The Relational Construction of Conceptual Patterns - Tools, Implementation and Theory

Piero Pagliani

Research Group on Knowledge and Communication Models
pier.pagliani@gmail.com

Abstract. Different conceptual ways to analyse information are here defined by means of the fundamental notion of a relation. This approach makes it possible to compare different mathematical notions and tools used in qualitative data analysis. Moreover, since relations are representable by Boolean matrices, computing the conceptual-oriented operators is straightforward. Finally, the relational-based approach makes it possible to conceptually analyse not only sets but relations themselves.

1 Relations, Concepts and Information

The world is made of relations. In a sense, every entity or thing is nothing else but a sheaf of relations which occur with other sheaves of relations. Without offending Immanuel Kant, from this point of view the concept of a "monads" seems to be an expedient to bypass some philosophical problem. On the contrary, our point is inspired by another monumental assumption of Kant's philosophy: relations hold between entities, but entities themselves are phenomenological relations. That is, they are relations between *noumena* (not observed entities or events) and their manifestation through observed properties, which transforms *noumena* into *phenomena* (observable entities or events). In this way we arrive at the pair *intension-extension* which is at the very heart of conceptual data analysis.

Relations connect entities with their properties, in what we call an *observation system* or *property system* $\mathbf{P} = \langle U, M, R \rangle$, where U is the universe of entities, M the set of properties, and $R \subseteq U \times M$ is the "manifestation" relation so that $\langle g, m \rangle \in R$ means that entity g fulfills property m. From now on, instead of "entity" we shall use the term "object" in the sense of the German term *Gegenstand* which means an object before interpretation. The symbol M is after *Merkmal*, which means "property" or "characteristic feature". Relations in property systems induce derived relations between objects themselves or between properties. Indeed, sometimes phenomena can be perceived by directly observing relations occurring within objects or within properties. What is important is the coherence of the entire framework. Thus, relations assemble *concepts*, for instance by associating together those properties which are observed of a given set of objects. Vice-versa, relations assemble *extensions of concepts* by grouping together the objects which fulfill a given set of properties. In the former case an

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intension is derived from an extension. In the latter an extension is derived from an intension. Therefore, we shall call *intensional* the operators which transform extensions into intensions, and, vice-versa, *extensional* if the construction operates in the opposite direction. The former kind of operators will be decorated by an "*i*" and the latter by an "*e*". Obviously, mutual constructions are in order and we shall explore the properties of intensions of extensions and extensions of intensions.

As much as Category Theory advocates that what is relevant are the *morphisms* between elements, and not the elements standing alone, one can maintain that in conceptual data analysis ontological commitments should be avoided because the fundamental ingredients of the analysis are relations.

However, some ontological feature comes into the picture if U or M are equipped with some relational structure (for instance a preference relation). Anyway, this structure is in principle given by some intensional or extensional operator derivable from other property systems.

The paper will discuss some fundamental topics related to the pair *intension-extension* as defined by relations:

- The logical schemata which define the basic extensional and intensional operators and, thus, their meanings.
- How to use these operators in conceptual data analysis (in particular, in approximation analysis).
- How to compute the operators.
- How to implement the above procedures by manipulating Boolean matrices.

The theses and/or the proofs of the proved results are new.

2 Relations, Closures, Interiors and Modalities

The first natural step is collecting together the properties fulfilled by a set of objects, and the objects which fulfill a given set of properties:

Definition 1. *Given a property system* $\mathbf{P} = \langle U, M, R \rangle$*,* $A \subseteq U, B \subseteq M$*:*

$$\langle i \rangle(A) = \{m : \exists g(\langle g, m \rangle \in R \wedge g \in A)\} \tag{1}$$

$$\langle e \rangle(B) = \{g : \exists m(\langle g, m \rangle \in R \wedge m \in B)\} \tag{2}$$

We call these operators *constructors*. Some observations are in order:

Observation 1. In Relation Algebra these two constructors are well known and are denoted by $R(A)$ and, respectively, $R^{\smile}(B)$. The first is called *the left Peirce product of* R *and* A, while the second is the *right Peirce product of* R *and* B, or the left Peirce product of $R^{\smile}$ and B, where $R^{\smile} = \{\langle m, g \rangle : \langle g, m \rangle \in R\}$ is the *inverse relation* of R. Indeed $\langle e \rangle$ is the same as $\langle i \rangle$ applied to the inverse relation: $\langle e \rangle(B) = \{g : \exists m(\langle m, g \rangle \in R^{\smile} \wedge m \in B)\}$. In this way the quantified variable takes the first place in the ordered pair of the definition of $\langle e \rangle$, as it happens in the definition of $\langle i \rangle$. The role of a quantified variable in a relation is a formality which will be useful to compare different definitions. For any singleton $\{x\}$, instead of $R(\{x\})$ we shall write $R(x)$.

OBSERVATION 2. The definition of the constructor $\langle e\rangle$ is the same as that of the operator $\Diamond$ (*possibility*) in Modal Logic. In fact, a *Kripke model* is a triple $\langle W, R, \models\rangle$, where W is a set of *possible worlds*, $R \subseteq W \times W$ is an *accessibility relation*, and for any formula α, $w \models \Diamond(\alpha)$ if and only if there exists a possible world w' which is accessible to w and such that $w' \models \alpha$, that is, $\exists w'(\langle w, w'\rangle \in R \wedge w' \models \alpha)$. If in Definition 1 we set $M = U$ and identify a subset A of U with the domain of validity of a formula α (i. e. $A = \{g \in U : g \models \alpha\}$), and if we denote the modal operator by $\langle R\rangle$, we obtain $\langle R\rangle(A) = \langle e\rangle(A) = R^{\smile}(A)$ while $\langle i\rangle(A) = \langle R^{\smile}\rangle(A) = R(A)$. Indeed, with respect to our constructors, one has the following modal reading: if $g \in A$ then it is *possible* that g fulfills properties in $\langle i\rangle(A)$, because if $m \in \langle i\rangle(A)$, then $R^{\smile}(m)$ has non void intersection with A. Analogously, if $b \in B$ then it is *possible* that m is fulfilled by entities in $\langle e\rangle(B)$.

