7 Introduction to Combinatorial Auctions

7.1 Introduction

In this chapter, we introduce the concept of **combinatorial auction** (CA), or **package auction**. In this model, the seller offers multiple items (usually heterogeneous but related) in a single auction, in which the bidders are allowed to bid for the items or combinations of items that they want. These auctions are particularly suitable when substitutes and complements items are auctioned, because the risk of aggregation or exposure is reduced.

In working with CAs, the winner determination problem (WDP), that is, finding the winning combination of bids, is of particular relevance. Once this allocation problem is solved, the payments to be done by the winners for the acquired items must be established, which will depend on the pricing rule selected by the seller. We will study all of these concepts throughout this chapter.

7.2 Substitutes and Complements Items

When multiple related items are offered, it is important to recognize how winning the first item affects the marginal value of the other items. In Chap. 4, we mentioned that two items may be **substitutes** (when the value of a combination of items is lower than the sum of the individual values) or **complements** (when the value of a combination of items is greater than the sum of the individual values).

To better understand these concepts, let us analyze the following example. In a CA a seller offers an umbrella, a rain cap, and rain boots. A bidder may bid for all of the items or combinations. Table 7.1 shows the values of bidder i for each item and combination. For bidder i, the umbrella and the rain cap are substitutes because the sum of the individual values is greater than the value of both items together: $v_{i,1} + v_{i,2} = 18 > 15 = v_{i,1+2}$. In other words, when the bidder obtains one of the two items, the marginal value of the second item is lower because one

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may be substituted for the other. However, for this bidder, the umbrella and the rain boots are complements; that is, the value of obtaining both items together is greater than the sum of the individual values: $v_{i,1} + v_{i,3} = 16 < 18 = v_{i,1+3}$; there are **synergies** between the items. The rain cap and boots are also complements: $v_{i,2} + v_{i,3} = 14 < 16 = v_{i,2+3}$.

There are multiple markets in which the items auctioned may either be substitutes or complements, such as radio spectrum licenses [7], cable television licenses [33], transportation services [15], construction services [76], among others. For example, a mobile telephone company may be willing to pay 10 million euros for a radio spectrum license for zone A and 15 million euros for the same license in zone B. These licenses would be substitutes if the bidder were not willing to pay an amount greater than or equal to 25 million euros for both. However, both licenses would be complements if, by operating in both zones, synergies were obtained. In this case, the operator would be willing to pay an amount greater than 25 million euros to obtain both licenses.

7.3 The Exposure Problem

If bidders have complex preference structures (substitutes or complements items) the seller must choose an auction design that allows bidders to fully express their preferences through their bidding strategies. With complements, the bidders have to deal with the **exposure problem** (or *aggregation risk*), which means that if they decide to bid aggressively for a package of items but only win some items, they may incur in losses because they do not get the super-additivity value of the complete package.

For example, the items in Table 7.1 are auctioned in a sealed-bid (single-round) simultaneous auction and the first-price rule is established (see Chap. 5). With this mechanism, each item is offered in an independent auction, but all of the auctions are performed at the same time. Therefore, the bidders may bid for the item(s) that they want. A possible strategy for bidder i, based on his values and assuming sincere bidding, is to attempt to win the combination umbrella and boots, for which his package value is equal to $v_{i,1+3} = 18$ euros. With this aim, he could submit a bid for the umbrella of $b_{i,1} = 10$ euros and $b_{i,3} = 8$ euros for the boots. However, in

Table 7.1 Values for substitutes and complements items

Items and packages	j	$v_{i,j}$	
Umbrella	$j = 1$	10	
Cap	$j = 2$	8	
Boots	$j = 3$	6	
Umbrella and cap	$j = 1 + 2$	15	Substitutes
Umbrella and boots	$j = 1 + 3$	18	Complements
Cap and boots	$j = 2 + 3$	16	Complements
Umbrella, cap, and boots	$j = 1 + 2 + 3$	20	

Table 7.2 Bids for substitutes and complements items

Items and packages	j	$b_{i,j}$
Umbrella	$j = 1$	10
Cap	$j = 2$	8
Boots	$j = 3$	6
Umbrella and cap	$j = 1 + 2$	15
Umbrella and boots	$j = 1 + 3$	18
Cap and boots	$j = 2 + 3$	16
Umbrella, cap, and boots	$j = 1 + 2 + 3$	20

this auction model, winning one item does not mean winning the other. If the bidder loses the umbrella and only wins the boots he would have to pay eight euros (two euros more than his value). In this example, he would have incurred losses because he did not get the complete package.

