3 The 1990s and Early 2000s: Development of Research Specifically Referred to as *Design Experiments*

The term *design experiment* came into prominence in 1992 with the psychologist Ann Brown's (1992) paper on design experiments (see also Collins, 1992). Brown emphasized that design experiments aim at increasing the relevance of earlier cognitive science laboratory studies to the real activity of classrooms and that this research is designed to inform practice, as well as benefit from the experience of practitioners. This attention to classrooms, teaching, and teaching practice was a reflection of the movement from cognitive to sociocultural perspectives on learning—a movement that had emerged when Vygotsky's works started to become better known in the West toward the end of the 1980s and one that had already begun to take hold within the mathematics education research community.

Brown's paper signaled a kind of tipping point with respect to interest in design in the mathematics education research community (Lesh, 2002). Several factors had fallen into place, including the maturing of the community over a 20-year period and an evolving desire to be able to study within one's research not just learning or not just teaching. Design experiments aimed at taking into account the entire learning picture. As Cobb et al. (2003) pointed out:

> Design experiments ideally result in greater understanding of a learning ecology. … Elements of a learning ecology typically include the tasks or problems that students are asked to solve, the kinds of discourse that are encouraged, the norms of participation that are established, the tools and related material means provided, and the practical means by which classroom teachers can orchestrate relations among these elements. (p. 9)

Within this conception of design experiments, the task or task sequence is considered but one of a larger set of design considerations involving the entire learning ecology—*task* or *task sequence* (which could take an entire lesson or more) being characterized in the ICMI Study-22 Discussion Document as "anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something … also anything that students decide to do for themselves in a particular situation" (Watson et al., 2013, p. 12).

Another central feature of design experiments is, according to Cobb et al. (2003), the role played by theory, as well as the nature of this theory:

> General philosophical orientations to educational matters—such as constructivism—are important to educational practice, but they often fail to provide detailed guidance in organizing instruction. The critical question that must be asked is whether the theory informs prospective design and, if so, in precisely what way? Rather than grand theories of learning that may be difficult to project into particular circumstances, design experiments tend to emphasize an intermediate theoretical scope. (pp. 10–11)

Cobb et al.'s argument that theories of intermediate scope do a better job of informing prospective design leads naturally to the question of the nature of such theories and the ways in which they can inform prospective design. A broad approach to answering this question suggests that it might be helpful to first draw upon an example from outside the field to see what kinds of theories educational designers who specialize in the work of design integrate into their work.

#### 2.2.1.4 An Example of Design Work by Educational Technologists

Jeroen van Merriënboer and his colleagues (van Merriënboer, Clark, & de Croock, 2002) are leading educational technologists who have developed a design model that consists of the following four components of instructional design for complex learning: (1) classes of learning tasks that are ordered and that promote schema construction, along with rule-oriented tasks for routine aspects, (2) supportive bridging information to link with prior knowledge, (3) just-in-time prerequisite information, and (4) part-task practice. The elaboration of these components is supported explicitly by major theoretical foundations in cognitive psychology, accompanied by a mix of different instructional approaches suitable for the different components of the design model. For example, in discussing the ordering of tasks with respect to their complexity, van Merriënboer et al. (2002) refer to cognitive load theory (Sweller, van Merriënboer, & Paas, 1998); in describing the amount and nature of learner support required, they refer to the framework of human problem-solving provided by Newell and Simon (1972); in noting the important role of cognitive feedback, they refer to the cognitive apprenticeship model (Collins, Brown, & Newman, 1989); and for just-in-time information that is best characterized as "how-to instruction or rule-based instruction", they cite Fisk and Gallini (1989). As an example of the kinds of instructional support useful in helping learners identify relevant relationships, van Merriënboer et al. distinguish between the inquiry method (e.g., "ask the learners to present a well-known, familiar example or counterexample for a particular idea") and the expository method (e.g., the instructor "presents a well-known, familiar example or counterexample for a particular idea") and note that "inquiry approaches are time-consuming, but because they directly build on learners' prior knowledge they are very appropriate for interconnecting new information and already existing cognitive schemata" (p. 48).

Van Merriënboer et al. state that their model was not developed for teaching conceptual knowledge or procedural skills per se nor is it very useful for designing very short learning programs that only take an instructional time of hours or a few days—it was generated with the aim of developing solutions to complex problems and has its roots in vocational education. Nevertheless, we consider it useful for illustrating how this team of professional task designers relies on a variety of theories of intermediate scope to underpin their design, as well as for pointing out how their suggested teaching approaches vary according to the nature of the given component of the instructional design.

#### 2.2.1.5 From Early 2000 Onward

Theorizing related to design in mathematics education research, and in educational research more broadly, continued to evolve during the 2000s (Kelly, Lesh, & Baek, 2008). In addition, the term *task design* came to be more clearly present. For example, at the 2005 PME conference, a research forum was dedicated to task design, having as its stated theme, "The significance of task design in mathematics education" (Ainley & Pratt, 2005). One of the presenting teams, Gravemeijer, van Galen, and Keijzer (2005), pointed out that, "in Realistic Mathematics Education, instructional design concerns series of tasks, embedded in a local instruction theory; this local instruction theory enables the teacher to adapt the task to the abilities and interests of the students, while maintaining the original end goals" (pp. 1–108)—a local instructional theory being described by Gravemeijer et al. as the rationale for the instructional sequence, a rationale that evolves over several design experiments that involve testing and revising the sequence. Gravemeijer et al.'s statement suggests another view of theory as one that not only informs prospective design but is also a product of instructional design—an issue to which we will return immediately below. These ideas continued to be explored at ICME-11 in 2008 where the scientific program, for the first time, included a Topic Study Group (TSG) on task design: "Research and development in task design and analysis". The excitement generated regarding this research area was such that a similar TSG was put on the program for ICME-12 in 2012, as well as for ICME-13 in 2016. This interest was further illustrated by the holding of the 2013 ICMI Study-22 Conference on the same theme, a conference whose scientific work and discussions are the subject of this very volume.

#### 2.2.1.6 Two Key Issues

In bringing to a close this first subsection devoted to a historical overview of the emergence of research related to design activity, we emphasize two central issues in need of clarification regarding the place of task design within design research. These issues underpin and run through much of the discussion that follows in the next subsection on frameworks and principles. In a recent article on design tools in didactical research, Ruthven, Laborde, Leach, and Tiberghien (2009) elaborated on the distinction between *design as intention* and *design as implementation* (Collins, Joseph, & Bielaczyc, 2004). *Design as implementation* focuses attention on the process by which a designed sequence is integrated into the classroom environment and subsequently is progressively refined, whereas *design as intention* addresses specifically the initial formulation of the design. While many studies address both, the distinction can be useful for understanding certain nuanced differences between one study and another. Ruthven et al. state that design as intention emphasizes the "original design and the clarity and coherence of the intentions it expresses" (p. 329). Design as intention makes use, in general, of theoretical frames that are well developed so as to provide this clarity and coherence. Although Ruthven et al. add that "the availability of design tools capable of identifying and addressing specific aspects of the situation under design can support both the initial formulation of a design and its subsequent refinement in the light of implementation" (p. 329), their examples cast light in particular on the *design as intention* orientation. In so doing, they illustrate clearly the role that theoretical tools play in the initial design.

In contrast to the front-end importance given to theory-based design tools by Ruthven et al. (2009), Gravemeijer and Cobb (2006) put the focus more toward the *development* of theory and its role as a product of the design research. In their design experiment studies, the initial theoretical base for the study and its accompanying instructional plan undergo successive refinements by means of the implementation process. The description of the entire process constitutes the development of the theory. Because of the centrality of the implementation process in the development of the resulting theory, such studies are characterized as *design as implementation* studies—even if their theoretical starting points could also qualify them as *design as intention* studies. For example, Gravemeijer and Cobb (2006) point out that

> from a design perspective, the goal of the preliminary phase of design research experiments is to formulate a local instruction theory that can be elaborated and refined while conducting the intended design experiment; … this local instruction theory encompasses both provisional instructional activities, and a conjectured learning process that anticipates how students' thinking and understanding might evolve when the instructional activities are employed in the classroom. (p. 48)

They emphasize that the "products of design experiments typically include sequences of activities and associated resources for supporting a particular form of learning, together with a domain-specific, instructional theory that underpins the instructional sequences and constitutes its rationale; a domain-specific, instructional theory consists of a substantiated learning process that culminates with the achievement of significant learning goals as well as the demonstrated means of supporting that learning process" (Cobb & Gravemeijer, 2008, p. 77).

More precisely, Cobb et al. (2003) insist that "design experiments are conducted to develop theories" (p. 9).

Put another way, theories are both a resource and a product. As a resource, they provide theoretical tools and principles to support the design of a teaching sequence (e.g., Ruthven et al., 2009) and, as a product of design research, theories inform us about both the processes of learning and the means that have been shown to support that learning (Cobb et al., 2003).

A second issue related to the role and nature of theory in design is the significance given to task design itself within the design process. When theory and its design tools are viewed as a front-end resource in the design process, the way in which task design is informed by these theory-based tools moves to center stage (e.g., Ruthven et al., 2009). By way of contrast, when theory development is viewed as the aim of design experiments, task design tends to be less central: "One of the primary aims of this type of research is *not* to develop the instructional sequence as such, but to support the constitution of an empirically grounded local instruction theory that underpins that instructional sequence" (Gravemeijer & Cobb, 2006, p. 77, emphasis added). This is not to say, in the latter case, that task design is unimportant (it clearly is) but rather the design of the teaching/instructional sequence is only one of several all-encompassing considerations within the whole interactive learning ecology. In practice, most design experiments combine both orientations: the design is based on a conceptual framework and upon theoretical propositions, while the successive iterations of implementation and retrospective analysis contribute to further theory building that is central to the research. In fact, both orientations will be seen to be present in most of the design studies exemplified below.

These two issues, that is, (1) *design as intention* and *design as implementation* and (2) the status given to the initial design of the set of tasks, point to central differences in the way in which the roles of theory and task design are considered within the design process in the mathematics education research community. In the presentation that follows—one that focuses on current frameworks and principles for task design—we shall attempt to interweave these distinctions into our discussion of the nature of the frames adopted, adapted, and developed within the activity of design. By so doing, we hope to be able to contribute to clarifying some of the ways in which theory and task design are related.

### *2.2.2 A Conceptualization of Current Theoretical Frameworks and Principles for Task Design in Mathematics Education Research*

#### 2.2.2.1 Introduction

Our historical look at the early research efforts related to task design revealed a mix of task and instructional considerations. However, the extent to which instructional aspects are factored into task design is but one of the ways in which design frameworks can vary. Frameworks can also differ according to the degree to which the learning environments are student centered, knowledge centered, or assessment centered (Bransford, Brown, & Cocking, 1999), as well as the manner in which they draw upon cognitive, sociological, sociocultural, discursive, or other theories. In addition, frameworks are distinguishable according to the extent to which they can be related to various task genres, that is, whether the tasks are geared toward (1) the development of mathematical knowledge (such as concepts, procedures, representations; see, e.g., Swan, 2008), (2) the development of the processes of mathematical reasoning (such as conjecturing, generalizing, proving, as well as fostering creativity, argumentation, and critical thinking; see, e.g., Leikin, 2013; Lin, Yang, Lee, Tabach, & Stylianides, 2012; Martinez & Castro Superfine, 2012), (3) the development of modeling and problem-solving activity (e.g., Lesh, Hoover, Hole, Kelly, & Post, 2000; Ponte, Mata-Pereira, Henriques, & Quaresma, 2013; Schoenfeld, 1985), (4) the assessment of mathematical knowledge, processes, and problem-solving (e.g., Swan & Burkhardt, 2012), (5) the context of mathematical team competitions (e.g., Goddijn, 2008), and so on. As well, some frameworks may be more suited to the design of specific tasks, others to the design of lesson flow (e.g., Corey, Peterson, Lewis, & Bukarau, 2010), and still others to the design of sequences involving the integration of particular artifacts (e.g., Kieran & Drijvers, 2006). Because several considerations enter into an overall design—considerations that include the specific genre of the task, its instructional support, the classroom milieu, the tools being used, and so on—each part of the design might call for different theoretical underpinnings. Thus, the resulting design can involve a *networking* of various theoretical frames and principles (Prediger, Bikner-Ahsbahs, & Arzarello, 2008) or a *bricolage* (Gravemeijer, 1994) or a *bridging* (Koedinger, 2002). Furthermore, the nature of the principles or heuristics associated with various frames and the way in which these heuristics are construed—according to whether they are viewed as illuminating, inspiring, guiding, systematizing, or even constraining—all have a part to play (see Sect. 2.4 for discussion of other factors, such as the artistic and value-related aspects of task design). A more holistic way of thinking about frames is to view them as being of different levels (e.g., Goldenberg, 2008) or types, for example, grand frames, intermediate-level frames, domain-specific frames (i.e., frames related to the learning of specific mathematical concepts and reasoning processes), and frames related to particular features of the learning environment (e.g., frames for tool use)—all of them together constituting any one theoretical base for the design of a given study (Gravemeijer & Cobb, 2006). This manner of conceptualizing design frames according to the levels of grand, intermediate, and domain specific (note that tool-related frames are treated in Chap. 6) will now be used as a backdrop for examining the nature of current theoretical frameworks and principles for task design in mathematics education research.

#### 2.2.2.2 Grand Theoretical Frames and Their Affordances

Mathematics education research has tended in large measure to adopt such grand theoretical perspectives as the cognitive-psychological, the constructivist, and the socioconstructivist. However, as pointed out by Lerman, Xu, and Tsatsaroni (2002), these are but three of the vast array of theoretical fields, in addition to those from educational psychology and/or mathematics that have backgrounded mathematics education research. In line with Cobb (2007), who has argued that such grand theories need to be adapted and interpreted in order to serve the needs of design research, our discussion of them will be brief and limited to a few selective aspects.

A cognitive-psychological theoretical perspective has dominated research on the learning of mathematics ever since the days of Piaget—at least up until the late 1980s and early 1990s when Vygotsky's work came to be better known among Western mathematics educators (Bartolini Bussi, 1991). Tightly linked with the cognitive-psychological perspective is the constructivist frame (von Glasersfeld, 1987) that stemmed mainly from Piaget's genetic epistemology (Steffe & Kieren, 1994). Learning came to be widely interpreted as a constructive process, a process in which students actively construct mathematical knowledge. While constructivism has always been with us, perhaps under another guise, its growing acceptance as an educational tenet during the 1980s (Cobb & Steffe, 1983) helped to oust the view of mathematical teaching as the transmission of the teacher's knowledge and mathematical learning as the reception of that knowledge. However, constructivist, cognitively oriented research soon became hard pressed to reconcile the notion that all learning is individually constructed with the evidence of commonalities found across individuals. Constructivists had to admit the social dimensions of learning, thereby paving the way for integrating the Soviet work of Vygotsky and Leont'ev. The view of learning as situated with respect to social and cultural practices, and thus a socioconstructivist frame of reference, soon became widely accepted (Lerman, 1996). This frame directly allowed for a focus on the role of teaching and of classroom interactions in the learning process.

