Chapter 2
Mathematical Modeling and Analysis of Nonlinear Time-Invariant RLC Circuits

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Abstract We give a basic and self-contained introduction to the mathematical description of electrical circuits that contain resistances, capacitances, inductances, voltage, and current sources. Methods for the modeling of circuits by differential–algebraic equations are presented. The second part of this paper is devoted to an analysis of these equations.

Keywords Electrical circuits · Modelling · Differential–algebraic equations · Modified nodal analysis · Modified loop analysis · Graph theory · Maxwell's equations

2.1 Introduction

It is in fact not difficult to convince scientists and nonscientists of the importance of electrical circuits; they are nearly everywhere! To mention only a few, electrical circuits are essential components of power supply networks, automobiles, television sets, cell phones, coffee machines, and laptop computers (the latter two items have been heavily involved in the writing process of this article). This gives a hint to their large economical and social impact to the today's society.

When electrical circuits are designed for specific purposes, there are, in principle, two ways to verify their serviceability, namely the "construct-trial-and-error approach" and the "simulation approach." Whereas the first method is typically cost-intensive and may be harmful to the environment, simulation can be done a priori on a computer and gives reliable impressions on the dynamic circuit behavior even before it is physically constructed. The fundament of simulation is the mathematical model. That is, a set of equations containing the involved physical quantities (these are typically voltages and currents along the components) is formulated, which is later on solved numerically. The purpose of this article is a detailed and self-contained introduction to mathematical modeling of the rather simple but nevertheless important class of time-invariant nonlinear RLC circuits. These are analog

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circuits containing voltage and current sources as well as resistances, capacitances, and inductances. The physical properties of the latter three components will be assumed to be independent of time, but they will be allowed to be nonlinear. Under some additional, physically meaningful, assumptions on the components, we will further depict and discuss several interesting mathematical features of circuit models and give back-interpretation to physics.

Apart from the high practical relevance, the mathematical treatment of electrical circuits is interesting and challenging especially due to the fact that various different mathematical disciplines are involved and combined, such as graph theory, ordinary and partial differential equations, differential–algebraic equations, vector analysis, and numerical analysis.

This article is organized as follows: In Sect. 2.3, we introduce the physical quantities that are involved in circuit theory. Based on the fact that every electrical phenomenon is ultimately caused by electromagnetic field effects, we present their mathematical model (namely *Maxwell's equations*) and define the physical variables voltage, current, and energy by means of electric and magnetic field and their interaction. We particularly highlight model simplifications that are typically made for RLC circuits. Section 2.4 is then devoted to the famous *Kirchhoff laws*, which can be mathematically inferred from the findings of the preceding section. It will be shown that graph theory is a powerful tool to formulate these equations and analyze their properties. Thereafter, in Sect. 2.5, we successively focus on mathematical description of sources, resistances, inductances, and capacitances. The relation between voltage and current along these components and their energetic behavior is discussed. Kirchhoff and component relations are combined in Sect. 2.6 to formulate the overall circuit model. This leads to the modeling techniques of *modified nodal analysis* and *modified loop analysis*. Both methods lead to *differential–algebraic equations* (DAEs), whose fundamentals are briefly presented as well. Special emphasis is placed on mathematical properties of DAE models of RLC circuits.

2.2 Nomenclature

Throughout this article we use the following notation.

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^{n,m}$	the set of real $n \times m$
I_n	identity matrix of size $n \times n$
$M^{\mathrm{T}} \in \mathbb{R}^{m,n}$, $x^{\mathrm{T}} \in \mathbb{R}^{1,n}$	transpose of the matrix $M \in \mathbb{R}^{n,m}$ and the vector $x \in \mathbb{R}^n$
$\operatorname{im} M$, $\ker M$	image and kernel of a matrix M, resp.
$M > (\geq) 0$,	the square real matrix M is symmetric positive (semi)definite
$\lVert x \rVert$	$= \sqrt{x^{\mathrm{T}} x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\mathcal{V}^{\perp}$	orthogonal space of $\mathcal{V} \subset \mathbb{R}^n$

$\mathrm{sign}(\cdot)$	sign function, i.e., $\mathrm{sign}: \mathbb{R} \to \mathbb{R}$ with $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(0) = 0$, and $\mathrm{sign}(x) = -1$ if $x < 0$
t	time variable ($\in \mathbb{R}$)
ξ	space variable ($\in \mathbb{R}^3$)
ξ_x, ξ_y, ξ_z	components of the space variable $\xi \in \mathbb{R}^3$
e_x, e_y, e_z	canonical unit vectors in $\mathbb{R}^3$
$\nu(\xi)$	positively oriented tangential unit vector of a curve $\mathcal{S} \subset \mathbb{R}^3$ in $\xi \in \mathcal{S}$
$n(\xi)$	positively oriented normal unit vector of an oriented surface $\mathcal{A} \subset \mathbb{R}^3$ in $\xi \in \mathcal{A}$
$u \times v$	vector product of $u, v \in \mathbb{R}^3$
$\mathrm{grad}\, f(t,\xi)$	gradient of the scalar-valued function f with respect to the spatial variable
$\mathrm{div}\, f(t,\xi)$, $\mathrm{curl}\, f(t,\xi)$	divergence and, respectively, curl of an $\mathbb{R}^3$-valued function f with respect to the spatial variable
$\partial\Omega$ ($\partial\mathcal{A}$)	boundary of a set $\Omega \subset \mathbb{R}^3$ (surface $\mathcal{A} \subset \mathbb{R}^3$)
$\int_{\mathcal{S}} f(\xi)\, ds(\xi)$ ($\oint_{\mathcal{S}} f(\xi)\, ds(\xi)$)	integral of a scalar-valued function f over a (closed) curve $\mathcal{A} \subset \mathbb{R}^3$
$\iint_{\mathcal{A}} f(\xi)\, dS(\xi)$ ($\oiint_{\mathcal{A}} f(\xi)\, dS(\xi)$)	integral of a scalar-valued function f over a (closed) surface $\mathcal{A} \subset \mathbb{R}^3$
$\iiint_{\Omega} f(\xi)\, dV(\xi)$	integral of a scalar-valued function f over a domain $\Omega \subset \mathbb{R}^3$

The following abbreviations will be furthermore used:

DAE	differential–algebraic equation (see Sect. 2.6)
KCL	Kirchhoff's current law (see Sects. 2.4 and 2.3)
KVL	Kirchhoff's voltage law (see Sects. 2.4 and 2.3)
MLA	Modified loop analysis (see Sect. 2.6)
MNA	Modified nodal analysis (see Sect. 2.6)
ODE	ordinary differential equation (see Sect. 2.6)

2.3 Fundamentals of Electrodynamics

We present some basics of classical electrodynamics. A fundamental role is played by *Maxwell's equations*. The concepts of voltage and current will be derived from these fundamental concepts and laws. The derivations will be done by using tools from vector calculus, such as the Gauss and Stokes theorems. Note that, in this section (as well as in Sect. 2.5, where the component relations will be derived), we will not present all derivations with full mathematical precision. For an exact presentation of smoothness properties on the involved surfaces, boundaries, curves, and functions to guarantee the applicability of the Gauss theorem and the Stokes theorem and interchanging the order of integration (and differentiation), we refer to textbooks on vector calculus, such as [1, 31, 37].

2.3.1 The Electromagnetic Field

The following physical quantities are involved in an electromagnetic field.

D: electric displacement,	B: magnetic flux intensity,
E: electric field intensity,	H: magnetic field intensity,
j: electric current density,	ρ: electric charge density.

