On the Oscillation Rigidity of a Lipschitz Function on a High-Dimensional Flat Torus

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Abstract Given an arbitrary 1-Lipschitz function f on the torus $\mathbb{T}^n$, we find a k-dimensional subtorus $M \subseteq \mathbb{T}^n$, parallel to the axes, such that the restriction of f to the subtorus M is nearly a constant function. The k-dimensional subtorus M is selected randomly and uniformly. We show that when $k \leq c \log n / (\log \log n + \log 1/\varepsilon)$, the maximum and the minimum of f on this random subtorus M differ by at most ε, with high probability.

1 Introduction

A uniformly continuous function f on an n-dimensional space X of finite volume tends to concentrate near a single value as n approaches infinity, in the sense that the ε-extension of some level set has nearly full measure. This phenomenon, which is called the *concentration of measure in high dimension*, is frequently related to a transitive group of symmetries acting on X. The prototypical example is the case of a 1-Lipschitz function on the unit sphere S^n, see [3, 4, 8].

One of the most important consequences of the concentration of measure is the emergence of *spectrum*, as was discovered in the 1970s by the third named author, see [5–7]. The idea is that not only does the distinguished level set have a large ε-extension in a sense of measure, but one may actually find structured subsets on which the function is nearly constant. When we have a group G acting transitively on X, this structured subset belongs to the orbit $\{gM_0\,;\, g \in G\}$ where $M_0 \subseteq X$ is a fixed subspace. The third named author also noted some connections with Ramsey theory, which were developed in two different directions: by Gromov [2] in the direction of metric geometry, and by Pestov [9, 10] in the unexpected direction of dynamical systems.

The phenomenon of spectrum thus follows from concentration, and it comes as no surprise that most of the results in Analysis which establish spectrum, have appeared as a consequence of concentration. In this note we demonstrate an instance

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where no concentration of measure is available, but nevertheless a geometrically structured level set arises.

To state our result, consider the standard flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$, which inherits its Riemannian structure from $\mathbb{R}^n$. We say that $M \subseteq \mathbb{T}^n$ is a *coordinate subtorus of dimension* k if it is the collection of all n-tuples $(\theta_j)_{j=1}^n \in \mathbb{T}^n$ with fixed $n-k$ coordinates. Given a manifold X and $f : X \to \mathbb{R}$ we denote the oscillation of f along X by

$$\mathrm{Osc}(f; X) = \sup_X f - \inf_X f.$$

Theorem 1. *There is a universal constant $c > 0$, such that for any $n \geq 1, 0 < \varepsilon \leq 1$ and a function $f : \mathbb{T}^n \to \mathbb{R}$ which is 1-Lipschitz, there exists a k-dimensional coordinate subtorus $M \subseteq \mathbb{T}^n$ with $k = \left\lfloor c \frac{\log n}{\log\log(5n) + \log|\varepsilon|} \right\rfloor$, such that $Osc(f; M) \leq \varepsilon$.*

Note that the collection of all coordinate subtori equals the orbit $\{gM_0 \, ; \, g \in G\}$ where $M_0 \subseteq \mathbb{T}^n$ is any fixed k-dimensional coordinate subtorus, and the group $G = \mathbb{R}^n \rtimes S_n$ acts on $\mathbb{T}^n$ by translations and permutations of the coordinates. Theorem 1 is a manifestation of *spectrum*, yet its proof below is inspired by proofs of the Morrey embedding theorem, and the argument does not follow the usual concentration paradigm. We think that the spectrum phenomenon should be much more widespread, perhaps even more than the concentration phenomenon, and we hope that this note will be a small step towards its recognition.