Within the tradition of cognitive psychology, two types of theories have been developed by mathematics education researchers (Cobb, 2007). One concerns theories of learning across mathematics in general (e.g., Pirie & Kieren's, 1994, recursive theory of mathematical understanding; Sfard's, 1991, theory of reification); the other concerns theories of the development of students' learning in specific mathematical areas (e.g., Filloy & Rojano's, 1989, theory of algebraic reasoning; Clements & Battista's, 1992, theory of geometric reasoning). As will be seen below, these theories that have been inspired by and are situated within the grander theories are key components of design in mathematics education research, even if, as Cobb (2007) insists, they are not instructional and require adaptation or combination with other theories, in order to serve the needs of instructional design. Cobb's point of view was also emphasized earlier by Bransford et al. (1999) in their volume *How People Learn*: "Learning theory provides no simple recipe for designing effective learning environments, but it constrains the design of effective ones" (p. xvi).

#### 2.2.2.3 Intermediate-Level Frames

Intermediate-level frames have a more specialized focus than the grand theories of socioconstructivism and the like, yet intermediate-level frames still tend to be situated within the perspective of one or the other of these grand frames. Even if their focus is more specialized, intermediate-level frames have the property that they can be applied across a wide variety of mathematical areas. In brief, intermediate-level frames are located between the grand theories and the more local, domain-specific frames that address particular mathematical concepts, procedures, or processes. A multitude of intermediate-level frames have been developed that are being applied in adaptive ways to design research in mathematics education. They include, for example, Realistic Mathematics Education theory (Treffers, 1987), the Theory of Didactical Situations (Brousseau, 1997), the Anthropological Theory of Didactics (ATD) (Chevallard, 1999), Lesson Study (Lewis, 2002), Cultural-Semiotics theory (Radford, 2003), Commognitive Theory (Sfard, 2008), and so on (see Sect. 2.3 for an elaboration of the ways in which some of the of various intermediate-level frames have been adopted and adapted for use in research on task design).

In general, intermediate-level frames can be characterized by explicit principles/heuristics/tools that can be applied to the design of tasks and task sequences. Because these frames tend to be highly developed, they are often used in *design-as-intention* approaches. In addition, intermediate-level frames can also be characterized according to whether their roots are primarily theoretical or whether they are based to a large extent on deep craft knowledge. The two examples of intermediate-level frames and their accompanying principles for task design that we offer immediately below reflect these two roots. The first is the Theory of Didactical Situations and the second is that of Lesson Study. For both types, we examine the framework and associated principles that support the process of task design.

### *An Example of a Theory-Based Intermediate-Level Frame: Theory of Didactical Situations*

The Theory of Didactical Situations (TDS) is generally associated with Guy Brousseau (1997); however, its development over the years has been contributed to by the French mathematical *didactique* community at large. A central characteristic of TDS research is its framing within a deep a priori analysis of the underlying mathematics of the topic to be learned, integrating the epistemology of the discipline, and supported by cognitive hypotheses related to the learning of the given topic. TDS is said to be an intermediate theory in that it draws upon the grand theory of Piaget's work in cognitive development. According to Ruthven et al. (2009), one of the central design tools provided by TDS is the *adidactical situation*, which mediates the development of students' mathematical knowledge through independent problem-solving. The term *adidactical* within TDS refers specifically to that part of the activity "between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, [a time when] the teacher refrains from interfering and suggesting the knowledge that she wants to see appear" (Brousseau, 1997, p. 30).

A situation includes both the task and the environment that is designed to provide for the adidactical activity of the student. According to the TDS frame, the adidactical situation tool furnishes guidelines as to: "the problem to be posed, the conditions under which it is to be solved, and the expected progression toward a strategy that is both valid and efficient; this includes the process of 'devolution' intended to lead students to directly experience the mathematical problem as such and the creation of a (material and social) 'milieu' that provides students with feedback conducive to the evolution of their strategies" (Ruthven et al., 2009, p. 331). During the early years of the development of the TDS, it was found that the frame needed some modification so as to take into account the necessary role played by the teacher in fostering the later institutionalization of the student's mathematical knowledge acquired during the adidactical phase—the term *institutionalization* referring specifically to the process whereby the teacher gives a certain status to the ideas developed by students by framing and situating them within the concepts and terminology of the broader cultural body of scientific knowledge (see also Chaps. 3 and 5).

Identifying a suitable set of problem situations that can support the development of new mathematical knowledge is absolutely central to the design of a TDS teaching sequence. The adidactical situation must be one for which students have a starting approach but one that turns out to be unsatisfactory. Students must be able to obtain feedback from the milieu that both lets them know that their approach is inappropriate and also provides the means to move forward. The "enlargement of a shape puzzle" (see Ruthven et al., 2009, pp. 332–334) is a fine example of the design of an adidactical situation. When students (who are working in small groups) are asked to make a larger puzzle of the same shape but with the edge whose length is 4 cm being enlarged to 7 cm, it is expected that they would use additive reasoning. But the feedback provided by the attempt to put the enlarged puzzle pieces together lets the students know that their way of solving the enlargement problem is incorrect. Eventually, "intellectual" feedback is provided by the teacher in order to help the students to arrive collectively at a multiplicative model.

In addition to the adidactical situation tool, TDS-based design is also informed by a second design heuristic, that of the *didactical variables* tool. This supplementary design tool allows for choices regarding particular aspects of the main task and how it is to be carried out (e.g., shape and dimensions of the pieces, the ratio of the enlargement, the various pieces of the puzzle being constructed by different students), aspects that are subject to modification as a result of successive cycles of the teaching sequence. Although certain modifications are made to those aspects of the task that are found to improve the learning potential of the situation (i.e., that students are more likely to learn what is intended), the initial design of the task is absolutely central to the TDS-framed *design-as-intention* process.

*An Example of a Craft-Based Intermediate-Level Frame: Lesson Study*

Lesson Study is typically associated with Japanese education where its roots can be traced back to the early 1900s (Fernandez & Yoshida, 2004). However, variants of Lesson Study have been developed in China (Huang & Bao, 2006; Yang & Ricks, 2013), as well as in other countries (Hart, Alston, & Murata, 2011). For example, in the Chinese version, according to Ding, Jones, and Pepin (2013), the role of the expert in the development and refinement of a lesson plan is of critical importance. This role consists of contributions that are said to go beyond that of deep craft knowledge—contributions that Ding, Jones, Pepin, and Sikko (2014) describe as consisting of a complex combination of considerable knowledge of mathematical didactics and general theories of learning and of students, as well as the "accumulated wisdom of practice." Thus, the distinction we are proposing between craft-based and theory-based intermediate-level frames may be rather blurry for certain versions of Lesson Study. Even within Japan, various types of Lesson Study exist: at the school level, at the local and prefectural level, and at the national level. Nevertheless, in that the majority of Lesson Studies in Japan occur at the school level and that school-based Lesson Study in Japan tends to be considered around the world as prototypical of Lesson Study practice, it is this latter version of Lesson Study that is the focus here.

Lesson Study is a culturally situated, collaborative, approach to design—one where teachers with their deep, craft-based knowledge are central to the process and which at the same time constitutes a form of professional development (Krainer, 2011; Ohtani, 2011). Fundamental to Japanese teachers' ability to design and implement high-quality mathematics lessons that are centered on high-quality mathematical tasks is a detailed, widely shared conception of what constitutes effective mathematics pedagogy (Jacobs & Morita, 2002). Thus, Lesson Study, with its cultural and collaborative foundations, could be said to be situated within the grand theory of socioculturalism.

Lesson Study consists of the following phases: (1) collaboratively planning a research lesson, (2) seeing the research lesson in action, (3) discussing the research lesson, (4) revising the lesson (optional), (5) teaching the new version of the lesson, and (6) sharing reflections on the new version of the lesson. Phases 4–6 are sometimes replaced with a single phase of "consolidating and reporting". In any case, it is the research lesson—its planning, its implementation, and its evaluation, but especially its planning—that is the focus here. As will be seen, Lesson Study is a frame devoted as much to *design as intention* as it is to *design as implementation*.

A typical lesson plan proposal contains the following seven items (Lewis, 2002):

1. Name of the unit
2. Unit objectives
3. Research theme
4. Current characteristics of students
5. Learning plan for the unit, which includes the sequence of lessons in the unit and the tasks for each lesson
6. Plan for the research lesson, which includes:

   - Aims of the lesson,
   - Teacher activities
   - Anticipated student thinking and activities
   - Points to notice and evaluate
   - Materials

- Strategies
- Major points to be evaluated
- Copies of lesson materials (e.g., blackboard plan, student handouts, visual aids)

7. Background information and data collection forms for observers (e.g., a seating chart).

While the unfolding of the research lesson and its evaluation takes only 1 day, its planning can occupy anywhere from 1 to 2 months. As can be seen from Lewis's (2002) list of seven items, the plan of the lesson includes not only the detailed design of the task itself, which constitutes the essence of the research lesson, but also the links with other tasks in the larger unit. Central to this planning is the process of *kyozaikenkyu. Kyozaikenkyu* means literally "instructional materials research" and constitutes a first principle for task design. The study of instructional materials goes beyond the textbook series being used in the classroom. As pointed out by Fujii (2013), *kyozaikenkyu* involves examining teaching materials and tasks from a mathematical point of view (mathematical content analysis), an educational point of view (considering broader values such as "skills for living"), as well as from the students' point of view (readiness, what students know, anticipated students' thinking and misconceptions, etc.). It includes studying other textbook series treating the same topic, thinking about the manipulatives being used, and analyzing what the curriculum standards and research have to say about the topic and its teaching and learning. If the decision is ultimately made to modify an existing textbook task, that decision is made with great care because the teachers know that the textbook task was designed with considerable thoughtfulness. Tasks that will lead to multiple strategies are crucial to the task design process of Lesson Study (for details, see the Lesson Study case that is illustrated in Sect. 2.3)—strategies that will ultimately comprise the basis for the classroom discussion phase of the research lesson.

Consequently, a second design principle concerns the actual form that the research lesson takes. Referred to as *structured problem-solving* by Stigler and Hiebert (1999), the research lesson involves a single task and the following four specific phases: (1) teacher presenting the problem (*donyu*, 5–10 min), (2) students working at solving the problem without the teacher's help (*jiriki-kaiketsu*, 10–20 min), (3) comparing and discussing solution approaches (*neriage*, 10–20 min), and (4) summing up by the teacher (*matome*, 5 min). During students' independent working, the teacher walks between the desks (*kikan-junshi*) and silently assesses students' work; she is in the process of making a provisional plan as to which student contribution should be presented first in order to make clear the progress and elaboration from simple idea to sophisticated one: this is the core of *neriage*, a phase during which students' shared ideas are analyzed, compared, and contrasted. During the fourth phase of the research lesson (*matome*), the teacher will usually comment as to the more efficient of the discussed strategies, as well as the task's and the lesson's mathematical and educational values. As an aside, it is noted that Japanese teachers use these specific didactical terms to discuss their teaching and that such didactical terms not only mediate the activity of the various participants involved in Lesson Study but also lead to the co-construction of deep craft knowledge.

After the research lesson has been observed by other teachers, school administrators, and sometimes by an outside expert, it is then discussed and evaluated in relation to its overall goals. This process of lesson evaluation, and in particular *task evaluation*, is considered a third design principle. The post-lesson discussion focuses to a large extent on the effects of the initial task design with respect to student thinking and learning. The teacher's thought-out key questioning receives much attention. Another of the main aspects discussed is whether the anticipated student solutions were in fact evoked by the task and its accompanying manipulative materials or whether improvements in specific parts of the task design are warranted.

Of the three design principles that are the core of the research lesson of Lesson Study, *kyozaikenkyu* is the most all encompassing. However, it is one that is often underrepresented or even overlooked in Lesson Study practice in other countries (Doig, Groves, & Fujii, 2011). *Kyozaikenkyu* is, in fact, central to Japanese teachers' everyday practice. As such, it is a key component of the Japanese Lesson Study example of craft-based frames for task design.

#### 2.2.2.4 Domain-Specific Frames

In contrast to intermediate-level frames whose characterizations do not specify any particular mathematical reasoning process or any particular mathematical content area, domain-specific frames for the design of tasks or task sequences do specify particular reasoning processes (e.g., conjecturing, arguing, proving) or particular content (e.g., geometry, integer numbers, numerical concepts, algebraic techniques). Task design research involving domain-specific frames typically draws upon past research findings in a given area, in addition to being situated within more general, intermediate-level, and grand-level frameworks. As such, domain-specific frames for task design research tend to be more eclectic than their intermediate-level counterparts. As an aside, note that Realistic Mathematics Education theory has at times been referred to by its adherents as a domain-specific instructional theory in that it is an instructional theory for the domain of mathematics education; however, in this chapter we reserve the term *domain specific* for frames dealing with specific mathematical content areas or reasoning processes. Some researchers use the term "local theories" or "local frames" for what we are referring to as domain-specific frames. In general, domain-specific frames are associated with *design as implementation* in that the main aim of the research is the further development of the domain-specific frame by means of the implementation process. However, this is not a hard distinction. As will be seen, for some examples of design research studies that make use of and develop domain-specific frames, the approach is as much *design as intention* as it is *design as implementation*.

### *A Domain-Specific Frame for Fostering Mathematical Argumentation Within Geometric Problem-Solving*

In a recent article, Prusak, Hershkowitz, and Schwarz (2013) reported on a yearlong, design research-based course with third graders in mathematical problem-solving that aimed at instilling inquiry learning and argumentative norms. The researchers investigated if, and in which ways, principled design is effective in promoting a problem-solving culture, mathematical reasoning, and conceptual learning. Their design was situated in a multifaceted framework that drew upon principles from the intermediate-level, educational theory of Cognitive Apprenticeship, as described in Schoenfeld (1994), and from domain-specific research in geometric reasoning (Hershkowitz, 1990) and argumentation (Arzarello & Sabena, 2011; Duval, 2006), as well as from multiple studies with a sociocultural orientation. The Prusak et al. study was, in fact, one that articulated explicitly two design components: one for the task and one for the learning environment.

The task that Prusak et al. (2013) discuss in their paper is the *sharing a cake* task:

> Yael, Nadav, and their friends Itai and Michele came home from school very, very hungry. On the kitchen table was a nice square piece of cake, leftover from their birthday. They wanted to be fair and divide the square into four equal pieces so that everyone would get a fourth (1/4) of the leftover cake. Suggest different ways in which the children can cut up and divide the square piece of cake. For each suggestion, explain why this would give each child exactly a fourth of the leftover cake. (p. 6)

Accompanying the text on the task worksheet was a set of nine square grids upon which the students, who worked first individually and then in groups, could draw their suggested cuttings of the cake and several blank lines per grid where they were to explain their thinking. Prusak et al. state that the design of this task, as well as that of the others used within their yearlong study, relied on the following five principles:

- Encourage the production of multiple solutions (Levav-Waynberg & Leikin, 2009).
- Create collaborative situations (Arcavi, Kessel, Meira, & Smith, 1998).
- Engage in socio-cognitive conflicts (Limón, 2001).
- Provide tools for checking hypotheses (Hadas, Hershkowitz, & Schwarz, 2001).
- Invite students to reflect on solutions (Pólya, 1945/1957).

