Chapter 6
The Loss Tangent of Visco-Elastic Models

Franz Konstantin Fuss

Keywords Visco-elasticity • Elasticity • Viscosity • Loss tangent • Standard linear solid • Power model • Log model • Stress strain • Transient part • Steady state • Laplace transform • Inverse Laplace transform • Convolution integral

6.1 Introduction

Visco-elastic models are characterised by phenomena such as

– Velocity dependency: increase of stiffness with strain rate,
– Stress relaxation: decay of stress with time at constant strain,
– Creep: increase of strain with time at constant stress, and
– Loss of stored (elastic) energy due to inner friction resulting in unequal loading and unloading stress–strain curves, the area between the two curves corresponding to the dissipated energy (hysteresis).

Visco-elastic models can be classified in various ways, e.g.

(1) Linear or non-linear models

  – Linear models with at least one elastic and one viscous element in parallel or in series;

F.K. Fuss (✉)
School of Aerospace, Mechanical, and Manufacturing Engineering, RMIT University, Melbourne, VIC, Australia
e-mail: franz.fuss@rmit.edu.au

L. Dai, R.N. Jazar (eds.), *Nonlinear Approaches in Engineering Applications*,
DOI 10.1007/978-3-319-09462-5_6

  - *QLV*/quasi-linear "viscous" models such as Prony series (Wiechert model; Wiechert 1889, 1893);
  - Non-linear models such as logarithmic and power law models.

  or

(2) According to the decay function of stress relaxation, which can be

  - Exponential: linear three-element model (standard linear solid),
  - Power: non-linear power law model, or
  - Logarithmic: non-linear logarithmic law model

Examples for the latter two non-linear models are:

- Power law model: biological tissues such as ligaments (Provenzano et al. 2001), foams with non-negative stiffness (Fuss 2009), cork in cricket balls (Fuss 2008a, 2012);
- Logarithmic law model: solid polymers (Findley et al. 1989), polymer golf balls (Fuss 2008b, 2012).

The loss tangent, tan δ, is defined as the tangent of the phase angle δ, which, in turn, is the ratio of loss modulus E'' to storage modulus E'.

$$\tan\delta = \frac{E''}{E'} \tag{6.1}$$

where

$$E' = \frac{\sigma_0}{\varepsilon_0}\cos\delta \tag{6.2}$$

$$E'' = \frac{\sigma_0}{\varepsilon_0}\sin\delta \tag{6.3}$$

and σ_0 and ε_0 are the peak amplitudes of stress σ and strain ε, respectively.

The complex modulus E^* is defined as

$$E^* = \frac{\sigma_0}{\varepsilon_0}\mathrm{e}^{i\delta} = \frac{\sigma_0}{\varepsilon_0}(\cos\delta + i\,\sin\delta) = E' + iE'' = |E^*|\,\mathrm{e}^{i\delta} \tag{6.4}$$

where $i = \sqrt{-1}$, and $|E^*|$ is the dynamic modulus, the magnitude of E^*, i.e. the resultant of loss modulus E'' and storage modulus E'

$$|E^*| = \frac{\sigma_0}{\varepsilon_0} \tag{6.5}$$

The loss tangent, tan δ, is usually determined by subjecting a material or structure to sinusoidal strain ε

$$\varepsilon = \varepsilon_0 \sin(2\pi f t) \tag{6.6}$$

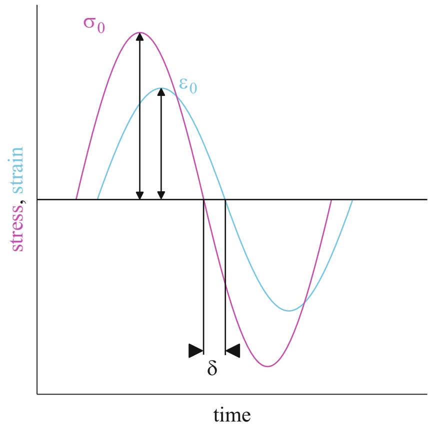

Fig. 6.1 Sinusoidal strain curve (*blue*) imposed on a visco-elastic material and resulting stress curve (*pink*); σ_0: stress amplitude, maximal stress; ε_0: strain amplitude, maximal strain; δ: phase shift

where f is the cyclic frequency (angular frequency $\omega = 2\pi f$). The resulting reaction stress σ is equally sinusoidal, but out of phase with respect to the strain by the phase angle δ

$$\sigma = \sigma_0 \sin(2\pi f t + \delta) \tag{6.7}$$

A positive phase angle δ causes the stress peak to occur earlier than the strain peak (Fig. 6.1), resulting in the typical hysteresis effect of visco-elastic materials when plotting stress against strain (Fig. 6.2). The area of the hysteresis loop corresponds to the energy dissipated into the material as thermal energy.

When subjecting a material to cyclic (sinusoidal) strain, the peak stress, σ_0, increases during the first cycles (*transient part*) until it reaches an equilibrium or *steady state* (Fig. 6.3). The phase angle δ is measured once the steady state has set in.

It is evident, that the energy dissipated by inner friction depends on the viscosity parameter η. However, as the loss tangent is the ratio of loss to storage modulus, the strain rate independent elasticity parameter E is expected to influence the loss tangent too. Lastly, as the modulus (Young's and tangent) increases with strain rate and thus with frequency f, the latter could contribute to the loss tangent as well.

The objective of this *Book Chapter* is to explore in how far the viscosity parameter η, the strain rate independent elasticity parameter E and the strain frequency f affect the loss tangent. The aim is to derive the loss tangent at steady state of the three visco-elastic models mentioned above, in order to understand the interaction between elasticity, viscosity and frequency and their effect on energy loss. The function of stress relaxation with time is not the only difference between the three models mentioned above. A further objective of this paper is to analyse the basic differences of these models and to understand their applicability and constraints.

