Towards a Conflicting Part of a Belief Function

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Abstract. Belief functions usually contain some internal conflict. Based on Hájek-Valdés algebraic analysis of belief functions, a unique decomposition of a belief function into its conflicting and non-conflicting part was introduced at ISIPTA'11 symposium for belief functions defined on a two-element frame of discernment.

This contribution studies the conditions under which such a decomposition exists for belief functions defined on a three-element frame. A generalisation of important Hájek-Valdés homomorphism f of semigroup of belief functions onto its subsemigroup of indecisive belief functions is found and presented. A class of quasi-Bayesian belief functions, for which the decomposition into conflicting and non-conflicting parts exists is specified. A series of other steps towards a conflicting part of a belief function are presented. Several open problems from algebra of belief functions which are related to the investigated topic and are necessary for general solution of the issue of decomposition are formulated.

Keywords: Belief function, Dempster-Shafer theory, Dempster's semigroup, conflict between belief functions, uncertainty, non-conflicting part of belief function, conflicting part of belief function.

1 Introduction

Belief functions represent one of the widely used formalisms for uncertainty representation and processing; they enable representation of incomplete and uncertain knowledge, belief updating, and combination of evidence [20].

When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear which are assigned to $\emptyset$ by the non-normalised conjunctive rule $\textcircled{\cap}$ or normalised by Dempster's rule of combination $\oplus$. Combination of conflicting BFs and interpretation of conflicts is often questionable in real applications; hence a series of alternative combination rules were suggested and a series of papers on conflicting BFs were published, e.g., [13, 16–18, 22].

In [5, 10, 11], new ideas concerning interpretation, definition, and measurement of conflicts of BFs were introduced. We presented three new approaches to interpretation and computation of conflicts: combinational conflict, plausibility conflict, and comparative conflict. Later, pignistic conflict was introduced [11].

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When analyzing mathematical properties of the three approaches to conflicts of BFs, there appears a possibility of expression of a BF Bel as Dempster's sum of a non-conflicting BF Bel_0 with the same plausibility decisional support as the original BF Bel has and of an indecisive BF Bel_S which does not prefer any of the elements of frame of discernment. A new measure of conflict of BFs based on conflicting and non-conflicting parts of BFs is recently under development.

A unique decomposition to such BFs Bel_0 and Bel_S was demonstrated for BFs on 2-element frame of discernment in [6]. The present study analyses possibility of its generalisation and presents three classes of BFs on a 3-element frame for which such decomposition exists; it remains an open problem for other BFs.

2 Preliminaries

2.1 General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* (BFs) [20] on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$, see also [4–9]. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; its values are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed, if it holds, we speak about *normalised bba*. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. There is a unique correspondence between m and the corresponding Bel and Pl; thus we often speak about m as a BF.

A *focal element* is $X \subseteq \Omega$, such that $m(X) > 0$. If all of the focal elements are *singletons* (i.e. one-element subsets of Ω), this is what we call a *Bayesian belief function* (BBF). If all of the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), this is what we call a *quasi-Bayesian belief function* (qBBF). If all focal elements have non-empty intersections (or all are nested), we call this a *consistent BF* (or a *consonant BF*, also a possibility measure).

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$ [20]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalised conjunctive rule of combination* $\circledcirc$, see, e. g., [21].

We say that BF Bel is *non-conflicting* (or conflict free, i.e., it has no internal conflict), when it is consistent; i.e., whenever $Pl(\omega_i) = 1$ for some $\omega_i \in \Omega_n$. Otherwise, BF is *conflicting*, i.e., it contains an internal conflict [5]. We can observe that Bel is non-conflicting if and only if the conjunctive combination of Bel with itself does not produce any conflicting belief masses[1] (when $(Bel \circledcirc Bel)(\emptyset) = 0$).

U_n is the *uniform Bayesian belief function*[2] [5], i.e., the uniform probability distribution on Ω_n. The *normalised plausibility of singletons*[3] of Bel is the BBF (prob. distrib.) $Pl\_P(Bel)$ such, that $(Pl\_P(Bel))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [1, 4].

[1] Martin calls $(m \circledcirc m)(\emptyset)$ autoconflict of the BF [18].

