Implications Satisfying the Law of Importation with a Given Uninorm

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Abstract. In this paper a characterization of all fuzzy implications with continuous e-natural negation that satisfy the law of importation with a given uninorm U is provided. The cases when the considered uninorm U is representable or a uninorm in $\mathcal{U}_{\min}$ are studied separately and detailed descriptions of those implications with continuous natural negation with respect to e that satisfy the law of importation with a uninorm in these classes are done. In the process some important examples are included.

Keywords: Fuzzy implication, law of importation, uninorm, fuzzy negation.

1 Introduction

Fuzzy implication functions are the generalization of binary implications in classical logic to the framework of fuzzy logic. Thus, they are used in fuzzy control and approximate reasoning to perform fuzzy conditionals [15, 20, 26] and also to perform forward and backward inferences in any fuzzy rules based system through the inference rules of modus ponens and modus tollens [17, 26, 30].

Moreover, fuzzy implication functions have proved to be useful in many other fields like fuzzy relational equations [26], fuzzy DI-subsethood measures and image processing [7, 8], fuzzy morphological operators [13, 14, 21] and data mining [37], among others. In each one of these fields, there are some additional properties that the fuzzy implication functions to be used should have to ensure good results in the mentioned applications.

The analysis of such additional properties of fuzzy implication functions usually reduces to the solution of specific functional equations. Some of the most studied properties are:

a) *The modus ponens*, because it becomes crucial in the inference process through the compositional rule of inference (CRI). Some works on this property are [23, 34–36].
b) *The distributivity properties* over conjunctions and disjunctions. In this case, these distributivities allow to avoid the combinatorial rule explosion in fuzzy systems (see [10]). They have been extensively studied again by many authors, see [1, 2, 4, 6, 31–33].

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c) *The law of importation.* This property is extremely related to the exchange principle (see [27]) and it has proved to be useful in simplifying the process of applying the CRI in many cases, see [3] and [16]. It can be written as

$$I(T(x,y),z) = I(x, I(y,z)) \quad \text{for all} \quad x,y,z \in [0,1],$$

where T is a t-norm (or a more general conjunction) and I is a fuzzy implication function. The law of importation has been studied in [3, 16, 24, 25, 27]. Moreover, in this last article the law of importation has also been used in new characterizations of some classes of implications like (S,N)-implications and R-implications. Finally, it is a crucial property to characterize Yager's implications (see [28]).

Although all these works devoted to the law of importation, there are still some open problems involving this property. In particular, given any t-norm T (conjunctive uninorm U), it is an open problem to find all fuzzy implications I such that they satisfy the law of importation with respect to this fixed t-norm T (conjunctive uninorm U). Recently, the authors have studied this problem, for implications with continuous natural negation, in the cases of the minimum t-norm and any continuous Archimedean t-norm (see [29]).

In this paper we want to deal with this problem but for the case of a conjunctive uninorm U lying in the classes of representable uninorms and uninorms in $\mathcal{U}_{\min}$. We will give some partial solutions (in the sense that we will find all solutions involving fuzzy implications with an additional property). Specifically, we will characterize all fuzzy implication functions with continuous natural negation with respect to e that satisfy the law of importation with any conjunctive uninorm U in the mentioned classes. Along the process, some illustrative examples as well as particular cases when the fixed conjunctive uninorm U is an idempotent uninorm in $\mathcal{U}_{\min}$ are presented separately.

2 Preliminaries

We will suppose the reader to be familiar with the theory of t-norms and t-conorms (all necessary results and notations can be found in [22]) and uninorms (see [12] and Chapter 5 in [3]). To make this work self-contained, we recall here some of the concepts and results used in the rest of the paper.

We will only focus on conjunctive uninorms in $\mathcal{U}_{\min}$ and representable uninorms.

Theorem 1 ([12]). *Let U be a conjunctive uninorm with neutral element $e \in]0,1[$ having functions $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ $(x \in [0,1])$ continuous except (perhaps) at the point $x = e$. Then U is given by*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{1}$$

where T is a t-norm and S is a t-conorm. In this case we will denote the uninorm by $U \equiv \langle T, e, S\rangle_{\min}$.

