On Fuzzy Polynomial Implications

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Abstract. In this work, the class of fuzzy polynomial implications is introduced as those fuzzy implications whose expression is given by a polynomial of two variables. Some properties related to the values of the coefficients of the polynomial are studied in order to obtain a fuzzy implication. The polynomial implications with degree less or equal to 3 are fully characterized. Among the implications obtained in these results, there are some well-known implications such as the Reichenbach implication.

Keywords: Fuzzy implication, polynomial implication, (S,N)-implication, exchange principle.

1 Introduction

Fuzzy implications have become one of the most important operations in fuzzy logic. Their importance lies on the fact that they perform an analogous function to the classical implication in binary logic. Fuzzy implications generalize the classical ones in the sense that restricted to $\{0,1\}^2$ both coincide. Nowadays, these operations are modelled by means of monotonic functions $I : [0,1]^2 \to [0,1]$ satisfying the aforementioned border conditions. In the last years, a great number of researchers have devoted their efforts to the study of these logical connectives. Thus, we can highlight the survey [8] and the books [2] and [3], entirely devoted to fuzzy implications. This peak of interest in fuzzy implications is induced by the wide range of applications where these operations are useful. They play an essential role in approximate reasoning, fuzzy control, fuzzy mathematical morphology and other fields where these theories are applied.

All these applications trigger the need of having a large bunch of different classes of implications. In [11] the relevance of having many different classes of implications is pointed out. The main reason is that any "If-Then" rule can be modelled by a fuzzy implication and therefore, depending on the context and the proper behaviour of the rule, different implications can be suitable in each case. In addition, fuzzy implications are used to perform backward and forward inferences and so the choice of the implication can not be independent from the inference rule it is going to model.

In order to answer adequately to this necessity, several classes of fuzzy implications have been introduced. There exist two main strategies to obtain new classes.

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The first one is based on the use of aggregation functions (t-norms, t-conorms, uninorms or aggregation functions in general) and other logical connectives, such as fuzzy negations. Some examples of this strategy are R and (S,N)-implications, QL and D-operations, among many others (see [3]). The second one is based on the use of univalued generators, obtaining the well-known Yager's implications or the h-implications. An exhaustive compilation of the different classes of fuzzy implications can be found in [10].

The implications obtained by means of these strategies can have very different expressions that will depend on the expressions of the aggregation functions or the generators used in their construction. However, the final expression of the fuzzy implication is important for its use in any application. It is well-known that some expressions of functions are tougher in order to compute their values and more propitious to spread possible errors caused by numerical approximations of the inputs. Consequently, operations with polynomial or rational expressions are more friendly than those which have a more complex expression from the computational point of view. Thus, in [1] and [7], all the rational Archimedean continuous t-norms are characterized. This family of t-norms is the well-known Hamacher class which contains the t-norms given by the following expression

$$T_\alpha(x,y) = \frac{xy}{\alpha + (1-\alpha)(x+y-xy)}, \quad x,y \in [0,1]$$

with $\alpha \geq 0$. Note that the only polynomial t-norm is the product t-norm $T_P(x,y) = xy$. Moreover, in [5], Fodor characterizes all the rational uninorms as those whose expression is given by

$$U_e(x,y) = \frac{(1-e)xy}{(1-e)xy + e(1-x)(1-y)}$$

if $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and, $U(1,0) = U(0,1) = 0$ or $U(0,1) = U(1,0) = 1$. In this case, there not exist any polynomial uninorm since they are never continuous.

So after recalling some definitions and results which will be used in this work, the main target is the introduction of fuzzy polynomial implications, those implications which have a polynomial of two variables as their expression. Some necessary conditions on the coefficients of a polynomial in order to be suitable to obtain a fuzzy implication are determined. After that, we will fully characterize all fuzzy polynomial implications of degree less or equal to 3 and we will study which additional properties they fulfil. From the derived results, the relationship of the obtained fuzzy polynomial implications with (S,N) and f-generated Yager's implications will be established. The paper ends with some conclusions and future work we want to develop.

2 Preliminaries

Let us recall some concepts and results that will be used throughout this paper. First, we give the definition of fuzzy negation.

Definition 1 ([6, Definition 1.1]). *A non-increasing function $N : [0,1] \to [0,1]$ is a* fuzzy negation*, if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is*

(i) strict*, if it is continuous and strictly decreasing.*
(ii) strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.*

Next, we recall the definition of fuzzy implication.

