Antitone L-bonds*

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Abstract. L-bonds represent relationships between fuzzy formal contexts. We study these intercontextual structures w.r.t. antitone Galois connections in fuzzy setting. Furthermore, we define direct $\triangleleft$-product and $\triangleright$-product of two formal fuzzy contexts and show conditions under which a fuzzy bond can be obtained as an intent of the product. This extents our previous work on isotone fuzzy bonds.

1 Introduction

Formal Concept Analysis (FCA) [10] is an exploratory method of analysis of relational data. The method identifies some interesting clusters (formal concepts) in a collection of objects and their attributes (formal context) and organizes them into a structure called concept lattice. Formal Concept Analysis in fuzzy setting [3] allows us to work with graded data.

In the present paper, we deal with intercontextual relationships in FCA in fuzzy setting. Particularly, our approach originated in relation to [16] on the notion of Chu correspondences between formal contexts, which led to obtaining information about the structure of **L**-bonds. In [15] we studied properties of **L**-bonds w.r.t. isotone concept-forming operators.

The present paper concerns with **L**-bonds with antitone character; We describe their properties and explain how these **L**-bonds relate to the structures studied in [16]. In addition, we also focus on the direct products of two formal fuzzy contexts and show conditions under which a bond can be obtained as an intent of the product.

The paper is structured as follows: in Section 2 we recollect some notions used in this paper; in Section 3 we define the **L**-bonds and direct products, and describe their properties. Our conclusions and related further research are summarized in Section 4.

2 Preliminaries

In this section, we recall some basic notions used in the paper.

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2.1 Residuated Lattices and Fuzzy Sets

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice [3,12,21] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1 \rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leqslant$. Throughout this paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of a (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \rightarrow 0. \tag{1}$$

$\mathbf{L}$-*sets and* $\mathbf{L}$-*relations* An $\mathbf{L}$-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted L^X, or $\mathbf{L}^X$ if the structure of $\mathbf{L}$ is to be emphasized.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. An $\mathbf{L}$-set $A \in L^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write

$$\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \ldots, {}^{A(x_n)}/x_n\}.$$

An $\mathbf{L}$-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An $\mathbf{L}$-set $A \in L^X$ is called empty (denoted by $\varnothing$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, the $\mathbf{L}$-sets $a \otimes A$, $a \rightarrow A$, $A \rightarrow a$, and $\neg A$ in X are defined by

$$(a \otimes A)(x) = a \otimes A(x), \tag{2}$$
$$(a \rightarrow A)(x) = a \rightarrow A(x), \tag{3}$$
$$(A \rightarrow a)(x) = A(x) \rightarrow a, \tag{4}$$
$$\neg A(x) = A(x) \rightarrow 0. \tag{5}$$

An a-complement is an $\mathbf{L}$-set A which satisfies $(A \rightarrow a) \rightarrow a = A$.

Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I).

$V \subseteq L^X$ is called an $\mathbf{L}$-*closure system* if

- V is closed under left $\rightarrow$-multiplication (or $\rightarrow$-shift), i.e. for every $a \in L$ and $C \in V$ we have $a \rightarrow C \in V$,
- V is closed under intersection, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigcap_{j \in J} C_j \in V$.

$V \subseteq L^X$ is called an **L**-*interior system* if

- V is closed under left $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$ we have $a \otimes C \in V$,
- V is closed under union, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigcup_{j \in J} C_j \in V$.

Relational products We use three relational product operators, $\circ$, $\triangleleft$, and $\triangleright$, and consider the corresponding products $R = S \circ T$, $R = S \triangleleft T$, and $R = S \triangleright T$ (for $R \in L^{X \times Z}, S \in L^{X \times Y}, T \in L^{Y \times Z}$). In the compositions, $R(x, z)$ is interpreted as the degree to which the object x has the attribute z; $S(x, y)$ as the degree to which the factor y applies to the object x; $T(y, z)$ as the degree to which the attribute z is a manifestation (one of possibly several manifestations) of the factor y. The composition operators are defined by

$$(S \circ T)(x, z) = \bigvee_{y \in Y} S(x, y) \otimes T(y, z), \tag{6}$$

$$(S \triangleleft T)(x, z) = \bigwedge_{y \in Y} S(x, y) \rightarrow T(y, z), \tag{7}$$

$$(S \triangleright T)(x, z) = \bigwedge_{y \in Y} T(y, z) \rightarrow S(x, y). \tag{8}$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [13]. They have natural verbal descriptions. For instance, $(S \circ T)(x, z)$ is the truth degree of the proposition "there is factor y such that y applies to object x and attribute z is a manifestation of y"; $(S \triangleleft T)(x, z)$ is the truth degree of "for every factor y, if y applies to object x then attribute z is a manifestation of y". Note also that for $L = \{0, 1\}$, $S \circ T$ coincides with the well-known composition of binary relations.