[2] U_n which is idempotent w.r.t. Dempster's rule $\oplus$, and moreover neutral on the set of all BBFs, is denoted as ${}_{nD}0'$ in [4], $0'$ comes from studies by Hájek & Valdés.

[3] Plausibility of singletons is called the *contour function* by Shafer [20], thus $Pl\_P(Bel)$ is in fact a normalisation of the contour function.

Let us define an *indecisive (indifferent) BF* as a BF which does not prefer any $\omega_i \in \Omega_n$, i.e., a BF which gives no decisional support for any ω_i, i.e., a BF such that $h(Bel) = Bel \oplus U_n = U_n$, i.e., $Pl(\{\omega_i\}) = const.$, that is, $(Pl\_P(Bel))(\{\omega_i\}) = \frac{1}{n}$. Let us further define an *exclusive BF* as a BF Bel such[4] that $Pl(X) = 0$ for a certain $\emptyset \neq X \subset \Omega$; the BF is otherwise *non-exclusive.*

2.2 Belief Functions on a Two-Element Frame of Discernment; Dempster's Semigroup

Let us suppose that the reader is slightly familiar with basic algebraic notions like *a group, semigroup, homomorphism*, etc. (Otherwise, see e.g., [3, 14, 15].)

We assume $\Omega_2 = \{\omega_1, \omega_2\}$, in this subsection. We can represent any BF on Ω_2 by a couple (a, b), i.e., by enumeration of its m-values $a = m(\{\omega_1\}), b = m(\{\omega_2\})$, where $m(\{\omega_1, \omega_2\}) = 1 - a - b$. This is called *Dempster's pair* or simply *d-pair* [14, 15] (it is a pair of reals such that $0 \leq a, b \leq 1, a + b \leq 1$).

The set of all non-extremal d-pairs (i.e., d-pairs different from $(1, 0)$ and $(0, 1)$) is denoted by D_0; the set of all non-extremal *Bayesian d-pairs* (where $a+b = 1$) is denoted by G; the set of d-pairs such that $a = b$ is denoted by S (set of indecisive d-pairs); the set where $b = 0$ by S_1 ($a = 0$ by S_2), simple support BFs. Vacuous BF is denoted as $0 = (0, 0)$ and $0' = U_2 = (\frac{1}{2}, \frac{1}{2})$, see Figure 1.

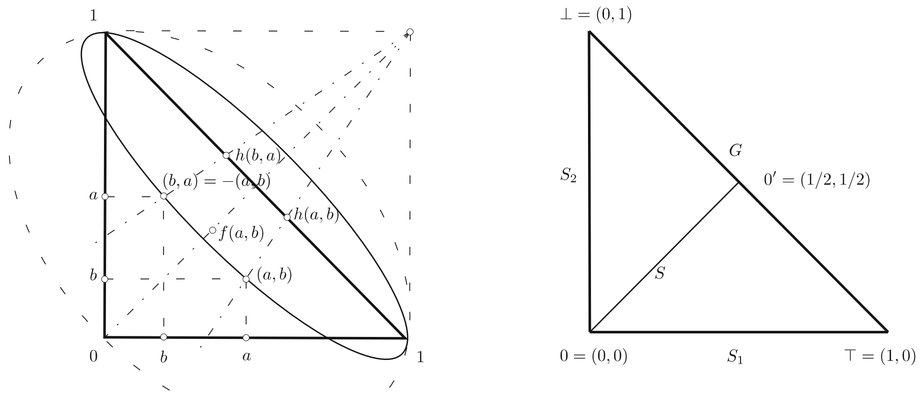

Fig. 1. Dempster's semigroup D_0. Homomorphism h is here a projection of the triangle D_0 to G along the straight lines running through $(1, 1)$. All of the d-pairs lying on the same ellipse are mapped by homomorphism f to the same d-pair in semigroup S.