Definition 2 ([6, Definition 1.15]). *A binary operator $I : [0,1]^2 \to [0,1]$ is called a* fuzzy implication*, if it satisfies:*

(I1) $I(x,z) \geq I(y,z)$ *when* $x \leq y$*, for all* $z \in [0,1]$.
(I2) $I(x,y) \leq I(x,z)$ *when* $y \leq z$*, for all* $x \in [0,1]$.
(I3) $I(0,0) = I(1,1) = 1$ *and* $I(1,0) = 0$.

From the definition, we can deduce that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ while the symmetrical values $I(x,0)$ and $I(1,x)$ are not determined. Some additional properties of fuzzy implications which will be used in this work are:

- The *left neutrality principle*,

$$I(1,y) = y, \quad y \in [0,1]. \tag{\textbf{NP}}$$

- The *exchange principle*,

$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,y,z \in [0,1]. \tag{\textbf{EP}}$$

- The *law of importation* with a t-norm T,

$$I(T(x,y),z) = I(x, I(y,z)), \quad x,y,z \in [0,1]. \tag{\textbf{LI}}$$

- The *ordering property*,

$$x \leq y \Longleftrightarrow I(x,y) = 1, \quad x,y \in [0,1]. \tag{\textbf{OP}}$$

Finally, we recall the definitions of (S,N)-implications and Yager's f-generated implications.

Definition 3 ([3, Definition 2.4.1]). *A function $I : [0,1]^2 \to [0,1]$ is called an* (S,N)-implication *if there exist a t-conorm S and a fuzzy negation N such that*

$$I_{S,N}(x,y) = S(N(x), y), \quad x,y \in [0,1].$$

Definition 4 ([3, Definition 3.1.1]). *Let $f : [0,1] \to [0,\infty]$ be a continuous and strictly decreasing function with $f(1) = 0$. The function $I_f : [0,1]^2 \to [0,1]$ defined by*

$$I_f(x,y) = f^{-1}(x \cdot f(y)), \quad x,y \in [0,1],$$

*understanding $0 \cdot \infty = 0$, is called an f-*generated implication*. The function f is an f-generator of the implication I_f.*

3 Polynomial Implications

In this section, we will introduce the concept of fuzzy polynomial implication and we will determine some necessary conditions on the coefficients of the polynomial in order to obtain a fuzzy implication from this expression.

Remark 1. Although in the introduction, the characterizations of rational Archimedean continuous t-norms and rational uninorms have been recalled (understanding a rational function as a quotient of two polynomials), in this work we will only focus on the fuzzy polynomial implications. This limitation is a direct consequence of the definition of a fuzzy implication. While uninorms and t-norms are associative functions and therefore, there exists a quite restrictive property in their definitions, fuzzy implications have a more flexible definition. This flexibility allows the existence of a great number of fuzzy polynomial implications and therefore, their study is worthy in itself.

Definition 5. *Let $n \in \mathbb{N}$. A binary operator $I : [0,1]^2 \to [0,1]$ is called a* fuzzy polynomial implication of degree n *if it is a fuzzy implication and its expression is given by*

$$I(x, y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$$

for all $x, y \in [0,1]$ where $a_{ij} \in \mathbb{R}$ and there exist some $0 \le i, j \le n$ with $i+j = n$ such that $a_{ij} \neq 0$.

The following example shows the existence of fuzzy polynomial implications of any degree $n \in \mathbb{N}$ with $n \geq 2$.

Example 1. Let us consider the parametrized family of fuzzy negations given by $N_n(x) = 1 - x^n$ for all $x \in [0,1]$ and $n \in \mathbb{Z}^+$, and the probabilistic sum t-conorm, whose expression is $S_P(x, y) = x + y - xy$ for all $x, y \in [0,1]$. It is straightforward to check that the probabilistic sum belongs to the family of Hamacher t-conorms (the dual t-conorms of the Hamacher t-norms) and moreover, it is the unique polynomial t-conorm. Then, if we consider these two operators, we can construct the following parametrized family of (S,N)-implications

$$I_{S_P,N_{n-1}}(x, y) = S_P(N_{n-1}(x), y) = 1 - x^{n-1} + x^{n-1}y$$

for all $x, y \in [0,1]$ and $n \geq 2$. As it can be observed, they are polynomial implications of degree n. In addition, they satisfy (LI) with $T_P(x, y) = xy$ and therefore, they are also Yager's f-generated implications with $f(x) = \sqrt[n-1]{1-x}$ (see [9]).

