Chapter 7
Monotone Operators Approached via Convex Analysis

7.1 Historical Overview and Motivation

The monotone operators started being intensively investigated during the 1960's by authors like Browder, Brézis or Minty, and it did not take much time until their connections with convex analysis were noticed by Rockafellar, Gossez and others. The fact that the (convex) subdifferential of a proper, convex and lower semicontinuous function is a maximally monotone operator was one of the reasons for connecting these at a first sight maybe unrelated research fields. One of the most important challenges of the next decades was to identify a function that could be associated to a monotone operator in order to help investigating it by means of convex analysis, in addition to the previously used methods belonging to fixed point theory and equilibrium problems. Such functions were proposed by Coodey, Simons or Krauss, but the real breakthrough was brought by Fitzpatrick's function, introduced in [86], neglected for more than a decade and independently rediscovered in the early 2000's by Martínez-Legaz and Théra, and Burachik and Svaiter, respectively. Shortly afterwards, the Fitzpatrick family of representative functions was introduced, offering new tools for approaching the monotone operators via convex analysis. Since then, the number of papers where different aspects of monotone operators were investigated, especially by means of convex analysis, has increased in a spectacular manner, due to authors like Bauschke, Borwein, Boţ, Marques Alves, Martínez-Legaz, Penot, Simons, Svaiter, Voisei, Yao, Zălinescu and some of the already mentioned ones, besides the new results many older statements being rediscovered or improved in this way.

Perhaps the most famous problem regarding monotone operators concerns the maximality of the sum of two maximally monotone operators. Different hypotheses that guarantee the mentioned outcome were successfully proposed for the case the space on which the mentioned monotone operators are defined on is reflexive, but it is still unknown whether they work or not if the space is a general Banach one. Other interesting problems involving monotone operators regard their surjectivity

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properties, the properties of their domains and ranges, the relations between different classes of them, their extensions etc. The investigations on monotone operators have led to advances back in convex analysis, too, let us mention here only the notions of Fenchel totally unstable functions (cf. [21, 190]) or sets that are closed regarding others (cf. [42, 45]). Moreover, the algorithms for finding zeros of (combinations of) monotone operators were successfully employed for solving convex optimization problems, too.

The sum of the ranges of two monotone operators defined on Banach spaces is usually larger than the range of their sum. Under some additional conditions these sets are almost equal, i.e. their interiors and closures coincide. Brézis and Haraux brought the first contributions in this directions in [60] and since then determining when the sum of the ranges of two monotone operators is almost equal in the sense mentioned above to the range of their sum is known as the *Brézis-Haraux approximation* problem, being treated in works like [9, 70, 72, 73, 171, 176, 190]. There is a rich literature on the applications of the Brézis-Haraux approximation, let us mention here only the ones for variational inequality problems, Hammerstein equations and Neumann problem (cf. [60]), complementarity problems (cf. [70]), generalized equations of maximally monotone type (cf. [171]) and Bregman and projection algorithms. Our contributions to this topic, summarized in Sect. 7.3 and originally published in [35, 40, 42, 44], concern Brézis-Haraux type approximation statements for the sum of a monotone operator with the composition with a linear mapping of another one, where the involved spaces are general Banach ones. When particularizing the involved operators to subdifferentials of proper, convex and lower semicontinuous functions, some statements from [70, 176] are corrected and extended, respectively.

Problems arising from fields like inverse problems, Fenchel-Rockafellar and Singer-Toland duality schemes, Clarke-Ekeland least action principle (cf. [5]), variational inequalities (cf. [19]), Schrödinger equations and others (cf. [4]) can be modelled to lead to the surjectivity or the identification of zeros of a combination of monotone operators. These, together with the known surjectivity properties of a monotone operator, let us mention just the classical ones due to Minty and Rockafellar (see, for instance, [190]), respectively, motivated the investigations regarding the ranges of combinations of monotone operators whose outcomes were published in recent works such as [162, 163, 177, 190, 222]. In Sect. 7.4 we present, following our paper [30], weak closedness type conditions involving representative functions that equivalently characterize or guarantee the surjectivity of a sum of a maximally monotone operator with a translation of another one. Particularizing then these results for the zeros of the mentioned sum and for the case when the involved monotone operators are subdifferentials, we improved several recent statements from the literature.

Similarities and connections between monotone operators and bifunctions were noticed in the seminal paper [12], followed by works like [116, 135, 160], where the latter were investigated mostly by means of equilibrium problems and different maximality or boundedness results for them were provided. On the other hand, we proposed in [33] a way to deal with the maximal monotonicity of the bifunctions by means of representative functions and this path was followed in very recent papers like [2, 136]. In Sect. 7.5 we attach to a bifunction two functions which are then used for approaching the maximal monotonicity of the bifunction by means of convex analysis. We succeeded to extend in this way to general Banach spaces some results known in the literature only for reflexive ones. Moreover, we provided positive answers to some recently posed conjectures from [135, 136].

7.2 Preliminaries on Monotone Operators

Before proceeding with our investigations on monotone operators, we present some notions and preliminary results used later in the exposition, following [19,21,65,86, 104, 161, 172, 173, 190, 221] and some of the references therein.

7.2.1 Monotone Operators

Within this chapter, unless otherwise mentioned, the involved spaces will be considered to be Banach spaces, equipped with norms usually denoted by $\|\cdot\|$, while the norm on its dual space is denoted by $\|\cdot\|_*$. Let X and Y be nontrivial real Banach spaces. We present first the definition of a monotone operator, followed by ones of different properties the latter can have.

Definition 7.1. A multifunction $T : X \rightrightarrows X^*$ is called a *monotone operator* provided that for any $x, y \in X$ one has $\langle y^* - x^*, y - x\rangle \geq 0$ whenever $x^* \in T(x)$ and $y^* \in T(y)$.

Having a monotone operator $T : X \rightrightarrows X^*$, its *domain* is the set $D(T) = \{x \in X : T(x) \neq \emptyset\}$, its *range* is $R(T) = \cup\{T(x) : x \in X\}$, while its *graph* is $G(T) = \{(x, x^*) : x \in X, x^* \in T(x)\}$. One can also consider the monotone operator $-T : X \rightrightarrows X^*$ whose graph is $G(-T) = \{(x, x^*) \in X \times X^* : (x, -x^*) \in G(T)\}$.

Definition 7.2. The monotone operator $T : X \rightrightarrows X^*$ is called *maximal* when its graph is not properly included in the graph of any other monotone operator $T' : X \rightrightarrows X^*$.

The next class of monotone operators was introduced in [104] and afterwards it was shown that it coincides in the maximality case with some other ones considered in various circumstances in the literature.

Definition 7.3. A monotone operator $T : X \rightrightarrows X^*$ is called of *type* (D) provided that each element of its *monotone closure* operator $\overline{T} : X^{**} \rightrightarrows X^*$,

$$G(\overline{T}) = \left\{(x^{**}, x^*) \in X^{**} \times X^* : \langle x^{**} - y, x^* - y^*\rangle \geq 0 \ \forall (y, y^*) \in G(T)\right\}$$

is the limit in the weak$^*\times$strong topology of $X^{**} \times X^*$ of a bounded net $\{(x_i, x_i^*)_i\} \subseteq G(T)$.

Remark 7.1. The monotone closure is not the only closure of a monotone operator considered in the literature. Another one can be found, for instance, in [62].

Remark 7.2. In reflexive Banach spaces every maximally monotone operator is of type (D) and coincides with its closure operator. On the other hand, not every monotone operator of type (D) is maximal, as the example presented in [104, Remarques 2, p.376] shows. Note also that according to [173], $\operatorname{cl} R(T) = \operatorname{cl} R(\overline{T})$ for any monotone operator $T : X \rightrightarrows X^*$.

Another class of monotone operators we consider within this work is the following one, originally introduced in [60], but mentioned in the literature under different names like *star-monotone operators* (see [171]), 3^**-monotone operators* (cf. [70, 176, 217]) and (BH)*-operators* (in [72, 73]).

Definition 7.4. A monotone operator $T : X \rightrightarrows X^*$ is said to be *rectangular* if for all $x^* \in R(T)$ and $x \in D(T)$ there is some $\beta(x^*, x) \in \mathbb{R}$ such that $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y\rangle \geq \beta(x^*, x)$.

Example 7.1. The subdifferential of a proper, convex and lower semicontinuous function defined on X is a classical example for all these classes of monotone operators. In [104, Théoréme 3.1] it was proven that it is a monotone operator of type (D), according to [217] (see also [190]) it is rectangular, while its maximal monotonicity was proven for the first time in [179]. However, one can find in the literature (see, for instance, [9, 10, 104, 190]) also examples of monotone operators belonging to the mentioned classes that are not subdifferentials. Moreover, in [10, Example 5.4] one can find a maximally monotone operator that is not rectangular, while in [10, Example 3.3] a rectangular monotone operator that is not maximal is mentioned.

Remark 7.3. One of the most important maximally monotone operators is the *duality map*

$$\mathcal{J} : X \rightrightarrows X^*,$$
$$\mathcal{J}(x) = \partial\Big(\frac{1}{2}\|\cdot\|^2\Big)(x) = \Big\{x^* \in X^* : \|x\|^2 = \|x^*\|_*^2 = \langle x^*, x\rangle\Big\}, x \in X,$$

that can be used, for instance, as noted below, for formulating a maximality criterium for a monotone operator.

The following statements from [19] and [176], respectively, will be used later in our investigations.

Lemma 7.1. *When X is a reflexive Banach space, a monotone operator $T : X \rightrightarrows X^*$ is maximal if and only if the mapping $T(x + \cdot) + \mathcal{J}(\cdot)$ is surjective for all $x \in X$.*

Lemma 7.2. *Given the monotone operator of type* (D) $T : X \rightrightarrows X^*$ *and the nonempty subset* $E \subseteq X^*$ *such that for any* $x^* \in E$ *there is some* $x \in X$ *fulfilling* $\inf_{(y,y^*)\in G(T)} \langle x^* - y^*, x - y\rangle > -\infty$, *one has* $E \subseteq \operatorname{cl}(R(T))$ *and* $\operatorname{int}(E) \subseteq R(\overline{T})$.

7.2.2 Representative Functions

In order to deal with monotone operators by means of convex analysis, different functions were attached to them in the literature. The one that has facilitated the most important progresses in this direction is the one introduced by Fitzpatrick in [86].

Definition 7.5. The *Fitzpatrick function* attached to the monotone operator $T : X \rightrightarrows X^*$ is

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}},\ \varphi_T(x, x^*) = \sup\big\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in Ty\big\}.$$

The Fitzpatrick function attached to any monotone operator is convex and weak-weak* lower semicontinuous. Moreover, using it one can show that a monotone operator $T : X \rightrightarrows X^*$ is rectangular if and only if $D(T) \times R(T) \subseteq \operatorname{dom}\varphi_T$. Note also that in [9] one can find interesting connections between rectangular monotone operators and almost convex sets (that are called there nearly convex). The function $\psi_T := \overline{\operatorname{co}}(c + \delta_{G(T)})$, where the closure is considered in the strong topology, is very well connected to the Fitzpatrick function. On $X \times X^*$ we have $\psi_T^{*\top} = \varphi_T$ and, when X is a reflexive Banach space, one also has $\varphi_T^{*\top} = \psi_T$. If $T : X \rightrightarrows X^*$ is maximally monotone, then $\varphi_T \geq c$ and $G(T) = \{(x, x^*) \in X \times X^* : \varphi_T(x, x^*) = \langle x^*, x\rangle\}$. These properties of the Fitzpatrick function motivate attaching to monotone operators other functions, as follows.

Definition 7.6. Given the monotone operator $T : X \rightrightarrows X^*$, a convex and strong-strong lower semicontinuous function $h_T : X \times X^* \to \overline{\mathbb{R}}$ fulfilling $h_T \geq c$ and $G(T) \subseteq \{(x, x^*) \in X \times X^* : h_T(x, x^*) = c(x, x^*)\}$ is said to be a *representative function* of T. The set $\mathscr{F}_T$ of all the representative functions of the monotone operator T is said to be the *Fitzpatrick family* of T.

Note that if $G(T) \neq \emptyset$ (in particular if T is maximally monotone), then every representative function of T is proper. It follows immediately that $\varphi_T, \psi_T \in \mathscr{F}_T$. If $f : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, then the function $(x, x^*) \mapsto f(x) + f^*(x^*)$ is a representative function of the maximally monotone operator $\partial f : X \rightrightarrows X^*$ and we call it the *Fenchel representative function* (cf. [30]). If f is moreover sublinear, the only representative function associated to ∂f is the Fenchel one, which coincides in this case with the Fitzpatrick function of ∂f. Some properties of maximally monotone operators and representative functions attached to them that we need further follow (cf. [65]).

Lemma 7.3. *Let $T : X \rightrightarrows X^*$ be a maximally monotone operator and $h_T \in \mathscr{F}_T$. Then*

(i) $\varphi_T(x,x^*) \leq h_T(x,x^*) \leq \psi_T(x,x^*)$ *for all* $(x,x^*) \in X \times X^*$*;*
(ii) The restriction of $h_T^{*\top}$ *to* $X \times X^*$ *is also a representative function of* T*;*
(iii) $\{(x,x^*) \in X \times X^* : h_T(x,x^*) = c(x,x^*)\} = \{(x,x^*) \in X \times X^* : h_T^{*\top}(x,x^*) = c(x,x^*)\} = G(T)$.

Given the monotone operator $T : X \rightrightarrows X^*$ with $G(T) \neq \emptyset$ and $h_T \in \mathscr{F}_T$, denote by $\hat{h}_T : X \times X^* \to \overline{\mathbb{R}}$ the function defined as $\hat{h}_T(x,x^*) = h_T(x,-x^*)$, $x \in X$, $x^* \in X^*$. Note that $\hat{f}_T$ is proper, convex and strong-strong lower semicontinuous, too and $\hat{h}_T(x,x^*) \geq -\langle x^*, x\rangle$ and $\hat{h}_T^*(x^*,x) = h_T^*(x^*,-x)$ for all $x \in X$ and all $x^* \in X^*$.

