Assigning Multiple Homogeneous Items in a Single Auction

4

4.1 Introduction

In previous chapters, the main single-unit auction models were discussed. However, in many instances auctions are used to award multiple related units. For example, if an olive oil factory wants to sell part of its stock, it can choose to conduct an auction. The items to be auctioned may be homogeneous (oil bottles of the same size and acidity) or heterogeneous (different sizes and acidities) and may be awarded in a single auction or in different auctions. This chapter presents situations in which multiple identical items (homogeneous) are available in the same auction. These auctions can be dynamic, sealed-bid (single-round), or hybrid processes. We also assume that the items to be auctioned are **substitutes**,[1] so the marginal value of the second item for bidder i cannot exceed the marginal value of the first item: $v_{i,1} \geq v_{i,2}$, where $v_{i,1}$ is bidder i's value for the first item.

4.2 Multi-unit Dynamic Auctions of Homogeneous Items

In a multi-unit auction the seller will have to choose between a dynamic or a sealed-bid process. If the first option is chosen, the main models are the ascending auction, the descending auction, and the Ausubel auction.

[1]In multi-unit auctions, items can be substitutes or **complements**. If the items are substitutes, the value of winning the second unit is less than the value of winning the first. Items are complements (**synergies**) if the value of winning the second unit is higher than that of winning the first, if the first item has already been purchased. In Chap. 6, we will explain these concepts in more detail.

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_4

4.2.1 Multi-unit Ascending Auction of Homogeneous Items

In a **multi-unit ascending auction**, also called **multi-unit English auction** a seller offers M identical items ($J = (1, 2, \ldots, M)$) to N bidders ($I = (1, 2, \ldots, N)$). The seller sets a minimum bid (p^s) and each bidder i indicates which quantity q_i^0 is willing to buy at that price (round $t = 0$). At the end of each round, the seller calculates the aggregate demand ($Q^t = \Sigma_{i=1}^N q_i^t$). If it is greater than the supply ($Q^t > M$), the seller raises the price in the next round, and the bidders submit new bids (q_i^{t+1}). As the price increases, the bidders' demand decreases, and the auction ends when the number of items demanded does not exceed the supply ($Q^T \leq M$), where T is the last round of the auction. All winners pay the price of the last round per acquired item ($p^* = p^T$). Bidder i's payment is:

$$P_i^* = p^* q_i^*, \tag{4.1}$$

where q_i^* is the number of items won by bidder i. The seller's revenue is equal to the payment made by all winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*, \tag{4.2}$$

where W is the set of winning bidders. The values of bidder i are represented by the vector $V_i = (v_{i,1}, v_{i,2}, \ldots v_{i,M})$, where $v_{i,j}$ is bidder i's marginal value for item j.[2] Bidder i's income for having won q_i^* items is equal to the sum of the private values of the items won:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}. \tag{4.3}$$

The surplus of bidder i is equal to the difference between the income and the total payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.4}$$

To better understand this model, consider an auction of seven identical items ($M = 7$) with an opening bid per item of $p^s = 2$ euros and a bid increment of one euro. The first bidder's valuation vector is $V_1 = (12, 10, 8, 6, 4, 0, 0)$, the second's vector is $V_2 = (6, 5, 4, 3, 2, 0, 0)$, and the third's vector is $V_3 = (6, 4, 3, 2, 1, 0, 0)$. We assume that all bidders bid on items whose prices are below their marginal valuation ($p^t < v_{i,j}$). In round $t = 0$ bidders submit the following demands: $q_1^0 = 5$ items, $q_2^0 = 4$ items, and $q_3^0 = 3$ items. The aggregate demand in this round is $\Sigma_{i=1}^N q_i^0 = Q^0 = 12$ items, which exceeds supply ($M = 7$); thus, the seller raises the price. In the following round ($t = 1$), the price rises to $p^1 = 3$ euros, and the bidders place the following bids: $q_1^1 = 5$ items, $q_2^1 = 3$ items, and $q_3^1 = 2$ items.

[2]In this chapter, we assume substitute items: decreasing marginal values, $v_{i,1} \geq v_{i,2} \geq \cdots \geq v_{i,M}$.

Table 4.1 Rounds in a multi-unit ascending auction of homogeneous items

Round (t)	Price (p^t)	q_1^t	q_2^t	q_3^t	Q^t
$t = 0$	$p^0 = 2$	5	4	3	12
$t = 1$	$p^1 = 3$	5	3	2	10
$t = 2$	$p^2 = 4$	4	2	1	7
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$	
	$j = 1$	12	6	6	
	$j = 2$	10	5	4	
	$j = 3$	8	4	3	
	$j = 4$	6	3	2	
	$j = 5$	4	2	1	
	$j = 6$	0	0	0	
	$j = 7$	0	0	0	

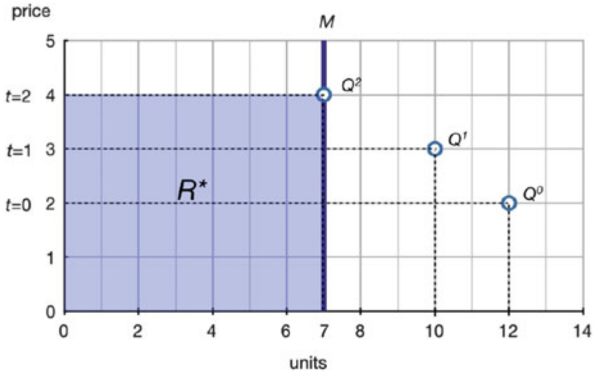


Fig. 4.1 Multi-unit ascending auction of homogeneous items

As total demand continues to exceed supply ($Q^1 = 10 > 7 = M$), the seller raises the price again. In round $t = 2$, the price is $p^2 = 4$ euros, and the bidders place the following bids: $q_1^2 = 4$ items, $q_2^2 = 2$ items, and $q_3^2 = 1$ item. In this round, demand equals supply ($M = Q^2 = 7$ items), and the auction ends. The auction is summarized in Table 4.1.

The auction yields the following outcome. The first bidder wins $q_1^* = 4$ items and pays $P_1^* = 4{\times}4 = 16$ euros, the second bidder wins $q_2^* = 2$ items for $P_2^* = 2{\times}4 = 8$ euros, and the third bidder obtains $q_3^* = 1$ item at a price of $P_3^* = 1 \times 4 = 4$ euros. The price per item is the same for all bidders, $p^* = 4$ euros. Each bidder's income is equal to the values for the items acquired: $\Gamma_1^* = \Sigma_{j=1}^{4} v_{1,j} = 36$ euros, $\Gamma_2^* = \Sigma_{j=1}^{2} v_{2,j} = 11$ euros, and $\Gamma_3^* = v_{3,1} = 6$ euros; thus, the surpluses obtained are $\Pi_1^* = \Gamma_1^* - P_1^* = 36 - 16 = 20$ euros, $\Pi_2^* = \Gamma_2^* - P_2^* = 11 - 8 = 3$ euros, and $\Pi_3^* = \Gamma_3^* - P_3^* = 6 - 4 = 2$ euros. Finally, the seller gets a revenue equal to the sum of the payments from all bidders, $R^* = P_1^* + P_2^* + P_3^* = 16 + 8 + 4 = 28$ euros.

Figure 4.1 shows the aggregate demand in each round which is greater than supply ($Q^t > M$) in rounds $t = 0$ and $t = 1$, so the price increases. However, demand equals supply in round $t = 2$, and the auction ends. The final price is $p^* = 4$ euros per item. The seller obtains a revenue equal to the shaded area $R^* = 7 \times 4 = 28$ euros.