The class of all uninorms with expression (1) will be denoted by $\mathcal{U}_{\min}$. Next, we give the definition of a conjunctive representable uninorm.

Definition 1 ([12]). *A conjunctive uninorm U with neutral element $e \in (0,1)$ is representable if there exists a continuous and strictly increasing function $h : [0,1] \to [-\infty, +\infty]$ (called additive generator of U), with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that U is given by*

$$U_h(x,y) = h^{-1}(h(x) + h(y))$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) = 0$.

Now, we give some definitions and results concerning fuzzy negations.

Definition 2 ([11, Definition 1.1]). *A decreasing function $N : [0,1] \to [0,1]$ is called a fuzzy negation, if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called*

(i) strict, if it is strictly decreasing and continuous,
(ii) strong, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Next lemma plays an important role in the results presented in this paper. Essentially, given a fuzzy negation, it defines a new fuzzy negation which in some sense can perform the role of the inverse of the original negation.

Lemma 1 ([3, Lemma 1.4.10]). *If N is a continuous fuzzy negation, then the function $\mathfrak{R}_N : [0,1] \to [0,1]$ defined by*

$$\mathfrak{R}_N(x) = \begin{cases} N^{(-1)}(x) & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0, \end{cases}$$

where $N^{(-1)}$ stands for the pseudo-inverse of N given by $N^{(-1)}(x) = \sup\{z \in [0,1] \mid N(z) > x\}$ for all $x \in [0,1]$, is a strictly decreasing fuzzy negation. Moreover,

(i) $\mathfrak{R}_N^{(-1)} = N$,
(ii) $N \circ \mathfrak{R}_N = id_{[0,1]}$,
(iii) $\mathfrak{R}_N \circ N|_{Ran(\mathfrak{R}_N)} = id|_{Ran(\mathfrak{R}_N)}$,

where $Ran(\mathfrak{R}_N)$ stands for the range of function $\mathfrak{R}_N$.

Now, we recall the definition of fuzzy implications.

Definition 3 ([11, Definition 1.15]). *A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a fuzzy implication if it satisfies:*

(I1) $I(x,z) \geq I(y,z)$ *when* $x \leq y$, *for all* $z \in [0,1]$.
(I2) $I(x,y) \leq I(x,z)$ *when* $y \leq z$, *for all* $x \in [0,1]$.
(I3) $I(0,0) = I(1,1) = 1$ *and* $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition. Fuzzy implications can satisfy additional properties coming from tautologies in crisp logic. In this paper, we are going to deal with the law of importation, already presented in the introduction.

The natural negation with respect to e of a fuzzy implication will be also useful in our study.

Definition 4 ([3, Definition 5.2.1]). *Let I be a fuzzy implication. If $I(1,e) = 0$ for some $e \in [0,1)$, then the function $N_I^e : [0,1] \to [0,1]$ given by $N_I^e(x) = I(x,e)$ for all $x \in [0,1]$, is called the* natural negation *of I with respect to e.*

Remark 1.
(i) If I is a fuzzy implication, N_I^0 is always a fuzzy negation.
(ii) Given a binary function $F : [0,1]^2 \to [0,1]$, we will denote by $N_F^e(x) = F(x,e)$ for all $x \in [0,1]$ its e-horizontal section. In general, N_F^e is not a fuzzy negation. In fact, it is trivial to check that N_F^e is a fuzzy negation if, and only if, $F(x,e)$ is a non-increasing function satisfying $F(0,e) = 1$ and $F(1,e) = 0$.

3 On the Satisfaction of (LI) with a Given Conjunctive Uninorm U

In this section, the main goal is the characterization of all fuzzy implications with a continuous natural negation with respect to $e \in [0,1)$ which satisfy the Law of Importation (LI) with a fixed conjunctive uninorm U.