Setting up a problem-solving culture in the classroom was an integral part of the Prusak et al. design study. More specifically, they brought into play Schoenfeld's (1994) use of the Cognitive Apprenticeship model by which he scaffolded students' problem-solving in a classroom culture that emphasized communication, reflective mathematical practice, and reasoning rather than results. In line with Schoenfeld, the following instructional-practice principles constituted a second overall design frame for the Prusak et al. study:

- Emphasize processes rather than solely results.
- Use a variety of social settings (individual, small group, and whole class).

- Develop a critical attitude toward mathematical arguments using prompts like, "Does it convince me?"
- Encourage students to listen and try to persuade each other and, thus, to develop ideas together.
- Have students learn to report on what they do, first verbally, then in written form, explaining their solutions to their teammates or to the entire class.

The authors argue that the findings of their study provided evidence that

> the meticulous design as well as the problem-solving culture triggered a general process according to which students capitalised on problem-solving heuristics and engaged in multimodal argumentation, subsequently reaching deep understanding of a geometrical property (the fact that non-congruent shapes may have equal areas). … The activity we described encourages the production of multiple solutions, which is an explicit instruction in the task. Also, students were arranged in small groups, and were asked to collaborate. Collaboration led students to compare solutions. Since they were asked to justify their solutions, these justifications naturally created socio-cognitive conflicts. The nine grids in the task provide a tool for checking hypotheses. (pp. 16–17)

The authors concluded their paper with a theoretical model for learning early geometry through multimodal argumentation in a problem-solving context—a model that includes the description of the learning process and the demonstrated means of supporting that learning process. They emphasize that the designed task served as a principle-based research tool, one that was central to the elaboration of their domain-specific model.

The Prusak et al. study presents an example of the use of well-defined, even if quite general, principles as a front-end resource for the design of the tasks. A second set of principles provided the frame for the design of the learning culture in which the tasks would unfold. Both sets of principles make their study one that could be described as *design as intention*. The empirical evidence that the initial design was effective in eliciting the aimed-for learning of specific geometrical notions through argumentation within a problem-solving setting led to the theoretical elaboration of a domain-specific model. In this sense, the study could be said to be also an example of *design as implementation*. Additionally, and of pivotal importance for design in mathematics education research, the design of the task activities was supported by the accompanying design of an instructional environment involving specific teaching practices that would nurture a collaborative problem-solving culture. This emphasizes the crucial interactive relation between the design of a task or task sequence and the design of the instructional culture in which the task is to be integrated—an emphasis that is also seen in design research involving intermediate-level frames.

### *A Domain-Specific Frame for Proof Problems with Diagrams*

The frame used by Komatsu and Tsujiyama (2013) in their design research, which centered on eighth grade proof and proving, was inspired by the notion of deductive guessing—a notion formulated by Lakatos (1976) as a heuristic rule for coping with counterexamples. In deductive guessing, after one proves conjectures and then faces their counterexamples or non-examples, one invents deductively more general conjectures that hold true even for these examples. Because deductive guessing is a mathematical notion, some adaptation with respect to pedagogical perspectives was necessary so as to use deductive guessing as a frame for task design. Its adaptation yielded the *proof problems with diagrams* frame—a proof problem with diagrams being a problem in which a statement is described with reference to particular diagrams with symbols (one diagram in most cases) and solvers are required to prove the statement and then to deal with related diagrams involving counterexamples and non-examples. The frame was also informed by the earlier research of Shimizu (1981) who had argued that, after students solve proof problems with diagrams, it is important for them to further inquire "of what (mathematical) relations the given diagram is a representative special case" (p. 36) by utilizing the already obtained proof.

As is the case with much of the current task design research in the field, Komatsu and Tsujiyama (2013) point out that, because "it is unrealistic to expect that only posing the designed problems will facilitate students' activities and mathematical learning, task design involves not only selection or development of problems but also teachers' instructional guidance related to the problems" (p. 472). In line with (a) deductive guessing in Lakatos's work, (b) the nature of proof problems with diagrams, and (c) the instructional guidance to be provided by the teacher, the researchers derived the following three task design principles:

- Educators and teachers should select or develop certain kinds of proof problems with diagrams where students can find counterexamples or non-examples and engage in deductive guessing through changing the attached diagrams.
- Teachers should encourage their students to change the attached diagrams while keeping the conditions of the statements, so that they find counterexamples or non-examples of the statements.
- After students face the counterexamples or non-examples, teachers should plan their instructional guidance by which students can utilize their proofs of initial problems to invent more general statements that hold true for these examples.

Komatsu and Tsujiyama illustrate their principles for task design by means of a problem involving parallelograms, drawn from Okamoto, Koseki, Morisugi, and Sasaki et al. (2012) (see also Komatsu, Tsujiyama, Sakamaki, & Koike, 2014). Their principle-based description of the design of the parallelogram task, accompanied by suggestions related to specific instructional guidance (see Komatsu & Tsujiyama, 2013, pp. 476–477), provides a detailed plan for the teaching of proof problems with diagrams, one that will eventually be subjected to further classroom implementation and possible revisions. Thus, the domain-specific frame crafted by Komatsu and Tsujiyama yielded, at this stage of their research, a primarily *design-as-intention* tool—a tool for task design that integrated earlier research on proof problems with diagrams, a novel theoretical frame based on Lakatosian deductive guessing, and a cultural tradition involving the role of the teacher.

*A Domain-Specific Frame for the Learning of Integer Concepts and Operations*

The design research of Stephan and Akyuz (2013) involved creating and implementing a hypothetical learning trajectory (HLT) and associated sequence of instructional tasks for teaching integers in a middle-grade classroom over a 5-week period. Grounded in the researchers' deep knowledge of past research on the learning of integers and integer operations, the design of their instructional sequence was underpinned by the following three heuristics of the intermediate-level frame of Realistic Mathematics Education (RME):

- Guided reinvention—"To start developing an instructional sequence, the designer first engages in a thought experiment to envision a learning route the class might invent with guidance of a teacher" (p. 510).
- Sequences experientially real for students—"Instructional tasks draw on realistic situations as a semantic grounding for students' mathematizations" (p. 510).
- Emergent models—"Instructional activities should encourage students to transition from reasoning with models of their informal mathematical activity to modeling their formal mathematical activity, also called *emergent modeling* (Gravemeijer & Stephan, 2002)" (p. 510).

The anticipated learning path (HLT) led to the generation of a six-phase instructional sequence involving various mathematical tools, which was then implemented in the classroom. The authors used a version of social constructivism, called the *emergent perspective* (Cobb & Yackel, 1996), to situate their interpretation of classroom events. In the emergent perspective, learning is considered both an individual, psychological process and a social process. Thus, two frames were used by Stephan and Akyuz to analyze their classroom data: (a) a framework for interpreting the evolving classroom learning environment, that is, the emergent perspective, and (b) a framework for interpreting student mathematical reasoning and learning of integer concepts, that is, a frame based on the instructional theory for Realistic Mathematics Education. After implementation and analysis of the collective learning of the class, the authors considered various possible revisions to the instructional sequence. The details of the design of the instructional sequence, its implementation, classroom analysis, suggested revisions, and reflective theoretical discussion can be found in Stephan and Akyuz (2012).

The description of the entire process, which constitutes an empirically sustained, domain-specific theoretical model for the teaching of integers and integer operations, is a classic example of *design as implementation*. In the spirit of Cobb and Gravemeijer (2008), Stephan and Akyuz generated a domain-specific, instructional theory that embodied the classroom-based, activity-oriented process of learning a specific mathematical content and which included a very detailed description of the representational tools, classroom interactions, and teacher interventions that sustained this learning. The elaboration of their domain-specific theoretical frame was supported explicitly in its design, implementation, and analysis by the two frames of Realistic Mathematics Education and the emergent perspective and implicitly by its reliance upon prior research and previous domain-specific design work on the learning of integers.

In describing their research, Stephan and Akyuz stress students' engagement with tasks: "In RME, … tasks are defined as problematic situations that are experientially real for students" (Stephan & Akyuz, 2013, p. 509), a perspective based on Freudenthal's assertion that "people need to see mathematics not as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (Stephan & Akyuz, 2012, p. 432). This emphasis on the way in which students engage with tasks, and the way in which teachers actually facilitate that activity, is central to *design as implementation*. It also helps to shed an explanatory light on Gravemeijer and Cobb's (2006) earlier statement that one of the primary aims of design research is *not* to develop an instructional sequence as such. More precisely, the description of the entire design process (including initial design, implementation, and revision) is intended to foster an understanding of why and how the final sequence is supposed to promote learning. The whole description supports others in implementing the sequence in other contexts and as such constitutes its theoretical role: that of a local instructional theory for a specific mathematical domain.

#### 2.2.2.5 In Drawing This Section to a Close

A main objective of this section of the chapter has been to examine the nature and roles of frameworks and principles in the design process and, at the same time, to draw out the relative centrality given to the design of the task or task sequence itself. Building upon the pioneering work of scholars during the early years of the growth of the mathematics education research community and its evolution through to design experiments and beyond, a double lens was used to explore the nature of current theoretical frameworks and principles for task design: (1) an analysis according to *levels of frames* that focused particular attention upon both intermediate and domain-specific frames and (2) a consideration of the constructs of *design as intention* and *design as implementation* within the design process. The lenses that were used, accompanied by a sampling of examples drawn from the international body of research literature related to design in mathematics education, helped to clarify some of the ways in which theory and task design are related. Among the relationships that emerged from the analysis of frames and their roles in design in general and task design in particular, one was particularly salient: it was the design consideration related to instructional support that was common to all the examples and central to each.

The examples all included attention to instructional support, some in the form of quite explicit principles. For instance, in the Prusak et al. (2013) example, a separate list of specific principles related to the design of the instructional environment was provided—principles that delineated a clear set of indices related to the way in which the instructional environment and the designed task were to mutually support each other. This example offers a viable model for further productive work in design in mathematics education and for its reporting. In fact, the way in which instructional principles were incorporated into the design of the studies exemplified so far in this chapter leads to suggesting that it might be more appropriate, terminology-wise, to refer to this field as *task design for instruction*. This more precise terminology could thereby give weight to the notion that the initial formulation of the design and its description include principles related to the design of the instruction and instructional environment, as well as to the design of the task. This terminology would also capture the spirit of the early research efforts in this area by Freudenthal, Bell, Wittman, and others. But even more importantly, integrating the terms *task design* and *for instruction* would allow us to emphasize that which would appear to be fundamental to design in mathematics education, a fundamental that was well expressed by Komatsu and Tsujiyama (2013, p. 472), namely: “It is unrealistic to expect that only posing the designed problems will facilitate students’ activities and mathematical learning; task design involves not only selection or development of problems but also teachers’ instructional guidance related to the problems.”

## 2.3 Case Studies Illustrating the Relation Between Frameworks and Task Design

### *2.3.1 Introduction*

The cases within this section illustrate the variety in types of frameworks for task design and the variety in relationships among the frameworks, design principles, and the actual design process. The heart of the discussion of each of the prototypical cases is guided by two questions:

(a) What do tasks look like when designed within a given theoretical frame or according to given design principles?
(b) Why do they look the way they do?

Using these questions as a lens, this section goes into some detail with respect to each example and thus extends the discussion that was initiated in the previous section. Seven cases are herein presented, most of them drawing upon aspects of grand theories and illustrating the use of intermediate-level theories. The cases are based on the contributions of the participants of Theme Group D (Frameworks and Principles for Task Design) of the ICMI Study-22 Conference (Margolinas, 2013). They are not intended to represent a sample of all possible design principles and frameworks that are currently used or investigated all over the world. The cases reflect the different levels of frames discussed in the previous section and illustrate how these frameworks can be applied across a variety of mathematical domains, as well as offer design approaches related to particular mathematical understandings. They also include cases that exemplify principles from design frames based on deep craft knowledge and from design related to various task genres, such as concept development and assessment. In sum, the seven cases being discussed, often too briefly to do justice to the richness of the underlying theory and the design, are intended to provide insight into the current state of the art of task design in mathematics education.

## *2.3.2 Cases*

### 2.3.2.1 Case 1: Anthropological Theory of Didactics

Within the ATD, mathematics is conceived as a human activity, institutionally situated, and modeled in terms of practices that go beyond learning "concepts" or "processes." This results in the need for a renewed paradigm of learning mathematics in school (Chevallard, 2012). The paradigm thus changes from *visiting mathematical concepts and skills* to *questioning the world* (motivated, functional encounters). This elaboration of the ATD has its roots in Chevallard's earlier ATD work of the 1990s, as well as in his collaborative research with Brousseau on the notions of didactic engineering, the didactic transposition, and the Theory of Didactical Situations.

The following example focuses on the application of an intermediate-level frame to the design of a mathematical activity involving young children. It illustrates design principles that are related to the previously mentioned paradigm shift. These principles were not extracted from this particular case but result from a collection of ATD study and research paths that have been designed in the last 10 years (e.g., Barquero & Bosch, 2015).

The aim of the task was to embed the emergence and use of numbers and addition in the study of a system that is real and that gives rise to a meaningful mathematical activity for (preschool) students (García & Ruiz-Higueras, 2013). The initial question for the students was *if we've got a box with silkworms, how many leaves do we need to feed them?*

Firstly, students would collect leaves by themselves. But after a few days, they would ask the gardener to collect the leaves for them, using a written message. That would provoke the need of being aware of quantities, as well as using codes to express them. Next, the biological system would start to evolve: silkworms turn into cocoons, then moths arise, and finally, they die. Students would have to control a heterogeneous collection made of silkworms, cocoons, and moths. As change happened, they would need techniques to record the evolution of the system. The teacher would prepare different tables to record and control the evolution of the system. She would introduce this tool so that students could take control of the evolution of the system under their own responsibility. This would widen students' activity, particularly toward addition, time control, and recording. At the end, when all the moths would have died, the system would disappear. However, students would have lots of information (models) about its evolution. Through the interpretation of these models, pupils would carry out the final task: reconstructing the system and its evolution. Figure 2.1 illustrates the unfolding of part of the task activity in class.

Designing tasks for a renewed paradigm of learning mathematics, from visiting mathematical works to questioning the world, is operationalized within the ATD by design principles for creating research paths for students (Table 2.1). The whole task, called a *study and research path* (SRP), is linked within the ATD to an epistemological conception of mathematics as a human activity and modeled in terms of practices.

