A New Approach to Economic Production Quantity Problems with Fuzzy Parameters and Inventory Constraint

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Abstract. In this paper, we will develop a new multi-item economic production quantity model with limited storage space. This new model will then be extended to allow for fuzzy demand and solved numerically with a non-linear programming solver for two cases: in the first case the optimization problem will be defuzzified with the signed distance measure and in the second case, the storage constraint needs to be fulfilled, only to a certain degree of possibility. Both cases are solved and illustrated with an example.

Keywords: Economic Production Quantity, Triangular fuzzy numbers, Inventory constraint, Signed distance, Chance constrained optimization.

1 Introduction

Even with more than 100 years of EOQ (Economic Order Quantity) development, current stream of new findings and results do not tend to decrease. Even if the first model by Harris [8] was very simple, it has been very popular in industry and also an inspiration to many researchers. In this basic model the order size needed to be determined given holding costs, order setup costs and annual demand. This model has been altered in many ways to capture more complex and realistic situations in the industry. For instance, the EPQ (Economic Production Quantity) solved a problem where the product is produced to stock, also multi item, storage capacity limitation and so on is further extensions of the basic model.

These additions may be very crucial, even to the extent of only having storage capacity for one weeks production (this was the case in a Nordic plywood production facility that we have collaborated with). It is obvious that we will produce to stock in the process industry environment. In these settings we need the EOQ-models with some proper extensions. The uncertainties in the process industry can sometimes be measured probabilistically, but sometimes data is not enough and therefore fuzzy measures may be needed, c.f [3,5]. There have also been a lot of research contributions in this line of research. For instance [3] solved an EOQ model with backorders and infinite replenishment lead time

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with fuzzy lead times. However, sometimes the environment may be more stable, and only a few things may be uncertain. These fuzzy uncertainties may come from the fact that the demand may be uncertain, but still reliable data is not found to make justified probabilistic statements. This case is tackled in our paper for a special case of inventory constraints. Often is desirable to try to solve the EOQ-models with their extensions analytically through the solution of the derivatives (as also done originally by Harris, [8]). There are also other optimization approaches used in the EOQ literature. If the uncertainties in the EOQ-models can be modeled stochastically (as done in [9]), the track of probabilistic models should be conducted, but this is not always possible in the process industry. For the uncertainties relevant to this paper it is better to use fuzzy numbers instead of probabilistic approaches ([17,18]). In the line of research of fuzzy EOQ-models, there are contributions for instance like Chang [6], who worked out fuzzy modifications of the model of [13], which took the defective rate of the goods into account. Ouyang and Wu and [11] Ouyang and Yao [12] solved an EOQ-model with the lead times as decision variables as well as the order quantities. Taleizadeh and Nematollahi [15] presented again an EOQ-model with a final time horizon, with perishable items, backordering and delayed payments. Sanni and Chukwu [14] did a EOQ-model with deteriorating items, ramp-type demand as well as shortages. This paper has a track of research development behind. Already Björk and Carlsson [3] solved analytically an EOQ problem with backorders, with a signed distance defuzzification method. Björk [1] solved again a problem with a finite production rate and fuzzy cycle time, which was extended in [2] to a more general fuzzy case. The approach used in this paper is novel since there are no papers (to our knowledge) that focus on the realistic modeling of inventory constraints. Our solution methodology is to one part similar also to Björk and Carlsson [4] and Björk [1], , where the fuzzy model is defuzzified using the signed distance method [16], however, the solution is here not found through the derivatives, but numerically, since our fuzzy problem is more difficult to solve. This paper extends the results in the recent publication by Björk [2] with the limited storage capacity restriction with a more complex, but much more realistic inventory space constraint model. In addition, we consider not only the crisp case, but also the case of chance constrained formulation (in the fuzzy sense) of the storage limitations. The rest of the paper is structured as follows. First we will explain some preliminaries, then we will present the crisp case, after which we allow the demand to be fuzzy. Finally we solve the model both with defuzzification method as well as introducing fuzzy necessity constraints. Finally we will show this with an example as well as make some concluding remarks.

2 Preliminaries

In this section we introduce the necessary definitions and notations that are necessary for developing and solving our new model. We focus on fuzzy numbers and possibilistic chance constrained programming.