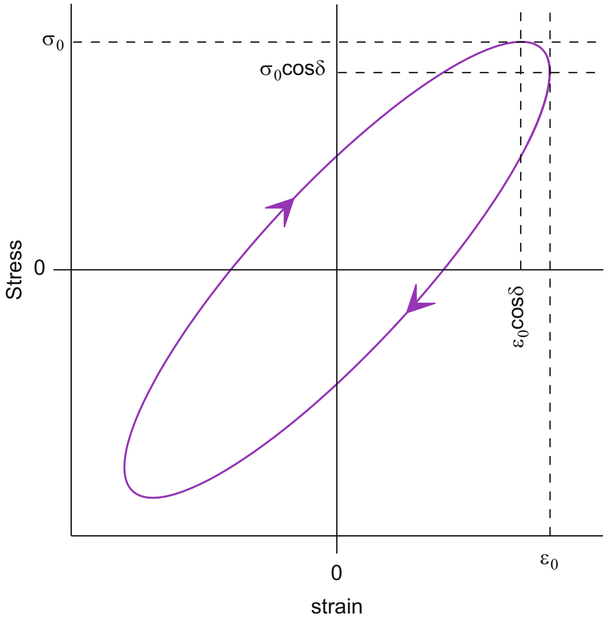

Fig. 6.2 Hysteresis loop of stress–strain ellipse; σ_0: maximal stress; ε_0: maximal strain; δ: phase angle

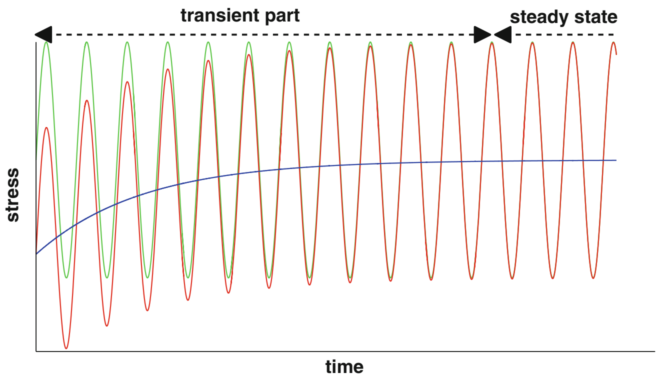

Fig. 6.3 Stress–time curve (*red*) during the first load cycles; *blue curve*: transient component; *green curve*: steadys state component; the stress–time curve (*red*) is the sum of transient and steady state components

6.2 Analysis

6.2.1 The Standard Linear Solid (Zener Model)

The standard linear solid (SLS of Voight form) consists of two Hookean springs and a viscous damper, where the spring with the modulus E_1 is connected in series with a Kelvin–Voight model, with spring of modulus E_2 and damper of viscosity constant η connected in parallel (Fig. 6.2).

From Fig. 6.4

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{6.8}$$

$$\sigma = \varepsilon_1 E_1 \tag{6.9}$$

$$\sigma = \varepsilon_2 E_2 + \dot{\varepsilon}_2 \eta \tag{6.10}$$

Taking the Laplace transform of Eqs. (6.8)–(6.10), eliminating ε_1 and ε_2 by substitution, and solving for $\widehat{\sigma}$ yields the constitutive equation of the standard linear solid

$$\widehat{\sigma} = \widehat{\varepsilon}\frac{E_1 E_2 + s\eta E_1}{E_1 + E_2 + s\eta} \tag{6.11}$$

where the caret (ˆ) denotes the transformed parameter.

The equation for stress relaxation results from applying a constant strain ε_c to the model through a Heaviside function H(t)

$$\varepsilon = \varepsilon_c \mathrm{H}(t) \tag{6.12}$$

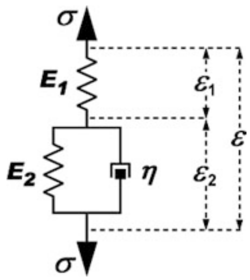

Fig. 6.4 Standard linear solid of Voight form; σ: stress; ε: strain; η: viscosity constant; E_1: modulus of series spring; E_2: modulus of parallel spring; ε_1: strain of series spring; ε_2: strain of Kelvin–Voight model

the Laplace transform of which is

$$\widehat{\varepsilon} = \frac{\varepsilon_c}{s} \tag{6.13}$$

By substituting Eq. (6.13) into Eq. (6.11), we obtain

$$\widehat{\sigma} = \varepsilon_c \frac{E_1 E_2 + s\eta E_1}{s(E_1 + E_2 + s\eta)} \tag{6.14}$$

the inverse Laplace transform of which yields the function of stress relaxation

$$\frac{\sigma}{\varepsilon_c} = E_1 \frac{E_2 + E_1 \mathrm{e}^{-t\frac{(E_1+E_2)}{\eta}}}{E_1 + E_2} \tag{6.15}$$

where the stress σ is normalised to the constant strain ε_c. Equation (6.15) proves the exponential decay of stress with time, as mentioned above in the Introduction.

The loss tangent results from applying the sinusoidal strain of Eq. (6.6) to the constitutive Eq. (6.11). The Laplace transform of Eq. (6.6) is

$$\widehat{\varepsilon} = \varepsilon_0 \frac{2\pi f}{s^2 + (2\pi f)^2} \tag{6.16}$$

where ε_0 is the peak amplitude of the strain.

By substituting Eq. (6.16) into the constitutive Eq. (6.11), we obtain

$$\widehat{\sigma} = \varepsilon_0 \left(\frac{2\pi f}{s^2 + (2\pi f)^2} \right) \left(\frac{E_1 E_2 + s\eta E_1}{E_1 + E_2 + s\eta} \right) \tag{6.17}$$

After rearranging

$$\widehat{\sigma} = \varepsilon_0 2\pi f E_1 \frac{E_2 + s\eta}{s^3\eta + s^2 E_1 + s^2 E_2 + s\eta(2\pi f)^2 + E_1(2\pi f)^2 + E_2(2\pi f)^2} \tag{6.18}$$

the inverse Laplace transform of Eq. (6.18) yields

$$\sigma = \varepsilon_0 E_1 \frac{-E_1\eta 2\pi f \mathrm{e}^{-t\frac{E_1+E_2}{\eta}} + E_1\eta 2\pi f \cos(2\pi f t) + \left(E_1 E_2 + {E_2}^2 + \eta^2(2\pi f)^2\right)\sin(2\pi f t)}{{E_1}^2 + 2E_1 E_2 + {E_2}^2 + \eta^2(2\pi f)^2} \tag{6.19}$$

The numerator of Eq. (6.19) comprises of a *transient* part (exponential function) and the *steady state* part (sine and cosine functions). At large times, if $t \to \infty$, the transient part, i.e. the exponential term of the numerator, vanishes and the steady state sets in:

$$\sigma = \frac{\varepsilon_0 E_1}{(E_1 + E_2)^2 + \eta^2(2\pi f)^2}\left[E_1\eta 2\pi f \cos(2\pi f t) + \left(E_1E_2 + E_2{}^2 + \eta^2(2\pi f)^2\right)\sin(2\pi f t)\right] \tag{6.20}$$

In order to obtain the loss tangent and the peak stress σ_0, we apply the addition rules to Eq. (6.7)

$$\sigma = \sigma_0 \sin(\delta)\cos(2\pi f t) + \sigma_0 \cos(\delta)\sin(2\pi f t) \tag{6.21}$$

where the unknowns are the two components of the peak stress σ_0: σ_0 sin δ and σ_0 cos δ.