In the presence of complements items, if a CA that allows bidding for complete packages is not used, the only possible strategy to avoid incurring losses would be not to bid above the individual value of each item. In other words, bidder i should at most bid $b_{i,1} = 10$ euros for the umbrella and $b_{i,3} = 6$ euros for the boots. Therefore, although he may not obtain both items, he will never be exposed to a loss.[1]

CAs are the best option to avoid the exposure problem. Continuing with the previous example in which the values of bidder i are summarized in Table 7.1, if the seller had chosen a sealed-bid combinatorial auction, bidder i could have bid individually for each item and combinations of items. Assuming sincere bidding, he would have made the seven bids shown in Table 7.2. With this mechanism, he could have bid up to 18 euros for the umbrella and rain boots combination and only six euros for the rain boots. With CAs, the bidders may fully express their preferences for the different combinations of items without the risk of incurring losses.[2]

7.4 The WDP in Combinatorial Auctions

In a CA, multiple items $J = (1, 2, \ldots, M)$ are offered among various players $I = (1, 2, \ldots, N)$. Each bidder i may submit as many bids as he likes for the items or combinations of these items. The combinations or packages are represented by $S \subseteq J$. Bidder i's value for the combination of items S is represented as $v_i(S)$,

[1]The **chopstick auction** is an example in which complements are auctioned and bidders have to face the exposure problem. The seller simultaneously offers three chopsticks, and the bidder with the highest bid wins two chopsticks. Therefore, the player with the second highest bid will be affected by the exposure problem because he will win one useless chopstick for which he must pay, see [29].

[2]In this chapter, we assume that bidders can only win with one of the bids; that is, the bids are mutually exclusive, XOR bidding language. With this rule, the bidders are assured that they will not face the exposure problem.

which is the maximum value that bidder i would be willing to pay for package S. Bidder i's bid for that package is represented as $b_i(S)$.

Among all bids submitted by all bidders, the seller will determine the winning bids that maximize his revenue, this is the feasible combination of bids that maximizes the sum of accepted bids under the constraint that each item is allocated, at most, to one bidder.[3] This allocation problem is known as the **winner determination problem** (WDP), which has the following mathematical formulation[4]:

$$\max \Sigma_{i \in I} \Sigma_{S \subseteq J} b_i(S) x_i(S), \tag{7.1}$$

subject to:

$$\begin{array}{ll} (1)\ \Sigma_{S \supseteq \{j\}} \Sigma_{i \in I} x_i(S) \leq 1 & \forall j \in J, \\ (2)\ \Sigma_{S \subseteq J} x_i(S) \leq 1 & \forall i \in I, \\ (3)\ x_i(S) \in \{0, 1\} & \forall S \subseteq J, \forall i \in I. \end{array}$$

Solving this problem implies determining, among all of the bids ($b_i(S)$), the combination that maximizes the seller's revenue. According to this formulation, $x_i(S)$ is a binary variable, which is equal to one when a bidder wins an item or combination of items and equal to zero when he does not win any items, restriction (3). Restriction (1) ensures that each item is awarded to, at most, one bidder; that is, that a feasible allocation of items is made. Finally, restriction (2) limits the solution of the problem such that each bidder obtains, at most, one winning bid, meaning that the bids are mutually exclusive, **XOR bidding language**.[5]

The following example shows how to solve the WDP in a combinatorial auction in which the following items are offered: A, B, and C. Each bidder may submit up to seven bids, one for each item and combinations of items: A, B, C, AB, AC, BC, and ABC.[6] Assuming that there are three bidders in this auction, Table 7.3 presents the bids made by each of them $b_i(S)$ in a sealed-bid (single-round) CA in which the first-price rule is established.

With the submitted bids, there are many possible ways to allocate the items. However, solving the WDP requires identifying, among all of the possible solutions, the one that maximizes the accepted bids. One possible combination would be to award the AC items to the third bidder and the B item to the second one. With this

[3]In the final allocation the seller may not sell all items.

[4]There are different ways of mathematically expressing this problem; in this manual, we use the formulation presented by Day and Raghavan [24].

[5]The bidding languages that are most common in CAs are the **OR bidding language** and XOR bidding. With OR bidding, each bidder may win multiple bids. However, with XOR bidding, each bidder may win, at most, one bid. The problem with the OR bidding language is that when there are complements items, the bidders are affected by exposure problem, which does not occur when XOR bidding is used. In this book, all of the CAs will be explained using the XOR bidding language.

[6]The bidders are not obliged to bid for all the items or combinations.

Table 7.3 The winner determination problem

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	**200**	150	200
B	100	**200**	100
C	100	100	**200**
AB	200	200	200
AC	250	300	275
ABC	300	400	300
BC	150	150	150

allocation and the first-price rule, the seller's revenue is equal to $R = b_3(AC) + b_2(B) = 275 + 200 = 475$ euros. Although this combination is feasible (the same item is not awarded to different bidders) and meets the condition of XOR bids (each bidder wins at most one bid), is it really the combination that maximizes the seller's revenue?