2.1 Fuzzy Numbers

Fuzzy sets have been introduced by Zadeh [17] to represent uncertainty different from randomness. In this paper, we employ fuzzy sets to model incomplete information inherent in many real world applications of inventory management. The most used special case of fuzzy sets is the family of triangular fuzzy numbers.

Definition 1 *Consider the fuzzy set $\tilde{A} = (a, b, c)$ where $a < b < c$ and defined on $\mathbb{R}$, which is called a triangular fuzzy number, if the membership function of $\tilde{A}$ is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{x-b} & \text{if } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

In order to find non-fuzzy values for the model, we need to use some distance measures, and we will use the signed distance [16]. Before the definition of this distance, we need to introduce the concept of α-cut of a fuzzy set.

Definition 2 *Let $\tilde{B}$ a fuzzy set on $\mathbb{R}$ and $0 \leq \alpha \leq 1$. The α-cut of $\tilde{B}$ is the set of all the points x such that $\mu_{\tilde{B}}(x) \geq \alpha$, i.e. $\tilde{B}(\alpha) = \{x | \mu_{\tilde{B}}(x) \geq \alpha\}$.*

Let Ω be the family of all fuzzy sets $\tilde{B}$ defined on $\mathbb{R}$ for which the α-cut $\tilde{B}(\alpha) = [\tilde{B}_l(\alpha), \tilde{B}_u(\alpha)]$ exists for every $0 \leq \alpha \leq 1$, and both $\tilde{B}_l(\alpha)$ and $\tilde{B}_u(\alpha)$ are continuous functions on $\alpha \in [0, 1]$.

Definition 3 *For $\tilde{B} \in \Omega$ define the signed distance of $\tilde{B}$ to $\tilde{0}$ as*

$$d(\tilde{B}, \tilde{0}) = \frac{1}{2} \int_0^1 (\tilde{B}_l(\alpha) + \tilde{B}_u(\alpha)) \mathrm{d}\alpha$$

2.2 Chance Constrained Programming

Chance constrained programming, originally introduced in probabilitic environment by Charnes and Cooper [7], is a widely-used method to handle uncertain parameters in optimization problems. The original approach was later modified to incorporate fuzzy parameters and possibility and necessity measures [10]. According to this approach, it is not necessary to use any defuzzification method, the extent to which the constraints of the models are satisfied in terms of possibility or necessity are calculated.

Possibility measure [19] is a maxitive normalized monotone measure, i.e.

$$Pos\left(\bigcup_i B_i\right) = \sup_i Pos(B_i).$$

where $\{B_i\}$ is any family of sets in the universe of discourse. The dual measure of possibility, termed as necessity, is defined as:

$$Nec(B) = 1 - Pos(B^C).$$

We can consider fuzzy numbers as possibility distributions on the real line using the formula

$$Pos(C \subset \mathbb{R}) = \sup_{x \in C} \mu_B(x),$$

where $\mu_B(x)$ is the membership function of the fuzzy number B. In this paper we will calculate the possibility of the fulfilment of constraint with the left-hand side being a fuzzy expression and the right-hand side as a crisp number (size of available storage). As crisp numbers are special cases of fuzzy numbers we can use the following formula for $A \in \mathbb{R}$:

$$Pos(B \leq A) = \sup \{\mu_B(x) \mid x \leq A\}.$$

3 EPQ Model with Fuzzy Parameters and Storage Constraint

In this section, we are first going to present the crisp and fuzzy models, and two approaches for solving the fuzzy formulation. The parameters and variables (can be assumed strictly greater than zero) in the classical multi-item EPQ model with shared cycle time and storage space limitation are the following (where the index $i \in I = \{1, 2, \ldots, n\}$ denotes the products):

- Q_i is the production batch size (variable)
- K_i is the fixed cost per production batch (parameter)
- D_i is the annual demand of the product (parameter)
- h_i is the unit holding cost per year (parameter)
- T is the cycle time (variable)
- P_i is the annual production rate (parameter)
- a_i is the storage area requirement per inventory unit (parameter)
- A is the maximum available storage area (parameter)

