Chapter 6
Plane Mixed Discriminants and Toric Jacobians

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Dedicated to the memory of our friend Andrei Zelevinsky (1953–2013)

6.1 Introduction

Polynomial algebra offers a standard and powerful approach to handle several problems in geometric modeling. In particular, the study and solution of systems of polynomial equations has been a major topic. Discriminants provide a key tool when examining well-constrained systems, including the case of one univariate polynomial. Their theoretical study is a thriving and fruitful domain today, but they are also very useful in a variety of applications.

The best studied discriminant is probably known since high school, where one studies the discriminant of a quadratic polynomial $f(x) = ax^2+bx+c = 0\,(a \neq 0)$. The polynomial f has a double root if and only if its discriminant $\Delta_2 = b^2 - 4ac$ is equal to zero. Equivalently, this can be defined as the condition for $f(x)$ and its derivative $f'(x)$ to have a common root:

$$\exists\, x \;:\; f(x) = ax^2 + bx + c = f'(x) = 2ax + b = 0 \;\Leftrightarrow\; \Delta_2 = 0. \tag{6.1}$$

One can similarly consider the discriminant of a univariate polynomial of any degree. If we wish to calculate the discriminant $\Delta_5(f)$ of a polynomial f of degree five in one variable, we consider the condition that both f and its derivative vanish:

$$\begin{aligned} f(x) &= ax^5 + bx^4 + cx^3 + dx^2 + ex + g = 0, \\ f'(x) &= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e = 0. \end{aligned}$$

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In this case, elimination theory reduces the computation of Δ_5 to the computation of a 9×9 Sylvester determinant, which equals $a\ \Delta_5(f)$. If we develop this determinant, we find out that the number monomials in the discriminant increases rapidly with the input degree:

$$\begin{aligned}\Delta_5 = &-2{,}050a^2g^2bedc + 356abed^2c^2g - 80b^3ed^2cg + 18dc^3b^2g\\ &e - 746agdcb^2e^2 + 144ab^2e^4c - 6ab^2e^3d^2 - 192a^2be^4d - 4d^2ac\\ &^3e^2 + 144d^2a^2ce^3 - 4d^3b^3e^2 - 4c^3e^3b^2 - 80abe^3dc^2 + 18b^3e^3\\ &dc + 18d^3acbe^2 + d^2c^2b^2e^2 - 27b^4e^4 - 128a^2e^4c^2 + 16ac^4e^3 - 27\\ &a^2d^4e^2 + 256a^3e^5 + 3{,}125a^4g^4 + 160a^2gbe^3c + 560a^2gdc^2e^2 + 1{,}020\\ &a^2gbd^2e^2 + 160ag^2b^3ed + 560ag^2d^2cb^2 + 1{,}020ag^2b^2c^2e - 192\\ &b^4ecg^2 + 24ab^2ed^3g + 24abe^2c^3g + 144b^4e^2dg - 6b^3e^2c^2g + 14\\ &4dc^2b^3g^2 - 630dac^3bg^2 - 630d^3a^2ceg - 72d^4acbg - 72dac^4e\\ &g - 4d^3c^2b^2g - 1{,}600ag^3cb^3 - 2{,}500a^3g^3be - 50a^2g^2b^2e^2 - 3{,}750a^3\\ &g^3dc + 2{,}000a^2g^3db^2 + 2{,}000a^3g^2ce^2 + 825a^2g^2d^2c^2 + 2{,}250a^2g^3b\\ &c^2 + 2{,}250a^3g^2ed^2 - 900a^2g^2bd^3 - 900a^2g^2c^3e - 36agb^3e^3 - 1{,}600\\ &a^3ge^3d + 16d^3ac^3g - 128d^2b^4g^2 + 16d^4b^3g - 27c^4b^2g^2 + 108ac^5\\ &g^2 + 108a^2d^5g + 256b^5g^3.\end{aligned}$$

In fact, if we compute the resultant of f and xf' by means of the 10×10 Sylvester determinant, we find the more symmetric output: $a\ g\ \Delta_5(f)$. This formula is very well known for univariate discriminants (Ch.12, [18]), and we generalize it in Theorem 3.

One univariate polynomial is the smallest well-constrained system. We are concerned with multivariate systems of sparse polynomials, in other words, polynomials with fixed support, or set of nonzero terms. *Sparse (or toric)* elimination theory concerns the study of resultants and discriminants associated with toric varieties. This theory has its origin in the work of Gel'fand, Kapranov and Zelevinsky on multivariate hypergeometric functions. Discriminants arise as singularities of such functions [17].

Gel'fand, Kapranov and Zelevinsky [18] established a general definition of sparse discriminant, which gives as special case the following definition of (sparse) mixed discriminant (see Sect. 6.2 for the relation with the discriminant of the associated Cayley matrix and with the notion of mixed discriminant in [3]). In case $n = 2$, the mixed discriminant detects tangencies between families of curves with fixed supports. In general, the *mixed discriminant* $\Delta_{A_1,\ldots,A_n}(f_1,\ldots,f_n)$ of n polynomials in n variables with fixed supports $A_1,\ldots,A_n \subset \mathbb{Z}^n$ is the irreducible polynomial (with integer coprime coefficients, defined up to sign) in the coefficients of the f_i which vanishes whenever the system $f_1 = \cdots = f_n = 0$ has a multiple root (that is, a root which is not simple) with non-zero coordinates, in case this discriminantal variety is a hypersurface (and equal to the constant 1 otherwise). The zero locus of the mixed discriminant is the variety of ill-posed systems [24].

For each $i = 1,\ldots,n$, pick an element $a_{i,0} \in A_i$ and denote by $\mathcal{L}_{A_1,\ldots,A_n}$ the lattice generated by $\{a - a_{i,0}, a \in A_i, i = 1,\ldots,n\}$. We shall work with the

polynomial defining the *discriminant cycle* (see Sect. 6.2), which is defined as the power $\Delta_{A_1,\ldots,A_n}^{i(A_1,\ldots,A_n)}$ of the mixed discriminant raised to the index

$$i(A_1,\ldots,A_n) = [\mathbb{Z}^n : \mathcal{L}_{A_1,\ldots,A_n}], \tag{6.2}$$

which we always assume to be finite. In most situations, this index equals 1 and so both concepts coincide.

Discriminants have many applications. Besides the classical application in the realm of differential equations to describe singularities, discriminants occur for instance in the description of the topology of real algebraic plane curves [19], in solving systems of polynomial inequalities and zero-dimensional systems [16], in determining the number of real roots of square systems of sparse polynomials [9], in studying the stability of numerical solving [6], in the computation of the Voronoi diagram of curved objects [13], or in the determination of cusp points of parallel manipulators [20].

