Orthopairs in the 1960s: Historical Remarks and New Ideas

Davide Ciucci

Dipartimento di Informatica, Sistemistica e Comunicazione
Università di Milano – Bicocca
Viale Sarca 336 – U14, I–20126 Milano, Italia
ciucci@disco.unimib.it

Abstract. Before the advent of fuzzy and rough sets, some authors in the 1960s studied three-valued logics and pairs of sets with a meaning similar to those we can encounter nowadays in modern theories such as rough sets, decision theory and granular computing. We revise these studies using the modern terminology and making reference to the present literature. Finally, we put forward some future directions of investigation.

1 Introduction

An orthopair on a universe U is a pair of subsets $A, B \subseteq U$ such that $A \cap B = \emptyset$. From a purely set-theoretical standpoint an orthopair is equivalent to a pair of nested sets $A \subseteq C$ once defined $C := B^c$ ($\cdot^c$ denoting the set complement with respect to the universe U). Clearly, an orthopair partitions the universe in three sets: A, B and $(A \cup B)^c$. So, a bijection between orthopairs and three-valued sets can be established. Given an orthopair (A, B), a three-valued set $f : U \mapsto \{0, \frac{1}{2}, 1\}$ can be defined as $f(x) = 1$ if $x \in A$, $f(x) = 0$ if $x \in B$ and $f(x) = \frac{1}{2}$ otherwise[1]. Vice versa, from a three-valued set f, an orthopair can be defined as the inverse of the previous mapping. It follows that we can equivalently study orthopairs, nested pairs or three-valued sets and in the following we will mix these three approaches keeping in mind their syntactical equivalence. Let us also notice that this tri-partition directly points to three-way decision [33] whose aim is to partition in three a universe and to the theory of opposition, in particular to a hexagon of oppositions, that can be naturally defined from a tri-partition [12,17].

Several interpretations can be attached to the two sets A, B, here is some example:

- They can be a set of true and false propositional variables;
- They can be a set of examples and counterexamples;
- A can represent the elements surely belonging to a given concept and B those surely not belonging;

[1] This translation reflects somehow the semantics usually assigned to orthopairs. Without giving any semantics to the two sets A, B and to the three-valued sets, any permutation in the assigned values defines a bijection.

C. Cornelis et al. (eds.): RSCTC 2014, LNAI 8536, pp. 1–12, 2014.

– A can represent the elements surely belonging to a given concept and B can represent a region of uncertainty: the elements about which either we cannot decide on their (Boolean) belongingness or that belong to a certain extent.

The last point emphasize a difference in the interpretation of the no-certainty zone: *unknown*, an epistemic notion, and *borderline*, an ontic notion. This difference is reflected to a greater extent by the third truth value ($\frac{1}{2}$) meaning: under the epistemic view it can stand for unknown, possible, inconsistent, etc. and under the ontic one, it can be interpreted as borderline (or half-true), undefined (in the sense of Kleene [23], that is outside the definition domain of a given function), irrelevant, etc. For reasoning purposes, it can be argued that three-valued truth-functional logics are suited in the ontic case whereas in the epistemic one non-truth functional, such as modal logic or possibility theory, are better placed [15].

Orthopairs (and nested pairs) appear in different contexts such as rough sets and ill-known sets, three-way decision, three-valued logic, shadowed sets, etc... They have been studied from a logical-algebraic standpoint by several authors and also in general contexts where the underlying structure is weaker than a Boolean algebra (for an overview see [8] and further historical remarks can be found in [28, Frame 10.11]). In Section 2, we will revise some of the operations that can be defined on these structures. For a more complete study on operations we refer to [8,24,11] for orthopairs, an overview on three values is given in [10] and about the relationship between orthopairs and three-valued operations see [13].

The aim of the paper is two-fold, as the title suggests. From one side, it presents an overview of some papers published before the coming of fuzzy sets and rough sets, which contain several ideas about concepts that will be developed in the following years. Then, we show that some ideas contained in those papers are still innovative and can give some insight to the paradigms connected with orthopairs. In particular, we will take into account the works by Fadini [19,18], Andreoli [1,2,3] and the work by Gentilhomme [21]. Only the last one is to some extent known in the fuzzy logic community and cited by some authors[2], the first two are almost unknown. On the other hand, we remark that it is out of the scope of the present work to survey all the operations existing in rough set theory, for this aspect we refer the interested reader to [4,5,13].

