Lovász and Schrijver N_+-Relaxation on Web Graphs

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Abstract. In this contribution we continue the study of the Lovász-Schrijver PSD-operator applied to the edge relaxation of the stable set polytope of a graph. The problem of obtaining a combinatorial characterization of graphs for which the PSD-operator generates the stable set polytope in one step has been open since 1990. In an earlier publication, we named these graphs N_+-perfect. In the current work, we prove that the only imperfect web graphs that are N_+-perfect are the odd-cycles and their complements. This result adds evidence for the validity of the conjecture stating that the only graphs which are N_+-perfect are those whose stable set polytope is described by inequalities with near-bipartite support. Finally, we make some progress on identifying some minimal forbidden structures on N_+-perfect graphs which are also rank-perfect.

1 Introduction

Perfect graphs were introduced by Berge in the early sixties [1]. A graph is *perfect* if each of its induced subgraphs has chromatic number equal to the cardinality of a maximum cardinality clique in the subgraph.

According to the results in [6] the family of perfect graphs constitute a class where the Maximum Weighted Stable Set Problem (MWSSP) can be solved in polynomial time. Some years later, the same authors proved a beautiful result [8]: for every graph G,

$$\begin{aligned} &G \text{ is perfect} \Leftrightarrow \text{TH}(G) = \text{STAB}(G) \Leftrightarrow \text{TH}(G) = \text{CLIQUE}(G) \Leftrightarrow \\ &\text{STAB}(G) = \text{CLIQUE}(G) \Leftrightarrow \text{TH}(G) \text{ is polyhedral,} \end{aligned} \tag{1}$$

where STAB(G) is the stable set polytope of G, CLIQUE(G) is its clique relaxation and TH(G) is the *theta body* of G defined by Lovász [10].

In the early nineties, Lovász and Schrijver introduced the PSD-operator N_+ which, applied over the edge relaxation of STAB(G), generates the positive semi-definite relaxation $N_+(G)$ stronger than TH(G) [11].

As it holds for perfect graphs, MWSSP can be solved in polynomial time for the class of graphs for which $N_+(G) = \text{STAB}(G)$. We will call these graphs N_+-*perfect*.

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Our main goal is to obtain a characterization of N_+-perfect graphs similar to the one given in (1) for perfect graphs. More precisely, we would like to find an appropriate polyhedral relaxation of STAB(G) playing the role of CLIQUE(G) in (1). Following this line, in a recent publication [3], we proposed the following conjecture:

Conjecture 1. The stable set polytope of every N_+-perfect graph can be described by facet inducing inequalities with near-bipartite support.

In [2] the validity of this conjecture on near-perfect graphs is established. In fact, the following theorem is proved.

Theorem 1 [2]. *Let G be an N_+-perfect and a properly near-perfect graph. Then, either G or its complement is an odd cycle.*

Later, in [3] we extended its validity to fs-*perfect* graphs, a superclass of near-perfect graphs defined as those graphs for which the stable set polytope is completely described by clique constraints and a single full-support inequality.

The main contribution of this paper is to prove the validity of the conjecture on one more infinite family of graphs, the web graphs.

2 Preliminaries

Given a graph $G = (V, E)$ a *stable set* is a subset of mutually non-adjacent nodes in G. The maximum cardinality of a stable set is denoted by $\alpha(G)$, the *stability number of G*. The *stable set polytope* is the convex hull of the incidence vectors of the stable sets in the graph G and it is denoted by STAB(G).

The polyhedron

$$\text{FRAC}(G) = \{x \in [0,1]^V : x_i + x_j \leq 1 \text{ for every } ij \in E\}$$

is the *edge relaxation* of STAB(G).

A *clique Q* is a subset of pairwise adjacent nodes in G. Every incidence vector of a stable set must satisfy clique constraints, i.e., $\sum_{i\in Q} x_i \leq 1$. These constraints define the *clique relaxation* of the stable set polytope, CLIQUE(G). In general, STAB(G) $\subset$ CLIQUE(G). Chvatal [5] showed that perfect graphs are exactly those graphs for which equality holds.

Minimally imperfect graphs are those graphs that are not perfect but after deleting any node they become perfect. The Strong Perfect Graph Theorem states that the only minimally imperfect graphs are the odd cycles and their complements [4].