2 Proof of the Theorem

We write $|\cdot|$ for the standard Euclidean norm in $\mathbb{R}^n$ and we write log for the natural logarithm. The standard vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_n$ on $\mathbb{R}^n$ are well-defined also on the quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. These n vector fields are the *coordinate directions* on the unit torus $\mathbb{T}^n$. Thus, the partial derivatives $\partial_1 f, \ldots, \partial_n f$ are well-defined for any smooth function $f : \mathbb{T}^n \to \mathbb{R}$, and we have $|\nabla f|^2 = \sum_{i=1}^n (\partial_i f)^2$. A k-dimensional subspace $E \subseteq T_x\mathbb{T}^n$ is a *coordinate subspace* if it is spanned by k coordinate directions. For $f : \mathbb{T}^n \to \mathbb{R}$ and $M \subseteq \mathbb{T}^n$ a submanifold, we write $\nabla_M f$ for the gradient of the restriction $f|_M : M \to \mathbb{R}$.

Throughout the proof, c, C will always denote universal constants, not necessarily having the same value at each appearance. Since the Riemannian volume of $\mathbb{T}^n$ equals one, Theorem 1 follows from the case $\alpha = 1$ of the following:

Theorem 2. *There is a universal constant $c > 0$ with the following property: Let $n \geq 1, 0 < \varepsilon \leq 1, 0 < \alpha \leq 1$ and $1 \leq k \leq c \frac{\log n}{\log\log(5n)+|\log\varepsilon|+|\log\alpha|}$. Let $f : \mathbb{T}^n \to \mathbb{R}$ be a locally-Lipschitz function such that, for $p = k(1+\alpha)$,*

$$\int_{\mathbb{T}^n} |\nabla f|^p \leq 1. \tag{1}$$

Then there exists a k-dimensional coordinate subtorus $M \subseteq \mathbb{T}^n$ with Osc $(f; M) \leq \varepsilon$.

The essence of the proof is as follows. First, for some large k we find a k-dimensional coordinate subtorus M where the derivative is small on average, in the sense that $\left(\int_M |\nabla_M f|^p \right)^{1/p}$ is small. The existence of such a subtorus is a consequence of the observation that at every point, most of the partial derivatives in the coordinate directions are small. We then restrict our attention to this subtorus and take any two points $\tilde{x}, \tilde{y} \in M$. Our goal is to show that $f(\tilde{x}) - f(\tilde{y}) < \varepsilon$.

To this end we construct a polygonal line from $\tilde{x}$ to $\tilde{y}$ which consists of intervals of length $1/2$. For every such interval $[x, y]$ we randomly select a point Z in a $(k-1)$-dimensional ball which is orthogonal to the interval $[x, y]$ and is centered at its midpoint. We then show that $|f(x) - f(Z)|$ and $|f(y) - f(Z)|$ are typically small, since $|\nabla_M f|$ is small on average along the intervals $[x, Z]$ and $[y, Z]$.

We proceed with a formal proof of Theorem 2, beginning with the following computation:

Lemma 3. *For any $n \geq 1, 0 < \varepsilon \leq 1, 0 < \alpha \leq 1$ and $1 \leq k \leq c \frac{\log n}{\log\log(5n) + |\log \varepsilon| + |\log \alpha|}$, we have that $k \leq n/2$ and*

$$\left(\frac{2k}{\delta^2 n} \right)^{1/p} \leq \sqrt{k} \cdot \delta \tag{2}$$

where $p = (1+\alpha)k$ and

$$\delta = \frac{\alpha}{16(1+\alpha)} \cdot \frac{\varepsilon}{k^{3/2}}. \tag{3}$$

Proof. Take $c = 1/200$. The desired conclusion (2) is equivalent to $4k^{2-p} \leq \delta^{2p+4} n^2$, which in turn is equivalent to

$$2^{8p+18} \cdot \left(\frac{\alpha+1}{\alpha} \right)^{2p+4} \cdot k^{2p+8} \leq \varepsilon^{2p+4} n^2. \tag{4}$$