The *(conjunctive) Dempster's semigroup* $\mathbf{D}_0 = (D_0, \oplus, 0, 0')$ is the set D_0 endowed with the binary operation $\oplus$ (i.e., with Dempster's rule) and two distinguished elements 0 and $0'$. Dempster's rule can be expressed by the formula $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ for d-pairs [14]. In D_0 it is defined further: $-(a, b) = (b, a),\ h(a, b) = (a, b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$, $h_1(a, b) = \frac{1-b}{2-a-b},\ f(a, b) = (a, b) \oplus (b, a) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2}, \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$; $(a, b) \leq (c, d)$ iff $[h_1(a, b) < h_1(c, d)$ or $h_1(a, b) = h_1(c, d)$ and $a \leq c]$ [5].

[4] BF Bel excludes all ω_i such, that $Pl(\{\omega_i\}) = 0$.

[5] Note that $h(a, b)$ is an abbreviation for $h((a, b))$, similarly for $h_1(a, b)$ and $f(a, b)$.

Theorem 1. *(i)* $\mathbf{G} = (G, \oplus, -, 0', \leq)$ *is an ordered Abelian group, isomorphic to the additive group of reals with the usual ordering.* $G^{\leq 0'}$, $G^{\geq 0'}$ *are its cones.*
(ii) The sets S, S_1, S_2 *with operation* $\oplus$ *and the ordering* $\leq$ *form ordered commutative semigroups with neutral element* 0*; all isomorphic to* $(Re, +, -, 0, \leq)^{\geq 0}$.
(iii) h *is ordered homomorphism:* $\mathbf{D}_0 \longrightarrow \mathbf{G}$*;* $h(Bel) = Bel \oplus 0' = Pl\_P(Bel)$.
(iv) f *is homomorphism:* $(D_0, \oplus, -, 0, 0') \longrightarrow (S, \oplus, -, 0)$*; (but, not ordered).*
(v) Mapping $- : \mathbf{D}_0 \longrightarrow \mathbf{D}_0$, $-(a, b) = (b, a)$ *is automorphism of* $\mathbf{D}_0$.

2.3 Dempster's Semigroup on a 3-Element Frame of Discernment

Analogously to d-pairs we can represent BFs by six-tuples $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}) = (m(\{\omega_1\}), m(\{\omega_2\}), m(\{\omega_3\}), m(\{\omega_1, \omega_2\}), m(\{\omega_1, \omega_3\}), m(\{\omega_2, \omega_3\}))$, i.e. by enumeration of its $2^3 - 2$ values, where the $(2^3 - 1)$-th value $m(\Omega_3) = 1 - \sum_i d_i$. Thus there is a significant increase of complexity considering 3-element frame of discernment Ω_3. While we can represent BFs on Ω_2 by a 2-dimensional triangle, we need a 6-dimensional simplex in the case of Ω_3. Further, all the dimensions are not equal: there are 3 independent dimensions corresponding to singletons from Ω_3, but there are other 3 dimensions corresponding to 2-element subsets of Ω_3, each of them somehow related to 2 dimensions corresponding to singletons.

Dempster's semigroup $\mathbf{D}_3$ on Ω_3 is defined analogously to $\mathbf{D}_0$. First algebraic results on $\mathbf{D}_3$ were presented at IPMU'12 [8] (a quasi-Bayesian case $\mathbf{D}_{3-0}$, the dimensions related to singletons only, see Figure 2) and a general case in [9].

Let us briefly recall the following results on $\mathbf{D}_3$ which are related to our topic.

