Smart Fuzzy Weighted Averages of Information Elicited through Fuzzy Numbers

Andrea Capotorti and Gianna Figá-Talamanca

Universitá degli Studi di Perugia, Italy
capot@dmi.unipg.it, gianna.figatalamanca@unipg.it

Abstract. We illustrate a preliminary proposal of weighted fuzzy averages between two membership functions. Conflicts, as well as agreements, between the different sources of information in the two new operators are endogenously embedded inside the average weights. The proposal is motivated by the practical problem of assessing the fuzzy volatility parameter in the Black and Scholes environment via alternative estimators.

Keywords: merging, aggregation, fuzzy mean.

1 Introduction and Motivation

In [2] we introduced a methodology for membership elicitation on parameters. The goal was to estimate the hidden volatility parameter σ of a risky asset through both the historical volatility estimator $\widehat{\sigma}$, based on a sample of log-returns of the asset itself, and the estimator $\nu = \mathrm{VIX}/100$, based on VIX which is a volatility index obtained through a set of prices for options written on the asset. Thanks to the interpretation of membership functions as coherent conditional probability assessments (see [3,4]) integrated with observational data and expert evaluations, we were able in some cases to elicit proper membership functions for the volatility parameter based on each single estimator, while in another case two memberships were considered possible. Moreover, results were obtained through probability-possibility transformation of specific simulation distributions. Thus, the peculiarity of the proposal was to deal with implicit and alternative sources of information, while one of the open problem was to find proper fusion operators.

In literature it is known that the choice of a fusion operator, given the variety of information items, is not unique and heavily context-dependent. Classes of aggregation functions covered include triangular norms and conorms, copulas, means and averages, and those based on nonadditive integrals [11]. A main characteristic of the aggregation functions is that they are used in a large number of areas and disciplines, leading to a strong demand for a wide variety of aggregation functions with predictable and tailored properties [12], [13]. Authors in [15] affirm that there are more than 90 different fuzzy operators proposed in the literature for fuzzy set operations. The role of fuzzy sets in merging information can be understood in two ways: either as a tool for extending estimation techniques to fuzzy data (this is done applying the extension principle to classical

A. Laurent et al. (Eds.): IPMU 2014, Part I, CCIS 442, pp. 466–475, 2014.

estimators, and methods of fuzzy arithmetics - see [5] for a survey); or as a tool for combining possibility distributions that represent imprecise pieces of information (then fuzzy set-theoretic operations are instrumental for this purpose - see [9] for a survey).

In view of this dichotomy, the role of standard aggregation operations like arithmetic mean is twofold. It is a basic operation for estimation and also a fuzzy set-theoretic connective. A bridge between the "estimation" and the "fusion" views of merging information is ensured using the concept of *constrained merging* [8, §6.6.2]. We borrow from it the motivation of including a "smart" component in the averaging process to address conflicts in the data to be fused, but, contrarily to the original "intelligent" proposal, without the introduction of an exogenous "combinability function". We need two different kinds of fusion operators: one for merging conjointly the values stemming from the different estimators; and another that disjointly considers different possibilities or distribution models. Our operators are weighted averages where conflicts, as well as agreements, are endogenously embedded on the average weights; for the reasons mentioned above the choices for the weights we suggest here are deeply motivated by the practical problem at hand. The difference between the two proposals is in which direction there is a deformation of the arithmetic mean: for merging of joint information distortion is toward canonical conjunction, i.e. min, while for merging of alternative information distortion is toward canonical disjunction, i.e. max.

The rest of the paper is organized as follows: next section briefly refreshes main fuzzy membership notions and introduces basic notations for our purposes, while subsequent Section 3 defines our weighted averages proposals. Section 4 illustrates the numerical applications to the original practical problem of elicitation of a single membership function for the fuzzy volatility parameter $\widetilde{\sigma}$ and its consequences on the option pricing. A similarity comparison with crisp bid-ask prices is also performed. Section 5 briefly concludes the contribution.