Context: Taking care of our silkworms
Characteristics of this rich context:

- Dynamic system (evolving over time)
- Many different quantities to be measured
- Communicative tasks can be naturally formulated (representing quantities with numbers & numerals)
- Increasing complexity

**Fig. 2.1** Children taking care of silkworms (García & Ruiz-Higueras, 2013)

**Table 2.1** Design principles for a *study and research path* in ATD

| Design principle | Illustrated by the case |
|---|---|
| Develop an epistemological reference model for the mathematical activity the task is aiming at. Investigate how the mathematical objects of study are related, how they are articulated and used in specific (out-of-school) practices, and how these can be transposed into the educational system. | Numbers (as mathematical objects) and codes to express them (numerals) emerge in communicative situations where the aim is not just to measure a discrete set but to communicate about it so that another person can understand the evolution of the system without having access to it (neither visually nor manipulatively). |
| Look for generating questions beyond school mathematics that are crucial and alive for the students, connected with society and its problems (questioning the world). | "When we've got a box of silkworms, how many leaves do we need to feed them?" |
| Generate questions that do not lead the study process to a dead end but that give rise to new questions that could expand it. | How to communicate the number of leaves needed?<br>How to keep track of the number of silkworms, cocoons, and moths? |
| Create a collaborative and shared study process with shared responsibilities and shared norms for justification. | The teacher introduced tools so that students could take control of the evolution of the system under their own responsibility. |
| Support the search for answers by stimulating the study of (extra) mathematical works or consulting other communities. | Students were stimulated to ask parents about the time needed for the cocoon phase. |

For tasks like these, designers need to leave the school, step out of traditional school mathematics, and question the meaning of the objects they want students to work with (their origin, evolution, and purpose in current society). This leads to a reference model for the design of a study path and will inform possible overarching generative questions. Piloting is an essential phase in the design process for checking conjectured teaching and learning processes and for improving the ecological and economic robustness of the task.

#### 2.3.2.2 Case 2: Variation Theory

Variation Theory (VT) focuses task designers on what varies and what remains invariant in a series of tasks in order to enable learners to experience and grasp the intended object of learning (Runesson, 2005). The learners' experiences depend on the critical features of the object to which their awareness is directed. Consequently, designing task sequences requires an analysis of possible variations so that learners "might observe regularities and differences, develop expectations, make comparisons, have surprises, test, adapt and confirm their conjectures within the exercise" (Watson & Mason, 2006, p. 109). Analyses of variation space, patterns in learners' experiences, and how these patterns are compatible with the intended object of learning are key elements in the intermediate-level frame of VT.

Three successive versions of a task for teachers illustrate how VT guided a cyclic process of task design, analysis, and redesign. The learning objective of the task was to facilitate the teachers' awareness of mathematics as a connected field of study by directing their attention to structural similarities and differences among the basic concepts of analytical geometry and loci of points (Koichu, Zaslavsky, & Dolev, 2013).

The first version of the task consisted of 24 representations of loci of points that had to be sorted by the teachers by creating groups of similar loci (see Fig. 2.2). It was created so that three types of controlled variation would be maximized:

- The first type of variation was related to the mathematical objects described in the cards for sorting (e.g., a straight line, circles, parabolas, ellipses, hyperbolas).
- The second type of variation was related to the type of representation (symbolic, graphical, and verbal).
- The third type of variation was related to the type of experience needed to handle the task (prior knowledge, information provided with the task).

During the trial of this first version of the task, it was found that a lot of time was devoted to technical work and to classifying the items by surface features. To reduce the amount of time and the attraction of surface features, the second version consisted of 18 items. The items that were approached in all the groups only algebraically were excluded (items 8–10, 15, and 19–21). In spite of a smaller intended variation space, it appeared that the enacted variation space became richer and the teachers more engaged. However, the presence of the well-familiar graphical and symbolic representations in the task postponed, and likely hindered, the learning experiences offered by the verbal items. For this reason, pictorial representations were eliminated and the third version of the task contained only 11 verbal items

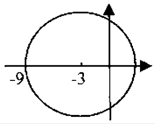
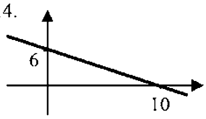
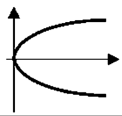
| | | |
|---|---|---|
| 1. Locus of points such that the ratio of distances from the points to point (8,0) to the distances from the points to straight line $x = 12.5$ equals 0.8. | 9. Locus of points resulting from multiplication of $y$-coordinate of the points of the circle $x^2 + y^2 = 100$ by 5/3. | 17. Locus of points such that the difference of the distances from the points to points (-8;0) and (8;0) is 20. |
| 2.  | 10. Locus of middles of the chords connecting the points on the parabola $y^2 = 40x$ with its vertex. | 18. Locus of points such that the difference of the distances from the points to points (-10;0) and (10;0) is 16. |
| 3. Locus of points such that the ratio of the distances from the points to point (10;0) to the distances from the points to straight line $x = 6.4$ is 1.25. | 11. Locus of points such that the ratio of the distances from the points to point (6;0) to the distances from the points to point (1;0) is 1.5. | 19. Locus of middles of the segments connecting the origin of the coordinate system with the points of hyperbola $\frac{x^2}{256} - \frac{y^2}{144} = 1$ |
| 4. $\frac{x^2}{10^2} + \frac{y^2}{6^2} = 1$ | 12. Locus of points such that the ratio between the distances from the points to point (3;4) to the distances from the points to point (-3;4) equals 1. | 20. Locus of points resulting from multiplication by 0.75 of $y$-coordinates of the points of hyperbola $\frac{x^2}{64} - \frac{y^2}{64} = 1$ |
| 5. Locus of points such that the ratio of distances from the points to point (5;0) to the distances from the points to straight line $x = -5$ equals 1. | 13. $\frac{x}{10} + \frac{y}{6} = 1$ | 21. Locus of points such that the ratio of the lengths of tangent lines from the points to circle $x^2 + y^2 = 64$ to the distances between the points and $x$-axis is 5/3. |
| 6. Locus of points such that the ratio of distances from the points to point (0;0) to the distances between the points to straight line $x = 0$ equals 1.25. | 14.  | 22. Locus of points such that the ratio of the lengths of tangent lines from the points to circle $x^2 + y^2 = 36$ to the distances between the points and $y$-axis is 0.8. |
| 7. Locus of points such that the ratio of distances from the points to point (0;0) to the distances between the points to straight line $x = 0$ equals 0.75. | 15. Locus of middles of the segments connecting points of circle $(x + 3)^2 + y^2 = 144$ with the center of the circle. | 23. $\frac{x^2}{8^2} - \frac{y^2}{6^2} = 1$ |
| 8. Locus of centers of circles belonging to the first quadrant, which are tangent to $y$-axis and circle $(x - 5)^2 + y^2 = 25$ | 16. Locus of points such that the sum of the distances from the points to points (-8;0) and (8;0) is 20. | 24.  |

**Fig. 2.2** The first version of the sorting task (Koichu et al., 2013, edu.technion.ac.il/docs/KoichuZaslavskyDolevThemeA_Supplementary_material.pdf)

[i.e., items 1, 3, 5–7, 11, 12, 16–18, and a new item 25: "Locus of points such that the distance from them to point (−3, 0) is 6."].

The intention of the third version of the task was to suppress the affordance of using sorting criteria based upon surface features, in favor of criteria related to the identification of structural similarities and differences. The experiences showed that the two main enacted subcategories of the by-keywords criterion in the third version of the task (i.e., by main operation and by main generating elements) were remarkably close to one of the intended types of variation of the task.

**Table 2.2** Design principles underlying a VT example

| Design principle | Illustrated by the case |
|---|---|
| Identify and analyze the object of learning and its critical features that constitute a variation space (Marton, Runesson, & Tsui, 2004). | The object of learning for the teachers is to facilitate their awareness of structural similarities and differences among the basic mathematical concepts of analytical geometry. A critical feature is to classify conics by names of loci of points, because this requires an understanding of structural similarities and differences. The variation space consisted of the mathematical objects, their various representations, and the types of prior experience needed to handle the task. |
| Create task(s) so as to have the learners discern critical aspects of the intended learning object and aim for coinciding the intended and enacted variation space. | Map the types of variation in the sorting task and connect them to the intended object of learning. The teacher-awareness facet of the study prompted a first version of the task where the space of variation was maximized. |
| Focus on the central role of the main intended activity (be careful with including mathematically challenging items and affording complementary mathematical techniques). | The central activity was discovering structural similarities and differences among the basic concepts of analytical geometry, but this was obscured by technical manipulations evoked by the first version of the task. |
| Carefully analyze whether the variation space of a task can be improved toward the intended object of learning. | The final version had a reduced variation space that was more engaging and resulted in richer learners' experiences. This version succeeded in suppressing sorting criteria by surface features, in favor of criteria related to the identification of structural similarities and differences. |

This example illustrates the process of task design guided by the interplay between analyzing and providing variation space and observing patterns in learners' experiences. Design principles drawn from this example in connection with VT orient the designer to what varies and what remains invariant in a series of tasks (Table 2.2).

This case, which exemplifies the use of the VT frame in task design, shows that design decisions can easily hinder or support affordances of a task with respect to the intended object of learning. The challenge for task design is to anticipate and organize learners' experiences so that they serve as reference points to more meaningful decisions.

#### 2.3.2.3 Case 3: Conceptual Change Theory

A particular issue for task design is the teaching of concepts that are known to be difficult for students because prior knowledge is in conflict with what is to be learned. Conceptual Change Theory (CCT) is an intermediate-level frame that allows researchers to specifically investigate this issue. The case, which is drawn from research into students' learning of nonnatural numbers (Van Dooren, Vamvakoussi, & Verschaffel, 2013), illustrates design principles that are derived from CCT, as well as from existing domain-specific research related to the learning of rational number.

Many difficulties that students have with nonnatural numbers are rooted in prior knowledge about whole (natural) numbers. The conceptual change perspective provides an explanatory framework for these difficulties as it analyzes them in terms of students' initial, intuitive theories that shape their predictions and explanations in a coherent way (Vosniadou, Vamvakoussi, & Skopeliti, 2008). This results in the following starting point for task design: How to deal with an incompatibility between students' initial theories and intended mathematical development that unavoidably will occur? The initial theories students rely on when encountering the ideas of nonnatural numbers are related to their understanding of whole numbers. Consequently, students see numbers as being discrete, used for counting, and grounded in additive reasoning (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2012). These initial theories easily lead to typical misconceptions like longer decimals are larger, for example, $2.12 > 2.2$; a fraction gets bigger when one of its parts is larger, for example, $2/5 < 2/7$; and the density misconception that between two non-equal numbers, there is a finite number of other numbers. Furthermore, students are often unaware of the background assumptions of their reasoning.

The sequence of tasks in this example takes these background assumptions into account and supports overcoming the incompatibility between the discreteness of whole numbers and the density of nonnatural numbers (see Table 2.3). The tasks were accompanied by the introduction of a tool-like representation that fostered

**Table 2.3** A sequence of tasks supporting conceptual change

| Task | Goal |
|---|---|
| 1. What do you know about the number line? Describe it as well as you can. Read and comment upon the answers of your fellow students. | Express prior knowledge about the number line. |
| 2. We often use the term "the set of all numbers". Suppose someone tries to understand what we mean by that. Could you draw a picture to help him/her understand? | Construct a representation for all numbers. |
| 3. Imagine the number line as a rubber band that can be stretched. Position 0 and 1 on the band and place numbers between them until it looks like you have used all the available points. If you stretch the rubber band, then you will find out there are more points, corresponding to more numbers. This procedure can be repeated infinitely many times—your imaginary rubber band never breaks! | Construct the imaginary rubber band as a representation for all numbers. |
| 4. We have been talking about two different representations of numbers: A "formal" one, which we usually use at school, and a second one, which was proposed in our discussion and you seem to find adequate. Could you find a solid reason why we should prefer one over the other? | Compare two different representations. |
| 5. Imagine that you can become as small as a point of the number line. Then you could see other points up close. Suppose that you are on the point that stands for the number 2.3. Can you define what point is the one closest to you? Describe in words or by drawing a picture. | Reason about density with the number line. |

**Table 2.4** Design principles for a Conceptual Change example

| Design principle | Illustrated by the case |
|---|---|
| Take students' prior knowledge and potential initial understandings into consideration (explore existing literature). | Build on students' prior knowledge of differences and similarities between natural and nonnatural numbers by explicitly addressing the number line (tasks 1 and 2). |
| Facilitate students' awareness of their background assumptions by creating opportunities for them to externalize their ideas, to compare them with peers' ideas, and to reflect on them. | Let students compare and discuss representations for all numbers (task 2). |
| Use models and external representations, know their power and their limitations. | The rubber band was introduced to prevent the number line from continuing to be interpreted as a ruler with a finite number of points (task 3, also task 4). |
| Foster analogical reasoning that supports conceptual restructuring. | The rubber band is a bridging analogy that fosters students' comparison between a continuous geometrical object and the real number line (task 5). |

reasoning with numbers in a geometrical analogy: an imaginary rubber band (Vamvakoussi & Vosniadou, 2012).

The sequence of tasks in this example illustrates how initial, usually largely unconscious assumptions can be elicited and made explicit. It also shows how cross-domain mapping between continuous magnitudes (points on the rubber band) and the set of numbers can be fostered (see Table 2.3). Design principles drawn from this conceptual change example are listed in Table 2.4.

CCT is primarily a cognitively oriented theory and therefore does not encompass all aspects related to instructional design. The design principles emerging in this case are intended as instructional tools to change, to move forward, students' cognitions. As such, this theory offers added value for task design when dealing with difficult concepts. The sequence of tasks in this example was designed on the basis of specific theoretical principles, was empirically tested, and appeared useful for teachers as well as for students. The resulting design principles have a wide field of application in that instructional design has to cope with similar prevailing misconceptions in many domains of mathematics and beyond.

#### 2.3.2.4 Case 4: Conceptual Learning Through Reflective Abstraction

This case (M. Simon, 2013) is derived from a one-on-one teaching experiment (M. Simon et al., 2010) with a prospective primary school teacher, Erin. The teaching experiment focused on developing a common-denominator algorithm for the division of fractions with conceptual understanding. Conceptual learning in this case is understood as the process of developing new and more powerful abstractions through activity. The approach draws task design principles from the grand-level frame of Piaget's construct of reflective abstraction.

*Solving Word Problems Using Rectangular Drawings*
1. I have seven-eighths of a gallon of ice cream, and I want to give each of my friends a one eighth portion. How many friends can I give ice cream to?
2. A scuba diver has two hours worth of air in her tank. If each dive to the bottom of the bay takes three-eighths of an hour, how many dives can she make with the air she has?
3. Each ticket at an amusement park in France is worth four fifths of a Euro. If a pack of tickets costs four Euros, how many tickets are in a pack?