The current density and flux and field intensities are $\mathbb{R}^3$-valued functions depending on time $t \in I \subset \mathbb{R}$ and spatial coordinate $\xi \in \Omega$, whereas the electric charge density $\rho : I \times \Omega \to \mathbb{R}$ is scalar-valued. The interval I expresses the time period, and $\Omega \subset \mathbb{R}^3$ is the spatial domain in which the electromagnetic field evolves. The dependencies of the above physical variables are expressed by *Maxwell's equations* [40, 57], which read

$$\operatorname{div} D(t,\xi) = \rho(t,\xi), \qquad \text{charge induces electrical fields,} \tag{1a}$$

$$\operatorname{div} B(t,\xi) = 0, \qquad \text{field lines of a magnetic flux are closed,} \tag{1b}$$

$$\operatorname{curl} E(t,\xi) = -\frac{\partial}{\partial t} B(t,\xi), \qquad \text{law of induction,} \tag{1c}$$

$$\operatorname{curl} H(t,\xi) = j(t,\xi) + \frac{\partial}{\partial t} D(t,\xi), \qquad \text{magnetic flux law.} \tag{1d}$$

Further algebraic relations between electromagnetic variables are involved. These are called *constitutive relations* and are material-dependent. That is, they express the properties of the medium in which electromagnetic waves evolve. Typical constitutive relations are

$$E(t,\xi) = f_e\big(D(t,\xi),\xi\big), \qquad H(t,\xi) = f_m\big(B(t,\xi),\xi\big), \tag{2a}$$

$$j(t,\xi) = g\big(E(t,\xi),\xi\big) \tag{2b}$$

for some functions $f_e, f_m, g : \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$. In the following, we collect some assumptions on f_e, f_m, and g made in this article. Their practical interpretation is subject of subsequent parts of this article.

Assumption 3.1 (Constitutive relations)

(a) There exists some function $V_e : \mathbb{R}^3 \times \Omega \to \mathbb{R}$ (electric energy density) with $V_e(D,\xi) > 0$ and $V_e(0,\xi) = 0$ for all $\xi \in \Omega$, $D \in \mathbb{R}^3$, which is differentiable with respect to D and satisfies

$$\frac{\partial}{\partial D} V_e^{\mathrm{T}}(D,\xi) = f_e(D,\xi) \quad \text{for all } D \in \mathbb{R}^3, \xi \in \Omega. \tag{3}$$

(b) There exists some function $V_m : \mathbb{R}^3 \times \Omega \to \mathbb{R}$ (magnetic energy density) with $V_m(B,\xi) > 0$ and $V_m(0,\xi) = 0$ for all $\xi \in \Omega$, $B \in \mathbb{R}^3$, which is differentiable with respect to B and satisfies

$$\frac{\partial}{\partial B} V_m^{\mathrm{T}}(B,\xi) = f_m(B,\xi) \quad \text{for all } B \in \mathbb{R}^3, \xi \in \Omega. \tag{4}$$

(c) $E^{\mathrm{T}} g(E,\xi) \geq 0$ for all $E \in \mathbb{R}^3$, $\xi \in \Omega$.

If f_e and f_m are linear, assumptions (a) and (b) reduce to

$$V_e(D,\xi) = D^{\mathrm{T}} M_e(\xi)^{-1} D, \qquad V_m(B,\xi) = B^{\mathrm{T}} M_m(\xi)^{-1} B$$

for some symmetric and matrix-valued functions $M_e, M_m : \Omega \to \mathbb{R}^{3,3}$ such that $M_e(\xi) > 0$ and $M_m(\xi) > 0$ for all $\xi \in \Omega$. The functional relations between field intensities, displacement, and flux intensity then read

$$D(t,\xi) = M_e(\xi) E(t,\xi) \quad \text{and} \quad B(t,\xi) = M_m(\xi) H(t,\xi).$$

A remarkable special case is *isotropy*. That is, M_e and M_m are pointwise scalar multiples of the unit matrix, that is,

$$M_e = \varepsilon(\xi) I_3, \qquad M_m = \mu(\xi) I_3$$

for positive functions $\varepsilon, \mu : \Omega \to \mathbb{R}$. In this case, electromagnetic waves propagate with velocity $c(\xi) = (\varepsilon(\xi) \cdot \mu(\xi))^{-1/2}$ through $\xi \in \Omega$. In vacuum, we have

$$\varepsilon \equiv \varepsilon_0 \approx 8.85 \cdot 10^{-12}\ \mathrm{A} \cdot \mathrm{s} \cdot \mathrm{V}^{-1} \cdot \mathrm{m}^{-1},$$

$$\mu \equiv \mu_0 \approx 1.26 \cdot 10^{-6}\ \mathrm{m} \cdot \mathrm{kg} \cdot \mathrm{s}^{-2} \cdot \mathrm{A}^{-2}.$$

Consequently, the quantity

$$c_0 = (\varepsilon_0 \cdot \mu_0)^{-1/2} \approx 3.00\ \mathrm{m} \cdot \mathrm{s}^{-1}$$

is the speed of light [30, 34].

As we will see soon, the function g has the physical interpretation of an *energy dissipation rate*. That is, it expresses energy transfer to thermodynamic domain. In the linear case, this function reads

$$g(E,\xi) = G(\xi) \cdot E,$$

where $G : \Omega \to \mathbb{R}^{3,3}$ is a matrix-valued function with the property that $G(\xi) + G^{\mathrm{T}}(\xi) \geq 0$ for all $\xi \in \Omega$. In perfectly isolating media (such as the vacuum), the electric current density vanishes; the dissipation rate consequently vanishes there.

Assuming that f_e, f_m, and g fulfill Assumptions 3.1, we define the electric energy at time $t \in I$ as the spatial integral of the electric energy density over Ω at time t. Consequently, the magnetic energy is the spatial integral of the magnetic energy density over Ω at time t, and the electromagnetic energy at time t is the sum over these two quantities, that is,

$$W(t) = \iiint_{\Omega} \big(V_e\big(D(t,\xi),\xi\big) + V_m\big(B(t,\xi),\xi\big)\big)\, dV(\xi).$$

We are now going to derive an energy balance for the electromagnetic field: First, we see, by using elementary vector calculus, that the temporal derivative of the total energy density fulfills

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(V_e\big(D(t,\xi),\xi\big) + V_m\big(B(t,\xi),\xi\big)\big)\\
&\quad = \frac{\partial}{\partial D} V_e\big(D(t,\xi),\xi\big) \cdot \frac{\partial}{\partial t} D(t,\xi) + \frac{\partial}{\partial B} V_m\big(B(t,\xi),\xi\big) \cdot \frac{\partial}{\partial t} B(t,\xi)\\
&\quad = E^{\mathrm{T}}(t,\xi) \cdot \frac{\partial}{\partial t} D(t,\xi) + H^{\mathrm{T}}(t,\xi) \cdot \frac{\partial}{\partial t} B(t,\xi)\\
&\quad = E^{\mathrm{T}}(t,\xi) \cdot \operatorname{curl} H(t,\xi) - E^{\mathrm{T}}(t,\xi) \cdot g\big(E(t,\xi)\big) - H^{\mathrm{T}}(t,\xi) \cdot \operatorname{curl} E(t,\xi)\\
&\quad = \operatorname{div}\big(E(t,\xi) \times H(t,\xi)\big) - E^{\mathrm{T}}(t,\xi) \cdot g\big(E(t,\xi)\big).
\end{aligned} \tag{5a}$$