We will need following lemma.

Lemma 1 ([3]). *For $R \in L^{W \times X}, S \in L^{X \times Y}, T \in L^{Y \times Z}$ we have*

$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T \quad \textit{and} \quad R \triangleright (S \circ T) = (R \triangleright S) \triangleright T.$$

2.2 Formal Concept Analysis in the Fuzzy Setting

An **L**-context is a triplet $\langle X, Y, I \rangle$ where X and Y are (ordinary) sets and $I \in L^{X \times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to degree a." An **L**-context is usually depicted as a table whose rows correspond to objects and whose columns correspond to attributes; entries of the table contain the degrees $I(x, y)$.

Concept-forming operators induced by an **L**-context $\langle X, Y, I\rangle$ are the following operators: First, the pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of operators ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\downarrow} : L^Y \to L^X$ is defined by

$$A^{\uparrow}(y) = \bigwedge_{x\in X} A(x) \to I(x, y), \quad B^{\downarrow}(x) = \bigwedge_{y\in Y} B(y) \to I(x, y). \tag{9}$$

Second, the pair $\langle {}^{\cap}, {}^{\cup}\rangle$ of operators ${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x\in X} A(x) \otimes I(x, y), \quad B^{\cup}(x) = \bigwedge_{y\in Y} I(x, y) \to B(y), \tag{10}$$

Third, the pair $\langle {}^{\wedge}, {}^{\vee}\rangle$ of operators ${}^{\wedge} : L^X \to L^Y$ and ${}^{\vee} : L^Y \to L^X$ is defined by

$$A^{\wedge}(y) = \bigwedge_{x\in X} I(x, y) \to A(x), \quad B^{\vee}(x) = \bigvee_{y\in Y} B(y) \otimes I(x, y), \tag{11}$$

for $A \in L^X$, $B \in L^Y$. When we need to emphasize that a pair of concept-forming operators is induced by a particular **L**-relation we write it as a subscript, for instance we write $\uparrow_I$ instead of just $\uparrow$.

Furthermore, denote the corresponding sets of fixed points by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$, $\mathcal{B}^{\cap\cup}(X, Y, I)$, and $\mathcal{B}^{\wedge\vee}(X, Y, I)$, i.e.

$$\begin{aligned}
\mathcal{B}^{\uparrow\downarrow}(X, Y, I) &= \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\, B^{\downarrow} = A\},\\
\mathcal{B}^{\cap\cup}(X, Y, I) &= \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\cap} = B,\, B^{\cup} = A\},\\
\mathcal{B}^{\wedge\vee}(X, Y, I) &= \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\wedge} = B,\, B^{\vee} = A\}.
\end{aligned}$$

The sets of fixpoints are complete lattices [1,11,20], called **L**-concept lattices associated to I, and their elements are called formal concepts.

For a concept lattice $\mathcal{B}^{\Delta\nabla}(X, Y, I)$, where $\mathcal{B}^{\Delta\nabla}$ is either of $\mathcal{B}^{\uparrow\downarrow}$, $\mathcal{B}^{\cap\cup}$, or $\mathcal{B}^{\wedge\vee}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}^{\Delta\nabla}(X, Y, I)$ and $\mathrm{Int}^{\Delta\nabla}(X, Y, I)$. That is,

$$\begin{aligned}
\mathrm{Ext}^{\Delta\nabla}(X, Y, I) &= \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}^{\Delta\nabla}(X, Y, I) \text{ for some } B\},\\
\mathrm{Int}^{\Delta\nabla}(X, Y, I) &= \{B \in L^Y \mid \langle A, B\rangle \in \mathcal{B}^{\Delta\nabla}(X, Y, I) \text{ for some } A\}.
\end{aligned}$$

The operators induced by an **L**-context and their sets of fixpoints have been extensively studied, see e.g. [1,2,4,11,20].