2 Preliminary Notions

As said in the introduction, several interpretations to three-valued sets can be given and this reflects also on the different operations that we are entitled to introduce. The same situation applies to orthopairs and nested pairs. As we will see in Section 4, according to which interpretation or representation we use, different operations naturally arise.

[2] The paper [21] has 95 citations on Google Scholar at 23 April 2014.

Indeed, let us consider a concept A that we want to describe and a pair of sets (A_1, A_2) that we use to effectively represent it. Usually, A_1 represents the elements that *surely belong* to A. This corresponds in rough set theory to the lower approximation and so, from now on, we will denote it as A_l. The second set can have three (mutually definable) interpretations that we are going to use through the paper:

- A_2 contains the objects *possibly belonging* to A, that is, it correponds to the upper approximation and we denote it as A_u.
- A_2 contains the objects *surely not belonging* to concept A, that is, the exterior region. Thus, we are going to denote it as A_e.
- A_2 represents the objects on which we are *undecided* about their belongingness to A, this is the boundary region and it will be named A_{bnd}.

Needless to stress (A_l, A_u) is a nested pair, whereas (A_l, A_e) and (A_l, A_{bnd}) are orthopairs. We are not going to consider here the case where the first element A_1 is different from A_l, which will generate other orthopairs, such as (A_u, A_e) and (A_e, A_{Bnd}). From now on, A and B will stand for orthopairs $(A_l, A_e), (B_l, B_e)$ or nested pairs $(A_l, A_u), (B_l, B_u)$.

We now introduce some operations and order relations that have been introduced on ortho (nested) pairs. This is not an exhaustive list, but a presentation of some operations already known in literature that are helpful for the following discussion. Other operations can be found in [8,25,24,11,10,14].

Let us first consider unary operations, usually meant to model a negation. We have on three values the involutive, the intuitionistic and the paraconsistent negations which extend the Boolean negation and thus differ only on the negation of the third-truth value, respectively defined as $\frac{1}{2}' = \frac{1}{2}$, $\sim \frac{1}{2} = 0$ and $-\frac{1}{2} = 1$. Once translated to orthopairs and nested pairs these negations are defined as follows:

$$(A_l, A_e)' := (A_e, A_l) \qquad (A_l, A_u)' := (A_u^c, A_l^c) \tag{1}$$

$$\sim (A_l, A_e) := (A_e, A_e^c) \qquad \sim (A_l, A_u) := (A_u^c, A_u^c) \tag{2}$$

$$-(A_l, A_e) := (A_l^c, A_l) \qquad -(A_l, A_u) := (A_l^c, A_l^c) \tag{3}$$

As far as binary operations are concerned, we are mainly interested in conjunction and disjunction, as well as related order relations. So, the basic meet and join on three-values are the min and max (Kleene conjunction and disjunction [23]) corresponding to the usual ordering on numbers: $0 \leq \frac{1}{2} \leq 1$. On orthopairs A and B, this ordering is known as the *truth ordering* [6] and it reads $A_l \subseteq B_l$ and $B_e \subseteq A_e$ or equivalently $A_u \subseteq B_u$ on nested pairs. The meet and join operations are respectively defined on orthopairs as

$$(A_l, A_e) \sqcap (B_l, B_e) := (A_l \cap B_l, A_e \cup B_e) \tag{4}$$

$$(A_l, A_e) \sqcup (B_l, B_e) := (A_l \cup B_l, A_e \cap B_e) \tag{5}$$

The other usually considered order relation on orthopairs is the *knowledge ordering* [6,32], also known as *semantic precision* [24]: $A \preceq_k B$ if $A_l \subseteq B_l$ and

$A_e \subseteq B_e$ (equiv., $B_u \subseteq A_u$). As its name reflects, this ordering means that A is less informative than B. It is a just a partial order which corresponds on three values to $\frac{1}{2} \leq \{0,1\}$. Thus, it does not generate a join operator but only a meet one, that is the min with respect to this ordering, and it corresponds on orthopairs to the *optimistic combination operator* [24].