*Solving Context-Free Problems Using Rectangular Drawings*
4. 3/4 ÷ ¼ =
5. 7/3 ÷ 2/3 =
6. 8/5 ÷ 3/5 =
7. 5/6 ÷ 4/6 =

*Solving Context-Free Problems Using Mental Runs of Rectangular Drawings*
8. 23/25 ÷ 7/25 =
9. 7/167 ÷ 2/167 =
10. 7/103 ÷ 2/103 =

**Fig. 2.3** A task sequence for learning to solve division problems with common denominators (M. Simon, 2013, p. 508)

The researcher engaged Erin in a sequence of tasks, probed her thinking, and allowed Erin to develop her understanding without input from the researcher. The task sequence began with division-of-fraction word problems whose dividend and divisors had common denominators. Erin was asked to solve them by drawing a diagram. She was able to solve the first task without difficulty ("I have $\frac{7}{8}$ of a gallon of ice cream and I want to give each of my friends a $\frac{1}{8}$-gallon portion. To how many friends can I give ice cream?"). The task sequence progressed to word problems in which the dividend and the divisor still had common denominators, but the divisor did not divide the dividend equally, and then to similar tasks presented using only number expressions (e.g., $\frac{8}{5} \div \frac{3}{5} =$). Erin still drew rectangles for solving the problem. For $\frac{8}{5} \div \frac{3}{5}$, she first drew two whole rectangles divided into fifths. Next, she shaded $\frac{2}{5}$ of one rectangle leaving $\frac{8}{5}$ unshaded. She circled each $\frac{3}{5}$, counted 2 groups, and was able to deduce that the remainder $\frac{2}{5}$ is $\frac{2}{3}$ of $\frac{3}{5}$, thereby finding the solution $2\frac{2}{3}$.

Next, Erin was asked to solve a more complex fraction division task $\frac{23}{25} \div \frac{7}{25}$ (see Fig. 2.3).

Erin made clear that she did not know the answer and the researcher encouraged her to talk through a diagram solution without actually drawing. Erin described the diagram process she would use and the result she would get. Erin easily solved the next task, $\frac{7}{167} \div \frac{2}{167}$, using the same approach, that is, narration of a diagram solution. However, when that task was followed by the task, $\frac{7}{103} \div \frac{2}{103}$, Erin gave the answer immediately. She realized that the change in the fractional units would not affect the quotient. Further, she was able to explain the invariance of the quotient across a range of denominators by creating a general diagram. Erin had made an

**Table 2.5** Design principles for Conceptual Learning by Reflective Abstraction

| Design principle | Illustrated by the case |
|---|---|
| Identify a potential activity that is already available to the learner and that can be the basis for the intended abstraction (the identified learning goal). | The student's informal diagram solutions supported anticipations toward a common-denominator algorithm for the division of fractions. This learning goal affected the identification of the solution strategy and the strategy affected the specific goal toward which the design was oriented. |
| Design tasks to elicit the available activity and to promote reflective abstraction (a learned anticipation supported by a shift from activities with external representations to mental runs). | The task sequence starts with word problems and context-free tasks to elicit and reinforce the diagram-drawing strategy. Once the student is using the intended strategy, the task sequence provokes the anticipated abstraction. For this purpose, larger numbers for the denominators and invited mental runs of diagram drawings were used. |

abstraction as a result of this task sequence. She perceived a commonality in her activity involving these mental diagram solutions.

In this example, two design principles fostering Conceptual Learning by Reflective Abstraction can be recognized (see Table 2.5).

When Erin was faced with a second task with the same pair of numerators and different common denominators, she realized that she was about to enact the same activity as in the previous task. At that moment, she also realized why the size of the common denominators did not change the quotient. This was an example of Erin's reflection on her (mental) activity. That is, she perceived the commonality in her activity in the two cases that led to an abstraction. These tasks helped her to foreground key quantitative relationships and to create a need to invoke a new concept and mental operations that are critical to the concept being developed.

This is an example of task design for concept development that does not depend on students making a leap through problem-solving. Rather, the task sequence affords them the opportunity to build an abstraction from already available activity. In this case, the abstraction was built from the activity of creating informal diagram solutions for solving simple sharing tasks. The approach illustrated by this example can serve to inform ongoing and future research work on the crafting of domain-specific frames for task design related to the process of mathematical abstraction.

#### 2.3.2.5 Case 5: Realistic Mathematics Education

Realistic Mathematics Education (RME) is an intermediate-level frame that has been developed in the Netherlands (see Van den Heuvel-Panhuizen & Drijvers, 2013). RME is rooted in the work of Freudenthal (1973, 1991) who argued for teaching mathematics that is relevant for students and instigated research in how students can be offered opportunities for guided reinvention of mathematics. This example illustrates design principles drawn from RME by presenting one task from a longitudinal sequence on the topic of percentage (Van den Heuvel-Panhuizen, 2003).

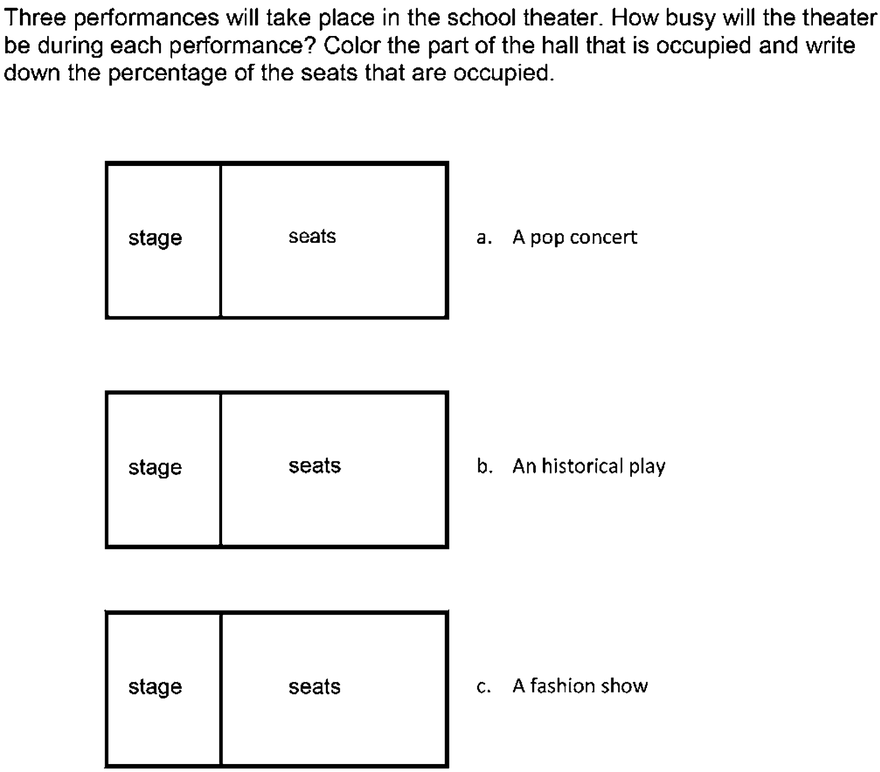



**Fig. 2.4** Percentage of occupied seats in a school theater (adapted from Van den Heuvel-Panhuizen, 2003, p. 19)

The learning of percentage is embedded within the domain of rational numbers and is strongly intertwined with learning fractions, decimals, and ratios. The example is taken from a sequence that starts with a qualitative informal introduction to percentages before it proceeds toward quantitative formal procedures. The underlying notion is that you first need to know what the procedures are about before you can perform and practice them. The introduction attempts to evoke the use of so-many-out-of-so-many reasoning in everyday situations. The design question from the RME perspective is *How to evoke and build on informal and outside-school knowledge of students when aiming at having them make sense of percentage?*

The introductory exploratory activities of the sequence are designed to make students aware of the daily life use of percentages, to evoke tentative representations, and to prepare for model building. The activities are cast in problem situations that "beg to be organized" by means of the mathematics under study (Freudenthal, 1983, p. 32). Some of these initial tasks are based on a school theater scenario. The students are asked to indicate for different performances how busy the theater will be. They can do this by coloring in the part that is occupied and then writing down the related percentage (see Fig. 2.4).

**Table 2.6** Design principles for an RME example

| Design principle | Illustrated by the case |
|---|---|
| Identify the fundamental concept, potential starting points, and models that support the learning of mathematics through a phenomenological didactical analysis, thought experiments, discussions with teachers, and working with students. | Starting points for the design of the sequence are the relative character in percentages (so many out of so many), the use of contexts like comparing the occupation of a school theater for various performances, and the bar model that supports the shift from intuitions to mathematical reasoning. |
| Model-eliciting activities are at the heart of an instructional sequence. They are cast in contexts that are familiar for students and provide relevant and challenging elements that need to be organized or schematized mathematically so as to have the potential to evoke their (informal) knowledge. | The theater context offers (limited) opportunities to be mathematically creative, to learn to solve problems for which the students do not have a standard solution procedure yet, and, at the same time, to learn about percentage. |
| A task sequence guides students from informal to formal mathematical reasoning. Models play a key role by shifting from a "model of" a particular situation to a "model for" mathematical reasoning (Streefland, 1993). | The drawings in the theater are expected to first become a "bar model of" so-many-out-of-so-many situations and later turn into a "model for" mathematical reasoning about percentages and fractions. |
| Take into account the design of skill development and connections with related mathematical topics to develop strong structures and procedures. | The notion of percentage is being taught in close connection with fractions, decimals, and ratios. A qualitative understanding precedes the development of quantitative skills. |
| Design whole-class and peer-to-peer interaction. | Whole-class discussion of students' answers, their drawings, and estimated percentages is essential for the progress of the teaching process (not included in the example task). |

This task is an example of an exploratory activity to support students in building models (i.e., the bar model) based upon their prior ideas and experiences. For the students, the coloring of theater halls is intended to lead to a way to express so-many-out-of-so-many situations. Furthermore, it is expected that students will spontaneously use fractions to "explain" the percentage of fullness. With a system of tasks, including more closed practicing tasks, students are guided to reinvent the mathematics of percentages.

This example illustrates core principles of RME that were articulated originally by Treffers (1987) but were reformulated over the years (see Table 2.6).

In recent years, new aspects, like mathematics in vocational education, in special education, and in linguistically diverse classrooms, have also been approached from an RME perspective. These projects enrich RME and enhance the robustness of the research that accompanies its further development.

#### 2.3.2.6 Case 6: Formative Assessment for Developing Problem-Solving Strategies

This case is drawn from the work at the Shell Centre at Nottingham University (UK) (i.e., Burkhardt & Swan, 2013; Mathematics Assessment Project[1]). The case illustrates how formative assessment can support the development of problem-solving strategies in mathematics. The power of formative assessment for enhancing learning in mathematics classrooms is well known (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Black & Wiliam, 1998).

Formative assessment includes "all those activities undertaken by teachers, and by their students in assessing themselves, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged; such assessment becomes 'formative assessment' when the evidence is actually used to adapt the teaching work to meet the needs" (Black & Wiliam, 1998, p. 140). Herein lies the real challenge: For assessment to be formative, the teacher must develop expertise in becoming aware of and *adapting to* the learning needs of students, both in planning lessons and in the moment-by-moment of the classroom.

These problem-solving lessons are not about developing understanding of new mathematical concepts but rather about students developing and comparing alternative approaches to nonroutine tasks. The structure of a typical lesson is illustrated with the *Counting Trees* task (Fig. 2.5). This task is intended to assess how well students are able to select an appropriate sampling method and use it, together with mathematical concepts such as area and proportion, to solve an unfamiliar problem.

In a preliminary lesson, students are invited to tackle the problem individually. They are told not to worry if they don't find an answer, that there are many ways to

The diagram shows some trees in a tree farm.
The circles ● show old trees and the triangles ▲ show young trees.
Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.
What method could Tom use to estimate the number of trees of each type?
Use your method to estimate the number of:
(a) Old trees (b) New trees.

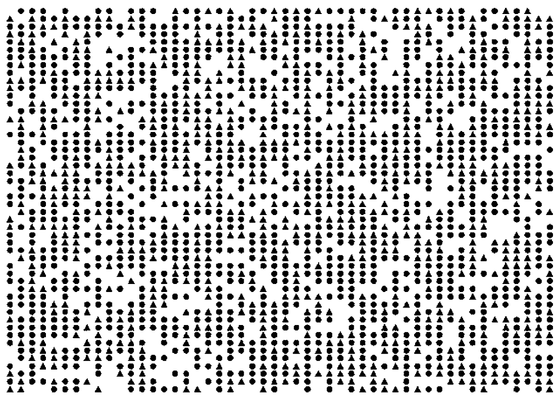


**Fig. 2.5** The counting trees task (adapted from MARS, 2012)

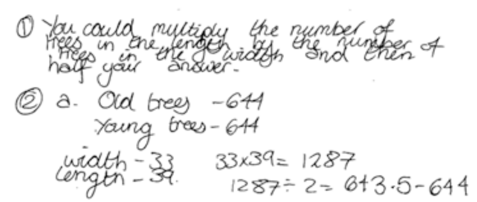
[1] http://map.mathshell.org

tackle the problem, and that there may be more than one correct answer. The task is used to expose students' different intuitive approaches to the problem. Students' responses are collected by the teacher and analyzed *before* the actual lesson. This gives the teacher time to plan well-considered responses to students.

The lesson itself begins with the teacher returning students' attempts along with questions (not explanations) that are intended to move their thinking forward. This role shift for students encourages them to reflect on their own strategy and to consider alternative methods. Instead of using the work of fellow students, the teacher introduces sample student work from materials provided. These samples are carefully chosen to highlight different approaches and common mistakes. Each piece of work is annotated with questions to focus students' attention. Figure 2.6 shows two examples of this work. The first (from Laura) contains some common mistakes that students make (ignoring gaps, assuming that there are an equal number of old and new trees), while the second (from Amber) introduces students to a sampling method they may not have considered. Introducing work from outside the classroom is helpful in that (1) students are able to critique it freely without fear of other students being hurt by criticism and (2) handwritten "student" work carries less status than printed or teacher-produced work and it is thus easier for students to challenge, extend, and adapt.

**Laura** attempts to estimate the number of old and new trees by multiplying the number along each side of the whole diagram and then halving. She does not account for gaps nor does she realize that there are an unequal number of trees of each kind.

*Can you explain why Laura halves her answer? What assumption is she making?*




**Amber** chooses a representative sample and carries through her work to get a reasonable answer. She correctly uses proportional reasoning. She checks her work as she goes along by counting the gaps in the trees. Her work is clear and easy to follow, although a bit inefficient.