Computing (mixed) discriminants is a (difficult) elimination problem. In principle, they can be computed with Gröbner bases, but this is very inefficient in general since these polynomials have a rich combinatorial structure [18]. Ad-hoc computations via complexes (i.e., via tailored homological algebra) are also possible, but they also turn out to be complicated. The tropical approach to compute discriminants was initiated in [8] and the tropicalization of mixed planar discriminants was described in [10]. Recently, in [12], the authors focus on computing the discriminant of a multivariate polynomial via interpolation, based on [11, 23]; the latter essentially offers an algorithm for predicting the discriminant's Newton polytope, hence its nonzero terms. This yields a new output-sensitive algorithm which, however, remains to be juxtaposed in practice to earlier approaches.

We mainly work in the case $n = 2$, where the results are more transparent and the basic ideas are already present, but all our results and methods can be generalized to any number of variables. This will be addressed in a subsequent paper [7]. Consider for instance a system of two polynomials in two variables and assume that, the first polynomial factors as $f_1 = f_1' \cdot f_1''$. Then, the discriminant also factors and we thus obtain a multiplicativity formula for it, which we make precise in Corollary 7. This significantly simplifies the discriminant's computation and generalizes the formula in [2] for the classical homogeneous case. This multiplicativity formula is a consequence of our main result (Theorem 3 in dimension 2, see also Theorem 4 in any dimension) relating the mixed discriminant and the resultant of the given polynomials and their *toric Jacobian* (see Sect. 6.3 for precise definitions and statements). As another consequence of Theorem 3, we reprove, in Corollary 6, the bidegree formula for planar mixed discriminants in [3].

The rest of this chapter is organized as follows. The next section overviews relevant existing work and definitions. In Sect. 6.3 we present our main results relating the mixed discriminant with the sparse resultant of the two polynomials and their toric Jacobian. In Sect. 6.4 we deduce the general multiplicativity formula for the mixed discriminant when one polynomial factors.

6.2 Previous Work and Notation

In this section we give a general description of discriminants and some definitions and notations that we are going to use in the following sections.

Given a set $A \subset \mathbb{R}^n$, let $Q = conv(A)$ denote the convex hull of A. We say that A is a *lattice configuration* if it is contained in $\mathbb{Z}^n$, whereas a polytope with integer vertices is called a lattice polytope. We denote by $\mathrm{Vol}(\cdot)$ the volume of a lattice polytope, normalized with respect to the lattice $\mathbb{Z}^n$, so that a primitive simplex has normalized volume equal to 1. Normalized volume is obtained by multiplying Euclidean volume by $n!$.

Given a non-zero Laurent polynomial

$$f = \sum_a c_a x^a,$$

the finite subset A of $\mathbb{Z}^n$ of those exponents a for which $c_a \neq 0$ is called the *support* of f. The *Newton polytope* $N(f)$ of f is the lattice polytope defined as the convex hull of A. We will assume that the coefficients c_a take values in an algebraically closed field K of characteristic 0.

A (finite) set A is said to be *full*, if it consists of all the lattice points in its convex hull. In [3], A is called *dense* in this case, but we prefer to reserve the word dense to refer to the classical homogeneous case. A subset $F \subseteq A$ is called a *face* of A, denoted $F \prec A$, if F is the intersection of A with a face of the polytope $conv(A)$.

As usual $Q_1 + Q_2$ denotes the Minkowski sum of sets Q_1 and Q_2 in $\mathbb{R}^n$. The *mixed volume* $MV(Q_1, \ldots, Q_n)$ of n convex polytopes Q_i in $\mathbb{R}^n$ is the multilinear symmetric function with respect to Minkowski sum that generalizes the notion of volume in the sense that $MV(Q, \ldots, Q) = \mathrm{Vol}(Q)$, when all Q_i are equal to a fixed convex polytope Q.

The following key result is due to Bernstein and Kouchnirenko. The mixed volume of the Newton polytopes of n Laurent polynomials $f_1(x), \ldots, f_n(x)$ in n variables is an integer that bounds the number of isolated common solutions of $f_1(x) = 0, \ldots, f_n(x) = 0$ in the algebraic torus $(K^*)^n$, over an algebraically closed field K of characteristic 0 containing the coefficients. If the coefficients of the polynomials are generic, then the common solutions are isolated and their number equals the mixed volume. This bound generalizes Bézout's classical bound to the sparse case: for homogeneous polynomials the mixed volume and Bézout's bound coincide.

Mixed volume can be defined in terms of Minkowski sum volumes as follows.

$$MV(Q_1, \ldots, Q_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{I \subset \{1,\ldots,n\}, |I|=k} \frac{1}{n!} \mathrm{Vol}\Big(\sum_{i \in I} Q_i \Big).$$

This implies, for $n = 2$:

$$2MV(Q_1, Q_2) = \mathrm{Vol}(Q_1 + Q_2) - \mathrm{Vol}(Q_1) - \mathrm{Vol}(Q_2).$$

Definition 1 *A family of finite lattice configurations $A_1, \ldots, A_k$ in $\mathbb{Z}^n$ is called essential if the affine dimension of the lattice $\mathcal{L}_{A_1,\ldots,A_n}$ equals $k-1$, and for all proper subsets $I \subset \{1, \ldots, k\}$ it holds that the dimension of the lattice generated by $\{a - a_i, a \in A_i, i \in I\}$ is greater or equal than its cardinality $|I|$.*

Definition/Theorem 1 [18, 25] *Fix a family of $n+1$ finite lattice configurations $A_1, \ldots, A_{n+1}$ which contains a unique essential subfamily $\{A_i, i \in I\}$. Given Laurent polynomials $f_1, \ldots, f_{n+1}$ in n variables with respective supports $A_1, \ldots, A_{n+1}$, the resultant $\mathrm{Res}_{A_1,\ldots,A_{n+1}}(f_1, \ldots, f_{n+1})$ is the irreducible polynomial with coprime integer coefficients (defined up to sign) in the coefficients of $f_1, \ldots, f_{n+1}$, that vanishes whenever $f_1, \ldots, f_{n+1}$ have a common root in the torus $(\mathbb{C}^*)^n$. In fact, in this case, the resultant only depends on the coefficients of f_i with $i \in I$.*

If there exist two different essential subfamilies, then the (closure of the) variety of solvable systems is not a hypersurface and in this case we set:

$$\mathrm{Res}_{A_1,\ldots,A_{n+1}}(f_1, \ldots, f_{n+1}) = 1.$$

In what follows, we consider n (finite) lattice configurations $A_1, \ldots, A_n$ in $\mathbb{Z}^n$ and we denote by $Q_1, \ldots, Q_n$ their respective convex hulls. Let $f_1, \ldots, f_n$ be Laurent polynomials with coefficients in K and support $A_1, \ldots, A_n$, respectively:

$$f_i(x) = \sum_{\alpha \in A_i} c_{i,\alpha} x^\alpha, \quad i = 1 \ldots, n.$$

In [3] the *mixed discriminantal variety*, is defined as closure of the locus of coefficients $c_{i,\alpha}$ for which the associated system $f_1 = \cdots = f_n = 0$ has a non-degenerate multiple root $x \in (K^*)^n$. This means that x is an isolated root and the n gradient vectors

$$\left(\frac{\partial f_i}{\partial x_1}(x), \ldots, \frac{\partial f_i}{\partial x_n}(x)\right)$$

are linearly dependent, but any $n-1$ of them are linearly independent.