Since $c \leq 1/12$ and $\alpha \leq 1$ we have that $6p \leq 12k \leq \log n / |\log \varepsilon|$ and hence $\varepsilon^{2p+4} n^2 \geq \varepsilon^{6p} n^2 \geq n$. Since $\alpha + 1 \leq 2$ then in order to obtain (4) it suffices to prove

$$\left(\frac{32}{\alpha} \cdot k \right)^{2p+8} \leq n. \tag{5}$$

Since $c \le 1/200$ and $k \le c\log n/(\log\log(5n))$ then $24k\log k \le \log n$. Since $k \le c\frac{\log n}{|\log\alpha|+\log(\log 5)}$ then $24k\log\left(\frac{32}{\alpha}\right) \le \log n$. We conclude that $12k\log\left(\frac{32}{\alpha}\cdot k\right) \le \log n$, and hence

$$\left(\frac{32}{\alpha}\cdot k\right)^{12k} \le n. \tag{6}$$

However, $p = (1+\alpha)k$ and hence $2p+8 \le 12k$. Therefore the desired bound (5) follows from (6). Since $k \le \frac{1}{2}\log n \le n/2$, the lemma is proven. □

Our standing assumptions for the remainder of the proof of Theorem 2 are that $n \ge 1, 0 < \varepsilon \le 1, 0 < \alpha \le 1$ and that

$$1 \le k \le c\frac{\log n}{\log\log(5n) + |\log\varepsilon| + |\log\alpha|} \tag{7}$$

where $c > 0$ is the constant from Lemma 3. We also denote

$$p = (1+\alpha)k \tag{8}$$

and we write $e_1,\dots,e_n$ for the standard n unit vectors in $\mathbb{R}^n$.

Lemma 4. *Let $v \in \mathbb{R}^n$ and let $J \subseteq \{1,\dots,n\}$ be a random subset of size k, selected uniformly from the collection of all $\binom{n}{k}$ subsets. Consider the k-dimensional subspace $E \subseteq \mathbb{R}^n$ spanned by $\{e_j;\, j \in J\}$ and let P_E be the orthogonal projection operator onto E in $\mathbb{R}^n$. Then,*

$$\left(\mathbb{E}|P_E v|^p\right)^{1/p} \le \frac{\alpha}{8(1+\alpha)}\cdot\frac{\varepsilon}{k}\cdot|v|.$$

Proof. We may assume that $v = (v_1,\dots,v_n) \in \mathbb{R}^n$ satisfies $|v| = 1$. Let $\delta > 0$ be defined as in (3). Denote $I = \{i;\, |v_i| \ge \delta\}$. Since $|v| = 1$, we must have $|I| \le 1/\delta^2$. We claim that

$$\mathbb{P}(I \cap J = \emptyset) \ge 1 - \frac{2k}{\delta^2 n}. \tag{9}$$

Indeed, if $\frac{2k}{\delta^2 n} \ge 1$ then (9) is obvious. Otherwise, $|I| \le \delta^{-2} \le n/2 \le n-k$ and

$$\mathbb{P}(I \cap J = \emptyset) = \prod_{j=0}^{k-1}\frac{n-|I|-j}{n-j} \ge \left(1 - \frac{|I|}{n-k+1}\right)^k \ge \left(1 - \frac{2}{\delta^2 n}\right)^k \ge 1 - \frac{2k}{\delta^2 n}.$$

Thus (9) is proven. Consequently,

$$\mathbb{E}|P_E v|^p = \mathbb{E}\left(\sum_{j\in J} v_j^2\right)^{p/2} \le \frac{2k}{\delta^2 n} + \mathbb{E}\left[1_{\{I\cap J=\emptyset\}}\cdot\left(\sum_{j\in J} v_j^2\right)^{p/2}\right] \le \frac{2k}{\delta^2 n} + \left(k\cdot\delta^2\right)^{p/2},$$

where 1_A equals one if the event A holds true and it vanishes otherwise. By using the inequality $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$ we obtain