*Can you explain why Amber multiplies by 25 in her method?*

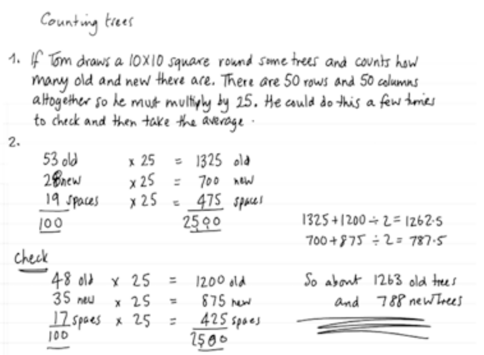



**Fig. 2.6** Sample student work with commentary for discussion (MARS, 2012© 2007–2012 Mathematics Assessment Resource Service, University of Nottingham reuse under Creative Commons License)

**Table 2.7** Design principles for Formative Assessment for Problem-Solving

| Design principle | Illustrated by the case |
|---|---|
| Tasks for formative assessment of problem-solving strategies need to be unfamiliar for students but at the same time offer opportunities to start the solving process in order to elicit students' different intuitive approaches. | The counting trees task is unfamiliar to students, but students can start reasoning using mathematical concepts related to area and proportion. |
| Follow-up activities are intended to support reflection on intuitions and to help all students to move their thinking forward. | The well-chosen sample work handed over to students encourages them to reflect on possible mistakes (Laura's work) and to consider more sophisticated methods (Amber's work). |
| Formative assessment includes offering students opportunities to revise and improve their initial responses (e.g., based upon individual feedback or feedback through sample work). | After evaluating and discussing sample work in small groups, the students get the opportunity to revise their initial responses to the Counting Trees task. |
| A sequence of formative assessment activities asks for an explicit reflection and conclusion on the content as well as on the problem-solving strategies. | The example lesson finishes with a whole-class discussion of students' revised responses so as to draw out lessons learned from the approaches used and about the power of sampling. |

After critiquing the work, students are offered the opportunity to refine their own approaches. This process of successive refinement in which methods are tried, critiqued, and adapted has been found to be extremely profitable for developing problem-solving strategies. The lesson concludes with a whole-class discussion that is intended to draw out some comparisons of the approaches used and the power of sampling.

Principles for task design underlying this example (Table 2.7) relate to both the design of the actual task and the supporting materials, including the student work and the lesson plan.

This example illustrates how a series of lesson activities (tackling the problem alone and then in groups, evaluating sample work, refining solutions, and whole-class discussion) may be designed to foster reflective, metacognitive behavior in which students step back from their own approaches and compare them with alternatives. The carefully designed nature of these lessons allows teachers to respond to student learning needs more sensitively and in a planned manner. The principles that are described for this example offer vital theoretical tools for designing formative assessment that can enhance the development of students' approaches to nonroutine problem-solving in mathematics.

#### 2.3.2.7 Case 7: Japanese Lesson Study

This case is drawn from a videotaped Lesson Study (Tejima, 1987) at an elementary school affiliated to the University of Tsukuba, the oldest normal experimental school in Japan. The case illustrates principles related to task design within Lesson Study, a "craft-based" intermediate-level frame, and shows how these principles

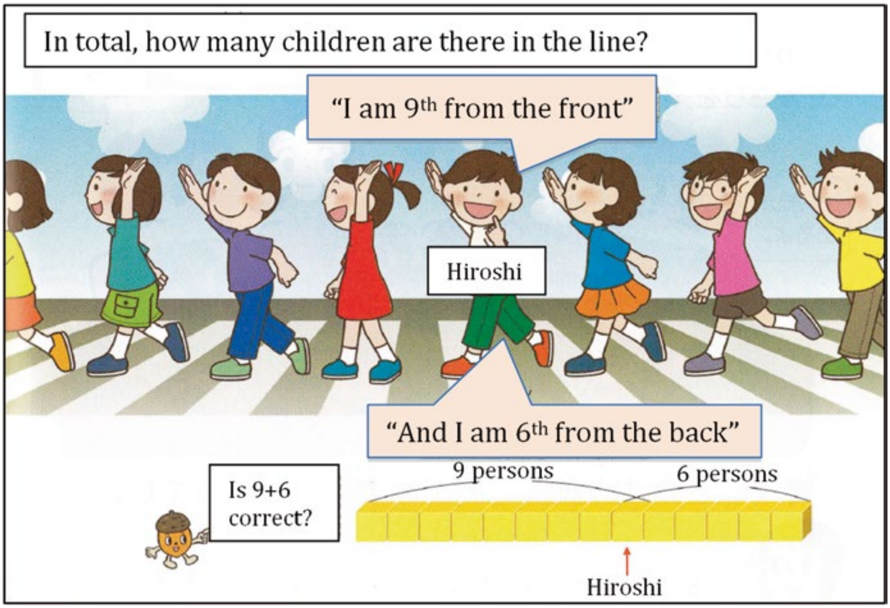



**Fig. 2.7** A textbook task (adapted from Sawada & Sakai, 2013)

encompass the various phases related to task design. These phases include an analysis of existing practices and instructional materials, a consideration of alternatives for reaching a new goal or for solving an educational problem, the actual task design, the teaching of the lesson, and the evaluation of the lesson and, in particular, the task (see also Sect. 2.2.2.3). As will be seen from this case, the task that is the basis for a Lesson Study can originate with a textbook task that is adapted for the research lesson.

This case focuses on how an expert mathematics teacher organized his research lesson. The task design challenge for this teacher was to support second-grade students in developing their understanding of addition and subtraction for ordinal numbers. So far, these children have learned to add and subtract in many contexts dealing with cardinal numbers. The first task in a textbook that uses a situation with ordinal numbers deals with a row of children (see Fig. 2.7).

The textbook task illustrated in Fig. 2.7 was originally preceded by the question: *There are twelve children standing in line. Hiroshi is the fifth from the front. How many children are standing after Hiroshi?* From a mathematical point of view, this question is easier than the task shown in Fig. 2.7. However, the teacher (Mr. Tejima) thought it important to start immediately with Fig. 2.7 task so as to challenge the children, to induce their naïve ideas (e.g., adding the numbers in the text), and to create opportunities for learning. The teacher also decided not to use the box representation of the textbook (i.e., the row of cubes with the double arches overlapping at the Hiroshi cube, in the bottom-right corner of Fig. 2.7). He wanted the students to think about this critical aspect of the problem for themselves.

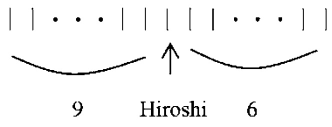



**Fig. 2.8** An intuitive (wrong) strategy and the emergence of a representation

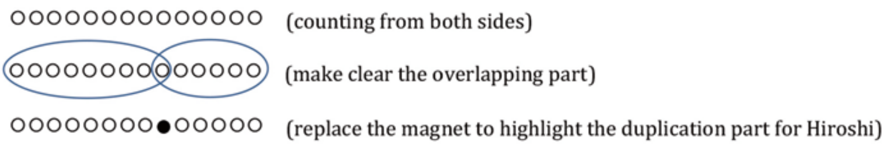



**Fig. 2.9** Several strategies resulting in 14 with variations on a row representation

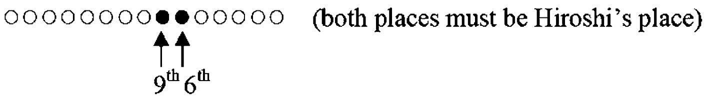



**Fig. 2.10** Explaining that 15 is wrong

Next, the teacher taught the lesson with the adapted task. He started by presenting the situation: *Children are standing in a line. Hiroshi is 9th from the front and 6th from the back*. Then the teacher asked the students to formulate a mathematical question for this situation. Next, the teacher took the question of the number of children in the row (the question emerged in the class) and asked all students to solve it. He assessed the students' answers as he circulated around the class, observing them working on the problem, and made a provisional plan for the following classroom discussion. The teacher intentionally picked up the dominant and wrong idea of $9+6=15$. He asked volunteers to explain their answer. One student explained the reason why 9 plus 6 equals 15. Then the teacher asked for an explanation from a student who thought the answer should be 16. The student illustrated his explanation on the blackboard (see Fig. 2.8).

In reaction to this idea, the proponents of 14 displayed their ideas in different representations with progressive sophistication (see Fig. 2.9).

Basically, the order in which the teacher nominated students to expose their reasoning was based on his provisional plan for the classroom discussion. In the course of this discussion, a student who usually struggled with mathematics said loudly: "I got it, I got it." He came to the blackboard by himself and explained the reason why the answer should be 14 (see Fig. 2.10): "Assume the answer is 15, there must be two Hiroshis and it is impossible."

This case illustrates how the alternate design generated by the teacher for the research lesson helped the students to come to understand the problem and to use a row representation for solving addition and subtraction tasks with ordinal numbers. The case also illustrates various design principles related to Japanese Lesson Study (Table 2.8).

**Table 2.8** Design principles related to Japanese Lesson Study

| Design principle | Illustrated by the case |
|---|---|
| An examination of existing practices and instructional materials. The identification of an issue worth studying and the design of an alternative task and a structured lesson plan (*kyozaikenkyu*). | An analysis of the textbook task and a consideration of possible alternatives. Rearrange the task to evoke multiple solutions and to support the students in developing correct conceptions of ordinal situations. |
| Teaching the lesson. Evoke students' naïve ideas and create opportunities for learning. Guide the students to critically analyze, compare, and contrast emerging ideas (*neriage*). | The teacher presents the task. After seeing how the students are solving the problem, he makes a provisional plan for them to share their work, starting with the more common, erroneous approach. Students present their reasoning at the blackboard and are encouraged to compare and discuss their ideas. |
| Lesson evaluation. | The lesson was videotaped and afterwards discussed with colleagues. During the discussion, the task and the effects of the initial task design were evaluated. |

The reorganization of the task sequence actually challenged the students and created opportunities for learning. The teacher's assessment of the students' individual naïve ideas allowed him to make a provisional plan for a whole-class discussion of possible strategies for tackling the task. His order of nominating students to expose and discuss their ideas supported all students in developing an understanding of the structure of the task situation and the emerging solution procedure.

### *2.3.3 Discussion*

The cases in this section illustrate a variety of design principles and frameworks for task design in relation to specific starting points or learning aims. Each case shows how task design can start from learners with particular characteristics and needs or from (new) knowledge, skills, attitudes, and competences that are aimed at. The application of the principles to a particular content area or mathematical topic renders specific the rather general principles and frameworks for task design. The resulting implementation of the starting frame is domain specific, and the design process tells the story of converting the general to the particular. As already remarked in the introduction of this section, it must be said that the cases are discussed too briefly to do justice to the richness of the underlying theory, to the whole design and design process, and to the context in which the task was designed and used. However, the cases reflect aspects of the current state of the art in task design and offer possibilities for reflecting on the two starting questions of this section:

(a) What do tasks look like when designed within a given theoretical frame or according to given design principles?
(b) Why do they look the way they do?

The cases describe a design challenge and how a framework for task design guides the design process and results in specific task characteristics. As such, these cases illuminate what is asked for with the first question. The second question is used to reflect on task characteristics and similarities and differences among the various cases. In some cases, the domain-specific implementation of a general framework is explicitly enriched by prior domain-specific findings. For example, the task design in the Conceptual Change case (case 3) is informed by previous research on students' conceptions of rational numbers.

All cases have in common a view of mathematics learning as being driven by doing mathematics. Especially, the ATD and RME cases (cases 1 and 5) emphasize the importance of interpreting mathematics as a human activity. Learning mathematics involves starting with students' current understandings and aiming at extending (e.g., by using rich realistic contextual problems in ATD and RME) their mathematical knowledge and skills in connection with their common sense understanding of everyday phenomena (e.g., questioning the world in ATD).

The size of the design problem being addressed is different in the presented cases. Some cases offer principles to solve a local problem in an existing task sequence. For example, the VT case (case 2) is oriented toward improving the anticipation and organization of learners' experiences in an existing task setting. The Japanese Lesson Study case (case 7) focuses on improving an existing textbook task. Still other cases describe the design of tasks as part of a task sequence that covers an entire topic. For instance, the ATD describes a task as part of a design for cardinal numbers (case 1) and the RME case describes one task in a task sequence on percentages (case 5).

With respect to characteristics of the resulting tasks, we can distinguish between the context of the task and the learning opportunities offered by the task. Two cases describe explicitly characteristics of the context for the task. The ATD case (case 1) stresses the importance of looking for a context that offers generating questions that go beyond school mathematics (e.g., how to take care of silkworms?). The context of the task in the RME case (case 5) has a slightly different focus. It stresses the notion that contexts should "beg to be organized" from a mathematical perspective and thus evoke solution strategies that have the potential to be mathematized (from "model of" a situation to "model for" mathematical reasoning). The case dealing with Formative Assessment for Problem-Solving (case 6) asks for unfamiliar contexts, contexts that do not immediately refer to well-known solution procedures but that also provide opportunities to start solving the problem and require processes like planning, representing, and collaborating.

All cases have similarities with respect to the learning opportunities offered by the task. They all stress the importance of tasks that create opportunities to build upon students' current understandings. For instance, the Conceptual Change example (case 3) and the Conceptual Learning example (case 4) both take students' current ways of reasoning into account. The Conceptual Change case takes an inevitable misconception as starting point, while the latter case starts from a potential activity that is already available for abstraction. In addition, most cases reflect the principle that tasks offer opportunities to share initial ideas and strategies. For instance, in the

Formative Assessment example (case 6), students are offered sample work done by other students; in the Lesson Study example (case 7), students are asked to share their ideas at the blackboard in an order that the teacher thinks will be most conducive to supporting all students in developing the intended understandings.

The cases show differences in the balance between learning opportunities aiming at mathematical content or aiming at more general regulative or motivational learning aims. For instance, the silkworm context in the ATD case (case 1) seems to be rather sophisticated in comparison with the mathematics involved, whereas the VT case (case 2) and the RME case (case 5) are explicitly oriented toward a specific mathematical object of learning. Design theories that focus on how an object of learning can be handled are important for helping teachers in their classrooms. In this respect, Variation Theory provides an effective instrument for design studies that also aim at promoting teachers' professional development (Cheng & Lo, 2013).

In several cases, tasks were designed so as to foster the development of representations and models that support the learning of mathematical concepts and skills, for instance, the rubber band for a number line in case 3, the diagram for reasoning with fractions in case 4, and the row representation for reasoning with ordinal numbers in case 7. This feature of tasks is further utilized in task sequences that foster the constitution of mathematics by exploiting didactical models that emerge from the activity of students, as in the development of the percentages concept in case 5.

Another aspect that arises when considering all cases is the degree of "challenge" offered to the students. Not all tasks have to be very challenging, but to foster students' learning, we need to provide them at some point with tasks that have (for them) some degree of challenge. This notion of challenge is not explicitly discussed in the cases presented in this section. It shows up in a somewhat incidental way in the discussion of VT (case 2) and of Japanese Lesson Study (case 7). Perhaps it is evident that tasks for students need to be challenging, but what "challenge" means for diverse classrooms with students having mixed abilities is not trivial at all. How to address this aspect in task design is an issue for future development related to design frameworks.