Definition 2 *If the mixed discriminantal variety is a hypersurface, the mixed discriminant of the previous system is the unique up to sign irreducible polynomial $\Delta_{A_1,\ldots,A_n}$ with integer coefficients in the unknowns $c_{i,a}$ which defines this hypersurface. Otherwise, the family is said to be defective and we set $\Delta_{A_1,\ldots,A_n} = 1$. The* mixed discriminant cycle *$\tilde{\Delta}_{A_1,\ldots,A_n}$ is equal to $i(A_1, \ldots, A_n)$ times the mixed discriminant variety, and thus its equation equals $\Delta_{A_1,\ldots,A_n}$ raised to this integer (defined by (6.2)).*

By [3, Theorem 2.1], when the family $A_1, \ldots, A_n$ is non defective, the mixed discriminant $\Delta_{A_1,\ldots,A_n}$ coincides with the A-discriminant defined in [18], where A is the Cayley matrix

$$A = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \\ A_1 & A_2 & \ldots & A_n \end{pmatrix}.$$

This matrix has $2n$ rows and $m = \sum_{i=1}^{n} |A_i|$ columns, so $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$ denote row vectors of appropriate lengths. We introduce n new variables $y_1, \ldots, y_n$ in order to encode the system $f_1 = \cdots = f_n = 0$ in one polynomial with support in A, via the *Cayley trick*: $\phi(x, y) = y_1 f_1(x) + \cdots + y_n f_n(x)$. Note that $i(A_1, \ldots, A_n)$ equals the index in $\mathbb{Z}^{2n}$ of the lattice generated by the columns of the matrix A.

In what follows, when we refer to resultants or discriminants, we shall refer to the *equations of the corresponding cycles* (as in Definition 2), but we will omit the tildes in our notation. More explicitly, we will follow the convention in the articles [5] by D'Andrea and Sombra and [14] by Esterov, which is faithful to intersection theory. This convention allows us to present cleaner formulas. For instance, when the family $A_1, \ldots, A_{n+1}$ is essential, our notion of resultant equals the resultant in [18, 25] raised to the index $i(A_1, \ldots, A_{n+1})$. In most examples these two lattices coincide, and so our resultant cycle equals the resultant variety and the associated resultant polynomial is irreducible.

Remark 6.2 Assume A_1 consists of a single point α and that $\{1\}$ is the only essential subfamily of a given family $A_1, \ldots, A_{n+1}$. Let $f_1(x) = cx^{\alpha}$. Then, for any choice of Laurent polynomials $f_2, \ldots, f_{n+1}$ with supports $A_2, \ldots, A_{n+1}$, it holds that (cf. [5, Example 3.14])

$$\mathrm{Res}_{A_1,\ldots,A_{n+1}}(f_1, \ldots, f_n) = c^{MV(A_2,\ldots,A_{n+1})}. \tag{6.3}$$

With this convention, the following multiplicativity formula holds:

Theorem 2 [5, Corollary 4.6],[22] *Let $A_1', A_1'', A_1, \ldots, A_{n+1}$ be finite subsets of $\mathbb{Z}^n$ with $A_1 = A_1' + A_1''$. Let $f_1, \ldots, f_{n+1}$ be polynomials with supports contained in $A_1, \ldots, A_{n+1}$ and assume that $f_1 = f_1' f_1''$ where f_1' has support A_1' and f_1'' has support A_1''. Then*

$$\begin{aligned} &\mathrm{Res}_{A_1,\ldots,A_{n+1}}(f_1, \ldots, f_{n+1}) \\ &\quad = \mathrm{Res}_{A_1',\ldots,A_{n+1}}(f_1', \ldots, f_{n+1}) \cdot \mathrm{Res}_{A_1'',\ldots,A_{n+1}}(f_1'', \ldots, f_{n+1}). \end{aligned}$$

Cattani, Cueto, Dickenstein, Di Rocco and Sturmfels in [3] proved that the degree of the mixed discriminant Δ is a piecewise linear function in the Plücker coordinates of a mixed Grassmanian. An explicit degree formula for plane curves is also presented in [3, Corollary 3.15]. In case A_1, A_2, they are two dimensional, full, and with the same normal fan, then the bidegree of Δ_{A_1,A_2} in the coefficients of f_1 and f_2 equals:

$$(\mathrm{Vol}(Q_1+Q_2)-area(Q_1)-perim(Q_2), \mathrm{Vol}(Q_1+Q_2)-area(Q_2)-perim(Q_1),$$

where $Q_i = conv(A_i)$, $i = 1, 2$, and $Q_1 + Q_2$ is their Minkowski sum. The area is normalized, so that a primitive triangle has area 1 and the perimeter $perim(Q_i)$ of Q_i is the cardinality of $\partial Q_i \cap \mathbb{Z}^2$. We will recover the general formula for this degree and present it in Corollary 6.

Busé and Jouanolou consider in [2] the following equivalent definition of the mixed discriminant, in case that $f_1, \ldots, f_n$ are dense homogeneous polynomials in $(x_0, \ldots, x_n)$ of degrees $d_1, \ldots, d_n$ respectively, that is, their respective supports $A_i = d_i\sigma$ are all the lattice points in the d_i-th dilate of the unit simplex σ in $\mathbb{R}^n$. It is the non-zero polynomial in the coefficients of $f_1, \ldots, f_n$ which equals

$$\frac{\mathrm{Res}_{d_1\sigma,\ldots,d_n\sigma,\delta_i\sigma}(f_1,\ldots,f_n,J_i)}{\mathrm{Res}_{d_1\sigma,\ldots,d_n\sigma,\sigma}(f_1,\ldots,f_n,x_i)}, \tag{6.4}$$

for all $i \in \{1, \ldots, n\}$, where J_i is the maximal minor of the Jacobian matrix associated to $f_1, \ldots, f_n$ obtained by deleting the i-th row and column and $\delta_i = \sum_{j\neq i}(d_j - 1)$. We give a more symmetric and general formula in Corollary 5 below.

