L-Fuzzy Context Sequences on Complete Lattices

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Abstract. This work studies the L-fuzzy context sequences when L is a complete lattice extending the results obtained in previous works with $L = [0, 1]$. To do this, we will use n-ary OWA operators on complete lattices. With the aid of these operators, we will study the different contexts values of the sequence using some new relations. As a particular case, we have the study when $L = \mathcal{J}([0, 1])$. Finally, we illustrate all the results by means of an example.

Keywords: L-fuzzy context, L-fuzzy concept, L-fuzzy context sequences, n-ary OWA operators.

1 Introduction

The L-Fuzzy Concept Analysis studies the information from an L-fuzzy context by means of the L-fuzzy concepts. These L-fuzzy contexts are tuples (L, X, Y, R), with L a complete lattice, X and Y sets of objects and attributes, and $R \in L^{X \times Y}$ an L-fuzzy relation between the objects and the attributes.

In some situations, we have a sequence formed by the L-fuzzy contexts (L, X, Y, R_i), $i = \{1, \ldots, n\}$, $n \in \mathbb{N}$, where R_i is the ith relation between the objects of X and the attributes of Y. The study of these L-fuzzy context sequences will be the main target of this work.

A particular case of this sequence is when it represents an evolution in time of an L-fuzzy context.

To start, we will see some important results about the L-Fuzzy Concept Analysis.

2 *L*-Fuzzy Contexts

The Formal Concept Analysis of R. Wille [23] extracts information from a binary table that represents a Formal context (X, Y, R) with X and Y finite sets of objects and attributes respectively and $R \subseteq X \times Y$. The hidden information

A. Laurent et al. (Eds.): IPMU 2014, Part III, CCIS 444, pp. 31–40, 2014.

consists of pairs (A, B) with $A \subseteq X$ and $B \subseteq Y$, called Formal concepts, verifying $A^* = B$ and $B^* = A$, where $(\cdot)^*$ is a derivation operator that associates the attributes related to the elements of A with every object set A, and the objects related to the attributes of B with every attribute set B. These Formal Concepts can be interpreted as a group of objects A that shares the attributes of B.

In previous works [8,9] we have defined the L-fuzzy contexts (L, X, Y, R), with L a complete lattice, X and Y sets of objects and attributes respectively and $R \in L^{X \times Y}$ a fuzzy relation between the objects and the attributes.

In our case, to work with these L-fuzzy contexts, we have defined the derivation operators $(\cdot)_1$ and $(\cdot)_2$ given by means of these expressions:

$$\forall A \in L^X, \forall B \in L^Y,\ A_1(y) = \inf_{x \in X}\{\mathcal{I}(A(x), R(x, y))\}$$
$$B_2(x) = \inf_{y \in Y}\{\mathcal{I}(B(y), R(x, y))\}$$

with $\mathcal{I}$ a fuzzy implication operator defined in the lattice $(L, \leq)$.

Some authors use a residuated implication operator in their definitions of derivation operators [7,19,20].

The information stored in the context is visualized by means of the L-fuzzy concepts that are pairs $(A, B) \in (L^X \times L^Y)$ fulfilling $A_1 = B$ and $B_2 = A$. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a group of objects that share a group of attributes in a fuzzy way.

On the other hand, given $A \in L^X$, (or $B \in L^Y$) we can obtain the associated L-fuzzy concept applying twice the derivation operators. In the case of using a residuated implication, as we do in this work, the associated L-fuzzy concept is (A_{12}, A_1) (or (B_2, B_{21})).

Other important results about this theory are in [6,10,7,2,19,15,16,20].