However, by transforming this order relation into a total one, two different orderings and two different conjunctions and disjunctions are generated. These orderings read on orthopairs as:

$$(A_l, A_e) \preceq_e (B_l, B_e) \quad \text{iff} \quad A_e \subseteq B_e \text{ and } A_l \cup A_e \subseteq B_l \cup B_e, \tag{6}$$

$$(A_l, A_e) \preceq_l (B_l, B_e) \quad \text{iff} \quad A_l \subseteq B_l, \text{ and } A_l \cup A_e \subseteq B_l \cup B_e, \tag{7}$$

So, in both cases A is less informative than B. Let us notice, indeed, that $A_l \cup A_e = A^c_{Bnd}$ and so the second condition of both relations can be equivalently stated as $B_{bnd} \subseteq A_{Bnd}$. In the first case A is at least as negative as B, whereas the second ordering means that A is at least as positive as B. From an information point of view these two orderings are, thus, less demanding than the information ordering and we have that $A \preceq_k B$ iff $A \preceq_e B$ and $A \preceq_l B$.

Other orders on three values are respectively $\frac{1}{2} \leq_e 1 \leq_e 0$ and $\frac{1}{2} \leq_l 0 \leq_l 1$ (we are using the symbol $\leq$ on numbers and $\preceq$ on pairs). The interpretation of these orderings on three values is not interesting per se, but they generate two important pairs of conjunction and disjunction:

– the Sobociński operations [31], corresponding to uninorms with neutral element $\frac{1}{2}$ [22]. The conjunction corresponds to the max with respect to the order $\leq_e$ and the disjunction is the max with respect to $\leq_l$. They are used in conditional events to fuse conditionals [16]. Their definition on orthopairs is given in equations 8.

$$(A_l, A_e) \sqcap_S (B_l, B_e) := (A_l, A_e) \sqcup_e (B_l, B_e) = (A_l \backslash B_e \cup B_l \backslash A_e, A_e \cup B_e) \tag{8a}$$

$$(A_l, A_e) \sqcup_S (B_l, B_e) := (A_l, A_e) \sqcup_l (B_l, B_e) = (A_l \cup B_l, A_e \backslash B_l \cup B_e \backslash A_l) \tag{8b}$$

– the weak Kleene meet and join [23], respectively corresponding to the min with respect to $\leq_l$ and $\leq_e$. In this case, the third value is interpreted as undefined. On orthopairs they generate the operations in equations 9.

$$(A_l, A_e) \sqcap_K (B_l, B_e) := (A_l, A_e) \sqcap_e (B_l, B_e) \tag{9a}$$
$$:= ((A_l \cap B_l) \cup [(A_l \cap B_e) \cup (B_l \cap A_e)], A_e \cap B_e))$$

$$(A_l, A_e) \sqcup_K (B_l, B_e) := (A_l, A_e) \sqcap_l (B_l, B_e) \tag{9b}$$
$$:= (A_l \cap B_l, (A_e \cap B_e) \cup [(A_e \cap B_l) \cup (B_e \cap A_l)])$$

Finally, we can introduce six different negations by using these two orderings similarly as was done in equations (1)–(3). Some of these negations will be discussed in Section 4.2.

3 Interpretation

We start now to analyze how the three authors under investigation approach orthopairs. Here, we consider the interpretation attached to a pair of sets and in the following section, we see which operations are defined on them.

Orthopairs are introduced by Fadini [18,19] under the name *complex classes*. They are defined as pairs of the kind (A_l, A_{Bnd}), so with the meaning lower-boundary. Using Fadini's terminology: A_l is the extension of a class and A_{Bnd} contains "all the elements whose belonging to the class has the third truth value"[3] [18]. The union $A_l \oplus A_{Bnd}$ is the *complex extension* of a class, and to distinguish between the two different parts of the extension a new unity i is introduced as in complex numbers and a complex class C is thus denoted as $C = A_l \cup iA_{Bnd}$.

Fadini is aware that different meanings can be attached to the third truth value and that operations have to be chosen accordingly to the interpretation. He gives the interpretation of *indeterminate* whose meaning is "clearly different from *unknown*", and for which the tertium non datur principle should not hold (contrary to the unknown case). The term "indeterminate" is taken from Reichenbach [29], that is, from quantum mechanics. So, it seems that the third value has an ontic nature and it is not a knowledge flaw. This point of view is clarified by Fadini himself in his later book on fuzzy sets, where he says that indeterminate stands for a third truth value which is neither true nor false and it is different from *unknown* or *unknowable* [20]. Moreover, in this book he also studies the case of *unknown* which represents "the indecision between true and false and so it is not a real third truth value". In order to manage this case he refers to a doxastic logic [26].