This combination is not an efficient allocation with respect to the bids that have been received because it does not maximize the sum of the accepted bids. In this example, the combination of winning bids that solves the WDP is $b_1^*(A)$, $b_2^*(B)$, $b_3^*(C)$. The revenue that the seller obtains with this combination is the maximum and is equal to $R^* = 600$ euros.

Another combination for which the seller obtains the same revenue is $R^* = b_3(A) + b_2(B) + b_3(C) = 600$ euros. However, this combination does not satisfy restriction (2). In other words, the bids are not XOR because the third bidder has won two different bids. Therefore, this combination would not be a possibility with the established bidding language.[7]

In a CA, as the number of bidders and items increases, the possible allocations grow exponentially, which means that solving the WDP may be complicated and may require significant computation time. According to Sandholm [70], it is an *NP*-complete problem, which often requires the use of advanced optimization techniques to be solved.[8]

7.5 Payments to Be Done in a Combinatorial Auction

In a CA the bidders submit their bids and then, the seller solves the WDP by obtaining the winning bids, $b_i^*(S)$, thus working out the allocation problem. The next step consists of determining the payments that the winning bidders will have to make, which will depend on the **pricing rule**. Next, we present the two basic rules: first-price and VCG mechanism.

[7] If the seller were to opt for an OR bidding language, then this allocation could be another solution to the WDP.

[8] Several studies related to solving the WDP have been presented by Sandholm and Suri [71], Sandholm et al. [72], Saez et al. [68], among others.

Table 7.4 First-price combinatorial auction

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	**20**	10		10
B		**20**	10	10
C	10		**20**	10
AB				28

7.5.1 First-Price Combinatorial Auction

If the seller sets the **first-price**, each bidder i will pay an amount equal to his bid for the item or combination that he has acquired:

$$P_i^{*1st} = b_i^*(S), \tag{7.2}$$

and the seller's revenue is equal to the sum of all of the payments made by the winning bidders:

$$R^{*1st} = \Sigma_{i \in W} P_i^{*1st}, \tag{7.3}$$

in which W is the set of winning bidders.

Table 7.4 shows the bids made by four bidders in a first-price sealed-bid CA.[9] The final allocation after solving the WDP is $b_1^*(A), b_2^*(B), b_3^*(C)$. The first bidder wins item A, the second item B, and the third item C. With the first-price rule, each bidder's payment is equal to his winning bid: $P_1^{*1st} = 20$ euros for the first bidder, $P_2^{*1st} = 20$ euros for the second bidder, and $P_3^{*1st} = 20$ euros for the third bidder. The seller's revenue is therefore equal to $R^{*1st} = 60$ euros.

The main drawback with this pricing rule is that the bidders tend to **underbid**, that is, bid below their values ($b_i(S) < v_i(S)$), in order to obtain a positive surplus. Hence this strategy yields inefficient allocation of the items because the bidder with the highest value is not always the winning bidder with the highest bid. To alleviate this problem, the seller may apply other pricing rules.

7.5.2 The Vickrey–Clarke–Groves Mechanism

An alternative way to solve the problem derived from the use of the first-price rule is to opt for the generalized Vickrey auction: the **Vickrey–Clarke–Groves mechanism** (**VCG mechanism**).[10] By implementing this mechanism, each winning bidder $i \in W$ pays an amount equal to the opportunity cost of the obtained item.

[9] With three items, there is a total of seven possible combinations. However, to simplify, in this example we have considered only four combinations and that all bidders do not bid for all of them.

[10] See Vickrey [79], Clarke [17], and Groves [35].

This amount depends only on the bids placed by the bidder's rivals and is calculated using the following formula:

$$P_i^{*\mathrm{VCG}} = \alpha_i - \Sigma_{k\neq i} b_k^*(S), \tag{7.4}$$

in which $\alpha_i = \max\{\Sigma_{k\neq i} b_i(S) | \Sigma_{k\neq i} S_k \leq J\}$ is the result of solving the WDP among all of the bids, ignoring those made by bidder i. The second term on the right side of Eq. (7.4) is equal to the sum of the initial winning bids ($b_k^*(S)$) made by all of the bidders except i. Once the payments to be made by each winning bidder have been calculated, the seller's revenue is equal to the sum of these payments:

$$R^{*\mathrm{VCG}} = \Sigma_{i\in W} P_i^{*\mathrm{VCG}}. \tag{7.5}$$

Using the data from the example included in Table 7.4, we will calculate the amount that each winning bidder would have to pay based on the VCG mechanism. Let us remember that the allocation of items is always the same, regardless of the pricing rule that has been chosen. In this case, the combination of winning bids is as follows: $b_1^*(A)$, $b_2^*(B)$, $b_3^*(C)$.