A first property which can be derived form the definition is the continuity of these implications.

Proposition 1. *All fuzzy polynomial implications are continuous implications.*

Remark 2. It is worthy to note that some usual implications whose expression is piecewise polynomial will not be considered as polynomial functions. Thus, for instance, among many others, the following implications are not polynomial since they do not satisfy the requirements of the Definition 5:

$$I_{GD}(x,y) = \begin{cases} 1 \text{ if } x \leq y, \\ y \text{ if } x > y, \end{cases} \quad I_{LK}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1-x+y & \text{if } x > y. \end{cases}$$

Note that both implications of the previous remark have a non-trivial one region. This fact is not a coincidence as the following result proves.

Proposition 2. *All fuzzy polynomial implications I have a trivial one region, i.e., $I(x,y)=1$ if, and only if, $x=0$ or $y=1$.*

This property is studied in detail in [4] where it is proved that this property is essential to generate strong equality indices. Furthermore, in particular, the polynomial implications never satisfy (OP). On the other hand, they can satisfy (EP) and in that case, they are (S,N)-implications.

Proposition 3. *Let $I(x,y) = \sum_{\substack{0\leq i,j\leq n \\ i+j\leq n}} a_{ij}x^i y^j$ be a polynomial implication of degree n. If I satisfies (EP), then I is an (S,N)-implication generated by the strict fuzzy negation $N(x) = \sum_{i=0}^{n} a_{i0}x^i$ and the t-conorm $S(x,y) = \sum_{\substack{0\leq i,j\leq n \\ i+j\leq n}} a_{ij}(N^{-1}(x))^i y^j$.*

However, the question of which polynomials can be fuzzy polynomial implications remains still unanswered. The problem relies on to characterize which coefficients $a_{ij} \in \mathbb{R}$ have to be chosen in order to generate a polynomial $p(x,y)$ satisfying the conditions of the Definition 2. We will partially answer this question in general for polynomials of degree n. First of all, the next result determines the necessary and sufficient conditions a polynomial must satisfy in order to be the expression of a fuzzy polynomial implication.

Theorem 1. *A polynomial $p(x,y) = \sum_{\substack{0\leq i,j\leq n \\ i+j\leq n}} a_{ij}x^i y^j$ of degree n is a fuzzy polynomial implication if, and only if, the following properties hold:*

(i) $p(0,y) = p(x,1) = 1$ *for all* $x,y \in [0,1]$.
(ii) $p(1,0) = 0$.
(iii) $\frac{\partial p(x,y)}{\partial x} \leq 0$ *for all* $x,y \in [0,1]$.
(iv) $\frac{\partial p(x,y)}{\partial y} \geq 0$ *for all* $x,y \in [0,1]$.
(v) $0 \leq p(1,y), p(x,0) \leq 1$.

The two first properties of the previous theorem provide some conditions on the coefficients a_{ij} of the polynomial $p(x,y)$ in a direct way.

Proposition 4. *Let* $p(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *be a polynomial of degree* n. *Then we have the following equivalences:*

(i) $p(0,y) = 1$ *if, and only if,* $a_{00} = 1$ *and* $a_{0j} = 0$ *for all* $0 < j \le n$.

(ii) $p(x,1) = 1$ *if, and only if,* $\sum_{j=0}^{n} a_{0j} = 1$ *and* $\sum_{j=0}^{n-i} a_{ij} = 0$ *for all* $0 < i \le n$.

(iii) $p(1,0) = 0$ *if, and only if,* $\sum_{i=0}^{n} a_{i0} = 0$.

Thus, the next result gives some necessary conditions on the coefficients of the fuzzy polynomial implications.

Corollary 1. *Let* $I(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *be a polynomial implication of degree* n. *Then the following properties hold:*

(i) $a_{00} = 1$.

(ii) $a_{0j} = 0$ *for all* $0 < j \le n$.

(iii) $\sum_{j=0}^{n} a_{0j} = 1$ *and* $\sum_{j=0}^{n-i} a_{ij} = 0$ *for all* $0 < i \le n$.

(iv) $\sum_{i=1}^{n} a_{i0} = -1$.

However, the transfer of the properties (iii)-(v) of Theorem 1 to properties on the coefficients of the polynomial is harder for polynomials of degree n. Consequently, and with the aim of characterizing some polynomial implications, from now on we will restrict the study to polynomial implications of degree less or equal to 3.