The fundamental theorem of calculus and the Gauss theorem then implies the energy balance

$$\begin{aligned}
W(t_2) - W(t_1) &= \int_{t_1}^{t_2} \iiint_{\Omega} \frac{\partial}{\partial t}\big(V_e\big(D(t,\xi),\xi\big) + V_m\big(B(t,\xi),\xi\big)\big)\, dV(\xi)\, dt\\
&= \int_{t_1}^{t_2} \iiint_{\Omega} \operatorname{div}\big(E(t,\xi) \times H(t,\xi)\big)\, dV(\xi)\, dt\\
&\quad - \int_{t_1}^{t_2} \iiint_{\Omega} E^{\mathrm{T}}(t,\xi) \cdot g\big(E(t,\xi)\big)\, dV(\xi)\, dt\\
&= \int_{t_1}^{t_2} \oiint_{\partial\Omega} n^{\mathrm{T}}(\xi) \cdot \big(E(t,\xi) \times H(t,\xi)\big)\, dS(\xi)\\
&\quad - \int_{t_1}^{t_2} \iiint_{\Omega} E^{\mathrm{T}}(t,\xi) \cdot g\big(E(t,\xi)\big)\, dV(\xi)\, dt\\
&\le \int_{t_1}^{t_2} \oiint_{\partial\Omega} n^{\mathrm{T}}(\xi) \big(E(t,\xi) \times H(t,\xi)\big)\, dS(\xi).
\end{aligned} \tag{5b}$$

A consequence of the above finding is that energy transfer is done by dissipation and via the outflow of the *Poynting vector field* $E \times H : I \times \Omega \to \mathbb{R}^3$.

The electromagnetic field is not uniquely determined by Maxwell's equations. Besides imposing suitable initial conditions on electric displacement and magnetic flux, that is,

$$D(0,\xi)=D_0(\xi), \qquad B(0,\xi)=B_0(\xi), \quad \xi\in\Omega. \tag{6}$$

To fully describe the electromagnetic field, we further have to impose physically (and mathematically) reasonable boundary conditions [40]. These are typically zero conditions if $\Omega=\mathbb{R}^3$ (that is, $\lim_{\|\xi\|\to\infty}E(t,\xi)=\lim_{\|\xi\|\to\infty}H(t,\xi)=0$) or, in case of bounded domain Ω with smooth boundary, tangential or normal conditions on electrical or magnetic field, such as, for instance,

$$\begin{aligned} &n(\xi)\times\big(E(t,\xi)-E_b(t,\xi)\big)=0, &&n(\xi)\times\big(H(t,\xi)-H_b(t,\xi)\big)=0,\\ &n^{\mathrm{T}}(\xi)\big(E(t,\xi)-E_b(t,\xi)\big)=0, &&n^{\mathrm{T}}(\xi)\big(H(t,\xi)-H_b(t,\xi)\big)=0, \quad \xi\in\partial\Omega. \end{aligned} \tag{7}$$

2.3.2 *Currents and Voltages*

Here we introduce the physical quantities that are crucial for circuit analysis.

Definition 3.2 (Electrical current) Let $\Omega\subset\mathbb{R}^3$ describe a medium in which an electromagnetic field evolves. Let $\mathcal{A}s\subset\Omega$ be an oriented surface. Then the *current through* $\mathcal{A}$ is defined by the surface integral of the current density, that is,

$$i(t)=\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot j(t,\xi)\,dS(\xi). \tag{8}$$

Remark 3.3 (Orientation of the surface) Reversing the orientation of the surface means changing the sign of the current. The indication of the direction of a current is therefore a matter of the orientation of the surface.

Remark 3.4 (Electrical current in the case of absent charges/stationary case) Let $\Omega\subset\mathbb{R}^3$ be a domain, and $\mathcal{A}\subset\Omega$ be a surface. If the medium does not contain any electric charges (i.e., $\rho\equiv 0$), then we obtain from Maxwell's equations that the current through A is

$$\begin{aligned} i(t)&=\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot j(t,\xi)\,dS(\xi)\\ &=\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot \operatorname{curl} H(t,\xi)\,dS(\xi)-\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot\frac{\partial}{\partial t}D(t,\xi)\,dS(\xi)\\ &=\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot \operatorname{curl} H(t,\xi)\,dS(\xi)-\frac{d}{dt}\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot D(t,\xi)\,dS(\xi). \end{aligned}$$

Elementary calculus implies that curl H is divergence free, that is,

$$\operatorname{div}\operatorname{curl} H(t,\xi) = 0.$$

The absence of electric charges moreover gives rise to

$$\operatorname{div} D(t,\xi) = 0.$$

We consider two case scenarios:

(a) $\Omega \in \mathbb{R}^3$ is star-shaped. Poincaré's lemma [1] and the divergence-freeness of the electric displacement implies the existence of an *electric vector potential* $F : I \times \Omega \to \mathbb{R}^3$ such that

$$D(t,\xi) = \operatorname{curl} F(t,\xi).$$

The Stokes theorem then implies that the current through $\mathcal{A}$ reads

$$\begin{aligned} i(t) &= \iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot \operatorname{curl} H(t,\xi)\, dS(\xi) - \frac{d}{dt}\iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot \operatorname{curl} F(t,\xi)\, dS(\xi) \\ &= \oint_{\partial\mathcal{A}} \nu^{\mathrm{T}}(\xi)\cdot H(t,\xi)\, ds(\xi) - \frac{d}{dt}\oint_{\partial\mathcal{A}} \nu^{\mathrm{T}}(\xi)\cdot F(t,\xi)\, ds(\xi). \end{aligned}$$

Consequently, the current through the surface $\mathcal{A}$ is solely depending on the behavior of the electromagnetic field on the boundary $\partial\mathcal{A}$. In other words, if $\partial\mathcal{A}_1 = \partial\mathcal{A}_2$ for $\mathcal{A}_1, \mathcal{A}_2 \subset \Omega$, then the current through $\mathcal{A}_1$ equals the current through $\mathcal{A}_2$.

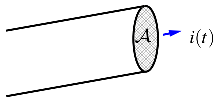
Note that the condition that $\Omega \subset \mathbb{R}^3$ is star-shaped can be relaxed to the *second de Rham cohomology* of Ω being trivial, that is, $H^2_{\mathrm{dR}}(\Omega) \tilde{=} \{0\}$ [1]. This is again a purely topological condition on Ω, that is, a continuous and continuously invertible deformation of Ω does not influence the de Rham cohomology.

It can be furthermore seen that the above findings are true as well if the topological condition on Ω, together with the absence of electric charges, is replaced with the physical assumption that the electric displacement is stationary, that is, $\frac{\partial}{\partial t} D \equiv 0$. This follows by

$$\begin{aligned} i(t) &= \iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot j(t,\xi)\, dS(\xi) \\ &= \iint_{\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot \operatorname{curl} H(t,\xi)\, dS(\xi) - \iint_{\partial\mathcal{A}} n^{\mathrm{T}}(\xi)\cdot \underbrace{\frac{\partial}{\partial t} D(t,\xi)}_{=0}\, dS(\xi) \\ &= \iint_{\partial\mathcal{A}} \nu^{\mathrm{T}}(\xi)\cdot H(t,\xi)\, dS(\xi). \end{aligned} \tag{9}$$

Now consider a wire as presented in Fig. 1, which is assumed to be surrounded by a perfect isolator (that is, the $n^{\mathrm{T}}(\xi)\, j(\xi) = 0$ at the boundary of the wire).