Andreoli [2,3] studies the generalizations of Boolean algebras and Boolean sets in two directions. The first one, is by extending the set of truth values (the membership function in case of sets). So, in the case of three values, he classifies objects as "interior", "exterior", "boundary" or also as "accepted", "rejected", "undecided". So, we can see that the former terminology coincide with the standard rough-set theory one whereas the second interpretation echoes the terminology of three-way decision theory [33]. The second direction is what he calls "levels", whose motivations arise from genetics. Indeed, he gives the example of a gene with a dominant allele A and a recessive one a. Then, we can have three different kinds of pair: AA, $Aa \equiv aA$ and aa with the order $AA > Aa > aa$ or if we want to distinguish between Aa and aA with a Boolean lattice structure (see Figure 1). We notice, however, that an operation of Boolean operators in this context is not provided. Perhaps, we could hazard that the min/max can give the combination with the minor/major number of recessive alleles possible.

This idea of levels reminds the granular computing approach to represent knowledge. This similarity is also supported by Andreoli's idea that two oper-

[3] All translations from Italian (and in case of Gentilhomme, from French) are under our complete responsibility.

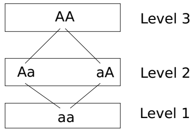

Fig. 1. Three levels of a Boolean algebra

ations named "refinement" and "attenuation" can be introduced on levels [3]. The first operation introduces a new level, that is, we can move from two to three by splitting one level into other two. The second operation acts in the opposite direction by fusing together two different levels. These operations can be encountered nowadays in granular computing acting on levels or on concepts, for instance in [7] we defined a refinement operation and an elimination one on an ontology's concepts. He also gives some hints on the fact that these two operations can be generalized to more generic structures, not only to three values. We postpone the investigation of this idea to a future work. We also notice that in [1] a general study on the algebraic structure of pairs from a Boolean algebra is given. These pairs, however, are not necessarily disjoint and an interpretation is not provided.

Finally, Gentilhomme pairs are in the form lower-upper (A_l, A_u) and they are named fuzzy ("flou") sets. The lower set is the "certainty zone", the upper the "maximal extension" and the boundary, the fuzzy zone. He also notices that a flou set can be equivalently expressed as the orthopair lower-exterior.

The interpretation of the boundary (equivalently, the third value) has an ontic nature (the value $\frac{1}{2}$ is named "maybe"). Indeed, it refers to linguistic problems where we are unable to correctly classify some linguistic object and this "hesitation" is a "matter of the language" [21]. So, the problem is intrinsic to the object under study, not to the observer. However, in some passages Gentilhomme also comments on the causes of this "partial failure" in classifying objects with certainty, opening the door to some problems also in the observers. Indeed, in the list of causes given by Gentilhomme, we have:

- the fact that different agents have different opinions. This could lead to an interpretation of the third truth value as *inconsistent* and thus to paraconsistent logic where two (or more) agents can be in accordance or not on the Boolean truth value to assign.
- the fact that data cannot be analyzed in a certain way or that we cannot apply some given criteria to the data.

Finally, we notice that he also uses the terminology of complex numbers when generalizing to not nested pairs (A, B). Analogously to the numerical case, the sense is that "all the symbols do not have an immediate interpretation [...] but they obey to similar formal rules". He also notices that such pairs can be defined by a set "completely" fuzzy $(\emptyset, B)$ and a set "totally imaginary" $(A, \emptyset)$ as $(A, B) = (A, \emptyset) \sqcup (\emptyset, B)$.

4 Operations

In this section, we look at the operations introduced by the three authors in their works making reference, when possible, to those introduced in Section 2 and pointing out which operations are new and the relations with other theories. Of course, these operations are defined in an abstract setting and they should be adapted to the context where they are used. For instance, in rough set theory, not all orthopairs are representable as rough sets, due to the partition generated by the equivalence relation; this entails a non truth-functional behaviour of rough sets [13].

4.1 Intersection, Union and Difference

Andreoli is interested in a general study on orthopairs (sometimes only pairs as in [1]) and he considers as plausible all the binary operations that are associative, commutative and idempotent. There are six operations of this kind and they are pairwise linked by a de Morgan property (using the standard involutive negation (1)). They are the Kleene min and max, Sobocinski conjunction and disjunction (equations 8) and weak Kleene conjunction and disjunction (equations 9). He also notices that they correspond to the min or max with respect to different orderings on the three values. So, for instance Sobocinski conjunction is the max with respect to the order $\leq_e$ (equiv., the min with respect to the opposite order $0 \leq 1 \leq \frac{1}{2}$).