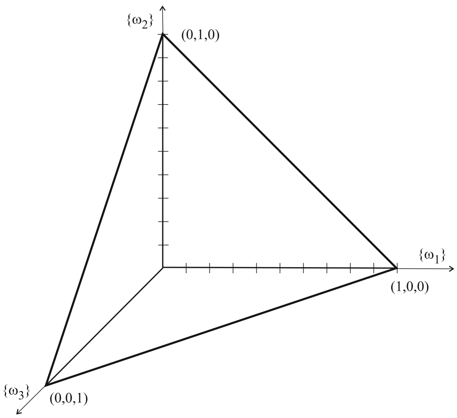

Fig. 2. Quasi-Bayesian BFs on Ω_3

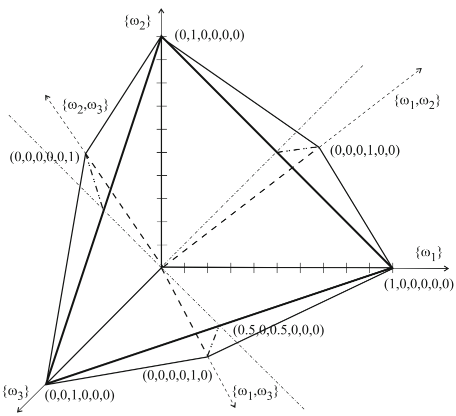

Fig. 3. General BFs on 3-elem. frame Ω_3

Theorem 2. *(i)* $\mathbf{D}_{3-0} = (D_{3-0}, \oplus, 0, U_3)$ *is subalgebra of* $\mathbf{D}_3 = (D_3, \oplus, 0, U_3)$, *where* D_{3-0} *is set of non-exclusive qBBFs* $\mathbf{D}_{3-0} = \{(a, b, c, 0, 0, 0)\}$, D_3 *is set of all non-exclusive BFs on* Ω_3, $0 = (0, 0, 0, 0, 0, 0)$, *and* $U_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$.

(ii) $G_3 = (\{(a, b, c, 0, 0, 0) \mid a+b+c = 1;\ 0 < a, b, c\}, \oplus, "-", U_3)$ *is a subgroup of* $\mathbf{D}_3$*, where* "$-$" *is given by* $-(a, b, c, 0,0,0) = (\frac{bc}{ab+ac+bc}, \frac{ac}{ab+ac+bc}, \frac{ab}{ab+ac+bc}, 0,0,0)$.

(iii a) $S_0 = (\{(a,a,a,\ 0,0,0)\,|\,0 \le a \le \frac{1}{3}\}, \oplus, 0)$*,* $S_1 = (\{(a,0,0,0,0,0)\,|\,0 \le a < 1\}, \oplus, 0), S_2, S_3$*, are monoids with neutral element* 0*, all are isomorphic to the positive cone of the additive group of reals* $\mathbf{Re}^{\ge 0}$ *(*S_0 *to* $\mathbf{Re}^{\ge 0+}$ *with* ∞*).*

(iii b) Monoids $S = (\{(a,a,a,b,b,b) \in D_3\}, \oplus, 0)$ *and* $S_{Pl} = (\{(d_1, d_2, ..., d_{23}) \in D_3\,|\,Pl(d_1, d_2, ..., d_{23}) = U_3\}, \oplus, 0)$ *are alternative generalisations of Hájek-Valdés* S*, both with neutral idempotent* 0 *and absorbing one* U_3*. (note that set of BFs* $\{(a,a,a,a,a,a) \in D_3\}$ *is not closed under* $\oplus$*, thus it does not form a semigroup).*

(iv) Mapping h is homomorphism: $(D_3, \oplus, 0, U_3) \longrightarrow (G_3, \oplus, "-", U_3)$*;* $h(Bel) = Bel \oplus U_3 = Pl\_P(Bel)$ *i.e., the normalised plausibility of singletons.*

See Theorems 2 and 3 in [8] and [9], assertion (iv) already as Theorem 3 in [6]. Unfortunately, a full generalisation either of $-$ or of f was not yet found [8, 9].

3 State of the Art

3.1 Non-conflicting and Conflicting Parts of Belief Functions on Ω_2

Using algebraic properties of group G, of semigroup S (including 'Dempster's subtraction' $(s,s) \oplus (x,x) = (s', s')$, and 'Dempster's half' $(x,x) \oplus (x,x) = (s,s)$, see [6]) and homomorphisms f and h we obtain the following theorem for BFs on Ω_2 (for detail and proofs see [6]):

Theorem 3. *Any BF* (a,b) *on a 2-element frame of discernment* Ω_2 *is Dempster's sum of its unique* non-conflicting part $(a_0, b_0) \in S_1 \cup S_2$ *and of its unique* conflicting part $(s,s) \in S$*, which does not prefer any element of* Ω_2*, that is,* $(a,b) = (a_0, b_0) \oplus (s,s)$*. It holds true that* $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ *and* $(a_0, b_0) = (\frac{a-b}{1-b}, 0) \oplus (s,s)$ *for* $a \ge b$*; and similarly that* $s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}$ *and* $(a_0, b_0) = (0, \frac{b-a}{1-a}) \oplus (s,s)$ *for* $a \le b$*.* (See Theorem 2 in [6].)