OBSERVATION 3. The logical structure of the definitions (1) and (2) is given by the combination $(\exists, \wedge)$. This is the logical core of a number of mathematical concepts. Apart from the above notion of "possibility" in Modal Logic, notably one finds it in the definition of a *closure* operator. Recalling that our framework is the Boolean lattice $\wp(U)$ or $\wp(M)$, we remind the following definitions:

Definition 2. *An operator ϕ on a lattice* $\mathbf{L}$ *is said to be a* closure *(resp.* interior*) operator if for any $x, y \in \mathbf{L}$ it is (i) increasing: $x \leq \phi(x)$ (resp. decreasing: $\phi(x) \leq x$), (ii) monotone: $x \leq y$ implies $\phi(x) \leq \phi(y)$, and (iii) idempotent: $\phi(\phi(x)) = \phi(x)$. Moreover, it is* topological *if it is (iv) additive: $\phi(x \vee y) = \phi(x) \vee \phi(y)$ (resp. multiplicative: $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$) and (v) normal: $\phi(0) = 0$ (resp, conormal: $\phi(1) = 1$).*

We call a property system such that $U = M$ a *square relational system*, SRS. Intuitively, in a SRS an object $g \in U$ is closed to a subset A of U with respect to R, if in A there exists a g' such that $\langle g, g'\rangle \in R$, so that g is linked to A in this way. We then call $R(g)$ the *R-neighborhood* of g and if $g' \in R(g)$ then g' will be called an *R-neighbor* of g. Thus g is closed to a set A if $R(g) \cap A \neq \emptyset$. The closure of A is the operation of embedding all the entities which are closed to A.

Now, we illustrate how simple is the computation of a closure. Any finite property system can be represented by a Boolean matrix such that the entry (g, m) is 1 if $\langle g, m\rangle \in R$, 0 otherwise. To compute $\langle i\rangle(\{x_1, x_2, ..., x_n\})$ one has just to collect the elements of U that display 1 in the rows $x_1, x_2, ..., x_n$. Vice-versa, to compute $\langle e\rangle(\{x_1, x_2, ..., x_n\})$ one has to collect the elements of U that display 1 in the columns $x_1, x_2, ..., x_n$. We shall denote the Boolean matrix corresponding to a relation R by $\mathbf{R}$. If $X \subseteq U$, $\mathbf{R}(\mathbf{X})$ shall denote the Boolean array corresponding to $R(X)$ and $\mathbf{R}^{\smile}(\mathbf{X})$ the array corresponding to $R^{\smile}(X)$. $\mathbf{R} \restriction \mathbf{X}$ is the matrix representing the subrelation $\{\langle x, y\rangle : x \in X \wedge y \in R(x)\}$ The example runs in a SRS, but with generic property systems the story is the same

EXAMPLE 1. $U = \{a, b, c, d\}$. Let us manipulate the subset $\{a, c\}$.

$\mathbf{R}$	a	b	c	d
a	1	0	0	1
b	0	1	1	1
c	0	1	0	0
d	0	1	0	1

$\mathbf{R} \restriction \{\mathbf{a},\mathbf{c}\}$	a	b	c	d
$\mathbf{a}$	1	0	0	1
b				
$\mathbf{c}$	0	1	0	0
d				

$\mathbf{R}^{\smile} \restriction \{\mathbf{a},\mathbf{c}\}$	$\mathbf{a}$	b	$\mathbf{c}$	d
a	1		0	
b	0		1	
c	0		0	
d	0		0	

$$\mathbf{R}(\mathbf{a},\mathbf{c}) = \overset{a\,b\,c\,d}{[1101]}$$
$$\langle i\rangle(\{a,c\}) = \{a,b,d\}$$

$$\mathbf{R}^{\smile}(\mathbf{a},\mathbf{c}) = \overset{a\,b\,c\,d}{[1100]}$$
$$\langle e\rangle(\{a,c\}) = \{a,b\}$$

Notice that $\mathbf{R}(\mathbf{X}) = \bigvee \mathbf{R}(\mathbf{x})_{x\in X}$, where $\vee$ is the element-wise Boolean sum of the arrays (for instance, $\mathbf{R}(\{\mathbf{a},\mathbf{c}\}) = \mathbf{R}(\mathbf{a}) \vee \mathbf{R}(\mathbf{c}) = [1\vee 0, 0\vee 1, 0\vee 0, 1\vee 0]$).

Since $\{g : g \in R^{\smile}(B)\} = \{g : R(g)\cap B \neq \emptyset\}$, $\langle e\rangle$ has a definition formally similar to that of a closure and, also, of an upper approximation operator. In fact, if R is an equivalence relation on U, $\langle e\rangle(X)$ is the upper approximation $(uR)(X)$ of Pawlak's Rough Set Theory. In turn, independently of the properties of R the *lower approximation* is defined as $(lR)(X) = \{x : R(x) \subseteq X\}$. That is, $(lR)(X) = \{x : \forall x'(\langle x,x'\rangle \in R \Longrightarrow x' \in X)\}$. Thus, we set in any property system $\mathbf{P} = \langle U,M,R\rangle, A \subseteq U, B \subseteq M$:

$$[i](A) = \{m : \forall g(\langle g,m\rangle \in R \Longrightarrow g \in A)\} \tag{3}$$
$$[e](B) = \{g : \forall m(\langle g,m\rangle \in R \Longrightarrow m \in B)\} \tag{4}$$

It is immediate to see that in a SRS, $(lR)(X) = [e](X)$.

Observation 4. As we shall see, if a set X is represented by a particular kind of relation (a *right cylinder*), then in Relation Algebra $[i](X)$ coincides with the *right residual* of R and X, while $[e](X)$ is the right residual of $R^{\smile}$ and X.

Observation 5. Again, one verifies a correspondence with Modal Logic. Given a Kripke model $\langle W,R,\models\rangle$ the forcing clause for a necessary formula $\Box(\alpha)$ is:

$$w \models \Box(\alpha) \;\; \textit{iff} \;\; \forall w'(\langle w,w'\rangle \in R \Longrightarrow w' \models \alpha) \tag{5}$$

Again, if $A = \{g : g \models \alpha\}$, then $[R](\alpha) = [e](A)$. Indeed, the modal reading of the above constructors is: in order to fulfill properties in $[i](A)$ it is necessary to be an object in A. Dually, in order to be fulfilled by objects in $[e](B)$, it is necessary to be a property of B.