It may happen that as the price increases, the aggregate demand decreases below the quantity supplied ($Q^T < M$). In order to avoid having unsold items, the seller can set a **rationing rule** to allocate the items. The appendix included at the end of the chapter explains one of the most common rationing rules: the proportional rationing rule.

4.2.2 Multi-unit Descending Auction of Homogeneous Items

In a **multi-unit descending auction**, also known as a **multi-unit Dutch auction**, a seller offers M identical items ($J = (1, 2, \ldots, M)$) to N bidders ($I = (1, 2, \ldots, N)$). The auction begins with a relatively high starting price per unit (p^s) set by the seller. This price decreases successively in each round, and when a bidder is interested in purchasing an item at the price in round t (p^t), a bid is placed. At this point, the bidder automatically wins the item. The total number of items assigned in round t to the bidders who have bid is equal to:

$$Q^{t*} = \Sigma_{i=1}^{N} q_i^{t*}, \tag{4.5}$$

where q_i^{t*} is the number of items won by bidder i in round t. The total number of items won from round 0 to round l by all bidders is equal to $\Sigma_{t=0}^{l} Q^{t*}$. The seller continues decreasing the price per item while the items assigned until round l is less than the total supply $\Sigma_{t=0}^{l} Q^{t*} < M$. The auction ends at round T when no items remain $\Sigma_{t=0}^{T} Q^{t*} = M$.[3] Bidder i's acquired items at the end of the auction is equal to the items acquired in all rounds:

$$q_i^* = \Sigma_{t=0}^{T} q_i^{t*}. \tag{4.6}$$

In contrast to the multi-unit ascending auction, each bidder pays a different amount for the items obtained in this model. For each item, they pay the standing price of the round in which each item is won. The final payment made by bidder i is equal to the sum of the items acquired in each round multiplied by the prices in the corresponding rounds (p^t):

$$P_i^* = \Sigma_{t=0}^{T} p^t q_i^{t*}. \tag{4.7}$$

[3]The aggregate demand in round T could exceed the remaining supply, $\Sigma_{t=0}^{T} Q^{t*} > M$. To solve this tie, the seller can set different rules to allocate the items among the bidders that have submitted offers in this last round. There are many options such as: allocating the items to those who bid first or in a random way.

The income for bidder i is equal to the sum of the values of the items won at the end of the auction:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.8}$$

and the surplus is the difference between the income and the total payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.9}$$

Finally, the seller's revenue is equal to the sum of the payments made by all winning bidders $i \in W$:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.10}$$

The following example describes a multi-unit descending auction of six identical items in which there are three bidders. The valuation vectors of the three bidders are $V_1 = (250, 200, 150, 100, 50, 25)$ for the first bidder, $V_2 = (200, 150, 125, 100, 75, 50)$ for the second, and $V_3 = (150, 140, 130, 120, 110, 100)$ for the third. To simplify we assume that bidders bid on any item for which the price equals the marginal value: sincere bidding ($p^t = v_{i,j}$).[4]

The starting price per item is $p^s = 300$ euros, a price at which nobody bids. The bid decrement is 50 euros, so the price in the following round is $p^1 = 250$ euros. The first bidder bids for one item ($q_1^{1*} = 1$) at 250 euros. In the next round ($p^2 = 200$ euros), the first and the second bidder bid on an item, $q_1^{2*} = 1$, and $q_2^{2*} = 1$, which they acquire at the price set in this round. The total number of items assigned until round $t = 2$ is less than the supply ($\Sigma_{t=0}^{2} Q^{2*} = 3 < 6 = M$), and thus, the price continues to decrease. The auction concludes in the third round when the price reaches $p^3 = 150$ euros and all three bidders bid on an item: $q_1^{3*} = 1$, $q_2^{3*} = 1$, and $q_3^{3*} = 1$. Table 4.2 summarizes the auction.

The first bidder wins 3 items ($q_1^* = 3$), i.e., one in round $t = 1$, another one in round $t = 2$, and a third in round $t = 3$; and his total payment is $P_1^* = (1 \times 250) + (1 \times 200) + (1 \times 150) = 600$ euros. The second bidder wins two items ($q_2^* = 2$): one in round $t = 2$ and another in $t = 3$. For these items he pays $P_2^* = (1 \times 200) + (1 \times 150) = 350$ euros. Finally, the third bidder only wins one item ($q_3^* = 1$) and pays $P_3^* = (1 \times 150) = 150$ euros. Each bidder's income is equal to the sum of the values of the items acquired: $\Gamma_1^* = \Sigma_{j=1}^{3} v_{1,j} = 250 + 200 + 150 = 600$ euros for the first bidder, $\Gamma_2^* = \Sigma_{j=1}^{2} v_{2,j} = 200 + 150 = 350$ euros for the second bidder, and $\Gamma_3^* = v_{3,1} = 150$ euros for the third bidder. In this example, because all bidders placed sincere bids, none of them have a positive surplus: $\Pi_1^* = \Gamma_1^* - P_1^* = 0$

[4] In multi-unit descending auction bidders will not bid sincerely because this strategy implies zero surplus. Bidders tend to wait until the price is less than the valuation to obtain a positive surplus: underbidding ($p^t < v_{i,j}$).

Table 4.2 Rounds in a Multi-unit descending auction of homogeneous items

Round (t)	Round price (p^t)	q_1^{t*}	q_2^{t*}	q_3^{t*}	$\Sigma_{t=0}^{l} Q^{t*}$
$t = 0$	$p^0 = 300$	0	0	0	0
$t = 1$	$p^1 = 250$	1	0	0	1
$t = 2$	$p^2 = 200$	1	1	0	3
$t = 3$	$p^3 = 150$	1	1	1	6
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$	
	$j = 1$	250	200	150	
	$j = 2$	200	150	140	
	$j = 3$	150	125	130	
	$j = 4$	100	100	120	
	$j = 5$	50	75	110	
	$j = 6$	25	50	100	

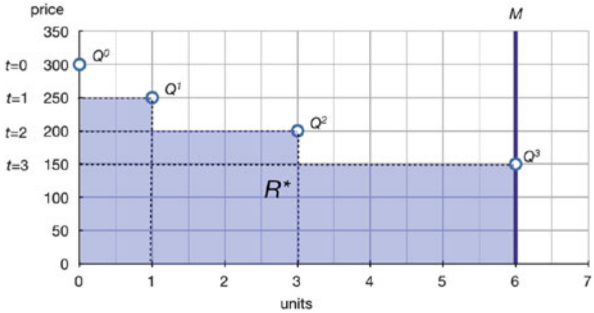


Fig. 4.2 Multi-unit descending auction of homogeneous items

euros, $\Pi_2^* = \Gamma_2^* - P_2^* = 0$ euros, and $\Pi_3^* = \Gamma_3^* - P_3^* = 0$ euros. Finally, the revenue obtained by the seller after allocating the six items is equal to the amount paid by all of the winning bidders: $R^* = P_1^* + P_2^* + P_3^* = 1100$ euros.

Figure 4.2 illustrates the aggregate demand in each round. At the starting price there are no bids. In round $t = 1$, a bidder demands and wins one item at that round's price $p^1 = 250$ euros. In round $t = 2$, bidders demand two items at a price of $p^2 = 200$ euros. The last three items are allocated in round $t = 3$ at a price of $p^3 = 150$ euros. In this auction model, each bidder pays a different price for the items won. The seller's revenue is equal to the shaded area $R^* = (1 \times 250) + (2 \times 200) + (3 \times 150) = 1100$ euros.