The *support* of a valid inequality for STAB(G) is the subgraph induced by the nodes having positive coefficient in it. We say that an inequality is a *full-support inequality* if its support is the whole graph.

In [14] Shepherd called a graph G *near-perfect* if its stable set polytope is defined only by non-negativity constraints, clique constraints and the *full-rank constraint*

$$\sum_{u\in V} x_u \leq \alpha(G).$$

Clearly, every node induced subgraph of a near-perfect graph is also near-perfect [14].

Due to results of Chvátal [5] near-perfect graphs constitute a superclass of perfect graphs. According to Padberg [13] minimally imperfect graphs are also near-perfect graphs.

Near-bipartite graphs, defined in [15], are those graphs such that removing all neighbours of an arbitrary node and the node itself, leaves the resulting graph bipartite.

Given integer numbers k and n such that $n \geq 2(k+1)$, the *web* graph, denoted by W_n^k, is the graph having node set $\{1, \ldots, n\}$ and such that ij is an edge if i and j differ by at most $k \pmod n$ and $i \neq j$.

If $k = 1$, W_n^1 is a cycle. If $k \geq 2$ and $n \leq 2k+2$ W_n^k is a perfect graph and W_{2k+3}^k is the complementary graph of the $(2k+3)$-cycle.

In [18] Wagler characterized all near-perfect web graphs:

Theorem 2 [18]. *A web graph is near-perfect if and only if it is perfect, an odd hole, the web W_{11}^2 or it has stability number 2.*

If $W_{n'}^{k'}$ is a node induced subgraph of W_n^k then it is a *subweb* of W_n^k. In [17] Trotter characterized for which values of n' and k', $W_{n'}^{k'}$ is a subweb of W_n^k.

Theorem 3 [17]. *If $k \geq 1$ and $n \geq 2(k+1)$ the graph $W_{n'}^{k'}$ is a subweb of W_n^k if and only if*

$$\frac{k'}{k} \leq \frac{n'}{n} \leq \frac{k'+1}{k+1}.$$

2.1 The N_+-Operator

As we have already mentioned, in this paper we focus on the behaviour of the N_+-operator defined by Lovász and Schrijver [11] on the edge relaxation of the stable set polytope.

We denote by $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ the vectors of the canonical basis of $\mathbb{R}^{n+1}$ (where the first coordinate is indexed zero), $\mathbf{1}$ the vector with all components equal to 1 and $\mathbb{S}_+^n$ the space of n-by-n symmetric and positive semidefinite matrices with real entries.

Given a convex set K in $[0,1]^n$, let

$$\text{cone}(K) = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} : x = x_0 y; \;\; y \in K \right\}.$$

Then, we define the polyhedral set

$$\begin{aligned} M(K) = \big\{ Y \in \mathbb{S}_+^{n+1} : \;\; & Y\mathbf{e}_0 = \text{diag}(Y), \\ & Y\mathbf{e}_i \in \text{cone}(K), \\ & Y(\mathbf{e}_0 - \mathbf{e}_i) \in \text{cone}(K), \; i = 1, \ldots, n \big\}, \end{aligned}$$

where $\text{diag}(Y)$ denotes the vector whose i-th entry is Y_{ii}, for every $i = 0, \ldots, n$.

Projecting this polyhedral lifting back to the space $\mathbb{R}^n$ results in

$$N_+(K) = \left\{ x \in [0,1]^n : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y\mathbf{e}_0, \text{ for some } Y \in M(K) \right\}.$$

In practice, we prove that a point $x \in [0,1]^n$ belongs to $N_+(K)$ by showing the existence of a symmetric PSD matrix Y of the form

$$Y = \left(\begin{array}{c|c} 1 & x^t \\ \hline x & \bar{Y} \end{array} \right) \tag{2}$$

where x^t stands for the transpose of column vector x and $\bar{Y}$ is an $n \times n$ matrix with columns $\bar{Y}_i$ for $i = 1, \ldots, n$, satisfying the following conditions:

1. $\bar{Y}_{ii} = x_i$,
2. If $x_i = 0$ then $\bar{Y}_i = \mathbf{0}$,
3. If $x_i = 1$ then $\bar{Y}_i = x$,
4. If $0 < x_i < 1$ then $\frac{1}{x_i}\bar{Y}_i \in K$ and $\frac{1}{1-x_i}(x - \bar{Y}_i) \in K$,

for every $i = 1, \ldots, n$.