Observation 6. The logical core of Definitions (3) and (4) is the combination $(\forall, \Longrightarrow)$. We remind that the logical core of the *possibility* operators is $(\exists, \wedge)$. To exploit these facts we need a strategic notion:

Definition 3. *Let* $\mathbf{O}$ *and* $\mathbf{O'}$ *be two preordered sets and* $\sigma : \mathbf{O} \longmapsto \mathbf{O}'$ *and* $\iota : \mathbf{O}' \longmapsto \mathbf{O}$ *be two maps such that for all* $p \in \mathbf{O}$ *and* $p' \in \mathbf{O}'$

$$\iota(p') \leq p \;\; \textit{iff} \;\; p' \leq' \sigma(p) \tag{6}$$

then σ *is called the* upper adjoint *of* ι *and* ι *is called the* lower adjoint *of* σ. *This fact is denoted by* $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$.

Now, in a Heyting algebra $\mathbf{H}$, $\wedge$ is lower adjoint to $\Longrightarrow$ in the sense that for all elements $x,y,z \in \mathbf{H}$, $\wedge_x(y) \leq z$ iff $y \leq \Longrightarrow_x (z)$, where $\wedge_x(y)$ is a parameterized formulation of $x \wedge y$ and $\Longrightarrow_x (z)$ of $x \Longrightarrow z$. Moreover, $\exists$ and $\forall$ are, respectively, lower and upper adjoints to the pre-image $f^{-1} : \wp(Y) \longmapsto \wp(X)$ of a function $f : X \mapsto Y$: for all $A \subseteq X, B \subseteq Y$ one has $\exists_f(A) \subseteq B$ iff $A \subseteq f^{-1}(B)$ and $B \subseteq \forall_f(A)$ iff $f^{-1}(B) \subseteq A$, where $\exists_f(A) = \{b \in B : \exists a(f(a) = b \wedge a \in A\}$ and $\forall_f(A) = \{b \in B : \forall a(f(a) = b \Longrightarrow a \in A\}$. Therefore, since i and e constructors operate in opposite directions, it is not surprise if it can be proved that the following adjointness properties hold in any property system $\mathbf{P} = \langle U, M, R \rangle$, for $\mathbf{M} = \langle \wp(M), \subseteq \rangle$ and $\mathbf{U} = \langle \wp(U), \subseteq \rangle$ (see [17]):

$$(a)\mathbf{M} \dashv^{\langle e \rangle, [i]} \mathbf{U} \quad (b)\mathbf{U} \dashv^{\langle i \rangle, [e]} \mathbf{M} \tag{7}$$

From this we immediately obtain that $\langle \cdot \rangle$ constructors are additive (as like as any lower adjoint), while $[\cdot]$ constructors are multiplicative (as any upper adjoint)[1]. Moreover, $\langle \cdot \rangle (A \cap B) \subseteq \langle \cdot \rangle (A) \cap \langle \cdot \rangle (B)$ and $[\cdot](A \cup B) \supseteq [\cdot](A) \cup [\cdot](B)$. Again, this is a consequence of adjointness, but can be easily verified using the distributive properties of quantifiers[2].

To compute the necessity constructors, the following result is exploited, which can be proved by means of the equivalences $\neg\exists \equiv \forall\neg$ and $\neg(A \wedge \neg B) \equiv A \Longrightarrow B$:

$$\forall X \subseteq U, [\cdot](X) = -\langle \cdot \rangle (-X) \tag{8}$$

Let us continue the previous example and compute $[\cdot](\{a, b\})$: $-\{a, b\} = \{c, d\}$.

$\mathbf{R}$	a	b	c	d
a	1	0	0	1
b	0	1	1	1
c	0	1	0	0
d	0	1	0	1

$\mathbf{R}^{\smile} \upharpoonright \{\mathbf{c}, \mathbf{d}\}$	a	b	$\mathbf{c}$	$\mathbf{d}$
a			0	1
b			1	1
c			0	0
d			0	1

$\mathbf{R} \upharpoonright \{\mathbf{c}, \mathbf{d}\}$	a	b	c	d
a				
b				
$\mathbf{c}$	0	1	0	0
$\mathbf{d}$	0	1	0	1

$$-\mathbf{R}^{\smile}(\mathbf{c}, \mathbf{d}) = \overset{a\,b\,c\,d}{[0010]}$$
$$[e](\{a, b\}) = \{c\}$$
$$-\mathbf{R}(\mathbf{c}, \mathbf{d}) = \overset{a\,b\,c\,d}{[1010]}$$
$$[i](\{a, b\}) = \{a, c\}$$

$-\mathbf{R}$ is the element-wise Boolean complement of the matrix $\mathbf{R}$.

3 Pretopologies, Topologies and Coincidence of Operators

It may sound surprising, but substantially the above procedures are all the machinery we need in order to compute the operators required by relation-based conceptual data analysis. Notice that the procedures to compute $\langle \cdot \rangle$ and $[\cdot]$ are independent of the properties of the relation R. On the contrary, the properties of these constructors strictly depend on those of R. For a generic binary relation R, $\langle \cdot \rangle$ may fail to be increasing or idempotent: $\{a, c\} \nsubseteq \langle i \rangle (\{a, c\})$ and $\langle i \rangle (\langle i \rangle (\{a, c\})) \neq \langle i \rangle (\{a, c\})$. Anyway, additivity gives monotonicity. In turn, notwithstanding the formal analogy with the definition of an interior operator, $[\cdot]$ may be neither decreasing nor idempotent: $[e](\{a, b\}) \nsubseteq \{a, b\}$ and

[1] Often, a lower adjoint is called "left adjoint" and an upper adjoint is called "right adjoint". We avoid the terms "right" and "left" because they could make confusion with the position of the arguments of the operations on binary relations.

[2] For instance one has $\forall x A(x) \vee \forall x B(x) \Longrightarrow \forall x (A(x) \vee B(x))$, but not the opposite. This proves that $\forall$ cannot have an upper adjoint, otherwise it should be additive.