4.2.3 The Ausubel Auction

The **Ausubel auction**, presented by Ausubel [4], is a particular mechanism of multi-unit ascending auctions. The seller sets a starting bid per item (p^s) in round $t = 0$. In subsequent rounds, the price increases, and the bidders indicate the number of items that they are willing to buy (q_i^t). With the bids submitted in round t, the seller calculates the aggregate demand $Q^t = \Sigma_{i=1}^{N} q_i^t$ and increases the price if the demand is greater than the supply ($Q^t > M$). Bidders indicate demand at the updated price of the next round, which can never be greater than their demand in the previous rounds.[5] The auction ends in round T when the aggregate demand of all bidders does not exceed the supply ($Q^T \leq M$).[6]

The difference between this auction and the multi-unit ascending auction lies in the price that bidder i must pay for items acquired at the end of the auction q_i^*. The Ausubel auction states that each bidder has to pay the price for each item in the round in which the item was obtained (*clinched*). A bidder obtains an item in round t when the aggregate demand of the other bidder's $k \neq i$ is less than the supply ($\Sigma_{k \neq i} q_k^t < M$) but the total demand of all bidders is greater than the supply ($Q^t > M$).[7] At this point, it holds that the residual supply of bidder i is greater than zero ($M_i^t > 0$). Bidder i's residual supply is calculated as the total supply minus the demand of the rivals:

$$M_i^t = M - \Sigma_{k \neq i} q_k^t. \tag{4.11}$$

As bidders cannot increase their bids in future rounds, if bidder i's residual supply is greater than zero ($M_i^t > 0$), this implies that he has already guaranteed to win an item (clinched). The number of items that bidder i has clinched until round t (C_i^t) is equal to the maximum value between zero and the residual supply of bidder i in that round, calculated as follows:

$$C_i^t = \max\{0, M_i^t\}. \tag{4.12}$$

The number of clinched items in round t is equal to:

$$c_i^t = C_i^t - C_i^{t-1}, \tag{4.13}$$

Therefore, we can also calculate the total number of items clinched by bidder i until round t as the sum of the items assured in each round:

$$C_i^t = \Sigma_{t=0}^T c_i^t. \tag{4.14}$$

When the auction ends, the number of items awarded to bidder i is equal to $q_i^* = C_i^T$. For each item allocated to bidder i, he will pay the corresponding price

[5] According to the rules of the auction, when the price increases, bidders are required to maintain or decrease their bid, but not increase it. Furthermore, bidders who decide to stop bidding in a round cannot bid again in later rounds.

[6] It could happen that at a certain increase of the price, the aggregate demand decreases below the supply ($Q^T < M$). To avoid unsold items, the seller can set a rationing rule to allocate all items, see Appendix at the end of this chapter.

[7] This procedure sequentially implements the **Vickrey rule**, which establishes that each bidder pays an amount equal to the opportunity cost of the item won.

Table 4.3 Rounds in an Ausubel auction

Round (t)	Round price (p^t)	q_1^t	q_2^t	q_3^t	Q^t
$t = 0$	$p^0 = 100$	3	4	4	11
$t = 1$	$p^1 = 150$	2	3	4	9
$t = 2$	$p^2 = 200$	1	2	3	6
$t = 3$	$p^3 = 250$	1	2	2	5
$t = 4$	$p^4 = 300$	1	1	2	4
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$	
	$j = 1$	350	350	400	
	$j = 2$	200	300	350	
	$j = 3$	150	200	250	
	$j = 4$	50	150	200	

to the round in which he clinched the item. Bidder i's payment for the acquired items is equal to:

$$P_i^* = \Sigma_{t=0}^T p^t c_i^t. \tag{4.15}$$

The income for bidder i is equal to the sum of the values of the acquired items:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.16}$$

and the surplus is difference between the income and the payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.17}$$

The seller's revenue is the sum of the payments made by all winning bidders, $i \in W$:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.18}$$

To depict the step-by-step operation of this auction, let us review the following example. A seller holds an Ausubel auction to award four identical items ($M = 4$) to three bidders with the following valuation vectors: $V_1 = (350, 200, 150, 50)$ for the first bidder, $V_2 = (350, 300, 200, 150)$ for the second bidder, and $V_3 = (400, 350, 250, 200)$ for the third bidder. The seller sets $p^s = 100$ euros and a bid increment of 50 euros. If we assume that the bidders bid while the price is below their marginal value, the auction develops as shown in Table 4.3. To determine how many items are allocated to each bidder, the Ausubel auction works like a multi-unit ascending auction. In this example, the auction ends in round $t = 4$. The first and second bidder win one item $q_1^* = q_2^* = 1$, and the third obtains two items $q_3^* = 2$.

The difference between a multi-unit ascending auction and an Ausubel auction is how to calculate the final payments. Table 4.4 shows the residual supply of

Table 4.4 Residual supply and clinched units

Round (t)	M_1^t	M_2^t	M_3^t	C_1^t	C_2^t	C_3^t
$t = 0$	−4	−3	−3	0	0	0
$t = 1$	−3	−2	−1	0	0	0
$t = 2$	−1	0	1	0	0	1
$t = 3$	0	1	1	0	1	1
$t = 4$	1	1	2	1	1	2

each bidder (M_i^t) and the accumulated clinched items per round (C_i^t), which is the maximum of 0 and the residual supply.

At the starting price, the aggregate demanded is greater than the supply ($Q^0 = 11 > 4 = M$). In this round, the residual supply of all bidders is less than zero ($M_1^0 = -4$, $M_2^0 = -3$, and $M_3^0 = -3$), so no one clinched an item ($C_1^0 = C_2^0 = C_3^0 = 0$). In the next round, the seller increases the price per item, and the bidders place new bids (equal to or lower than the previous round). Still, no bidder clinches an item. In round $t = 2$, the total demand is greater than the supply ($Q^2 = 6 > 4 = M$), but the demand of all bidders except the third is less than the total supply ($\Sigma_{k \neq 3} q_k^2 = 3 < 4 = M$). In this round, the residual supply of the third bidder is greater than zero, i.e., $M_3^2 = 1$. The third bidder thus clinches the first unit ($C_3^2 = 1$) at the price of the current round ($p^2 = 200$ euros). In the next round, the total demand is again greater than the total supply, and the second bidder clinches one item. In this round, the demand of the second bidder's rivals is lower than the total supply ($\Sigma_{k \neq 2} q_k^3 = 3 < 4 = M$), and his residual supply is $M_2^3 = 1$; therefore, the second bidder clinches one item $C_2^3 = 1$. For this item, the second bidder pays $p^3 = 250$ euros. The auction ends in round $t = 4$, in which the total demand equals the supply ($Q^4 = 4 = M$). In this round, the first bidder wins his first item ($M_1^4 = 1$, thus clinching one item $C_1^4 = 1$) and pays $p^4 = 300$ euros. The third bidder wins his second item in this round ($M_3^4 = 2$) and has accumulated two items $C_3^4 = 2$ (the first item clinched in round $t = 2$). For this second item, the third bidder pays $p^4 = 300$ euros.

The final outcome of the auction is the following. The first bidder wins one item ($q_1^* = 1$) and pays $P_1^* = 1 \times 300 = 300$ euros. The second bidder also wins one item ($q_2^* = 1$) and pays $P_2^* = 1 \times 250 = 250$ euros. Finally, the third bidder wins two items ($q_3^* = 2$) and pays $P_3^* = (1 \times 200) + (1 \times 300) = 500$ euros. The surplus obtained by the bidders are equal to $\Pi_1^* = \Gamma_1^* - P_1^* = 350 - 300 = 50$ euros for the first bidder, $\Pi_2^* = \Gamma_2^* - P_2^* = 350 - 250 = 100$ euros for the second bidder, and $\Pi_3^* = \Gamma_3^* - P_3^* = (400 + 350) - 500 = 250$ euros for the third bidder. The seller's revenue is equal to $R^* = P_1^* + P_2^* + P_3^* = 300 + 250 + 500 = 1050$ euros. Figure 4.3 illustrates the aggregate demand per round. As it shows, the auction ends in round $t = 4$ when the total demand is equal to the supply $Q^4 = M$.