$$\left(\mathbb{E}|P_E v|^p\right)^{1/p} \le \left(\frac{2k}{\delta^2 n}\right)^{1/p} + \sqrt{k}\cdot\delta \le 2\sqrt{k}\cdot\delta = \frac{\alpha}{8(1+\alpha)}\cdot\frac{\varepsilon}{k},$$

where we utilized (3) and Lemma 3. □

Corollary 5. *Let $f : \mathbb{T}^n \to \mathbb{R}$ be a locally-Lipschitz function with $\int_{\mathbb{T}^n} |\nabla f|^p \le 1$. Then there exists a k-dimensional coordinate subtorus $M \subseteq \mathbb{T}^n$ such that*

$$\left(\int_M |\nabla_M f|^p\right)^{1/p} \le \frac{\alpha}{8(1+\alpha)}\cdot\frac{\varepsilon}{k}. \tag{10}$$

Proof. The set of all coordinate k-dimensional subtori admits a unique probability measure, invariant under translations and coordinate permutations. Let M be a random coordinate k-subtorus, chosen with respect to the uniform distribution. All the tangent spaces $T_x\mathbb{T}^n$ are canonically identified with $\mathbb{R}^n$, and we let $E \subseteq \mathbb{R}^n$ denote a random, uniformly chosen k-dimensional coordinate subspace. Then we may write

$$\mathbb{E}_M \int_M |\nabla_M f|^p = \int_{\mathbb{T}^n} \mathbb{E}_E |P_E \nabla f|^p \le A^p \int_{\mathbb{T}^n} |\nabla f|^p \le A^p,$$

where $A = \frac{\alpha}{8(1+\alpha)}\cdot\frac{\varepsilon}{k}$ and we used Lemma 4. It follows that there exists a subtorus M which satisfies (10). □

The following lemma is essentially Morrey's inequality (see [1, Sect. 4.5]).

Lemma 6. *Consider the k-dimensional Euclidean ball $B(0,R) = \{x \in \mathbb{R}^k \,;\, |x| \le R\}$. Let $f : B(0,R) \to \mathbb{R}$ be a locally-Lipschitz function, and let $x, y \in B(0,R)$ satisfy $|x-y| = 2R$. Recall that $p = (1+\alpha)k$. Then,*

$$|f(x)-f(y)| \le 4\frac{1+\alpha}{\alpha}\cdot k^{\frac{1}{2(1+\alpha)}}\cdot R^{1-\frac{k}{p}}\left(\int_{B(0,R)} |\nabla f(x)|^p dx\right)^{1/p}. \tag{11}$$

Proof. We may reduce matters to the case $R = 1$ by replacing $f(x)$ by $f(Rx)$; note that the right-hand side of (11) is invariant under such replacement. Thus x is a unit vector, and $y = -x$. Let Z be a random point, distributed uniformly in the $(k-1)$-dimensional unit ball

$$B(0,1) \cap x^{\perp} = \{v \in \mathbb{R}^k \,;\, |v| \le 1,\ v \cdot x = 0\},$$

where $v \cdot x$ is the standard scalar product of $x, v \in \mathbb{R}^k$. Let us write

$$\begin{aligned} \mathbb{E}|f(x) - f(Z)| &\le \mathbb{E}|x - Z| \int_0^1 |\nabla f((1-t)x + tZ)|\, dt \qquad (12) \\ &\le 2\mathbb{E}|\nabla f((1-T)x + TZ)| = 2\int_{B(0,1)} |\nabla f(z)|\rho(z)dz, \end{aligned}$$

where T is a random variable uniformly distributed in $[0, 1]$, independent of Z, and where ρ is the probability density of the random variable $(1-T)x + TZ$. Then,

$$\rho((1-r)x + rz) = \frac{c_k}{r^{k-1}}$$

when $z \in B(0,1) \cap x^{\perp}, 0 < r < 1$. We may compute c_k as follows:

$$1 = c_k \int_0^1 \frac{1}{r^{k-1}} V_{k-1}(r)dr = c_k V_{k-1}(1) = c_k \frac{\pi^{k-1}}{\Gamma\left(\frac{k+1}{2}\right)},$$