Let us now give two maximality criteria for monotone operators involving convex functions, the first one, following [65, Theorem 3.1] and [172, Proposition 2.1], in reflexive spaces, the other one originally given in [161, Theorem 3.1] with the hypothesis $0 \in \operatorname{sqri}\left(\Pr_X\ (\operatorname{dom} h)\right)$ and generalized by translation arguments as given below in [33].

Lemma 7.4. *Let X be reflexive. If $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function with $h \geq c$, then the monotone operator $\{(x,x^*) \in X \times X^* : h(x,x^*) = c(x,x^*)\}$ is maximal if and only if $h^{*\top} \geq c$.*

Lemma 7.5. *Let $h : X \times X^* \to \overline{\mathbb{R}}$ be a proper and convex function with $h \geq c$ and $h^{*\top} \geq c$ on $X \times X^*$. If* $\operatorname{sqri}\Pr_X(\operatorname{dom} h) \neq \emptyset$*, then the operator $\{(x,x^*) \in X \times X^* : h^*(x^*,x) = c(x,x^*)\}$ is maximally monotone.*

7.3 Brézis-Haraux Type Approximations

We give in this section some results concerning the so-called *Brézis-Haraux type approximation* of the range of the sum of a monotone operator with a monotone operator composed with a linear continuous mapping, following our papers [35, 40, 42, 44]. These results are then particularized by taking for the monotone operators the subdifferentials of some proper, convex and lower semicontinuous functions.

7.3.1 Brézis-Haraux Type Approximations for Sums of Rectangular Monotone Operators

Consider two monotone operators $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ and a linear continuous mapping $A : X \to Y$. It is known that $S + A^* \circ T \circ A$ is a monotone operator and under certain conditions it is maximally monotone (see [42, 44, 171, 172], for instance). The construction $S + A^* \circ T \circ A$ encompasses at least two important special cases. Taking S to be the *zero operator* defined as $S(x) = \{0\}$ for all $x \in X$, the results we give provide their counterparts for the composition of a monotone operator with a linear continuous mapping, while when $X = Y$ and A is the identity mapping of X one obtains corresponding results regarding the sum of two monotone operators. We show first that $S + A^* \circ T \circ A$ is rectangular when S and T are rectangular monotone operators.

Theorem 7.1. *If the monotone operators S and T are rectangular, then $S + A^* \circ T \circ A$ is rectangular, too.*

Proof. If $D(S + A^* \circ T \circ A) = \emptyset$, the conclusion arises trivially. Otherwise take $w^* \in R(S + A^* \circ T \circ A)$, i.e. there are some $w \in X$ and $x^*, z^* \in X^*$ such that $x^* \in S(w)$, $z^* \in A^* \circ T \circ A(w)$ and $w^* = x^* + z^*$. Let $x \in D(S + A^* \circ T \circ A)$. We have

$$\inf_{(y,y^*)\in G(S+A^*\circ T\circ A)} \langle w^* - y^*, x - y\rangle = \inf_{\substack{(y,u^*)\in G(S),\\ (y,v^*)\in G(A^*\circ T\circ A),\\ u^*+v^*=y^*}} \langle x^* + z^* - (u^* + v^*), x - y\rangle$$

$$\geq \inf_{(y,u^*)\in G(S)} \langle x^* - u^*, x - y\rangle + \inf_{(y,v^*)\in G(A^*\circ T\circ A)} \langle z^* - v^*, x - y\rangle. \tag{7.3.1}$$

As $z^* \in A^* \circ T \circ A(w)$, there is some $r^* \in T \circ A(w)$ such that $z^* = A^* r^*$. Clearly, $r^* \in R(T)$. Denote $u = Ax \in D(T)$. When $v^* \in A^* \circ T \circ A(y)$ there is some $s^* \in T \circ A(y)$ such that $v^* = A^* s^*$. We have

$$\inf_{(y,v^*)\in G(A^*\circ T\circ A)} \langle z^* - v^*, x - y\rangle = \inf_{(y,s^*)\in G(T\circ A)} \langle A^* r^* - A^* s^*, x - y\rangle$$

$$= \inf_{(y,s^*)\in G(T\circ A)} \langle r^* - s^*, A(x - y)\rangle \geq \inf_{(v,s^*)\in G(T)} \langle r^* - s^*, u - v\rangle \geq \beta(r^*, u) \in \mathbb{R},$$

since T is rectangular. As S is also rectangular, (7.3.1) yields that $S + A^* \circ T \circ A$ is rectangular, too. □

Remark 7.4. Taking $X = Y$ and A to be the identity mapping of X, one rediscovers as a special case of Theorem 7.1 the result given in [9, Lemma 11], i.e. that the sum of two rectangular monotone operators is rectangular, too.

The next statement provides a Brézis-Haraux type approximation of the range of $S + A^* \circ T \circ A$ through the ranges of the monotone operators S and T.

Theorem 7.2. *If the monotone operators S and T are rectangular and $S + A^* \circ T \circ A$ is of type (D), one has*

(i) $\operatorname{cl} R(S + A^* \circ T \circ A) = \operatorname{cl}(R(S) + A^*(R(T))) = \operatorname{cl} R(\overline{S + A^* \circ T \circ A})$;
(ii) $\operatorname{int} R(S + A^* \circ T \circ A) \subseteq \operatorname{int}(R(S) + A^*(R(T))) \subseteq \operatorname{int} R(\overline{S + A^* \circ T \circ A})$.

Proof. As the monotone operator $S + A^* \circ T \circ A$ is of type (D) its domain is nonempty, thus $D(S) \cap D(A^* \circ T \circ A) \neq \emptyset$. By Theorem 7.1 we obtain that it is rectangular, too.

Take $x^* \in R(S + A^* \circ T \circ A)$. Then there exist $x \in D(S + A^* \circ T \circ A)$ and $y^*, z^* \in X^*$ such that $x^* = y^* + z^*$, $y^* \in S(x)$ and $z^* \in A^* \circ T \circ A(x)$. Obviously $z^* \in A^*(R(T))$, thus $x^* = y^* + z^* \in R(S) + A^*(R(T))$. Consequently $R(S + A^* \circ T \circ A) \subseteq R(S) + A^*(R(T))$ and the same inclusion exists also between the closures, respectively the interiors, of these sets.

Let now $x^* \in R(S) + A^*(R(T))$, thus there are some $x_1^* \in R(S)$, $x_2^* \in R(A^* \circ T \circ A)$ and $z^* \in R(T)$ such that $x^* = x_1^* + x_2^*$ and $x_2^* = A^* z^*$. Taking an $x \in D(S + A^* \circ T \circ A)$ there holds

$$\inf_{(y,y^*) \in G(S+A^* \circ T \circ A)} \langle x^* - y^*, x - y \rangle = \inf_{\substack{(y,u^*) \in G(S), \\ (y,v^*) \in G(A^* \circ T \circ A), \\ u^* + v^* = y^*}} \langle x_1^* + x_2^* - (u^* + v^*), x - y \rangle$$

$$\geq \inf_{(y,u^*) \in G(S)} \langle x_1^* - u^*, x - y \rangle + \inf_{(y,v^*) \in G(A^* \circ T \circ A)} \langle x_2^* - v^*, x - y \rangle > -\infty,$$

as both S and $A^* \circ T \circ A$ are rectangular. Applying Lemma 7.2 for $E = R(S) + A^*(R(T))$ and $S + A^* \circ T \circ A$, we obtain that $R(S) + A^*(R(T)) \subseteq \operatorname{cl} R(S + A^* \circ T \circ A)$ and $\operatorname{int}(R(S) + A^*(R(T))) \subseteq R(\overline{S + A^* \circ T \circ A})$. Taking into consideration what we have already proven above, (i) and (ii) follow. □

Remark 7.5. Taking $X = Y$ and A to be the identity mapping of X, one rediscovers as a special case of Theorem 7.2 the result given in [70, Theorem 3.1] and [176, Theorem 1].

When X is moreover reflexive the inequalities in Theorem 7.2(ii) turn into equalities and we get a more accurate Brézis-Haraux approximation of the range of $S + A^* \circ T \circ A$.

Theorem 7.3. *If the Banach space X is moreover reflexive, the monotone operators S and T are rectangular and $S + A^* \circ T \circ A$ is maximally monotone, one has*

(i) $\operatorname{cl}(R(S) + A^*(R(T))) = \operatorname{cl} R(S + A^* \circ T \circ A)$;
(ii) $\operatorname{int} R(S + A^* \circ T \circ A) = \operatorname{int}(R(S) + A^*(R(T)))$.

Proof. As X is reflexive, the maximally monotone operator $S + A^* \circ T \circ A$ is of type (D), too, and $S + A^* \circ T \circ A = \overline{S + A^* \circ T \circ A}$. The conclusion follows via Theorem 7.2. □

Remark 7.6. Taking $X = Y$ and A to be the identity mapping of X, one rediscovers as a special case of Theorem 7.3 the result given in [70, Corollary 3.1] and [176, Corollary 1].

7.3.2 Brézis-Haraux Type Approximations for Sums of Subdifferentials

Now we turn our attention to the most famous example for many classes of monotone operators, namely the subdifferential of a proper, convex and lower semicontinuous function. Let the proper, convex and lower semicontinuous functions $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, and the linear continuous mapping $A : X \to Y$ fulfilling the feasibility condition $A(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$. Like in Sects. 2.2.3, 2.3.3, and the other places where we dealt with unconstrained optimization problems, let us note that valuable special cases of the results presented in the following can be obtained by taking $X = Y$ and A to be the identity mapping of X and, respectively, when f is the zero function. Before giving a Brézis-Haraux type statement involving ranges of subdifferentials, we introduce the following regularity condition inspired from (RC_4^U)

$$(RCM^{BH}) \;\Big|\; \operatorname{epi} f^* + (A^* \times \operatorname{id}_{\mathbb{R}})(\operatorname{epi} g^*) \text{ is closed in the topology } \omega(X^*, X) \times \mathscr{R}.$$

Theorem 7.4. *If* (RCM^{BH}) *is valid, then one has*

(i) $\operatorname{cl}(R(\partial f) + A^*(R(\partial g))) = \operatorname{cl} R(\partial f + A^* \circ \partial g \circ A) = \operatorname{cl} R(\partial(f + g \circ A))$;
(ii) $\operatorname{int} R(\partial(f + g \circ A)) = \operatorname{int} R(\partial f + A^* \circ \partial g \circ A) \subseteq \operatorname{int}(R(\partial f) + A^*(R(\partial g))) \subseteq \operatorname{int} D(\partial(f^* \Box A^* g^*)) = \operatorname{int} D(\partial(f + g \circ A)^*)$.

Proof. As f, g and $f + g \circ A$ are proper, convex and lower semicontinuous, by Example 7.1 we know that $\partial(f + g \circ A)$ is a monotone operator of type (D), while ∂f and ∂g are rectangular.

By Corollary 2.14 we know that (RCM^{BH}) implies $\partial f + A^* \circ \partial g \circ A = \partial(f + g \circ A)$, therefore $\partial f + A^* \circ \partial g \circ A$ is maximally monotone operator of type (D), too.

Applying Theorem 7.2 for $S = \partial f$ and $T = \partial g$ we get

$$\operatorname{cl}(R(\partial f) + A^*(R(\partial g))) = \operatorname{cl} R(\partial f + A^* \circ \partial g \circ A) = \operatorname{cl} R(\partial(f + g \circ A)),$$

i.e. (i), and

$$\operatorname{int} R(\partial f + A^* \circ \partial g \circ A) \subseteq \operatorname{int}(R(\partial f) + A^*(R(\partial f))) \subseteq \operatorname{int} R(\overline{\partial f + A^* \circ \partial g \circ A}),$$

which becomes

$$\operatorname{int} R(\partial(f + g \circ A)) \subseteq \operatorname{int}(R(\partial f) + A^*(R(\partial g))) \subseteq \operatorname{int} R(\overline{\partial(f + g \circ A)}). \tag{7.3.2}$$

From Sect. 2.2.3 one can deduce that under (RCM^{BH}) it holds $(f + g \circ A)^* = f^* \Box A^* g^*$, by [104, Théoréme 3.1] we get $R(\overline{\partial(f + g \circ A)}) = D(\partial(f + g \circ A)^*) = D(\partial(f^* \Box A^* g^*))$. Combining this with (7.3.2) one gets (ii). $\square$

Remark 7.7. Similar results to the ones in Theorem 7.4 have been obtained for the case when $X = Y$ and A is the identity mapping of X in [176, Corollary 2] and [70, Corollary 3.2], under the hypothesis that $\bigcup_{t>0} t(\text{dom}\, f - \text{dom}\, g)$ is a closed linear subspace of X. However, some of the results obtained there are not true in general Banach spaces. In [176] it is claimed that the mentioned hypotheses yield $\text{int}(R(\partial f) + R(\partial g)) = \text{int}\, D(\partial(f^* \Box g^*))$, while according to [70] they imply that $\text{int}(R(\partial f) + R(\partial g)) = \text{int}\, D(\partial(f + g)^*)$. However, as the situation depicted in Example 7.2, which is due to Fitzpatrick and was brought into our attention by [173, Example 2.21], shows, these conclusions can be false when working in nonreflexive Banach spaces.

Example 7.2. Take $X = c_0$, the space of the real sequences converging to 0, which is a nonreflexive Banach space with the usual norm $\|x\| = \sup_{n\geq 1} |x_n|$ for $x = (x_n)_{n\geq 1} \in c_0$, and let $f, g : c_0 \to \mathbb{R}$, with f taking everywhere the value 0 and $g(x) = \|x\| + \|x - e_1\|$, for all $x \in c_0$, where $e_1 = (1, 0, 0, \dots) \in c_0$. Both functions f and g are proper, convex and continuous and the regularity condition required in [70, 176] is fulfilled. Moreover for any $x \in c_0$ one has $\partial g(x) = \partial\| \cdot \|(x) + \partial\| \cdot - e_1\|(x)$. The dual space of c_0 is ℓ^1, which consists of all the sequences $y = (y_n)_{n\geq 1}$ such that $\|y\|_* = \sum_{n=1}^{+\infty} |y_n| < +\infty$. Denote by F the set of sequences in ℓ^1 having finitely many nonzero entries and by B^* the closed unit ball in ℓ^1.