In this auction model, a graph can be used to calculate the amount paid by each bidder for the acquired items. Let us focus on the third bidder as a reference point. Figure 4.4 shows his bids in each round (q_3^t) and the number of accumulated clinched items per round (C_3^t). The bidder clinches an item in round t when the

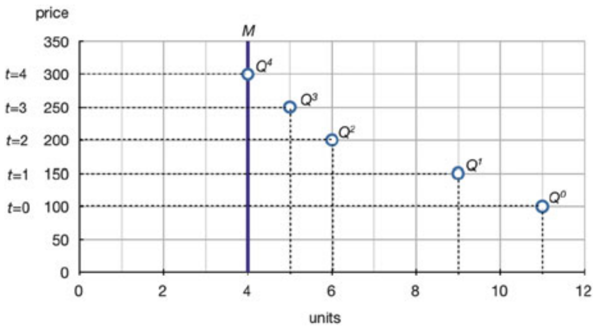


Fig. 4.3 Ausubel auction of homogeneous items

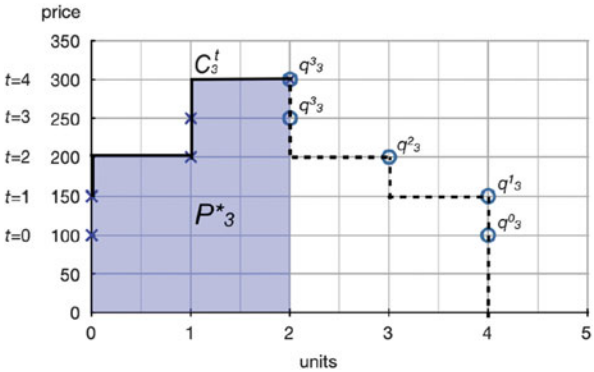


Fig. 4.4 Demand and clinched items per round for bidder three in an Ausubel auction

residual supply is greater than zero and must pay the price at that round. In this example, his final payment is equal to the shaded area ($P_3^* = (1\times200)+(1\times300) =$ 500 euros), i.e., the area below the clinched items (C_3^t) until the intersection with his demand (q_3^t). To calculate the seller's revenue, this analysis would have to be performed for all winning bidders and add the shaded areas.

The main advantage of this auction model is that with private values and diminishing marginal values, **sincere bidding** by every bidder constitutes a perfect equilibrium (*ex post*), which implies an efficient allocation of the items (**efficient auction**).[8]

4.3 Multi-unit Sealed-Bid Auction of Homogeneous Items

Instead of using dynamic auctions, the seller may choose to conduct a sealed-bid (single-round) auction. In a **multi-unit sealed-bid auction**, the seller offers M homogeneous items and bidders are asked to submit M bids. Each bidder submits a bid vector indicating the price that he is willing to pay for each unit.

[8]See demonstration in [4].

$B_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,M})$ is the vector of bidder i, where $b_{i,j}$ indicates how much bidder i is willing to pay for item j.[9] We can consider a bid vector as an inverse demand function which can thus be inverted to obtain bidder i's demand function (d_i). Given a price p, bidder i demands j units for which the bids placed are equal to or higher than the price $p \leq b_{i,j}$. We obtain the demand function via the following equation:

$$d_i(p) \equiv \max\{j : p \leq b_{i,j}\}. \tag{4.19}$$

For example, in an auction of three identical items, bidder i places the following bids: $B_i = (b_{i,1}, b_{i,2}, b_{i,3}) = (6, 4, 2)$. To obtain his demand function we have to find the number of items he will be interested in at each price. If the unit price were $p = 6$ euros, the bidder would demand one item. If the unit price were $p = 4$ euros, the bidder would demand two items. Finally, if the price were $p = 2$ euros, the bidder would be interested in acquiring three items, as all bids are greater than or equal to two euros.

In this auction model, the seller receives a total of $N \times M$ bids because each of the N bidders places M bids, one for each item. The seller lists the bids in decreasing order such that M items are awarded to the M highest bids. The winning bidders are those who have made the M highest bids.[10]

In an auction with two bidders and three items, the following bids are submitted: $B_1 = (6, 4, 2)$ and $B_2 = (5, 3, 1)$, Table 4.5. The seller lists all bids in descending order $(6, 5, 4, 3, 2, 1)$, and the aggregate demand is obtained. Figure 4.5 shows the aggregate demand of the bidders and its intersection with the fixed supply: $M = 3$. The winning bids, which correspond to the M highest bids, are those that fall to the left of the supply line. In this example, the winning bids are $(b^*_{1,1}, b^*_{2,1}, b^*_{1,2}) = (6, 5, 4)$. The first bidder wins two items and the second one wins one. The losing bids are to the right of the supply.[11]

Table 4.5 Bids in a multi-unit sealed-bid auction of homogeneous items

	$b_{1,j}$	$b_{2,j}$
$j = 1$	6	5
$j = 2$	4	3
$j = 3$	2	1

[9]We have assumed that the marginal values are decreasing (substitute items). Therefore, the bid vector must satisfy the following condition: $b_{i,1} \geq b_{i,2} \geq \cdots \geq b_{i,M}$.

[10]To calculate the winners, the aggregate demand function must first be obtained by horizontally adding the N individual demand functions of the bidders. The supply will then determine the winner bidders.

[11]If there is a tie among bidders, the seller can set a tie-breaking rule such as: by order of submission and randomly.

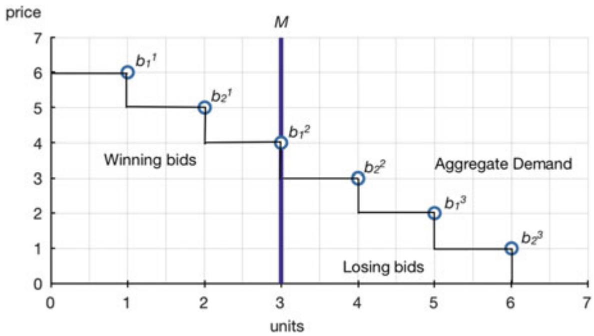


Fig. 4.5 Multi-unit sealed-bid auction of homogeneous items

With this mechanism, the seller solves the allocation problem. The price paid by each winner will depend on the pricing rule selected. The main options are the discriminatory auction, uniform-price auction, and Vickrey auction.[12]

4.3.1 Discriminatory Auction

In a **discriminatory auction**, also called **pay-your-bid auction**, the winning bidders pay an amount equal to the sum of the bids placed on the items acquired. If bidder i has q_i^* winning bids (among the M highest bids) and obtains q_i^* items, his total payment is equal to:

$$P_i^* = \Sigma_{j=1}^{q_i^*} b_{i,j}^*, \tag{4.20}$$

where $b_{i,j}^*$ represents the winning bid submitted by bidder i for item j. The surplus of bidder i is equal to the difference between the income and the cost:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.21}$$

Bidder i's income is equal to the sum of the values of the items acquired:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}. \tag{4.22}$$

Finally, the seller's revenue is equal to the sum of the payments done by all of the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.23}$$

[12]The specific auction chosen affects the bidding strategy of the bidders, which makes it difficult to determine the model that generates the greatest revenue for the seller.