3.1 Degree Less or Equal to One

First, we are going to study the existence of polynomial implications of degree less or equal to 2, i.e., fuzzy implications given by the following expression

$$\begin{aligned} I(x,y) &= a_{00}, && \text{with } a_{00} \in \mathbb{R}, \\ I(x,y) &= a_{00} + a_{10}x + a_{01}y, && \text{with } a_{10} \neq 0 \text{ or } a_{01} \neq 0. \end{aligned}$$

It is easy to check that by Corollary 1 in the first case, it must hold that $a_{00} = 0$ and $a_{00} = 1$. Therefore, there not exist fuzzy polynomial implications of degree less or equal to 1. Let us recall again that there exist constant piecewise fuzzy implications. Two well-known examples of these implications are the least I_{Lt} and the greatest I_{Gt} fuzzy implications defined as follows

$$I_{Lt}(x,y) = \begin{cases} 1 \text{ if } x = 0 \text{ or } y = 1, \\ 0 \text{ otherwise,} \end{cases} \qquad I_{Gt}(x,y) = \begin{cases} 0 \text{ if } x = 1 \text{ and } y = 0, \\ 1 \text{ otherwise.} \end{cases}$$

Furthermore, there are also no fuzzy polynomial implications of degree 1. In this case, Corollary 1 states that the coefficients must satisfy $a_{00} = 1$, $a_{01} = 0$, $a_{10} = 0$ and also $a_{10} = -1$. Therefore, there is no feasible solution. However, there exist again fuzzy implications which are polynomial of degree less or equal to 1 piecewise. For instance, among many others, we have the fuzzy implications presented in Remark 2. To sum up, from the previous discussion, the following result is immediate.

Proposition 5. *There are no fuzzy polynomial implications of degree less or equal to 1.*

3.2 Degree 2

Now we deal with the characterization of all polynomial implications of degree 2, i.e., those whose expression is given by

$$I(x, y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$$

with $a_{11}^2 + a_{20}^2 + a_{02}^2 \neq 0$. First of all, using Corollary 1, we obtain that there exists only a value for each coefficient, namely $a_{00} = 1$, $a_{10} = -1$, $a_{11} = 1$ and $a_{01} = a_{20} = a_{02} = 0$. Replacing these values into the expression of the polynomial, we get

$$p(x, y) = 1 - x + xy = I_{RC}(x, y),$$

i.e., we obtain the Reichenbach implication. Since it is a well-known fuzzy implication, it satisfies the remaining conditions of Theorem 1. Therefore, there exists only one fuzzy polynomial implication of degree 2.

Proposition 6. *There exists only one fuzzy polynomial implication of degree 2, the Reichenbach implication $I(x, y) = 1 - x + xy$.*

Note that the implication I_{RC} is an (S,N)-implication obtained using the method introduced in Example 1 as I_{S_P,N_1}. It is well-known that this implication is also a Yager's f-generated implication with $f(x) = 1 - x$ and so, it satisfies (LI) with T_P.

3.3 Degree 3

Finally, in this section, we will analyse the fuzzy polynomial implications of degree 3. These implications have the following expression

$$I(x, y) = \sum_{\substack{0 \leq i,j \leq 3 \\ i+j \leq 3}} a_{ij} x^i y^j$$

where $a_{ij} \in \mathbb{R}$ and there exist $0 \leq i, j \leq 3$ with $i + j = 3$ such that $a_{ij} \neq 0$. Corollary 1 in this case provides some relations between the different coefficients.

Corollary 2. *Let* $I(x,y) = \sum_{\substack{0\le i,j\le 3\\ i+j\le 3}} a_{ij}x^i y^j$ *be a fuzzy polynomial implication of degree 3. Then the following properties hold:*

- $a_{00} = 1$ *and* $a_{01} = a_{02} = a_{03} = a_{30} = 0$.
- $a_{12} = -a_{10} - a_{11}$.
- $a_{20} = -1 - a_{10}$.
- $a_{21} = 1 + a_{10}$.
- $a_{10} \neq -1$ *or* $a_{10} \neq -a_{11}$.

The previous result reduces the candidate polynomials to be the expression of a polynomial implication to

$$p(x,y) = 1 + a_{10}x + (-1-a_{10})x^2 + a_{11}xy + (1+a_{10})x^2y + (-a_{10}-a_{11})xy^2 \quad (1)$$

where $a_{10} \neq -1$ or $a_{10} \neq -a_{11}$. However, not all these polynomials satisfy the properties (iii)-(v) of Theorem 1 and therefore, not all are fuzzy implications. The next result fully characterizes all fuzzy polynomial implications of degree 3.