In [11], Lovász and Schrijver proved that $N_+(K)$ is a relaxation of the convex hull of integer solutions in K.

If we let $N_+^0(K) = K$ then k-th application of the N_+-operator is $N_+^k(K) = N_+(N_+^{k-1}(K))$ for every $k \geq 1$. The authors in [11] showed that $N_+^n(K) = \text{conv}(K \cap \{0,1\}^n)$.

In this work we focus on the behaviour of a single application of the N_+-operator on the edge relaxation of the stable set polytope of a graph. Then, in order to simplify the notation we write $N_+(G) = N_+(\text{FRAC}(G))$.

In [11] it is shown that

$$\text{STAB}(G) \subset N_+(G) \subset \text{TH}(G) \subset \text{CLIQUE}(G).$$

Also from results in [11], we know that graphs for which every facet defining inequality of $\text{STAB}(G)$ has a near-bipartite support is N_+-perfect. Then, Conjecture 1 establishes that these graphs are the only N_+-perfect graphs.

In particular, perfect and near-bipartite graphs are N_+-perfect. In addition, it can be proved that every subgraph of an N_+-perfect graph is also N_+-perfect. A graph G that is not N_+-perfect is called *N_+-imperfect*.

Using the properties of the N_+-operator, if G' is an N_+-imperfect subgraph of G then G is also N_+-imperfect.

In [7] and [9] it was proved that all the imperfect graphs with at most 6 nodes are N_+-perfect graphs, except for the two imperfect near-perfect graphs depicted in Fig. 1. The graph on the left is denoted by G_{LT} and the other one is denoted by G_{EMN}.

A graph G' is an *odd subdivision* of a graph G if it is obtained by replacing an edge of G by a path of odd length.

As a consequence of the results in [9] we have the following:

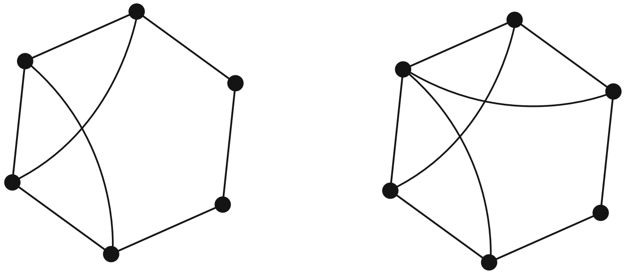

Fig. 1. The graphs G_{LT} and G_{EMN}.

Lemma 1. *If G is N_+-imperfect and G' is obtained after the odd subdivision of an edge in G, then G' is also N_+-imperfect.*

This result becomes relevant in the proof of the validity of the conjecture on web graphs since there we show that most of the web graphs have an odd subdivision of the graph G_{LT} as a node induced subgraph.

3 The Conjecture on Web Graphs

The fact that the conjecture holds on web graphs will follow after proving that the only N_+-perfect webs are either perfect or minimally imperfect webs.

Theorem 1 [2] asserts that every near-perfect graph satisfies the conjecture, therefore, from Theorem 2, we only need to consider web graphs with stability number at least three. It is known that the stability number of W_n^k is $\alpha(W_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor$. Then, if $n \leq 3k+2$, W_n^k is near-perfect and from Theorem 1 the conjecture holds on these web graphs. Therefore, from now on we can consider web graphs W_n^k with $n \geq 3k+3$.

Now we are able to present the following result:

Theorem 4. *If $n \geq 9$ and $n \neq 10$, W_n^2 has an odd subdivision of G_{LT} as a node induced subgraph.*

Proof. Let $n \geq 9$ and $\{1, \ldots, n\}$ be the node set of W_n^2. Assume that we delete the six consecutive nodes in the set $\{n-5, n-4, \ldots, n\}$. Note that if we find a subset $T^s = \{v_1, \ldots, v_{2s}\}$ of $\{1, \ldots, n-6\}$, with $s \geq 1$, $v_1 = 1$, $v_{2s} = n-6$ and such that T^s induces a path in W_n^2, then $T^s \cup \{n-4, n-3, n-2, n-1\}$ induces in W_n^2 an odd subdivision of G_{LT}.

For example, in the web W_{14}^2 the set $T^3 = \{1, 2, 4, 5, 7, 8\}$ induces a path and $T^3 \cup \{10, 11, 12, 13\}$ induces an odd subdivision of G_{LT}. See Fig. 2.