On the other hand, both Fadini and Gentilhomme consider as intersection and union the standard Kleene operations (min and max on three values). This is coherent with respect to the ontic interpretation they have in mind as explained in the previous sections.

Besides conjunction and disjunction, Gentilhomme also devotes some efforts to define a difference between two nested pairs. He considers as the correct definition of a difference the following one ($\cdot'$ is the involutive negation on pairs as defined in equation (1)):

$$A \setminus B := A \sqcap B' = (A_l \setminus B_u, A_u \setminus B_l) \tag{10}$$

that corresponds to an "experimental reality" where the certainty zone is exactly $A_l \setminus B_u$. However, he also introduces two other differences: the greatest, that consists in accepting the maximum of the risk, and the smallest, that is accepting no risk. They are respectively defined as

$$A \setminus_g B := (A_l \setminus B_l, A_u \setminus B_l) \tag{11}$$

$$A \setminus_l B := (A_l \setminus B_u, A_u \setminus B_u) \tag{12}$$

It is interesting to notice that these difference operators can be obtained as in equation (10) but using the intuitionistic and paraconsistent negations of equations (2) and (3), that is, we have $A \setminus_g B = A \sqcap -B$ and $A \setminus_l B = A \sqcap \sim B$.

Of course, this reference to the risk occurring to using one operation instead of another points to decision theory and so to the possibility to introduce these (and other) operations on three-way decision theory. This aspect deserves a further investigation in the future.

All these differences are not equal to the difference on orthopairs defined in [25] as $(A_l \setminus B_e, A_e \setminus B_l)$. This last corresponds to making the orthopair "consistent", that is removing from the positive/negative part of A the negative/positive part of B so as to avoid conflict between A and B. The corresponding operation on nested pairs reads $(A_l \cap B_u, A_u \cup B_u^c)$ whose interpretation is not so clear. Thus, in the difference case, it is evident that even though mathematically equivalent, nested and orthopairs give birth to different operations which make sense more in one case than in the other.

4.2 Negations

All the three authors consider the involutive negation as one possibility. Gentilhomme just considers this one, whereas Andreoli and Fadini give more solutions.

Indeed, in order to obtain a negation, Fadini considers what happens in negating the Boolean part, the complex part or both. As a result, not all these operations are an extension of the Boolean negation on two values. By the negation of the Boolean part we get the standard involutive negation. By negating the complex part, we obtain a swapping of 0 with $\frac{1}{2}$: on nested pairs, $(A_l, A_u)^* := (A_l, A_l \cup A_e)$ and on orthopairs, $(A_l, A_e)^* := (A_l, A_{Bnd})$. That is, in the case of orthopairs the negation consists in a switch of the interpretation from necessity-impossibility to necessity-unknown. Let us notice that it corresponds to an involutive negation based on the order $\leq_e$ (i.e., the "middle" value is 1).

Finally, by negating both the Boolean and the complex part we get two negations: $\overline{(A_l, A_u)} := (A_l^c, U)$ and $\underline{(A_l, A_u)} := (A_{Bnd}, U)$. On three values, the first negation is such that $\overline{1} = \frac{1}{2}$, $\overline{0} = \overline{\frac{1}{2}} = 1$, which is the *complete negation* introduced by Reichenbach and has its reasons[4] in quantum mechanics [29]. We remark that it can be obtained by a paraconsistent-like negation based on the order $\leq_l$. On the same ordering, the intuitionistic-like negation corresponds to the second negation by Fadini, and it is defined as $\overline{1} = \overline{0} = \frac{1}{2}$, $\overline{\frac{1}{2}} = 1$. In both cases, nothing is false, the difference lies in what is true: in the first case it corresponds to what was not true (that is, false or unknown), in the second to what was not known (i.e., in the boundary).

Moreover, he also considers the possibility that a negation of a Boolean or complex part is allowed to contain *only* elements outside the class itself instead of *all and only* the elements. In this way, a further negation is introduced by swapping the Boolean and complex part. In terms of nested pairs: $(A_l, A_u)^\circ = (A_u \setminus A_l, A_u)$ and of three-values $0^o = 0$, $\frac{1}{2}^o = 1$ and $1^o = \frac{1}{2}$. This corresponds to an involutive negation based on the order $\leq_l$ ($\frac{1}{2} \leq_l 0 \leq_l 1$).