To compute the price that the first bidder has to pay, we must first calculate the value of α_1. In other words, we must omit the bids made by the first bidder and again calculate the WDP. Table 7.5 shows the winning combination if we do not take into account the bids of the first bidder: $b_4(A)$, $b_2(B)$, $b_3(C)$. Therefore, $\alpha_1 = 10 + 20 + 20 = 50$ euros. The second term on the right side of Eq. (7.4), calculated for the first bidder, is equal to $\Sigma_{k\neq i} b_i^*(S) = b_2^*(B) + b_3^*(C) = 20 + 20 = 40$ euros. Hence, the amount that the first player has to pay with this mechanism is equal to $P_1^{*\mathrm{VCG}} = 50 - 40 = 10$ euros.

We follow the same steps to calculate the final payments that the other winning bidders have to make. The second bidder's payment is equal to $P_2^{*\mathrm{VCG}} = (20 + 10 + 20) - (20 + 20) = 10$ euros, and the third bidder's payment is equal to $P_3^{*\mathrm{VCG}} = (20 + 20 + 10) - (20 + 20) = 10$ euros. The revenue that the seller obtains upon applying this pricing rule is equal to $R^{*\mathrm{VCG}} = 10 + 10 + 10 = 30$ euros.

There is also another way to compute VCG prices. The VCG price for bidder i is equal to the sum of the difference between the losing and winning bids per bidder for all bidders except him, $k \neq i$. The losing bid is the bid that would have become winning if bidder i would have not participated in the auction. Table 7.6 shows the

Table 7.5 VCG price for the first bidder

S	~~$b_1(S)$~~	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	~~20~~	10		**10**
B		**20**	10	10
C	~~10~~		**20**	10
AB				28

Table 7.6 VCG price for the first bidder

Bidder	Losing bid	Winning bid	Losing–winning
b_2	20	20	0
b_3	20	20	0
b_4	10	0	10
			10

winning and losing bids for the second, third, and fourth bidder and computes the VCG price for the first bidder.

As we mentioned in Chap. 2, this is an **incentive compatible** mechanism; that is, the dominant strategy for all of the bidders is to bid according to their true values ($b_i(S) = v_i(S)$), which yields an efficient allocation of the items (see **efficient auction**, Chap. 3).

Despite the advantages of the VCG mechanism, this pricing rule may also have significant drawbacks, such as: low revenues for the seller, non-monotonicity of the seller's revenues in the set of bidders and bids,[11] vulnerability to **collusion** among the bidders, and vulnerability to **shill bids**.[12] Because of these drawbacks, the VCG mechanism is not frequently used in real auctions, see the work done by Ausubel and Milgrom [10].[13]

Implications of the VCG mechanism

The main feature of the VCG mechanism is that the winners' payments depend on the bids submitted by their rivals. To understand the implications of this pricing rule, consider the following example. In a CA items A, B, B, C, C are offered and the first bidder submits a single bid of 100 euros for the combination (A, B, C). The following scenarios can happen depending on his rival's bids, see Table 7.7.

1. **Without rivals:** In this scenario, bidder one is the only bidder, so he wins the combination (A, B, C) for zero euros.
2. **A rival with a matching bid:** The bid of the second bidder is compatible with the first bidder's bid, i.e., both bidders can win the items they have requested. In this scenario the combination that solves the WDP is $b_1^*(ABC) + b_2^*(BC) = 180$. Both bidders get their packages and pay nothing: $P_1^{*VCG} = 80 - 80 = 0$ euros and $P_2^{*VCG} = 100 - 100 = 0$ euros.

[11] An auction has **bidder monotonicity** if, upon including another bidder, the bidders' surplus always decreases (weakly) and the seller's revenue increases (weakly).

[12] Multiple bidding identities by a single bidder.

[13] These weaknesses do not surface in environments in which all of the items are **substitutes** for all of the bidders. However, when this condition is violated, even for a single bidder, these problems can occur. The fact that the bidders have budget restrictions may also affect the auction result when applying the VCG mechanism.

Table 7.7 Different scenarios

Scenario	Bidders	A	BB	CC	Bid	P_i^{*VCG}
1	b_1^*	A	B	C	100	0
2	b_1^*	A	B	C	100	0
	b_2^*		B	C	80	0
3	b_1^*	A	B	C	100	80
	b_2	A	B		80	
4.a	b_1	A	B	C	100	
	b_2^*	A	B		90	30
	b_3^*		B	C,C	70	10
4.b	b_1^*	A	B	C	100	90
	b_2	A	B		50	
	b_3		B	C,C	40	