Then, in order to prove the result we show the existence of such a set T^s with the above required properties for every $n \geq 9$ and $n \neq 10$.

We divide the rest of the proof into four different cases according to the value of $n-6$.

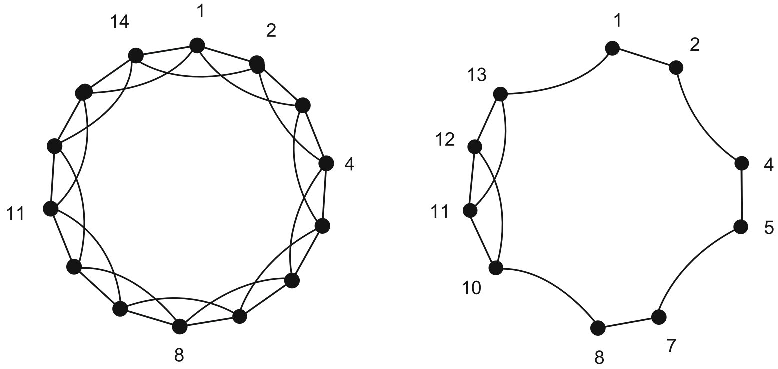

Fig. 2. The web graph W_{14}^2 and a node induced odd subdivision of G_{LT}.

– If $n-6=4r+3$ for some $r \geq 0$, then

$$T^{r+1} = \{2t-1 : 1 \leq t \leq 2r+2\}.$$

– If $n-6=4r+2$ then $r \geq 1$ since $n \geq 9$. In this case, we consider

$$T^{r+1} = \{2t-1 : 1 \leq t \leq 2r+1\} \cup \{n-6\}.$$

– If $n-6=4r+1$ again $r \geq 1$. In this case, we find

$$T^{r+1} = \{2t : 1 \leq t \leq 2r\} \cup \{1, n-6\}.$$

– If $n-6=4r$ we have $r \geq 2$ since $n \geq 9$ and $n \neq 10$. In this case we consider

$$T^{r+1} = \{3+2t : 1 \leq t \leq 2r-2\} \cup \{1, 2, 4, n-6\}.$$

□

Corollary 1. *If $n \geq 8$ and $n \neq 10$ then W_n^2 is N_+-imperfect.*

Proof. The web graph W_8^2 is an imperfect near-perfect graph and then it is N_+-imperfect.

Since every odd subdivision of G_{LT} is N_+-imperfect and the N_+-imperfection of a subgraph implies the N_+-imperfection of the graph itself, the result follows directly from the previous theorem. □

In order to complete the analysis of the family of web graphs W_n^2 we need to prove that W_{10}^2 is N_+-imperfect. Note that it does not have an odd subdivision of G_{LT} as a node induced subgraph. Instead, we make use of the definition of $N_+(W_{10}^2)$.

Lemma 2. *The web graph W_{10}^2 is N_+-imperfect.*

Proof. The proof is based on finding a point $\bar{x} \in N_+(W_{10}^2) \setminus \text{STAB}(W_{10}^2)$.

Let us consider the point $\bar{x} = \lambda\mathbf{1} \in \text{FRAC}(W_{10}^2)$ for $\lambda = \frac{31}{100}$. Clearly it violates the full-rank constraint and therefore, $\bar{x} \notin \text{STAB}(W_{10}^2)$.

In order to prove that $\bar{x} \in N_+(W_{10}^2)$ we present a matrix Y as in (2) which represents the point in the higher dimensional space.

For this purpose we make use of the following definition.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be such that $T(v_1, \ldots, v_n) = (v_n, v_1, \ldots, v_{n-1})$. The matrix $\text{circ}(u)$ is the $n \times n$-matrix whose first row is $T^0(u) = u$ and whose j-th row is given by $T^{j-1}(u) = T(T^{j-2}(u))$, for every $j \geq 2$.