The loss tangent thus results from the ratio of the two components

$$\tan\delta = \frac{\sigma_0 \sin\delta}{\sigma_0 \cos\delta} \tag{6.22}$$

and the resultant of the peak stress σ_0 is obtained from

$$\sigma_0 = \sqrt{(\sigma_0 \sin\delta)^2 + (\sigma_0 \cos\delta)^2} \tag{6.23}$$

From Eq. (6.20) it follows that

$$\sigma_0 \sin\delta = \frac{\varepsilon_0 E_1}{(E_1 + E_2)^2 + \eta^2(2\pi f)^2} E_1\eta 2\pi f \tag{6.24}$$

and

$$\sigma_0 \cos\delta = \frac{\varepsilon_0 E_1}{(E_1 + E_2)^2 + \eta^2(2\pi f)^2}\left(E_1E_2 + E_2{}^2 + \eta^2(2\pi f)^2\right) \tag{6.25}$$

The loss tangent thus is

$$\tan\delta = \frac{E_1\eta 2\pi f}{E_1E_2 + E_2{}^2 + \eta^2(2\pi f)^2} \tag{6.26}$$

and the peak stress σ_0 results from

$$\sigma_0 = \frac{\varepsilon_0 E_1}{(E_1 + E_2)^2 + \eta^2(2\pi f)^2}\sqrt{E_1{}^2\eta^2(2\pi f)^2 + \left(E_1E_2 + E_2{}^2 + \eta^2(2\pi f)^2\right)^2} \tag{6.27}$$

In both Eqs. (6.26) and (6.27) f and η are linked together and always occur as the product $f\eta$.

As the maximal strain rate $\dot{\varepsilon}_0$ of a sinusoidal strain function equals $\varepsilon_0 2\pi f$, the strain rate dependency of σ_0 is given by

$$\sigma_0 = \frac{\varepsilon_0 E_1}{(E_1 + E_2)^2 + \eta^2\left(\frac{\dot{\varepsilon}_0}{\varepsilon_0}\right)^2} \sqrt{{E_1}^2\eta^2\left(\frac{\dot{\varepsilon}_0}{\varepsilon_0}\right)^2 + \left[E_1 E_2 + {E_2}^2 + \eta^2\left(\frac{\dot{\varepsilon}_0}{\varepsilon_0}\right)^2\right]^2} \tag{6.28}$$

The peak stress σ_0 increases with f and η. If f or $\eta \to \infty$, σ_0 asymptotes to $\varepsilon_0 E_1$. If f or $\eta \to 0$, $\sigma_0 \to \varepsilon_0({E_1}^{-1} + {E_2}^{-1})^{-1}$. The limits of σ_0 are evident when considering that the peak strain rate changes with f. Thus, at zero strain rate or zero η, the standard linear solid reduces to two springs in series and the effective modulus of the model becomes $({E_1}^{-1} + {E_2}^{-1})^{-1}$. At infinite strain rate or infinite η, the damper becomes rigid and the effective modulus of the model is just E_1.

The peak stress σ_0 increases with E_1. If f or $\eta \to \infty$ and $E_1 \to 0$ or ∞, both the effective modulus and σ_0 become equally 0 or ∞, respectively. If f or $\eta \to 0$ and $E_1 \to 0$ or ∞, the effective modulus and σ_0 become zero in the former case, and E_2 and $\varepsilon_0 E_2$ in the latter.

If $E_2 \to 0$, the standard linear solid reduces to a Maxwell model (with a spring and a damper in series), the modulus of which is (1) ∞ or (2) E_1 or (3) 0, if (1) E_1, and η or f, are ∞ or (2) η or f are ∞ or (3) E_1, and η or f are 0, respectively.

At $\eta > 0$ and $f < \infty$, and $E_2 \to 0$, the peak stress reaches

$$\sigma_0 = \frac{2\pi f \varepsilon_0 E_1 \eta}{\sqrt{{E_1}^2 + \eta^2(2\pi f)^2}} \tag{6.29}$$

If $E_2 \to \infty$, the modulus of the standard linear solid reduces to E_1. σ_0 shows a local minimum at a certain E_{2_0} (Fig. 6.5)

$$E_{2_0} = \frac{\sqrt{{E_1}^2 + 4\eta^2(2\pi f)^2} - E_1}{2} \tag{6.30}$$

resulting from equating the first E_2 derivative of Eq. (6.27) with zero and solving for E_2. Equation (6.30) is the only positive and real result of the fourth order polynomial nature of the first E_2 derivative of Eq. (6.27).

The loss tangent, Eq. (6.26), reveals that the phase angle δ depends on all four parameters, E_1, E_2, η, and f, where the latter two always occur as the product $f\eta$.

If the frequency $f \to \infty$ or 0, the phase angle $\delta \to 0$ in both cases. Thus, we expect a local maximum of δ at a certain frequency f_0 (actually shown by Findley et al. 1989). Taking the first derivative of tan δ with respect to f in Eq. (6.26), equating it with zero and solving for f provides

$$f_0 = \frac{\sqrt{E_1 E_2 + {E_2}^2}}{2\pi\eta} \tag{6.31}$$

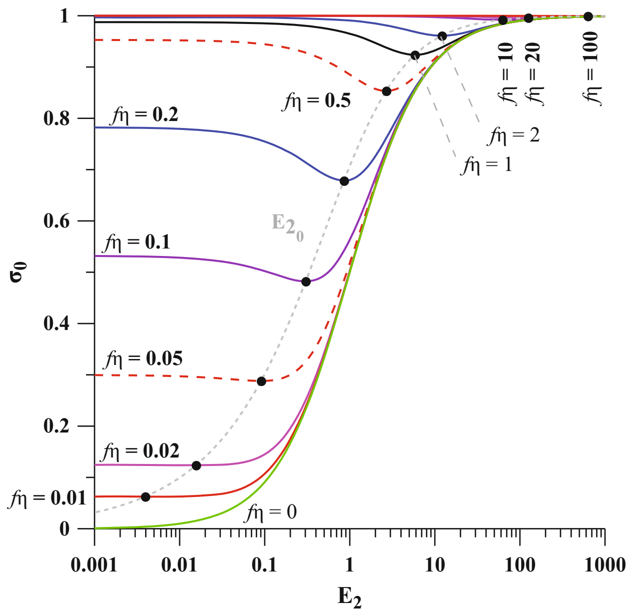

Fig. 6.5 Maximal stress σ_0 against modulus E_2 of parallel spring; f: frequency; η: viscosity constant; E_{2_0}: E_2 at which a local minimum of σ_0 is found