A theoretically important aspect that we can draw from the cases in this section is that the distinction between task design and lesson design is indeed blurred. In fact, the boundaries between them have been found to be extremely fluid. Almost all the cases presented in this section illustrated principles for task design that extended considerably beyond that of a task narrowly conceived as a question or sequence of questions proposed by a teacher or alternatively by a student. The task or task sequence, while treated as the main focus, was clearly conceived of within an orchestrated classroom activity—one where principles related to the actual classroom processes and instructional support that would make it possible to experience the potential of the task(s) were explicitly included as part of the task design. Design as intention was inherent to these cases, even if some of them could also be characterized by design as implementation. The current state of the art of task design in mathematics education would appear to suggest that designing a task or task sequence in isolation from consideration of the design of the instructional culture in which the task is to be integrated may be of quite limited value—somewhat analogous to expecting a bird to fly with just one wing.

Finally, in the process of moving from frameworks and principles to their actual application in the design of tasks, a great many decisions need to be made by the designer. How tasks look is largely determined by the hand of the artist! Nevertheless, these cases have shed some light upon the relation between these rather general starting points and the resulting tasks with their aimed-at learning processes.

## 2.4 Frameworks and Principles Do Not Tell the Whole Story in Task Design

### *2.4.1 Introduction*

The two previous sections have examined the design process, and task design in particular, from the perspective of the frameworks and principles that have underpinned much of the design-oriented research in mathematics education. The particular perspective that was used was that of grand, intermediate, and domain-specific levels of frames—a perspective that aimed at elaborating the ways in which frames and task design are related. However, such frameworks and principles do not tell the whole story of task design. Some of the scholars and reflective practitioners who engage in task design see it as being a much more eclectic activity than has been suggested thus far. In addition, several factors have not yet been accounted for in the design process, such as its artistic and value-laden aspects. With the aim of providing a more balanced picture of the state of the art, this section of the chapter addresses task design from a variety of other perspectives, including the tension between *design as science* and *design as art* and the lens of "basic" versus "applied" research. Elaborating on these various perspectives opens up the idea of the value of collaborative work across different groups and leads to a discussion of some of the recent collaborative efforts in task design across professional communities.

### *2.4.2 Additional Factors Related to Task Design*

#### 2.4.2.1 The Tension Between *Design as Science* and *Design as Art*

In the educational design community at large, there is a tension between centripetal and centrifugal forces. Centripetal forces urge stasis and system and desire consistency. Centrifugal forces urge change and feed the need for diversity (Clark & Holquist, 1986). The yin and yang of such forces are at work in the world of the educational design community. Schein (1972) characterized science and practice as "convergent" and "divergent", respectively, and remarked that there is a gap between the two. Schoenfeld (2009) saw a similar gap between the theoretical aims of educational research and the practical aims of designers and consequently recommended the unpacking of designers' productive practices and a sharpening of the notion of *professional vision*—an elaboration that would be of value not only to the design community itself but also to educational design researchers. In addition, he pointed to the mutual benefits to be derived from the collaboration of the educational researcher and the designer-practitioner.

In contrast to proponents for delineating the explicit and rational frameworks and principles for task design, some colleagues insist that educational design research cannot help to sharpen the notion of professional vision. According to de Lange (2012), educational design research hardly offers usable knowledge for designers and practical suggestions for design nor does it offer a theoretical underpinning for educational design at the microlevel. De Lange (2013; also Chap. 10) argues that this limitation of theory is due to the artistic aspects of task creation. He underlines his reasoning by quoting Hilton (1976):

> Since mathematics [analogous to educational design theory/science] incorporates a systematic body of knowledge and involves cumulative reasoning and understanding, it is to that extent a science. And since applied mathematics (analogous to the actual practice of designers) involves *choices which must be made on the basis of experience, intuition, and even inspiration, it partakes the quality of art*. (p. 95, emphasis added)

It thus comes as no surprise that some educational designers prefer the powers that inhere in centrifugal forces: design activity in flux, simultaneity, diversity, and heterogeneity. Moreover, according to Schunn (2008), the educational design community has no communal mechanisms for codifying craft knowledge. Codifying design thinking is said to threaten its central value of flexibility (Collopy, 2009). Schön (1983) explicitly challenged the positivist doctrine underlying much of the *design science* movement and offered instead a constructivist paradigm. He criticized H.A. Simon's view of a *science of design* for being based on approaches to solving well-formed problems, whereas professional practice throughout design and technology and elsewhere has to face and deal with "messy, problematic situations". As pointed out by Cross (2001), Schön proposed instead to search for "an epistemology of practice implicit in the artistic, intuitive processes which some practitioners do bring to situations of uncertainty, instability, uniqueness, and value conflict" and which he characterized as "reflective practice" (p. 54).

Based on his personal reflection, including experiences with the HEWET project (1981–1985), which involved designing a quite new Dutch secondary mathematics course (*Wiskunde A—Mathematics A*) for humanity and social majors, de Lange (2013) describes what he refers to as a "slow design process." A *slow design process* involves several cyclic stages with rich partnership among researchers, designers, teachers, and students. It includes: selecting the subject, duration, and level; designing a mental sketch of flow while using intuitions; choosing a context for mathematization with the help of inspirations gained from random search; refining the design for a classroom experiment; discussing with experienced teachers; observing classroom activities and checking students' reactions while walking around; taking discrepancies between "intended" and "achieved" seriously; and concentrating on essential conceptual development. According to de Lange, slow design is possible under the following conditions: freedom of choice of what to design, freedom in time, freedom of thought, and freedom to explore with certain restrictions according to contextual and theoretical conditions.

To illustrate the slow design process, de Lange (1979) has described an example related to the topic of "Exponentials and Logarithms" for humanity and social majors, the design of which was guided by the philosophy of Realistic Mathematics Education. He progressively designed a task situation that functions as a model for a mathematical concept (de Lange, 1987). In this situation, "propagation of water plants" is chosen as the introductory task situation and the concept of logarithm is defined by growth factor and time: $\log_3 10$ is defined as the time needed to get 10 times the spread of water plants when the growth factor per month is 3 (i.e., a bit more than 2 months). With this situation and language in mind, students can interpret basic logarithmic relations such as $\log_3 10 + 1 = \log_3 30$ as follows: with this 1 extra month, you get 3 times more than 10, which equals 30. Experimental textbooks were developed to elaborate, try out, and optimize this approach (de Lange & Kindt, 1984); eventually, the approach entered Dutch curriculum descriptions and was adopted by commercial textbooks (e.g., Boer et al., 2004, p. 30).

De Lange (2013) argues that such a design and implementation process asks for slow design. It illustrates the need for extensive design processes that can do justice to both scientific and artistic aspects of task design. The tension between *design as science* and *design as art* is not easily solved, if it can be solved at all, and emphasizes a reconsideration of the time allocated for task design in educational research and in curriculum innovation projects.

#### 2.4.2.2 Values in Task Design

Frameworks and principles for task design will vary relative to philosophies of mathematics education. Different philosophies of mathematics education mediate different values with respect to task design. Ernest (1991) distinguishes four sets of issues related to one's philosophy of mathematics education: the philosophy of mathematics, the nature of learning, the aims of education, and the nature of teaching. In this regard, Burkhardt (2014) points out that different groups of people have different priorities with respect to curricular aims or goals in mathematics: "basic skills people", "mathematical literacy people", "technology people", and "investigation people". Likewise, Treffers (1987) distinguished four trends in instructional approaches to mathematics in terms of "horizontal" and "vertical" mathematization: mechanistic, empiricist, structuralist, and realistic, with each instructional approach drawing upon different psychological backgrounds—Gagné's cumulative learning for the mechanistic, Piaget's constructivism for the empiricist, Bruner's modes of representation for the structuralist, and Gestalt psychology for the realistic.

The role of values in task design is illuminated by contrasting the approaches described in two recent studies that were presented at the ICMI Study-22 Conference on Task Design: *Concept-development task design* by Koichu et al. (2013) and *Competence-based task design* by Aizikovitsh-Udi, Clarke, and Kuntze (2013).

In the first study, based on Variation Theory, the task designer values delineating a variation space for the intended object of learning by eliminating or excluding hindering experience factors. This is done to direct the learner's attention to certain aspects that constitute the defining characteristic of the concept. In the second study, the competence-based task design proposes the idea of a "hybrid task" that stimulates different forms of thinking through a single task: the discipline-specific thinking of statistics and more generic forms of higher-order thinking, such as critical thinking. A hybrid task is characterized as having a structure that can offer to the learner superfluous and sometimes contradictory information. These two examples serve to illustrate that the frames and principles used in task design are intimately related to aims of mathematics education, which can in turn prioritize either the acquisition of conceptual and procedural knowledge or competence in dealing critically with information. A somewhat different perspective on the role of values in task design is exemplified by the explicit integration of "educating for values" into the teaching of mathematics. For instance, Movshovitz-Hadar and Edri (2013) conducted a multifaceted study to investigate the possibilities of combining social and personal values like equity, tolerance, social justice, rationality, and achievement and reaching one's intellectual potential—all within a designed approach to learning mathematics (see Chap. 5 for an elaborated example).

### *2.4.3 Diversity of Design Approaches Through the Lens of Basic Versus Applied Research*

#### 2.4.3.1 A Two-Dimensional Scheme for Classifying Research

In describing diverse approaches to task design, a useful perspective is offered by Schoenfeld's (1999) text on the synergy between theory and applications. Here he discusses the productive dialectical relationship between pure and applied work in education and makes use of Stokes's (1997) two-dimensional scheme of research in science and technology (see Fig. 2.11). Despite its formulation for the field of science, its applicability to the area of task design in mathematics education makes it of interest, especially with respect to situating the purely artistic position on task design as well as informing a potential bridging between the design-as-science and design-as-art tensions that were previously discussed.

In the two-dimensional representation, Niels Bohr and Thomas Edison are located as paradigmatic figures of pure basic and pure applied research. Louis Pasteur is located differently, as the paradigmatic figure of "use-inspired basic research": he not only engaged in germ theory for solely basic biological interests but was also motivated by problems of spoilage of drinks and curing diseases.

By its nature, educational research generally and educational design research especially aim at conducting "use-inspired basic research." According to McKenney and Reeves (2012): "Educational design research describes a family of approaches that strive toward the dual goal of developing theoretical understanding that can be of

Research is inspired by:

| | | Considerations of Use? | |
|---|---|---|---|
| | | No | Yes |
| Quest for Fundamental Understanding? | Yes | Pure Basic Research (Bohr) | Use-Inspired Basic Research (Pasteur) |
| | No | | Pure Applied Research (Edison) |

**Fig. 2.11** Stokes's (1997, p. 73) quadratic scheme for categorizing science research (Copyright 1997 by the Brookings Institution)

use to others while also designing and implementing interventions to address problems in practice" (p. 17). However, in the development of educational research, it may be difficult for any work to contribute simultaneously to both theory and practice: "Sometimes the state of theory is such that it may best be nurtured, temporarily, aside from significant considerations of use; sometimes the need to solve practical problems seems so urgent that theoretical considerations may be given secondary status" (Schoenfeld, 1999, p. 9). In his seminal work on problem-solving, Schoenfeld (1985) describes the dialectical relationship of give and take between theory and practice. Purely theoretical research in a laboratory setting can suggest some substantial ideas for designing practical courses in problem-solving and vice versa; actually teaching a course can raise theoretical issues to be pursued in an experimental setting. In order to have a relatively comprehensive "use-inspired basic research", it is necessary to move between such carefully designed laboratory settings and those settings that represent daily teaching practice.

#### 2.4.3.2 Design Frames in the Light of Distinctions Between Basic and Applied Research

The principles and frameworks for task design that were described in earlier sections of this chapter in terms of levels of, or rootedness in, theory could alternatively be characterized according to their situatedness with respect to basic and applied research. In this spirit, we explore the use of the Stokes two-dimensional scheme as a lens for reflecting upon some of the various task design frames that were presented in Sect. 2.3 but in an alternative, albeit complementary, light.

Adherents of Variation Theory (VT) can be associated with the "basic research" cell of Stokes's two-dimensional scheme. A paradigmatic example is the research aimed at identifying the critical features of designated objects of learning and at ensuring that the designed task situations impart these critical features (Koichu, 2013). As pointed out by Cheng and Lo (2013), designers "must first identify a worthwhile object of learning and the critical features that the students must discern in order to see the object of learning in the intended way; they would then design patterns of variation (what to vary and what to keep invariant) to help the students to discern the critical features/aspects" (p. 10). Clinical observations in a laboratory setting are often used for identifying the affordances of task variations (see the VT case in Sect. 2.3). Such studies illustrate the complex relationships among the intended, enacted, and lived objects of learning and the need for clinical research settings to investigate these relationships. Task design for Conceptual Change (Van Dooren et al., 2013) and Conceptual Learning through Reflective Abstraction (M. Simon, 2013) are also associated with "basic research." These two frames rely on laboratory and clinical settings for studying the kind of experiences a task affords and the extent to which those experiences are beneficial for conceptual change and salient abstraction, respectively.

The Japanese Lesson Study (LS) approach to task design, which is based on deep craft knowledge and expertise, can be located in the "pure applied research" cell. It does not aim at developing substantial theoretical understanding. Rather, LS aims at teachers' professional development through *kyozaikenkyu* (Fujii, 2013): building up insights into children's learning trajectories, decision-making competency with respect to carrying out tactical interventions during classroom interactions, organizing provocative discourse, establishing a productive classroom microculture, and so forth. The setting of LS in daily teaching practice mediates the activity of the various participants involved in LS and leads to the co-construction of deep craft knowledge.

Research frames such as the ATD and Realistic Mathematics Education (RME) can be viewed as paradigmatic examples of the "use-inspired basic research". Both paradigms can contribute simultaneously to theory and practice—the contributions to theory occurring especially during the early period of development of these intermediate-level research frames, as well as during their later application phases when the intermediate-level frame is particularized to the learning of domain-specific concepts and processes. ATD-based task sequences, or study research paths, are developed and implemented across many years of schooling from pre-primary to university. Many of the ATD-based tasks are characterized as open-ended mathematical modeling activities that address social issues. Likewise, RME has served in the development of different types of curriculum projects at all school levels varying from kindergarten (e.g., van Nes & Doorman, 2011) to upper secondary education (e.g., Doorman & Gravemeijer, 2009). As well, rich RME-based, problem-solving, assessment tasks (A-lympiade and Math B-day) are elaborated and implemented annually by teachers in their daily practice (Goddijn, 2008; Goris, 2006).