It is known that $\| \cdot \|^*(y) = 0$ if $\|y\|_* \leq 1$ and $\| \cdot \|^*(y) = +\infty$ otherwise, which leads to $\partial\| \cdot \|(x) = B^*$ if $x = 0$, $\partial\| \cdot \|(e_1) = \{e_1\}$, $\partial\| \cdot \|(-e_1) = \{-e_1\}$ and $\partial\| \cdot \|(x) = \{y \in \ell^1 : \|y\|_* \leq 1, \langle y, x\rangle = \|x\|\} \subseteq F$, otherwise, where we note that $e_1 \in \ell^1$, too. Moreover, we have $\partial\| \cdot - e_1\|(x) = \partial\| \cdot \|(x - e_1)$ for any $x \in c_0$. Further one gets $\partial g(0) = -e_1 + B^*$ and $\partial g(e_1) = e_1 + B^*$. Otherwise, i.e. if $x \in c_0 \setminus \{0, e_1\}$, $\partial g(x) \subseteq F$. Therefore

$$R(\partial g) \subseteq (-e_1 + B^*) \cup (e_1 + B^*) \cup F. \tag{7.3.3}$$

Since $\text{int}\, R(\partial g)$ includes $\text{int}\, B^* \pm e_1$, assuming it convex yields $0 = 1/2(e_1 - e_1) \in \text{int}\, R(\partial g)$. Hence there exists a neighborhood of 0, say U, completely included in $R(\partial g)$. Take some $\lambda > 0$ sufficiently small such that

$$\nu(\lambda) = \left(0, \frac{\lambda}{2^2}, \frac{\lambda}{2^3}, \frac{\lambda}{2^4}, \dots\right) \in U.$$

Thus $\nu(\lambda) \in R(\partial g)$. One can check that $\|\nu(\lambda) \pm e_1\|_* = 1 + \frac{\lambda}{2} > 1$, so, taking into consideration (7.3.3), $\nu(\lambda)$ must be in F. It is clear that this does not happen, thus we reached a contradiction. Therefore $\text{int}\, R(\partial g)$ is not convex, unlike $\text{int}\, \overline{R(\partial g)}$, whose convexity follows via [189, Theorem 20].

On the other hand, the relations claimed in [70, 176] to be valid and mentioned in Remark 7.7 become both now $\text{int}\, R(\partial g) = \text{int}\, D(\partial g^*)$, which is equivalent, via [104, Théoréme 3.1], to $\text{int}\, R(\partial g) = \text{int}\, \overline{R(\partial g)}$. But, as we have seen above, this does not happen for f and g as selected above, thus the allegations concerning the interior of the sum of the ranges of two subdifferentials in [70, 176] are false.

In the light of Remark 7.7 and Example 7.2, let us give below the consequence of Theorem 7.4 for the case $X = Y$ and A is the identity mapping of X which corrects and generalizes, by asking the fulfillment of a weaker regularity condition, [176, Corollary 2] and [70, Corollary 3.2].

Corollary 7.1. *Let f and g be two proper, convex and lower semicontinuous functions on the Banach space X with extended real values such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Assuming that*

$$\operatorname{epi} f^* + \operatorname{epi} g^* \textit{is closed in the product topology } \omega(X^*, X) \times \mathscr{R},$$

one has

(i) $\operatorname{cl}(R(\partial f) + R(\partial g)) = \operatorname{cl} R(\partial f + \partial g) = \operatorname{cl} R(\partial(f + g))$*;*
(ii) $\operatorname{int} R(\partial f + \partial g) = \operatorname{int} R(\partial(f + g)) \subseteq \operatorname{int}(R(\partial f) + R(\partial g)) \subseteq \operatorname{int} D(\partial(f^* \Box g^*)) = \operatorname{int} D(\partial((f + g)^*))$.

Remark 7.8. Considering moreover that the Banach space X is reflexive, Theorem 7.3 yields that the inclusions in Corollary 7.1(*ii*) turn into equalities.

7.3.3 Applications of the Brézis-Haraux Type Approximations

Besides the fields of applications of the Brézis-Haraux type approximations mentioned before (see, for instance, [60, 171]), we present below two concrete ways to apply the results we provided within this section.

7.3.3.1 Existence of a Solution to an Optimization Problem

Let the proper, convex and lower semicontinuous functions $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ and the linear continuous mapping $A : X \to Y$ such that $A(\operatorname{dom} f) \cap \operatorname{dom} g \neq \emptyset$.

Theorem 7.5. *Assume that* (RCM^{BH}) *is satisfied and moreover that* $0 \in \operatorname{int}(R(\partial f) + A^*(R(\partial g)))$. *Then there exists a neighborhood V of* 0 *in X^* such that for all $x^* \in V$ there exists an $\bar{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$ for which*

$$f(\bar{x}) + g(A\bar{x}) - \langle x^*, \bar{x} \rangle = \min_{x \in X} \big[f(x) + g(Ax) - \langle x^*, x \rangle \big].$$

Proof. By Theorem 7.4 we have $\operatorname{int}(R(\partial f) + A^*(R(\partial g))) \subseteq \operatorname{int} D(\partial(f^* \Box A^* g^*))$, thus $0 \in \operatorname{int} D(\partial(f^* \Box A^* g^*))$, i.e. there is a neighborhood V of 0 in X^* such that $V \subseteq D(\partial(f^* \Box A^* g^*)) = D(\partial((f + g \circ A)^*))$.

Let $x^* \in V$. The properties of the subdifferential yield that there is an $\bar{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g)$ such that $(f + g \circ A)^*(x^*) + (f + g \circ A)^{**}(\bar{x}) = \langle x^*, \bar{x} \rangle$.

As $f + g \circ A$ is a proper, convex and lower semicontinuous function we have $(f + g \circ A)^{**} = f + g \circ A$, hence the equality stated above becomes

$$f(\bar{x})+g(A\bar{x})-\langle x^*,\bar{x}\rangle = -(f+g\circ A)^*(x^*) = -\max_{x\in X}\{\langle x^*,x\rangle - f(x)-g(Ax)\},$$

yielding thus the conclusion. □

Remark 7.9. Under the hypotheses of Theorem 7.5, (RCM^{BH}) is equivalent to

$$\inf_{x\in X}\left[f(x)+g(Ax)-\langle x^*,x\rangle\right] = \max_{y^*\in Y^*}\left\{-f^*(x^*-A^*y^*)-g^*(y^*)\right\}\ \forall x^*\in X^*.$$

Thus one may notice that the conclusion of the mentioned statement can be refined in the sense that the outcome is something that may be called *locally stable total Fenchel duality*, i.e. the situation where both the primal and the dual problem have optimal solutions and their values coincide for small enough linear perturbations of the objective function of the primal problem. Let us notice moreover that as $0 \in V$, for $x^* = 0$ we obtain also the Fenchel total duality statement, too.

7.3.3.2 Existence of a Solution to a Complementarity Problem

Consider now X to be a reflexive Banach space, let $C \subseteq X$ be a closed convex cone and $S : X \rightrightarrows X^*$ a maximally monotone operator. In the following we will show that Theorem 7.3 can guarantee under certain hypotheses the existence of a solution to the complementarity problem (cf. [70])

$$(CP) \qquad \begin{cases} x \in C,\ x^* \in C^*,\\ \langle x^*,x\rangle = 0,\\ x^* \in S(x).\end{cases}$$

But before we can prove the mentioned statement we have to mention a recent result of ours, originally given in [42, 44]. Recall that the sum of two maximally monotone operators is always a monotone operator that in general fails to be maximal and the problem of finding hypotheses that guarantee its maximality has been firstly solved in [180].

Lemma 7.6. *Given two maximally monotone operators $S, T : X \rightrightarrows X^*$, if the condition*

$$(RCM^M)\left|\begin{array}{l}\{(x^*+y^*,x,y,r) : \varphi_S^*(x^*,x)+\varphi_T^*(y^*,y)\le r\}\ \textit{is closed}\\ \textit{regarding the subspace } X^*\times\Delta_X\times\mathbb{R},\end{array}\right.$$

is fulfilled then $S + T$ is a maximally monotone operator, too.

Proof. Fix first some $z \in X$ and $z^* \in X^*$. We prove that there is always an $\bar{x} \in X$ such that $z^* \in (S+T)(\bar{x}+z) + \mathscr{J}(\bar{x})$. Consider the functions $f, g : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$f(x, x^*) = \inf_{y^* \in X^*} \left[\varphi_S(x + z, x^* + z^* - y^*) + \varphi_T(x + z, y^*)\right] - \langle x^* + z^*, z\rangle$$

and

$$g(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle,\ (x, x^*) \in X \times X^*.$$

Let us calculate the conjugates of f and g. For any $(w^*, w) \in X^* \times X$ we have

$$f^*(w^*, w) = \sup_{\substack{x \in X,\\ x^* \in X^*}} \Big\{\langle w^*, x\rangle + \langle x^*, w\rangle - \inf_{y^* \in X^*} \left[\varphi_S(x + z, x^* + z^* - y^*)\right.$$

$$\left.+\varphi_T(x + z, y^*)\right] + \langle x^* + z^*, z\rangle\Big\} = \sup_{\substack{x \in X,\\ x^*, y^* \in X^*}} \{\langle w^*, x\rangle + \langle x^*, w\rangle + \langle x^* + z^*, z\rangle$$

$$-\varphi_S(x + z, x^* + z^* - y^*) - \varphi_T(x + z, y^*)\} = \sup_{\substack{u \in X,\\ u^*, y^* \in X^*}} \{\langle w^*, u - z\rangle + \langle u^* + y^*$$

$$-z^*, w\rangle + \langle u^* + y^*, z\rangle - \varphi_S(u, u^*) - \varphi_T(u, y^*)\} = \sup_{\substack{u \in X,\\ u^*, y^* \in X^*}} \{\langle w^*, u\rangle + \langle u^* + y^*,$$

$$w + z\rangle - \varphi_S(u, u^*) - \varphi_T(u, y^*)\} - \langle w^*, z\rangle - \langle z^*, w\rangle.$$

Considering the function $F : X \times X \times X^* \times X^* \to \overline{\mathbb{R}}$, $F(a, b, a^*, b^*) = \varphi_S(a, a^*) + \varphi_T(b, b^*)$ and the linear mappings $A : X \times X^* \times X^* \to X \times X \times X^* \times X^*$, $A(a, a^*, b^*) = (a, a, a^*, b^*)$ and $M : X^* \times X \to X^* \times X \times X, M(a^*, a) = (a^*, a, a)$, we have that

$$f^*(w^*, w) = (F \circ A)^*(M(w^*, w + z)) - \langle w^*, z\rangle - \langle z^*, w\rangle\ \forall (w^*, w) \in X^* \times X.$$

Because $F^* : X^* \times X^* \times X \times X \to \overline{\mathbb{R}}$, $F^*(a^*, b^*, a, b) = \varphi_S^*(a^*, a) + \varphi_T^*(b^*, b)$ and $A^* : X^* \times X^* \times X \times X \to X^* \times X \times X$, $A^*(a^*, b^*, a, b) = (a^* + b^*, a, b)$, one has

$$A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(F^*)) = \{(a^* + b^*, a, b, r) : \varphi_S^*(a^*, a) + \varphi_T^*(b^*, b) \le r\}.$$

Knowing that $\mathrm{Im}\, M \times \mathbb{R} = X^* \times \Delta_X \times \mathbb{R}$, the regularity condition (RCM^M) is equivalent to saying that $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(F^*))$ is closed regarding the subspace $\mathrm{Im}\, M \times \mathbb{R}$. So, by Theorem 2.10, we have that for any $(w^*, w) \in X^* \times X$ it holds

$$(F \circ A)^*(M(w^*, w + z))$$
$$= \min\left\{F^*(a^*, b^*, a, b) : (a^* + b^*, a, b) = (w^*, w + z, w + z)\right\}.$$

Back to f^*, one gets immediately that for any $(w^*, w) \in X^* \times X$

$$f^*(w^*, w) = \min_{a^*+b^*=w^*} \left[\varphi_S^*(a^*, w + z) + \varphi_T^*(b^*, w + z)\right] - \langle w^*, z\rangle - \langle z^*, w\rangle.$$

Regarding g^*, the conjugate of g, for any $(w^*, w) \in X^* \times X$ one has

$$\begin{aligned} g^*(w^*, w) &= \sup_{\substack{x\in X,\\ x^*\in X^*}} \left\{\langle w^*, x\rangle + \langle x^*, w\rangle - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x^*\|_*^2 + \langle z^*, x\rangle\right\} \\ &= \sup_{x\in X} \left\{\langle w^* + z^*, x\rangle - \frac{1}{2}\|x\|^2\right\} + \sup_{x^*\in X^*} \left\{\langle x^*, w\rangle - \frac{1}{2}\|x^*\|_*^2\right\} \\ &= \frac{1}{2}\|w^* + z^*\|_*^2 + \frac{1}{2}\|w\|^2. \end{aligned}$$

For any $(x, x^*) \in X \times X^*$ and $y^* \in X^*$, by Lemma 7.3 one gets

$$\begin{gathered} \varphi_S(x + z, x^* + z^* - y^*) + \varphi_T(x + z, y^*) - \langle x^* + z^*, z\rangle + g(x, x^*) \geq \\ \langle x^* + z^* - y^*, x + z\rangle + \langle y^*, x + z\rangle - \langle x^* + z^*, z\rangle + \\ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 + \langle x^*, x\rangle \geq 0. \end{gathered}$$