Replacing f by f_0 in Eq. (6.26),

$$(\tan\delta)_{f_0} = \frac{E_1}{2\sqrt{E_1 E_2 + {E_2}^2}} \tag{6.32}$$

and equating its denominator with zero yields the scenario at which $\delta = 0.5\,\pi$, which is $E_1 = -E_2$ and $E_2 = 0$. In both cases, however, $f_0 = 0$, which means that a standard linear solid can never reach $\delta = 0.5\pi$ or an infinite loss tangent.

Rearranging Eqs. (6.31) and (6.32) shows that f_0 depends on two ratios, R_1 and R_2, whereas tan δ at f_0 (but also any other f at the same R_1) depends only on R_2 (Fig. 6.6)

$$f_0 = \frac{R_1}{2\pi} \tag{6.33}$$

$$(\tan\delta)_{f_0} = \frac{R_2}{2\sqrt{1 + R_2}} \tag{6.34}$$

where

$$R_1 = \frac{E_2\sqrt{1 + R_2}}{\eta} \tag{6.35}$$

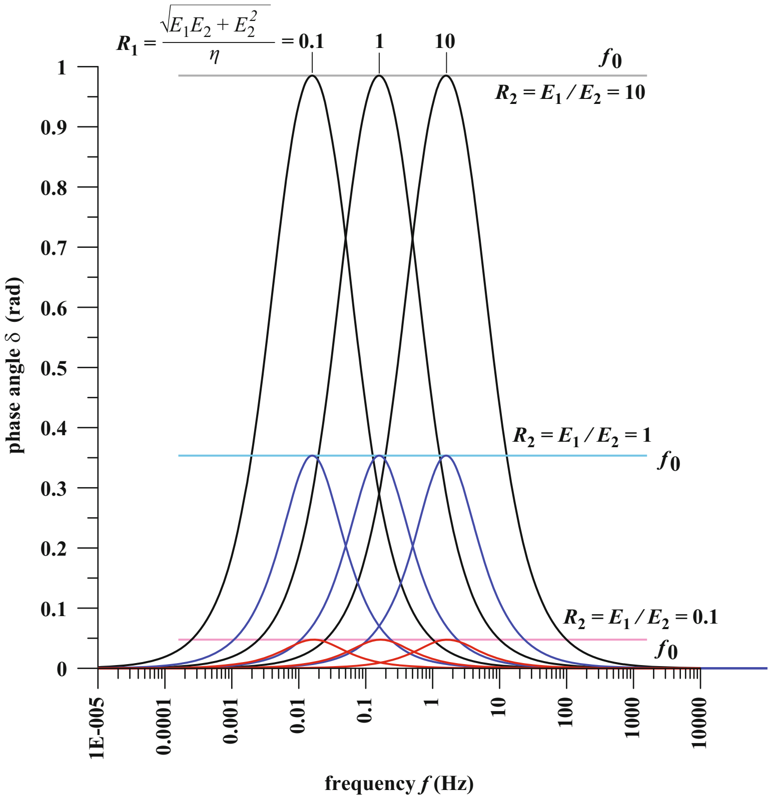

Fig. 6.6 Phase angle δ against frequency f; η: viscosity constant; E_1 and E_2: moduli of series and parallel spring; f_0: f at which a local maximum of δ is found

and

$$R_2 = \frac{E_1}{E_2} \tag{6.36}$$

Equation (6.34) shows that tan δ at f_0 is a constant and independent of f_0 (Fig. 6.6).

Changing η has the same effect as changing f has. If $\eta \to \infty$ or 0, the phase angle $\delta \to 0$ in both cases and we obtain a local maximum of δ at a certain η_0

$$\eta_0 = \frac{\sqrt{E_1 E_2 + E_2{}^2}}{2\pi f} \tag{6.37}$$

Equation (6.37) is similar to Eq. (6.31) and the same principles apply.

The phase angle δ and tan δ increase with E_1. If $E_1 \to 0$, tan $\delta \to 0$. If $E_1 \to \infty$, tan δ asymptotes to

$$\lim_{E_1 \to \infty} (\tan \delta) = \frac{\eta 2\pi f}{E_2} \tag{6.38}$$

The phase angle δ and tan δ decrease with increasing E_2, asymptoting to 0 if $E_2 \to \infty$. If $E_2 \to 0$, tan δ reaches a limit of

$$\lim_{E_2 \to 0} (\tan \delta) = \frac{E_1}{\eta 2\pi f} \tag{6.39}$$

Lakes (2009) derived the loss tangent of the SLS of Maxwell form (with a spring connected in parallel with a Maxell model).

6.2.2 The Power Law Model

The power law model is characterised by a power decay of stress σ with time t:

$$\frac{\sigma}{\varepsilon_c} = Et^{-\eta} \tag{6.40}$$

Stress σ is normalised to the constant strain ε_c applied by the Heaviside function H(t) of Eq. (6.12).

Taking Laplace transform of Eq. (6.40) yields

$$\widehat{\sigma} = \varepsilon_c E \frac{\Gamma(-\eta + 1)}{s^{-\eta+1}} \tag{6.41}$$

where Γ denotes the Gamma function (Fuss 2008a, 2012).

By substituting Eq. (6.13) into Eq. (6.41), we obtain the constitutive equation of the power law of non-linear visco-elasticity (Fuss 2008a, 2012):

$$\widehat{\sigma} = s^{\eta} \widehat{\varepsilon} E \Gamma(-\eta + 1) \tag{6.42}$$

By substituting the sinusoidal strain of Eq. (6.16) into the constitutive Eq. (6.42), we obtain

$$\widehat{\sigma} = \varepsilon_0 E \Gamma(-\eta + 1) s^{\eta} \frac{2\pi f}{s^2 + (2\pi f)^2} \tag{6.43}$$

Taking the inverse Laplace transform of Eq. (6.43), we obtain a general fractional derivative of the sine function

$$\sigma = \varepsilon_0 E \Gamma(-\eta + 1) \frac{\mathrm{d}^{\eta}}{\mathrm{d}t^{\eta}} \sin(2\pi f t) \tag{6.44}$$

where σ is the ηth time derivative of the strain function of Eq. (6.6) times a constant.