2 Notation

Given our goal of parameter estimation, for the sequel we will consider real valued quantities. We recall that a membership function $\mu : \mathbb{R} \to [0, 1]$ of the fuzzy set of possible values of a random variable X can be viewed as a possibility distribution (see e.g. [17]). In particular, the subset $\mu^S = \{x \in \mathbb{R} : \mu(x) > 0\}$ is named the "support" of the membership while the subset $\mu^1 = \{x \in \mathbb{R} : \mu(x) = 1\}$ is its "core". Membership functions are fully characterized by their (dual) representation through α-cuts $\mu^\alpha = \{x \in \mathbb{R} : \mu(x) \geq \alpha\}, \quad \alpha \in [0, 1]$. The α value can be conveniently interpreted as 1 minus the lower bound of the probability that quantity X hits μ^α. Then the possibility distribution is viewed as the family of probability measures ([6]): $\mathcal{P} = \{\text{prob. distr. } P : P(X \in \mu^\alpha) \geq 1 - \alpha\}$. In [2] we were able to elicit membership functions through probability-possibility transformations ([7]) induced by confidence intervals around the median of specific simulating distributions; we got so called "fuzzy numbers", i.e. unimodal membership functions with nested α-cuts. Hence, each μ we consider has an increasing left branch μ_l and a decreasing right one μ_r and each α-cut is identified by an interval $[\mu_l^\alpha, \mu_r^\alpha]$ in the extended reals $\widetilde{\mathbb{R}}$.

Aggregations are performed between α-cuts, so we always deal with two intervals, possibly degenerate, $[\mu 1_l^\alpha, \mu 1_r^\alpha]$ and $[\mu 2_l^\alpha, \mu 2_r^\alpha]$. From these it is immediate to define their four characteristic values

$$\mu_{lO}^\alpha = \min\{\mu 1_l^\alpha, \mu 2_l^\alpha\} \quad \mu_{lI}^\alpha = \max\{\mu 1_l^\alpha, \mu 2_l^\alpha\}; \tag{1}$$

$$\mu_{rI}^\alpha = \min\{\mu 1_r^\alpha, \mu 2_r^\alpha\} \quad \mu_{rO}^\alpha = \max\{\mu 1_r^\alpha, \mu 2_r^\alpha\}, \tag{2}$$

and their lengths

$$\Delta_1^\alpha = \mu 1_r^\alpha - \mu 1_l^\alpha \qquad \Delta_2^\alpha = \mu 2_r^\alpha - \mu 2_l^\alpha \qquad \Delta^\alpha = \mu_{rO}^\alpha - \mu_{lO}^\alpha; \tag{3}$$

where the subscript O refers to the "outer" values, while the subscript I to the "inner" ones (see e.g. Fig.1).

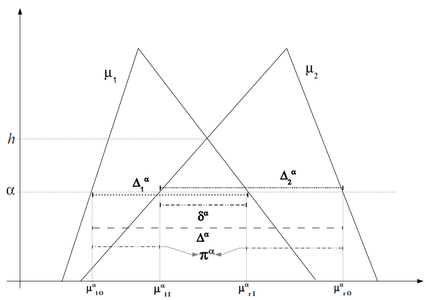

Fig. 1. Characteristic values for the merging of two α-cuts

A crucial value for our proposal is the "height" h of the intersection between $\mu 1$ and $\mu 2$, i.e.

$$h = \max\{\alpha \,:\, [\mu 1_l^\alpha, \mu 1_r^\alpha] \cap [\mu 2_l^\alpha, \mu 2_r^\alpha] \neq \emptyset\} \tag{4}$$

whenever the two memberships overlaps, while $h = 0$ if $\mu 1$ and $\mu 2$ are incompatible. Other involved quantities are

$$\delta^\alpha = |\mu_{rI}^\alpha - \mu_{lI}^\alpha| \tag{5}$$

that measures the width of the intersection if the two α-cuts overlaps or, otherwise, the minimal distance between them; and

$$\pi^\alpha = \Delta^\alpha - \delta^\alpha \tag{6}$$

that measure the length of the parts outside the (possible) intersection.