Let $z = (\lambda, 0, 0, \beta, \gamma, \delta, \gamma, \beta, 0, 0)$ where

$$\gamma = \frac{853}{10000}, \quad \delta = \frac{336}{10000} \quad \text{and} \quad \beta = \frac{2234}{10000}.$$

If $\bar{Y} = \text{circ}(z)$ then it is not difficult to check that $Y \in \mathbb{R}^{11}$ defined as in (2) is PSD and it satisfies that

$$\frac{1}{\lambda}\bar{Y}_i \in \text{FRAC}(W_{10}^2) \quad \text{and} \quad \frac{1}{1-\lambda}(\lambda\mathbf{1} - \bar{Y}_i) \in \text{FRAC}(W_{10}^2).$$

□

The following result implies that Conjecture 1 holds for web graphs.

Theorem 5. *If the web graph W_n^k is N_+-perfect then it is either a perfect or a minimally imperfect graph.*

Proof. Due to the fact that the conjecture is proved for near-perfect graphs (Theorem 1) we only need consider those webs which are not near-perfect and prove that none of them are N_+-perfect.

If the web W_n^k is not near-perfect then $k \geq 2$ and $n \geq 3k + 3$.

If $k = 2$ the result follows from Corollary 1 and Lemma 2.

Let $k \geq 3$ and $n \geq 3k + 3$. We will prove that every web W_n^k has a subweb of the form $W_{n'}^2$ for some $n' \geq 8$.

After Trotter's result (Theorem 3) $W_{n'}^2$ is a subweb of W_n^k if

$$\frac{2n}{k} \leq n' \leq \frac{3n}{k+1}.$$

Let $\Delta^k(n) = \frac{3n}{k+1} - \frac{2n}{k} = n\frac{k-2}{k(k+1)}$.

Observe that $\Delta^k(n)$ assumes its minimum value when n is minimum, i.e. when $n = 3k + 3$. In this case, $\Delta^k(n) = \frac{3(k-2)}{k} \geq 1$. Then, for every value of $n \geq 3k + 3$ we can find an integer n' satisfying

$$\frac{2n}{k} \leq n' \leq \frac{3n}{k+1}$$

and then $W_{n'}^2$ is a subweb of W_n^k.

Moreover, since $n \geq 3k + 3$ we have that $\left\lfloor \frac{3n}{k+1} \right\rfloor \geq 9$ and W_n^k has a subweb $W_{n'}^2$ for some $n' \geq 8$.

Finally, Corollary 1 and Lemma 2 prove the result. □

4 On minimally N_+-imperfect subgraphs

In [2] we proved the validity of the conjecture on the family of near-perfect graphs. Rank-perfect graphs constitute a superclass of near-perfect graphs, then it seems natural to continue our work towards proving the conjecture on this family. In fact, while studying the N_+-perfect graphs which are also near-perfect graphs we could identify some minimal forbidden structures on this family.

We say that a graph is *minimally N_+-imperfect* if it is N_+-imperfect but deleting any node leaves an N_+-perfect graph. The results in [2] give the minimally N_+-imperfect graphs in the family of near-perfect graphs. In order to present them let us introduce some more definitions.

We denote by C_{2k+1} the cycle having node set $\{1,\dots,2k+1\}$ and edge set $\{i(i+1) : i \in \{1,\dots,2k\} \cup \{1(2k+1)\}$.

In [2] we consider two families of near-perfect graphs, named $\mathcal{W}^k$ and $\mathcal{H}^k$ for each $k \geq 2$.

Let $\mathcal{H}^2 = \mathcal{W}^2 = \{G_{LT}, G_{EMN}\}$. For $k \geq 3$, $\mathcal{W}^k$ is the family of graphs with node set $\{0,1,\dots,2k+1\}$ such that:

- $G - 0 = C_{2k+1}$;
- there is no pair of consecutive nodes (in C_{2k+1}) with degree 2;
- the degree of node 0 is $k+2$.

For $k \geq 3$, $\mathcal{H}^k$ is the family of graphs having node set $\{0,1,\dots,2k+1\}$ such that:

- $G - 0$ is the complement of C_{2k+1};
- there is no pair of consecutive nodes (in C_{2k+1}) with degree $2k-2$;
- the degree of node 0 is at most $2k$.

In Fig. 3 we have represented one of the graphs in the family $\mathcal{W}^9$ and one of the graphs in the family $\mathcal{H}^9$.