Andreoli, apart from the standard involutive negation, introduces on three values an interesting approach, which is different from what we have seen until

[4] Reichenbach justifies the name *complete* by the fact that $a \vee \overline{a}$ is a tautology and this form of excluded middle is required by quantum mechanics.

now. In a certain sense, it can be seen as a Boolean negation on the set of truth values. Indeed, if $t \in \{0, \frac{1}{2}, 1\}$, then the negation of $\{t\}$ is $\{t\}^c$. This is justified by the interpretation of three values as *chosen*, *rejected*, *undecided*. So, for instance, the negation of undecided is *decided* which can mean either chosen or refused. With respect to decision theory he then suggests that one can arrive at a final decision with a two steps "Boolean" procedure. That is, at a first step there is the division of the world in decided and undecided and in the second step decided is further classified as rejected or chosen. This is a sort of sequential reasoning which however works in the opposite direction with respect to the three-way sequential decision theory [34] (see Figure 2).

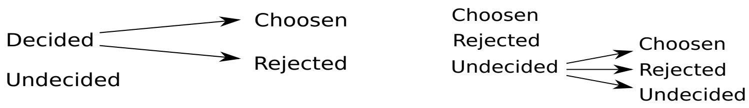

Fig. 2. Sequential decision making (Andreoli left, Yao right)

Indeed, this last consists in developing at the second step what is left undecided in the first one. In this way, through a sequence of three-way decision it is possible to arrive at a Boolean decision, classifying all objects as accepted or rejected. This strategy is pointed out (but not developed) by Gentilhomme with respect to classification: "we classify what we are able to classify and we devote a "cesspool" to the rebel elements" [21].
Going back to the negation, when applied to orthopairs we get that the negation of (A_l, A_e) is $((A_l)^c, A_e^c) = ((A^c)_u, A_u)$ which is no more an orthopair and the intersection of A_l^c and A_u is the boundary A_{Bnd}. Pairs of this kind make sense when we want to model conflicting information, such as in Belnap or paraconsistent logics, so that an object can be in both sets of the pair [11,14]. The role of this negation in decision theory and conflicting information should be better investigated.

Finally, let us notice that the fact that at a first step we do not take a definite position but just exclude one possibility among three, that is we wonder if a proposition is false (either true) or unknown has been also discussed in terms of orthopairs arising from formal contexts in [27]. Similarly, in the translation from three-valued to epistemic logics [9] we admit the possibility to represent that a valuation of a formula is $\leq \frac{1}{2}$ (false or unknown) or $\geq \frac{1}{2}$ (unknown or true).

4.3 Inclusion

According to Fadini [19], the main property to define an inclusion is based on the two parts (the Boolean and boundary ones) of his complex-class notion. Indeed, the inclusion between two complex classes $A_l \oplus iA_{Bnd}$ and $B_l \oplus iB_{Bnd}$ holds if it holds for the union of the two parts $A_l \cup A_{Bnd} \subseteq B_l \cup B_{Bnd}$. This, on nested pairs, reduces to $A_u \subseteq B_u$. Fadini outlined five ways to fulfill this condition, namely

(sub1) $A_l \subseteq B_l$, $A_u \subseteq B_u$ it is named *proper inclusion* since it is the only one to be an order relation. We can see that it is the standard inclusion relation on nested pairs, corresponding to the standard order relation on three-values;
(sub2) $A_u \subseteq B_l$ named *total inclusion* for the intuitive reason that the concept A it totally contained in B, both in its certainty and possibility parts;
(sub3) $A_{bnd} \subseteq B_l$, $A_l \subseteq B_{bnd}$ it is called *improper* or *inverted inclusion*;
(sub4) $A_u \subseteq B_{Bnd}$, the *total improper inclusion*;
(sub5) A_{bnd} and A_l are both contained in B_u and they have a non-empty intersection with both B_l and B_{bnd}, this is the *mixed inclusion.*

As can be seen his notions of inclusion are greatly influenced by his interpretation of the orthopair as lower-boundary. This view generates original (pseudo) order relations that not always can find an easy interpretation outside this framework.