With such quantities, for the levels $\alpha \leq h$ we can define the relative contributions ϵ_l^α and ϵ_r^α of the inner memberships μ_{lI}, μ_{rI} to the intersection as:

$$\epsilon_l^\alpha = \frac{\frac{\delta^\alpha}{\Delta_{lI}^\alpha}}{\frac{\delta^\alpha}{\Delta_{lI}^\alpha} + \frac{\delta^\alpha}{\Delta_{lO}^\alpha}} = \frac{\Delta_{lO}^\alpha}{\Delta_1^\alpha + \Delta_2^\alpha}; \quad (7)$$

$$\epsilon_r^\alpha = \frac{\frac{\delta^\alpha}{\Delta_{rI}^\alpha}}{\frac{\delta^\alpha}{\Delta_{rI}^\alpha} + \frac{\delta^\alpha}{\Delta_{rO}^\alpha}} = \frac{\Delta_{rO}^\alpha}{\Delta_1^\alpha + \Delta_2^\alpha}; \quad (8)$$

with

$$\Delta_{lI}^\alpha = \begin{cases} \Delta_1^\alpha & \text{if } \mu_{lI}^\alpha = \mu 1_l^\alpha \\ \Delta_2^\alpha & \text{if } \mu_{lI}^\alpha = \mu 2_l^\alpha \end{cases} \quad \Delta_{lO}^\alpha = \begin{cases} \Delta_1^\alpha & \text{if } \mu_{lO}^\alpha = \mu 1_l^\alpha \\ \Delta_2^\alpha & \text{if } \mu_{lO}^\alpha = \mu 2_l^\alpha \end{cases} \quad (9)$$

and, similarly,

$$\Delta_{rI}^\alpha = \begin{cases} \Delta_1^\alpha & \text{if } \mu_{rI}^\alpha = \mu 1_r^\alpha \\ \Delta_2^\alpha & \text{if } \mu_{rI}^\alpha = \mu 2_r^\alpha \end{cases} \quad \Delta_{rO}^\alpha = \begin{cases} \Delta_1^\alpha & \text{if } \mu_{rO}^\alpha = \mu 1_r^\alpha \\ \Delta_2^\alpha & \text{if } \mu_{rO}^\alpha = \mu 2_r^\alpha \end{cases}. \quad (10)$$

3 A Proposal of Two Smart Weighted Averages

We propose two new binary operations $\bar{\wedge}$ and $\veebar$ to average in a conjunctive or in a disjunctive way, respectively, different information $\mu 1$ and $\mu 2$. Both generalize, by deformation, the usual arithmetic mean between two fuzzy numbers: $\bar{\wedge}$ deforms the arithmetic mean toward the min conjunction operator, while $\veebar$ toward the max disjunction operator.

Hence we define our generalized conjunction level-wise by setting as α-cut $(\mu 1 \,\bar{\wedge}\, \mu 2)^\alpha$ the interval

$$[(\mu 1 \,\bar{\wedge}\, \mu 2)_l^\alpha, (\mu 1 \,\bar{\wedge}\, \mu 2)_r^\alpha] = [wl^\alpha \mu_{lI}^\alpha + (1 - wl^\alpha)\mu_{lO}^\alpha \,,\, wr^\alpha \mu_{rI}^\alpha + (1 - wr^\alpha)\mu_{rO}^\alpha] \quad (11)$$

with weights:
for $\alpha \leq h$

$$wl^\alpha = \frac{1}{2} + \frac{\epsilon_l^\alpha}{2} \quad , \quad wr^\alpha = \frac{1}{2} + \frac{\epsilon_r^\alpha}{2}, \quad (12)$$

for $\alpha > h$,

$$wl^\alpha = \frac{(\mu 1 \,\bar{\wedge}\, \mu 2)_l^h - \mu_{lO}^\alpha + k(M_l^\alpha - M_l^h) + \theta_l(\alpha)}{(\mu_{lI}^\alpha - \mu_{lO}^\alpha)} \quad (13)$$

$$wr^\alpha = \frac{(\mu 1 \,\bar{\wedge}\, \mu 2)_r^h - \mu_{rI}^\alpha + k(M_r^\alpha - M_r^h) - \theta_r(\alpha)}{(\mu_{rO}^\alpha - \mu_{rI}^\alpha)} \quad (14)$$

with $[M_l^\alpha, M_r^\alpha]$ the α-cut of the arithmetic fuzzy mean; θ_l and θ_r specific quadratic functions used to emphasize the deformation, and $k = \frac{(\mu 1 \bar{\wedge} \mu 2)_r^h - (\mu 1 \bar{\wedge} \mu 2)_l^h}{M_r^h - M_l^h}$ a scale factor. It is important to remark that, since the intersections between the α-cuts are empty for $\alpha > h$, the choice of the weights in that case just resumes our operator to the arithmetic mean, but shifted and deformed to be "glued" with the lower levels and to emphasize the contradiction between the two sources.