The total cost function (TCU), including production setup costs, the inventory holding costs, and the constraint concerning the limitation on the storage area for all products are given by

$$\begin{aligned} \min \quad & TCU(Q_1, \ldots, Q_n) = \sum_{i=1}^{n} \frac{K_i D_i}{Q_i} + \sum_{i=1}^{n} \frac{h_i Q_i \rho_i}{2} \\ \text{s. t.} \quad & a_i Q_i \rho_i + \sum_{j>i} a_j \left(Q_j \rho_j - I D_j - \left(\sum_{k>j} \frac{Q_k D_j}{P_k} \right) - \left(\sum_{k \leq i} \frac{Q_k D_j}{P_k} \right) \right) \\ & + \sum_{j<i} a_j \left(Q_j \rho_j - \left(\sum_{k=j+1}^{i} \frac{Q_k D_j}{P_k} \right) \right) \leq A, \; i = 1, \ldots, n \end{aligned} \qquad (1)$$

where $I = T - \sum_{i=1}^{n} \frac{Q_i}{P_i}$ (the idle time of the machine, we suppose it takes place in the end of a cycle, between the production of item n is finished and production of item 1 starts) and $\rho_i = 1 - \dfrac{D_i}{P_i}$. Here we assume that $\sum_{i=1}^{n} \dfrac{D_i}{P_i} \leq 1$.The production batch size Q_i can also be described with the cycle time T according the formula $Q_i = TD_i$.

The storage constraint can be justified in the following way. First, we have to notice that the maximum storage requirement occurs at one of the n time points $(t_i, i = 1, , n)$ when the production of one of the n items is finished. This follows from the observation that for any $i \leq n$, if $a_i \rho_i Q_i \geq \sum_{k \neq i} \dfrac{a_k Q_i D_k}{P_i}$ (the required storage place for item i during production is bigger than the storage that becomes available because of all the other product units sold based on the predicted demand), then we need more storage place at time point $t_(i+1)$ than t_i (and the storage requirement continuously increases between the two time points); otherwise we need more storage place at time point t_i than $t_(i+1)$ (with continuous decrease between the two points).

After using the $\rho_i = 1 - \dfrac{D_i}{P_i}$ substitution and replacing the cycle time in the constraint, we obtain the following form of the objective function:

$$TCU(Q_1, \ldots, Q_n) = \sum_{i=1}^{n} \frac{K_i D_i}{Q_i} + \sum_{i=1}^{n} \frac{h_i Q_i}{2} - \sum_{i=1}^{n} \frac{h_i Q_i D_i}{2P_i} \tag{2}$$

and the constraint

$$\begin{aligned} a_i Q_i - \frac{a_i Q_i D_i}{2P_i} + \sum_{j>i} a_j D_j \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k} \right) \\ + \sum_{j<i} a_j \left(Q_j - \left(\sum_{k=j}^{i} \frac{Q_k D_j}{P_k} \right) \right) \leq A, \; i = 1, ..., n \end{aligned} \tag{3}$$

To incorporate the uncertainty related to the estimation of demand as an input parameter for this model, we assume that the demand is uncertain but it is possible to describe it with a triangular fuzzy number (asymmetric). The fuzzy demand ($\tilde{D}_i$) will then be represented as an asymmetrical triangular fuzzy number:

$$\tilde{D}_i = (D_i - \delta_i, D_i, D_i + \eta_i)$$

The Total Annual Cost in the fuzzy sense will be

$$T\tilde{C}U(Q_1, \ldots, Q_n) = \sum_{i=1}^{n} \frac{K_i \tilde{D}_i}{Q_i} + \sum_{i=1}^{n} \frac{h_i Q_i}{2} - \sum_{i=1}^{n} \frac{h_i Q_i \tilde{D}_i}{2P_i} \tag{4}$$

and the storage limitation with fuzzy demand can be written as

$$\begin{aligned} a_iQ_i - \frac{a_iQ_i\tilde{D}_i}{2P_i} + \sum_{j>i} a_j\tilde{D}_j \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k}\right) \\ + \sum_{j<i} a_j \left(Q_j - \left(\sum_{k=j}^{i} \frac{Q_k\tilde{D}_j}{P_k}\right)\right) \leq A,\ i = 1, ..., n \end{aligned} \tag{5}$$

We will employ two different approaches to find the optimal solution to this problem:

1. We calculate the signed distance for the total cost function and the constraint to obtain the defuzzified version of the model and then solve it as a crisp problem.
2. We use necessity measure to specify the required degree of fulfilment for the storage constraint and solve the problem based on this new constraint.