$[e]([e](\{a,b\})) \neq [e](\{a,b\})$, although multiplicativity guarantees monotonicity. We shall see that given a generic R, $\langle\cdot\rangle$ and $[\cdot]$ behave like pretopological closure and, respectively, interior operators induced by neighborhood families which are lattice filters (with respect to $\subseteq$ and $\cap$). Actually, in real-world situations R is derived from observations (for instance data collected by sensors) and one cannot expect R to enjoy "nice properties" necessarily. Unfortunately, in view of the failure of the increasing property, $\langle\cdot\rangle$ cannot in general be used to compute any sort of upper approximation and $[\cdot]$ cannot provide any lower approximation because of the failure of the decreasing property. Thus, we need more structured operators. One approach is equipping R with particular properties. Modal Logic, then, tells us the new behaviors of the possibility and necessity operators. But one obtains very interesting operators if adjoint constructors are combined. Let us then set, for all $A \subseteq U, B \subseteq M$:

$$(a)\ \ int(A) = \langle e\rangle([i](A)) \qquad (b)\ cl(A) = [e](\langle i\rangle(A)). \tag{9}$$

$$(c)\ \ \mathcal{C}(B) = \langle i\rangle([e](B)) \qquad (d)\ \mathcal{A}(B) = [i](\langle e\rangle(B)). \tag{10}$$

Notice that int and cl map $\wp(U)$ on $\wp(U)$, while $\mathcal{A}$ and $\mathcal{C}$ map $\wp(M)$ on $\wp(M)$.

Observation 7.(see [17]) Since these operators are combinations of adjoint functors, they fulfill a number of properties: (i) int and $\mathcal{C}$ are interior operators; (ii) cl and $\mathcal{A}$ are closure operators. This means that $int(A) \subseteq A \subseteq cl(A)$, any $A \subseteq U$ and $\mathcal{C}(B) \subseteq B \subseteq \mathcal{A}(B)$, any $B \subseteq M$. Thus, they are veritable approximations. However, they are not topological: int and $\mathcal{C}$ are not multiplicative, because the external constructor $\langle\cdot\rangle$ is not, and cl and $\mathcal{A}$ are not additive, because the external constructor $[\cdot]$ is not.

The interpretation of the above operators is intuitive in property systems. $\mathcal{C}(B)$ displays those properties which are fulfilled by objects which fulfills at most properties in B. That is, first we select from U the objects which fulfill at most properties in B, then we check the properties from B which are effectively fulfilled by the selected objects. Dually, $\mathcal{A}(B)$ displays all the properties which are fulfilled only by the elements which fulfill some property in B. So to say, one selects the objects which fulfill at least one property of B, and, after that, the properties which are exclusively fulfilled by the selected objects. For the opposite direction just substitute "objects" for "properties" and "fulfill" for "fulfilled".

On the contrary, the interpretation for SRSs is not that clear. For instance, $\mathcal{C}(B)$ displays all and only the elements which are R-related to some element whose R-neighborhood is included in B. And so on. We do not go further into this interpretation, but just list in parallel the set-theoretic shapes of pair-wise related operators:

Modal constructors	Pre-topological operator
$[e](X) = \{x : R(x) \subseteq X\}$	$\mathcal{C}(X) = \bigcup\{R(x) : R(x) \subseteq X\}$
$[i](X) = \{x : R^{\smile}(x) \subseteq X\}$	$int(X) = \bigcup\{R^{\smile}(x) : R^{\smile}(x) \subseteq X\}$
$\langle e\rangle(X) = \{x : x \in R^{\smile}(X)\}$	$\mathcal{A}(X) = \{x : R^{\smile}(x) \subseteq R^{\smile}(X)\}$
$\langle i\rangle(X) = \{x : x \in R(X)\}$	$cl(X) = \{x : R(x) \subseteq R(X)\}$

In our Example 1, $\mathcal{C}(\{a,b\}) = \langle i\rangle(\{c\}) = \{b\}$ and $\mathcal{A}(\{a,d\}) = [i](\{a,b,d\}) = \{a,c,d\}$. Notice that $\mathcal{C}$ and $\mathcal{A}$ are dual, that is, $\mathcal{C}(X) = -\mathcal{A}(-X)$. This is due to the fact that $\langle i\rangle$ and $[i]$ are dual. Moreover, $\mathcal{A}(X) = \{x : \forall y(x \in R(y) \Longrightarrow R(y) \cap X \neq \emptyset)\}$ and $cl(X) = \{x : \forall y(x \in R^{\smile}(y) \Longrightarrow R^{\smile}(y) \cap X \neq \emptyset)\}$.

Now we want to understand when the operators on the right column are equivalent to the corresponding constructors on the left column. We shall deal with *int* and $[i]$. But first an interesting brief excursus is in order.

Excursus: coverings. Generalisations of Pawlak's approximation operators have been introduced that are based on coverings instead of partitions. The relational machinery can simplify this approach. Given a set U, a *covering* is a family $C \subseteq \wp(U)$ such that $\bigcup C = U$. If one assigns all the elements of a component K of C to a unique element m from a set M, through a relation R, one obtains a property system $\langle U, M, R\rangle$ such that $\{R^{\smile}(m) : m \in M\} = C$. It is straightforward to prove that the operator $\underline{C_1}(X) = \bigcup\{K \in C : K \subseteq X\}$ introduced in [24], coincides with *int*. Therefore, its dual operator $\overline{C_1}$ coincides with *cl* and all the properties of $\underline{C_1}$ and $\overline{C_1}$ are provided for free by the adjunction properties. For further considerations see the final remarks.

Let us came back to our goal. We start with noticing that in view of Observation 7, $[i]$ must be an interior operator, in order to coincide with *int*. Moreover, it must be topological because of multiplicativity (similarly, if $\langle i\rangle$ is a closure operator, it is necessarily topological because of additivity). Indeed, we now prove that $[i] = int$ if and only if R is a preorder. It is well-known that in this case Kripkean necessity modalities are topological interior operators. However, if part of the result is well-known, the proof will be developed in a novel way which provides relevant information about the operators.