Theorem 2. *Let* $I : [0,1]^2 \to [0,1]$ *be a binary operator. Then* I *is a fuzzy polynomial implication of degree 3 if, and only if,* I *is given by*

$$I(x,y) = 1 + \alpha x + (-1-\alpha)x^2 + \beta xy + (1+\alpha)x^2y + (-\alpha-\beta)xy^2 \quad (2)$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq -1$, $\alpha \neq -\beta$, *and one of these cases hold:*

- $-2 \le \alpha \le -1$ *and* $-1-\alpha \le \beta \le 2$.
- $-1 < \alpha < 0$ *and* $0 \le \beta \le -2\alpha$.
- $\alpha = \beta = 0$.

At this stage, let us study some properties of these implications in order to determine after that, the class of fuzzy implications which these operations belong to.

Proposition 7. *Let* I *be a fuzzy polynomial implication of degree 3 given by Expression (2). Then the following statements are equivalent:*

- I *satisfies (EP).*
- I *satisfies (NP).*
- $\alpha = -\beta$ *with* $-2 \le \alpha \le 0$.

In this case, the implication I *is given by*

$$I(x,y) = 1 + \alpha x + (-1-\alpha)x^2 - \alpha xy + (1+\alpha)x^2y. \quad (3)$$

Since the most usual fuzzy implications satisfy (NP), there exist fuzzy polynomial implications of degree 3 which are neither (S,N), R, QL nor D-implications. For example, the following fuzzy polynomial implications do not satisfy (NP)

$$I_1(x,y) = 1 - 2x + x^2 + xy - x^2y + xy^2, \quad I_2(x,y) = \frac{1}{2}(2 - x - x^2 + x^2y + xy^2).$$

On the other hand, using Proposition 3, the fuzzy polynomial implications of degree 3 satisfying (EP) are (S,N)-implications obtained from the unique polynomial t-conorm S_P.

Theorem 3. *Let $I : [0,1]^2 \to [0,1]$ be a binary operator, S a t-conorm and N a fuzzy negation. Then the following assertions are equivalent:*

(i) I is a fuzzy polynomial implication of degree 3 and an (S,N)-implication obtained from S and N.
(ii) $S = S_P$ and $N(x) = 1 + \alpha x + (-1-\alpha)x^2$ with $-2 \leq \alpha \leq 0$.

Finally, and using the recent characterization of Yager's f-generated implications in [9], the next result determines which fuzzy polynomial implications of degree 3 are Yager's implications.

Theorem 4. *Let $I : [0,1]^2 \to [0,1]$ be a binary operator. Then the following assertions are equivalent:*

(i) I is a fuzzy polynomial implication of degree 3 and a Yager's f-generated implication.
(ii) I is a fuzzy polynomial implication of degree 3 satisfying (LI) with $T_P(x,y) = xy$.
(iii) $I(x,y) = 1 - x^2 + x^2y$, the f-generated implication with $f(x) = \sqrt{1-x}$.

As one might expect, the obtained implication belongs to the family constructed in Example 1 taking I_{S_P,N_2}.

4 Conclusions and Future Work

In this paper, we have started the study of fuzzy implications according to their final expression instead of the usual study on the construction methods of these operators using aggregation functions or generators. As a first step, we have studied the fuzzy polynomial implications, presenting some general results for polynomial implications of any degree and characterizing all fuzzy polynomial implications of degree less or equal to 3. The family of polynomial implications has a non-empty intersection with (S,N)-implications and f-generated implications, although there are also implications of this family which do not belong to any of the most usual families of implications. From the obtained results, some questions remain unanswered and must be tackled as future work. First,

Problem 1. Characterize all fuzzy polynomial implications of any degree.

For this purpose, it will be a vital requirement to determine which conditions on the coefficients of the polynomial imply the properties 3-5 of Theorem 1. Finally, from the results obtained in Proposition 6 and Theorem 3, we can conclude that all fuzzy polynomial implications of degree 2 or 3 which are also (S,N)-implications satisfy $S = S_P$. From this previous discussion, the next question emerges:

Problem 2. Is there any fuzzy polynomial implication which is also an (S,N)-implication obtained from a t-conorm $S \neq S_P$?

Finally, it would be interesting to check the advantages of using polynomial fuzzy implications instead of other implications in a concrete application in terms of computational cost saving and the reduction of the spreading of possible errors caused by numerical approximations of the inputs.

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