3. **A rival with a non matching bid:** In the third scenario bids are not compatible, as there is only one A item and both bidders have bid for it. $b_2(AB)$ turns to be the *losing bid* and $b_1^*(ABC) = 100$ the winning bid. The first bidder gets his package for $P_1^{*VCG} = 80 - 0 = 80$ euros (the value of the losing bid).
4. **Two rivals with matching bids among them but non matching bids respect to the first bidder:** The winning combination is that which maximizes the value of the bids.
   - **Scenario 4.a:** $b_2^*(AB) + b_3^*(BCC) = 160$ is the winning combination of bids. The second bidder gets the package AB and pays $P_2^{*VCG} = 100 - 70 = 30$ euros. The third bidder wins the combination BCC for $P_3^{*VCG} = 100 - 90 = 10$ euros. The first bidder does not win his package.
   - **Scenario 4.b:** $b_1^*(ABC) = 100$ is the winning bid, and the first bidder pays $P_1^{*VCG} = 90 - 0 = 90$ euros.

7.6 Core-Selecting Package Auctions

One of the problems that may emerge when using the VCG mechanism is that the seller's revenue may be very low (or even zero). Let us analyze the example presented by Ausubel and Milgrom [10], as shown in Table 7.8. With these bids, the winning combination is $b_2^*(A)$, $b_3^*(B)$.[14] With this allocation, the second bidder's payment is equal to $P_2^{*VCG} = 2000 - 2000 = 0$ euros and the third bidder's payment is equal to $P_3^{*VCG} = 2000 - 2000 = 0$ euros. This example illustrates that the use of this mechanism may generate an unacceptable outcome because the seller's revenue is zero although the bidders had positive values for the items offered.

The question that emerges at this point is, how low do the seller's revenues have to be for the outcome to be considered unacceptable? In auction literature, a solution

[14] In this example, the winning combination could be either $b_2^*(A)$, $b_3^*(B)$ or $b_3^*(A)$, $b_2^*(B)$, but the result would be the same.

Table 7.8 Combinatorial auction (example 1)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	0	**2000**	2000
B	0	2000	**2000**
AB	2000	0	0

Table 7.9 Combinatorial auction (example 2)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	**20**			10
B		**25**		10
C			**30**	10
AB	20			
AC				
ABC	25	25	25	50
BC				

is considered to be acceptable if the payments are the result of a **core allocation** with respect to the bids that have been received. Given several bids, an auction generates a **core outcome** if and only if there is no group of bidders that would strictly prefer an alternative outcome that would also be strictly preferred by the seller. This group of bidders would form a **blocking coalition** against the original outcome.

To understand this concept, let us examine the following example. Table 7.9 shows the bids that have been received and the winning bidders after solving the WDP.

The efficient allocation that maximizes the value with respect to the submitted bids, that is, the combination of bids after solving the WDP, is as follows: $b_1^*(A)$, $b_2^*(B)$, and $b_3^*(C)$. Under the VCG mechanism, the bidders' payment for the acquired items is equal to $P_1^{*VCG} = (25 + 30 + 10) - (25 + 30) = 10$ euros for the first bidder, $P_2^{*VCG} = (20 + 30 + 10) - (20 + 30) = 10$ euros for the second bidder, and $P_3^{*VCG} = (20 + 25 + 10) - (20 + 25) = 10$ euros for the third bidder. In other words, the seller awards all of the items and obtains a revenue equal to $R^{*VCG} = 30$ euros.

However, the result obtained cannot be considered a core outcome because there is a bidder who strictly prefers another outcome, which would also benefit the seller. The fourth bidder is willing to pay 50 euros for the combination ABC, such that he would form a coalition that blocks the outcome obtained with the VCG mechanism. Therefore, the initial result is considered socially unacceptable because there is another solution that would be better for one or more bidders in addition to the seller.

As an alternative to the VCG mechanism in situations in which complementary items are offered, Day and Milgrom [23] proposed the **core-selecting package auction**. In core-selecting package auctions, first the feasible allocation of items that maximizes the revenue of the seller is calculated. In other words, the WDP is solved. Then a pricing rule that ensures that a result is achieved in the core with respect to the bids that have been received is implemented. The final payments are obtained by increasing the VCG prices in a way that a core outcome is obtained.[15] Several authors highlight core-selecting package auctions, such as Day and Raghavan [24], Day and Cramton [22], Erdil and Klemperer [30], and Ausubel and Baranov [5], among others.

We now continue with the example in Table 7.9, but instead of using the VCG mechanism, the seller opts for the core-selecting auction proposed by Day and Raghavan [24]. The combination of winning bids will still be $b_1^*(A)$, $b_2^*(B)$ and $b_3^*(C)$, but now each winning bidder i would have to increase his payments with respect to $P_i^{*\text{VCG}}$ to obtain a core outcome. With this core-selecting auction, the final payments of the bidders are as follows: $P_1^{*\text{CORE}} = P_2^{*\text{CORE}} = P_3^{*\text{CORE}} = 16.67$ euros. As can be observed, with these amounts, the seller obtains a revenue equal to $R^{*\text{CORE}} = 50$ euros, meaning that the fourth bidder stops blocking the outcome because there is no other combination in which both the bidders and the seller are better off.