3 L-bonds

This section introduces antitone **L**-bonds, namely a-bonds and c-bonds, and describes their properties.

Definition 1. *(a) An* a-bond *from* **L***-context* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *to* **L***-context* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *is an* **L***-relation* $\beta \in L^{X_1\times Y_2}$ *s.t.*

$$\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1) \textit{ and } \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2).$$

(b) A c-bond *from* **L**-*context* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *to* **L**-*context* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *is an* **L**-*relation* $\beta \in L^{X_1 \times Y_2}$ *s.t.*

$$\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1) \text{ and } \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, \beta) \subseteq \mathrm{Int}^{\wedge\vee}(X_2, Y_2, I_2).$$

Remark 1. 1) The terms—a-bond and c-bond—were chosen to match with notions of a-morphism and c-morphism [7,14,9]. We show in Theorem 2 that the a-bonds and c-bonds are in one-to-one correspondence of a-morphisms and c-morphisms, respectively, on sets of intents of associated concept lattices.

2) Note that all considered sets of extents and intents in Definition 1 are **L**-closure systems. From this point of view, the condition of subsethood is natural.

Theorem 1. *(a)* $\beta \in L^{X_1 \times Y_2}$ *is an a-bond between* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *and* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *iff there exist* **L**-*relations* $S_{\mathrm{i}} \in L^{Y_1 \times Y_2}$ *and* $S_{\mathrm{e}} \in L^{X_1 \times X_2}$*, such that*

$$\beta = I_1 \triangleleft S_{\mathrm{i}} = S_{\mathrm{e}} \triangleleft I_2. \tag{12}$$

(b) $\beta \in L^{X_1 \times Y_2}$ *is a c-bond between* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *and* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *iff there exist* **L**-*relations* $S_{\mathrm{i}} \in L^{Y_1 \times Y_2}$ *and* $S_{\mathrm{e}} \in L^{X_1 \times X_2}$*, such that*

$$\beta = I_1 \triangleright S_{\mathrm{i}} = S_{\mathrm{e}} \triangleright I_2. \tag{13}$$

Proof. Follows from results in [9]. □

3.1 Morphisms

This section explains correspondence of **L**-bonds with morphisms of **L**-interior/**L**-closure spaces. First, we recall notions of c-morphisms and a-morphisms. These morphisms were previously studied in [7,9,14].

Definition 2. *(a) A mapping* $h : V \to W$ *from an* **L**-*interior system* $V \subseteq L^X$ *into an* **L**-*closure system* $W \subseteq L^Y$ *is called an* a-morphism *if*

- $h(a \otimes C) = a \to h(C)$ *for each* $a \in L$ *and* $C \in V$*;*
- $h(\bigvee_{k\in K} C_k) = \bigwedge_{k\in K} h(C_k)$ *for every collection of* $C_k \in V$*.*

An a-morphism $h : V \to W$ *is called an* extendable a-morphism *if* h *can be extended to an a-morphism of* L^X *to* L^Y*, i.e. if there exists an a-morphism* $h' : L^X \to L^Y$ *such that for every* $C \in V$ *we have* $h'(C) = h(C)$*.*

(b) A mapping $h : V \to W$ *from an* **L**-*closure system* $V \subseteq L^X$ *into an* **L**-*closure system* $W \subseteq L^Y$ *is called a* c-morphism *if it is a* $\to$*- and* $\bigwedge$*-morphism and it preserves a-complements, i.e. if*

- $h(a \to C) = a \to h(C)$ *for each* $a \in L$ *and* $C \in V$*;*
- $h(\bigwedge_{k\in K} C_k) = \bigwedge_{k\in K} h(C_k)$ *for every collection of* $C_k \in V$ $(k \in K)$*;*
- *if* C *is an a-complement then* $h(C)$ *is an a-complement.*

A c-morphism $h : V \to W$ is called an extendable c-morphism *if h can be extended to a c-morphism of L^X to L^Y, i.e. if there exists a c-morphism $h' : L^X \to L^Y$ such that for every $C \in V$ we have $h'(C) = h(C)$.*

In this paper we consider only extendable {a,c}-morphims.

Theorem 2. *(a) The a-bonds between $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ are in one-to-one correspondence with*

- *a-morphisms from* $\mathrm{Int}^{\cap\cup}(X_1, Y_1, I_1)$ *to* $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$;
- *c-morphisms from* $\mathrm{Ext}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ *to* $\mathrm{Ext}^{\cap\cup}(X_1, Y_1, I_1)$.