The generalised solution of $\frac{d^{\eta}}{dt^{\eta}}\sin(t)$ results from applying the inverse operation of the Riemann–Liouville fractional integration (with lower limit = 0),

$$\frac{d^{\eta}}{dt^{\eta}}\sin(t) = \sin\left(t + \eta\frac{\pi}{2}\right) + \left[\frac{t^{-1-\eta}}{\Gamma(-\eta)} - \frac{t^{-3-\eta}}{\Gamma(-\eta-2)} + \ldots\right], \tag{6.45}$$

i.e. Eq. 6.10.3 of Oldham and Spanier (1974)
consisting of *steady state* (sine function) and *transient* parts (Maclaurin series in square brackets). If $\eta \to 0$ or 1, the denominators of the transient part approach $\pm\infty$ (Gamma function of negative integers), the transient part reduces to 0, and a sine or cosine function, respectively, remains.

By replacing the lower limit of the inverse operation of the Riemann–Liouville fractional integration by $-\infty$, which equals the inverse operation of the Weyl integral (Weyl 1917), we obtain the steady state part of the fractional derivative of Eq. (6.44)

$$\sigma = \varepsilon_0 E\Gamma(-\eta+1)(2\pi f)^{\eta}\sin\left(2\pi f t + \eta\frac{\pi}{2}\right) \tag{6.46}$$

Comparing Eq. (6.7) with the steady state Eq. (6.46), it becomes evident that the phase angle δ is

$$\delta = \eta\frac{\pi}{2} \tag{6.47}$$

the loss tangent is

$$\tan\delta = \tan\left(\eta\frac{\pi}{2}\right) \tag{6.48}$$

and the peak stress σ_0 is

$$\sigma_0 = \varepsilon_0 E\Gamma(-\eta+1)(2\pi f)^{\eta} \tag{6.49}$$

The loss tangent depends solely on η, independent of E and f. If $\eta \to 1$, both $\tan\delta$ and $\sigma_0 \to \infty$, as both $\tan(0.5\pi)$ and $\Gamma(0)$ are infinite. Thus, $0 \leq \eta < 1$.

The peak stress σ_0 increases non-linearly with f and linearly with E and f^{η}. The peak stress σ_0 decreases non-linearly with η, reaching a limit of $\varepsilon_0 E$, i.e. a Hookean spring, if $\eta \to 0$. If $\eta \to 1$, σ_0 reaches infinity.

As the maximal strain rate $\dot{\varepsilon}_0$ of a sinusoidal strain function equals $\varepsilon_0 2\pi f$, the strain rate dependency of σ_0 of Eq. (6.49) is given by

$$\sigma_0 = \varepsilon_0\frac{E\Gamma(-\eta+1)}{(2\pi f)^{1-\eta}}(2\pi f)^{\eta}(2\pi f)^{1-\eta} = \dot{\varepsilon}_0\frac{E\Gamma(-\eta+1)\,{\varepsilon_0}^{1-\eta}}{\dot{\varepsilon}_0^{1-\eta}} = \dot{\varepsilon}_0^{\eta}E\Gamma(-\eta+1)\,{\varepsilon_0}^{1-\eta} \tag{6.50}$$

In contrast to the fractional derivative approach shown above, Lakes (2009) solved the loss tangent of the power model from the ratio of the constitutive equations of loss to storage modulus.

6.2.3 The Logarithmic Law Model

The logarithmic law model is characterised by a logarithmic decay of stress σ with time t:

$$\frac{\sigma}{\varepsilon_c} = E - \eta \log(t) \tag{6.51}$$

where "log" denotes the natural logarithm. Stress σ is normalised to the constant strain ε_c applied by the Heaviside function H(t) of Eq. (6.12).

Taking Laplace transform of Eq. (6.51) yields

$$\widehat{\sigma} = \varepsilon_c \frac{E}{s} - \varepsilon_c \eta \left(-\frac{\gamma}{s} - \frac{\log s}{s} \right) \tag{6.52}$$

where γ denotes the Euler–Mascheroni constant, i.e. 0.577215665 ... (Fuss 2008b, 2012).

By substituting Eq. (6.13) into Eq. (6.52), we obtain the constitutive equation of the logarithmic law of non-linear visco-elasticity (Fuss 2008b, 2012):

$$\widehat{\sigma} = \widehat{\varepsilon} E + \widehat{\varepsilon} \eta \left(\gamma + \log s \right) \tag{6.53}$$

By substituting the sinusoidal strain of Eq. (6.16) into the constitutive Eq. (6.53), we obtain

$$\widehat{\sigma} = \frac{2\varepsilon_0 \pi f}{s^2 + (2\pi f)^2} \left(E + \gamma\eta + \eta \log s \right) \tag{6.54}$$

after rearranging

$$\widehat{\sigma} = \varepsilon_0 E \frac{2\pi f}{s^2 + (2\pi f)^2} - \varepsilon_0 \eta \left(-\frac{\gamma}{s} - \frac{\log s}{s} \right) \left(\frac{s 2\pi f}{s^2 + (2\pi f)^2} \right) \tag{6.55}$$

and taking inverse Laplace transform, we obtain

$$\sigma = \varepsilon_0 E \sin(2\pi f t) - \varepsilon_0 \eta 2\pi f \left[\log t \right]^* \left[\cos(2\pi f t) \right] \tag{6.56}$$

where * denotes a convolution.

Applying the convolution integral, the convolution log (t) * cos (ωt), where $\omega = 2\pi f$, is solved accordingly:

$$[\log t] * [\cos(\omega t)] = \int_0^t \log(\tau)\cos[\omega(t-\tau)]\mathrm{d}\tau \tag{6.57}$$

where τ is the dummy variable of the convolution integral.

Decomposition of $\cos(\omega t - \omega\tau)$ according to the addition rules and subsequent partial integration yields:

$$[\log t] * [\cos(\omega t)] = \omega^{-1}[\mathrm{Ci}(\omega\tau)\sin(\omega t) - \mathrm{Si}(\omega\tau)\cos(\omega t) - \log(\tau)\sin(\omega t - \omega\tau)]_0^t \tag{6.58}$$

where Ci and Si denote cosine and sine integrals respectively (definition of Ci and Si according to Abramowitz and Stegun 1972).