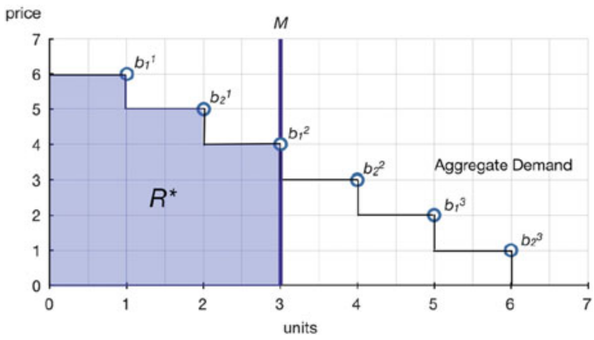


Fig. 4.6 Discriminatory auction

Continuing with the example of the previous section, whose bids are contained in Table 4.5, the amount that the first bidder pays for the winning items is $P_1^* = b_{1,1}^* + b_{1,2}^* = 6 + 4 = 10$ euros. The amount that the second bidder pays for the winning item is $P_2^* = b_{2,1}^* = 5$ euros. The total revenue that the seller obtains is $R^* = P_1^* + P_2^* = 15$ euros. As shown in Fig. 4.6, the seller's revenue is equal to the shaded area below the winning bids: $R^* = (1 \times 6) + (1 \times 5) + (1 \times 4) = 15$ euros.

The discriminatory auction is an extension of the **first-price sealed-bid auction** when there is a supply of multiple identical items.

4.3.2 Uniform-Price Auction

In an **uniform-price auction**, all bidders pay the same price for the acquired items (p^*), the market-clearing price at which demand equals supply. Demand equals supply at any value between the *highest-rejected-bid* (*HRB*) and the *lowest-accepted-bid* (*LAB*).[13] In this book, we will use the HRB to fix the clearing price.

If bidder i wins q_i^* items, his payment is equal to:

$$P_i^* = p^* \times q_i^*. \tag{4.24}$$

The income of bidder i is equal to the sum of the values of the q_i^* items obtained:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.25}$$

and his surplus is equal to the difference between income and payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.26}$$

[13]Cramton and Sujarittanonta [19] analyzed the effect of choosing the HRB or the LAB in a multi-unit ascending auction.

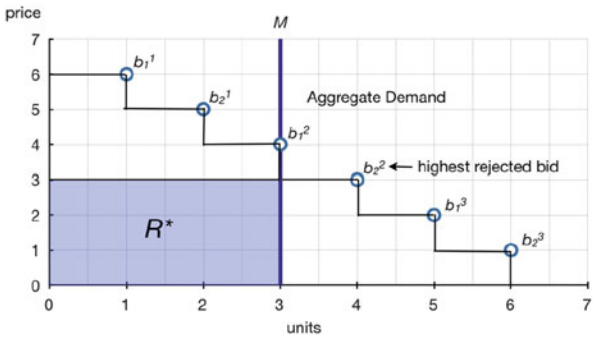


Fig. 4.7 Uniform-price auction

The seller's revenue is equal to the payments made by the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*, \tag{4.27}$$

where W is the group of winning bidders.

Continuing with the data of the previous exercise contained in Table 4.5, the bid made by the second bidder for the second item is the HRB, $b_{2,2} = 3$ euros, the market-clearing price at which the demand equals supply ($p^* = 3$ euros). Thus, the first bidder who wins two items pays $P_1^* = 3 \times 2 = 6$ euros. The second pays $P_2^* = 3 \times 1 = 3$ euros. The seller's revenue is the sum of both payments: $R^* = P_1^* + P_2^* = 9$ euros, represented by the shaded area in Fig. 4.7 ($R^* = 3 \times 3 = 9$ euros).[14]

4.3.3 Multi-unit Vickrey Auction of Homogeneous Items

A **Vickrey auction** of M identical items is a sealed-bid auction (single-round) in which the winning bidders pay the opportunity cost for the items obtained. If bidder i wins q_i^* units, he will pay the q_i^* highest rejected bids of the other bidders (the q_i^* highest bids not including his own).[15]

To compute this payment, we first find the *vector of competing bids* facing bidder i, $C_{-i} = (c_{-i,1}, c_{-i,2}, \ldots, c_{-i,M})$ which is the M-vector of the highest bids submitted by his rivals. We denote $c_{-i,1}$ as the highest bid of the others, $c_{-i,2}$ is the second highest, and so on. This vector is obtained by rearranging the $(N-1) \times M$

[14]If the price were calculated according to the LAB, it would be equal to $p = 4$ euros ($b_{1,2} = 4$); therefore, the first bidder would have paid $P_1^* = 4 \times 2 = 8$ euros for the two items and the second bidder $P_2^* = 4 \times 1 = 4$ euros. In this case, the seller's revenue would have been $R^* = 4 \times 3 = 12$ euros.

[15]The Vickrey auction is a **VCG mechanism**, in which each winning bidder pays an amount equal to the opportunity cost of the acquired item.

bids of all bidders except i in decreasing order and selecting the M highest bids. These would be the M winning bids if bidder i would have not placed a bid.

Then we have to determine whether the highest bid of bidder i exceeds the lowest bid included in C_{-i}, i.e., if $b_{i,1} > c_{-i,M}$. In the event that this condition is met, bidder i wins an item and pays an amount equal to $c_{-i,M}$, which is the highest rejected bid submitted by the others. For bidder i to win the second item, his second highest bid must exceed the second lowest competing bid ($b_{i,2} > c_{-i,M-1}$). If so, bidder i pays $c_{-i,M-1}$ for the second item. To calculate the number of items that bidder i wins, the process is continued until the jth highest bid placed by bidder i is lower than the jth lowest competing bid.

According to the Vickrey rule, bidder i pays the q_i^* highest rejected bids placed by the rivals for the q_i^* items obtained. Bidder i's total payment is calculated as follows:

$$P_i^* = \Sigma_{j=1}^{q_i^*} c_{-i,M-q_i^*+j}. \tag{4.28}$$

The income of bidder i is equal to the sum of the values of the q_i^* items won:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.29}$$

and the surplus is the difference between the income and the cost:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.30}$$

The seller obtains a revenue equal to the sum of payments done by the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.31}$$

Consider the following example in which there are three bidders and three items to be auctioned. Suppose that each bidder submits the following bids: $B_1 = (10, 8, 6)$, $B_2 = (12, 7, 5)$, and $B_3 = (15, 13, 9)$. Table 4.6 depicts bids done by all bidders in decreasing order.

Table 4.7 shows the vector of competing bids facing each bidder and the auction outcome. To determine if the first bidder wins an item, first we obtain his vector

Table 4.6 Bids in decreasing order

Bids	15	13	12	10	9	8	7	6	5
Bidders	3	3	2	1	3	1	2	1	2

Table 4.7 Multi-unit Vickrey auction of homogeneous items

Bidders	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$	$c_{-i,1}$	$c_{-i,2}$	$c_{-i,3}$	q_i^*	P_i^*
$i = 1$	10	8	6	15	13	12	0	0
$i = 2$	12	7	5	15	13	10	1	10
$i = 3$	15	13	9	12	10	8	2	18

of competing bids, the highest three bids placed by the others: $C_{-1} = (15, 13, 12)$. Next, we check if the highest bid made by the first bidder is greater than the lowest bid of the rivals, that is, if $b_{1,1} > c_{-1,3}$ holds. This condition does not hold because $b_{1,1} = 10 < 12 = c_{-1,3}$, so the first bidder does not win an item ($q_1^* = 0$).