The multiplicativity property of the discriminant in the case of dense homogeneous polynomials was already known to Sylvester in the multivariate case [26] and generalized by Busé and Jouanolou in [2], where they develop a formalism for discriminants for polynomials with coefficients in a ring. In particular $A_1 = d_1\sigma = (d_1' + d_1'')\sigma$ and f_1 is equal to the product $f_1' \cdot f_1''$ of two polynomials with respective degrees d_1', d_1'', the following factorization holds:

$$\begin{aligned}\Delta_{d_1\sigma,\ldots,d_n\sigma}(f_1,\ldots,f_n) &= \Delta_{d_1'\sigma,\ldots,d_n\sigma}(f_1',\ldots,f_n)\cdot\Delta_{d_1''\sigma,\ldots,d_n\sigma}(f_1'',\ldots,f_n)\\ &\quad\cdot \mathrm{Res}_{d_1'\sigma,d_1''\sigma,\ldots,d_n\sigma}(f_1', f_1'', \ldots, f_n)^2.\end{aligned} \tag{6.5}$$

It is straightforward to see from the definition, that in case $\Delta_{A_1',\ldots,A_n}(f_1',\ldots,f_n) = 0$ or $\Delta_{A_1'',\ldots,A_n}(f_1'',\ldots,f_n) = 0$ or $\mathrm{Res}_{A_1',A_1'',\ldots,A_n}(f_1', f_1'', \ldots, f_n) = 0$ then,

$$\Delta_{A_1'+A_1'',\ldots,A_n}(f_1'f_1'', f_2, \ldots, f_n) = 0.$$

It follows from [14] that when each support configuration A_i is full, the Newton polytope of the discriminant $\Delta_{A_1'+A_1'',A_2,\ldots,A_n}(f_1'f_1'', f_2, \ldots, f_n)$ equals the Minkowski sum of the Newton polytopes of the discriminants $\Delta_{A_1',A_2,\ldots,A_n}(f_1', f_2, \ldots, f_n)$ and $\Delta_{A_1'',A_2,\ldots,A_n}(f_1'', f_2, \ldots, f_n)$ plus two times the

Newton polytope of the resultant $\mathrm{Res}_{A_1',A_1'',A_2,\ldots,A_n}(f_1', f_1'', f_2, \ldots, f_n)$. So, a first guess would be that the factorization into the three factors in (6.5) above holds for general supports. We will see in Corollary 7 that other factors may occur, which we describe explicitly.

This behaviour already occurs in the univariate case:

Example 1 *Let* $A_1' = \{0, i_1, \ldots, i_m, d_1\}, A_1'' = \{0, j_1, \ldots, j_l, d_2\}$ *be the support sets of* $f_1' = a_0 + a_{i_1}x^{i_1} + \cdots + a_{i_m}x^{i_m} + a_{d_1}x^{d_1}$, $f_1'' = b_0 + b_{j_1}x^{j_1} + \cdots + b_{j_l}x^{j_l} + b_{d_2}x^{d_2}$ *respectively. Then*

$$\Delta(f_1' f_1'') = \Delta(f_1') \cdot \Delta(f_1'') \cdot R(f_1', f_1'')^2 \cdot E,$$

where $E = a_0^{i_1-m_0}\, b_0^{j_1-m_0}\, a_{d_1}^{d_1-i_m-m_1}\, b_{d_2}^{d_2-j_l-m_1}$, *with* $m_0 := \min\{i_1, j_1\}$ *and* $m_1 := \min\{d_1 - i_m, d_2 - j_l\}$. *On the other hand, in the full case* $i_1 = j_1 = 1, i_m = d_1 - 1, j_l = d_2 - 1$, *thus* $E = 1$ *because its exponents are equal to zero.*

6.3 A General Formula

The aim of this section is to present a formula which relates the mixed discriminant with the resultant of the given polynomials and their toric Jacobian, whose definition we recall.

Definition 3 *Let* $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ *be n Laurent polynomials in n variables. The associated toric Jacobian* J_f^T *equals* $x_1 \cdots x_n$ *times the determinant of the Jacobian matrix of f, or equivalently, the determinant of the matrix:*

$$\begin{bmatrix} x_1 \dfrac{\partial f_1}{\partial x_1} & \cdots & x_n \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ x_1 \dfrac{\partial f_n}{\partial x_1} & \cdots & x_n \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Note that the Newton polytope of J_f^T is contained in the sum of the Newton polytopes of $f_1, \ldots, f_n$.

As we remarked before, we will mainly deal in this chapter with the case $n = 2$. Also, to avoid excessive notation and make the main results cleaner, we assume below that A_1, A_2 are two finite lattice configurations whose convex hulls satisfy

$$\dim(Q_1) = \dim(Q_2) = 2.$$

Let f_1, f_2 be polynomials with respective supports A_1, A_2:

$$f_i(x) = \sum_{\alpha \in A_i} c_{i,\alpha} x^\alpha, \quad i = 1, 2,$$

where $x = (x_1, x_2)$. We denote by Σ the set of primitive inner normals $\eta \in (\mathbb{Z}^2)^*$ of the edges of the convex hull of $A_1 + A_2$. We call A_i^η the face of A_i where the inner product with η is minimized. We call this minimum value v_i^η. We also denote by f_i^η the subsum of terms in f_i with exponents in this face

$$f_i^\eta(x) = \sum_{\alpha \in A_i^\eta} c_{i,\alpha} x^\alpha, \quad i = 1, 2,$$

which is η-homogeneous of degree v_i^η. Now, A_i^η is either a vertex of A_i (but not of both A_1, A_2 since two vertices do not give a Minkowski sum edge), or its convex hull is an edge of A_i (with inner normal η), which we denote by e_i^η. Note that if the face of $A_1 + A_2$ associated to η is a vertex, both polynomials f_i^η are monomials and their resultant locus has codimension two.

We denote by $\mu_i(\eta)$ $(i = 1, 2)$ the integer defined by the following difference:

$$\mu_i(\eta) = \min\{\langle \eta, m \rangle, m \in A_i - A_i^\eta\} - v_i^\eta \tag{6.6}$$

and by

$$\mu(\eta) = \min\{\mu_1(\eta), \mu_2(\eta)\}, \tag{6.7}$$

the minimum of these two integers. Note that by our assumption that $\dim(Q_i) = 2$, we have that $\mu(\eta) \geq 1$.

Without loss of generality, we can translate the support sets A_1^η, A_2^η to the origin and consider the line L^η containing them. The resultant (cycle) $\mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)$ is considered as before, with respect to the lattice $L^\eta \cap \mathbb{Z}^2$.