Taking in the left-hand side the infimum subject to all $y^* \in X^*$, we get $f(x, x^*)+g(x, x^*) \geq 0$. Thus $\inf_{(x,x^*)\in X\times X^*}[f(x, x^*)+g(x, x^*)] \geq 0$. Because of the convexity of f and g and since the latter is continuous Fenchel's duality theorem (cf. [48, Theorem 3.3.7]) guarantees the existence of a pair $(\bar{x}^*, \bar{x}) \in X^* \times X$ such that

$$\begin{aligned} \inf_{(x,x^*)\in X\times X^*}[f(x, x^*) + g(x, x^*)] &= \max_{(x^*,x)\in X^*\times X}\{-f^*(x^*, x) - g^*(-x^*, -x)\} \\ &= -f^*(\bar{x}^*, \bar{x}) - g^*(-\bar{x}^*, -\bar{x}). \end{aligned}$$

Using the result from above, one gets $f^*(\bar{x}^*, \bar{x}) + g^*(-\bar{x}^*, -\bar{x}) \leq 0$. So there are some $\bar{a}^*$ and $\bar{b}^*$ in X^* such that $\bar{a}^* + \bar{b}^* = \bar{x}^*$ and

$$\varphi_S^*(\bar{a}^*, \bar{x} + z) + \varphi_T^*(\bar{b}^*, \bar{x} + z) - \langle\bar{x}^*, z\rangle - \langle z^*, \bar{x}\rangle + \frac{1}{2}\|-\bar{x}^* + z^*\|_*^2 + \frac{1}{2}\|-\bar{x}\|^2 \leq 0.$$

Taking into account that $\bar{a}^* + \bar{b}^* = \bar{x}^*$, we get

$$\begin{aligned} 0 \geq{}& \left(\varphi_S^*(\bar{a}^*, \bar{x} + z) - \langle\bar{a}^*, \bar{x} + z\rangle\right) + \left(\varphi_T^*(\bar{b}^*, \bar{x} + z) - \langle\bar{b}^*, \bar{x} + z\rangle\right) \\ &+ \left(\langle\bar{x}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|\bar{x}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2\right) \geq 0, \end{aligned}$$

where the last inequality comes from Lemma 7.3. Thus the inequalities above must hold as equalities, hence

$$\varphi_S^*(\bar{a}^*, \bar{x}+z) = \langle \bar{a}^*, \bar{x}+z\rangle, \ \ \varphi_T^*(\bar{b}^*, \bar{x}+z) = \langle \bar{b}^*, \bar{x}+z\rangle,$$

and

$$\langle \bar{a}^* + \bar{b}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|\bar{a}^* + \bar{b}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2 = 0.$$

These three equalities are equivalent, due to Lemma 7.3, to $\bar{a}^* \in S(\bar{x}+z)$, $\bar{b}^* \in T(\bar{x}+z)$ and, respectively,

$$z^* - \bar{a}^* - \bar{b}^* \in \partial \frac{1}{2}\|\cdot\|^2(\bar{x}) = \mathscr{J}(\bar{x}).$$

Summing these three relations up, one gets

$$z^* - \bar{a}^* - \bar{b}^* + \bar{a}^* + \bar{b}^* \in (S+T)(\bar{x}+z) + \mathscr{J}(\bar{x}).$$

As z and z^* have been arbitrarily chosen, the conclusion follows via Lemma 7.1. □

Remark 7.10. The regularity condition (RCM^M) we gave in Lemma 7.6 is the weakest in the literature that guarantees the maximal monotonicity of the sum of two maximally monotone operators. For a review on more restrictive regularity conditions that deliver the same outcome the reader is referred to [44]. Note moreover that in [21, Theorem 25.4] one can find another weak regularity condition for this, that is formulated via arbitrary representative functions attached to the involved maximally monotone operators, while in [42, Theorem 1] and [21, Theorem 25.1] (see also [38]) weak hypotheses that guarantee the maximal monotonicity of the sum of a maximally monotone operator with another one that is composed with a linear continuous mapping are provided.

Now we are ready to formulate the announced assertion regarding the existence of a solution to (CP).

Theorem 7.6. *Suppose that the monotone operator S is maximal and rectangular, the regularity condition*

$$(RCM^C)\left|\begin{array}{l}\{(x^*+y^*, x, y, r) : (x^*, x, r) \in \operatorname{epi}(\varphi_S^*), y \in C, y^* \in -C^*\} \text{ is closed} \\ \text{regarding the subspace } X^* \times \Delta_X \times \mathbb{R},\end{array}\right.$$

is satisfied and $0 \in \operatorname{int}(R(S) - C^*)$. *Then the complementarity problem* (CP) *admits a solution.*

Proof. Recall first that $\delta_C^* = \delta_{-C^*}$ and $N_C(x) = \{y^* \in -C^* : \langle y^*, x\rangle = 0\}$ for all $x \in C$. Moreover, $R(N_C) = -C^*$ since $R(N_C) \subseteq -C^* = N_C(0)$.

The Fitzpatrick function attached to N_C is, when $(x, x^*) \in X \times X^*$,

$$\varphi_{N_C}(x, x^*) = \sup_{(y,y^*)\in G(N_C)} \{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle\}$$

$$= \sup_{\substack{y\in C, y^*\in -C^*,\\ \langle y^*, y\rangle = 0}} \{\langle y^*, x\rangle + \langle x^*, y\rangle\} = \begin{cases} 0, & \text{if } x \in C, x^* \in -C^*,\\ +\infty, & \text{otherwise,}\end{cases}$$

while its conjugate at $(z^*, z) \in X^* \times X$ is

$$\varphi_{N_C}^*(z^*, z) = \sup_{\substack{x\in C,\\ x^*\in -C^*}} \{\langle z^*, x\rangle + \langle x^*, z\rangle\} = \begin{cases} 0, & \text{if } z \in C,\ z^* \in -C^*,\\ +\infty, & \text{otherwise.}\end{cases}$$

As (RCM^C) is actually (RCM^M) for S and N_C, the maximality of the monotone operator $S + N_C$ is secured via Lemma 7.6, so by Theorem 7.3 one gets

$$\operatorname{int}(R(S) - C^*) = \operatorname{int}(R(S) + R(N_C)) = \operatorname{int} R(S + N_C).$$

Then we get $0 \in \operatorname{int} R(S + N_C)$, thus $0 \in R(S + N_C)$, i.e. there exists an $x \in C$ such that $0 \in (S + N_C)(x)$. Thus we found an $x^* \in S(x)$ such that $-x^* \in -N_C(x)$, which, since $N_C(x) \subseteq C^*$, yields that (x, x^*) is a solution to (CP). □

7.4 Surjectivity Results Involving the Sum of Two Maximally Monotone Operators

In this section we approach by means of convex analysis different surjectivity problems involving maximally monotone operators defined on a reflexive Banach space, following our paper [30]. First we deliver characterizations via closedness type regularity conditions involving representative functions of the surjectivity of the sum of a maximally monotone operator with a translation of another one. Besides particularizing them for some valuable special cases, we derive from these equivalences regularity conditions for guaranteeing the surjectivity of the sum of two maximally monotone operators and different particular instances of it that are weaker than their previous counterparts from the literature.

7.4.1 Surjectivity Results for the Sum of Two Maximally Monotone Operators

Let X be a reflexive Banach space and S and T be two maximally monotone operators defined on X. Before giving the first main statement of this subsection, the following observation is necessary.

Remark 7.11. Let $p \in X$ and $p^* \in X^*$. Then $p^* \in R(S(p+\cdot)+T(\cdot))$ if and only if $(p, p^*) \in G(S) - G(-T)$.

Theorem 7.7. *Let $p \in X$ and $p^* \in X^*$. The following statements are equivalent*

(i) $p^* \in R(S(p+\cdot)+T(\cdot))$*;*
(ii) for all $h_S \in \mathscr{F}_S$ and all $h_T \in \mathscr{F}_T$ one has $\operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (p, p^*)) \neq \emptyset$ *and the function $h_S^* \Box \big(\hat{h}_T^* + \langle (p^*, p), (\cdot,\cdot)\rangle\big)$ is lower semicontinuous at (p^*, p) and exact at (p^*, p);*
(iii) there exist $h_S \in \mathscr{F}_S$ and $h_T \in \mathscr{F}_T$ fulfilling $\operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (p, p^*)) \neq \emptyset$ *such that the function $h_S^* \Box \big(\hat{h}_T^* + \langle (p^*, p), (\cdot,\cdot)\rangle\big)$ is lower semicontinuous at (p^*, p) and exact at (p^*, p).*

Proof. Note first that the assertion "$(ii) \Rightarrow (iii)$" is immediate and one also has

$$\big(\hat{h}_T(\cdot - p, \cdot - p^*)\big)^* = \hat{h}_T^* + \langle p^*, \cdot\rangle + \langle \cdot, p\rangle. \tag{7.4.4}$$

"$(iii) \Rightarrow (i)$" Proposition 2.1 yields the equivalence of (iii) to

$$\big(h_S + \hat{h}_T(\cdot - p, \cdot - p^*)\big)^*(p^*, p) = \min_{u^* \in X^*, u \in X} \big[h_S^*(p^* - u^*, p - u) + \hat{h}_T^*(u^*, u) + \langle p^*, u\rangle + \langle u^*, p\rangle\big] \tag{7.4.5}$$

Denoting by $(\bar{u}^*, \bar{u}) \in X^* \times X$ the point where this minimum is attained, we obtain, via Lemma 7.3,

$$\begin{aligned}\big(h_S + \hat{h}_T(\cdot - p, \cdot - p^*)\big)^*(p^*, p) &= h_S^*(p^* - \bar{u}^*, p - \bar{u}) + \hat{h}_T^*(\bar{u}^*, \bar{u}) + \langle p^*, \bar{u}\rangle \\ &+ \langle \bar{u}^*, p\rangle \geq \langle p^* - \bar{u}^*, p - \bar{u}\rangle - \langle \bar{u}^*, \bar{u}\rangle + \langle p^*, \bar{u}\rangle + \langle \bar{u}^*, p\rangle = \langle p^*, p\rangle.\end{aligned} \tag{7.4.6}$$

But Lemma 7.3 also yields for every $x \in X$ and $x^* \in X^*$

$$\begin{aligned}(h_S + \hat{h}_T(\cdot - p, \cdot - p^*))(x, x^*) &\geq \langle x^*, x\rangle + \langle -(x^* - p^*), x - p\rangle \\ &= \langle x^*, p\rangle + \langle p^*, x\rangle - \langle p^*, p\rangle,\end{aligned}$$

thus $\langle p^*, p\rangle \geq \langle x^*, p\rangle + \langle p^*, x\rangle - (h_S + \hat{h}_T(\cdot - p, \cdot - p^*))(x, x^*)$. Consequently,

$$\big(h_S + \hat{h}_T(\cdot - p, \cdot - p^*)\big)^*(p^*, p) \leq \langle p^*, p\rangle. \tag{7.4.7}$$

Together with (7.4.6) this yields

$$\big(h_S + \hat{h}_T(\cdot - p, \cdot - p^*)\big)^*(p^*, p) = \langle p^*, p\rangle,$$

and consequently the inequalities invoked to obtain (7.4.6) must be fulfilled as equalities. Therefore

$$h_S^*(p^* - \bar{u}^*, p - \bar{u}) = \langle p^* - \bar{u}^*, p - \bar{u}\rangle \text{ and } \hat{h}_T^*(\bar{u}^*, \bar{u}) = \langle -\bar{u}^*, \bar{u}\rangle. \tag{7.4.8}$$

Having these, Lemma 7.3 yields then $p^* - \bar{u}^* \in S(p - \bar{u})$ and $\bar{u}^* \in T(-\bar{u})$, followed by $p^* \in S(p - \bar{u}) + T(-\bar{u})$, i.e. $p^* \in R(S(p + \cdot) + T(\cdot))$.

"$(i) \Rightarrow (ii)$" Whenever $h_S \in \mathscr{F}_S$, $h_T \in \mathscr{F}_T$, (i) yields, via Remark 7.11, $(p, p^*) \in \operatorname{dom} h_S - \operatorname{dom} \hat{h}_T$, i.e. $\operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (p^*, p)) \neq \emptyset$.

For every $h_S \in \mathscr{F}_S$, $h_T \in \mathscr{F}_T$, $u \in X$ and $u^* \in X^*$ we have $h_S^*(p^* - u^*, p - u) + \hat{h}_T^*(u^*, u) + \langle (p^*, p), (u, u^*)\rangle \geq \langle p^* - u^*, p - u\rangle - \langle u^*, u\rangle + \langle p^*, u\rangle + \langle u^*, p\rangle = \langle p^*, p\rangle$, consequently, $h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)\,(p^*, p) \geq \langle p^*, p\rangle$ and, since the function in the right-hand side is strong-strong continuous its value at (p^*, p) must be also smaller than $\overline{h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)}(p^*, p)$. But from [21, Theorem 7.6] we know, via (7.4.4), that one has $\overline{h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)} = \underline{(h_S + \hat{h}_T(-(p^*, p) + (\cdot, \cdot)))^*}$ and since (7.4.7) always holds, it follows that $\overline{h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)}\,(p^*, p) \leq \langle p^*, p\rangle$. Consequently,

$$h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)(p^*, p) \geq \overline{h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)}(p^*, p) = \langle p^*, p\rangle. \tag{7.4.9}$$

Since $p^* \in R(S(p + \cdot) + T(\cdot))$, there exist $(\bar{u}^*, \bar{u}) \in X^* \times X$ fulfilling (7.4.8). Then $h_S^*(p^* - \bar{u}^*, p - \bar{u}) + \hat{h}_T^*(\bar{u}^*, \bar{u}) + \langle (p^*, p), (\bar{u}, \bar{u}^*)\rangle = \langle p^*, p\rangle$, i.e. $h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big)(p^*, p) = h_S^*(p^* - \bar{u}^*, p - \bar{u}) + \hat{h}_T^*(\bar{u}^*, \bar{u}) + \langle (p^*, p), (\bar{u}, \bar{u}^*)\rangle = \langle p^*, p\rangle$, therefore the exactness of the infimal convolution in (ii) is proven, while its lower semicontinuity follows via (7.4.9). □

From Theorem 7.7 we obtain immediately the following surjectivity result.