We perform the same process for the second bidder, whose vector of competing bids is equal to $C_{-2} = (15, 13, 10)$. This bidder wins an item because his highest bid is greater than the lowest bid submitted by his rivals, $b_{2,1} = 12 > 10 = c_{-2,3}$. For this first item, the bidder pays an amount equal to the highest rejected bid made by the others: 10 euros. This bidder earns no more items, as $b_{2,2} = 7 < 13 = c_{-2,2}$. Finally, the vector of competing bids for the third bidder is $C_{-3} = (12, 10, 8)$. This bidder wins two items because his two highest bids are greater than the two lowest bids included in his vector of competing bid: $b_{3,1} = 15 > 8 = c_{-3,3}$, and $b_{3,2} = 13 > 10 = c_{-3,2}$. This bidder pays 8 euros for the first item and 10 euros for the second one. The final outcome is summarized as follows. The first bidder does not win any items, the second bidder wins one item and pays $P_2^* = 10$ euros, and the third bidder acquires two items for $P_3^* = 8 + 10 = 18$ euros. The seller's revenue is $R^* = 10 + 18 = 28$ euros.

The **second-price sealed-bid auction** is a particular case of the Vickrey auction in which only one item ($M = 1$) is auctioned.

4.4 Bidding Strategies and Equivalences of Multi-unit Auction Formats

When a seller offers M homogeneous items, the decision of the auction model is not trivial. Bidders will have different strategies depending on the type of auction chosen, thereby affecting the final result. In a multi-unit descending auction and in a discriminatory auction, in which winning bidders pay an amount equal to the bids placed, the bidders will always tend to bid below their private valuations to obtain a surplus. In contrast, in a multi-unit ascending auction, bidder i will not stop bidding on item j as long as the unit price is below the valuation ($v_{i,j}$). Finally, in an Ausubel auction and in a Vickrey auction, the bidders have incentives to submit sincere bids because the final price paid depends only on the bids made by the rivals.

In the specific case in which the bidders have **private values**, the following equivalences hold: the descending auction is equivalent to the discriminatory auction, the ascending auction is equivalent to the uniform price auction, and the Ausubel auction is equivalent to the Vickrey auction. To further explore the behavior of bidders in these models see [41].

4.5 Multi-unit Anglo-Dutch Auction of Homogeneous Items

The multi-unit **Anglo-Dutch auction**, developed by Klemperer [38], is a hybrid auction in two phases. The first phase is a multi-unit ascending auction and the second one is a sealed-bid auction. This two-stage auction was used to allocate 3G spectrum licenses in Europe.

Phase I is an ascending auction in which M items are available. The seller sets $p^{\mathrm{I,s}}$ as the starting price per unit, and the bidders indicate which quantities they are willing to buy, $q_i^{\mathrm{I},0}$. At the end of each round, the seller obtains the aggregate demand $Q^{\mathrm{I},t} = \Sigma_{i=1}^{N} q_i^{\mathrm{I},t}$ and incrementally increases the price whenever the supply is greater than the demand. This phase ends when the demand is greater than the supply by only one unit, $Q^{\mathrm{I},t} = M + 1$.[16] Bidders who were active in the last round of the ascending phase have the option to bid in Phase II, which is a one round, sealed-bid auction in which the bidders indicate the price that they are willing to pay for the items they bid on in the previous phase ($B_i^{\mathrm{II}} = (b_{i,1}^{\mathrm{II}}, b_{i,2}^{\mathrm{II}}, \ldots, b_{i,j}^{\mathrm{II}})$). The bids of this second phase cannot be less than the price per item established in the last round of the previous phase; thus, the opening bid of Phase II ($p^{\mathrm{II,s}}$) is equal to the closing price of Phase I. The bidders with the M highest bids are the winners.

Consider the following example in which three items are auctioned to six bidders and each bidder is only allowed to win one. The opening bid per item of Phase I is $p^{\mathrm{I,s}} = 5$ euros, a price at which all bidders demand one unit. In the following round, the price is $p^{\mathrm{I},1} = 10$ euros, and the first bidder stops bidding. In round $t = 2$, the price increases to $p^{\mathrm{I},2} = 15$ euros, and the second bidder stops bidding. In this round, the Phase I ends because the demand is greater than the supply by one unit ($M + 1 = 4 = Q^{\mathrm{I},2}$). Only bidders three, four, five, and six, who were active in the final round of the ascending phase, participate in Phase II. These bidders submit their final sealed-bid in one round. The minimum bid in Phase II is 15 euros per item, which is the closing price of the previous phase. The active bidders submit the following bids: $b_3^{\mathrm{II}} = 50$ euros, $b_4^{\mathrm{II}} = 15$ euros, $b_5^{\mathrm{II}} = 30$ euros, and $b_6^{\mathrm{II}} = 40$ euros. The winners are the third, fifth, and sixth bidder, who each acquires one item. Table 4.8 summarizes the two phases of the auction.

This model allows to implement different pricing rules. The simplest is the **first-price**, under which each bidder pays the bid placed. In this example, the payments would be $P_3^* = 50$ euros, $P_5^* = 30$ euros, and $P_6^* = 40$ euros, and the seller's revenue $R^* = P_3^* + P_5^* + P_6^* = 120$ euros.

Table 4.8 Rounds in a multi-unit Anglo-Dutch auction of homogeneous items

Phase I: ascending								
Rounds	$p^{\mathrm{I},t}$	$q_1^{\mathrm{I},t}$	$q_2^{\mathrm{I},t}$	$q_3^{\mathrm{I},t}$	$q_4^{\mathrm{I},t}$	$q_5^{\mathrm{I},t}$	$q_6^{\mathrm{I},t}$	$Q^{\mathrm{I},t}$
$t = 0$	$p^{\mathrm{I},0} = 5$	1	1	1	1	1	1	6
$t = 1$	$p^{\mathrm{I},1} = 10$	0	1	1	1	1	1	5
$t = 2$	$p^{\mathrm{I},2} = 15$	0	0	1	1	1	1	4
Phase II: sealed-bid		$b_{1,j}^{\mathrm{II}}$	$b_{2,j}^{\mathrm{II}}$	$b_{3,j}^{\mathrm{II}}$	$b_{4,j}^{\mathrm{II}}$	$b_{5,j}^{\mathrm{II}}$	$b_{6,j}^{\mathrm{II}}$	
$b_{i,j}^{\mathrm{II}}$ ($j = 1$)		0	0	50	15	30	40	

[16]According to the rules used for the 3G spectrum auction in the UK, each bidder could not earn more than one item.