3 *L*-Fuzzy Context Sequences

In this section, we are interested in the study of the L-fuzzy context sequences where L is a complete lattice. We have analyzed these sequences when $L = [0, 1]$ in [4,5]. To do this, we have given the formal definition:

Definition 1. *An L-fuzzy context sequence is a tuple* $(L, X, Y, R_i), i = \{1, \ldots, n\}$, $n \in \mathbb{N}$, *with L a complete lattice, X and Y sets of objects and attributes respectively and* $R_i \in L^{X \times Y}, \forall i = \{1, \ldots, n\}$, *a family of L-fuzzy relations between X and Y.*

In previous works [12,11], we have done some studies in order to aggregate the information of different contexts with the same set of objects and attributes. The use of weighted averages [13,14] (with L=[0,1]) in order to summarize the information stored in the different relations allows us to associate different weights to the L-fuzzy contexts highlighting some of them. However, it is possible that some observations of an L-fuzzy context of the sequence are interesting whereas others not so much. For instance, in [3] we study that the used methods to obtain the L-fuzzy concepts do not give good results when we have very low values in some relations. Moreover, we want now to do different studies based on different exigency levels. This is one of the new contributions of this work.

In order to introduce this subject, let us see the following example.

Example 1. Let $(L, X, Y, R_i), i = \{1, \ldots, n\}$, be an L-fuzzy context sequence that represents the sales of sports articles (X) in some establishments (Y) throughout a period of time (I), and we want to study the places where the main sales hold taking into account that there are seasonal sporting goods (for instance skies, bathing suits) and of a certain zone (it is more possible to sale skies in Colorado than in Florida).

In this case, the weighted average model is not valid since it is very difficult to associate a weight to an L-fuzzy context (in some months more bath suits are sold whereas, in others, skies are). To analyze this situation, it could be interesting the use of the OWA operators [21,17] with the most of the weights near the largest values. In this way, we give more relevance to the largest observations, independently of the moment when they have taken place and, on the other hand, we would avoid some small values in the resulting relations (that can give problems in the calculation of the L-fuzzy concepts as was studied in [3]).

The next section summarizes the main results about these operators.

4 $n-$ary OWA Operators

This is the definition of these operators given by Yager [21]:

Definition 2. *A mapping F from $L^n \longrightarrow L$, where $L = [0, 1]$ is called an OWA operator of dimension n if associated with F is a weighting $n-$tuple $W = (w_1, w_2, \ldots, w_n)$ such that $w_i \in$[0,1] and $\sum_{1 \leq i \leq n} w_i = 1$, where $F(a_1, a_2, \ldots, a_n) = w_1.b_1 + w_2.b_2 + \cdots + w_n.b_n$, with b_i the ith largest element in the collection $a_1, a_2, \ldots, a_n$.*

To study the fuzzy context sequence, we are interested in the use of operators close to *or*. To measure this proximity we can use the orness degree [21].

However, Yager's OWA operators are not easy to be extended to any complete lattice L. The main difficult is that Yager's construction is based on a previous arrangement of the real values which have to be aggregated, which is not always possible in a partially ordered set. In order to overcome this problem Lizasoain and Moreno [18] have built an ordered vector for each given vector in the lattice. This construction allows to define the n-ary OWA operator on any complete lattice which has Yager's OWA operator as a particular case.
Their contribution involves the construction, for each vector $(a_1, \ldots, a_n) \in L^n$ of a totally ordered vector $(b_1, \ldots, b_n)$ as shown in the following proposition:

Proposition 1. *Let $(L, \leq_L)$ be a complete lattice. For any $(a_1, a_2, \ldots, a_n) \in L^n$, consider the values*

- $b_1 = a_1 \vee \cdots \vee a_n \in L$
- $b_2 = [(a_1 \wedge a_2) \vee \cdots \vee (a_1 \wedge a_n)] \vee [(a_2 \wedge a_3) \vee \cdots \vee (a_2 \wedge a_n)] \vee \cdots \vee [a_{n-1} \wedge a_n] \in L$

  $\vdots$
- $b_k = \bigvee \{a_{j_1} \wedge \cdots \wedge a_{j_k} | \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}\} \in L$

  $\vdots$
- $b_n = a_1 \wedge \cdots \wedge a_n \in L$

Then $a_1 \wedge \cdots \wedge a_n = b_n \leq_L b_{n-1} \leq \cdots \leq_L b_1 = a_1 \vee \cdots \vee a_n$.

Moreover, if the set $\{a_1, \ldots, a_n\}$ *is totally ordered, then the vector* $(b_1, \ldots, b_n)$ *agrees with* $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ *for some permutation* σ *of* $\{1, \ldots, n\}$.