(b) The c-bonds between $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ are in one-to-one correspondence with

- *c-morphisms from* $\mathrm{Int}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ *to* $\mathrm{Int}^{\wedge\vee}(X_2, Y_2, I_2)$;
- *a-morphisms from* $\mathrm{Ext}^{\wedge\vee}(X_2, Y_2, I_2)$ *to* $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$.

Proof. Follows from Theorem 1 and results in [9,14].

Theorem 3. *(a) The system of all a-bonds is an* **L***-closure system.*
(b) The system of all c-bonds is an **L***-closure system.*

Proof. (a) Consider a collection of a-bonds β_i. By Theorem 1 the β_is are in the form $\beta_i = I_1 \triangleleft S_{\mathrm{i}} = S_{\mathrm{e}} \triangleleft I_2$. We have

$$\begin{aligned}
\bigcap_{j\in J} \beta_j &= \bigcap_{j\in J} (I_1 \triangleleft S_{\mathrm{i}j}) = I_1 \triangleleft (\bigcap_{j\in J} S_{\mathrm{i}j}) \\
&= \bigcap_{j\in J} (S_{\mathrm{e}j} \triangleleft I_2) = (\bigcup_{j\in J} S_{\mathrm{e}j}) \triangleleft I_2; \\
a \to \beta &= a \to (I_1 \triangleleft S_{\mathrm{i}}) = I_1 \triangleleft (a \to S_{\mathrm{i}}) \\
&= a \to (S_{\mathrm{e}} \triangleleft I_2) = (a \otimes S_{\mathrm{e}}) \triangleleft I_2.
\end{aligned}$$

Thus, $\bigcap_{j\in J} \beta_j$ and $a \to \beta$ are a-bonds. Proof of (b) is similar. □

3.2 Direct Products

In this part, we focus on direct products of **L**-contexts related to a-bonds and c-bonds.

Definition 3. *Let $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle, \mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ be* **L***-contexts.*

(a) A direct $\triangleleft$-product *of $\mathbb{K}_1$ and $\mathbb{K}_2$ is defined as the* **L***-context $\mathbb{K}_1 \boxminus \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$ with $\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_1(x_1, y_1) \to I_2(x_2, y_2)$ for all $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$.*
(b) A direct $\triangleright$-product of $\mathbb{K}_1$ and $\mathbb{K}_2$ is defined as the **L***-context $\mathbb{K}_1 \boxminus \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$ with $\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_2(x_2, y_2) \to I_1(x_1, y_1)$ for all $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$.*

The following theorem shows that $\mathbb{K}_1 \boxed{\rightarrow} \mathbb{K}_2$ (resp. $\mathbb{K}_1 \boxed{\leftarrow} \mathbb{K}_2$) induces a-bonds (resp. c-bonds) as its intents.

Theorem 4. *(a) The intents of $\mathbb{K}_1 \boxed{\rightarrow} \mathbb{K}_2$ w.r.t $\langle \uparrow, \downarrow \rangle$ are a-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$, i.e for each $\phi \in L^{X_2 \times Y_1}$, $\phi^{\uparrow}$ is an a-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.*
(b) The intents of $\mathbb{K}_1 \boxed{\leftarrow} \mathbb{K}_2$ w.r.t $\langle \uparrow, \downarrow \rangle$ are c-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$, i.e for each $\phi \in L^{X_2 \times Y_1}$, $\phi^{\uparrow}$ is a c-bond from $\mathbb{K}_1$ to $\mathbb{K}_2$.