Given a set of bids, the seller's revenue with a core-selecting CA is at an intermediate point between the revenue generated by a first-price and a VCG mechanism. In other words, the following holds:

$$R^{*\text{VCG}} \le R^{*\text{CORE}} \le R^{*\text{1st}}.$$

With the bids included in Table 7.4, the seller's revenue with the three aforementioned price mechanisms is as follows:

$$R^{*\text{VCG}} = 30 \le R^{*\text{CORE}} = 50 \le R^{*\text{1st}} = 60.$$

It is mentioning that this relationship between revenues only holds when the comparison is made for bids already submitted. However, when the seller sets a price mechanism, this decision significantly affects the bidding strategy of the bidders, and the bids made with each mechanism cease to be equal, meaning that, before the auction, it cannot be assured that a mechanism will generate more or less revenue to the seller.

7.7 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Items (homogeneous or heterogeneous).
- S: Package or combination of item, $S \subseteq J$.
- $v_i(S)$: Value of bidder i for the combination S in a CA.

[15]Given the complexity of the core-selecting package auction, the calculation of the final payments made by the winning bidders is beyond the scope of this book.

- $b_i(S)$: Bidder i's bid for the combination S in a CA.
- x_i: A binary variable that is equal to one when a bidder wins an item or combination of items and zero when he does not win any items.
- $b_i^*(S)$: Bidder i's winning bid for the combination S in a CA.
- P_i^{*1st}: Bidder i's payment for the items won under the first-price rule.
- P_i^{*VCG}: Bidder i's payment for the items won under the VCG mechanism.
- P_i^{*CORE}: Bidder i's payment for the items won under a core-selecting package auction.
- R^*: Seller's revenue.
- R^{*1st}: Seller's revenue with the first-price rule.
- R^{*VCG}: Seller's revenue with the VCG mechanism.
- R^{*CORE}: Seller's revenue with a core-selecting package auction.
- α_i: The result of solving the WDP among all of the bids, ignoring those submitted by bidder i.

7.8 Exercises

1. In a sealed-bid, first-price simultaneous auction, a seller offers a plane ticket to Paris, a train ticket to Paris, and lodging in a hotel in Paris. Bidder i's values for each item and combination are shown in Table 7.10.
   (a) Which items are substitutes?
   (b) Which items are complements?
   (c) Provide an example of a bidding strategy in which the bidder could be affected by the exposure problem upon bidding for complements if they are offered in simultaneous auctions.
   (d) Point out the maximum bid that the bidder may submit for complements items so as not to be affected by the exposure problem in a simultaneous auction.
   (e) If the seller opts for a CA, point out the maximum bid that the bidder may submit for the combinations of complements items and still not incur losses.
2. In a CA, a seller receives the offers that appear in Table 7.11. Indicate the following:

Table 7.10 Values for substitutes and complements items

Items and packages	j	$v_{i,j}$
Plane ticket	$j = 1$	100
Train ticket	$j = 2$	80
Lodging	$j = 3$	150
Plane and train ticket	$j = 1 + 2$	120
Plane ticket and lodging	$j = 1 + 3$	300
Train ticket and lodging	$j = 2 + 3$	250
Plane ticket, train ticket, and lodging	$j = 1 + 2 + 3$	320

Table 7.11 Combinatorial auction (exercise 2)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	10	4	1
B	3	9	3
C	2	3	9
AB	18	15	11
AC	18	12	16
ABC	20	20	25
BC	10	19	17

Table 7.12 Combinatorial auction (exercise 3)

S	$b_1(S)$	$b_2(S)$
A	30	10
B	20	15
AB	50	20

Table 7.13 Combinatorial auction (exercise 4)

S	$b_1(S)$	$b_2(S)$
A	30	10
B	20	15
AB	40	20

(a) A feasible allocation of items that does not maximize the total value of the accepted bids.
(b) A feasible allocation of items that does not satisfy the XOR bidding language.
(c) The efficient allocation of items that satisfies the WDP with XOR bidding.

3. Table 7.12 shows the bids received in a CA of two items with two bidders. Calculate the following:
   (a) All of the possible allocations with the XOR bidding language.
   (b) The combination of bids that solves the WDP.
   (c) The payments that the winning bidders must make and the revenue that the seller obtains if the first-price rule is established.
   (d) The payments that the winning bidders must make and the revenue that the seller obtains if the VCG mechanism is employed.
4. Table 7.13 shows the bids received in a CA of two items with two bidders. Calculate the following:
   (a) All of the possible allocations with the XOR bidding language.
   (b) The combination of bids that solves the WDP.
   (c) The payments that the winning bidders must make and the revenue that the seller obtains if the first-price rule is established.
   (d) The payments that the winning bidders must make and the revenue that the seller obtains if the VCG mechanism is employed.
5. Given the bids shown in Table 7.14, solve the following:
   (a) The combination of feasible bids that maximizes the seller's revenue.
   (b) Are the payments of the winning bidders with VCG mechanism a core outcome? If not, identify a coalition that will block this result.