On the other hand, it is very easy to see that if $\{a_1, \ldots, a_n\}$ is a chain, b_k is the $k-$th order statistic.

This proposition allows us to generalize Yager's $n-$ary OWA operators from $[0, 1]$ to any complete lattice. To do this, Lizasoain and Moreno give the following:

Definition 3. *Let* $(L, \leq_L, T, S)$ *be a complete lattice endowed with a t-norm* T *and a t-conorm* S*. We will say that* $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in L^n$ *is a*

(i) weighting vector in $(L, \leq_L, T, S)$ *if* $S(\alpha_1, \ldots, \alpha_n) = 1_L$ *and*
(ii) distributive weighting vector in $(L, \leq_L, T, S)$ *if it also satisfies that* $a = T(a, S(\alpha_1, \ldots, \alpha_n)) = S(T(a, \alpha_1), \ldots T(a, \alpha_n))$ *for any* $a \in L$.

Definition 4. *Let* $(\alpha_1, \ldots, \alpha_n) \in L^n$ *be a distributive weighting vector in* $(L, \leq_L, T, S)$*. For each* $(a_1, \ldots, a_n) \in L^n$*, call* $(b_1, \ldots, b_n)$ *the totally ordered vector constructed in Proposition 1. The function* $F_\alpha : L^n \longrightarrow L$ *given by*

$$F_\alpha(a_1, \ldots, a_n) = S(T(\alpha_1, b_1), \ldots, T(\alpha_n, b_n)),$$

$(a_1, \ldots, a_n) \in L^n$*, is called* $n-$*ary OWA operator.*

We will use these $n-$ary OWA operators in the following sections.

5 *L*-Fuzzy Context Sequences General Study

Returning to the initial situation, we can give a definition that summarizes the information stored in the L-fuzzy context sequence:

Definition 5. *Let* $(L, \leq_L, T, S)$ *be a complete lattice endowed with a t-norm* T *and a t-conorm* S*. Let* $(L, X, Y, R_i), i = \{1, \ldots, n\}$*, be the L-fuzzy context sequence,* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *a distributive weighting vector and* F_α *the* $n-$*ary OWA operator associated with* α*. We can define an L-fuzzy relation* R_{F_α} *that aggregates the information of the different L-fuzzy contexts by means of this expression:*

$$\begin{aligned} R_{F_\alpha}(x, y) =& F_\alpha(R_1(x, y), R_2(x, y), \ldots, R_n(x, y)) = \\ =& S(T(\alpha_1, b_1(x, y)), T(\alpha_2, b_2(x, y)), \ldots, T(\alpha_n, b_n(x, y)), \\ & \forall x \in X, y \in Y \end{aligned}$$

with $(b_1(x, y), b_2(x, y), \ldots, b_n(x, y))$ *the totally ordered vector constructed in Proposition 1 for* $(R_1(x, y), R_2(x, y), \ldots, R_n(x, y))$.

In general, the election of the distributive weighting vector will be very important in order to obtain different results.

On the other hand, we want also to establish different demand levels for a more exhaustive study of the L-fuzzy context sequence. To do this, we are going to define n relations using $n-$ary OWA operators where the distributive weighting vector α has just one non-null value $\alpha_k = 1$, for a certain $k \leq n$.

Relevant Case 1. *Let $(L, X, Y, R_i), i = \{1, \dots, n\}$, be an L-fuzzy context sequence with $(L, \leq_L, T, S)$ a complete lattice, X and Y sets of objects and attributes respectively and $R_i \in L^{X \times Y}, \forall i = \{1, \dots, n\}$, and consider $k \in \mathbb{N}, k \leq n$. We define the relation $R_{F_{\alpha^k}}$ using the $n-$ary OWA operator F_{α^k} with the distributive weighting vector $\alpha^k = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha^k_k = 1_L$ and $\alpha^k_i = 0_L, \forall i \neq k$:*

$$\begin{aligned} &\forall x \in X, y \in Y, \quad R_{F_{\alpha^k}}(x, y) = F_{\alpha^k}(R_1(x, y), R_2(x, y), \dots, R_n(x, y)) = \\ &= S(T(0_L, b_1(x, y)), T(0_L, b_2(x, y)), \dots, T(1_L, b_k(x, y)), \dots, T(0_L, b_n(x, y)) \end{aligned}$$

The value $R_{F_{\alpha^k}}(x, y)$ is the minimum of the k largest values associated with the pair (x, y) in the relations R_i. So, this new relation measures the degree in which the object x is at least k times related with the attribute y.