4.6 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Identical items.
- $V_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,M})$: Vector of bidder i's values for each item.
- $v_{i,j}$: Value of bidder i for the item j.
- q_i^t: Number of items demanded by bidder i in round t of a dynamic auction (bidder i's individual demand in round t).
- $q_i^{\mathrm{I},t}$: Number of items demanded by bidder i in round t in Phase I of a hybrid auction whose Phase I corresponds to a dynamic auction.
- q_i^*: Number of items won by bidder i.
- q_i^{t*}: Number of items won by bidder i in round t of a dynamic auction.
- Q^t: Number of items demanded by all bidders in round t of a dynamic auction (aggregate demand in round t).
- $Q^{\mathrm{I},t}$: Number of items demanded by all bidders in round t in Phase I of a hybrid auction whose Phase I corresponds to a dynamic auction (aggregate demand in round t in Phase I).
- Q^{t*}: Number of items assigned in round t of a dynamic auction.
- $B_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,M})$: Vector of bidder i's bids for each item in a sealed-bid auction (single-round).
- $b_{i,j}$: Bidder i's bid for item j in a sealed-bid auction (single-round).
- $b_{i,j}^{\mathrm{II}}$: Bidder i's bid for item j in Phase II of a hybrid auction whose Phase II corresponds to a sealed-bid auction (single-round).
- $b_{i,j}^*$: Winning bid submitted by bidder i for item j in a sealed-bid auction (single-round).
- $d_i(p)$: Bidder i's demand function.
- p^{s}: Starting bid (minimum bid) per item.
- $p^{\mathrm{I,s}}$: Starting bid (minimum bid) per item in Phase I of a hybrid auction.
- $p^{\mathrm{II,s}}$: Starting bid (minimum bid) per item in Phase II of a hybrid auction.
- p^*: Selling price in a uniform-price auction (clearing price).
- P_i^*: Bidder i's payment.
- M_i^t: Bidder i's residual supply in round t of an Ausubel auction.
- C_i^t: Number of units *clinched* by bidder i up to round t of an Ausubel auction.
- c_i^t: Number of units *clinched* by bidder i in round t of an Ausubel auction.
- C_{-i}: Vector of competing bids facing bidder i in a Vickrey auction.
- $c_{-i,j}$: Highest bid of the other bidders except bidder i for item j in a Vickrey auction.
- Γ_i^*: Income of bidder i from the acquired items.
- Π_i^*: Surplus of bidder i from the acquired items.
- R^*: Seller's revenue.

4.7 Exercises

1. A seller offers four identical items to two bidders. Values of the first bidder are $V_1 = (10, 8, 6, 4)$, and of the second bidder are $V_2 = (15, 10, 5, 0)$. The seller implements an ascending auction and sets a starting price per item of $p^s = 3$ euros. We assume that bidders bid for each item as long as the price is below his value. Calculate:
   (a) The quantity of items won by each bidder and at what price, assuming a price increment of one euro.
   (b) The final payment done by the winning bidders.
   (c) The seller's revenue.
   (d) The surplus of each bidder.
2. Using the same data as the previous exercise, if the seller had chosen a descending auction, set a starting price of $p^s = 16$ euros and the bidders decided to place bids when the price equals the marginal value of each unit. Calculate:
   (a) The quantity of items won by each bidder and at what price, assuming a price decrement of one euro.
   (b) The final payments of the winning bidders.
   (c) The seller's revenue.
   (d) The surplus of each bidder.
3. Using the same data as the previous exercise, if the seller had done an Ausubel auction with a starting price of $p^s = 3$ euros and the bidders bid for an item while the price is less than their marginal value, calculate:
   (a) The quantity of items won by each bidder and at what price, assuming a price increment of one euro.
   (b) The final payment done by the winning bidders.
   (c) The seller's revenue.
   (d) The surplus of each bidder.
4. A seller offers three identical items in a sealed-bid auction. The auction has three bidders who submit the following bid vectors in a single round: $B_1 = (200, 150, 50)$, $B_2 = (250, 100, 50)$, and $B_3 = (200, 50, 0)$. Solve the allocation process as follows:
   (a) Graphically represent the aggregate demand of all bidders, and distinguish winning bids from losing bids.
   (b) Indicate the winning bidders and how many items were won by each.
5. Using the data from the previous exercise, calculate the total payments of each winning bidder and the seller's revenue under the following assumptions.
   (a) The seller chooses a discriminatory auction.
   (b) The seller chooses an uniform-price auction.
   (c) The seller chooses a Vickrey auction.
   (d) Describe the effect of choosing one auction model over another on the seller's revenue and the bidder's strategies.

4.8 Solutions to Exercises

1. The ascending auction outcome is shown in Table 4.9. The first bidder has a marginal value for all items greater than the starting price, so he demands four units. The second bidder is not interested in the fourth item but his marginal value for the first, second, and third item is greater than three euros, so he demands three units in round $t = 0$. As the aggregate demand in this round is greater than supply ($Q^0 = 7 > 4 = M$), the seller increases the price. In the next round, $t = 1$, the first bidder reduces the demand by one unit because the marginal value of the fourth item is equal to the price set in this round $v_{1,4} = p^1$. The seller continues to increase the price in successive rounds, and the bidders decrease the quantity demanded until demand equal supply in round $t = 3$.
   (a) Each bidder wins two items ($q_1^* = q_2^* = 2$), and each pays $p^* = 6$ euros per item.
   (b) Both bidders have to do the same payments for the acquired items: $P_1^* = P_2^* = 2 \times 6 = 12$ euros.
   (c) The seller's revenue is equal to the sum of the payments made by all bidders, $R^* = P_1^* + P_2^* = 24$ euros.
   (d) The surplus of each bidder is equal to the difference between the income and the payment. Thus, $\Pi_1^* = \Gamma_1^* - P_1^* = (10 + 8) - 12 = 6$ euros and $\Pi_2^* = \Gamma_2^* - P_2^* = (15 + 10) - 12 = 13$ euros.
2. The descending auction in which all bidders bid when the price of each round equals the marginal value of the item ($p^t = v_{i,j}$) is presented in Table 4.10.
   (a) The first bidder wins two items ($q_1^* = 2$), one in round $t = 6$ at a price of $p^6 = 10$ euros and another in round $t = 8$ at a price of $p^8 = 8$ euros. The second bidder wins one item in round $t = 1$ at a price of $p^1 = 15$ euros and another item in round $t = 6$ at a price of $p^6 = 10$ euros.
   (b) The first bidder's payment is $P_1^* = \Sigma_{t=0}^{T} p^t q_1^{t*} = (1 \times 10) + (1 \times 8) = 18$ euros. The second bidder's payment is $P_2^* = \Sigma_{t=0}^{T} p^t q_2^{t*} = (1 \times 15) + (1 \times 10) = 25$ euros.
   (c) The seller's revenue is equal to the sum of the payments made by the winning bidders, $R^* = P_1^* + P_2^* = 18 + 25 = 43$ euros.

Table 4.9 Multi-unit ascending auction of homogeneous items (exercise 1)

Rounds (t)	Price (p^t)	q_1^t	q_2^t	Q^t
$t = 0$	$p^0 = 3$	4	3	7
$t = 1$	$p^1 = 4$	3	3	6
$t = 2$	$p^2 = 5$	3	2	5
$t = 3$	$p^3 = 6$	2	2	4
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	
	$j = 1$	10	15	
	$j = 2$	8	10	
	$j = 3$	6	5	
	$j = 4$	4	0	

Table 4.10 Multi-unit descending auction of homogeneous items (exercise 2)

Round (t)	Round price (p^t)	q_1^{t*}	q_2^{t*}	$\Sigma_{t=0}^{l} Q^{t*}$
$t = 0$	$p^0 = 16$	0	0	0
$t = 1$	$p^1 = 15$	0	1	1
$t = 2$	$p^2 = 14$	0	0	1
$t = 3$	$p^3 = 13$	0	0	1
$t = 4$	$p^4 = 12$	0	0	1
$t = 5$	$p^5 = 11$	0	0	1
$t = 6$	$p^6 = 10$	1	1	3
$t = 7$	$p^7 = 9$	0	0	3
$t = 8$	$p^8 = 8$	1	0	4
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	
	$j = 1$	10	15	
	$j = 2$	8	10	
	$j = 3$	6	5	
	$j = 4$	4	0	

Table 4.11 Ausubel auction (exercise 3)

Round (t)	Price (p^t)	q_1^t	q_2^t	Q^t	C_1^t	C_2^t
$t = 0$	$p^0 = 3$	4	3	7	1	0
$t = 1$	$p^1 = 4$	3	3	6	1	1
$t = 2$	$p^2 = 5$	3	2	5	2	1
$t = 3$	$p^3 = 6$	2	2	4	2	2
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$			
	$j = 1$	10	15			
	$j = 2$	8	10			
	$j = 3$	6	5			
	$j = 4$	4	0			