For the first approach, we need to calculate first the signed distance of an asymmetric triangular fuzzy number (representing the demand) from 0 as

$$\begin{aligned} d(\tilde{D}_I, \tilde{0}) &= \frac{1}{2}\int_0^1 ((\tilde{D}_i)_l(\alpha) + (\tilde{D}_i)_u(\alpha))\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 [(D_i - \delta_i + \delta_i\alpha) + (D_i + \eta_i - \eta_i\alpha)]\mathrm{d}\alpha = D_i + \frac{\delta_i + \eta_i}{4} \end{aligned} \tag{6}$$

The defuzzified total cost function can be obtained as

$$\begin{aligned} TCU(Q_1, \ldots, Q_n) = \sum_{i=1}^{n} \frac{K_iD_i}{Q_i} + \sum_{i=1}^{n} \frac{K_i(\eta_i - \delta_i)}{4Q_i} + \sum_{i=1}^{n} \frac{h_iQ_i}{2} \\ - \sum_{i=1}^{n} \frac{h_iQ_iD_i}{2P_i} - \sum_{i=1}^{n} \frac{h_iQ_i(\eta_i - \delta_i)}{8P_i} \end{aligned} \tag{7}$$

and the defuzzified storage constraint can be written as

$$\begin{aligned} a_iQ_i - \frac{a_iQ_iD_i}{2P_i} - \frac{a_iQ_i(\eta_i - \delta_i)}{8P_i} + \sum_{j>i} a_jD_j \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k}\right) \\ + \sum_{j>i} \frac{a_j(\eta_i - \delta_i)}{4} \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k}\right) \\ + \sum_{j<i} a_j \left(Q_j - \left(D_j + \frac{\eta_i - \delta_i}{4}\right)\left(\sum_{k=j}^{i} \frac{Q_k}{P_k}\right)\right) \leq A,\ i = 1, ..., n \end{aligned} \tag{8}$$

As for the second approach, we need to notice that the left hand side of the fuzzy constraint for every i is a linear combination of triangular fuzzy numbers and as a result of this, the whole expression also represents an asymmetric triangular fuzzy number for every i. According to this, we can define a triangular fuzzy number for every i with the center

$$\begin{aligned} C_i =& a_i Q_i - \frac{a_i Q_i D_i}{2P_i} + \sum_{j>i} a_j D_j \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k} \right) \\ &+ \sum_{j<i} a_j \left(Q_j - D_j \left(\sum_{k=j}^{i} \frac{Q_k}{P_k} \right) \right) \end{aligned} \tag{9}$$

with left end-point of the support as

$$\begin{aligned} \vartheta_i =& a_i Q_i - \frac{a_i Q_i (D_i + \eta_i)}{2P_i} + \sum_{j>i} a_j (D_j - \delta_j) \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k} \right) \\ &+ \sum_{j<i} a_j \left(Q_j - (D_j + \eta_j) \left(\sum_{k=j}^{i} \frac{Q_k}{P_k} \right) \right) \end{aligned} \tag{10}$$

and right end-point of the support as

$$\begin{aligned} \nu_i =& a_i Q_i - \frac{a_i Q_i (D_i - \delta_i)}{2P_i} + \sum_{j>i} a_j (D_j + \eta_j) \left(\sum_{k=i+1}^{j-1} \frac{Q_k}{P_k} \right) \\ &+ \sum_{j<i} a_j \left(Q_j - (D_j - \delta_j) \left(\sum_{k=j}^{i} \frac{Q_k}{P_k} \right) \right) \end{aligned} \tag{11}$$

To use the possibility measure for evaluating the storage constraint, we have to define first to which extent we require the constraint to be satisfied (what should be the possibility), as $0 \leq \omega \leq 1$, and we require that for every i, the fuzzy number $\tilde{C}_i = (\vartheta_i, C_i, \nu_i)$ satisfies that $C_1 - (1-\omega)(C_i - \vartheta_i) \leq A$.