As a means of fostering further development of design principles that might contribute simultaneously to theory and practice, as well as exploring whether in fact some of the more theory-oriented and more practice-oriented frames for task design are in fact amenable to joint articulation, some researchers (e.g., Artigue, Cerulli, Haspekian, & Maracci, 2009; Artigue & Mariotti, 2014; Kieran, Krainer, & Shaughnessy, 2013) have already begun to study the ways in which different groups, cultures, and communities work together productively on advancing such issues. As has been suggested by the examples in the previous paragraphs, basic laboratory research in VT, Conceptual Change, and Concept Learning can reveal some substantial ideas of potential interest to designers with a practical research orientation. Conversely, difficulties and dilemmas that have emerged in actual Lesson Studies raise salient theoretical issues that could be pursued in less complex clinical settings. Immediately below, we address collaborative work across professional communities, work that has begun to grapple with the theoretical-practical interface of design activity.

## *2.4.4 Design Activity Across Professional Communities*

### 2.4.4.1 The Stakeholder Approach as a Foundation for Thinking About Collaborative Design Activity

In order to realize collaboration across the research and teaching communities (as well as collaboration involving the educational researcher and the educational designer-practitioner), a wise strategy is to establish a transparent context between researchers and practitioners, not forgetting that some practitioners are researchers themselves. A provisional theoretical perspective would be Krainer's (2011) notion of the *stakeholder approach*, which avoids privileging theory over practice in the design process. According to Krainer, the term *stakeholder approach* is intended to capture the idea that teachers are key stakeholders in the design research enterprise, not mere users of research. It is teachers who are in a position to achieve one of the main purposes of that enterprise, which is the improvement of students' learning of mathematics. Developing a stakeholder approach is central to establishing the kind of collaboration between these two communities that will facilitate mathematical learning with rich task design. Krainer asserts that researchers should highlight teachers' reflective and creative practice and offer viable opportunities that encourage them to get interested in being involved in such research. The stakeholder approach asks of task design not *what* but *where* it is. With this approach, task design is situated in the interaction between practitioners and researchers.

### 2.4.4.2 Task Design Involving Practitioners and Researchers

In Sects. 2.2 and 2.3, our attention was fixed on the nature of the frameworks and principles used in the activity of task design research, without focusing on the nature of collaborative work in this area. We now take a closer look at this aspect and discuss some recent design efforts involving cross-community collaboration. A few of the research papers presented at the ICMI Study-22 Conference on Task Design reflected a rethinking of the boundaries between theory and practice and the relative roles of researchers and practitioners (e.g., Ding et al., 2013; Morselli, 2013; Ponte et al., 2013; Stephan & Akyuz, 2013). Ding and her colleagues report on the process of design and implementation of tasks within a team consisting of academic researchers, teachers, and a teacher educator who was also an expert teacher, in a school-based teacher professional development program in Shanghai, China. In their report (Ding et al., 2013), they highlight, in particular, the role played by the expert teacher, who contributed to the development of a "hypothetical learning structure" for a particular topic (decimal value) and to creating tasks within a web-like structure of knowledge constructions.

Morselli (2013) describes a collaborative project aiming at designing, experimenting, and refining task sequences for a smooth and meaningful approach to proof in lower secondary school in Genoa, Italy. The project was supported by an initiative of the Italian Ministry of Education aimed at fostering and stimulating young students' interest in studying science. Within this context, collaborative work between university researchers and school teachers was set up. Teams were created for each school level and these teams met regularly in order to share theoretical references on argumentation and to discuss theoretical tools and their didactical and methodological potential. Productive cycles of task design, experimentation, analysis, refinement, and modification emerged.

Ponte et al. (2013) address the design of exploratory tasks that were developed and implemented in collaboration between researchers and a group of teachers in Lisbon, Portugal. A new mathematics curriculum for basic education required teachers to develop and use exploratory tasks designed to support students' mathematical reasoning and the growth of their problem-solving abilities. With such an institutional context, developmental work on task design was conducted by using a combination of research expertise and classroom teaching expertise. The team started with an overall plan for a teaching unit, which included the formulation of the learning objectives, assumed previous knowledge of students, time available, and organization of a schedule. Tasks were later selected to fit the overall planning of the teaching unit, followed by a dialectical movement of adjustments at the macrolevel of the unit and at the specific level of the tasks. Usually, the first idea for an exploratory task was provided by a classroom teacher, and the subsequent refinement was carried out in interaction with the other teachers and researchers.

Stephan and Akyuz (2013) describe a design study involving a collaborative community in one middle school, consisting of two mathematics teachers, a special education teacher, a researcher, and a graduate student. After the researcher introduced the main idea of the hypothetical learning trajectory (HLT), the members of the collaborative group worked together to create a six-phase instructional sequence, based on RME heuristics, for the learning of integers and integer operations (see also Sect. 2.2.2.4). The community met on several occasions before instruction began and then almost daily throughout the implementation. The pivotal contributions of various members of the group included anticipating supportive mathematical imagery, creating challenging formative assessment, using their mathematical knowledge to alter the instructional sequence, and working and revising already created tasks or the sequence of the instruction. This collaborative research showed that, "during the implementation of the instruction, the practitioners began to discuss more theoretical issues while the researchers began to think more about teaching practices" (Stephan & Akyuz, 2013, p. 515).

In introducing this discussion on cross-community design activity, we were reminded of Krainer's (2011) elaboration of the stakeholder approach and the related query that it is "not *what* task design is, but *where* it is situated" that needs to be considered. The design projects that have just been exemplified and which integrated the cross-professional communities of practitioners, researchers, and, in some cases, teacher educators allow us to respond more fully to this query. Sections 2.2 and 2.3 situated task design in the nexus between principles and frameworks and their application to particular content areas or mathematical topics. The above collaborative projects have succeeded in showing that task design is also situated both in the interactions among its cross-community participants and in the interface between theory and practice.

### *2.4.5 Toward the Resolving of Perceived Tensions*

By examining a variety of alternative perspectives, this section of the chapter has touched upon a range of additional factors affecting task design and its diversity—factors that might suggest a certain inherent tension between opposing forces—but at the same time has offered avenues for resolving the perceived tensions. These alternative perspectives have allowed us to see that the structured frameworks and principles that characterize much of the design research in the mathematics education research community do not capture the eclectic nature of design activity as engaged in by some of its scholars. For example, some of the proponents of *design as art* espouse a quite different set of starting points from the proponents of *design as science*. At the heart of this tension, as we have noted, is Schön's (1983) criticism of H.A. Simon's view of a *science of design* as being based on approaches to solving well-formed problems, whereas professional design practice has to deal with "messy" problems. With positivist approaches to design practice being found to be of limited utility during the 1980s (Lincoln & Guba, 1985), design had to be reconsidered as a process in which uncertainty must be grappled with. Artigue (2009) reminded us of this when she remarked that theory-based intervention programs have faced difficulty because of the many design factors that are not under theoretical control.

As was seen, one of the ways of approaching this dilemma is to consider the practitioner as a full actor in the design process. The stakeholder approach (Krainer, 2011) embodies this perspective but also acknowledges that there are significant differences in the guiding principles specific to different communities. Stokes's (1997) two-dimensional scheme enabled us to situate various orientations in design research, including design activity that is motivated as much by artistic as by theoretical concerns. As was also seen in this section, frameworks and principles constitute a communal practice of task design, with innovation and use of specific frameworks and principles for task design being a reflexive activity. By means of interaction among diverse cultures and communities, frameworks and principles are progressively developed in the light of task implementation. Therefore, the interactions between diverse communities and the concomitant grappling with diverse design principles would seem crucial to moving forward in the area of design. However, such interactions may not be straightforward or easy to orchestrate. As pointed out by Artigue and Mariotti (2014) in their discussion of the networking efforts engaged in by researchers from different cultures in the ReMath project: "[When] the possibilities of networking are examined in terms of potential for guiding design, … the activity is much more demanding … [but can be] especially insightful; … such advances are especially important considering that design in mathematics education lies at the interface between theory and practice" (pp. 350–351). Although collaboration involving the diverse actors engaged in the enterprise of task design in mathematics education may be challenging, the process can yield not only an enhancement of the quality of the designed tasks and task sequences but also a narrowing of the perceived divide between design as artful practice and design as theory building. Recall Stephan and Akyuz's (2013) earlier remark: "It is interesting to note that during the implementation of the instruction, the practitioners began to discuss more theoretical issues while the researchers began to think more about teaching practices" (p. 515).

## 2.5 Concluding Discussion: Progress Thus Far and Progress Still Needed

The objective of this chapter was to give an overview of the current state of the art related to frameworks and principles for task design so as to provide a better understanding of the design process and the various interfaces between teaching, researching, and designing. The chapter started with a description of the history of task design in mathematics education. The 1970s reflected the beginnings of the new community of mathematics education researchers' efforts to grapple with the interaction between curriculum materials and the quality of mathematical teaching and learning. We noted, for example, that Alan Bell was one of the first colleagues who explicitly referred to the importance of design principles for the transition from a situation that embodies the concepts and relations of the conceptual field to the design of tasks that bring into play these concepts and relations.

What progress have we—as a community—made over the past four decades? This chapter has described in which directions we have made some progress in understanding and articulating aspects related to task design. These aspects have included aims, levels, communities, and values that influence and are influenced by frameworks and principles for task design in educational practice and in educational research. Topics that were addressed related to levels of frameworks for task design, the distinction between theories as resource for and as product of design research, the tension between design as science and design as art, and the relations among the professional communities that develop and use specific frameworks for task design. So what have we learned about this field, about the topics, and about ourselves, as a result of coming together at the ICMI Study-22 Conference and developing this chapter on frameworks and principles?

One aspect that became more clear concerns the nature and levels of the frames that guide the process of task design. Using the lens of grand, intermediate, and domain-specific levels allowed us to see that our frames tend to be either holistic or multidimensional in nature. That is, the inspiration for our designs can come primarily from one quite global, intermediate-level framework (e.g., TDS, ATD, VT) or from a constellation of theories of different levels and different types (e.g., the various examples of domain-specific frames for the learning of particular concepts or processes). We saw that drawing from a combination of theoretical foundations can present advantages that may not be available when we rely on just one overall frame and its design tools—advantages such as being able to delineate not only a broad set of principles for the design of tasks or task sequences but also a related set of principles for the design of the instructional culture in which the task is to be integrated. In fact, a significant number of the task design studies presented within the conference theme group on principles and frameworks relied upon principles for task design that extended considerably beyond that of a task narrowly conceived. While the task or task sequence was seen as being central, it was clearly viewed as taking place within an orchestrated classroom activity—one where principles related to the actual classroom processes and instructional support that would make it possible to experience the potential of the task were explicitly included as part of the task design. Thus, the distinction between task design and lesson design was found, indeed, to be quite blurred.

Another aspect that has emerged is that theories are both a resource for and a product of the design process. As a resource, they provide theoretical tools and principles to support the design of a teaching sequence. As a product of design research, theories inform us about both the processes of learning and the means that have proven to be effective for supporting that learning. Related to this dual role of theory is the distinction between *design as intention* and *design as implementation*—*design as intention* addressing specifically the initial formulation of the design and *design as implementation* focusing attention on the process by which a designed sequence is integrated into the classroom environment, subsequently refined, and then theorized about. This distinction highlights the relative nature of the significance given to the design of the task sequence or task itself within the design process.

Although the major part of this chapter has been devoted to the theoretical frames that underlie task design, not all design is based on theory. The Lesson Study frame is a classic example of craft-based task design based on teaching practice, one where teachers with their deep, experiential knowledge are central to the process. Fundamental to teachers' ability to design, implement, and study high-quality mathematics lessons is a detailed, widely shared conception of what constitutes effective mathematics pedagogy and professional development. The planning of the research lesson, which is the main component of Lesson Study, includes not only the task and its materials but also anticipated student thinking, the teacher's planned questioning and intervention activities, and the points to be noticed and evaluated.

At the same time, we have become aware that the grain size for describing principles for task design is an area for further reflection and development. While the cases presented in this chapter took account of principles related to grand, intermediate, and domain-specific levels of theories, as well as instructional and tool-related principles, the work of the educational designer Kali (2008) suggests the feasibility of considering, and possibly integrating, a much finer grain size of levels of principles into our design work. We are reminded of the critical question that, according to Cobb et al. (2003), must be asked of our frames, that is, whether their principles inform prospective design and, if so, in precisely what way. As seen in Sect. 2.3, current work in task design indicates that there is a great deal of variation in the nature of the principles and heuristics being adopted for task design, with only a few points of convergence across the broad set of principles informing task design. Many of the principles tend to be phrased in rather general terms that are subject to broad interpretation and thus cannot be said to inform prospective design in highly specific ways. Clearly, further theoretical work on grain size of principles for task design is needed. For example, in applying general task design principles to the learning of particular mathematical content areas or reasoning processes, being more explicit with respect to the way in which past research in that area is being woven into the design of the task or task sequence would surely be useful.

Tasks play a crucial role in forwarding the process of improving the educational system. While, for instance, in the mathematics education community, competencies like creativity, critical thinking, and problem-solving are highly valued, tasks presented by high-stakes examinations tend to address basic skills. Such examination tasks largely determine the types of tasks that are used in classrooms. Curriculum innovation can be moved forward with illustrative alternative tasks and explicit attention to the underlying principles and frameworks used to design them, without losing consideration of skill development, fluency, and flexibility. A vital component often missing in curriculum innovation documents is the vivid exemplification that is necessary to show exactly what tasks might look like and how they relate to improving teaching and learning.

In addition, current changes in educational systems and trends in mathematics education ask for a reconsideration of design principles. Trends in education that are related to task design are, for instance, beginning to show an increasing focus on interdisciplinarity and authentic practices. Trying to better connect mathematics education to other subjects like physics, biology, and economics requires a reconsideration of the role of contexts and bridging concepts. A serious consideration of the use of authentic practices and the world of work in mathematics education calls for tasks with a purpose and utility, shifting from solving a school mathematics problem to asking for a product as a final result. New task characteristics emerge and others might become less relevant in the near future.

From this Study conference and its follow-up exchanges and research for the preparation of this chapter, we have also learned that knowledge about design grows in the community as design principles are explicitly described, discussed, and refined. Although the papers presented within the Principles and Frameworks theme group of this conference all specified the frames and principles underlying their designs and illustrated how these were being implemented in the resulting tasks, such is not common in the majority of papers presented at mathematics education research conferences (Sierpinska, 2003). Despite the recent growth spurt of design studies within mathematics education, the specificity of the principles that inform task design in a precise way remains both underdeveloped and, even when somewhat developed, underreported. A possible obstacle that stands in the way of specificity can be traced to length constraints on published papers and the extended amount of space that the provision of specific details requires. Were it not for websites such as *Educational Designer* (http://www.educationaldesigner.org), there are few avenues for presenting the explicit and detailed thinking that lies behind the final versions of designed tasks. Nevertheless, it seems reasonable to expect that mathematics education researchers could be more explicit in their published research papers about the principles that underlie the tasks they design for their research studies. Clearly, more work remains to be done in encouraging such practice. This chapter provides a starting point for future efforts that aim at a further and deeper investigation of task design, its frameworks, and its principles, so that design might become a mature element in mathematics education research and practice.

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