Similarly, our generalized disjunction α-cuts $(\mu 1 \,\underline{\vee}\, \mu 2)^\alpha$ is defined as

$$[W^\alpha \mu_{lO}^\alpha + (1 - W^\alpha)\mu_{lI}^\alpha \,,\, W^\alpha \mu_{rO}^\alpha + (1 - W^\alpha)\mu_{rI}^\alpha] \tag{15}$$

with weights

$$W^\alpha = \frac{1 + \frac{(1-\alpha)\,\pi^\alpha}{\Delta^\alpha}}{2}\,. \tag{16}$$

We have already underlined that the main goal of the averaging operators $\overline{\wedge}$ and $\underline{\vee}$ is to deform usual fuzzy arithmetic mean toward min and max connectives, respectively. This realizes by the inter-change in equations (11) and (15) among the extremes. In fact in (11) weights wl^α and wr^α deform the results toward the "inner" part, through (12), until there is an overlap between the α-cuts, and abroad from it, through (13,14), otherwise. On the contrary in (15) weights W^α always deform the average towards the "outer" part, as much as there is "contradiction" between the two memberships. Other properties of $\overline{\wedge}$ and $\underline{\vee}$ are the closure (both averages of two fuzzy numbers produce a fuzzy number), the idempotence and the symmetry. It is easy to find examples of non-associativity of $\overline{\wedge}$ and $\underline{\vee}$, but virtually no averaging operation is associative because it is known [10] that the only associative averaging operations are of the form median.

4 Applied Example

As already stressed in the Introduction, the proposed weighted averages $\overline{\wedge}$ and $\underline{\vee}$ were motivated by the need left unresolved in [2] of an implicit assessment of fuzzy volatility in the Black and Scholes environment based on two different estimators $\widehat{\sigma}$ and ν, and on different simulating models for searched parameter σ. In particular, for each estimator, different scenarios are considered on the base of historical data and experts evaluations. For each scenario it was possible to build a pseudo-membership for the considered estimator by coherent extension of a-priori information and likelihood values stemming from specific simulation distribution of the unknown parameter. At the end, observed values of the estimators permitted to select most plausible scenarios, that could be a single one if there were sure dominance of one scenario over the others, or more than one if dominance was partial. For each scenario a probability-possibility transformation of the associated simulating distributions gave as results different membership functions. The adopted simulating distributions for σ were the uniform, the log-normal and the gamma densities, with parameters determined by the different scenarios characteristics. Hence, we have to face several merging requirements: among memberships associated to different selected scenarios, among memberships stemming from different simulating functions and between memberships associated to the two different estimators $\widehat{\sigma}$ and ν.

4.1 Elicitation of Fuzzy Volatility

As preliminary illustrative results let us show a prototypical situation (corresponding to "Case 2" in [2]) where for at least one estimator there is more than one plausible scenario and different simulating models produce quite different outputs, though at the end the two sources give quite agreeing results. In particular, for such "Case 2", associated to $\widehat{\sigma}_{obs} = 0.16$ the Log-Normal simulating model furnished two alternative scenarios (the "medium" or the "high") among the five considered, while the other two models agreed in selecting only the "medium" one. Here, by transforming the three simulating distributions we obtain the memberships reported in Fig.2 (a) where for the Log-Normal the two alternative memberships has been already merged through (15). About the

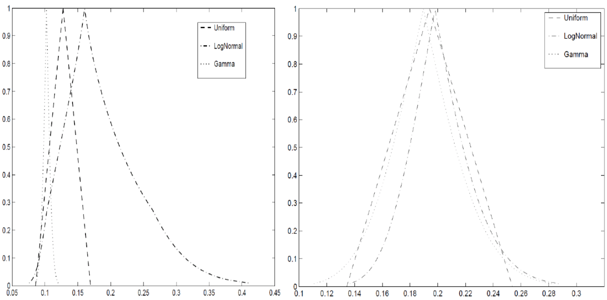

Fig. 2. Membership functions for Case 2 representing scenarios stemming from different simulating distributions, as selected by $\widehat{\sigma}_{obs} = 0.16$ (a) or by $\nu_{obs} = 0.19$ (b)

other estimator ν, its observed value $\nu_{obs} = 0.19$ always led to the selection of the "medium" scenario, obtaining the three memberships plotted in Fig.2 (b).