Corollary 7.2. *For $p \in X$, one has $R(S(p + \cdot) + T(\cdot)) = X^*$ if and only if*

$$\left| \begin{array}{l} \forall p^* \in X^* \forall h_S \in \mathscr{F}_S \forall h_T \in \mathscr{F}_T \text{ one has } \operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (p, p^*)) \neq \emptyset \text{ and} \\ h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot)\rangle\big) \text{ is lower semicontinuous at } (p^*, p) \text{ and exact at } (p^*, p), \end{array} \right.$$

and this is further equivalent to

$$\left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \exists h_T \in \mathscr{F}_T \text{ with } \operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (p, p^*)) \neq \emptyset \text{ such that} \\ h_S^* \square \big(\hat{h}_T^* + \langle\langle (p^*, p), (\cdot, \cdot) \rangle\rangle\big) \text{ is lower semicontinuous at } (p^*, p) \text{ and exact at } (p^*, p). \end{array} \right.$$

Inspired by Corollary 7.2 we are able to introduce a sufficient condition that guarantees the surjectivity of $S(p + \cdot) + T(\cdot)$ for a given $p \in X$.

Theorem 7.8. *Let $p \in X$. Then $R(S(p + \cdot) + T(\cdot)) = X^*$ if*

$$(RCM^S) \left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \exists h_T \in \mathscr{F}_T \text{ with } \operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (p, p^*)) \neq \emptyset \\ \text{such that } h_S^* \square \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot) \rangle\big) \text{ is lower semicontinuous} \\ \text{on } X^* \times \{p\} \text{ and exact at } (p^*, p). \end{array} \right.$$

Next we characterize the surjectivity of the monotone operator $S + T$ via a condition involving representative functions. The first statement follows directly from Theorem 7.7, while the second one is a direct consequence.

Theorem 7.9. *Let $p^* \in X^*$. The following statements are equivalent*

(i) $p^* \in R(S + T)$;
(ii) *for all $h_S \in \mathscr{F}_S$ and $h_T \in \mathscr{F}_T$ one has* $\operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (0, p^*)) \neq \emptyset$ *and the function $h_S^* \square \big(\hat{h}_T^* + \langle p^*, \cdot \rangle\big)$ is lower semicontinuous at $(p^*, 0)$ and exact at $(p^*, 0)$;*
(iii) *there exist $h_S \in \mathscr{F}_S$ and $h_T \in \mathscr{F}_T$ with* $\operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (0, p^*)) \neq \emptyset$ *such that the function $h_S^* \square \big(\hat{h}_T^* + \langle p^*, \cdot \rangle\big)$ is lower semicontinuous at $(p^*, 0)$ and exact at $(p^*, 0)$.*

Corollary 7.3. *One has $R(S + T) = X^*$ if and only if*

$$\left| \begin{array}{l} \forall p^* \in X^* \forall h_S \in \mathscr{F}_S \forall h_T \in \mathscr{F}_T \text{ one has } \operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (0, p^*)) \neq \emptyset \text{ and} \\ h_S^* \square \big(\hat{h}_T^* + \langle p^*, \cdot \rangle\big) \text{ is lower semicontinuous at } (p^*, 0) \text{ and exact at } (p^*, 0), \end{array} \right.$$

and this is further equivalent to

$$\left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \exists h_T \in \mathscr{F}_T \text{ with } \operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (0, p^*)) \neq \emptyset \text{ such that} \\ h_S^* \square \big(\hat{h}_T^* + \langle p^*, \cdot \rangle\big) \text{ is lower semicontinuous at } (p^*, 0) \text{ and exact at } (p^*, 0). \end{array} \right.$$

Inspired by Corollary 7.3 we are able to introduce a sufficient condition that guarantees the surjectivity of $S + T$.

Theorem 7.10. *One has $R(S + T) = X^*$ if*

$$(RCM^J) \left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \exists h_T \in \mathscr{F}_T \text{ with } \operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (0, p^*)) \neq \emptyset \\ \text{such that } h_S^* \square \big(\hat{h}_T^* + \langle p^*, \cdot \rangle\big) \text{ is lower semicontinuous on } X^* \times \{0\} \\ \text{and exact at } (p^*, 0). \end{array} \right.$$

Remark 7.12. In the literature there were given other regularity conditions guaranteeing the surjectivity of $S+T$, namely, for fixed $h_S \in \mathscr{F}_S$ and $h_T \in \mathscr{F}_T$,

– (cf. [163, Corollary 2.7]) $\operatorname{dom} h_T = X \times X^*$;
– (cf. [190, Theorem 30.2]) $\operatorname{dom} h_S - \operatorname{dom} \hat{h}_T = X \times X^*$;
– (cf. [222, Corollary 4]) $\{0\} \times X^* \subseteq \operatorname{sqri}(\operatorname{dom} h_S - \operatorname{dom} \hat{h}_T)$.

It is obvious that the first one implies the second, whose fulfillment yields the validity of the third condition. This one yields that for any $x^*, p^* \in X^*$ one has

$$\big(h_S + \hat{h}_T(\cdot, \cdot - p^*)\big)^*(x^*, 0) = \min_{u^* \in X^*, u \in X} \big[h_S^*(x^* - u^*, -u) + \hat{h}_T^*(u^*, u) + \langle p^*, u\rangle\big],$$

which is equivalent, when $\operatorname{dom} h_S \cap (\operatorname{dom} \hat{h}_T + (0, p^*)) \neq \emptyset$ (condition automatically fulfilled when any of the three regularity conditions given above is satisfied), to the fact that whenever $p^* \in X^*$ the function $h_S^* \square \big(\hat{h}_T^* + \langle p^*, \cdot\rangle\big)$ is lower semicontinuous at $(x^*, 0)$ and exact at $(x^*, 0)$ for all $x^* \in X^*$. It is obvious that this implies (RCM^J) and below we present a situation where (RCM^J) holds, unlike the conditions cited from the literature for the surjectivity of $S+T$.

Example 7.3. Let $X = \mathbb{R}$ and consider the maximally monotone operators $S, T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$S(x) = \begin{cases} \{0\}, & \text{if } x > 0, \\ (-\infty, 0], & \text{if } x = 0, \\ \emptyset, & \text{otherwise}, \end{cases} \quad \text{and } T(x) = \begin{cases} \mathbb{R}, & \text{if } x = 0, \\ \emptyset, & \text{otherwise}, \end{cases} \quad x \in \mathbb{R}.$$

They are actually subdifferentials of proper, convex and lower-semicontinuous functions, which are also sublinear, namely $S = N_{[0,+\infty)}$ and $T = N_{\{0\}}$. Obviously, $R(S+T) = \mathbb{R}$ and the Fitzpatrick families of both S and T contain only the corresponding Fitzpatrick function, i.e. $\varphi_S = \delta_{[0,+\infty)\times(-\infty,0]} = \varphi_S^{*\top}$ and $\varphi_T = \delta_{\{0\}\times\mathbb{R}} = \varphi_T^{*\top}$.

Then $\operatorname{dom} \varphi_S - \operatorname{dom} \hat{\varphi}_T = \mathbb{R}_+ \times \mathbb{R}$, where $\mathbb{R}_+ = [0, +\infty)$, and it is obvious that $\{0\} \times \mathbb{R}$ is not included in $\operatorname{sqri}(\operatorname{dom} \varphi_S - \operatorname{dom} \hat{\varphi}_T) = (0, +\infty) \times \mathbb{R}$. Consequently, the three conditions mentioned in Remark 7.12 fail in this situation. On the other hand, for $p^*, x, x^* \in \mathbb{R}$ one has

$$\varphi_S^* \square \big(\hat{\varphi}_T^* + \langle p^*, \cdot\rangle\big)(x^*, x) = \begin{cases} 0, & \text{if } x \geq 0, \\ +\infty, & \text{if } x < 0, \end{cases}$$

and this function is lower semicontinuous on $\mathbb{R} \times \mathbb{R}_+$ and exact at all $(x^*, x) \in \mathbb{R} \times \mathbb{R}_+$. Consequently, (RCM^J) is valid in this case.

Remark 7.13. When one of h_S and h_T is continuous, the condition (RCM^J) is automatically fulfilled. It is known (see for instance [190]) that the domain of the Fitzpatrick function attached to the duality map $\mathscr{J}$, which is a maximally monotone operator, is the whole product space $X \times X^*$. By [221, Theorem 2.2.20] it follows that $\varphi_{\mathscr{J}}$ is continuous, thus by Corollary 7.2 we obtain that $S(p + \cdot) + \mathscr{J}(\cdot)$ is surjective, whenever $p \in X$. In this way we rediscover a known property of the maximally monotone operators, already mentioned in Lemma 7.1, used for instance for verifying the maximal monotonicity of the sum of two monotone operators under certain hypotheses, as done for instance in [42, 44]. Moreover, via Corollary 7.3 one gets that $S + \mathscr{J}$ is surjective, rediscovering Rockafellar's classical surjectivity theorem for maximally monotone operators (see for instance [190, Theorem 29.5]).

Remark 7.14. One can notice via (7.4.4) that (7.4.5) can be rewritten when $p^* = 0$ and $p = 0$ as

$$\inf_{x \in X, x^* \in X^*} \big[h_S(x, x^*) + \hat{h}_T(x, x^*)\big] = \max_{u^* \in X^*, u \in X} \big\{ - h_S^*(-u^*, -u) - \hat{h}_T^*(u^*, u)\big\}, \tag{7.4.10}$$

i.e. there is strong duality for the convex optimization problem formulated above in the left-hand side of (7.4.10) and its Fenchel dual problem. When $(\bar{u}, \bar{u}^*) \in X \times X^*$ is an optimal solution to the dual problem, i.e. the point where the maximum in the right-hand side of (7.4.10) is attained, one obtains $\bar{u}^* \in S(\bar{u})$ and $-\bar{u}^* \in T(\bar{u})$. Employing now Lemma 7.3, we obtain $h_S(\bar{u}, \bar{u}^*) = h_S^*(-\bar{u}^*, -\bar{u}) = \langle \bar{u}^*, \bar{u} \rangle$ and $\hat{h}_T(\bar{u}, \bar{u}^*) = \hat{h}_T^*(\bar{u}^*, \bar{u}) = -\langle \bar{u}^*, \bar{u} \rangle$, therefore

$$h_S(\bar{u}, \bar{u}^*) + \hat{h}_T(\bar{u}, \bar{u}^*) = h_S^*(-\bar{u}^*, -\bar{u}) + \hat{h}_T^*(\bar{u}^*, \bar{u}) = 0.$$

Thus, the infimum in the left-hand side of (7.4.10) is attained, i.e. the primal optimization problem given there has an optimal solution, too, so total duality holds for the primal-dual pair of optimization problems in discussion. Therefore we can note for this special kind of optimization problems the coincidence of the strong and total Fenchel duality.

Remark 7.15. Given $p \in X$ and $p^* \in X^*$, the function $h_S^* \Box \big(\hat{h}_T^* + \langle (p^*, p), (\cdot, \cdot) \rangle\big)$ can be replaced in Theorem 7.7(*ii*)–(*iii*) with $\big(h_S^* - \langle (p^*, p), (\cdot, \cdot) \rangle\big) \Box \hat{h}_T^*$ without altering the statement. The other conditions considered afterwards within this section can be correspondingly rewritten, too.

Remark 7.16. The results given within this subsection can be extended for the sum of a maximally monotone operator with another one composed with a linear mapping, as considered in Sect. 7.3. However, because even in the case treated here the results are quite complicated we chose to work in the present framework. Another possible direction of generalization of the results provided in this subsection is for the situation when the involved Banach spaces are not necessarily reflexive, possibly by exploiting ideas and techniques from [161, 162]. Last but not least, it should be possible to obtain Lemma 7.6 as a consequence of Theorem 7.10 and taking into consideration Remark 7.13.

7.4.2 Special Cases

7.4.2.1 Zeros of Sums of Monotone Operators

An important consequence of Theorem 7.9 is the following statement, where we provide equivalent characterizations by means of representative functions of the situation when 0 lies in the range of $S + T$.

Corollary 7.4. *One has* $0 \in R(S + T)$ *if and only if*

$$\left| \begin{array}{l} \forall h_S \in \mathscr{F}_S \forall h_T \in \mathscr{F}_T \text{ one has } \operatorname{dom} h_S \cap \operatorname{dom} \hat{h}_T \neq \emptyset \text{ and the function} \\ h_S^* \Box \hat{h}_T^* \text{ is lower semicontinuous at } (0,0) \text{ and exact at } (0,0), \end{array} \right.$$

and this is further equivalent to

$$\left| \begin{array}{l} \exists h_S \in \mathscr{F}_S \exists h_T \in \mathscr{F}_T \text{ with } \operatorname{dom} h_S \cap \operatorname{dom} \hat{h}_T \neq \emptyset \text{ such that the function} \\ h_S^* \Box \hat{h}_T^* \text{ is lower semicontinuous at } (0,0) \text{ and exact at } (0,0). \end{array} \right.$$

From Corollary 7.4 one can deduce a sufficient condition which ensures that $0 \in R(S + T)$.

Corollary 7.5. *One has* $0 \in R(S + T)$ *if*

$$(RCM^Z) \left| \begin{array}{l} \exists h_S \in \mathscr{F}_S \exists h_T \in \mathscr{F}_T \text{ with } \operatorname{dom} h_S \cap \operatorname{dom} \hat{h}_T \neq \emptyset \text{ such that} \\ h_S^* \Box \hat{h}_T^* \text{ is lower semicontinuous on } X^* \times \{0\} \text{ and exact at } (0,0). \end{array} \right.$$

Remark 7.17. The problem of guaranteeing that $0 \in R(S + T)$ and furthermore of finding a solution of this equation has received a large interest in the literature because of both theoretical and practical reasons. In [19, Theorem 4.5] the condition $(0,0) \in \operatorname{core}(\operatorname{co} G(S) - \operatorname{co} G(-T))$ is shown to imply $0 \in R(S+T)$, while in [222, Lemma 1] the same result is achieved under the assumption $(0,0) \in \operatorname{sqri}(\operatorname{dom} h_S - \operatorname{dom} \hat{h}_T)$. Following similar arguments to the ones in Remark 7.12 one can show that both these conditions yield the validity of (RCM^Z). Checking the situation from Example 7.3, we see that the second condition fails, while (RCM^Z) is fulfilled. As $\operatorname{core}(\operatorname{co} G(S) - \operatorname{co} G(-T)) = \operatorname{int}(\mathbb{R}_+ \times (-\mathbb{R}_+) - \{0\} \times \mathbb{R}) = (0, +\infty) \times \mathbb{R}$ does not contain $(0,0)$, it is straightforward that (RCM^Z) is indeed weaker than both conditions mentioned above.