where $V_{k-1}(r)$ is the $(k-1)$-dimensional volume of a $(k-1)$-dimensional Euclidean ball of radius r. Denote $q = p/(p-1)$. Then,

$$\int_{B(0,1)} \rho^q = \int_0^1 \left(\frac{c_k}{r^{k-1}}\right)^q V_{k-1}(r)dr = \frac{c_k^q V_{k-1}(1)}{(k-1)(1-q)+1} = \frac{p-1}{p-k} \left(\frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{k-1}}\right)^{q-1},$$

and hence

$$\begin{aligned} \left(\int_{B(0,1)} \rho^q\right)^{1/q} &= \left(\frac{p-1}{p-k}\right)^{1/q} \left(\frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{k-1}}\right)^{1/p} \qquad (13) \\ &\le \left(\frac{1+\alpha}{\alpha}\right)^{1/q} \left(\frac{k^{k/2}}{\pi^{k-1}}\right)^{1/p} \le \frac{1+\alpha}{\alpha} \cdot k^{\frac{1}{2(1+\alpha)}}. \end{aligned}$$

Denote $C_{\alpha,k} = \frac{1+\alpha}{\alpha} \cdot k^{\frac{1}{2(1+\alpha)}}$. From (12), (13) and the Hölder inequality,

$$\mathbb{E}|f(x) - f(Z)| \le 2\left(\int_{B(0,1)} |\nabla f|^p\right)^{\frac{1}{p}} \left(\int_{B(0,1)} \rho^q\right)^{\frac{1}{q}} \le 2C_{\alpha,k} \left(\int_{B(0,1)} |\nabla f|^p\right)^{\frac{1}{p}}. \qquad (14)$$

A bound similar to (14) also holds for $\mathbb{E}|f(y) - f(Z)|$, since $y = -x$. By the triangle inequality,

$$|f(x)-f(y)| \leq \mathbb{E}|f(y)-f(Z)|+\mathbb{E}|f(Z)-f(x)| \leq 4C_{\alpha,k}\left(\int_{B(0,1)} |\nabla f|^p\right)^{1/p}.$$

□

Proof of Theorem 2. According to Corollary 5 we may select a coordinate subtorus $M = \mathbb{T}^k$ so that

$$\left(\int_M |\nabla_M f|^p\right)^{1/p} \leq \frac{\alpha}{8(1+\alpha)} \cdot \frac{\varepsilon}{k}. \tag{15}$$

Given any two points $x, y \in M$, let us show that

$$|f(x) - f(y)| \leq \varepsilon. \tag{16}$$

The distance between x and y is at most $\sqrt{k}/2$. Let us construct a curve, in fact a polygonal line, starting at x and ending at y which consists of at most $\sqrt{k} + 1$ intervals of length $1/2$. For instance, we may take all but the last two intervals to be intervals of length $1/2$ lying on a minimizing geodesic between x to y. The last two intervals need to connect two points whose distance is at most $1/2$, and this is easy to do by drawing an isosceles triangle whose base is the segment between these two points.

Let $[x_j, x_{j+1}]$ be any of the intervals appearing in the polygonal line constructed above. Let $B \subset \mathbb{T}^k = M$ be a geodesic ball of radius $R = 1/4$ centered at the midpoint of $[x_j, x_{j+1}]$. This geodesic ball on the torus is isometric to a Euclidean ball of radius $R = 1/4$ in $\mathbb{R}^k$. Lemma 6 applies, and implies that

$$\begin{aligned}|f(x_j) - f(x_{j+1})| &\leq 4\frac{1+\alpha}{\alpha} \cdot k^{\frac{1}{2(1+\alpha)}} \left(\frac{1}{4}\right)^{1-\frac{k}{p}} \left(\int_B |\nabla_M f|^p\right)^{\frac{1}{p}} \\ &\leq 4\frac{1+\alpha}{\alpha} \cdot \sqrt{k} \left(\int_M |\nabla_M f|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Since the number of intervals in the polygonal line is at most $\sqrt{k} + 1 \leq 2\sqrt{k}$, then