Gentilhomme presents six different inclusion relations, defined according to different mutual behaviours of lower and upper sets. These six relations form a lattice where the order relation is given by "implies", that is an inclusion is smaller than another if the second one implies the first[5]. Only three of these relations are considered meaningful, named *normal*, since "it is best suited for calculations": $A_l \subseteq B_l$ and $A_u \subseteq B_u$ (again the standard inclusion on orthopairs); *strong* and *weak* defined, respectively, as $A_u \subseteq B_l$ and $A_l \subseteq B_u$, the justification of the name being intuitive.

Finally, Andreoli does not study directly the inclusion relations on three values/orthopairs but just on four values in [2]. On three values, they are indirectly considered when studying the join and meet operations (see section 4.1). As can be seen no one considered the knowledge ordering but they have defined order relations never encountered before.

With respect to the relationship with implications, the only reference is by Fadini in [18]. He does not consider his inclusion operations to be the corresponding of an inference but of a conditional *if a, then b* since they hold only when a is true. Oddly, no further discussion on implication is present in other Fadini's works nor in Andreoli and Gentilhomme.

5 Conclusion

In the present paper, we reviewed some (old) works on orthopairs, related them to modern theories such as rough sets, decision theory and granular computing. We saw how different interpretations of an orthopair can influence the definition of operations. Some new operations (with respect to what is usually considered nowadays) have been found: the difference operations in Gentilhomme and the negations in Fadini and Andreoli. In particular these operations are often given an interpretation in decision theory. So, as a future work, it is worth considering the possibility to study orthopair operations in three-way decision. More generally, the role of the negation in orthopairs and conflicting information needs

[5] The relation of this lattice with implication lattices in rough sets [30] should be studied in the future.

a thorough understanding. Further, the role of levels in generalized structures outlined by Andreoli could be of some interest in granular computing. Finally, an algebraic study of these new operations deserves some attention and an ad hoc study.

References

1. Andreoli, G.: Struttura delle algebre di Boole e loro estensione quale calcolo delle classi. Giornale di Matematiche di Battaglini LXXXV, 141–171 (1957)
2. Andreoli, G.: Algebre di Boole - algebre di insieme - algebre di livelli. Giornale di Matematiche di Battaglini LXXXVII, 3–22 (1959)
3. Andreoli, G.: Dicotomie e tricotomie (anelli booleani triadici ed algebre booleane a tre valori). La Ricerca, pp. 1–10 (1961)
4. Banerjee, M., Chakraborty, K.: Algebras from rough sets. In: Pal, S., Skowron, A., Polkowski, L. (eds.) Rough-Neural Computing, pp. 157–188. Springer (2004)
5. Banerjee, M., Khan, M. A.: Propositional logics from rough set theory. In: Peters, J.F., Skowron, A., Düntsch, I., Grzymała-Busse, J.W., Orłowska, E., Polkowski, L. (eds.) Transactions on Rough Sets VI. LNCS, vol. 4374, pp. 1–25. Springer, Heidelberg (2007)
6. Belnap, N.D.: A useful four-valued logic. In: Dunn, J.M., Epstein, G. (eds.) Modern Uses of Multiple-Valued Logic, pp. 8–37. D. Reidel Publishing Company (1977)
7. Calegari, S., Ciucci, D.: Granular computing applied to ontologies. International Journal of Approximate Reasoning 51(4), 391–409 (2010)
8. Ciucci, D.: Orthopairs: A simple and widely used way to model uncertainty. Fundam. Inform. 108(3-4), 287–304 (2011)
9. Ciucci, D., Dubois, D.: A modal theorem-preserving translation of a class of three-valued logics of incomplete information. Journal of Applied Non-Classical Logics 23(4), 321–352 (2013)
10. Ciucci, D., Dubois, D.: A map of dependencies among three-valued logics. Information Sciences 250, 162–177 (2013), Corrigendum: Information Sciences 256, 234–235 (2014)
11. Ciucci, D., Dubois, D., Lawry, J.: Borderline vs. unknown: a comparison between three-valued valuations, partial models, and possibility distributions. In: Proceedings of the Workshop Weighted Logics for AI at ECAI 2012, pp. 83–90 (2012), `http://www2.lirmm.fr/ecai2012/images/stories/ecai_doc/pdf/workshop/W34_wl4ai-working-notes.pdf`
12. Ciucci, D., Dubois, D., Prade, H.: Oppositions in rough set theory. In: Li, T., Nguyen, H.S., Wang, G., Grzymala-Busse, J., Janicki, R., Hassanien, A.E., Yu, H. (eds.) RSKT 2012. LNCS, vol. 7414, pp. 504–513. Springer, Heidelberg (2012)
13. Ciucci, D., Dubois, D.: Three-valued logics, uncertainty management and rough sets. In: Peters, J.F., Skowron, A. (eds.) Transactions on Rough Sets XVII. LNCS, vol. 8375, pp. 1–32. Springer, Heidelberg (2014)
14. Ciucci, D., Dubois, D., Lawry, J.: Borderline vs. unknown - comparing three-valued representation of imperfect information. International Journal of Approximate Reasoning (2013) (submitted)
15. Dubois, D.: Reasoning about ignorance and contradiction: many-valued logics versus epistemic logic. Soft Comput. 16(11), 1817–1831 (2012)
16. Dubois, D., Prade, H.: Conditional objects as nonmonotonic consequence relationships. IEEE Transaction of Sysyems, Man, and Cybernetics 24(12), 1724–1740 (1994)