3.2 Non-Conflicting Part of BFs on General Finite Frames

We would like to verify that Theorem 3 holds true also for general finite frames:

Hypothesis 1 *We can represent any BF Bel on an n-element frame of discernment* $\Omega_n = \{\omega_1, ..., \omega_n\}$ *as Dempster's sum* $Bel = Bel_0 \oplus Bel_S$ *of non-conflicting BF* Bel_0 *and of indecisive conflicting BF* Bel_S *which has no decisional support, i.e. which does not prefer any element of* Ω_n *to the others, see Figure 4.*

Using algebraic properties of Bayesian BFs and homomorphic properties of h we have a partial generalisation of mapping "$-$" to sets of Bayesian and consonant BFs, thus we have $-h(Bel)$ and $-Bel_0$.

Theorem 4. *(i) For any BF Bel on* Ω_n *there exists unique consonant BF* Bel_0 *such that,* $h(Bel_0 \oplus Bel_S) = h(Bel)$ *for any* Bel_S *such that* $Bel_S \oplus U_n = U_n$*.*

(ii) If for $h(Bel) = (h_1, h_2, ..., h_n, 0, 0, ..., 0)$ *holds true that,* $0 < h_i < 1$*, then unique BF* $-Bel_0$ *and* $-h(Bel)$ *exist, such that,*$(h(-Bel_0 \oplus Bel_s) = -h(Bel)$ *and* $h(Bel_0) \oplus h(-Bel_0) = U_n$ (See Theorem 4 in [6].)

Corollary 1. *(i) For any consonant BF Bel such that* $Pl(\{\omega_i\}) > 0$ *there exists a unique BF* $-Bel$; $-Bel$ *is consonant in this case.*

Let us notice the importance of the consonance here: a stronger statement for general non-conflicting BFs does not hold true on Ω_3, for detail see [6].

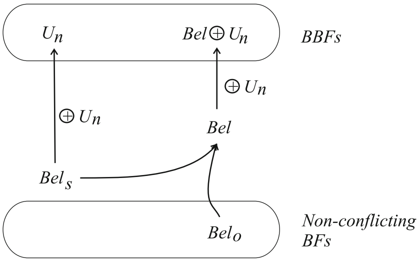

Fig. 4. Schema of Hypothesis 1

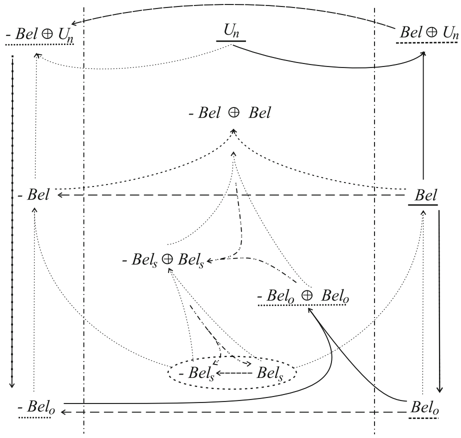

Fig. 5. Detailed schema of a decomposition of BF Bel

Including Theorem 4 into the schema of decomposition we obtain Figure 5. We have still partial results, as we have only underlined BFs; to complete the diagram, we need a definition of $-Bel$ for general BFs on Ω_3 to compute $Bel \oplus -Bel$; we further need an analysis of indecisive BFs to compute $Bel_S \oplus -Bel_S$ and resulting Bel_S and to specify conditions under which a unique Bel_S exists.

4 Towards Conflicting Parts of BFs on Ω_3

4.1 A General Idea

An introduction to the algebra of BFs on a 3-element frame was performed, but a full generalisation of basic homomorphisms of Dempster's semigroup $-$ and f is still missing [6–9]. We need $f(Bel) = -Bel \oplus Bel$ to complete the decomposition diagram (Figure 5) according to the original idea from [6].