Since the simulating models are alternative, for both groups we can apply level-wise the weighted average (15) just between the two most contradictory memberships, since the third remains fully covered by the others. At this point we have two fuzzy numbers representative of the two sources $\mu_{\widehat{\sigma}_{obs}}$ and $\mu_{\nu_{obs}}$ which can be merged in a conjunctive way obtaining the final result $\mu_\sigma = \mu_{\widehat{\sigma}_{obs}} \,\overline{\wedge}\, \mu_{\nu_{obs}}$ reported in Fig.3.

4.2 From Elicitation of the Fuzzy Volatility to Fuzzy Option Pricing

It is well known that under the assumptions in Black and Scholes ([1]), a closed formula is available for the price of European Call Options, given by

$$C(t, S, r, \sigma, K, T) = SN(d_1) - e^{-r(T-t)}KN(d_2), \tag{17}$$

with

$$d_1 = \frac{\log(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}, \tag{18}$$

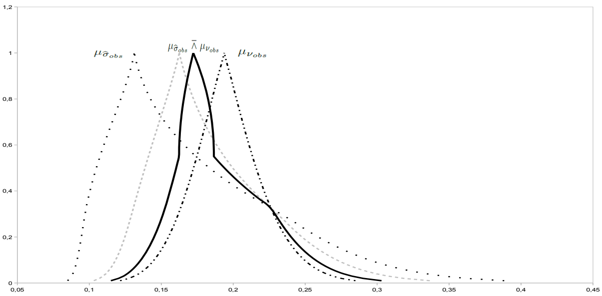

Fig. 3. Final membership function (solid) for σ in Case 2 obtained as $\mu_\sigma = \mu_{\widehat{\sigma}_{obs}} \overline{\wedge} \mu_{\nu_{obs}}$ of the information coming from $\widehat{\sigma}$ (dotted) and ν (dashed-dotted). Arithmetic mean (grey dashed) has been also reported for comparison.

where K, T are the strike price and the maturity of the Option, respectively, S is the price in t of the underlying asset and r, σ are model parameters representing the constant risk free continuously compounded rate and the volatility of the asset, and $N(\cdot)$ is the standard Normal cumulative function. Let us consider function C as a function of the volatility parameter only, assuming the other inputs as constant values i.e. $c = C(t, S, r, \sigma, K, T) = C(\sigma)$. Assuming that parameter σ is modeled as a a fuzzy number $\widetilde{\sigma}$, it is possible to detect the propagation of uncertainty from the volatility parameter to the option price by defining the fuzzy extension $\widetilde{c} = C(\widetilde{\sigma})$ of $c = C(\sigma)$; if the volatility σ is a fuzzy number $\widetilde{\sigma}$ described through its α-cuts $[\widetilde{\sigma}_l^\alpha, \widetilde{\sigma}_r^\alpha]$, for each level α, then the option price c is still a fuzzy number $\widetilde{c}$, also described by its α-cuts $[\widetilde{c}_l^\alpha, \widetilde{c}_r^\alpha]$. To obtain the fuzzy extension of C to normal upper semi-continuous fuzzy intervals one may apply the methodology as in [14], based on the solution of the box-constrained optimization problems

$$\begin{cases} \widetilde{c}_l^\alpha = \min\{C(\sigma) | \sigma \in [\widetilde{\sigma}_l^\alpha, \widetilde{\sigma}_r^\alpha]\} \\ \widetilde{c}_r^\alpha = \max\{C(\sigma) | \sigma \in [\widetilde{\sigma}_l^\alpha, \widetilde{\sigma}_r^\alpha]\}. \end{cases} \tag{19}$$

Since C is a strictly increasing function in σ we easily obtain

$$\begin{cases} \widetilde{c}_l^\alpha = C(\widetilde{\sigma}_l^\alpha) \\ \widetilde{c}_r^\alpha = C(\widetilde{\sigma}_r^\alpha). \end{cases} \tag{20}$$