The proof is made of two parts. In the first part we prove that R must be a preorder to make *int* and $[i]$ coincide. After that, we complete the proof in a more specific manner: it will be proved that if R is a preorder, then $\mathcal{C}$ (thus $[e]$) is the interior operator of a particular topology induced by R.

Lemma 1. *Let $\langle U, U, R\rangle$ be a SRS. Then $\forall x \in U, x \in [i](R^{\smile}(x))$.*

Proof. Trivially, $x \in [i](R^{\smile}(x))$ iff $R^{\smile}(x) \subseteq R^{\smile}(x)$.

Theorem 1. *Let $\langle U, U, R\rangle$ be a SRS. Then for all $A \subseteq U, int(A) = [i](A)$ if and only if R is a preorder (i.e. R is reflexive and transitive).*

Proof. A) If $\exists A \subseteq U, int(A) \neq [i](A)$ then R is not a preorder (either reflexivity or transitivity fail). Proof. The antecedent holds in two cases: (i) $\exists x \in [i](A), x \notin int(A)$; (ii) $\exists x \in int(A), x \notin [i](A)$. In case (i) $\forall y \in [i](A), x \notin R^{\smile}(y)$. In particular, $x \notin R^{\smile}(x)$, so that reflexivity fails. In case (ii) $\exists y \in [i](A)$ such that $x \in R^{\smile}(y)$. Therefore, since $\langle x, y\rangle \in R$ and $y \in [i](A)$, x must belong to A. Moreover, it must exists $z \notin A, \langle z, x\rangle \in R$, otherwise $x \in [i](A)$. If R were transitive, $\langle z, y\rangle \in R$, so that $y \notin [i](A)$. Contradiction.

B) If R is not a preorder, then $\exists A \subseteq U, int(A) \neq [i](A)$. Proof. (i) Take $A = R^{\smile}(x)$. From Lemma 1, $x \in [i](R^{\smile}(x))$. Suppose R is not reflexive with $\langle x, x\rangle \notin R$. Thus $x \notin R^{\smile}(x)$. Hence, it cannot exists an y such that $x \in R^{\smile}(y)$ and $R^{\smile}(y) \subseteq R^{\smile}(x)$. So, $x \notin int(R^{\smile}(x))$. (ii) Suppose transitivity fails, with $\langle x, y\rangle, \langle y, z\rangle \in R, \langle x, z\rangle \notin R$. From Lemma 1, $z \in [i](R^{\smile}(z))$, but $y \notin [i](R^{\smile}(z))$, because $x \in R^{\smile}(y)$ while $x \notin R^{\smile}(z)$ so that $R^{\smile}(y) \nsubseteq R^{\smile}(z)$. On the contrary, $y \in R^{\smile}(z)$ and $R^{\smile}(z) \subseteq R^{\smile}(z)$. Therefore, $y \in int(R^{\smile}(z))$. We conclude that $int(R^{\smile}(z)) \neq [i](R^{\smile}(z))$.

Corollary 1. *In a SRS $\langle U, U, R\rangle$, the following are equivalent: (i) R is a preorder, (ii) $\mathcal{C} = [e]$, (iii) $int = [i]$, (iv) $int, [i], [e]$ and $\mathcal{C}$ are topological interior operators.*

We recall that a topological interior operator $\mathcal{I}$ on a set X induces a *specialisation preorder* defined as follows: $\forall x, y \in X, x \preceq y$ iff $\forall A \subseteq X, x \in \mathcal{I}(A)$ implies $y \in \mathcal{I}(A)$. However, in what follows we extend this definition to any monadic operator on sets. If $R \subseteq X \times X$ is a preorder, then the topology with bases the family $\{R(x) : x \in X\}$ is called the *Alexandrov topology induced by R*. The specialisation preorder induced by such a topology coincides with R itself.

Lemma 2. *If $R \subseteq X \times X$ is transitive, then $\forall x, y \in X, \langle x, y\rangle \in R$ implies $R(y) \subseteq R(x)$. If R is reflexive, then $R(y) \subseteq R(x)$ implies $\langle x, y\rangle \in R$.*

Proof. Suppose $\langle x, y\rangle \in R$ and $z \in R(y)$. Then $\langle y, z\rangle \in R$ and by transitivity $\langle x, z\rangle \in R$ so that $z \in R(x)$. Thus, $R(y) \subseteq R(x)$. Vice-versa, if $R(y) \subseteq R(x)$ then for all $z, \langle y, z\rangle \in R$ implies $\langle x, z\rangle \in R$. In particular $\langle y, y\rangle \in R$ by reflexivity. Hence $\langle x, y\rangle \in R$.

Theorem 2. *Let $\langle U, U, R\rangle$ be a SRS such that R is preorder. Then the specialization preorder induced by $[i]$ coincides with $R^{\smile}$ and that induced by $[e]$ coincides with R.*

Proof. If $x \preceq y$ then for all $A \subseteq X, x \in [i](A)$ implies $y \in [i](A)$. Therefore, $R^{\smile}(x) \subseteq A$ implies $R^{\smile}(y) \subseteq A$, all A. In particular, $R^{\smile}(x) \subseteq R^{\smile}(x)$ implies $R^{\smile}(y) \subseteq R^{\smile}(x)$. But the antecedent is true, so the consequence must be true, too, so that $R^{\smile}(y) \subseteq R^{\smile}(x)$. Since R is reflexive, so is $R^{\smile}$ and from Lemma 2, $\langle x, y\rangle \in R^{\smile}$. The opposite implication is proved analogously by transitivity. The thesis for $[e]$ and R is a trivial consequence.

Corollary 2. *Let $\mathcal{C}$ be a topological interior operator induced by a SRS $\langle U, U, R\rangle$. Then $\mathcal{C}$ is the interior operator of the Alexandrov topology induced by R.*

Proof. If $\mathcal{C}$ is a topological interior operator, then from Corollary 1, R is a preorder and $\mathcal{C} = [e]$. Therefore, from Theorem 2, the specialisation preorder induced by $\mathcal{C}$ coincides with R which, in turn, coincides with the specialisation preorder of the Alexandrov topology induced by R.

Obviously, if R is symmetric (as for equivalence relations, thus in Pawlak Rough Set Theory), then $R = R^{\smile}$, with all the simplifications due to this fact.