Solving Eq. (6.58) from 0 to t:

$$[\log t]^* [\cos(\omega t)] = \omega^{-1}\Big[\mathrm{Ci}(\omega t)\sin(\omega t) - \mathrm{Si}(\omega t)\cos(\omega t) - \log(t)\sin(0) - \\ \mathrm{Ci}(\omega 0)\sin(\omega t) + \mathrm{Si}(\omega 0)\cos(\omega t) + \log(0)\sin(\omega t)\Big] \tag{6.59}$$

Apparently, Eq. (6.59) contains an indeterminate form, as both log(0) and Ci(0) are $-\infty$, and thus the term log(0) sin(ωt) – Ci(ω0) sin(ωt), or sin(ωt) [log(0) – Ci(0)], delivers sin(ωt) (∞–∞).

However,

$$\mathrm{Cin}(t) = \gamma + \log(t) - \mathrm{Ci}(t) \tag{6.60}$$

where Cin denotes an alternative cosine integral (definition of Cin according to Schelkunoff 1944). As Cin(0) = 0, log(0) – Ci(0) = –γ.

Yet, the argument of the cosine integral in Eq. (6.58) is $\omega\tau$, in contrast to the one of the natural logarithm, which is just τ. Thus we have to consider the multiplier ω. This multiplier leads to a convergence constant other than simply –γ, if $t \to 0$.

$$\mathrm{Ci}(\omega t) = \gamma + \log(\omega t) - \mathrm{Cin}(\omega t) = \gamma + \log(t) + \log(\omega) - \mathrm{Cin}(\omega t) \tag{6.61}$$

As Cin(0) = 0,

$$\log(0) - \mathrm{Ci}(0) = -\gamma - \log(\omega) \tag{6.62}$$

Thus,

$$\lim_{t\to 0}[\log(t) - \mathrm{Ci}(\omega t)] = -\log(\omega) - \gamma \tag{6.63}$$

Equation (6.59) can now be solved, considering that $\mathrm{Si}(0) = 0$

$$[\log t]^{*}[\cos(\omega t)] = \omega^{-1}\{\mathrm{Ci}(\omega t)\sin(\omega t) - \mathrm{Si}(\omega t)\cos(\omega t) - \sin(\omega t)[\log(\omega) + \gamma]\} \tag{6.64}$$

The solution of Eq. (6.56) after taking inverse Laplace follows

$$\sigma = \varepsilon_0 E \sin(\omega t) - \varepsilon_0 \eta\{\mathrm{Ci}(\omega t)\sin(\omega t) - \mathrm{Si}(\omega t)\cos(\omega t) - \sin(\omega t)[(\log(\omega) + \gamma)]\} \tag{6.65}$$

The steady state of Eq. (6.65) sets in at large times, or $t \to \infty$. The *transient* part of Eqs. (6.64) and (6.65) comprises of the cosine and sine integrals. When considering the values at infinity of cosine and sine integrals, which are 0 and 0.5 π respectively, the convolution of Eqs. (6.64) and (6.65) yields at large times (*steady state* equation)

$$\lim_{t\to\infty}[\log(t) * \cos(\omega t)] = \omega^{-1}\left\{-\frac{\pi}{2}\cos(\omega t) - \sin(\omega t)[\log(\omega) + \gamma]\right\} \tag{6.66}$$

and

$$\sigma = \varepsilon_0 E \sin(\omega t) + \varepsilon_0 \eta \frac{\pi}{2}\cos(\omega t) + \varepsilon_0 \eta \sin(\omega t)[\log(\omega) + \gamma] \tag{6.67}$$

respectively.

After rearranging according to the procedure applied for determining the loss tangent of the standard linear solid, we obtain

$$\tan\delta = \left(\frac{\pi}{2}\right)\frac{\eta}{E + \eta[\log(2\pi f) + \gamma]} \tag{6.68}$$

and

$$\sigma_0 = \varepsilon_0\sqrt{0.25\eta^2\pi^2 + [E + \eta\log(2\pi f) + \eta\gamma]^2} \tag{6.69}$$

As the maximal strain rate $\dot{\varepsilon}_0$ of a sinusoidal strain function equals $\varepsilon_0 2\pi f$, the strain rate dependency of σ_0 after rearranging Eq. (6.69) is given by

$$\sigma_0 = \varepsilon_0\sqrt{0.25\eta^2\pi^2 + \left[E + \eta\log\left(\frac{\dot{\varepsilon}_0}{\varepsilon_0}\right) + \eta\gamma\right]^2} \tag{6.70}$$

The stress amplitude σ_0 increases with E, η, and f. This fact is obvious, as E is the strain rate independent elasticity parameter, i.e. the modulus or stiffness; the viscosity parameter η is linked to the strain rate, i.e. at a given strain rate, σ increases with η; and the frequency f is linearly proportional to the strain rate applied periodically to the model, i.e. at a given η, σ increases with f.

If $\eta \to 0$, $\sigma_0 \to \varepsilon_0 E$, the stress equation of a Hookean spring. If $\eta \to \infty$, $\sigma_0 \to \infty$.

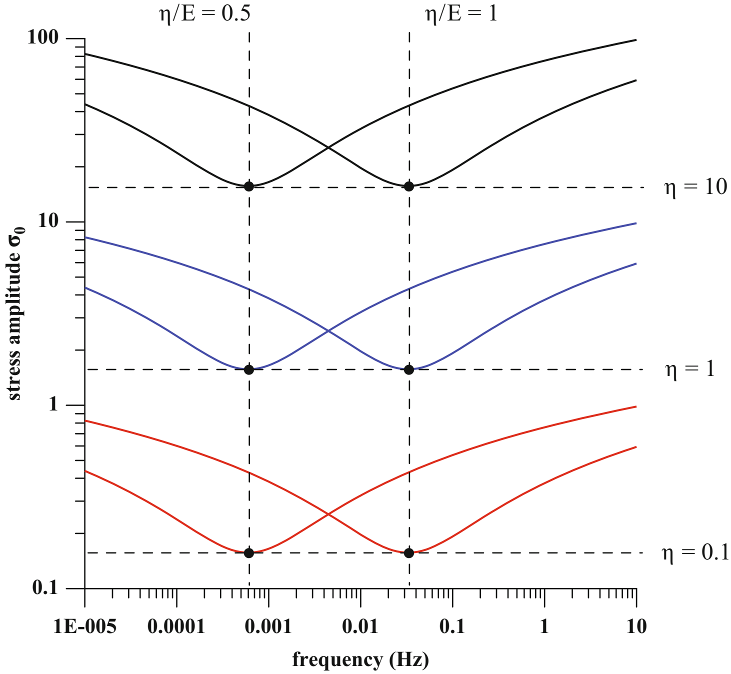

Fig. 6.7 Peak stress σ_0 against frequency f; η: viscosity constant; E: velocity independent elasticity parameter