Fig. 1 Electrical current through surface $\mathcal{A}$



Let $\mathcal{A}$ be a cross-sectional area across the wire. If the wire does not contain any charges or the electric field inside the wire is stationary, an application of the above argumentation implies that the current of a wire is well-defined in the sense that it does not depend on the particular choice of a cross-sectional area. This enables to speak about the *current through a wire*.

(b) Now assume that $\mathcal{V} \subset \Omega$ is a domain with sufficiently smooth boundary and consider the current though $\partial\mathcal{V}$. Applying the Gauss theorem, we obtain that, under the assumption $\rho \equiv 0$, the integral of the outward component of the current density vanishes for any closed surface, that is,

$$\begin{aligned}
&\oiint_{\partial\mathcal{V}} n^{\mathrm{T}}(\xi) \cdot j(t,\xi)\, dS(\xi) \\
&= \oiint_{\partial\mathcal{V}} n^{\mathrm{T}}(\xi) \cdot \operatorname{curl} H(t,\xi)\, dS(\xi) - \frac{d}{dt} \oiint_{\partial\mathcal{V}} n^{\mathrm{T}}(\xi) \cdot D(t,\xi)\, dS(\xi) \\
&= \iiint_{\mathcal{V}} \underbrace{\operatorname{div}\operatorname{curl} H(t,\xi)}_{=0}\, dV(\xi) - \frac{d}{dt} \iiint_{\mathcal{V}} \underbrace{\operatorname{div} D(t,\xi)}_{=0}\, dV(\xi) = 0.
\end{aligned}$$

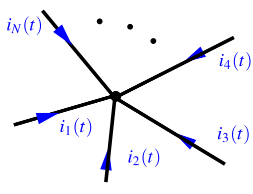
Further note that, again, under the alternative assumption that the field of electric displacement is stationary, the surface integral of the current density over $\partial\Omega$ vanishes as well (compare (9)).

In each of the above two cases, we have

$$\oiint_{\partial\Omega} n^{\mathrm{T}}(\xi) \cdot j(t,\xi)\, dS(\xi) = 0.$$

Now we focus on a conductor node as presented in Fig. 2 and assume that no charges are present or that the electric field inside the conductor node is stationary. Again assuming that all wires are surrounded by perfect isolators, we can choose a domain $\Omega \subset \mathbb{R}^3$ such that, for $k = 1, \ldots, N$, the boundary $\partial\Omega$ intersects with the kth wire to the cross-sectional area $\mathcal{A}_k$. Define the number $s_k \in \{1, -1\}$ to be positive if $\mathcal{A}_k$ has the same orientation of $\partial\Omega$ (that is, $i_k(t)$ is an outflowing current) and $s_k = -1$ otherwise (that is, $i_k(t)$ is an inflowing current). Then, by making use

Fig. 2 Conductor node



of the assumption that the current density is trivial outside the wires we obtain

$$0=\iint_{\partial\Omega} n^{\mathrm{T}}(\xi)\cdot\operatorname{curl} H(t,\xi)\,dS(\xi)=\sum_{k=1}^{N} s_k \iint_{\mathcal{A}_k} n^{\mathrm{T}}(\xi)\cdot\operatorname{curl} H(t,\xi)\,dS(\xi)$$
$$=\sum_{k=1}^{N} s_k \iint_{\mathcal{A}_k} n^{\mathrm{T}}(\xi)\cdot j(t,\xi)\,dS(\xi)=\sum_{k=1}^{N} s_k i_k(t),$$

where i_k is the current of the kth wire. This is known as *Kirchhoff's current law*.

Theorem 3.5 (Kirchhoff's current law (KCL)) *Assume that a conductor node is given that is surrounded by a perfect isolator. Further assume that the electric field is stationary or the node does not contain any charges. Then the sum of inflowing currents equals to the sum of inflowing currents.*

Next, we introduce the concept of electric voltage.

Definition 3.6 (Electrical voltage) Let $\Omega\subset\mathbb{R}^3$ describe a medium in which an electromagnetic field evolves. Let $\mathcal{S}\subset\Omega$ be a path (see Fig. 3). Then the *voltage along* $\mathcal{S}$ is defined by the path integral

$$u(t)=\int_{\mathcal{S}} \nu^{\mathrm{T}}(\xi)E(t,\xi)\,ds(\xi). \tag{10}$$

Remark 3.7 (Orientation of the path) The sign of the voltage is again a matter of the orientation of the path. That is, a change of the orientation of $\mathcal{S}$ results in replacing $u(t)$ be $-u(t)$ (compare Remark 3.3).

Remark 3.8 (Electrical current in the stationary case) If the field of magnetic flux intensity is stationary ($\frac{\partial}{\partial t}B\equiv 0$), then the Maxwell equations give rise to $\operatorname{curl} E\equiv 0$. Moreover, assuming that the spatial domain in which the stationary electromagnetic field evolves is simply connected [31], the electric field intensity is a gradient field, that is,

$$E(t,\xi)=\operatorname{grad}\Phi(t,\xi)$$

for some differentiable scalar-valued function Φ, which we call *an electric potential*. For a path $S_s\subset\Omega$ from ξ_0 to ξ_1, we have

$$\int_{S_s} \nu^{\mathrm{T}}(\xi)\cdot E(t,\xi)\,ds(\xi)=\Phi(t,\xi_1)-\Phi(t,\xi_0). \tag{11}$$

In particular, the voltage along S_s is solely depending on the initial and end point of S_s. This enables to speak about the *voltage between the points* ξ_0 *and* ξ_1.

Note that the electric potential is unique up to addition of a function independent on the spatial coordinate ξ. It can therefore be made unique by imposing the

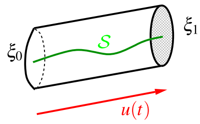


Fig. 3 Voltage along $\mathcal{S}$

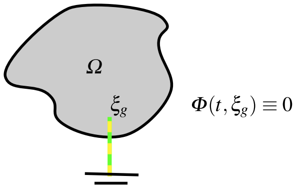


Fig. 4 Grounding of ξ_g

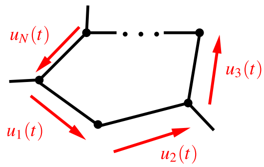


Fig. 5 Conductor loop

additional relation $\Phi(t, \xi_g) = 0$ for some prescribed position $\xi_g \in \Omega$. In electrical engineering, this is called *grounding of* ξ_g (see Fig. 4).