Let us forget for a moment a meaning of $'-'$ and its relation to group 'minus' in subgroups G and G_3; and look at its construction $-(a,b) = (b,a)$. It is a simple transposition of m-values of ω_1 and ω_2 in fact. Generally on Ω_3 we have:

Lemma 1. *Any transposition* τ *of a 3-element frame of discernment* Ω_3 *is an automorphism of* D_3. $\tau_{12}(\omega_1, \omega_2, \omega_3) = (\omega_2, \omega_1, \omega_3)$, $\tau_{23}(\omega_1, \omega_2, \omega_3) = (\omega_1, \omega_3, \omega_2)$, $\tau_{13}(\omega_1, \omega_2, \omega_3) = (\omega_3, \omega_2, \omega_1)$.

Theorem 5. *Any permutation π of a 3-element frame of discernment Ω_3 is an automorphism of D_3.*

For proofs of statements in this section see [12] (Lems 2–5 and Thms 6–9).

Considering function '$-$' as transposition (permutation), we have $f(a,b) = (a,b) \oplus (b,a)$ a Dempster's sum of all permutations of Bel given by (a,b) on Ω_2. Analogously we can define

$$f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel),$$

where $\Pi_3 = \{\pi_{123}, \pi_{213}, \pi_{231}, \pi_{132}, \pi_{312}, \pi_{321}\}$, i.e., $f(a,b,c,d,e,f;g) = \bigoplus_{\pi \in \Pi_3} \pi(a,b,c,d,e,f;g) = (a,b,c,d,e,f;g) \oplus (b,a,c,d,f,e;g) \oplus (b,c,a,f,d,e;g) \oplus (a,c,b,e,d,f;g) \oplus (c,a,b,e,f,d;g) \oplus (c,b,a,f,e,d;g)$.

Theorem 6. *Function $f : D_3 \longrightarrow S$, $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel)$ is homomorphism of Dempster's semigroup $\mathbf{D}_3$ to its subsemigroup $S = (\{(a,a,a,b,b,b;1-3a-3b)\}, \oplus)$.*

Having homomorphism f, we can leave a question of existence $-Bel$ such that $h(-Bel) = -h(Bel)$, where '$-$' from group of BBFs G_3 is used on the right hand side. Unfortunately, we have not an isomorphism of S subsemigroup of $\mathbf{D}_3$ to the additive group of reals as in the case of semigroup S of $\mathbf{D}_0$, thus we still have an open question of subtraction there. Let us focus, at first, on the subsemigroup of quasi-Bayesian BFs for simplification.

4.2 Towards Conflicting Parts of Quasi-Bayesian BFs on Ω_3

Let us consider qBBFs $(a,b,c,0,0,0;1–a–b–c) \in \mathbf{D}_{3-0}$ in this section. Following Theorem 6 we obtain the following formulation for qBBFs:

Theorem 7. *Function $f : D_{3\text{–}0} \longrightarrow S$, $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel)$ is homomorphism of Dempster's semigroup $\mathbf{D}_{3-0}$ to its subsemigroup $S_0 = (\{(a,a,a,0,0,0;1–3a)\}, \oplus)$.*

S_0 is isomorphic to the positive cone of the additive group of reals, see Theorem 2, thus there is subtraction which is necessary for completion of diagram from Figure 5. Utilising isomorphism with reals, we have also existence of 'Dempster's sixth'[6] which is needed to obtain preimage of $f(Bel)$ in S_0. Supposing Hypothesis 1, $Bel = Bel_0 \oplus Bel_S$, thus $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel_0) \oplus \bigoplus_{\pi \in \Pi_3} \pi(Bel_S)$; all 6 permutations are equal to identity for any qBBF $Bel_S \in S_0$), thus we have $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel_0) \oplus \bigoplus_{(6\text{-}times)} Bel_S$ in our case):

Lemma 2. *'Dempster's sixth'.*
Having $f(Bel_S)$ in S_0, there is unique $f^{-1}(f(Bel_S)) \in S_0$, such that $\bigoplus_{(6\text{-}times)} f^{-1}(f(Bel_S)) = f(Bel_S)$. If $Bel_S \in S_0$ then $f^{-1}(f(Bel_S)) = Bel_S$.