4 Approximation by Means of Neighborhoods

Let us now see the relationships between the operators so far discussed and those induced by neighborhoods. Consider a relational structure $\mathbf{N} = \langle U, \wp(U), R\rangle$, with $R \subseteq U \times \wp(U)$. We call it a *relational neighborhood structure*. If $u' \in N \in R(u)$, we say that u' is a *neighbor* and N a *neighborhood* of u. We set $\mathcal{N}_u = R(u)$ and call it the *neighborhood family* of u. The family $\mathcal{N}(U) = \{\mathcal{N}_u : u \in U\}$ is called a *neighborhood system*. Let us define on $\wp(U)$:

$$\text{(a) } G(X) = \{u : X \in \mathcal{N}_u\}; \text{ (b) } (X) = -G(-X) = \{u : -X \notin \mathcal{N}_u\}.$$

Consider the following conditions on $\mathcal{N}(U)$, for any $x \in U$, $A, N, N' \subseteq U$:
1: $U \in \mathcal{N}_x$; **0**: $\emptyset \notin \mathcal{N}_x$; **Id**: if $x \in G(A)$ then $G(A) \in \mathcal{N}_x$;
N1: $x \in N$, for all $N \in \mathcal{N}_x$; **N2**: if $N \in \mathcal{N}_x$ and $N \subseteq N'$, then $N' \in \mathcal{N}_x$;
N3: if $N, N' \in \mathcal{N}_x$, then $N \cap N' \in \mathcal{N}_x$. **N4**: $\exists N, \mathcal{N}_x = \uparrow N = \{N' : N \subseteq N'\}$.
They induce the following properties of the operators G and F (see [14] or [17]):

Condition	Equivalent properties of G	Equivalent properties of F
1	$G(U) = U$	$F(\emptyset) = \emptyset$
0	$G(\emptyset) = \emptyset$	$F(U) = U$
Id	$G(X) \subseteq G(G(X))$	$F(F(X)) \subseteq F(X)$
N1	$G(X) \subseteq X$	$X \subseteq F(X)$
N2	$X \subseteq Y \Rightarrow G(X) \subseteq G(Y)$ $G(X \cap Y) \subseteq G(X) \cap G(Y)$	$X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$ $F(X \cup Y) \supseteq F(X) \cup F(Y)$
N3	$G(X \cap Y) \supseteq G(X) \cap G(Y)$	$F(X \cup Y) \subseteq F(X) \cup F(Y)$

A neighborhood system can be defined by means of a property system $\mathbf{P} = \langle U, M, R\rangle$, in different ways. For instance by setting, for all $g \in U, \mathcal{N}_g = \{R^{\smile}(m) : m \in R(g)\}$. The properties of the operators G and F induced by such neighborhood systems will be studied in another paper. Here we just notice that the philosophy behind this choice is intuitive: we consider neighbors the elements which fulfill the same property. Hence, the extension of a property is a neighborhood, so that a neighborhood family $\mathcal{N}_g$ groups the neighborhoods determined by the properties fulfilled by g. However, in application contexts in which SRSs are involved, it is natural to consider $R(x)$ as the basic neighborhood of x. In [14] (see also [17]), families $\{R_i\}_{i\in I}$ of relations on the same domain are considered, so that one gathers neighborhood families by setting $\mathcal{N}_x = \{R_i(x)\}_{i\in I}$.

If one deals with just one SRS, an obvious way to obtain a neighborhood family is setting $\mathcal{N}^R_x = \uparrow R(x) = \{A : R(x) \subseteq A\}$. The family $\mathcal{N}_{F(R)}(U) = \{\mathcal{N}^R_x : x \in U\}$ will be called *principal neighborhood system generated by* R, by analogy with "principal filter", or *R-neighborhood system*, briefly. We now prove that the operator G induced by an R-neighborhood system coincides with the operator $[e]$ induced by R itself.

Theorem 3. *Let* $\mathbf{P} = \langle U, U, R\rangle$ *be a SRS and* $\mathcal{N}_{F(R)}(U)$ *its* R*-neighborhood system. Let* G *be the operator induced by* $\mathcal{N}_{F(R)}(U)$ *and* $[e]$ *the constructor defined by* $\mathbf{P}$*. Then for all* $A \subseteq U, G(A) = [e](A)$*.*

Proof. By definition, in $\mathcal{N}_{F(R)}(U)$ **N4** holds. In any neighborhood system with this property, if $\mathcal{N}_x = \uparrow Z_x$, then $x \in G(A)$ iff $Z_x \subseteq A$, because in this case $A \in \mathcal{N}_x$, too. But in $\mathcal{N}_{F(R)}(U)$, $Z_x = R(x)$. Hence, $G(A) = \{x : R(x) \subseteq A\} = [e](A)$.

An alternative proof runs as follows if R is a preorder:

Lemma 3. *Let* $\mathbf{P} = \langle U, U, R\rangle$ *be a SRS such that* R *is a preorder, and let* $\mathcal{N}_{F(R)}(U)$ *be its* R*-neighborhood system. Let* $\preceq$ *be the specialisation preorder induced by* G*. Then* $\preceq$ *coincides with* R*.*

Proof. If $x \preceq y$, then $\forall A \subseteq U, x \in G(A)$ implies $y \in G(A)$, so that $A \in \mathcal{N}^R_x$ implies $A \in \mathcal{N}^R_y$. In particular, $R(x) \in \mathcal{N}^R_x$. Hence, $R(x) \in \mathcal{N}^R_y$, so that $R(y) \subseteq R(x)$, because $\mathcal{N}^R_y = \uparrow R(y)$. Since R is a preorder, from Lemma 2 $\langle x, y\rangle \in R$. We omit the obvious reverse implication.

Then from Lemma 3 and Theorem 2 one obtains Theorem 3. Notice, however, that Theorem 3 holds independently of the properties of R. One can verify that in Example 1, $\{a, b\}$ belongs just to $\mathcal{N}^R_c = \uparrow \{b\}$. Hence, $G(\{a, b\}) = \{c\} = [e](\{a, b\})$. On the contrary, $\mathcal{C}(\{a, b\}) = \{b\}$. In fact, R is not a preorder ($\langle c, b\rangle, \langle b, c\rangle \in R$ but $\langle c, c\rangle \notin R$).