Table 7.14 Combinatorial auction (exercise 5)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	100	50	50
B	100	50	300
C	100	50	50
AB	600	100	100
AC	600	500	100
ABC	1000	300	300
BC	600	100	100

Table 7.15 Combinatorial auction (exercise 6)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	200	150	200	100
B	100	250	100	100
C	100	100	300	100
AB	200	200	200	200
AC	250	300	275	200
ABC	500	400	400	500
BC	150	150	150	200

6. Given the bids shown in Table 7.15 compute the following:
   (a) The combination of feasible bids that maximizes the seller's revenue.
   (b) Are the payments of the winning bidders with VCG mechanism a core outcome? If not, identify a coalition that will block this result.

7.9 Solutions to Exercises

1. With the data from the exercise, we obtain the following results.
   (a) Plane and train tickets are substitutes: $v_{i,1} + v_{i,2} = 180 > 120 = v_{i,1+2}$.
   (b) Plane tickets and the lodging are complements: $v_{i,1} + v_{i,3} = 250 < 300 = v_{i,1+3}$, as are the train tickets and the lodging: $v_{i,2} + v_{i,3} = 230 < 250 = v_{i,2+3}$.
   (c) If the bidder decides to bid for the plane ticket and lodging combination, he could be affected by the exposure problem in a simultaneous auction if $b_{i,1} > 100$ euros or $b_{i,3} > 150$ euros, because he may not win the two items for which he is bidding and may only obtain one of them. Similarly, if he opts for the train tickets and lodging combination, he may incur losses if $b_{i,2} > 80$ euros or $b_{i,3} > 150$ euros.
   (d) The bidder will not incur losses if his bids in a simultaneous auction are as follows: $b_{i,1} \leq 100$ euros, $b_{i,2} \leq 80$ euros, and $b_{i,3} \leq 150$ euros. However, this strategy implies not including the synergy value in his bids.
   (e) If, instead of a simultaneous auction, the seller were to opt for a CA, bidder i could include the value of the synergies without the fear of incurring losses.

Table 7.16 The winner determination problem (exercise 3)

S	$b_1(S)$	$b_2(S)$
A	30	10
B	20	15
AB	**50**	20

Table 7.17 VCG mechanism for the first bidder (exercise 3)

S	~~$b_1(S)$~~	$b_2(S)$
A	~~30~~	10
B	~~20~~	15
AB	~~50~~	**20**

In this case, he may bid as much as $b_{i,1+3} \leq 300$ euros and $b_{i,2+3} \leq 250$ euros for each package.

2. With the bids submitted by the bidders:
   (a) There are several feasible combinations, that is, combinations in which an item is not awarded to more than one bidder. Some of these combinations include the following: $b_1(AB) + b_3(C)$ or $b_3(ABC)$, in which the total values of the accepted bids are 27 euros and 25 euros, respectively. However, neither of these combinations solves the WDP because there is another feasible combination in which the value of the accepted bids is maximized.
   (b) The combination $b_1(A) + b_1(BC)$ would also be feasible but would not be valid as a solution as defined in this chapter because the bids are not XOR. With this allocation, the first bidder would win two bids.
   (c) The feasible combination of bids that maximizes the seller's revenue with XOR bids and that therefore implies an efficient allocation of items with respect to the received bids is $b_1^*(A) + b_2^*(BC)$, for which the total value of the accepted bids is maximized and is equal to 29 euros.
3. With the bids received in this auction, we obtain the following results.
   (a) The feasible allocations using XOR bidding language are:
   $b_1(A) + b_2(B) = 45$ euros.
   $b_2(A) + b_1(B) = 30$ euros.
   $b_1(AB) = 50$ euros.
   $b_2(AB) = 20$ euros.
   (b) The efficient allocation after solving the WDP, that is, the combination in which the value of the accepted bids is maximized, is $b_1^*(AB) = 50$ euros, see Table 7.16. Therefore, the first bidder wins both items.
   (c) If the seller establishes the first-price rule, the winning bidder will pay $P_1^{*1st} = 50$ euros, which will coincide with the seller's revenue: $R^{*1st} = 50$ euros (there is only one winning bidder).
   (d) The payment of the winning bidder under the VCG mechanism is calculated with Eq. (7.4). The value of α_1 is equal to the winning combination after eliminating the bids of the first bidder. As can be observed in Table 7.17, the winning allocation would be $b_2(AB) = 20$ euros, so $\alpha_1 = 20$ euros. The second term of the right side of Eq. (7.4) is equal to 0 because there are no