Proof. (a) For $\phi \in L^{X_2 \times Y_1}$ we have

$$
\begin{aligned}
\phi^{\uparrow}(x_1, y_2) &= \bigwedge_{\langle x_2, y_1 \rangle \in X_2 \times Y_1} \phi(x_2, y_1) \to \Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} \phi(x_2, y_1) \to (I_1(x_1, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} (I_1(x_1, y_1) \to (\phi(x_2, y_1) \to I_2(x_2, y_2))) \\
&= \bigwedge_{y_1 \in Y_1} (I_1(x_1, y_1) \to \bigwedge_{x_2 \in X_2} (\phi(x_2, y_1) \to I_2(x_2, y_2))) \\
&= \bigwedge_{y_1 \in Y_1} (I_1(x_1, y_1) \to \bigwedge_{x_2 \in X_2} (\phi^{\mathrm{T}}(y_1, x_2) \to I_2(x_2, y_2))) \\
&= \bigwedge_{y_1 \in Y_1} I_1(x_1, y_1) \to (\phi^{\mathrm{T}} \triangleleft I_2)(y_1, y_2) \\
&= (I_1 \triangleleft (\phi^{\mathrm{T}} \triangleleft I_2))(x_1, y_2) \\
&= ((I_1 \circ \phi^{\mathrm{T}}) \triangleleft I_2)(x_1, y_2).
\end{aligned}
$$

Thus $\phi^{\uparrow}$ is an a-bond by Theorem 1. Proof of (b) is similar. □

Not all a-bonds are intents of the direct product as the following examples shows.

Example 1. Consider **L**-context $\mathbb{K} = \langle \{x\}, \{y\}, \{^{0.5}/\langle x, y \rangle\} \rangle$ with **L** being the three-element Łukasiewicz chain. Obviously, $\{^{0.5}/\langle x, y \rangle\}$ is an a-bond from $\mathbb{K}$ to $\mathbb{K}$. We have $\mathbb{K} \boxed{\rightarrow} \mathbb{K} = \langle \{\langle x, y \rangle\}, \{\langle x, y \rangle\}, \{\langle x, y \rangle, \langle x, y \rangle\} \rangle$. The only intent of $\mathbb{K} \boxed{\rightarrow} \mathbb{K}$ is $\{\langle x, y \rangle\}$; thus the a-bond $\{^{0.5}/\langle x, y \rangle\}$ is not among its intents.

Example 2. Consider following **L**-context with **L** being three-element Łukasiewicz chain.

$$
\mathbb{K}_1 = \begin{array}{|cccc} \hline 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \mathbb{K}_2 = \begin{array}{|ccc} \hline 0 & 1 & 1 \\ 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array}.
$$

There are 11 a-bonds from $\mathbb{K}_1$ to $\mathbb{K}_2$, but $\mathbb{K}_1 \boxed{\rightarrow} \mathbb{K}_2$ has only 9 concepts; see Figure 1.

Since the definition of direct $\triangleleft$-product and direct $\triangleright$-product differ only in the direction of residuum, we can make the following corollary.

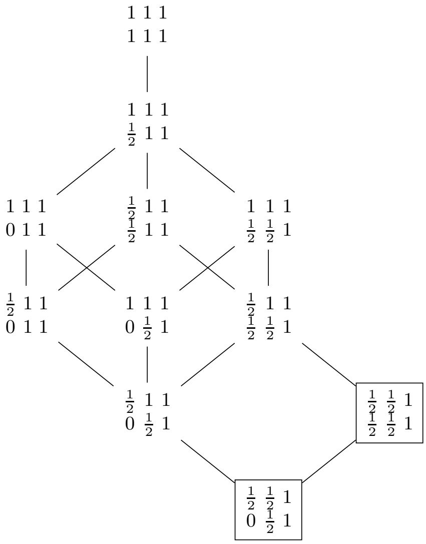

Fig. 1. System of a-bonds between $\mathbb{K}_1$ and $\mathbb{K}_2$ from Example 2. Boxed a-bonds are those which are not intents of $\mathbb{K}_1 \boxminus \mathbb{K}_2$.

Corollary 1. *(a) The extents of the direct* $\triangleleft$*-product of* $\langle X_1, Y_1, I_1 \rangle$ *and* $\langle X_2, Y_2, I_2 \rangle$ *are a-bonds from* $\langle X_2, Y_2, I_2 \rangle$ *to* $\langle X_1, Y_1, I_1 \rangle$.
(b) The extents of the direct $\triangleright$*-product of* $\langle X_1, Y_1, I_1 \rangle$ *and* $\langle X_2, Y_2, I_2 \rangle$ *are c-bonds from* $\langle X_2, Y_2, I_2 \rangle$ *to* $\langle X_1, Y_1, I_1 \rangle$.