7.4.2.2 Surjectivity Results Involving Normal Cones

Let $U \subseteq X$ be a nonempty closed convex set. Its normal cone N_U is a maximally monotone operator whose only representative function (cf. [8, Corollary 5.9]) is the Fenchel one, namely $h_{N_U}(x, x^*) = \delta_U(x) + \sigma_U(x^*)$, $(x, x^*) \in X \times X^*$.

From Theorem 7.7 and its consequences we obtain by taking $T = N_U$ the following results.

Corollary 7.6. *Let $p \in X$. Then $R(S(p+\cdot) + N_U(\cdot)) = X^*$ if and only if*

$$\left| \begin{array}{l} \forall p^* \in X^* \forall h_S \in \mathscr{F}_S \text{ one has } \operatorname{dom} h_S \cap (U \times \operatorname{dom} \sigma_{-U} + (p,p^*)) \neq \emptyset \text{ and the} \\ \text{function } (y^*,y) \mapsto \inf_{x \in -U, x^* \in X^*} \left[(h_S^* - \langle (p^*,p),(\cdot,\cdot)\rangle)(y^*-x^*, y-x) + \sigma_U(x^*) \right] \\ \text{is lower semicontinuous at } (p^*,p) \text{ and the infimum within} \\ \text{is attained when } (y^*,y) = (p^*,p), \end{array} \right.$$

and this is further equivalent to

$$\left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \text{ with } \operatorname{dom} h_S \cap (U \times \operatorname{dom} \sigma_{-U} + (p,p^*)) \neq \emptyset \text{ the} \\ \text{function } (y^*,y) \mapsto \inf_{x \in -U, x^* \in X^*} \left[(h_S^* - \langle (p^*,p),(\cdot,\cdot)\rangle)(y^*-x^*, y-x) + \sigma_U(x^*) \right] \\ \text{is lower semicontinuous at } (p^*,p) \text{ and the infimum within is attained} \\ \text{when } (y^*,y) = (p^*,p). \end{array} \right.$$

Corollary 7.7. *Let $p \in X$. Then $R(S(p+\cdot) + N_U(\cdot)) = X^*$ if*

$$\left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \text{ with } \operatorname{dom} h_S \cap (U \times \operatorname{dom} \sigma_{-U} + (p,p^*)) \neq \emptyset \text{ the} \\ \text{function } (y^*,y) \mapsto \inf_{x \in -U, x^* \in X^*} \left[(h_S^* - \langle (p^*,p),(\cdot,\cdot)\rangle)(y^*-x^*, y-x) + \sigma_U(x^*) \right] \\ \text{is lower semicontinuous on } X^* \times \{p\} \text{ and the infimum within} \\ \text{is attained when } (y^*,y) = (p^*,p). \end{array} \right.$$

Corollary 7.8. *One has $0 \in R(S + N_U)$ if*

$$(RCM^N) \left| \begin{array}{l} \exists h_S \in \mathscr{F}_S \text{ with } \operatorname{dom} h_S \cap (U \times \operatorname{dom} \sigma_{-U}) \neq \emptyset \text{ such that the function} \\ (y^*,y) \mapsto \inf_{x \in U} [(h_S^*(\cdot, y+x) \square \sigma_U)(y^*)] \text{ is lower semicontinuous} \\ \text{on } X^* \times \{0\} \text{ and the infimum within is attained when } (y^*,y) = (0,0). \end{array} \right.$$

Remark 7.18. In [19, Corollary 5.7] it is stated that the regularity condition $0 \in \operatorname{core} \operatorname{co}(D(S) - U)$ yields $0 \in R(S + N_U)$. Similarly to the considerations from Remarks 7.12 and 7.17 one can notice that this condition is indeed stronger than (RCM^N).

Not without importance is the question how can one equivalently characterize the surjectivity of a maximally monotone operator via its representative functions. To proceed to answering it, take $U = X$. Then $T = N_X$, i.e. $T(x) = \{0\}$ for all $x \in X$, and the Fenchel representative function of N_X is $(x,x^*) \mapsto \delta_X(x) + \sigma_X(x^*) = \delta_{\{0\}}(x^*)$. Then $S + T = S$ and the surjectivity of S can be characterized, via Corollary 7.6, as follows.

Corollary 7.9. *One has $R(S) = X^*$ if and only if*

$$\left| \begin{array}{l} \forall p^* \in X^* \forall h_S \in \mathscr{F}_S \text{ the function } y^* \mapsto -\big(h_S^*(y^*,\cdot)\big)^*(p^*) \text{ is lower} \\ \text{semicontinuous at } p^* \text{ and } \exists x \in X \text{ such that } p^* \in (\partial h_S^*(p^*,\cdot))(x), \end{array} \right.$$

and this is further equivalent to

$$\left| \begin{array}{l} \forall p^* \in X^* \exists h_S \in \mathscr{F}_S \textit{ the function } y^* \mapsto -\big(h_S^*(y^*,\cdot)\big)^*(p^*) \textit{ is lower} \\ \textit{semicontinuous at } p^* \textit{ and } \exists x \in X \textit{ such that } p^* \in (\partial h_S^*(p^*,\cdot))(x). \end{array} \right.$$

Proof. Corollary 7.6 asserts the equivalence of the surjectivity of the maximally monotone operator S to the lower semicontinuity at $(p^*, 0)$ of the function

$$(y^*, y) \mapsto \inf_{x\in X, x^*\in X^*} \left[(h_S^* - \langle p^*, \cdot\rangle)(y^* - x^*, y + x) + \sigma_X(x^*)\right]$$

concurring with the attainment of the infimum within when $(y^*, y) = (p^*, 0)$, for every $p^* \in X^*$. Taking a closer look at this function, we note that it can be simplified to $(y^*, y) \mapsto \inf_{x\in X} \left[h_S^*(y^*, y + x) - \langle p^*, y + x\rangle\right]$, which can be further reduced to $y^* \mapsto -\big(h_S^*(y^*, \cdot)\big)^*(p^*)$.

For $p^* \in X^*$, the attainment of the infimum from above when $(y^*, y) = (p^*, 0)$ means actually the existence of an $x \in X$ such that $h_S^*(p^*, x) - \langle p^*, x\rangle = -\big(h_S^*(p^*, \cdot)\big)^* (p^*)$, which is nothing but $p^* \in (\partial h_S^*(p^*, \cdot))(x)$. □

Remark 7.19. In [163, Corollary 2.2] it is shown that S is surjective if $\operatorname{dom}(\varphi_S) = X \times X^*$. This result can be obtained as a consequence of Corollary 7.9 knowing that the characterizations provided there for $R(S) = X^*$ are fulfilled when φ_S is continuous.

Remark 7.20. Since determining when $0 \in R(S)$ is important even beyond optimization, using Corollary 7.9 one can provide the following regularity condition for guaranteeing this

$$\left| \begin{array}{l} \exists h_S \in \mathscr{F}_S \textit{ the function } y^* \mapsto -\big(h_S^*(y^*,\cdot)\big)^*(0) \textit{ is lower} \\ \textit{semicontinuous and } \exists x \in X \textit{ such that } p^* \in (\partial h_S^*(0,\cdot))(x). \end{array} \right.$$

7.4.2.3 Surjectivity Results Involving Subdifferentials

Let now the proper, convex and lower semicontinuous functions $f, g : X \to \overline{\mathbb{R}}$. Take first $T = \partial g$ and consider for it the Fenchel representative function. Then Corollary 7.2 yields the following statement.

Corollary 7.10. *Let $p \in X$. Then $R(S(p + \cdot) + \partial g(\cdot)) = X^*$ if and only if*

$$\left| \begin{array}{l} \forall p^* \in X^* \forall h_S \in \mathscr{F}_S \textit{ one has } \operatorname{dom} h_S \cap (\operatorname{dom} g \times (-\operatorname{dom} g^*) + (p, p^*)) \neq \emptyset \\ \textit{and the function } h_S^* \Box \big(g(-\cdot) + g^*(\cdot) + \langle (p^*, p), (\cdot,\cdot)\rangle\big) \textit{ is} \\ \textit{lower semicontinuous at } (p^*, p) \textit{ and exact at } (p^*, p), \end{array} \right.$$

and this is further equivalent to

$$\left|\begin{array}{l}\forall p^* \in X^* \exists h_S \in \mathscr{F}_S \text{ with } \operatorname{dom} h_S \cap (\operatorname{dom} g \times (-\operatorname{dom} g^*) + (p, p^*)) \neq \emptyset \\ \text{such that the function } h_S^* \Box \big(g(-\cdot) + g^*(\cdot) + \langle (p^*, p), (\cdot, \cdot)\rangle\big) \text{ is} \\ \text{lower semicontinuous at } (p^*, p) \text{ and exact at } (p^*, p).\end{array}\right.$$

Remark 7.21. In [163, Proposition 2.9] it was proven that when g and g^* are real valued the monotone operator $S(p + \cdot) + \partial g(\cdot)$ is surjective whenever $p \in X$. This statement can be rediscovered as a consequence of Corollary 7.10, too. Using [221, Proposition 2.1.6] one obtains that g and g^* are continuous under the mentioned hypotheses. Then the Fenchel representative function of ∂g is continuous and this yields the fulfillment of the regularity condition from Corollary 7.10. Consequently, $S(p + \cdot) + \partial g(\cdot)$ is surjective whenever $p \in X$.

The other statements involving two maximally monotone operators given above can be particularized for this special case, too. However, we give here only a consequence of Corollary 7.5.

Corollary 7.11. *One has* $0 \in R(S + \partial g)$ *if*

$$\left|\begin{array}{l}\exists h_S \in \mathscr{F}_S \text{ with } \operatorname{dom} h_S \cap (\operatorname{dom} g \times (-\operatorname{dom} g^*)) \neq \emptyset \text{ such that the function} \\ h_S^* \Box \big(g(-\cdot) + g^*(\cdot)\big) \text{ is lower semicontinuous on } X^* \times \{0\} \text{ and exact at } (0,0).\end{array}\right.$$

Take now also $S = \partial f$, to which we associate the Fenchel representative function, too. Let the function $\hat{g} : X \to \overline{\mathbb{R}}$, $\hat{g}(x) = g(-x)$. Corollary 7.2 yields the following result.

Corollary 7.12. *Let* $p \in X$. *If* $\operatorname{dom} f \cap (p + \operatorname{dom} g) \neq \emptyset$, *then* $R(\partial f(p + \cdot) + \partial g(\cdot)) = X^*$ *if and only if*

$$\left|\begin{array}{l}\forall p^* \in X^* \text{ one has } \operatorname{dom} f^* \cap (p^* - \operatorname{dom} g^*) \neq \emptyset, \text{ the function } f \Box (\hat{g} + p^*) \\ \text{is lower semicontinuous at } p \text{ and exact at } p \text{ and the function} \\ f^* \Box (g^* + p) \text{ is lower semicontinuous at } p^* \text{ and exact at } p^*.\end{array}\right.$$

Moreover, from Corollary 7.11 one can deduce the following statement.

Corollary 7.13. *One has* $0 \in R(\partial f + \partial g)$ *if* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, $\operatorname{dom} f^* \cap (-\operatorname{dom} g^*) \neq \emptyset$ *and*

$$\left|\begin{array}{l}f \Box \hat{g} \text{ is lower semicontinuous at } 0 \text{ and exact at } 0 \text{ and the} \\ \text{function } f^* \Box g^* \text{ is lower semicontinuous and exact at } 0.\end{array}\right.$$

7.5 Dealing with the Maximal Monotonicity of Bifunctions via Representative Functions

The study of the maximal monotonicity of bifunctions began with the seminal paper [12], followed by works like [116, 135, 160], and in all of them the investigations were based on the theory of equilibrium problems. However, motivated by the recent results on maximally monotone operators, obtained almost exclusively by means of representative functions, we involved the latter in the new approach of the maximal monotonicity of bifunctions proposed in [33]. In this way we succeeded in extending some statements from the literature and, moreover, in proving some recent conjectures. This section is dedicated to presenting these results, but before stating them some preliminaries on monotone bifunctions are necessary.

7.5.1 Monotone Bifunctions

We begin with some preliminaries on bifunctions, following [116,135]. Take further X to be a normed space. Let the nonempty set $C \subseteq X$. A function $F : C \times C \to \mathbb{R}$ is called *bifunction*. The bifunction F is called *monotone* if $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$. To the bifunction F one can attach the *diagonal subdifferential operators* $A^F : X \rightrightarrows X^*$ and ${}^F\!A : X \rightrightarrows X^*$ defined by

$$A^F(x) = \begin{cases} \{x^* \in X^* : F(x, y) - F(x, x) \geq \langle x^*, y - x\rangle \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and, respectively,

$${}^F\!A(x) = \begin{cases} \{x^* \in X^* : F(x, x) - F(y, x) \geq \langle x^*, y - x\rangle \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

When $F(x, x) = 0$ for all $x \in C$ and F (respectively $-F$) is monotone, then A^F (${}^F\!A$) is a monotone operator. When F is monotone and $F(x, x) = 0$ for all $x \in C$ one has $G\big(A^F\big) \subseteq G\big({}^F\!A\big)$.