Observe that the defined α^k is a distributive weighting vector. Moreover, notice that if $L = [0, 1]$, then we are using step-OWA operators [22].

Proposition 2. *For any t-norm T and a t-conorm S, it is verified that*

$$R_{F_{\alpha^k}}(x, y) = b_k(x, y), \forall (x, y) \in X \times Y.$$

Proof. Taking into account the basic properties of a t-norm T and a t-conorm S, and the definition of the distributive weighting vector. □

Another interesting case is obtained when we want to analyze the average of the k largest values associated with the pair (x, y) in the relations R_i. In order to do it, we can consider the following relation:

Relevant Case 2. *If $L = [0, 1]$, $T(a, b) = ab$ and $S(a, b) = \min\{a+b, 1\}, \forall a, b \in [0, 1]$, using a distributive weighting vector $\widehat{\alpha}^k$ such that $\widehat{\alpha}^k_i = 1/k, i \leq k$ and $\widehat{\alpha}^k_i = 0, i > k$, the obtained relation $R_{F_{\widehat{\alpha}^k}}$ is given by:*

$$R_{F_{\widehat{\alpha}^k}}(x, y) = \sum_{i=1}^{k} \frac{b_i(x, y)}{k}, \quad \forall (x, y) \in X \times Y \tag{1}$$

Remark 1. It is immediate to prove that if $h \leq k$, then $R_{F_{\alpha^h}} \geq R_{F_{\alpha^k}}$ and $R_{F_{\widehat{\alpha}^h}} \geq R_{F_{\widehat{\alpha}^k}}$.

The observation of the L-fuzzy contexts defined from these new relations gives the idea for the following propositions:

Proposition 3. *Consider $k \in \mathbb{N}$, such that $k \leq n$. If (A,B) is an L-fuzzy concept of the L-fuzzy context $(L,X,Y,R_{F_{\alpha^k}})$, then $\forall h \in \mathbb{N}, h \leq k$, there exists an L-fuzzy concept (C,D) of the L-fuzzy context $(L,X,Y,R_{F_{\alpha^h}})$ such that $A \leq C$ and $B \leq D$.*

Proof. If $h \leq k$, then $R_{F_{\alpha^k}}(x,y) \leq R_{F_{\alpha^h}}(x,y) \quad \forall (x,y) \in X \times Y$.

Let (A,B) be an L-fuzzy concept of the L-fuzzy context $(L,X,Y,R_{F_{\alpha^k}})$. Then, as any implication operator is increasing on its second argument,

$$\forall y \in Y,\ B(y) = \inf_{x \in X}\{\mathcal{I}(A(x), R_{F_{\alpha^k}}(x,y))\} \leq \inf_{x \in X}\{\mathcal{I}(A(x), R_{F_{\alpha^h}}(x,y))\} = D(y)$$

Thus, the L-fuzzy set B derived from A in $(L,X,Y,R_{F_{\alpha^k}})$ is a subset of the L-fuzzy set D derived from A in $(L,X,Y,R_{F_{\alpha^h}})$. Therefore, $B \leq D$.

As the used implication operator $\mathcal{I}$ is residuated, if we derive the set D in $(L,X,Y,R_{F_{\alpha^h}})$, we obtain the set $C{=}D_2$ and the pair (C,D) is an L-fuzzy concept of the L-fuzzy context $(L,X,Y,R_{F_{\alpha^h}})$. Now, applying the properties of this closure operator formed by the composition of the derivation operators in $(L,X,Y,R_{F_{\alpha^h}})$ [6]: $A \leq A_{12} = D_2 = C$. Therefore, the other inequality also holds.