Table 7.18 The winner determination problem (exercise 4)

S	$b_1(S)$	$b_2(S)$
A	**30**	10
B	20	**15**
AB	40	20

Table 7.19 VCG mechanism for the first bidder (exercise 4)

S	~~$b_1(S)$~~	$b_2(S)$
A	~~30~~	10
B	~~20~~	15
AB	~~40~~	**20**

Table 7.20 VCG mechanism for the second bidder (exercise 4)

S	$b_1(S)$	~~$b_2(S)$~~
A	30	~~10~~
B	20	~~15~~
AB	**40**	~~20~~

other winning bidders. Therefore, the payment of the winning bidder with the VCG mechanism is equal to $P_1^{*VCG} = R^{*VCG} = 20$ euros. This is also the seller's revenue.

4. With the bids submitted in this auction, we obtain the following results.
   (a) The feasible allocations using XOR bidding language are:
   $b_1(A) + b_2(B) = 45$ euros.
   $b_2(A) + b_1(B) = 30$ euros.
   $b_1(AB) = 40$ euros.
   $b_2(AB) = 20$ euros.
   (b) The efficient allocation of items that solves the WDP, in which the value of the accepted bids is maximized, is $b_1^*(A) + b_2^*(B) = 45$ euros, as shown in Table 7.18. The first bidder wins item A, and the second bidder wins item B.
   (c) If the seller establishes the first-price rule, the payments of the winning bidders are as follows: $P_1^{*1st} = 30$ euros the first bidder and $P_2^{*1st} = 15$ euros the second bidder. The seller's revenue is the sum of both amounts: $R^{*1st} = 45$ euros.
   (d) Under the VCG mechanism, the first bidder's payment according to Eq. (7.4) is equal to $P_1^{*VCG} = \alpha_1 - b_2^*(B)$. As shown in Table 7.19, the winning combination after eliminating the bids of the first bidder is $\alpha_1 = b_2(AB) = 20$ euros. Therefore, the payment of the first bidder for the item that he acquires is equal to $P_1^{*VCG} = 20 - 15 = 5$ euros.
   In the same way, we calculate the second bidder's payment for the item that he acquires: $P_2^{*VCG} = \alpha_2 - b_1^*(A)$. Table 7.20 shows the winning combination after omitting the bids of the second bidder: $\alpha_2 = b_1(AB) = 40$ euros. Therefore, $P_2^{*VCG} = 40 - 30 = 10$ euros.
   The seller's income with this mechanism is equal to $R^{*VCG} = P_1^{*VCG} + P_2^{*VCG} = 15$ euros.

Table 7.21 The winner determination problem (exercise 5)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	100	50	50
B	100	50	300
C	100	50	50
AB	600	100	100
AC	600	500	100
ABC	**1000**	300	300
BC	600	100	100

Table 7.22 The winner determination problem (exercise 6)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	**200**	150	200	100
B	100	**250**	100	100
C	100	100	**300**	100
AB	200	200	200	200
AC	250	300	275	200
ABC	500	400	400	500
BC	150	150	150	200

5. With the bids received in this auction, we obtain the following outcome.
   (a) The efficient allocation of items that solves the WDP implies that the first bidder wins all three items $b_1^*(ABC)$, see Table 7.21.
   (b) If the VCG pricing rule is established, the payment that the winning bidder will have do is equal to $P_1^{*VCG} = (500 + 300) - 0 = 800$ euros, so $R^{*VCG} = 800$ euros. In this example, this is a core outcome, there is no blocking coalition.
6. With the bids submitted in this auction, we obtain the following outcome.
   (a) The efficient allocation of items after solving the WDP is $b_1^*(A)$, $b_2^*(B)$, and $b_3^*(C)$; in other words, the first bidder wins item A, the second wins item B, and the third wins item C, see Table 7.22.
   (b) If the VCG mechanism is established, the payment of each winning bidder is equal to $P_1^{*VCG} = (250 + 300 + 100) - (250 + 300) = 100$ euros for the first, $P_2^{*VCG} = (200+300+100)-(200+300) = 100$ euros for the second, and $P_3^{*VCG} = (200 + 250 + 100) - (200 + 250) = 100$ euros for the third. The seller obtains a revenue equal to $R^{*VCG} = 300$ euros. However, this outcome is not a core outcome because there is a blocking coalition: there is a bidder who would strictly prefer an alternative result that would also be strictly preferred by the seller. The fourth bidder is willing to pay 500 euros for the three items ($b_4(ABC) = 500$ euros), an outcome that the seller would also prefer.