4.3 Empirical Application

According to the fuzzy number obtained by suitably merging information on volatility as in Subsection 4.1, we compute the corresponding fuzzy option prices for SPX options written on the S&P500 Index. We considered options traded on October 21st, 2010: a maximum of 50 strike prices were available (for the one-month to maturity options) as well as 11 different expiration dates for a total of 168 options. The underlying price on October, 21st 2010 was S=1180.26. In order to asses the empirical significance of fuzzy option prices computed via our approach we need a proper comparison with the market bid and ask prices for the corresponding options. Besides, a selection criteria is needed to identify a set of more representative options on which to base our empirical exercise; it is well known that the more an option is traded the more its price may be interpreted as as an equilibrium price between supply and demand. For this reason, we compute, for each expiration date available, the mean trading volume obtained as the ratio of the total trading volume on options with that maturity and the total number of options with that maturity. We select, this way, 37 options.

To the end of comparing fuzzy option prices to market prices, we compute a suitably defined measure of fuzzy distances between the Black and Scholes fuzzy prices and the market bid-ask prices thought as crisp intervals, where the prices can be located with a step membership function with value 1 in the bid-ask interval and value 0 otherwise. For such purpose we consider two different fuzzy distance measures:

(a) the well known modified Bhattacharyya distance (see e.g. [18]) and
(b) the usual fuzzy similarity defined through min as t-norm and max as t-conorm (obtained also in [16] as special case of general similarities based on coherent conditional probabilities).

Further, to evaluate the added value of our merging approach with respect to usual fuzzy merging, we compute the distance/similarity measures also for the fuzzy option prices obtained applying the extension principle in the case when the fuzzy volatility parameter is modeled as the fuzzy arithmetic mean between memberships of $\widehat{\sigma}$ and ν (see a comparison of the two pricing techniques in Fig. 4). In 24 cases out of 37 the distance (a) is smaller for the fuzzy option prices obtained by our proper merging rather than by fuzzy arithmetic mean (see Fig. 5 (a)). Consistent results are obtained through the computation of the fuzzy similarity (b) which is larger for our fuzzy merging in 23 out of 24 cases above (see Fig. 5 (b)).

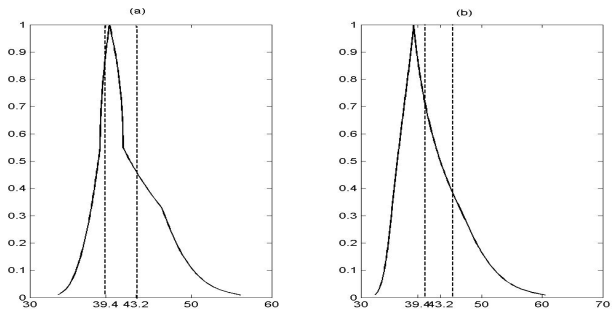

Fig. 4. Memberships of the SPX option price consequent of fuzzy volatility obtained by our merging (a) or by fuzzy arithmetic mean (b) of $\widehat{\sigma}$ and ν; both for T=30 days and K=1150 and compared to the Bid-Ask crisp interval

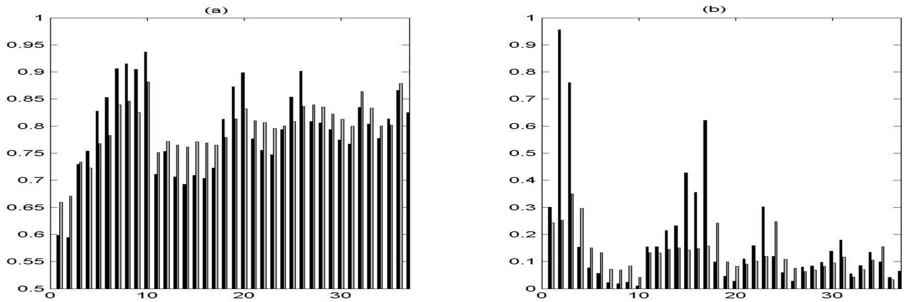

Fig. 5. Bhattacharyya distances (a) and fuzzy similarities (b) between the Bid_Ask crisp intervals and Fuzzy Option Prices obtained by our merging (black) or through the fuzzy arithmetic mean (grey)

5 Conclusion

We have illustrated a preliminary study of two weighted averages between membership functions that try to encompass in the usual fuzzy arithmetic mean the agreement or the contradiction of two heterogeneous sources of information. Formal properties of the two proposed operators $\overline{\wedge}$ and $\underline{\vee}$ must be fully investigated and practical consequences fully analyzed. Anyhow the first empirical results we have shown here seem to be promising, in particular with respect to the applicability in very different situations and the capability of conciliating quite different information.