First of all, the first question which arises concerns if fixed a concrete conjunctive uninorm U, any fuzzy negation can be the natural negation with respect to some $e \in [0,1)$ of a fuzzy implication satisfying (LI) with U. The answer is negative since as the following result shows, there exists some dependence between the conjunctive uninorm U and the natural negation of the fuzzy implication I with respect to some $e \in [0,1)$. To characterize which fuzzy negations are compatible with a conjunctive uninorm U in this sense, the following property will be considered:

$$\text{if } N(y) = N(y') \text{ for some } y, y' \in [0,1], \text{ then } N(U(x,y)) = N(U(x,y')) \quad \forall x \in [0,1]. \tag{2}$$

Note that any strict negation obviously satisfies the previous equation. However, there are many other negations, not necessarily strict, which satisfy this property as we will see in next sections. Note also that similar conditions on a negation N as (2) were considered in [9].

On the other hand, the following proposition is straightforward to check.

Proposition 1. *Let $I : [0,1]^2 \to [0,1]$ be a binary function such that N_I^e is a fuzzy negation for some $e \in [0,1)$. If I satisfies (LI) with a conjunctive uninorm U, then N_I^e and U satisfy Property (2).*

Next result gives the expression of any binary function with N_I^e a continuous fuzzy negation for some $e \in [0,1)$ satisfying (LI) with a conjunctive uninorm U. Note that the binary function only depends on the uninorm U and its natural negation with respect to $e \in [0,1)$.

Proposition 2. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I^e a continuous fuzzy negation for some $e \in [0,1)$ satisfying (LI) with a conjunctive uninorm U. Then*

$$I(x,y) = N_I^e(U(x, \mathfrak{R}_{N_I^e}(y))).$$

From now on, we will denote these implications generated from a conjunctive uninorm U and a fuzzy negation N by $I_{N,U}(x,y) = N(U(x, \mathfrak{R}_N(y)))$.

Remark 2. Instead of $\mathfrak{R}_{N_I^e}$, we can consider any function N_1 such that $N_1^{(-1)} = N_I^e$ and $N_I^e \circ N_1 = \mathrm{id}_{[0,1]}$. This is a straightforward consequence of the satisfaction of Property (2) in this case. Since $N_I^e(\mathfrak{R}_{N_I^e}(y)) = N_I^e(N_1(y))$, then using the aforementioned property, $N_I^e(U(x, \mathfrak{R}_{N_I^e}(y))) = N_I^e(U(x, N_1(y)))$ and therefore, $I_{N_I^e,U}$ can be computed using either $\mathfrak{R}_{N_I^e}$ or N_1.

Moreover, this class of implications satisfies (LI) with the same conjunctive uninorm U from which they are generated.

Proposition 3. *Let N be a continuous fuzzy negation and U a conjunctive uninorm satisfying Property (2). Then $I_{N,U}$ satisfies (LI) with U.*

Now, we are in condition to fully characterize the binary functions I with N_I^e a continuous fuzzy negation for some $e \in [0,1)$ satisfying (LI) with a conjunctive uninorm U.

Theorem 2. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I^e a continuous fuzzy negation for some $e \in [0,1)$ and U a conjunctive uninorm. Then*

I satisfies (LI) with $U \Leftrightarrow N_I^e$ and U satisfy Property (2) and $I = I_{N_I^e,U}$.

Note that it remains to know when N_I^e and U satisfy Property (2). From now on, we will try given a concrete conjunctive uninorm U, to determine which fuzzy negations satisfy the property with U.

4 On the Satifaction of Property 2 for Some Uninorms

In the previous section, Proposition 1 shows that the conjunctive uninorm and the natural fuzzy negation with respect to some $e \in [0,1)$ of the fuzzy implication must satisfy Property (2). Consequently, given a fixed conjunctive uninorm U, in order to characterize all fuzzy implications with a continuous natural negation with respect to some $e \in [0,1)$ satisfying (LI) with U, we need to know which fuzzy negations are compatible with the conjunctive uninorm U. In this section, we will answer this question for some conjunctive uninorms presenting for each one, which fuzzy negations can be considered and then finally, using the characterization given in Theorem 2, the expressions of these fuzzy implications can be retrieved easily.