4 Example

In this section we will present a numerical example to compare three different approaches to solve the problem defined in (1). We will calculate the optimal solutions for the:

– crisp model
– fuzzy model through signed distance-based defuzzification
– chance constrained formulation.

This problem is a fictive one, even if the numbers are in the likely range of a real Finnish paper producer. The parameters of the model are described in Table 1.

The optimal solutions for the crisp and fuzzy case are given in Table 2. As we can observe, the approach using signed distance as the defuzzification approach

Table 1. Parameters for the example

Product	K_i	D_i	h_i	a_i	P_i	η_i	δ_i
1	1500	900	3	10	5500	135	225
2	1000	400	3	15	5500	60	100
3	1200	700	3	8	5600	105	175
4	1300	700	3	12	5500	105	175
5	1400	500	3	9	5500	75	125
6	900	800	3	11	5800	120	200
7	1300	800	3	13	6200	120	200
8	1100	900	3	14	6000	135	225

results in a slightly lower total cost value. The optimal batch size is lower for every type of product. The possible explanation is that, since we accounted for the uncertainty and the membership functions are specified in a way that the right width is larger than the left (there is more uncertainty concerning the upper bound for demand), we need to produce less units at a time and have shorter cycle times in order to be able to react changes in the demand according to the uncertainty.

As for the chance constrained formulation, we used $\omega = 0.8$, i.i. a 80 % assurance that the available storage is enough at any given point. According to this, the total cost value decreases significantly: by accepting a specific amount of risk of running out of storage space, we can decrease the overall cost of the company. The main change in the cost is the consequence of the higher production batches and as a result the lover setup costs. As we are accept the risk related to the storage space availability, we allow for larger number of units to be produced.

Table 2. Optimal solutions for the example with the different approaches

Product	Crisp model	Signed-distance approach	Chance constrained
TCU	7969.37	7923.99	7373.93
Q_1	1082.81	1063.67	1172.33
Q_2	640.52	627.48	701.52
Q_3	1005.00	981.82	1104.13
Q_4	1134.30	1106.32	1176.15
Q_5	1158.12	1124.61	1228.94
Q_6	1184.33	1192.60	1224.46
Q_7	1162.32	1132.67	1298.00
Q_8	1068.35	1018.69	1314.65

To perform a simple sensitivity analysis, we considered a parameter that plays an important role in the final decision, the available storage space A. The results of the optimal TCU values for the three considered models are listed in Table 3.

The results show that the total cost value increases for all methods as we decrease the total available storage space. Additionally, we can observe that the difference between the crisp solution and the signed-distance solution increases with the storage space, while the difference with respect to the chance constrained solution decreases.

Table 3. The results for different values of A

A	Crisp	Signed distance	Chance constrained
3000	9792.47	9752.93	9611.49
3500	8725.48	8683.22	8567.16
4000	7969.37	7923.99	7827.64
4500	7420.31	7371.56	7291.09

5 Conclusions

Inventory optimization can have a positive impact for a company both on responsiveness and cost as well as the environment. The model introduced in this paper is a variation of the EPQ model with many items, one manufacturing machine and limited storage space, with the demand represented as a triangular fuzzy number to incorporate uncertainty in the model. This allows for taking expert opinion into account when modeling uncertainties, especially when new suppliers and/or products are introduced. The model provides an optimal solution that takes these uncertainties into account.

The first main contribution of the model presented in the paper is the formulation of the storage constraint. Although there exists previous models incorporating storage capacity in EPQ models, they are usually to restrictive as they use in many cases simply the sum of the batch sizes which is clearly an overestimation of the required storage. We provided a formula that specifies the exact storage requirement that can occur. As a second contribution, we used this formula to extend the traditional and fuzzy EPQ model with uncertain demand. We solved the fuzzy model using two different approaches: defuzzification using the signed distance measure and chance constrained programming.

Our model is particularly suitable for solving optimization problems in a process industry context. An example resembling Finnish paper industry was used to illustrate the effect of limited storage space on the different solution approaches. Future research tracks will include increasing the presented model to several machines with shared inventory space. Different defuzzification methods will be needed to be used. Finally a complete sensitivity analysis of the different models would need to be done within specific problem domains (such as the Nordic paper industry).

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