To understand the role of the properties of a neighborhood system, as to idempotence we notice that **Id** does not hold in the R-neighborhood system of Example 1: $c \in G(\{a, b\}) = \{c\}$, but $\{c\} \notin \mathcal{N}^R_c = \uparrow \{b\}$. As to deflation, notice that **N1** does not hold: $\{b\} \in \mathcal{N}^R_c$ but $c \notin \{b\}$.

Coming back to relational neighborhood systems one can notice that $\mathbf{N} = \langle U, \wp(U), R\rangle$ is a property system. So it is possible to define *int* and *cl*. What are the relations between the operator G defined on $\mathbf{N}$ *qua* relational neighborhood structure, and the operators *int* and $\mathcal{C}$ defined on $\mathbf{N}$ *qua* property system?

Lemma 4. *(see [17]) Let* $\mathbf{N} = \langle U, \wp(U), R\rangle$ *be a relational neighborhood structure. For all* $X \subseteq U, G(X) = R^{\smile}(\{X\})$*.*

Proof. $G(X) = \{x : X \in \mathcal{N}_x\}$. But $X \in \mathcal{N}_x$ iff $\langle x, X\rangle \in R$ iff $x \in R^{\smile}(\{X\})$. Hence, $G(X) = \{x : x \in R^{\smile}(\{X\})\} = R^{\smile}(\{X\})$.

Theorem 4. *Let* $\mathbf{N} = \langle U, \wp(U), R\rangle$ *induce a neighborhood system such that* **Id**, **N1** *and* **N2** *hold. Then for any* $A \subseteq U, int(A) = G(A)$*.*

Proof. If $x \in int(A)$, $x \in \langle e\rangle(\{X : R^{\smile}(\{X\}) \subseteq A\})$. Thus $x \in \bigcup\{R^{\smile}(\{X\}) : R^{\smile}(\{X\}) \subseteq A\} = \bigcup\{G(X) : G(X) \subseteq A\}$. Now we prove that $\bigcup\{G(X) : G(X) \subseteq A\} = G(A)$, provided the three conditions of the hypothesis hold. Let $x \in \bigcup\{G(X) : G(X) \subseteq A\}$. Then $x \in G(N)$ for some $N \subseteq U$. Since **Id** holds, $G(N) \in \mathcal{N}_x$. From **N2**, $A \in \mathcal{N}_x$, too, so that $x \in G(A)$. Vice-versa, suppose $x \in G(A)$. But $G(A) \subseteq A$, because **N1** holds. We conclude that $x \in \bigcup\{G(X) : G(X) \subseteq A\}$.

Actually, this is a simplified proof of Lemma 15.14.4 of [17].

5 The Full Relational Environment

We have mentioned that given a SRS $\langle U, U, R\rangle$ if a subset $X \subseteq U$ is represented as a particular relation, then the entire computational machinery can be embedded in the Algebra of Relations.

Definition 4. *A* full algebra of binary relations *over a set U, is an algebra*

$$fullREL(U) = (\wp(U \times U), \cup, \cap, -, \mathbf{1}, \otimes, ^{\smile}, \mathbf{1}')$$

where $(\wp(U \times U), \cup, \cap, -, \mathbf{1})$ is a Boolean algebra of sets, $\otimes$ is the relational composition, $^{\smile}$ is the inverse and $\mathbf{1}'$ is the identity relation.

Clearly, all elements of $\wp(U \times U)$ are binary relations. The unit $\mathbf{1}'$ is represented by the identity matrix, where the element at row i - column j is 1 if and only if $i = j$. Let $R, S \in \wp(U \times U)$, the composition is defined as follows:
$R \otimes S = \{\langle x, y\rangle \in U \times U : \exists z(\langle x, z\rangle \in R \text{ and } \langle z, y\rangle \in S)\}$.
Composition is simply the Boolean multiplication of matrices. Thus to obtain $\mathbf{R} \otimes \mathbf{S}$ we multiply pointwise row i with column j; if the pointwise Boolean multiplication gives 1 for at least one point, then element at row i and column j of $\mathbf{R} \otimes \mathbf{S}$ is 1. It is 0 otherwise.

EXAMPLE 2. Let $U = \{a, b, c, d\}$.

R	a	b	c	d
a	1	1	1	1
b	0	1	1	0
c	0	0	1	0
d	0	0	0	1

S	a	b	c	d
a	1	1	1	0
b	0	0	0	1
c	0	1	0	0
d	1	0	0	1

$\mathbf{R} \otimes \mathbf{S}$	a	b	c	d
a	1	1	1	1
b	0	1	0	1
c	0	1	0	0
d	1	0	0	1

To compute, for instance, the element at row c column b of $\mathbf{R} \otimes \mathbf{S}$, first we take row $\mathbf{R}(\mathbf{c})$, [0010], and column $\mathbf{S}^{\smile}(\mathbf{b})$, [1010]. Then we apply component-wise the logical multiplication to these two Boolean arrays obtaining [0010]. Finally we apply the logical summation to the resulting array and obtain 1.

We have enough instruments to introduce two fundamental operations. We define them on arbitrary binary relations. Assume $R : W \times W'$ and $S : U \times U'$.

$$R \longrightarrow S = -(R^{\smile} \otimes -S), \text{ right residuation of } S \text{ with respect to } R. \quad (11)$$

$$S \longleftarrow R = -(-S \otimes R^{\smile}), \text{ left residuation of } S \text{ with respect to } R. \quad (12)$$

The operation (11) is defined only if $|W| = |U|$; (12) is defined only if $|W'| = |U'|$. In particular, if R and S are binary relations on a set U, then (see [17]):

$$R \longrightarrow S = \{\langle a, b\rangle \in U \times U : \forall c \in U(\langle c, a\rangle \in R \Longrightarrow \langle c, b\rangle \in S)\} \quad (13)$$

$$S \longleftarrow R = \{\langle a, b\rangle \in U \times U : \forall c \in U(\langle b, c\rangle \in R \Longrightarrow \langle a, c\rangle \in S)\} \quad (14)$$

It can be shown that $R \longrightarrow S$, is the largest relation Z on U such that $R \otimes Z \subseteq S$, while $S \longleftarrow R$, is the largest relation Z such that $Z \otimes R \subseteq S$.