The proposition is analogously proved using the relations $R_{F_{\widehat{\alpha}^k}}$ and $R_{F_{\widehat{\alpha}^h}}$. □

The following result sets up relations between the L-fuzzy concepts associated with the same starting set (see section 2) in the different L-fuzzy contexts.

Proposition 4. *Consider $h,k \in \mathbb{N}$, such that $h \leq k \leq n$, and $A \in L^X$. If the L-fuzzy concepts associated with the set A in the contexts $(L,X,Y,R_{F_{\alpha^k}})$ and $(L,X,Y,R_{F_{\alpha^h}})$ are denoted by (A^k,B^k) and (A^h,B^h), then $B^k \leq B^h$.*

The same result is obtained if we consider the L-fuzzy contexts associated with the relations $R_{F_{\widehat{\alpha}^k}}$ and $R_{F_{\widehat{\alpha}^h}}$.

Proof. Consider $A \in L^X$ and the L-fuzzy contexts associated with the relations $R_{F_{\alpha^k}}$ and $R_{F_{\alpha^h}}$. Unfolding the fuzzy extensions of both L-fuzzy concepts, and taking into account that a fuzzy implication operator is increasing on its second argument, $\forall y \in Y$:

$$B^k(y) = \inf_{x \in X}\{\mathcal{I}(A(x), R_{F_{\alpha^k}}(x,y))\} \leq \inf_{x \in X}\{\mathcal{I}(A(x), R_{F_{\alpha^h}}(x,y))\} = B^h(y)$$

This inequality holds for every $A \in L^X$ and for every implication $\mathcal{I}$.

The result can be similarly proved considering the L-fuzzy contexts associated with the relations $R_{F_{\widehat{\alpha}^k}}$ and $R_{F_{\widehat{\alpha}^h}}$. □

6 L-Fuzzy Context Sequences on $\mathcal{J}([0,1])$

One of the most interesting situations is when we use interval-valued L-Fuzzy contexts. We have previously published some works [11,2] in which the chosen lattice is $L = \mathcal{J}([0,1])$.

In this case, notice that $(\mathcal{J}([0,1]), \leq)$ with the usual order ($[a_1,c_1] \leq [a_2,c_2] \iff a_1 \leq a_2$ and $c_1 \leq c_2$) is a complete but not totally ordered lattice.

Then, we can give the following definition:

Definition 6. *Let $(\mathcal{J}([0,1]), \leq, \boldsymbol{T}, \boldsymbol{S})$ be the complete lattice of the closed intervals in $[0,1]$ endowed with the t-norm $\boldsymbol{T}$ and the t-conorm $\boldsymbol{S}$ and consider the sequence of interval-valued L-fuzzy contexts $(\mathcal{J}([0,1]), X, Y, R_i), i = \{1, \ldots, n\}$. If $[\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_n, \beta_n])$ is a distributive weighting vector of intervals and $F_{[\alpha,\beta]}$ the n-ary OWA operator associated with $[\alpha, \beta]$, then the interval-valued L-fuzzy relation $R_{F_{[\alpha,\beta]}}$ that aggregates the information of the different L-fuzzy contexts can be defined $\forall (x,y) \in X \times Y$ as:*

$$\begin{aligned} R_{F_{[\alpha,\beta]}}(x,y) =& F_{[\alpha,\beta]}(R_1(x,y), R_2(x,y), \ldots, R_n(x,y)) = \\ =& \boldsymbol{S}(\, \boldsymbol{T}([\alpha_1, \beta_1], [b_1(x,y), d_1(x,y)]), \ldots, \boldsymbol{T}([\alpha_n, \beta_n], [b_n(x,y), d_n(x,y)]) \end{aligned}$$

where $([b_1(x,y), d_1(x,y)], [b_2(x,y), d_2(x,y)], \ldots, [b_n(x,y), d_n(x,y)])$ is the totally ordered vector constructed from $(R_1(x,y), R_2(x,y), \ldots, R_n(x,y))$.