17. Dubois, D., Prade, H.: From Blanché's hexagonal organization of concepts to formal concept analysis and possibility theory. Logica Universalis 6(1-2), 149–169 (2012)
18. Fadini, A.: Il calcolo delle classi in una logica a tre valori di verità. Giornale di Matematiche di Battaglini XC, 72–91 (1962)
19. Fadini, A.: Teoria degli elementi complessi nelle Algebre di Boole. Annali del Pontificio Istituto Superiore di Scienze e Lettere "S. Chiara" 12, 223–243 (1962)
20. Fadini, A.: Introduzione alla Teoria degli Insiemi Sfocati. Liguori Editore, Napoli (1979)
21. Gentilhomme, M.Y.: Les ensembles flous en linguistique. Cahiers de linguistique théorique et appliquée. Bucarest 47, 47–65 (1968)
22. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge University Press (2009)
23. Kleene, S.C.: Introduction to metamathematics. North–Holland Pub. Co., Amsterdam (1952)
24. Lawry, J., Dubois, D.: A bipolar framework for combining beliefs about vague propositions. In: Brewka, G., Eiter, T., McIlraith, S.A. (eds.) Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference, pp. 530–540. AAAI Press (2012)
25. Lawry, J., Tang, Y.: On truth-gaps, bipolar belief and the assertability of vague propositions. Artif. Intell. 191-192, 20–41 (2012)
26. Nola, A.D., Gerla, G.: A three-valued doxastic logic. La Ricerca XXXII, 19–33 (1981)
27. Pagliani, P.: Information gaps as communication needs: A new semantic foundation for some non-classical logics. Journal of Logic, Language and Information 6(1), 63–99 (1997)
28. Pagliani, P., Chakraborty, M.: A Geometry of Approximation. Springer (2008)
29. Reichenbach, H.: Philosophic Foundations of Quantum Mechanics. University of California Press, Berkeley (1954)
30. Samanta, P., Chakraborty, M.K.: Generalized rough sets and implication lattices. In: Peters, J.F., Skowron, A., Sakai, H., Chakraborty, M.K., Slezak, D., Hassanien, A.E., Zhu, W. (eds.) Transactions on Rough Sets XIV. LNCS, vol. 6600, pp. 183–201. Springer, Heidelberg (2011)
31. Sobociński, B.: Axiomatization of a partial system of three-value calculus of propositions. J. of Computing Systems 1, 23–55 (1952)
32. Yao, Y.: Interval sets and interval-set algebras. In: Proceedings of the 8th IEEE International Conference on Cognitive Informatics, pp. 307–314 (2009)
33. Yao, Y.: An Outline of a Theory of Three-way Decisions. In: Yao, J., Yang, Y., Słowiński, R., Greco, S., Li, H., Mitra, S., Polkowski, L. (eds.) RSCTC 2012. LNCS, vol. 7413, pp. 1–17. Springer, Heidelberg (2012)
34. Yao, Y.: Granular computing and sequential three-way decisions. In: Lingras, P., Wolski, M., Cornelis, C., Mitra, S., Wasilewski, P. (eds.) RSKT 2013. LNCS, vol. 8171, pp. 16–27. Springer, Heidelberg (2013)