First of all, we want to stress again that the goal of this paper is to characterize all fuzzy implications with a continuous natural negation with respect to some $e \in [0, 1)$ satisfying (LI) with a concrete conjunctive uninorm U. Therefore, there are other implications satisfying (LI) with a conjunctive uninorm U than those given in the results of this section. Of course, these implications must have non-continuous natural negations with respect to any $e \in [0, 1)$ such that $I(1, e) = 0$. An example of a fuzzy implication having non-continuous natural negations with respect to any $e \in [0, 1)$ such that $I(1, e) = 0$ is the least fuzzy implication.

Proposition 4. *Let I_{Lt} be the greatest fuzzy implication given by*

$$I_{Lt}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then I_{Lt} satisfies (LI) with any conjunctive uninorm U.

Consequently, although I_{Lt} satisfies (LI) with any conjunctive uninorm U, we will not obtain this implication in the next results since it has no continuous natural negation at any level $e \in [0, 1)$.

4.1 Representable Uninorms

The first class of uninorms we are going to study is the class of representable uninorms. The following result shows that the fuzzy negation must be strict in order to satisfy Property (2) with a uninorm of this class.

Proposition 5. *If U is a representable uninorm, then Property (2) holds if, and only if, N is an strict fuzzy negation.*

At this point, we can characterize all fuzzy implications with a continuous natural negation with respect to some $e \in [0, 1)$ satisfying (LI) with a representable uninorm U.

Theorem 3. *Let $I : [0, 1]^2 \to [0, 1]$ be a binary function with N_I^e a continuous fuzzy negation for some $e \in [0, 1)$ and let $h : [0, 1] \to [-\infty, +\infty]$ an additive generator of a representable uninorm. Then the following statements are equivalent:*

(i) I satisfies (LI) with the conjunctive representable uninorm U_h.
(ii) N_I^e is strict and I is given by $I(x, y) = N_I^e(U(x, (N_I^e)^{-1}(y))) =$

$$\begin{cases} N_I^e(h^{-1}(h(x) + h((N_I^e)^{-1}(y)))) & \text{if } (x, y) \notin \{(0, 0), (1, 1)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Note that the implications obtained in the previous theorem are in fact (U, N)-implications derived from the negation $(N_I^e)^{-1}$ and the uninorm $(N_I^e)^{-1}$-dual of U_h. Moreover, in the case that $(N_I^e)^{-1}$ coincides with the negation associated to the representable uninorm U_h, the implication is also the RU-implication derived from U_h (see [5]). Similar results with implications derived from t-norms were also obtained in [29], see also [18, 19].

4.2 Uninorms in $\mathcal{U}_{\min}$

The second class of uninorms which we want to study is the class of uninorms in $\mathcal{U}_{\min}$. We will restrict ourselves to the cases where the underlying t-norm and t-conorm are continuous Archimedean or idempotent. Therefore, we will consider four different cases.

▶ $\boldsymbol{U \equiv \langle T_M, e, S_M \rangle_{\min}}$**.** In a first step, we will consider the uninorm $U \equiv \langle T_M, e, S_M \rangle_{\min}$ where $T_M(x,y) = \min\{x,y\}$ and $S_M(x,y) = \max\{x,y\}$, which in addition to the class of $\mathcal{U}_{\min}$, it belongs also to the class of idempotent uninorms, those satisfying $U(x,x) = x$ for all $x \in [0,1]$. In contrast with the representable uninorms, in this case we will have continuous non-strict negations satisfying Property (2) with these uninorms.

Proposition 6. *If $U \equiv \langle T_M, e, S_M \rangle_{\min}$, then Property (2) holds if, and only if, N is a continuous fuzzy negation satisfying the following two properties:*

1. *There exists $\alpha \in (0,1]$ such that $N(x) = 0$ for all $x \geq \alpha$.*
2. *If $N(x) = k$ for all $x \in [a,b]$ for some constant $k > 0$ then $a \geq e$ or $b \leq e$.*

Note that any strict fuzzy negation satisfies the previous properties. In addition, those non-strict continuous fuzzy negations whose non-zero constant regions do not cross $x = e$ satisfy also Property (2) with $U \equiv \langle T_M, e, S_M \rangle_{\min}$. From this result and using Theorem 2, the expressions of the fuzzy implications we are looking for can be easily obtained.