Also in this case two relevant situations can be highlighted. In the first one we will establish an exigence level k and in order to measure the degree in which an object is at least k times related to an attribute we will use the following relation:

Relevant Case 3. *Consider $k \in \mathbb{N}$ such that $k \leq n$. If we represent by $[\![\alpha]\!]^k$ the distributive weighting vector $[\![\alpha]\!]^k = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \ldots, [\alpha_n, \beta_n])$ such that $[\alpha_k, \beta_k] = [1,1]$, and $[\alpha_i, \beta_i] = [0,0], \forall i \neq k$, we can define the relation $R_{F_{[\alpha]^k}}$ as:*

$$\begin{aligned} &\forall (x,y) \in X \times Y, \\ &\quad R_{F_{[\![\alpha]\!]^k}}(x,y) = \boldsymbol{S}(\, \boldsymbol{T}([0,0], [b_1(x,y), d_1(x,y)]), \ldots, \\ &\qquad\qquad \boldsymbol{T}([1,1], [b_k(x,y), d_k(x,y)]), \ldots, \boldsymbol{T}([0,0], [b_n(x,y), d_n(x,y)]) \end{aligned}$$

It is immediate to prove that, also in this case, for any t-norm $\boldsymbol{T}$ and t-conorm $\boldsymbol{S}$, the following proposition holds:

Proposition 5. *$R_{F_{[\![\alpha]\!]^k}}(x,y) = [b_k(x,y), d_k(x,y)], \ \forall (x,y) \in X \times Y$.*

The second interesting family of relations is associated with the average of the observations:

Relevant Case 4. *In the complete lattice $(\mathcal{J}([0,1]), \leq)$, consider the t-norm $\boldsymbol{T}$ and the t-conorm $\boldsymbol{S}$ given for any $[a_1, c_1], [a_2, c_2] \in \mathcal{J}([0,1])$ by*

$$\begin{aligned} &\boldsymbol{T}([a_1, c_1], [a_2, c_2]) = [a_1 a_2, c_1 c_2] \\ &\boldsymbol{S}([a_1, c_1], [a_2, c_2]) = [min\{a_1 + a_2, 1\}, min\{c_1 + c_2, 1\}] \end{aligned}$$

If $k \leq n$ and we use the weighting vector $[\![\widehat{\alpha}]\!]^k = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \in \mathcal{J}([0,1])^n$ verifying $[\alpha_i, \beta_i] = [\frac{1}{k}, \frac{1}{k}]$ for every $i \leq k$, and $[\alpha_i, \beta_i] = [0,0]$ for every $i > k$, we can define the relation $R_{F_{[\widehat{\alpha}]^k}}$ as follows:

$$R_{F_{[\![\widehat{\alpha}]\!]^k}}(x,y) = \left[\sum_{i=1}^{k} \frac{b_i(x,y)}{k}, \sum_{i=1}^{k} \frac{d_i(x,y)}{k} \right], \quad \forall (x,y) \in X \times Y$$

Let see an example to better understand the difference of using both ways of aggregated information in $L = \mathcal{J}([0,1])$:

Example 2. We come back to the L-fuzzy context sequence $(L, X, Y, R_i), i = \{1, \ldots, 5\}$, of Example 1 that represents the sports articles $X = \{x_1, x_2, x_3\}$ sales in some establishments $Y = \{y_1, y_2, y_3\}$ during 5 months. Every interval-valued observation of the relations $R_i \in \mathcal{J}([0,1])^{X \times Y}$, represents the variation of the percentage of the daily product sales in each establishment along a month.

$$R_1 = \begin{pmatrix} [0.7, 0.8] & [1,1] & [0.8,1] \\ [0,0] & [0.1,0.4] & [0.1,0.3] \\ [0,0.2] & [0.1,0.3] & [0,0.6] \end{pmatrix} R_2 = \begin{pmatrix} [1,1] & [0.8,1] & [1,1] \\ [0.8,0.9] & [0.4,0.5] & [0.1,0.3] \\ [0,0] & [0,0.2] & [0.2,0.4] \end{pmatrix}$$

$$R_3 = \begin{pmatrix} [1,1] & [1,1] & [1,1] \\ [0.6,0.8] & [0.5,0.5] & [0.7,0.8] \\ [0,0] & [0.1,0.2] & [0.2,0.4] \end{pmatrix} R_4 = \begin{pmatrix} [0.5,0.5] & [0.4,0.6] & [0.6,0.8] \\ [0.1,0.3] & [0.5,0.6] & [0.3,0.5] \\ [0.6,0.6] & [0.8,0.9] & [0.8,1] \end{pmatrix}$$

$$R_5 = \begin{pmatrix} [0.1,0.4] & [0,0.2] & [0,0.2] \\ [0,0] & [0.1,0.3] & [0,0.2] \\ [0.8,1] & [1,1] & [0.9,0.9] \end{pmatrix}$$