In [11] (see also [17]), it is possible to see how useful these operations are to compute and analyse, for instance, dependency relations between properties or choices based upon a set of properties. In this paper we want just to show how to use them to compute the operators $[\cdot]$, $\langle\cdot\rangle$.

First, we remind that a set $A \subseteq U$ may be represented by a *right cylinder* $A^c = A \times U = \{\langle a, x\rangle : a \in A, x \in U\}$. Its matrix $\mathbf{A^c}$ has dimensions $|U| \times |U|$ and $\mathbf{R}(\mathbf{a}) = [1, 1, ..., 1]$ only if $a \in A$, otherwise $\mathbf{R}(\mathbf{a}) = [0, 0, ..., 0]$. In this way sets turn into elements of full algebras of binary relations.

Let us now reconsider definitions (1), (2), (3) and (4). Since $-C$ and $R \otimes C$ output right cylinders whenever C is a right cylinder, we can turn any element x of the sets which appear in the definitions into a pair $\langle x, z\rangle$, where z is a dummy variable representing any element of U. For instance, $\{m : ...\}$, turns into $\{\langle m, z\rangle ...\}$ and $g \in X$ turns into $\langle g, z\rangle \in X^C$. In this way one obtains:

$$\langle i\rangle(X^c) = \{\langle m, z\rangle : \exists g(\langle g, m\rangle \in R \wedge \langle g, z\rangle \in X^c)\} = R^{\smile} \otimes X^c \quad (15)$$

$$\langle e\rangle(X^c) = \{\langle g, z\rangle : \exists m(\langle g, m\rangle \in R \wedge \langle m, z\rangle \in X^c)\} = R \otimes X^c \quad (16)$$

$$[i](X^c) = \{\langle m, z\rangle : \forall g(\langle g, m\rangle \in R \Longrightarrow \langle g, z\rangle \in X^c)\} = R \longrightarrow X^c \quad (17)$$

$$[e](X^c) = \{\langle g, z\rangle : \forall m(\langle g, m\rangle \in R \Longrightarrow \langle m, z\rangle \in X^c)\} = R^{\smile} \longrightarrow X^c \quad (18)$$

After that, we can compute for instance $[e](\{a, b\})$ in Example 2 using (11):
$[e](\{a, b\}^c) = -((\mathbf{R}^{\smile})^{\smile} \otimes -\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}}) = -(\mathbf{R} \otimes -\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}})$:

$\mathbf{R}$	a	b	c	d
a	1	0	0	1
b	0	1	1	1
c	0	1	0	0
d	0	1	0	1

$\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}}$	a	b	c	d
a	**1**	**1**	**1**	**1**
b	**1**	**1**	**1**	**1**
c	0	0	0	0
d	0	0	0	0

$-\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}}$	a	b	c	d
a	0	0	0	0
b	0	0	0	0
c	**1**	**1**	**1**	**1**
d	**1**	**1**	**1**	**1**

$\mathbf{R} \otimes -\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}}$	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	0	0	0
d	1	1	1	1

$-(\mathbf{R} \otimes -\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}})$	a	b	c	d
a	0	0	0	0
b	0	0	0	0
c	**1**	**1**	**1**	**1**
d	0	0	0	0

$(-(\mathbf{R} \otimes -\{\mathbf{a}, \mathbf{b}\}^{\mathbf{c}}))^{\smile}(U) = \{c\}$
To obtain $[e](\{a, b\})$, one applies the right Pierce product of the resulting relation to U (or any subset of U).

Relations and manipulations of relations provide all the ingredients for qualitative data analysis: the operations to perform the analysis and the object to be analysed. Central to this task are the implications (residuations) between relations and the composition of relations, which make it possible to define the basic analytic tools. Moreover, the objects to be analysed can be relations themselves. Indeed, right cylinders are just particular instances of relations. For some additional considerations see the next section.

6 Final Remarks and Bibliographic Notes

The bibliography on Modal Logic, Intuitionistic Logic, Adjointness, Kripke models, and Topology is huge. Thus we prefer to address the reader to the comprehensive bibliography and historical notes that can be found in [17].

The basic constructors have been introduced in different fields. We were inspired by the works on formal topology by G. Sambin (see for instance [20]). Together with the sufficiency constructors they have been analysed in [6] and in the context of property systems and neighborhood systems in ([13]). Moreover in [5] two of them where used to define "property oriented concepts", while in [22] the other two have been used to define "object oriented concepts". Eventually, they were fully used in approximation theory in [16]. In the present paper, however, we have not considered the *sufficiency constructors* which are used in R. Wille's Formal Concept Analysis. They are obtained by swapping the positions in the implicative parts of the $[\cdot]$ constructors. Since the $[\cdot]$ and $\langle\cdot\rangle$ constructors form a square of duality, application direction, isomorphisms and adjointness (see for instance [17]), by adding the sufficiency operators one enters into the cube of oppositions discussed in [3].

A survey on covering-based approximation operators is [23]. In [19] these operators are studied from the point of view of duality and adjoint pairs. In [15] twenty one covering-based approximation operators are interpreted exclusively by means of the four basic constructors. In this way duality and adjointness properties are immediate consequences of the properties of the constructors. Moreover, in view of the clear logical and topological meaning of the four constructors, the meaning of these approximation operators is explained as well.

Finally, the operators introduced in Section 5 have been extended to deal with conceptual patterns within *multi-adjoint formal contexts* in [4] (see also [10] and [2]). This is a promising generalization which on one side is linked to the problem of multi property systems (see [14] and [9] for a first look at the topic), and on the other side to the problem of approximation of relations, which was introduced in [21] and solved for the case of two relations in [12], together with a comparison of rough sets and formal concepts developed within relation algebra (see also [17], Chap. 15.18). Our approach, however, was inspired by the notions of a *weakest pre-specification* and a *weakest post-specification* introduced in [7], by Lambek Calculus and Non Commutative Linear Logic (see [1]).

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