3.3 Strong Antitone L-bonds

As classical bonds connect contexts with antitone Galois connections, we also consider **L**-bonds from $\langle X_1, Y_1, I_1 \rangle$ to $\langle X_2, Y_2, I_2 \rangle$ defined as **L**-relations $J \in L^{X_1 \times Y_2}$ such that

$$\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_2, J) \subseteq \mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1) \quad \text{and} \quad \mathrm{Int}^{\uparrow\downarrow}(X_1, Y_2, J) \subseteq \mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2). \tag{14}$$

In what follows, we call the **L**-relations defined by (14) *strong antitone* **L***-bonds.*

Using the double negation law. If the double negation law holds true in L, each pair of concept-forming operators (9)–(11) is definable by any of other two. As a consequence, we have

$$\mathcal{B}^{\uparrow\downarrow}(X, Y, I) \text{ and } \mathcal{B}^{\cap\cup}(X, Y, \neg I) \text{ are isomorphic as lattices} \tag{15}$$

with $\langle A, B \rangle \mapsto \langle A, \neg B \rangle$ being an isomorphism. In addition, we have

$$\mathrm{Ext}^{\uparrow\downarrow}(X, Y, \neg I) = \mathrm{Ext}^{\cap\cup}(X, Y, I) \text{ and, dually, } \mathrm{Int}^{\uparrow\downarrow}(X, Y, \neg I) = \mathrm{Int}^{\wedge\vee}(X, Y, I).$$

Theorem 5. *Let the double negation law hold true in* **L**. *the strong antitone* **L**-*bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are exactly a-bonds from* $\langle X_1, Y_1, \neg I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$*; and c-bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, \neg I_2\rangle$.

Note that the incidence relation Δ in direct product $\mathbb{K}_1 \boxed{\rightarrow} \mathbb{K}_2$ then becomes

$$\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = \neg I_1(x_1, y_1) \rightarrow I_2(x_2, y_2);$$

that is in agreement with results in [16]. Similarly, the incidence relation Δ in direct product $\mathbb{K}_1 \boxed{\leftarrow} \mathbb{K}_2$ becomes

$$\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_1(x_1, y_1) \rightarrow \neg I_2(x_2, y_2).$$

Using an alternative notion of complement. The mutual reducibility of concept-forming operators (9)–(11) does not hold generally. In [8], we proposed a new notion of complement of **L**-relation to overcome that. Using this notion we showed that each for each $I \in L^{X\times Y}$, one can define $\invneg I \in L^{X\times(Y\times L)}$ as

$$\invneg I(x, \langle y, a\rangle) = I(x, y) \rightarrow a,$$

and obtain

$$\mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, \invneg I) = \mathrm{Ext}^{\cap\cup}(X, Y, I)$$

and, similarly,

$$\mathrm{Int}^{\uparrow\downarrow}(X, Y \times L, \invneg I) = \mathrm{Int}^{\wedge\vee}(X \times L, Y, (\invneg I^{\mathrm{T}})^{\mathrm{T}})$$

Unfortunately, the opposite direction holds true only for those **L**-contexts $\langle X, Y, I\rangle$ whose set $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ (resp. $\mathrm{Int}^{\uparrow\downarrow}(X, Y, I)$) is a c-closure system [7]; i.e. an **L**-closure system generated by a system of all a-complements of some $\mathcal{T} \subseteq L^X$.

Theorem 6. *If* $\mathrm{Ext}^{\uparrow\downarrow}(X_1, Y_1, I_1)$ *is a c-closure system, the strong antitone* **L**-*bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are exactly a-bonds from* $\langle X_1, Y_1 \times L, \invneg I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$. *If* $\mathrm{Int}^{\uparrow\downarrow}(X_2, Y_2, I_2)$ *is a c-closure system, the antitone* **L**-*bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2, Y_2, I_2\rangle$ *are exactly c-bonds from* $\langle X_1, Y_1, I_1\rangle$ *to* $\langle X_2 \times L, Y_2, (\invneg I_2^{\mathrm{T}})^{\mathrm{T}}\rangle$.

We omit further details due to the lack of space.

4 Conclusions and Further Research

We studied bonds between fuzzy contexts related to mutually different types of concept-forming operators and their relationship to antitone fuzzy bonds.

Our future research includes:

- Covering the **L**-bonds described above and isotone **L**-bonds in [15] by a general framework. The isotone and antitone concept-forming operators are one type of operators in [6,5]; also in [18].
- Generalizing the described theory to bond **L**-contexts which each use different residuated lattice as the structure of truth-degrees. Results described in [17] seem to be promising for this goal.

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