Remark 6.3 As in Remark 6.2, if f_1^η is a monomial, the resultant equals the coefficient of f_1^η raised to the normalized length $\ell(e_2^\eta)$ of the edge e_2^η of A_2 (that is, the number of integer points in the edge, minus 1). If η is an inner normal of edges A_1^η and A_2^η, pick points $a_{i,0}^\eta \in A_i^\eta, i = 1, 2$, the resultant we consider equals the irreducible resultant raised to the index of the lattice generated by $\{a - a_{i,0}^\eta, a \in A_i^\eta, i = 1, 2\}$ in $L^\eta \cap \mathbb{Z}^2$. Note that the exponent $\mu(\eta) = 1$ if at least one of the configurations is full.

The following is our main result. We present a rather complete sketch of the proof; a full proof requires further technical tools related to the notions in [15], which will be given for the general case in [7]. We recall our convention that resultants and discriminants are defined as the irreducible equations raised to the lattice indices that define the corresponding cycles.

Theorem 3 *Let f_1, f_2 be generic Laurent polynomials with respective supports A_1, A_2. Then, we have the following equality of polynomials up to a nonzero rational number:*

$$\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T) = \Delta_{A_1,A_2}(f_1, f_2) \cdot E,$$

where the factor E equals the finite product:

$$E = \prod_{\eta \in \Sigma} \mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}.$$

Proof. Let X be the projective toric variety associated to $A_1 + A_2$. This compact variety consists of an open dense set T_X isomorphic to the torus $(\mathbb{C}^*)^2$ plus one toric Weil divisor D_η for each $\eta \in \Sigma$. The Laurent polynomials f_1, f_2, J_f^T define sections L_1, L_2, L_J of globally generated line bundles on X. The resultant $\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)$ vanishes if and only if L_1, L_2, L_J have a common zero on X, which could be at T_X or at any of the D_η. This indicates the only possible factors of the resultant.

There is an intersection point at T_X if and only if there is a common zero of f_1, f_2 and J_f^T in the torus $(\mathbb{C}^*)^2$. In this case, the discriminant $\Delta_{A_1,A_2}(f_1, f_2)$ vanishes. It follows that $\Delta_{A_1,A_2}(f_1, f_2)$ divides $\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)$ (the indices $[\mathbb{Z}^2 : \boldsymbol{\mathcal{L}}_{A_1,A_2}]$ and $[\mathbb{Z}^2 : \boldsymbol{\mathcal{L}}_{A_1,A_2,A_1+A_2}]$ are equal).

If instead there is a common zero at some D_η, this translates into the fact that f_1^η, f_2^η and $(J_f^T)^\eta = J_{f^\eta}^T$ (with obvious definition) have a common solution. But as f_i^η are η-homogeneous, they satisfy the weighted Euler equalities:

$$\eta_1 x_1 \frac{\partial f_i^\eta}{\partial x_1} + \eta_2 x_2 \frac{\partial f_i^\eta}{\partial x_2} = \nu_i^\eta f_i^\eta, \quad i = 1, 2, \tag{6.8}$$

from which we deduce that $J_{f^\eta}^T$ lies in the ideal $I(f_1^\eta, f_2^\eta)$ and so, the three polynomials will vanish exactly when there is a nontrivial common zero of f_1^η and f_2^η. This implies that all facet resultants $\mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)$ divide $\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)$.

Now, we wish to see that the resultant $\mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)$ raised to the power $\mu(\eta)$ occurs as a factor. The following argument would be better written in terms of the multihomogeneous polynomials in the Cox coordinates of X which represent L_1, L_2, L_J [4]. Fix a primitive inner normal direction $\eta \in \Sigma$ of $A_1 + A_2$, let t be a new variable and define the following polynomials

$$F_i(t, x) = \sum_{\alpha \in A_i} c_{i,\alpha} t^{\langle \eta, \alpha \rangle - \nu_i^\eta} x^\alpha, \quad i = 1, 2, \tag{6.9}$$

so that

$$F_i(1, x) = f_i(x), \quad F_i(0, x) = f_i^\eta(x), \quad i = 1, 2,$$

and we can write

$$F_i(t,x) = f_i^{\eta}(t,x) + t^{\mu_i(\eta)} g_i(x) + t^{\mu_i(\eta)+1} h_i(t,x), \quad i = 1,2, \tag{6.10}$$

where the polynomials $g_i(x)$ and $h_i(t,x)$ are defined by these equalities. Note that our assumption that the convex hulls Q_1, Q_2 have dimension two implies that $g_1, g_2 \neq 0$.

For each t, we deduce from the bilinearity of the determinant, that there exists a polynomial $H(t,x)$ such that the toric Jacobian of F_1, F_2 can be written as $J_F^T = J_{f^\eta}^T + t^{\mu(\eta)} H(t,x)$. But, as we remarked, $J_{f^\eta}^T$ lies in the ideal $I(f_1^\eta, f_2^\eta)$. Note that if for instance $\eta_1 \neq 0$, then the power of x_1 in each monomial of F_i can be obtained from the power of t and the power of x_2, that is, we could use t and x_2 as "variables" instead. We will denote by Res^X the resultant defined over X [4]. Therefore,

$$\mathrm{Res}_{A_1,A_2,A_1+A_2}(F_1, F_2, J_F^T) = \mathrm{Res}^X_{A_1,A_2,A_1+A_2}(F_1, F_2, t^{\mu(\eta)} H_2).$$

Now, it follows from Theorem 2 that

$$\mathrm{Res}^X_{A_1,A_2,A_1+A_2}(F_1, F_2, t^{\mu(\eta)}) = \mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}.$$

Setting $t = 0$ we see that $\mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}$ is a factor of $\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)$.

If we prove that no positive power of t divides H_2 for generic coefficients, we get the desired factorization considering all possible $\eta \in \Sigma$. To see this, first note that up to multiplying each f_i by a monomial (that is, after translation of each A_i) we can assume without loss of generality that $\nu_1^\eta = \nu_2^\eta = 0$. It follows from (6.8) that $J_f^T = 0$. The polynomials g_i in (6.10) are η-homogeneous of respective degrees $\mu_1(\eta), \mu_2(\eta)$. Assume $\mu(\eta) = \mu_1(\eta) \leq \mu_2(\eta)$. In case $\mu_1(\eta) < \mu_2(\eta)$, the coefficient of t^{μ_1} in J_F^T equals the toric Jacobian $J^T_{g_1, f_2^\eta}$ of g_1 and f_2^η, which are η-homogeneous polynomials with *different* η-degrees (equal to $\mu_1(\eta) > \nu_2 = 0$). It is easy to check that $J^T_{g_1, f_2^\eta}$ is a nonzero polynomial in the coefficients of g_1, f_2^η. In case $\mu_1 = \mu_2$, we get another term which is the toric Jacobian $J^T_{f_1^\eta, g_2}$ of f_1^η and g_2, which is nonzero by the same arguments and depends on different coefficients than $J^T_{g_1, f_2^\eta}$. Thus, their sum is not the zero polynomial, as wanted.