If $f \to 0$ or ∞, $\sigma_0 \to \infty$. This means that σ_0 has a local minimum at a certain frequency f_0 (Fig. 6.7). As the first frequency derivative of Eq. (6.69) contains the arguments f, $\log(2\pi f)$, and $\log^2(2\pi f)$, there is no closed-form analytical solution, and the frequency, at which σ_0 is at a minimum, can only be obtained numerically. Figure 6.7 shows that σ_0 increases with η, and f_0 increases with the ratio η/E. The ratio η/E is identical to the viscosity of a log law model, not to be confused with the viscosity constant η.

The peak stress σ_0 increases almost linearly with E.

If $E \to \infty$, $\sigma_0 \to \infty$. If $E \to 0$ (purely viscous material)

$$\lim_{E \to 0} \sigma_0 = \varepsilon_0 \eta \sqrt{0.25\pi^2 + [\log(2\pi f) + \gamma]^2} \tag{6.71}$$

and σ_0 increases with η and f.

The loss tangent, Eq. (6.68), reveals that the phase angle δ depends on E, η, and f.

Increasing the elasticity parameter E reduces the phase angle δ, thereby asymptoting to 0, when E approaches infinity. This result is evident, as a perfectly rigid solid ($E = \infty$) does not deform and thus there are no losses ($\delta = 0$).

Reducing E increases $\tan \delta$. If E approaches zero (purely viscous material), $\tan \delta$ reaches a limit of

$$\lim_{E \to 0} (\tan \delta) = \left(\frac{\pi}{2}\right) \frac{1}{\log(2\pi f) + \gamma} \tag{6.72}$$

a constant, which is a function of f but independent of η. The magnitude of the viscosity η does not matter in this case, as the material is anyway purely viscous. The phase angle δ becomes 0.5 π if

$$f = \frac{1}{2\pi\, \mathrm{e}^{\gamma}} \tag{6.73}$$

Thus, if $E = 0$ and the loss tangent is positive, f must be $\geq 0.08936\ldots$ Hz.

If $f < 0.08936\ldots$ Hz, $\delta > 0.5$ π, and E' (storage modulus) and tan δ are negative, which is impossible, as the stored energy is zero (as $E = 0$), and the energy dissipated cannot be negative.

If the frequency $f \to \infty$ or 0, the loss tangent $\to 0$ in both cases, and the phase angle $\delta \to 0$ or π, respectively. Thus, δ crosses π/2 at a certain frequency f_0. Equating the denominator of Eq. (3.20) with zero yields f_0 at which $\delta = 0.5$ π:

$$f_0 = \frac{\mathrm{e}^{-\frac{E}{\eta} - \gamma}}{2\pi} \tag{6.74}$$

The ratio E/η in Eq. (6.68) equals the reciprocal value of the viscosity of a log law model. The higher the viscosity, the smaller is the ratio E/η. Figure 6.8 shows E/η against the frequency, the f_0 curve at which $\delta = 0.5$ π, and the E/η and frequency ranges at which δ is < or > 0.5 π. If $E/\eta \to 0$, f_0 approaches the value of Eq. (6.73), which is $f_0 = 0.089359$ Hz, i.e. a cycle period of 11.2 s. From Fig. 6.8, the loss tangent, and this the storage modulus, is negative at small E/η (high viscosity) and small frequencies (large cycle periods with small strain rates). This is in accordance with a negative (elastic) modulus of the log law model at very small strain rates (Eq. 5.73 of Fuss 2012).

The reciprocal of Eq. (6.68)

$$\cot \delta = \left(\frac{2}{\pi}\right) \left(\frac{E}{\eta} + \log(2\pi f) + \gamma\right) \tag{6.75}$$

shows that the higher E/η, the higher is cot δ. Thus, the higher η/E, the higher is tan δ. This explains that the viscosity of a log law model corresponds to the ratio η/E, and not to the viscosity constant η. Figure 6.9 shows that the higher η/E, the higher is the phase angle δ. The viscosity constant η alone has no influence on δ.

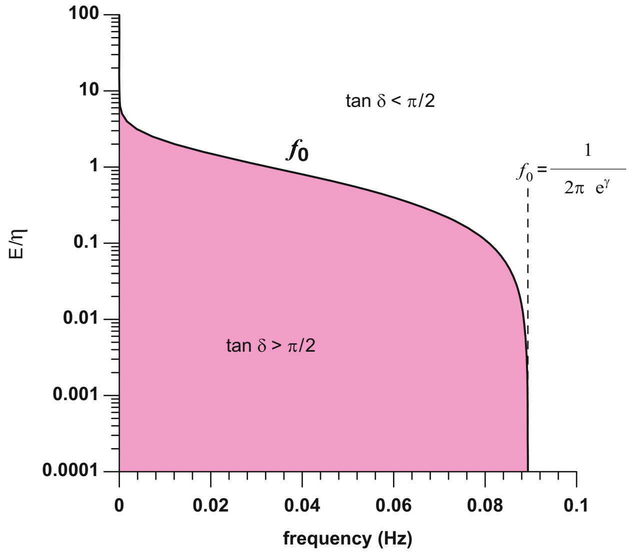

Fig. 6.8 Ratio of E/η against frequency f; η: viscosity constant; E: velocity independent elasticity parameter; tan δ: loss tangent; f_0: frequency at which tan $\delta = \pi/2$