Now we take a closer look at a loop of conductors (see Fig. 5) in which the field of magnetic flux is assumed to be stationary:

For $k = 1, \ldots, N$, assume that $\mathcal{S}_k$ is a path in the kth conductor connecting its nodes. Assume that the field of magnetic flux intensity is stationary and let $u_k(t)$ be the voltage between the initial and terminal point of $\mathcal{S}_k$. Define the number $s_k \in \{1, -1\}$ to be positive if $\mathcal{S}_k$ is in the direction of the loop and $s_k = -1$ otherwise. Taking a surface $\mathcal{A} \subset \Omega$ that is surrounded by the path

$$\mathcal{S}_1 \dot{\cup} \cdots \dot{\cup} \mathcal{S}_N = \partial \mathcal{A},$$

we can apply the Stokes theorem to see that

$$\begin{aligned} \sum_{k=0}^{N} s_k \cdot u_k(t) &= \sum_{k=0}^{N} s_k \cdot \int_{\mathcal{S}_k} \nu^{\mathrm{T}}(\xi) \cdot E(t, \xi)\, ds(\xi) \\ &= \oint_{\partial A} \nu^{\mathrm{T}}(\xi) \cdot E(t, \xi)\, ds(\xi) \\ &= \iint_{\mathcal{A}} n^{\mathrm{T}}(\xi) \cdot \operatorname{curl} E(t, \xi)\, dS(\xi) = 0. \end{aligned}$$

Theorem 3.9 (Kirchhoff's voltage law (KVL)) *In an electromagnetic field in which the magnetic flux is stationary, each conductor loop fulfills that the sum of voltages in direction of the loop equals the sum of voltages in the opposite direction to the loop.*

In the following, we will make some further considerations concerning energy and power transfer in stationary electromagnetic fields ($\frac{\partial}{\partial t}D \equiv \frac{\partial}{\partial t}B \equiv 0$) evolving in simply connected domains. Assuming that we have some electrical device in the domain $\Omega \subset \mathbb{R}^3$ that is physically closed in the sense that no current leaves the device (i.e., $n^{\mathrm{T}}(\xi)j(t,\xi) = 0$ for all $\xi \in \partial\Omega$), an application of the multiplication rule

$$\operatorname{div}\big(j(t,\xi)\Phi(t,\xi)\big) = \operatorname{div} j(t,\xi)\cdot\Phi(t,\xi) + j^{\mathrm{T}}(t,\xi)\cdot\operatorname{grad}\Phi(t,\xi)$$

and the Gauss theorem lead to

$$\begin{aligned}
&\iiint_\Omega j^{\mathrm{T}}(t_1,\xi)\cdot E(t_2,\xi)\,dV(\xi)\\
&\quad= \iiint_\Omega j^{\mathrm{T}}(t_1,\xi)\cdot\operatorname{grad}\Phi(t_2,\xi)\,dV(\xi)\\
&\quad= -\iiint_\Omega \operatorname{div} j(t_1,\xi)\cdot\Phi(t_2,\xi)\,dV(\xi) + \iiint_\Omega \operatorname{div}\big(j(t_1,\xi)\cdot\Phi(t_2,\xi)\big)\,dV(\xi)\\
&\quad= -\iiint_\Omega \underbrace{\operatorname{div} j(t_1,\xi)}_{=0}\cdot\Phi(t_2,\xi)\,dV(\xi)\\
&\qquad+ \oint\!\!\oint_{\partial\Omega} \underbrace{n^{\mathrm{T}}(\xi)j(t_1,\xi)}_{=0}\cdot\Phi(t_2,\xi)\,dV(\xi) = 0.
\end{aligned} \tag{12}$$

In other words, the spatial L_2-inner product [17] between $j(t_1,\cdot)$ and the field $E(t_1,\cdot)$ vanishes for all times t_1, t_2 in which the stationary electrical field evolves.

Theorem 3.10 (Tellegen's law for stationary electromagnetic fields) *Let a stationary electromagnetic field inside the simply connected domain $\Omega \subset \mathbb{R}^3$ be given, and assume that no electrical current leaves Ω. Then for all times t_1, t_2 in which the field evolves, the current density field $j(t_1,\cdot)$ and the electrical field density field $E(t,\cdot)$ are orthogonal in the L_2-sense.*

The concluding considerations in this section are concerned with energy inside conductors in which stationary electromagnetic fields evolve. Consider an electrical wire as displayed in Fig. 3. Assume that $\mathcal{S}$ is a path connecting the incidence nodes ξ_0, ξ_1. Furthermore, for each $\xi \in S$, let $\mathcal{A}_\xi$ be a cross-sectional area containing ξ and assume the additional property that the spatial domain of the wire Ω is the disjoint union of the surfaces $\mathcal{A}_\xi$, that is,

$$\Omega = \dot{\bigcup_{\xi\in\mathcal{S}}} \mathcal{A}_\xi.$$

The KCL implies that the current through $\mathcal{A}_\xi$ does not depend on $\xi \in \mathcal{S}$. Now making the (physically reasonable) assumptions that the voltage is spatially constant in each cross-sectional area $\mathcal{A}_\xi$ and using the Gauss theorem and the multiplication rule, we obtain

$$(\operatorname{curl} E)^{\mathrm{T}}(t,\xi)\cdot H(t,\xi) - E^{\mathrm{T}}(t,\xi)\cdot \operatorname{curl} H(t,\xi) = \operatorname{div}\bigl(E(t,\xi)\times H(t,\xi)\bigr).$$

From this we see that the following holds for the product between the voltage along and the current through the wire:

$$\begin{aligned}
u(t)i(t) &= \int_{\mathcal{S}} \nu^{\mathrm{T}}(\xi)\cdot E(t,\xi)\,ds(\xi)\cdot \iint_{\mathcal{A}_{\xi_1}} n^{\mathrm{T}}(\xi)\cdot j(t,\zeta)\,dS(\zeta)\\
&= \int_{\mathcal{S}} \nu^{\mathrm{T}}(\xi)\cdot E(t,\xi)\cdot \iint_{\mathcal{A}_{\xi}} n^{\mathrm{T}}(\xi)\cdot j(t,\zeta)\,dS(\zeta)\,ds(\xi)\\
&= \iiint_{\Omega} E^{\mathrm{T}}(t,\xi)\cdot j(t,\xi)\,dV(\xi)\\
&= \iiint_{\Omega} E^{\mathrm{T}}(t,\xi)\cdot \operatorname{curl} H(t,\xi)\,dV(\xi)\\
&= \iiint_{\Omega} (\operatorname{curl} E)^{\mathrm{T}}(t,\xi)\cdot H(t,\xi) - E^{\mathrm{T}}(t,\xi)\cdot \operatorname{curl} H(t,\xi)\,dV(\xi)\\
&= \iiint_{\Omega} \operatorname{div}\bigl(E(t,\xi)\times H(t,\xi)\bigr)\,dV(\xi)\\
&= \oiint_{\partial\Omega} n^{\mathrm{T}}(\xi)\bigl(E(t,\xi)\times H(t,\xi)\bigr)\,dV(\xi).
\end{aligned}$$

In other words, the product between $u(t)$ and $i(t)$ therefore coincides with the outflow of the Poynting vector field of the wire, whence the integral

$$W = \int_I u(t)\cdot i(t)\,dt$$

is the energy consumed by the wire.

2.3.3 Notes and References

(i) The constitutive relations with properties as in Assumptions 3.1 directly constitute an energy balance via (5a), (5b). Further types of constitutive relations can be found in [30].

(ii) The existence of global (weak, classical) solutions of Maxwell's equations in the general nonlinear case seems to be not fully worked out so far. A functional analytic approach to the linear case is, with boundary conditions sightly different from (7), in [66].

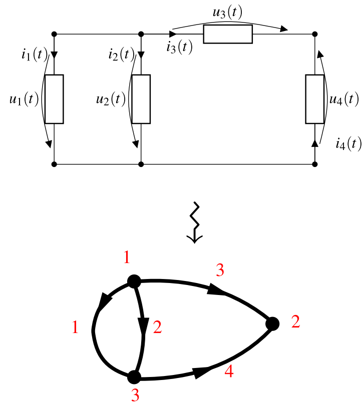


Fig. 6 Circuit as a graph

2.4 Kirchhoff's Laws and Graph Theory

In this part, we will approach the systematic description of Kirchhoff's laws inside a conductor network. To achieve this aim, we will regard an electrical circuit as a graph. Each branch of the circuit connects two nodes. To each branch of the circuit we assign a direction, which is not a physical restriction but rather a definition of the *positive direction* of the corresponding voltage and current. This definition is arbitrary, but it has to be however done in advance (compare Remarks 3.3 and 3.7). We assume that the voltage and current of each branch are equally directed. This is known as a *load reference-arrow system* [34]. This allows us to speak about an *initial node* and a *terminal node* of a branch.