Theorem 3 and the proof will be extended to the general n-variate setting in a forthcoming paper [7]. We only state here the following general version without proof. Recall that a lattice polytope P of dimension n in $\mathbb{R}^n$ is said to be *smooth* if at each every vertex there are n concurrent facets and their primitive inner normal directions form a basis of $\mathbb{Z}^n$. In particular, integer dilates of the unit simplex or the unit (hyper)cube are smooth.

Theorem 4 *Let $P \subset \mathbb{R}^n$ be a smooth lattice polytope of dimension n. Let $A_i = (d_i P) \cap \mathbb{Z}^n$, $i = 1, \ldots, n$, $d_1, \ldots, d_n \in \mathbb{Z}_{>0}$, and $f_1, \ldots, f_n$ polynomials with these supports, respectively. Then, we have the following factorization*

$$\mathrm{Res}_{A_1,\ldots,A_n,A_1+\cdots+A_n}(f_1, \ldots, f_n, J_f^T) = \Delta_{A_1,\ldots,A_n}(f_1, \ldots, f_n) \cdot E,$$

where the factor E equals the finite product:

$$E = \prod_{\eta \in \Sigma} \mathrm{Res}_{A_1^\eta,\ldots,A_n^\eta}(f_1^\eta, \ldots, f_n^\eta).$$

Note that all the exponents in E equal 1 and all the lattice indices equal 1.

When the given lattice configurations A_i are the lattice points $d_i\sigma$ of the d_i-th dilate of the standard simplex σ in $\mathbb{R}^n$ (that is, in the homogeneous case studied in [2]), formula (6.4) gives, for any n in our notation:

$$\mathrm{Res}_{d_1\sigma,\ldots,d_n\sigma,\delta\sigma}(f_1, \ldots, f_n, J_i) =$$
$$\Delta_{d_1\sigma,\ldots,d_n\sigma}(f_1, \ldots, f_n) \cdot \mathrm{Res}_{(d_1\sigma)^{e_i},\ldots,(d_n\sigma)^{e_i}}(f_1^{e_i}, \ldots, f_n^{e_i}),$$

where $e_0, \ldots, e_n$ are the canonical basis vectors (or $e_0 = -e_1 - \cdots - e_n$, if we consider the corresponding dehomogenized polynomials, by setting $x_0 = 1$). Note that Theorem 4 gives the following more symmetric formula:

Corollary 5 *With the previous notation, it holds:*

$$\mathrm{Res}_{d_1\sigma,\ldots,d_n\sigma,(d_1+\cdots+d_n)\sigma}(f_1, \ldots, f_n, J_f^T) =$$
$$\Delta_{d_1\sigma,\ldots,d_n\sigma}(f_1, \ldots, f_n) \cdot \prod_{i=0}^{n} \mathrm{Res}_{(d_1\sigma)^{e_i},\ldots,(d_n\sigma)^{e_i}}(f_1^{e_i}, \ldots, f_n^{e_i}).$$

It is straightforward to deduce from this expression the degree of the homogeneous mixed discriminant, obtained independently in [1, 2, 21]. Similar formulas can be obtained, for instance, in the multihomogeneous case.

We recall the following definition from [3]. If v is a vertex of A_i, we define its *mixed multiplicity* as

$$mm_{A_1,A_2}(v) := MV(Q_1, Q_2) - MV(C_i, Q_j), \quad \{i, j\} = \{1, 2\}, \tag{6.11}$$

where $C_i = conv(A_i - \{v\})$.

Let $\Sigma' \subset \Sigma$ be the set of inner normals of $A_1 + A_2$ that cut out, or define, edges e_i^η in both Q_1, Q_2. The factorization formula in Theorem 3 can be written as follows, and allows us to recover the bidegree formulas for planar mixed discriminants in [3].

Corollary 6 *Let A_1, A_2 be two lattice configurations of dimension 2 in the plane, and let f_1, f_2 be polynomials with these respective supports. Then, the resultant*

of f_1, f_2 and their toric Jacobian, namely $\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)$*, factors as follows:*

$$\Delta_{A_1,A_2}(f_1, f_2) \cdot \prod_{v \text{ vertex of } A_1 \text{ or } A_2} c_v^{mm_{A_1,A_2}(v)} \cdot \prod_{\eta \in \Sigma'} \mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}. \tag{6.12}$$

The bidegree (δ_1, δ_2) of the mixed discriminant $\Delta_{A_1,A_2}(f_1, f_2)$ in the coefficients of f_1 and f_2, respectively, is then given by the following:

$$Vol(Q_j) + 2 \cdot MV(Q_1, Q_2) - \sum_{\eta \in \Sigma'} \ell(e_j^\eta) \cdot \mu(\eta) - \sum_{v \text{ vertex of } (A_i)} mm_{A_1,A_2}(v), \tag{6.13}$$

where $i \in \{1, 2\}$, $i \neq j$.

Proof. To prove equality (6.12), we need to show by Theorem 3 that the factor

$$E = \prod_{\eta \in \Sigma} \mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}$$

equals the product

$$\prod_{v \text{ vertex of } A_1 \text{ or } A_2} c_v^{mm_{A_1,A_2}(v)} \cdot \prod_{\eta \in \Sigma'} \mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}.$$

When $\eta \in \Sigma'$, i.e. η is a common inner normal to edges of both Q_1 and Q_2, we get the same factor on both terms, since that our quantity $\mu(\eta)$ coincides with the index $\min\{u(e_1(\eta), A_1), u(e_2(\eta), A_2)\}$, in the notation of [3].

Assume then that η is only an inner normal to Q_2. So, A_1^η is a vertex v, $f_1^\eta = cx^v$ is a monomial (with coefficient c) and f_2^η is a polynomial whose support equals the edge e_2^η of A_2 orthogonal to η. In this case, $\mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta) = c^{\ell(f_\eta)}$ by Remark 6.2.

For such a vertex v, denote by $\boldsymbol{\mathcal{E}}(v)$ the set of those $\eta' \notin \Sigma'$ for which $v + e_2^{\eta'}$ is an edge of $Q_1 + Q_2$. Note that it follows from the proof of [3, Prop.3.13] (cf. in particular Figure 1 there), that there exist non negative integers $\mu'(\eta')$ such that

$$\mathrm{mm}(v) = \sum_{\eta' \in \boldsymbol{\mathcal{E}}(v)} \ell(e_2^{\eta'}) \cdot \mu'(\eta').$$

Indeed, $\mu(\eta') = \mu'(\eta')$.