On the other hand there is a complication considering qBBFs on Ω_3 that their non-conflicting part is a consonant BF frequently out of $\mathbf{D}_{3-0}$. Hence we can simply use the advantage of properties of S_0 only for qBBFs with singleton simple support non-conflicting parts.

[6] Analogously we can show existence of general 'Demspter's k-th' for any natural k and any BF Bel from S_0, but we are interested in 'Dempster's sixth' in our case.

Lemma 3. *Quasi-Bayesian belief functions which have quasi-Bayesian non-conflicting part are just BFs from the following sets* $Q_1 = \{(a, b, b, 0, 0, 0) \mid a \geq b\}$, $Q_2 = \{(b, a, b, 0, 0, 0) \mid a \geq b\}$, $Q_3 = \{(b, b, a, 0, 0, 0) \mid a \geq b\}$. Q_1, Q_2, Q_3 *with oplus are subsemigroups of* $\mathbf{D}_{3-0}$*; their union* $Q = Q_1 \cup Q_2 \cup Q_3$ *is not closed w.r.t.* $\oplus$. $(Q_1, \oplus)$ *is further subsemigroup of* $D_{1-2=3} = (\{(d_1, d_2, d_2, 0, 0, 0)\}, \oplus, 0, U_3)$, *for detail on* $D_{1-2=3}$ *see [8, 9], following this, we can denote* $(Q_i, \oplus)$ *as* $D^{i \geq j=k}_{i-j=k}$.

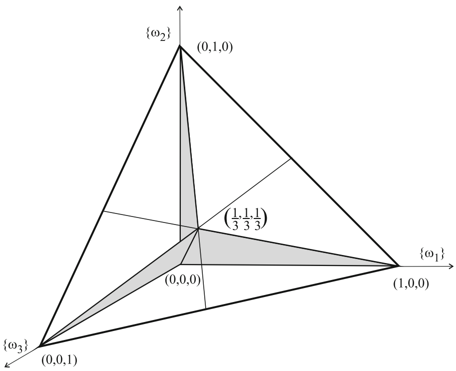

Fig. 6. Quasi-Bayesian BFs with unique decomposition into $Bel_0 \oplus Bel_S$

For quasi-Bayesian BFs out of Q (i.e. BFs from $\mathbf{D}_{3-0} \setminus Q$) we have not yet decomposition into conflicting and non-conflicting part according to Hypothesis 1, as we have not $f(Bel_0) \in S_0$ and have not subtraction in S in general. BFs from $\mathbf{D}_{3-0} \setminus Q$ either have their conflicting part in $S_{Pl} \setminus S_0$ or in $S_{Pl} \setminus S$ or have not conflicting part according to Hypothesis 1 (i.e. their conflicting part is a pseudo belief function out of D_3). Solution of the problem is related to a question of subtraction in subsemigroups S and S_{Pl}, as $f(Bel_0)$ is not in S_0 but in $S \setminus S_0$ for qBBFs out of Q. Thus we have to study these qBBFs together with general BFs from the point of view of their conflicting parts.

Theorem 8. *Belief functions Bel from* $Q = D^{1 \geq 2=3}_{1-2=3} \cup D^{2 \geq 1=3}_{2-1=3} \cup D^{3 \geq 1=2}_{3-1=2}$ *have unique decomposition into their conflicting part* $Bel_S \in S_0$ *and non-conflicting part in* S_1 *(*S_2 *or* S_3 *respectively).*

4.3 Towards Conflicting Parts of General Belief Functions on Ω_3

There is a special class of general BFs with singleton simple support non-conflicting part, i.e. BFs with $f(Bel_0) \in S_0$. Nevertheless due to the generality of Bel, we have $f(Bel) \in S$ in general, thus there is a different special type of belief 'subtraction' ($(a, a, a, b, b, b) \ominus (c, c, c, 0, 0, 0, 0)$).