Such a collection of branches can, in an abstract way, be formulated as a directed graph (see Fig. 6).

2.4.1 Graphs and Matrices

We present some mathematical fundamentals of directed graphs.

Definition 4.1 (Graph concepts) A *directed graph* (or *graph* for short) is a triple $\mathcal{G} = (V, E, \varphi)$ consisting of a *node set* V and a *branch set* E together with an *incidence map*

$$\varphi : E \to V \times V, \quad e \mapsto \varphi(e) = \big(\varphi_1(e), \varphi_2(e)\big).$$

If $\varphi(e) = (v_1, v_2)$, we call e to be *directed from* v_1 *to* v_2; v_1 is called the *initial node*, and v_2 the *terminal node* of e. Two graphs $\mathcal{G}_a = (V_a, E_a, \varphi_a)$ and $\mathcal{G}_b = (V_b, E_b, \varphi_b)$ are called *isomorphic* if there exist bijective mappings $\iota_E : E_a \to E_b$ and $\iota_V : V_a \to V_b$, such that $\varphi_{a,1} = \iota_V^{-1} \circ \varphi_{b,1} \circ \iota_E$ and $\varphi_{a,2} = \iota_V^{-1} \circ \varphi_{b,2} \circ \iota_E$.

Let $V' \subset V$, and let E' be a set of branches fulfilling

$$E' \subset E|_{V'} := \big\{e \in E : \varphi(e) \in V' \times V'\big\}.$$

Further, let $\varphi|_{E'}$ be the restriction of φ to E'. Then the triple $\mathcal{K} := (V', E', \varphi|_{E'})$ is called *a subgraph of* $\mathcal{G}$. In the case where $E' = E|_{V'}$, we call $\mathcal{K}$ the *induced subgraph on* V'. If $V' = V$, then $\mathcal{K}$ is called a *spanning subgraph*. A *proper subgraph* is that with $E \neq E'$.

$\mathcal{G}$ is called *finite* if both the node and the branch set are finite.

For each branch e, define an additional branch $-e$, which is directed from the terminal to the initial node of e, that is, $\varphi(-e) = (\varphi_2(e), \varphi_1(e))$ for $e \in E$. Now define the set $\tilde{E} = \{e, -e : e \in E\}$. A tuple $w = (w_1, \dots, w_r) \in \tilde{E}^r$ where

$$v_{k_i} := \varphi_2(w_i) = \varphi_1(w_{i+1}) \quad \text{for } i = 1, \dots, r-1$$

is called *a path from* v_{k_0} *to* v_{k_r}; w is called *an elementary path* if $v_{k_1}, \dots, v_{k_r}$ are distinct. A *loop* is an elementary path with $v_{k_0} = v_{k_r}$. A *self-loop* is a loop consisting of only one branch. Two nodes v, v' are called *connected* if there exists a path from v to v'. The graph itself is called connected if any two nodes are connected. A subgraph $\mathcal{K} := (V', E', \varphi|_{E'})$ is called *connected component* if it is connected and $\mathcal{K}^c := (V \setminus V', E \setminus E', \varphi|_{E \setminus E'})$ is a subgraph.

A *tree* is a minimally connected (spanning sub)graph, that is, it is connected without having any connected proper spanning subgraph.

For a spanning subgraph $\mathcal{K} = (V, E', \varphi|_{E'})$, we define the *complementary spanning subgraph* by $\mathcal{G} - \mathcal{K} := (V, E \setminus E', \varphi|_{E \setminus E'})$. The complementary spanning subgraph of a tree is called *a cotree*. A spanning subgraph $\mathcal{K}$ is called a *cutset* if its branch set is nonempty, $\mathcal{G} - \mathcal{K}$ is a disconnected graph, and additionally, $\mathcal{G} - \mathcal{K}'$ is connected for any proper spanning subgraph $\mathcal{K}'$ of $\mathcal{K}$.

We can set up special matrices associated to a finite graph. These will be useful to describe Kirchoff's laws.

Definition 4.2 Let a finite graph $\mathcal{G} = (V, E, \varphi)$ with n branches $E = \{e_1, \dots, e_n\}$ and m nodes $V = \{v_1, \dots, v_m\}$ be given. Assume that the graph does not contain any self-loops. The *all-node incidence matrix* of $\mathcal{G}$ is defined by $A_0 = (a_{jk}) \in \mathbb{R}^{m,n}$, where

$$a_{jk} = \begin{cases} 1 & \text{if branch } k \text{ leaves node } j, \\ -1 & \text{if branch } k \text{ enters node } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $L = \{l_1, \dots, l_b\}$ be the set of loops of $\mathcal{G}$. Then the *all-loop matrix* $B_0 = (b_{jk}) \in \mathbb{R}^{l,n}$ is defined by

$$b_{jk} = \begin{cases} 1 & \text{if branch } k \text{ belongs to loop } j \text{ and has the same orientation,} \\ -1 & \text{if branch } k \text{ belongs to loop } j \text{ and has the contrary orientation,} \\ 0 & \text{otherwise.} \end{cases}$$

2.4.2 Kirchhoff's Laws: A Systematic Description

Let $A_0 \in \mathbb{R}^{m,n}$ be the all-node incidence matrix of a graph $\mathcal{G} = (V, E, \varphi)$ with n branches $E = \{e_1, \dots, e_n\}$ and m nodes $V = \{v_1, \dots, v_m\}$ and no self-loops. The jth row of A_0 is, by definition, at the kth position, equal to 1 if the kth branch leaves the jth node. On the other hand, this entry equals to -1 if the kth branch enters the jth node. If the kth node is involved in the jth node, then this entry vanishes. Hence, defining $i_k(t)$ to be the current through the kth branch in the direction to its terminal node and defining the vector

$$i(t) = \begin{pmatrix} i_1(t) \\ \vdots \\ i_n(t) \end{pmatrix}, \tag{13}$$

the kth row vector $a_k \in \mathbb{R}^{1,n}$ gives rise to Kirchhoff's current law of the kth node via $a_k i(t) = 0$. Consequently, the collection of all Kirchhoff laws reads, in compact form,

$$A_0 i(t) = 0. \tag{14}$$

For $k \in \{1, \dots, n\}$, let $u_k(t)$ be the voltage between the initial and terminal nodes of the kth branch, and define the vector

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}. \tag{15}$$

By the same argumentation as before, the construction of the all-loop matrix gives rise to

$$B_0 u(t) = 0. \tag{16}$$

Since any column of A_0 contains exactly two nonzero entries, namely 1 and -1, we have

$$A_0^{\mathrm{T}} \cdot \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{\in \mathbb{R}^m} = 0. \tag{17}$$

This give rise to the fact that the KCL system $A_0 i(t) = 0$ contains redundant equations. Such redundancies occur more than ever in the KVL $B_0 u = 0$.

Remark 4.3 (Self-loops in electrical circuits) Kirchhoff's voltage law immediately yields that the voltage along a branch with equal incidence nodes vanishes. Kirchhoff's current law further implies that the current from a self-loop flows into the corresponding node and also flows out of this node. A consequence is that self-loops are physically neutral: Their removal does not influence the behavior of the remaining circuit. The assumption of their absence is therefore no loss of generality.