Using the results in [2] we have the following

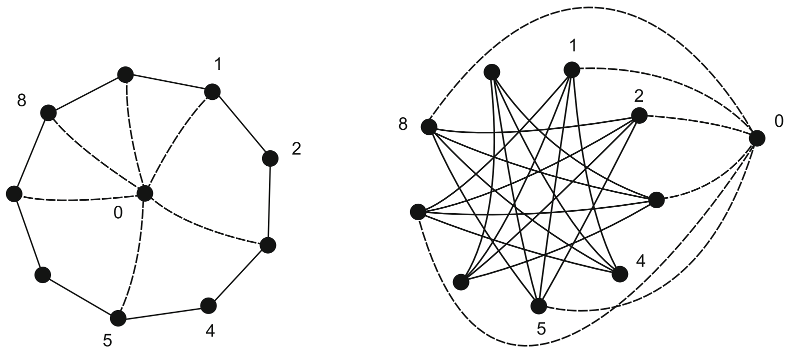

Fig. 3. A graph in the family $\mathcal{W}^9$ and $\mathcal{H}^9$.

Lemma 3. *Let G be a minimally N_+-imperfect graph. If G is a near-perfect graph then it is an odd subdivision of a graph in $\mathcal{H}^k \cup \mathcal{W}^k$ for some $k \geq 2$.*

In this contribution it was important to identify the odd subdivisions of G_{LT} as minimally N_+-imperfect structures in the webs, except for W_{10}^2.

In fact, we have proved that the web W_{10}^2 is a minimally N_+-imperfect rank-perfect graph which is not near-perfect. This shows some advance in order to characterize N_+-perfect rank-perfect graphs and also a line for our future research.

References

1. Berge, C.: Perfect graphs. In: Six Papers on Graph Theory, pp. 1–21. Indian Statistical Institute, Calcutta (1963)
2. Bianchi, S., Escalante, M., Nasini, G., Tunçel, L.: Near-perfect graphs with polyhedral $N_+(G)$. Electron. Notes Discrete Math. **37**, 393–398 (2011)
3. Bianchi, S., Escalante, M., Nasini, G., Tunçel, L.: Lovász-Schrijver PSD-operator and a superclass of near-perfect graphs. Electron. Notes Discrete Math. **44**, 339–344 (2013)
4. Chudnovsky, M., Robertson, N., Seymour, P., Thomas, R.: The strong perfect graph theorem. Ann. Math. **164**, 51–229 (2006)
5. Chvátal, V.: On certain polytopes associated with graphs. J. Comb. Theory B **18**, 138–154 (1975)
6. Grötschel, M., Lovász, L., Schrijver, A.: The ellipsoid method and its consequences in combinatorial optimization. Combinatorica **1**, 169–197 (1981)
7. Escalante, M., Montelar, M.S., Nasini, G.: Minimal N_+-rank graphs: progress on Lipták and Tunçel's conjecture. Oper. Res. Lett. **34**(6), 639–646 (2006)
8. Grötschel, M., Lovász, L., Schrijver, A.: Geometric Algorithms and Combinatorial Optimization. Springer, New York (1988)
9. Lipták, L., Tunçel, L.: Stable set problem and the lift-and-project ranks of graphs. Math. Program. **98**, 319–353 (2003)
10. Lovász, L.: On the Shannon capacity of a graph. IEEE Trans. **25**, 1–7 (1979)
11. Lovász, L., Schrijver, A.: Cones of matrices and set-functions and 0–1 optimization. SIAM J. Optim. **1**, 166–190 (1991)
12. Nasini, G.: Caracterización de los grafos fs-perfectos. Reunión Anual de Comunicaciones Científicas de la Unión Matemática Argentina, Tucumán, Argentina (2011)
13. Padberg, M.: On the facial structure of set packing polyhedra. Math. Program. **5**, 199–215 (1973)
14. Shepherd, F.B.: Near-Perfect matrices. Math. Program. **1–3**(64), 295–323 (1994)
15. Shepherd, F.B.: Applying Lehman's theorems to packing problems. Math. Program. **71**, 353–367 (1995)
16. Stauffer, G.: On the stable set polytope of claw-free graphs, Ph.D. thesis, EPF Lausanne (2005)
17. Trotter, L.: A class of facet producing graphs for vertex packing polyhedra. Discrete Math. **12**, 373–388 (1975)
18. Wagler, A.: Relaxing perfectness: which graphs are "almost perfect"? In: Groestchel, M.(ed.) The Sharpest Cut: The Impact of Manfred Padberg and His Work. SIAM, Philadelphia (2002)