The monotone bifunction F fulfilling $F(x, x) = 0$ for all $x \in C$ is said to be *maximally monotone* if A^F is maximally monotone and, respectively, BO-*maximally monotone* (where BO stands for Blum-Oettli, as this type of monotone bifunction was introduced in [12]) when for every $(x, x^*) \in C \times X^*$ it holds

$$F(y, x) + \langle x^*, y - x\rangle \leq 0 \ \forall y \in C \ \Rightarrow F(x, y) \geq \langle x^*, y - x\rangle \ \forall y \in C.$$

When F is monotone and $F(x, x) = 0$ for all $x \in C$, its BO-maximal monotonicity is equivalent to ${}^F\!A = A^F$. Any maximally monotone bifunction is BO-maximally monotone, but the opposite implication is not always valid, as the situation in [116, Example 2.2] shows.

In order not to overcomplicate the presentation, when $x \in C$ we denote by a slight abuse of notation by $F(x,\cdot) + \delta_C$ the function defined on X with extended real values which is equal to $F(x,\cdot)$ on C and takes the value $+\infty$ otherwise. Analogously, when $y \in C$ we denote by $-F(\cdot,y) + \delta_C$ the function defined on X with extended real values which is equal to $-F(\cdot,y)$ on C and takes the value $+\infty$ otherwise. Hence, when $F(x,x) = 0$ for all $x \in C$, one can write $A^F(x) = \partial(F(x,\cdot) + \delta_C)(x)$ and ${}^F\!A(x) = \partial(-F(\cdot,x) + \delta_C)(x)$ for all $x \in X$. Note that A^F and ${}^F\!A$ are not subdifferentials of functions, being at each point the subdifferential of another function.

We close this preliminary subsection by presenting a statement which holds in a more general framework than originally considered in [12, Lemma 3], followed by a consequence needed later in our investigations.

Lemma 7.7. *Let F and G be two bifunctions defined on the nonempty and convex set $C \subseteq X$, satisfying $F(x,x) = G(x,x) = 0$ for all $x \in C$, such that F is monotone, $F(x,\cdot)$ and $G(x,\cdot)$ are convex for all $x \in C$ and $F(\cdot,y)$ is upper hemicontinuous for all $y \in C$. Then the following statements are equivalent*

(i) $\bar{x} \in C$ *and* $F(y,\bar{x}) \le G(\bar{x},y)$ *for all* $y \in C$;
(ii) $\bar{x} \in C$ *and* $0 \le F(\bar{x},y) + G(\bar{x},y)$ *for all* $y \in C$.

Remark 7.22. The monotonicity of F is required only for proving the implication "$(ii) \Rightarrow (i)$" in Lemma 7.7, which actually holds even if the convexity and topological hypotheses are removed.

Lemma 7.8. *Let F be a bifunction defined on the nonempty and convex set $C \subseteq X$, satisfying $F(x,x) = 0$ for all $x \in C$. If $F(x,\cdot)$ is convex for all $x \in C$ and $F(\cdot,y)$ is upper hemicontinuous for all $y \in C$, then $G\left({}^F\!A\right) \subseteq G\left(A^F\right)$.*

Proof. Let $(x,x^*) \in G\left({}^F\!A\right)$. Then $x \in C$ and $F(y,x) \le \langle x^*, x-y\rangle$ for all $y \in C$. By Lemma 7.7$(i) \Rightarrow (ii)$ one gets $0 \le F(x,y) + \langle x^*, x-y\rangle$ for all $y \in C$, thus $(x,x^*) \in G\left(A^F\right)$. □

Remark 7.23. If in addition to the assumptions of Lemma 7.8 F is taken moreover monotone, one also gets that F is BO-maximally monotone.

7.5.2 *Maximal Monotone Bifunctions*

Let $F : C \times C \to \mathbb{R}$ be a bifunction, where $C \subseteq X$ is nonempty. In order to deal with its maximal monotonicity, we attach to F the functions $h_F, g_F : X \times X^* \to \overline{\mathbb{R}}$, defined at $(x,x^*) \in X \times X^*$ by

$$h_F(x,x^*) = \sup_{y\in C}\left\{\langle x^*, y\rangle - F(x,y)\right\} + \delta_C(x) = (F(x,\cdot) + \delta_C)^*(x^*) + \delta_C(x)$$

and

$$g_F(x,x^*) = \sup_{y\in C}\big\{\langle x^*,y\rangle + F(y,x)\big\} + \delta_C(x) = (-F(\cdot,x)+\delta_C)^*(x^*) + \delta_C(x).$$

Regarding their conjugates, for $(x^*,x)\in X^*\times X$ one has

$$h_F^*(x^*,x) = \sup_{y\in C}\big\{\langle x^*,y\rangle + (F(y,\cdot)+\delta_C)^{**}(x)\big\}$$

and

$$g_F^*(x^*,x) = \sup_{y\in C}\big\{\langle x^*,y\rangle + (-F(\cdot,y)+\delta_C)^{**}(x)\big\}.$$

Other properties of these functions are given in the following statements, whose proofs are trivial hence skipped.

Proposition 7.1. *(a) For all* $(x,x^*)\in X\times X^*$*, it holds* $g_F(x,x^*)\geq h_F^*(x^*,x)$*.*
(b) If $F(x,x)=0$ *for all* $x\in C$*, then* $h_F\geq c$ *and* $g_F\geq c$*.*
(c) If F *is monotone, then* $h_F(x,x^*)\geq g_F(x,x^*)$ *and* $\overline{\text{co}}h_F(x,x^*)\geq h_F^*(x^*,x)$ *for all* $(x,x^*)\in X\times X^*$*.*

Remark 7.24. If $F(x,x)=0$ for all $x\in C$, one has that $h_F(x,x^*)=c(x,x^*)$ if and only if $(x,x^*)\in G\big(A^F\big)$ and, respectively, $g_F(x,x^*)=c(x,x^*)$ if and only if $(x,x^*)\in G\big({}^FA\big)$. However, g_F and h_F are in general neither convex nor lower semicontinuous, therefore they are not always representative functions for A^F in case this is monotone. Note also that in [2] a function that slightly extends g_F is called the *Fitzpatrick transform* of the monotone bifunction F.

In the next statements we provide sufficient conditions for the maximal monotonicity of A^F. We begin with an assertion where F is not even asked to be monotone.

Theorem 7.11. *Let* C *be convex and closed and* F *be fulfilling* $F(x,x)=0$ *for all* $x\in C$*. If* $\text{sqri}\,C\neq\emptyset$*,* $F(x,\cdot)$ *is convex and lower semicontinuous for all* $x\in C$ *and* $F(\cdot,y)$ *concave and upper semicontinuous for all* $y\in C$*, then* A^F *is maximally monotone and* $A^F={}^FA$*.*

Proof. The convexity and topological assumptions on C and $F(x,\cdot)$, for $x\in C$, yield that the function $F(x,\cdot)+\delta_C$ is proper, convex and lower semicontinuous whenever $x\in C$. Then $(F(x,\cdot)+\delta_C)^{**}(z)=F(x,z)+\delta_C(z)$ whenever $x\in C$ and $z\in X$, consequently, via Proposition 7.1, $h_F^{*\top}=g_F\geq c$ on $X\times X^*$. Analogously, the convexity and topological assumptions on C and $-F(\cdot,y)$, $y\in C$, imply $h_F=g_F^{*\top}\geq c$ on $X\times X^*$. Obviously, h_F and g_F are in this case convex functions, whose properness follows immediately, too.

One gets $\text{Pr}_X(\text{dom}\,h_F)\subseteq\text{Pr}_X(\text{dom}\,g_F)\subseteq C$. Taking an $x\in C$, since $F(x,\cdot)+\delta_C$ is proper, convex and lower semicontinuous, its conjugate is proper (cf. [221, Theorem 2.3.3]), so there exists an $x^* \in X^*$ such that $(F(x,\cdot) + \delta_C)^*(x^*) < +\infty$. Consequently, $h_F(x,x^*) < +\infty$, i.e. $C \subseteq \mathrm{Pr}_X(\mathrm{dom}\, h_F)$. Therefore $\mathrm{Pr}_X(\mathrm{dom}\, h_F) = \mathrm{Pr}_X(\mathrm{dom}\, g_F) = C$. We are now ready to apply Lemma 7.5 for h_F and g_F, obtaining that the operators (identified through their graphs)

$$\{(x,x^*)\in X \times X^* : h_F^*(x^*,x) = c(x,x^*)\}$$
$$= \{(x,x^*)\in X \times X^* : g_F(x,x^*) = c(x,x^*)\},$$

which is actually $G\left({}^F\!A\right)$, and

$$\{(x,x^*)\in X \times X^* : g_F^*(x^*,x) = c(x,x^*)\}$$
$$= \{(x,x^*)\in X \times X^* : h_F(x,x^*) = c(x,x^*)\},$$

that is $G\left(A^F\right)$, are maximally monotone.

Using Lemma 7.8, it follows $G\left({}^F\!A\right) \subseteq G\left(A^F\right)$, consequently, $A^F = {}^F\!A$, since both are maximally monotone operators. □

Remark 7.25. If X is reflexive, the hypothesis $\mathrm{sqri}\, C \neq \emptyset$ is no longer needed in Theorem 7.11, since one can use in its proof in this case Lemma 7.4 instead of Lemma 7.5.

If $C = X$ the condition $\mathrm{sqri}\, C \neq \emptyset$ is automatically satisfied and Theorem 7.11 yields the following statement, noting that the lower/upper semicontinuity of a real valued convex/concave function on the entire space is equivalent to its continuity (cf. [221, Proposition 2.1.6]).

Corollary 7.14. *Let $F(x,x) = 0$ for all $x \in X$, $F(x,\cdot)$ be convex and continuous for all $x \in X$ and $F(\cdot,y)$ concave and continuous for all $y \in X$. Then A^F is maximally monotone and $A^F = {}^F\!A$.*

Remark 7.26. In Theorem 7.12 we prove one of the conjectures formulated at the end of [135], actually slightly weakening its hypotheses since instead of taking F continuous we ask it to be continuous in each of its variables. If X is reflexive, Theorem 7.12 slightly improves [135, Theorem 3.6(i)], by bringing the mentioned weakening of its hypotheses.

Taking F to be monotone, here are some hypotheses that guarantee its maximality even in the absence of convexity assumptions in its first variable.

Theorem 7.12. *Let C be convex and closed and F be monotone and fulfilling $F(x,x) = 0$ for all $x \in C$. If $\mathrm{sqri}\, C \neq \emptyset$, $F(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$ and $F(\cdot,y)$ upper hemicontinuous for all $y \in C$, then F is maximally monotone.*

Proof. The convexity and topological assumptions on C and $F(x,\cdot)$, for $x \in C$, yield that the function $F(x,\cdot) + \delta_C$ is proper, convex and lower semicontinuous whenever $x \in C$. Then $(F(x,\cdot)+\delta_C)^{**}(z) = F(x,z)+\delta_C(z)$ whenever $x \in C$ and $z \in X$, hence $h_F^*(x^*,x) = g_F(x,x^*)$ for all $(x,x^*) \in X \times X^*$. Consequently, via Proposition 7.1 and taking into consideration the properties of the conjugate function, one has

$$h_F(x,x^*) \geq \overline{\text{co}} h_F(x,x^*) \geq h_F^*(x^*,x) \geq c(x,x^*) \ \forall (x,x^*) \in X \times X^*. \tag{7.5.11}$$

Assuming that h_F were improper leads to a contradiction with (7.5.11), consequently h_F, $\overline{\text{co}} h_F$ and h_F^* are all proper. Like in the proof of Theorem 7.11 one can show that $\text{Pr}_X(\text{dom}\, h_F) = C$. Then

$$\text{Pr}_X(\text{dom}\, h_F) \subseteq \text{Pr}_X(\text{dom}\, \overline{\text{co}} h_F) \subseteq \overline{\text{co}}\text{Pr}_X(\text{dom}\, h_F) \tag{7.5.12}$$

and, since C is convex and closed, we get $\text{Pr}_X\left(\text{dom}\left(\overline{\text{co}} h_F\right)\right) = C$.

In the following we show that

$$\begin{aligned} G\left(A^F\right) &= \left\{(x,x^*) \in X \times X^* : \overline{\text{co}} h_F(x,x^*) = c(x,x^*)\right\} \\ &= \left\{(x,x^*) \in X \times X^* : h_F^*(x^*,x) = c(x,x^*)\right\}. \end{aligned} \tag{7.5.13}$$

If $(x,x^*) \in G\left(A^F\right)$, (7.5.11) yields $h_F^*(x^*,x) = c(x,x^*)$.

Let now $(x,x^*) \in X \times X^*$ for which $h_F^*(x^*,x) = c(x,x^*)$. Then $(x,x^*) \in G\left({}^F A\right)$, so Lemma 7.8 yields $(x,x^*) \in G\left(A^F\right)$. This implies that $\overline{\text{co}} h_F(x,x^*) = c(x,x^*)$ holds if and only if $(x,x^*) \in G\left(A^F\right)$. Applying Lemma 7.5 for $\overline{\text{co}} h_F$, it follows that A^F is maximally monotone, i.e. F is maximally monotone, too. □

Remark 7.27. In Theorem 7.12 we provide a positive answer to the conjecture formulated at the end of [136]. When the space X is reflexive, the regularity condition $\text{sqri}\, C \neq \emptyset$ is no longer necessary in the hypotheses of Theorem 7.12 and this statement rediscovers [116, Proposition 3.1], by means of representative functions, employing tools of convex analysis and without renorming the space X.

Corollary 7.15. *Let X be reflexive, C be convex and closed and F be monotone and fulfilling $F(x,x) = 0$ for all $x \in C$. If $F(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$ and $F(\cdot,y)$ upper hemicontinuous for all $y \in C$, then F is maximally monotone.*

Proof. Things work in the lines of the proof of Theorem 7.12, noticing that (7.5.11) and (7.5.13) are fulfilled. Then we apply Lemma 7.4. □

When $C = X$ we obtain from Theorem 7.12 the following statement.