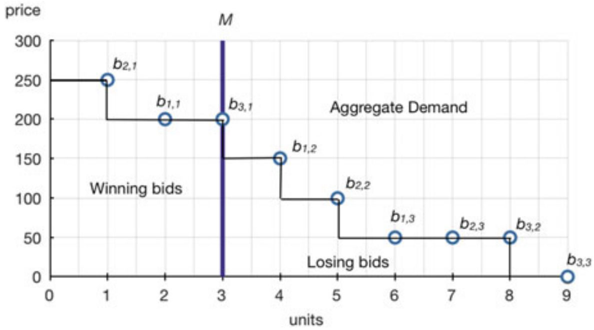
(d) The surpluses obtained by the winning bidders are: $\Pi_1^* = \Gamma_1^* - P_1^* = (10+8) - 18 = 0$ euros and $\Pi_2^* = \Gamma_2^* - P_2^* = (15+10) - 25 = 0$ euros.

In this auction model, the price to be paid by the winning bidders is equal to their bid (first-price rule), so they will never place bids equal to their valuations because they would obtain zero surplus, as in this example. The bidders thus tend to underbid.

3. In an Ausubel auction, items are awarded using the same method as a multi-unit ascending auction. However, final prices depend on the corresponding rounds in which each of the items was clinched. Table 4.11 presents the auction per round.

(a) At the starting price (round $t = 0$) the first bidder clinches one item ($M_1^0 = C_1^0 = 1$) for three euros and in round $t = 2$ he clinches the second one ($M_1^2 = C_1^2 = 2$) for five euros. The second bidder clinches his first item in round $t = 1$ ($M_2^1 = C_2^1 = 1$) and the second one in round $t = 3$ ($M_2^3 = C_2^3 = 2$). He pays four and six euros, respectively, for each of them.

Fig. 4.8 Multi-unit sealed-bid auction of homogeneous items



(b) The first bidder's payment is $P_1^* = (1\times3)+(1\times5) = 8$ euros. For each item, the first bidder pays the price of the round in which the item was clinched. The second bidder's payment is $P_2^* = (1 \times 4) + (1 \times 6) = 10$ euros.
(c) The seller's revenue is equal to the sum of the payments made by the winning bidders $R^* = P_1^* + P_2^* = 8 + 10 = 18$ euros.
(d) The surplus of the first bidder is $\Pi_1^* = \Gamma_1^* - P_1^* = (10 + 8) - 8 = 10$ euros and the surplus of the second one is equal to $\Pi_2^* = \Gamma_2^* - P_2^* = (15 + 10) - 10 = 15$ euros.

4. The allocation process for this multi-unit sealed-bid auction follows below.
(a) After all bids are received, the seller orders them from highest to lowest and calculates the aggregate demand, which is shown in Fig. 4.8. The vertical line represents the supply, and the bids on the left side correspond to the M highest bids, which are the winning bids. In this example, the three highest bids are $(b_{2,1}^*, b_{1,1}^*, b_{3,1}^*) = (250, 200, 200)$. The bids on the right side are the losing bids.
(b) At the end of the auction, each bidder wins one item.

5. In the previous exercise, the allocation process for the sealed-bid auction was solved. However, the payments for each bidder depend on the pricing rule selected.
(a) *Discriminatory auction:*
If a discriminatory auction is chosen, each bidder pays his winning bids (*pay-your-bid rule*). In this example, the first bidder pays $P_1^* = 200$ euros, the second bidder pays $P_2^* = 250$ euros, and the third bidder pays $P_3^* = 200$ euros. The seller's revenue is $R^* = P_1^* + P_2^* + P_3^* = 650$ euros.
(b) *Uniform-price auction:*
If the seller chooses an uniform-price auction, all winning bidders pay the HRB for each item. In this example, the HRB is $b_{1,2} = p^* = 150$ euros, so $P_1^* = P_2^* = P_3^* = 150$ euros. The seller's revenue is thus $R^* = P_1^* + P_2^* + P_3^* = 450$ euros.
(c) *Vickrey auction:*
To calculate the price paid by each bidder in a Vickrey auction, the vector of competing bids facing each bidder must be obtained. For the first bidder $C_{-1} = (250, 200, 100)$, for the second bidder $C_{-2} = (200, 200, 150)$, and for the third bidder $C_{-3} = (250, 200, 150)$. When we compare the highest

bid of each bidder to the lowest competing bid, we find that $b_{i,1} > c_{-i,3}$ for all bidders: $b_{1,1} = 200 > 100 = c_{-1,3}$ for the first bidder, $b_{2,1} = 250 > 150 = c_{-2,3}$ for the second, and $b_{3,1} = 200 > 150 = c_{-3,3}$ for the third. Thus, each bidder wins one item and pays: $P_1^* = 100$ euros, $P_2^* = 150$ euros and $P_3^* = 150$ euros, respectively. The seller's revenue is $R^* = P_1^* + P_2^* + P_3^* = 400$ euros.

(d) The seller's revenues vary depending on the chosen auction model. Assuming that the bidders keep the same bids in the three models, the revenues obtained by the seller are $R^* = 650$ euros in a discriminatory auction, $R^* = 450$ euros in an uniform-price auction and $R^* = 400$ euros in a Vickrey auction. However, it is important to note that these results do not demonstrate that the discriminatory auction necessarily generates higher revenues than the other models. The selection of an auction model directly affects the bidding strategy. Bidders in discriminatory auctions will always tend to underbid to have a positive surplus.

Appendix

In dynamic multi-unit auctions, a price change may involve a decrease in the demanded quantity such that the demand does not cover the supply. In these circumstances, the seller may set a **rationing rule** to allocate all items. There are many different rationing rules but we are going to explain the proportional rationing rule with an example.[17]

A seller offers ten items in an ascending auction in which there are three bidders. Table 4.12 shows the submitted bids round by round. In rounds $t = 0$ and $t = 1$, the aggregate demand is greater than the supply $Q^t > 10 = M$, so the seller raises the price. However, as a result of this price increase, the aggregate demand in the final round is lower than the supply $Q^T = 8 < 10 = M$ for $T = 2$. Bidders are only willing to buy eight items, even though the seller is offering ten.

In this situation, the seller may return to round $T - 1$ and award all the items by applying a rationing rule. If the seller implements the proportional rationing rule, we compute the items obtained by bidder i according to the following equation[18]:

$$q_i^* = q_i^T + \frac{q_i^{T-1} - q_i^T}{Q^{T-1} - Q^T}(M - Q^T). \tag{4.32}$$

Table 4.12 Multi-unit ascending auction of homogeneous items

Round (t)	Price (p^t)	q_1^t	q_2^t	q_3^t	Q^t
$t = 0$	$p^0 = 5$	8	8	4	20
$t = 1$	$p^1 = 10$	6	6	0	12
$t = 2$	$p^2 = 15$	4	4	0	8

[17] To explore other possible rationing rules, see [34] or [69], among others.

[18] If an integer number is not obtained, the seller may round up.

In this example, the final quantities allocated to the first and second bidders are:

$$q_1^* = q_2^* = 4 + \frac{6 - 4}{12 - 8}(10 - 8) = 5.$$

Going back to round $T - 1 = 1$ and applying the proportional rationing rule, the seller allocates the two unallocated items, which ensures that all supplied items are sold. Finally, both bidders win five items, and the price per unit is equal to $p^* = 10$ euros.