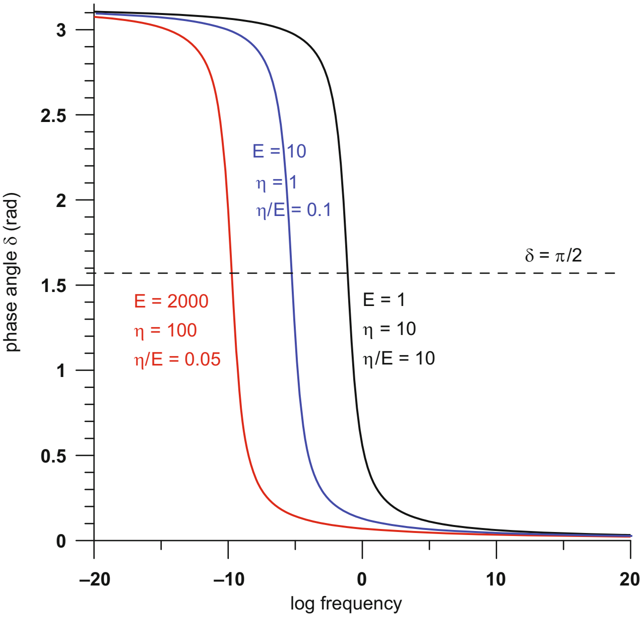

Fig. 6.9 Phase angle δ against log frequency; η: viscosity constant; E: velocity independent elasticity parameter

6.3 Summary

Material models are always simplified descriptions to assist calculations and assessment of mechanical behaviour. This means that a material does not necessarily have to follow the behaviour of its model. For example, the negative storage modulus of a log law model at small E/η and small frequencies may well be a mathematical phenomenon but does not necessarily exist in reality. The standard linear solid is certainly oversimplified with three elements (2 springs, 1 damper) and is usually expanded to more elements for better linear material characterisation (Prony series; Wiechert model; Wiechert 1889, 1893; Fuss 2012).

6.3.1 Loss Tangent and Viscosity

SLS: tan δ and δ depend on E, η, and f; but at the same f, tan δ and δ at the same R_1 depend only on R_2 (Eqs. 6.35 and 6.36), i.e. the ratio of moduli of series to parallel spring, but not on the viscosity constant η.

Power law model: tan δ and δ depend on η only; $0 \leq \eta < 1$.

Log law model: tan δ and δ depend on E, η, and f; but at the same f, larger E/η have larger tan δ and δ.

6.3.2 Relationship Between Frequency and Viscosity Constant

SLS: f and η are linked together and always appear as the product $f\eta$. Equations (6.26) and (6.27).

Power law model: η has no relationship with f in tan δ (as f does not influence tan δ), whereas for σ_0, η appears in the gamma function, and is the exponent of f, i.e. f^η.

Log law model: the viscosity constant appears as a stand alone η, and as the product of η and log $2\pi f$.

6.3.3 Transient and Steady State Parts

SLS: transient part: exponential function; steady state part: sine and cosine functions.

Power law model: transient part: Maclaurin series; steady state part: sine function with $\eta\pi/2$ phase shift (resulting in sine and cosine functions after applying addition rules).

Log law model: transient part: cosine and sine integrals; steady state part: sine and cosine functions.

6.3.4 Negative Storage Modulus if tan δ > π/2

SLS: tan $\delta < \pi/2$

Power law model: tan $\delta < \pi/2$

Log law model: tan δ can be $> \pi/2$ at small E/η (high viscosity) and small frequencies (large cycle periods with small strain rates).

Key Symbols

Ci	Cosine integral function
Cin	Alternative cosine integral function
cos	Cosine function
cot	Cotangent function
d	Differential operator
E	Modulus (velocity independent elasticity parameter)
E_1, E_2	Moduli of springs of a standard linear solid
E'	Storage modulus
E''	Loss modulus
E*	Complex modulus
|E*|	Dynamic modulus
e	Exponential function
f	Frequency
H	Heaviside (unit step) function
i	$\sqrt{-1}$
lim	Limit
log	Natural logarithm
R_1, R_2	Parameter ratios of a standard linear solid
s	Complex variable of transformed functions
Si	Sine integral function
sin	Sine function
t	Time
tan	Tangent function
tan δ	Loss tangent
Γ	Gamma function
γ	Euler–Mascheroni constant (0.577215665)
δ	Phase shift angle
ε	Strain
ε_c	Constant strain

ε_0	Amplitude of strain, peak strain
$\widehat{\varepsilon}$	Transformed strain
$\dot{\varepsilon}_0$	Peak strain rate
η	Viscosity constant
π	pi (3.14159 . . .)
σ	Stress
$\widehat{\sigma}$	Transformed stress
σ_0	Amplitude of stress, peak stress
τ	Dummy variable of convolution integral
ω	Angular frequency $\omega = 2\pi f$
$*$	Convolution operator
∞	Infinity

References

Abramowitz M, Stegun IA (1972) Handbook of mathematical functions, 9th edn. Dover Publishing, Mineola

Findley WN, Lai JS, Onaran K (1989) Creep and relaxation of nonlinear viscoelastic materials. Dover Publications, New York

Fuss FK (2008a) Cricket balls: construction, non-linear visco-elastic properties, quality control and implications for the game. Sports Technol 1(1):41–55. doi:10.1002/jst.8

Fuss FK (2008b) Logarithmic visco-elastic impact modelling of golf balls. In: Estivalet M, Brisson B (eds) The engineering of sport 7. Springer, Paris, pp 45–51

Fuss FK (2009) The collapse timing and rate of closed cell foams: revealing a conceptual misunderstanding of foam mechanics. In: Alam F, Smith LV, Subic A, Fuss FK, Ujihashi S (eds) The impact of technology on sport III. RMIT Press, Melbourne, pp 659–664

Fuss FK (2012) Nonlinear visco-elastic materials: stress relaxation and strain rate dependency. In: Dai L, Jazar RN (eds) Nonlinear approaches in engineering applications. Springer, New York, pp 135–170. doi:10.1007/978-1-4614-1469-8_5

Lakes R (2009) Viscoelastic materials. Cambridge University Press, Cambridge

Oldham KB, Spanier J (1974) The fractional calculus. Academic, New York

Provenzano P, Lakes R, Keenan T, Vanderby R (2001) Nonlinear ligament viscoelasticity. Ann Biomed Eng 29(10):908–14

Schelkunoff SA (1944) Proposed symbols for the modified cosine and exponential integral. Q Appl Math 2:90

Weyl H (1917) Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich 62:296–302

Wiechert E (1889) Ueber elastische Nachwirkung. Dissertation, Königsberg University, Königsberg

Wiechert E (1893) Gesetze der elastischen Nachwirkung für constante Temperatur. Ann Phys 286:335–348, 546–570