To compute the bidegree, we use the multilinearity of the mixed volume with respect to Minkowski sum. Observe that the toric Jacobian has bidegree $(1, 1)$ in the coefficients of f_1, f_2, from which we get that the bidegree of the resultant $\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)$ is equal to

$$(2MV(A_1, A_2) + \mathrm{Vol}(Q_2),\ 2MV(A_1, A_2) + \mathrm{Vol}(Q_1)). \tag{6.14}$$

Substracting the degree of the other factors and taking into account that the bidegree of the resultant $\mathrm{Res}_{A_1^\eta, A_2^\eta}(f_1^\eta, f_2^\eta)$ equals $(\ell(e_2^\eta), \ell(e_1^\eta))$, we deduce the formula (6.13), as desired.

6.4 The Multiplicativity of the Mixed Discriminant

This section studies the factorization of the discriminant when one of the polynomials factors. We make the hypothesis that f_1', f_1'', f_2 have fixed support sets $A_1', A_1'', A_2 \subseteq \mathbb{Z}^2$. So $f_1 = f_1' \cdot f_1''$ has support in the Minkowski sum $A_1 := A_1' + A_1''$; in fact, its support is generically equal to A_1. We will denote by $\mu'(\eta)$ (resp. $\mu''(\eta)$) the integer defined in (6.7), with A_1 replaced by A_1' (resp. A_1'').

Corollary 7 *Assume A_1', A_1'' and A_2 are full planar configurations of dimension 2. Let f_1', f_1'', f_2 be generic polynomials with these supports and let $f_1 = f_1' \cdot f_1''$. Then,*

$$\Delta_{A_1,A_2}(f_1, f_2) = \Delta_{A_1',A_2}(f_1', f_2) \cdot \Delta_{A_1'',A_2}(f_1'', f_2) \cdot \mathrm{Res}_{A_1',A_1'',A_2}(f_1', f_1'', f_2)^2 \cdot E,$$

where E equals the following product:

$$\prod_{\eta\in\Sigma} \mathrm{Res}_{(A_1')^\eta,A_2^\eta}((f_1')^\eta, f_2^\eta)^{\mu'(\eta)-\mu(\eta)} \cdot \mathrm{Res}_{(A_1'')^\eta,A_2^\eta}((f_1'')^\eta, f_2^\eta)^{\mu''(\eta)-\mu(\eta)}. \tag{6.15}$$

Proof. By Theorem 3, we get that

$$\Delta_{A_1,A_2}(f_1, f_2) = \frac{\mathrm{Res}_{A_1,A_2,A_1+A_2}(f_1, f_2, J_f^T)}{\prod_{\eta\in\Sigma} \mathrm{Res}_{A_1^\eta,A_2^\eta}(f_1^\eta, f_2^\eta)^{\mu(\eta)}}, \tag{6.16}$$

and similarly for $\Delta_{A_1',A_2}(f_1', f_2)$ and $\Delta_{A_1'',A_2}(f_1'', f_2)$. Let us write the numerator of (6.16) as follows:

$$\mathrm{Res}_{A_1'+A_1'',A_2,A_1'+A_1''+A_2}(f_1' f_1'', f_2, J^T_{f_1' f_1'', f_2}),$$

where $J^T_{f_1' f_1'', f_2} = f_1' J^T_{f_1'', f_2} + f_1'' J^T_{f_1', f_2}$. We now apply Theorem 2 to re-write it as follows:

$$\begin{aligned}&\mathrm{Res}_{A_1',A_2,A_1'+A_1''+A_2}(f_1', f_2, J^T_{f_1' f_1'', f_2}) \,\mathrm{Res}_{A_1'',A_2,A_1'+A_1''+A_2}(f_1'', f_2, J^T_{f_1' f_1'', f_2})\\ &= \mathrm{Res}_{A_1',A_2,A_1'+A_1''+A_2}(f_1', f_2, f_1'' J^T_{f_1',f_2}) \,\mathrm{Res}_{A_1'',A_2,A_1'+A_1''+A_2}(f_1'', f_2, f_1' J^T_{f_1'',f_2}),\end{aligned}$$

because the resultant of $\{h_1, h_2 + gh_1, \dots\}$ equals the resultant of $\{h_1, h_2, \dots\}$, for any choice of polynomials h_1, h_2, g (with suitable supports). We employ again

Theorem 2 to finalize the numerator as follows:

$$\mathrm{Res}_{A_1',A_2,A_1'+A_2}(f_1', f_2, J^T_{f_1',f_2}) \cdot \mathrm{Res}_{A_1'',A_2,A_1''+A_2}(f_1'', f_2, J^T_{f_1'',f_2}) \cdot \mathrm{Res}_{A_1',A_1'',A_2}(f_1', f_1'', f_2)^2.$$

For the denominator of (6.16), we use again Theorem 2 to write:

$$\prod_{\eta\in\Sigma'} \mathrm{Res}_{A_1'^{\eta},A_2^{\eta}}(f_1'^{\eta}, f_2^{\eta})^{\mu'(\eta)} \cdot \prod_{\eta\in\Sigma''} \mathrm{Res}_{A_1''^{\eta},A_2^{\eta}}(f_1''^{\eta}, f_2^{\eta})^{\mu''(\eta)} =$$

$$\prod_{\eta\in\Sigma} \mathrm{Res}_{A_1'^{\eta}+A_1''^{\eta},A_2^{\eta}}(f'^{\eta}_1 f''^{\eta}_1, f_2^{\eta})^{\mu(\eta)} \cdot E,$$

because the products

$$\prod_{\eta\in\Sigma\setminus\Sigma'} \mathrm{Res}_{A_1'^{\eta},A_2^{\eta}}(f_1'^{\eta}, f_2^{\eta})^{\mu'(\eta)} = \prod_{\eta\in\Sigma\setminus\Sigma''} \mathrm{Res}_{A_1''^{\eta},A_2^{\eta}}(f_1''^{\eta}, f_2^{\eta})^{\mu''(\eta)} = 1,$$

since $f_1'^{\eta}, f_2^{\eta}$ (resp. $f_1''^{\eta}, f_2^{\eta}$) are both monomials. To conclude the proof, simply assemble the above equations.

As a consequence, we have $\deg_{A_1,A_2} \Delta(f_1, f_2) =$

$$= \deg_{A_1',A_2} \Delta(f_1', f_2) + \deg_{A_1'',A_2} \Delta(f_1'', f_2) + 2 \cdot \deg_{A_1',A_1'',A_2} \mathrm{Res}(f_1', f_1'', f_2) - \deg(E).$$

When all the configurations are full and with the same normal fan, all the exponents $\mu(\eta) = \mu'(\eta) = \mu''(\eta) = 1$. Therefore, $E = 1$ and no extra factor occurs.