Corollary 7.16. *Let F be monotone and fulfilling $F(x,x) = 0$ for all $x \in X$. If $F(x,\cdot)$ is convex and continuous for all $x \in X$ and $F(\cdot,y)$ upper hemicontinuous for all $y \in X$, then F is maximally monotone.*

Remark 7.28. In [135, Theorem 3.6(ii)] the same conclusion as in Corollary 7.16 is obtained when X is reflexive for a monotone bifunction F that fulfills $F(x,x) = 0$ for all $x \in X$, by assuming $F(x,\cdot)$ only convex for all $x \in X$ and $F(\cdot,y)$ continuous for all $y \in X$. However, we doubt that this result holds without any topological assumption on the functions $F(x,\cdot)$, $x \in X$, since in its proof is used [135, Theorem 3.4(ii)], whose hypotheses should contain also the lower semicontinuity of $F(x,\cdot)$ for all $x \in X$. A similar comment can be made also for [135, Theorem 3.6(iii)] and for the conjectures extending the two mentioned statements to nonreflexive spaces given at the end of [135].

Whenever a monotone bifunction F fulfills $F(x,x) = 0$ for all $x \in C$ is BO-maximally monotone, one has $A^F = {}^F\!A$, so Lemma 7.7 is not longer needed in the proof of Theorem 7.12. Hence we rediscover, in the reflexive case, and extend, when X is a general Banach space, [160, Proposition 3.2], as follows.

Corollary 7.17. *Let C be convex and closed with $\operatorname{sqri} C \neq \emptyset$ and F be BO-maximally monotone. If $F(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$, then F is maximally monotone.*

Corollary 7.18. *Let X be reflexive, C convex and closed and F be BO-maximally monotone. If $F(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$, then F is maximally monotone.*

When $C = X$ one can formulate another maximality criterium for a monotone bifunction, extending [116, Proposition 3.5] to general Banach spaces.

Theorem 7.13. *Let F be monotone and fulfilling $F(x,x) = 0$ for all $x \in X$. If $D\big(A^F\big) = X$ and $F(\cdot,y)$ is upper hemicontinuous for all $y \in X$, then F is maximally monotone.*

Proof. As $D\big(A^F\big) = X$, for all $x \in X$ one has $\partial F(x,\cdot)(x) \neq \emptyset$, which yields $\overline{\text{co}}F(x,\cdot)\ (x) = F(x,x) = 0$. On the other hand, for all $x \in X$ it holds $X = \operatorname{dom} F(x,\cdot) \subseteq \operatorname{dom} \overline{\text{co}}F(x,\cdot)$, which implies $\operatorname{dom} \overline{\text{co}}F(x,\cdot) = X$ and via [221, Proposition 2.2.5], as $\overline{\text{co}}F(x,\cdot)(x) = 0$, also the properness of $\overline{\text{co}}F(x,\cdot)$. Then, for any $(x,x^*) \in X \times X^*$, one has

$$h_F^*(x^*,x) = \sup_{y\in X}\big\{\langle x^*,y\rangle + (F(y,\cdot))^{**}(x)\big\} = \sup_{y\in X}\big\{\langle x^*,y\rangle + \overline{\text{co}}F(y,\cdot)(x)\big\} \geq$$

$$\langle x^*,x\rangle + \overline{\text{co}}F(x,\cdot)(x) = \langle x^*,x\rangle,$$

consequently, $h_F \geq \overline{\text{co}}h_F \geq h_F^{*\top} \geq c$ on $X \times X^*$. As $D\big(A^F\big) = X$, $\Pr_X(\operatorname{dom} h_F) = X$, using (7.5.12) it follows $\Pr_X(\operatorname{dom}\overline{\text{co}}h_F) = X$. Applying Lemma 7.5 for $\overline{\text{co}}h_F$, the operator having the graph $\big\{(x,x^*) \in X \times X^* : h_F^*(x^*,x) = c(x,x^*)\big\}$ turns out to be maximally monotone. This graph includes $G\big(A^F\big)$. To show that the opposite inclusion holds, too, let $(x,x^*) \in X \times X^*$ for which $h_F^*(x^*,x) = c(x,x^*)$. Then $h_F^*(x^*,x) \leq c(x,x^*)$, so for all $y \in X$ it holds $\overline{\text{co}}F(y,\cdot)(x) \leq \langle x^*, x-y\rangle$. This means nothing but $(x,x^*) \in G\big({}^H\!A\big)$, where the bifunction $H : X \times X \to \mathbb{R}$ is defined by $H(x, y) := \overline{\text{co}} F(x, \cdot)(y)$. It follows immediately that $H(z, z) = 0$ for all $z \in X$. As $H(z, \cdot) = \overline{\text{co}} F(z, \cdot)$ is convex for all $z \in X$ and for all $y \in X$ one can verify that $H(\cdot, y)$ is upper hemicontinuous, Lemma 7.8 yields $(x, x^*) \in G\big(A^H\big)$. This means that for all $y \in X$ one has $\overline{\text{co}} F(x, \cdot)(y) \geq \langle x^*, y - x\rangle$, followed by $F(x, y) \geq \langle x^*, y - x\rangle$. Thus $(x, x^*) \in G\big(A^F\big)$, therefore (7.5.13) holds. Consequently, F is maximally monotone. □

Remark 7.29. One can see in the proofs of Theorems 7.11–7.13 that not only $\overline{\text{co}} h_F$ (which coincides with h_F under the hypotheses of the first of them), but also the restriction to $X \times X^*$ of $h_F^{*\top}$ are representative functions of the maximally monotone operator A^F.

In Theorems 7.11–7.13 we have shown with the help of the theory of representative functions that under some hypotheses A^F is maximally monotone. Now let us show that the representative functions of it identified there are actually representative to A^F whenever it is maximally monotone.

Theorem 7.14. *Let F be maximally monotone. Then $\overline{\text{co}} h_F$ and the restriction to $X \times X^*$ of $h_F^{*\top}$ are representative functions of A^F.*

Proof. The maximal monotonicity of F implies via Lemma 7.3 that

$$\begin{aligned} G\big(A^F\big) &= \big\{(x, x^*) \in X \times X^* : \psi_{A^F}(x, x^*) = c(x, x^*)\big\} \\ &= \big\{(x, x^*) \in X \times X^* : \varphi_{A^F}(x, x^*) = c(x, x^*)\big\}. \end{aligned}$$

On the other hand, the way h_F is constructed implies $(c + \delta_{A^F})(x, x^*) \geq h_F(x, x^*)$ for all $(x, x^*) \in X \times X^*$, which yields

$$h_F^*(x^*, x) \geq (c + \delta_{A^F})^*(x^*, x) = \psi_{A^F}^*(x^*, x) = \varphi_{A^F}(x, x^*)\ \forall (x, x^*) \in X \times X^*.$$

Since the monotonicity of F implies, via Proposition 7.2, $h_F(x, x^*) \geq \overline{\text{co}} h_F\ (x, x^*) \geq h_F^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$, it follows immediately that for all $(x, x^*) \in X \times X^*$ it holds

$$\psi_{A^F}(x, x^*) \geq \overline{\text{co}} h_F(x, x^*) \geq h_F^*(x^*, x) \geq \varphi_{A^F}(x, x^*) \geq c(x, x^*).$$

Consequently,

$$\begin{aligned} G\big(A^F\big) &= \big\{(x, x^*) \in X \times X^* : \overline{\text{co}} h_F(x, x^*) = c(x, x^*)\big\} \\ &= \big\{(x, x^*) \in X \times X^* : h_F^*(x^*, x) = c(x, x^*)\big\}, \end{aligned}$$

which implies that $\overline{\text{co}} h_F$ and $h_F^{*\top}$ restricted to $X \times X^*$ are representative functions of A^F. □

Remark 7.30. One can easily see that, when F is maximally monotone with $F(x,x) = 0$ for all $x \in C$, then $\overline{\text{co}} g_F$ and the restriction to $X \times X^*$ of $g_F^{*\top}$ are representative functions of A^F, too.

Remark 7.31. In the lines of the proof of Theorem 7.14, one can show that if $T : X \rightrightarrows X^*$ is a maximally monotone operator and $h : X \times X^* \to \overline{\mathbb{R}}$ is a function fulfilling $h(x,x^*) \geq h^*(x^*,x)$ for all $(x,x^*) \in X \times X^*$ and $h(x,x^*) \leq c(x,x^*)$ whenever $(x,x^*) \in G(T)$, then $\overline{\text{co}} h_F$ and the restriction to $X \times X^*$ of $h_F^{*\top}$ are representative functions of T.

7.5.3 *The Sum of Two Monotone Bifunctions*

One of the most dealt with questions regarding maximally monotone operators is what guarantees that the sum of two of them remains maximally monotone. This issue was extended for maximally monotone bifunctions in [116], by means of equilibrium problems. We provide another answer in this matter, preceded by a preliminary result.

Proposition 7.2. *Let F and G be monotone bifunctions defined on a nonempty set $C \subseteq X$. Then $A^F(x) + A^G(x) \subseteq A^{F+G}(x)$ for all $x \in X$ and $F+G$ is monotone.*

Proof. Let $x \in X$, $y^* \in A^F(x)$ and $z^* \in A^G(x)$. Then $x \in C$ and for all $y \in C$ one has $F(x,y) \geq \langle y^*, y-x\rangle$ and $G(x,y) \geq \langle z^*, y-x\rangle$. Adding these inequalities, one gets $F(x,y)+G(x,y) \geq \langle y^*+z^*, y-x\rangle$ for all $y \in C$, i.e. $y^*+z^* \in A^{F+G}(x)$.

Analogously, writing what the monotonicity of F and G means and adding the obtained inequalities one gets that $F+G$ is monotone. □

For the following statement we need to introduce the bivariate infimal convolution of two functions defined on a cartesian product of sets. Let A and B be two nonempty sets. When $f, g : A \times B \to \overline{\mathbb{R}}$ are proper, their *bivariate infimal convolution* is the function $f \square_2 g : A \times B \to \overline{\mathbb{R}}$, $f \square_2 g(a,b) = \inf\{f(a,c) + g(a, b-c) : c \in B\}$.

Theorem 7.15. *Let X be reflexive and F and G two maximally monotone bifunctions defined on a nonempty set $C \subseteq X$ with f_F and f_G their corresponding representative functions. If $0 \in \text{sqri}\big(D\big(A^F\big) - D(A^G)\big)$ (or, equivalently, $0 \in \text{sqri}\big(\text{Pr}_X(\text{dom } f_F) - \text{Pr}_X(\text{dom } f_G)\big)$), then $F+G$ is maximally monotone, $A^F + A^G = A^{F+G}$ and $f_F \square_2 f_G$ is a representative function of A^{F+G}.*

Proof. By [172, Corollary 3.6] we obtain that the hypotheses yield the maximal monotonicity of $A^F + A^G$, to which $f_F \square_2 f_G$ is a representative function. Then Proposition 7.2 implies that $A^F(x) + A^G(x) = A^{F+G}(x)$ for all $x \in X$. Consequently, $F+G$ is maximally monotone and $f_F \square_2 f_G$ is a representative function of A^{F+G}, too. □

Remark 7.32. Note that under the hypotheses of Theorem 7.15 also the function $(f_F \Box_2 f_G)^{*\top}$ is a representative function of A^{F+G}. If one takes $f_F := \overline{\text{co}} h_F$ and $f_G := \overline{\text{co}} h_G$, then it holds

$$(f_F \Box_2 f_G)^*(x^*, x) = \sup_{y \in C} \left\{ \langle x^*, y \rangle + (F(y, \cdot) + \delta_C)^{**}(x) + (G(y, \cdot) + \delta_C)^{**}(x) \right\}$$

and this is less than $h^*_{F+G}(x^*, x)$ for all $(x, x^*) \in X \times X^*$. Thus the just identified representative function of A^{F+G} is smaller than the ones obtained for it via Theorem 7.14.

Remark 7.33. If both F and G satisfy the hypotheses of one of Theorems 7.11–7.12, Corollary 7.15 or, when $C = X$, Theorem 7.15, then $F + G$ fulfills them, too, and this has as consequence its maximal monotonicity.

Now let us present a situation, different from the one displayed in Theorem 7.15, when the inclusion proven in Proposition 7.2 turns out to be actually an equality. Note that the space X needs not be reflexive for this statement.

Proposition 7.3. *Let F and G be monotone bifunctions defined on the convex and closed set C fulfilling $F(x, x) = G(x, x) = 0$ for all $x \in C$, such that for all $x \in C$ the functions $F(x, \cdot)$ and $G(x, \cdot)$ are convex and lower semicontinuous. If $0 \in \text{sqri}(C - C)$, then $A^F + A^G = A^{F+G}$.*

Proof. Let $x \in C$. One has $\text{dom}(F(x, \cdot) + \delta_C) = \text{dom}(G(x, \cdot) + \delta_C) = \text{dom}((F + G)(x, \cdot) + \delta_C) = C$. By definition, $A^F(x) = \partial(F(x, \cdot) + \delta_C)(x)$. Note also that $(F(x, \cdot) + \delta_C) + (G(x, \cdot) + \delta_C) = (F + G)(x, \cdot) + \delta_C$. By [221, Theorem 2.8.7], the hypotheses imply

$$\partial(F(x, \cdot) + \delta_C)(x) + \partial(G(x, \cdot) + \delta_C)(x) = \partial(F(x, \cdot) + G(x, \cdot) + \delta_C)(x).$$

Consequently, $A^F(x) + A^G(x) = A^{F+G}(x)$ and since $x \in C$ was arbitrarily chosen, the conclusion follows. □

Remark 7.34. Note that the hypotheses of Proposition 7.3 ensure that $\overline{\text{co}} h_{F+G}(x, x^*) \geq h^*_{F+G}(x^*, x) \geq c(x, x^*)$ for all $(x, x^*) \in X \times X^*$. Unfortunately, this is not enough in order to guarantee the maximality of $F + G$, which would follow for instance provided the BO-maximal monotonicity of this bifunction. However, checking also Remark 7.33, this additional assumption would make, at least in the reflexive case, the condition $0 \in \text{sqri}(C - C)$ redundant. Therefore, it remains as an open question what should one add to the hypotheses of Proposition 7.3 in order to obtain the maximality of $F + G$ under no stronger hypotheses than the ones in Remark 7.33.