$$|f(x) - f(y)| \leq \sum_j |f(x_j) - f(x_{j+1})| \leq 8\frac{1+\alpha}{\alpha} \cdot k \left(\int_M |\nabla_M f|^p\right)^{1/p} \leq \varepsilon,$$

where we used (15) in the last passage. The points $x, y \in M$ were arbitrary, and hence $\mathrm{Osc}(f; M) \leq \varepsilon$. □

Remarks. 1. It is evident from the proof of Theorem 2 that the subtorus M is selected randomly and uniformly over the collection of all k-dimensional coordinate subtori. It is easy to obtain that with probability at least 9/10, we have that $Osc(M; f) \leq \varepsilon$.

2. The assumption that f is locally-Lipschitz in Theorem 2 is only used to justify the use of the fundamental theorem of calculus in (12). It is possible to significantly weaken this assumption; it suffices to know that f admits weak derivatives $\partial_1 f, \dots, \partial_n f$ and that (1) holds true, see [1, Chap. 4] for more information.

   It is quite surprising that the conclusion of the theorem also holds for non-continuous, unbounded functions, with many singular points, as long as (1) is satisfied in the sense of weak derivatives. The singularities are necessarily of a rather mild type, and a variant of our proof yields a subtorus M on which the function f is necessarily continuous with $\mathrm{Osc}(f; M) \leq \varepsilon$.
3. Another possible approach to the problem would be along the lines of the proof of the classical concentration theorems—namely, finding an ε-net of points in a subtorus, where all the coordinate partial derivatives of the function are small. However, this approach requires some additional a-priori data about the function, such as a uniform bound on the Hessian.
4. We do not know whether the dependence on the dimension in Theorem 1 is optimal. Better estimates may be obtained if the subtorus $M \subseteq \mathbb{T}^n$ is permitted to be an arbitrary k-dimensional *rational* subtorus, which is not necessarily a coordinate subtorus. Recall that a rational torus is a quotient of $\mathbb{R}^n$ by a lattice which is spanned by n vectors with rational coordinates.

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References

1. L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1992)
2. M. Gromov, Filling Riemannian manifolds. J. Differ. Geom. **18**(1), 1–147 (1983)
3. M. Gromov, Isoperimetry of waists and concentration of maps. Geom. Funct. Anal. (GAFA) **13**(1), 178–215 (2003)
4. M. Ledoux, *The Concentration of Measure Phenomenon.* Mathematical Surveys and Monographs, vol. 89 (American Mathematical Society, Providence, RI, 2001)
5. V.D. Milman, The spectrum of bounded continuous functions which are given on the unit sphere of a B-space. Funkcional. Anal. i Priložen. **3**(2), 67–79 (1969) (Russian)
6. V.D. Milman, Geometric theory of Banach spaces. II. Geometry of the unit ball. Uspehi Mat. Nauk **26**(6)(162), 73–149 (1971) (Russian)
7. V.D. Milman, Asymptotic properties of functions of several variables that are defined on homogeneous spaces. Soviet Math. Dokl. **12**, 1277–1281 (1971). Translated from Dokl. Akad. Nauk SSSR **199**, 1247–1250 (1971) (Russian)

8. V.D. Milman, G. Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces. With an Appendix by M. Gromov.* Lecture Notes in Mathematics, vol. 1200 (Springer, Berlin, 1986)
9. V. Pestov, Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups. Israel J. Math. **127**, 317–357 (2002)
10. V. Pestov, *Dynamics of Infinite-Dimensional Groups. The Ramsey-Dvoretzky-Milman Phenomenon.* University Lecture Series, vol. 40 (American Mathematical Society, Providence, RI, 2006)