Following the idea from Figure 5, what do we already have?
We have the entire right part: given Bel, $Bel \oplus U_3$, and non-conflicting part Bel_0 (Theorem 4 (i)); in the left part we have $-Bel \oplus U_3 = -(Bel \oplus U_3)$ using G_3 group '$-$' (Theorem 2 (ii)) and $-Bel_0 = (-Bel \oplus U_3)_0$ (a non-conflicting part of $-Bel \oplus U_3$). In the central part of the figure, we only have U_3 and $-Bel_0 \oplus Bel_0$ in fact. As we have not $-Bel$ we have not $-Bel \oplus Bel$, we use $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel)$ instead of it; $f(Bel) \in S$ in general.

We can compute $-Bel_0 \oplus Bel_0$; is it equal to $f(Bel_0)$? We can quite easily find a counter-example, see [12]. Thus neither $-Bel \oplus Bel$ is equal to $f(Bel) = \bigoplus_{\pi \in \Pi_3} \pi(Bel)$ in general. What is a relation of these two approaches? What is their relation to the decomposition of Bel?

Lemma 4. $-Bel_0 \oplus Bel_0$ *is not equal to* $\bigoplus_{\pi \in \Pi_3} \pi(Bel)$ *in general. Thus there are two different generalisations of homomorphism* f *to* $\mathbf{D}_3$.

Learning this, we can update the diagram of decomposition of a BF Bel into its conflicting and non-conflicting part as it is in Figure 8.

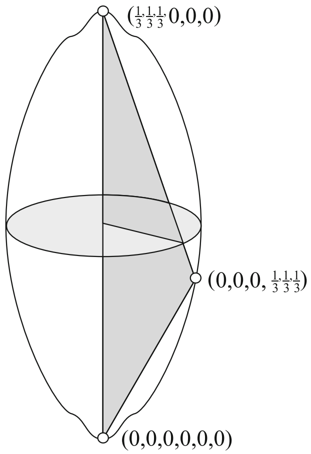

Fig. 7. S_{Pl} — subsemigroup of general indecisive belief functions

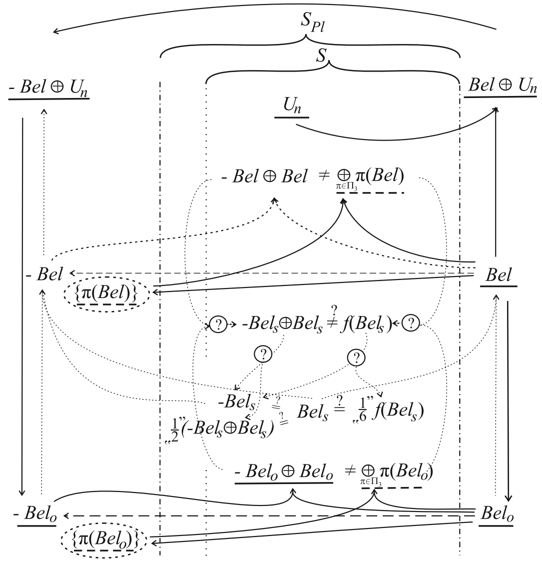

Fig. 8. Updated schema of decomposition of Bel

5 Open Problems for a Future Research

There are three main general open problems from the previous section:

- Elaboration of algebraic analysis, especially of sugbroup S_{Pl} (indecisive BFs).
- What are the properties of two different generalisations of homomorphism f; what is their relationship?
- Principal question of the study: verification of Hypothesis 1; otherwise a specification of sets of BFs which are or are not decomposable ($Bel_0 \oplus Bel_S$).

Open is also a question of relationship to and of possible utilisation of Cuzzolin's geometry [2] and Quaeghebeur-deCooman extreme lower probabilities [19].

6 Summary and Conclusions

New approach to understanding operation '−' and homomorphism f from $\mathbf{D}_0$ (a transposition of elements instead of some operation related to group 'minus' of G, G_3) is introduced in this study.

The first complete generalisation of Hájek-Valdés important homomorphism f is presented. Specification of several classes of BFs (on Ω_3) which are decomposable into $Bel_0 \oplus Bel_S$, and several other partial results were obtained.

The presented results improve general understanding of conflicts of BFs and of the entire nature of BFs. These results can be also used as one one of the mile-stones to further study of conflicts between BFs. Correct understanding of conflicts may consequently improve a combination of conflicting belief functions.

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