Acknowledgement. This work was partially supported by Italian Ministry of Education, University and Research funding of Research Projects of National Interest (PRIN 2010-11) under grant `2010FP79LR_003` *"Logical methods of information management"*.

References

1. Black, F., Scholes, M.: The pricing of options and corporate liabilities. Journal of Political Economy 81(3), 637–654 (1973)
2. Capotorti, A., Figá-Talamanca, G.: On an implicit assessment of fuzzy volatility in the Black and Scholes environment. Fuzzy Sets and Systems 223, 59–71 (2013)
3. Coletti, G., Scozzafava, R.: Conditional probability, fuzzy sets, and possibility: a unifying view. Fuzzy Sets and Systems 144, 227–249 (2004)
4. Coletti, G., Scozzafava, R.: Conditional probability and fuzzy information. Computational Statistics and Data Analysis 51(1), 115–132 (2006)
5. Dubois, D., Kerre, E., Mesiar, R.: Fuzzy interval analysis. In: Dubois, D., Prade, H. (eds.) Fundamentals of Fuzzy Sets. Handbook Series of Fuzzy Sets, pp. 483–582 (2000)

6. Dubois, D., Prade, H.: When Upper Probabilities Are Possibility Measures. Fuzzy Sets and Systems 49, 65–74 (1992)
7. Dubois, D., Prade, H., Sandri, S.: On Possibility/Probability Transformations. In: Lowen, R., Roubens, M. (eds.) Fuzzy Logic, pp. 103–112. Kluwer Academic Publishers, Dordrecht (1993)
8. Dubois, D., Prade, H., Yager, R.R.: Merging fuzzy information. In: Bezdek, J.C., Dubois, D., Prade, H. (eds.) Fuzzy Sets in Approximate Reasoning and Information Systems. The Handbooks of Fuzzy Sets Series, pp. 335–401. Kluwer Academic Publishers, Dordrecht (1999)
9. Fodor, J., Yager, R.: Fuzzy set-theoretic operators and quantifiers. In: Dubois, D., Prade, H. (eds.) Fundamentals of Fuzzy Sets. Handbook Series of Fuzzy Sets, pp. 125–193 (2000)
10. Fung, L.W., Fu, K.S.: An axiomatic approach to rational decision-making in a fuzzy environment. In: Zadeh, L.A., Fu, K.S., Tanaka, K., Shimura, M. (eds.) Fuzzy Sets and Their Application to Cognitive and Decision Processes, pp. 227–256. Academic Press, New York (1975)
11. Grabisch, M., Marichal, J.-L., Mesiar, R., Pap, E.: Aggregation Functions. Encyclopedia of Mathematics and its Applications, vol. 127. Cambridge University Press (2009)
12. Grabisch, M., Marichal, J.-L., Mesiar, R., Pap, E.: Aggregation functions: Means. Information Sciences 181, 1–22 (2011)
13. Grabisch, M., Marichal, J.-L., Mesiar, R., Pap, E.: Aggregation functions: construction methods, conjunctive, disjunctive and mixed classes. Information Sciences 181, 23–43 (2011)
14. Guerra, M.L., Sorini, L., Stefanini, L.: Parametrized Fuzzy Numbers for Option Pricing. In: 2007 IEEE Conference on Fuzzy Systems, vol. 1, pp. 728–733 (2007)
15. Li, H.-X., Yen, V.C.: Fuzzy Sets and Fuzzy Decision-making. CDC Press, Boca Raton (1995)
16. Scozzafava, R., Vantaggi, B.: Fuzzy Inclusion and Similarity Through Coherent Conditional Probability. Fuzzy Sets and Systems 160(3), 292–305 (2009)
17. Zadeh, L.A.: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems 1, 3–28 (1978)
18. Zwick, R., Carlstein, E., Budescu, D.V.: Mesures of Symilarity among Fuzzy Concepts: A Comparative Analysis. Int. J. of Approximate Reasoning 1, 221–242 (1987)