The next aim is to determine a set of (linearly) independent equations out of the so far constructed equations. To achieve this, we present several connections between some properties of the graph and its matrices A_0, B_0. We generalize the results in [7] to directed graphs. As a first observation, we may reorder the branches and nodes of $\mathcal{G} = (V, E, \varphi)$ into according to connected components such that we end up with

$$A_0 = \begin{bmatrix} A_{0,1} & & \\ & \ddots & \\ & & A_{0,k} \end{bmatrix}, \qquad B_0 = \begin{bmatrix} B_{0,1} & & \\ & \ddots & \\ & & B_{0,k} \end{bmatrix}, \tag{18}$$

where $A_{0,i}$ and $B_{0,i}$ are, respectively, the all-node incidence matrix and all-loop matrix of the ith connected component.

A spanning subgraph $\mathcal{K}$ of the finite graph $\mathcal{G}$ has an all-node incidence matrix $A_{\mathcal{K}}$, which is constructed by deleting rows of A_0 corresponding to the branches of the complementary spanning subgraph $\mathcal{G} - \mathcal{K}$. By a suitable reordering of the branches, the incidence matrix has a partition

$$A_0 = \begin{bmatrix} A_{0,\mathcal{K}} & A_{0,\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{19}$$

Theorem 4.4 *Let a finite graph $\mathcal{G} = (V, E, \varphi)$ with n branches $E = \{e_1, \ldots, e_n\}$ and m nodes $V = \{v_1, \ldots, v_m\}$ and no self-loops. Let $A_0 \in \mathbb{R}^{m,n}$ be the all-node incidence matrix of $\mathcal{G}$. Then*

(a) $\operatorname{rank} A_0 = m - k$.
(b) *$\mathcal{G}$ contains a cutset if and only if* $\operatorname{rank} A_0 = m - 1$.
(c) *$\mathcal{G}$ is a tree if and only if $A_0 \in \mathbb{R}^{m,m-1}$ and* $\ker A_0 = \{0\}$.
(d) *$\mathcal{G}$ contains loops if and only if* $\ker A_0 = \{0\}$.

Proof

(a) Since all-loop incidence matrices of nonconnected graphs allow a representation (18), the general result can be directly inferred if we prove the statement for the case where $\mathcal{G}$ is connected. Assume that A_0 is the incidence matrix of a connected graph, and assume that $A_0^{\mathrm{T}} x = 0$ for some $x \in \mathbb{R}^m$. Utilizing (17), we need to show that all entries of x are equal for showing that $\operatorname{rank} A_0 = m - 1$. By a suitable reordering of the rows of A_0 we may assume that the first k entries of x are nonzero, whereas the last $m - k$ entries are zero, that is, $x = [x_1^{\mathrm{T}}\ 0]^{\mathrm{T}}$, where all entries of x_1 are nonzero. By a further reordering of the columns we may assume that A_0 is of the form

$$A_0 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where each column vector of A_{11} is not the zero vector. This gives $A_{11}^{\mathrm{T}} x_1 = 0$.

Now take an arbitrary column vector $a_{21,i}$ of A_{21}. Since each column vector of A_0 has exactly two nonzero entries, $a_{21,i}$ either has no, one, or two nonzero entries. The latter case implies that the ith column vector of A_{11} is the zero vector, which contradicts the construction of A_{21}. If $a_{21,i}$ has exactly one nonzero entry at the jth position, the relation $x_1 A_{11} = 0$ gives rise to the fact that the jth entry of x_1 vanishes. Since this is a contradiction, the whole matrix A_{21} vanishes. Therefore, the all-node incidence matrix is block-diagonal. This however implies that none of the last $m-k$ nodes is connected to the first k nodes, which is a contradiction to $\mathcal{G}$ being connected.

(b) This result follows from (a) by using the fact that a graph contains cutsets if and only if it is connected.

(c) By definition, $\mathcal{G}$ is a tree if and only if it is connected and the deletion of an arbitrary branch results in a disconnected graph. By (a) this means that the deletion of an arbitrary column A_0 results in a matrix with rank smaller than $m-1$. This is equivalent to the columns of A_0 being linearly independent and spanning an $(n-1)$-dimensional space, in other words, $\operatorname{rank} A_0 = m-1$ and $\ker A_0 = \{0\}$.

(d) Assume that the kernel of A_0 is trivial. Seeking for a contradiction, assume that $\mathcal{G}$ contains a loop l. Define the vector $b_l = [b_{l1}, \ldots, b_{ln}] \in \mathbb{R}^{1,n} \setminus \{0\}$ with

$$b_{lk} = \begin{cases} 1 & \text{if branch } k \text{ belongs to } l \text{ and has the same orientation,} \\ -1 & \text{if branch } k \text{ belongs to } l \text{ and has the contrary orientation,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $a_1 \ldots, a_n$ be the column vectors of A_0. Then, by construction of b_l, each row of the matrix

$$\begin{bmatrix} b_{l1} a_1 & \ldots & b_{ln} a_n \end{bmatrix}$$

contains exactly one entry 1 and one entry -1 and zeros elsewhere. This implies $A_0 b_l^{\mathrm{T}} = 0$.

Conversely, assume that $\mathcal{G}$ contains no loops. By separately considering the connected components and the consequent structure (18) of A_0, it is again no loss of generality to assume that $\mathcal{G}$ is connected. Let e be a branch of $\mathcal{G}$, and let $\mathcal{K}$ be the spanning subgraph whose only branch is e. Then $\mathcal{G} - \mathcal{K}$ results in a disconnected graph (otherwise, $(e, e_{l1}, \ldots, e_{lv})$ would be a loop, where $(e_{l1}, \ldots, e_{lv})$ is an elementary path in $\mathcal{G} - \mathcal{K}$ from the terminal node to the initial node of e). This however implies that the deletion of an arbitrary column of A_0 results in a matrix with rank smaller than $n-1$, which means that the columns of A_0 are linearly independent, that is, $\ker A_0 = \{0\}$. $\square$

Since, by the dimension formula, $\dim\ker A_0^{\mathrm{T}} = k$, we can infer from (14) and (17) that $\ker A_0^{\mathrm{T}} = \operatorname{span}\{c_1, \ldots, c_k\}$, where

$$c_i = \begin{pmatrix} c_{1i} \\ \vdots \\ c_{mi} \end{pmatrix} \quad \text{with } c_{ji} = \begin{cases} 1 & \text{if branch } j \text{ belongs to the } i\text{-th connected component,} \\ 0 & \text{else.} \end{cases} \tag{20}$$

Furthermore, using the argumentation of the first part in the proof of (d), we obtain that

$$A_0 B_0^{\mathrm{T}} = 0. \tag{21}$$

We will show that the row vectors of B_0 even generate the kernel of A_0.

Based on a spanning subgraph $\mathcal{K}$ of $\mathcal{G}$, we may, by a suitable reordering of columns, perform a partition of the loop matrix according to the branches of $\mathcal{K}$ and $\mathcal{G} - \mathcal{K}$, that is,

$$B_0 = \begin{bmatrix} B_{0\mathcal{K}} & B_{0\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{22}$$

If a subgraph $\mathcal{T}$ is a tree, then any branch e in $\mathcal{G} - \mathcal{T}$ defines a loop in $\mathcal{G}$ via $(e, e_{l1}, \ldots, e_{lv})$, where $(e_{l1}, \ldots, e_{lv})$ is an elementary path in $\mathcal{T}$ from the terminal node to the initial node of e. Consequently, we may reorder the rows of $B_{\mathcal{T}}$ and $B_{\mathcal{G}-\mathcal{T}}$ to obtain the form

$$B_{0\mathcal{T}} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \qquad B_{0\mathcal{G}-\mathcal{T}} = \begin{bmatrix} I_{n-m+1} \\ B_{22} \end{bmatrix}. \tag{23}$$

Such a representation will be crucial for the proof of the following result.