▶ $\boldsymbol{U \equiv \langle T, e, S_M \rangle_{\min}}$ **with a Continuous Archimedean t-norm $\boldsymbol{T}$.** Now we focus on the case when we consider an underlying continuous Archimedean t-norm in addition to the maximum t-conorm. In this case, many of the fuzzy negations which were compatible with the uninorm of the first case are not compatible now with the uninorm of the current case.

Proposition 7. *If $U \equiv \langle T, e, S_M \rangle_{\min}$ with T a continuous Archimedean t-norm, then Property (2) holds if, and only if, N is a continuous fuzzy negation satisfying that there exists some $\alpha \in [0,e]$ such that $N(x) = 1$ for all $x \leq \alpha$ and N is strictly decreasing for all $x \in (\alpha, e)$.*

Of course, as we already know, strict fuzzy negations are compatible with these uninorms. Furthermore, when it is continuous but non-strict, the only constant region allowed in $[0,e]$ is a one region while in $[e,1]$, the fuzzy negation can have any constant region. Again, using Theorem 2, we can obtain the expressions of the fuzzy implications with a continuous natural negation with respect to some $e \in [0,1)$ satisfying (LI) with some of these uninorms.

▶ $\boldsymbol{U \equiv \langle T_M, e, S \rangle_{\min}}$ **with a Continuous Archimedean t-conorm $\boldsymbol{S}$.** In this third case, we analyse the case when we consider an underlying continuous

Archimedean t-conorm in addition to the minimum t-norm. In this case, in contrast to the second case, now the main restrictions on the constant regions are located in $[e, 1]$.

Proposition 8. *If $U \equiv \langle T_M, e, S\rangle_{\min}$ with S a continuous Archimedean t-conorm, then Property (2) holds if, and only if, N is a continuous fuzzy negation satisfying:*

1. *There exists $\alpha \in (0, 1]$ such that $N(x) = 0$ for all $x \geq \alpha$.*
2. *If $N(x) = k$ for all $x \in [a, b]$ for some constant $k > 0$ then $b \leq e$.*

As always strict fuzzy negations are compatible with these uninorms. Moreover, when the fuzzy negation is continuous but non-strict, the only constant region which could cross $x = e$ is the zero region while in $[0, e]$, the fuzzy negation can have any constant region. Finally, Theorem 2 can be applied to obtain the expressions of these implications.

▸ **$U \equiv \langle T, e, S\rangle_{\min}$ with a Continuous Archimedean t-norm T and t-conorm S.** In this last case, we analyse the case when we consider an underlying continuous Archimedean t-norm T and t-conorm S. This is the case where fewer fuzzy negations are compatible with the considered uninorm. In fact, only two special constant regions are allowed.

Proposition 9. *If $U \equiv \langle T, e, S\rangle_{\min}$ with T a continuous Archimedean t-norm and S a continuous Archimedean t-conorm, then Property (2) holds if, and only if, N is a continuous fuzzy negation satisfying the following two properties:*

1. *There exist $\alpha \in [0, e]$ and $\beta \in [e, 1]$ with $\alpha < \beta$ such that $N(x) = 1$ for all $x \leq \alpha$ and $N(x) = 0$ for all $x \geq \beta$.*
2. *N is strict for all $x \in (\alpha, \beta)$.*

Clearly, we retrieve strict fuzzy negations when $\alpha = 0$ and $\beta = 1$. As we can see, continuous non-strict fuzzy negations are also possible but only two constant regions (zero and one regions) are allowed. In order to get the expressions of these implications, Theorem 2 must be used.

5 Conclusions and Future Work

In this paper, we have characterized all fuzzy implications satisfying (LI) with a conjunctive uninorm U when the natural negation of the implication with respect to some $e \in [0, 1)$ is continuous. Moreover, we have determined in particular the expression of these implications when the conjunctive uninorm U belongs to the class of $\mathcal{U}_{\min}$ with some underlying continuous Archimedean or idempotent t-norm and t-conorm and to the class of representable uninorms.

As a future work, we want to study the remaining uninorms of the class of $\mathcal{U}_{\min}$ and some other classes such as idempotent uninorms. In addition, we want to establish the relation between the new class of implications introduced in this paper $I_{N,U}$ and (U, N)-implications.

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