We define $\mu_1'(\eta), \mu_1''(\eta)$ as in (6.6). Indeed, we now fix η and will simply write $\mu_1', \mu_1'', \mu_1, \mu_2$. It happens that only one of the factors associated to η can occur in E with non zero coefficient. More explicitly, we have the following corollary, whose proof is straightforward.

Corollary 8 *With the notations of Corollary 7, for any $\eta \in \Sigma$ it holds that:*

- *If $\mu_1' = \mu_1''$, then $\mu' = \mu'' = \mu$ and there is no factor in E "coming from η".*
- *If $\mu_1' \neq \mu_1''$, assume wlog that $\mu_1 = \mu_1' < \mu_1''$. There are three subcases:*
  - *If $\mu_2 \leq \mu_1$, again there is no factor in E "coming from η".*
  - *If $\mu_1 = \mu_1' < \mu_2 < \mu_1''$, then the resultant $\mathrm{Res}_{(A_1')^{\eta},A_2^{\eta}}((f_1')^{\eta}, f_2^{\eta})$ does not occur, but $\mathrm{Res}_{(A_1'')^{\eta},A_2^{\eta}}((f_1'')^{\eta}, f_2^{\eta})$ has nonzero exponent (this resultant could just be the coefficient of a vertex raised to the mixed multiplicity).*
  - *If $\mu_1 = \mu_1' < \mu_1'' \leq \mu_2$, the situation is just the opposite than in the previous case.*

Conclusion and Future Work

The intent of this book chapter was to present our main results relating the mixed discriminant of two bivariate Laurent polynomials with fixed support, with the sparse resultant of two bivariate Laurent polynomials with fixed support and their toric Jacobian. On our way, we deduced a general multiplicativity formula for the mixed discriminant when one polynomial factors as $f = f' \cdot f''$. This formula occurred as a consequence of our main result, Theorem 3, and generalized known formulas in the homogeneous case to the sparse setting. Furthermore, we obtained a new proof of the bidegree formula for planar mixed discriminants, which appeared in [3].

The generalization of our formulas to any number of variables will allow us to extend our applications and to develop effective computational techniques for sparse discriminants based on software for the computation of resultants.

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References

1. O. Benoist, Degrés d'homogénéité de l'ensemble des intersactions complètes singulières. Annales de l'Institut Fourier **62**(3), 1189–1214 (2012)
2. L. Busé, J.-P. Jouanolou, On the discriminant scheme of homogeneous polynomials. Math. Comput. Sci. **2**(8), 175–234 (2014)
3. E. Cattani, M.A. Cueto, A. Dickenstein, S. Di Rocco, B. Sturmfels, Mixed discriminants. Math. Z. **274**, 761–778 (2013)
4. E. Cattani, A. Dickenstein, B. Sturmfels, Residues and resultants. J. Math. Sci. **5**, 119–148 (1998). University of Tokyo
5. C. D'Andrea, M. Sombra, A Poisson formula for the sparse resultant (2013, preprint). arXiv:1310.6617
6. A. Dickenstein, A world of binomials, in *Foundations of Computational Mathematics*, Hong Kong, 2008, ed. by F. Cucker, A. Pinkus, M. Todd. London Mathematical Society Lecture Note Series, vol. 363 (Cambridge University Press, Cambridge/New York, 2009), pp. 42–66
7. A. Dickenstein, A. Esterov, Mixed discriminants and toric jacobians (2014, manuscript in progress)
8. A. Dickenstein, E.M. Feichtner, B. Sturmfels, Tropical discriminants. J. Am. Math. Soc. **20**, 1111–1133 (2007)
9. A. Dickenstein, J.M. Rojas, K. Rusek, J. Shih, Extremal real algebraic geometry and A-discriminants. Mosc. Math. J. **7**(3), 425–452 (2007)

10. A. Dickenstein, L.F. Tabera, Singular tropical hypersurfaces. Discret. Comput. Geom. **47**(2), 430–453 (2012)
11. I.Z. Emiris, V. Fisikopoulos, C. Konaxis, L. Penaranda, An Oracle-Based, Output-Sensitive algorithm for projections of resultant polytopes. Int. J. Comput. Geom. Appl. Spec. Issue **23**(4), 397–423 (2014)
12. I.Z. Emiris, T. Kalinka, C. Konaxis, T. Luu Ba, Sparse implicitization by interpolation: characterizing non-exactness and an application to computing discriminants. J. Comput. Aided Des. Spec. Issue **45**, 252–261 (2013)
13. I.Z. Emiris, E. Tsigaridas, G. Tzoumas, The predicates for the exact Voronoi diagram of ellipses under the Euclidean metric. Int. J. Comput. Geom. Appl. **18**(6), 567–597 (2008)
14. A. Esterov, Newton polyhedra of discriminants of projections. Discret. Comput. Geom. **44**(1), 96–148 (2010)
15. A. Esterov, The discriminant of a system of equations. Adv. Math. **245**, 534–572 (2013)
16. J.-C. Faugére, G. Moroz, F. Rouillier, M. Safey El Din, Classification of the perspective-three-point problem: discriminant variety and real solving polynomial systems of inequalities, in *Proceedings of ACM ISSAC*, Hagenberg (2008), pp. 79–86
17. I.M. Gel'fand, M.M. Kapranov, A.V. Zelevinsky, Hypergeometric functions and toric varieties. Funktsional. Anal. i Prilozhen. **23**(2), 12–26 (1989)
18. I.M. Gel'fand, M.M. Kapranov, A.V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants* (Birkhäuser, Boston, 1994)
19. L. González-Vega, I. Nacula, Efficient topology determination of implicitly defined algebraic plane curves. Comput. Aided Geom. Des. **19**(9), 719–743 (2002)
20. G. Moroz, F. Rouiller, D. Chablat, P. Wenger, On the determination of cusp points of 3-RPR parallel manipulators. Mech. Mach. Theory **45**(11), 1555–1567 (2010)
21. J. Nie, Discriminants and non-negative polynomials. J. Symb. Comput. **47**, 167–191 (2012)
22. P. Pedersen, B. Sturmfels, Product formulas for resultants and Chow forms. Math. Z. **214**, 377–396 (1993)
23. E.F. Rincón, Computing tropical linear spaces. J. Symb. Comput. **51**, 86–93 (2013)
24. M. Shub, S. Smale, Complexity of Bézout's theorem I. Geometric aspects. J. Am. Math. Soc. **6**(2), 459–501 (1993)
25. B. Sturmfels, The Newton polytope of the resultant. J. Algebr. Comb. **3**, 207–236 (1994)
26. J.J. Sylvester, Sur l'extension de la théorie des résultants algébriques. Comptes Rendus de l'Académie des Sciences **LVIII**, 1074–1079 (1864)