We want to study in what establishments are the highest sales for each product, no matter when the sale has been carried out, taking into account that there are seasonal sporting goods that are sold in certain periods of time and not in others (skies, bathing suits ...).

If we fix the demand level, for instance to $k = 3$, and we want to analyze if the products have been sold at least during three months, then associated with the distributive weighting vector $[\![\alpha]\!]^3$, we have the relation:

$$R_{F_{[\![\alpha]\!]^3}} = \begin{pmatrix} [0.7,0.8] & [0.8,1] & [0.8,1] \\ [0.1,0.3] & [0.4,0.5] & [0.1,0.3] \\ [0,0.2] & [0.1,0.3] & [0.2,0.6] \end{pmatrix}$$

Now, we take the L-fuzzy context $(L, X, Y, R_{F_{[\![\alpha]\!]^3}})$ and obtain the interval-valued L-fuzzy concept derived from the crisp singleton $\{x_2\}$ using the interval-valued implication operator defined from the Brouwer-Gödel implication ($\mathcal{I}(a,b) = 1, a \leq b$ and $\mathcal{I}(a,b) = b$ in other case) [1]:

$$(\{x_1/[1,1], x_2/[1,1], x_3/[0,0.2]\}, \{y_1/[0.1,0.3], y_2/[0.4,0.5], y_3/[0.1,0.3]\})$$

In this case, we can say that x_1 and x_2 have been important sales mainly in establishment y_2 at least during three months.

On the other hand, we can analyze the average sale of each article in the three months with highest sales. To do this, we will use the weighting vector $[\![\widehat{\alpha}]\!]^3$ and the obtained relation is:

$$R_{F_{[\![\widehat{\alpha}]\!]^3}} = \begin{pmatrix} [0.9,0.93] & [0.93,1] & [0.93,1] \\ [0.5,0.67] & [0.46,0.53] & [0.36,0.53] \\ [0.46,0.6] & [0.63,0.73] & [0.63,0.83] \end{pmatrix}$$

In this case, we will take the interval-valued implication operator obtained from the fuzzy implication $\mathcal{I}(a,b) = \min\{1, b/a\}$ associated with the t-norm $T(a,b) = ab$ [1]. Then, taking as a starting point x_2, we obtain the interval-valued L-fuzzy concept:

$$(\{x_1/[1,1], x_2/[1,1], x_3/[0.89, 0.89]\}, \{y_1/[0.5, 0.67], y_2/[0.46, 0.53], y_3/[0.36, 0.53]\})$$

We can say that, taking the average of the sales in the three months with highest sales, all the articles have been acceptable sales in all the establishments (the sales in y_1 and y_3 that were not important with the previous definition, now are because they are compensated using the values of the three months).

The use of the $n-$ary OWA operators allow us to ignore the small values of the relations (the sales of a non-seasonal sporting goods are close to 0) since, in this case, if we take the average of all the relations, the results will be biased.

7 Conclusions and Future Work

In this work, we have used OWA operators to study the L-fuzzy context sequence and the derived information by means of the L-fuzzy contexts.

A more complete study can be done when we work with L-fuzzy context sequences that represent the evolution in time of an L-fuzzy context.

On the other hand, these L-fuzzy contexts that evolve in time can be generalize if we study L-fuzzy contexts where the observations are other L-fuzzy contexts. This is the task that we will study in the future.

Acknowledgments. This paper is supported by the Research Group "Intelligent Systems and Energy (SI+E)" of the Basque Government, under Grant IT677-13, and by the Research Group "Artificial Intelligence and Approximate Reasoning" of the Public University of Navarra.

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