Theorem 4.5 *Let $\mathcal{G} = (V, E, \varphi)$ be a finite graph with no self-loops, n branches $E = \{e_1, \ldots, e_n\}$, and m nodes $V = \{v_1, \ldots, v_m\}$, and let the all-node incidence matrix $A_0 \in \mathbb{R}^{m,n}$ and b loops $\{l_1, \ldots, l_b\}$ be given. Furthermore, let k be the number of connected components of $\mathcal{G}$. Then*

(a) $\operatorname{im} B_0^{\mathrm{T}} = \ker A_0$;
(b) $\operatorname{rank} B_0 = n - m + k$.

Proof The relation $\operatorname{im} B_0^{\mathrm{T}} \subset \ker A_0$ follows from (21). Therefore, the overall result follows if we prove that $\operatorname{rank} B_0 \geq n - m + k$. Again, by separately considering the connected components and using the block-diagonal representations (18), the overall result immediately follows if we prove the case $k = 1$. Assuming that $\mathcal{G}$ is connected, we consider a tree $\mathcal{T}$ in $\mathcal{G}$. Then we may assume that the all-loop matrix is of the form $B_0 = [B_{0\mathcal{T}}\ B_{0\mathcal{G}-\mathcal{T}}]$ with submatrices as is (23). However, since the latter submatrix has full column rank and $n - m + 1$ columns, we have

$$\operatorname{rank} B_0 \geq \operatorname{rank} B_{0\mathcal{G}-\mathcal{T}} = n - m + 1,$$

which proves the desired result. □

Statement (a) implies that the orthogonal spaces of $\operatorname{im} B_0^{\mathrm{T}}$ and $\ker A_0$ coincide as well. Therefore,

$$\operatorname{im} A_0^{\mathrm{T}} = \ker B_0.$$

To simplify verbalization, we arrange that, by referring to connectedness, the incidence matrix, loop matrix, etc. of an electrical circuit, we mean the corresponding notions and concepts for the graph describing the electrical circuit.

It is a reasonable assumption that an electrical circuit is connected; otherwise, since the connected components do not physically interact, they can be considered separately.

Since the rows of A_0 sum up to the zero row vector, one might delete an arbitrary row of A_0 to obtain a matrix A having the same rank as A_0. We call A the *incidence matrix* of $\mathcal{G}$. The property $\operatorname{rank} A_0 = \operatorname{rank} A$ implies $\operatorname{im} A_0^{\mathrm{T}} = \operatorname{im} A^{\mathrm{T}}$. Consequently, the following holds.

Theorem 4.6 (Kirchhoff's current law for electrical circuits) *Let a connected electrical circuit with n branches and m nodes and no self-loops be given. Let $A \in \mathbb{R}^{m-1,n}$, and let, for $j = 1, \dots, n$, $i_j(t)$ be the current in branch e_j in the direction of initial to terminal node of e_j. Let $i(t) \in \mathbb{R}^n$ be defined as in (13). Then for all times t,*

$$Ai(t) = 0. \tag{24}$$

We can furthermore construct the *loop matrix* $B \in \mathbb{R}^{n-m+1,n}$ by picking $n - m + 1$ linearly independent rows of B_0. This implies $\operatorname{im} B_0^{\mathrm{T}} = \operatorname{im} B^{\mathrm{T}}$, and we can formulate Kirchhoff's voltage law as follows.

Theorem 4.7 (Kirchhoff's voltage law for electrical circuits) *Let a connected electrical circuit with n branches and m nodes be given. Let $B \in \mathbb{R}^{n-m+1,n}$, and let, for $j = 1, \dots, n$, $u_j(t)$ be the voltage in branch e_j between the initial and terminal node of e_j. Let $u(t) \in \mathbb{R}^n$ be defined as in (15). Then for all times t,*

$$Bu(t) = 0. \tag{25}$$

A constructive procedure for determining the loop matrix B can be obtained from the findings in front of Theorem 4.5: Having a tree $\mathcal{T}$ in the graph $\mathcal{G}$ describing an electrical circuit, the loop matrix can be determined by

$$B = \begin{bmatrix} B_{\mathcal{T}} & I_{n-m+1} \end{bmatrix},$$

where the jth row of $B_{\mathcal{T}}$ contains the information on the path in $\mathcal{T}$ between the initial and terminal nodes of the $(m - 1 + j)$th branch of $\mathcal{G}$.

The formulations (24) and (25) of Kirchhoff's laws give rise to the fact that a connected circuit includes $n = (m - 1) + (n - m + 1)$ linearly independent Kirchhoff equations. Using Theorem 4.5 and $\operatorname{im} A_0^{\mathrm{T}} = \operatorname{im} A^{\mathrm{T}}$, $\operatorname{im} B_0^{\mathrm{T}} = \operatorname{im} B^{\mathrm{T}}$, we further have

$$\operatorname{im} B^{\mathrm{T}} = \ker A.$$

Kirchhoff's voltage law may therefore be rewritten as $u(t) \in \operatorname{im} A^{\mathrm{T}}$. Equivalently, there exists some $\phi(t) \in \mathbb{R}^{m-1}$ such that

$$u(t) = A^{\mathrm{T}}\phi(t). \tag{26}$$

The vector $\phi(t)$ is called the *node potential*. Its ith component expresses the voltage between the ith node and the node corresponding to the deleted row of A_0. This relation can therefore be interpreted as a lumped version of (11). The node potential of the deleted row is set to zero, whence the deletion of a row of A_0 can therefore be interpreted as grounding (compare Sect. 2.3).

Equivalently, Kirchhoff's current law may be reformulated in the way that there exists a *loop current* $\iota(t) \in \mathbb{R}^{n-m+1}$ such that

$$i(t) = B^{\mathrm{T}}\iota(t). \tag{27}$$

The so far developed graph theoretical results give rise to a lumped version of Theorem 3.10.

Theorem 4.8 (Tellegen's law for electrical circuits) *With the assumption and notation of Theorems 4.6 and 4.7, for all times t_1, t_2, the vectors $i(t_1)$ and $u(t_2)$ are orthogonal in the Euclidean sense, that is,*

$$i^{\mathrm{T}}(t_1)u(t_2) = 0.$$

Proof For the incidence matrix A of the graph describing the electrical circuit, let $\Phi(t_2) \in \mathbb{R}^{m-1}$ be the corresponding vector of node potentials at time t_2. Then

$$i^{\mathrm{T}}(t_1)u(t_2) = i^{\mathrm{T}}(t_1)A^{\mathrm{T}}\phi(t_2) = \big(Ai(t_1)\big)^{\mathrm{T}}\phi(t_2) = 0 \cdot \phi(t_2) = 0. \tag{28}$$

□

2.4.3 Auxiliary Results on Graph Matrices

This section closes with some further results on the connection between properties of subgraphs and linear algebraic properties of the corresponding submatrices of incidence and loop matrices. Corresponding for undirected graphs can be found in [7]. First, we declare some manners of speaking.

Definition 4.9 Let $\mathcal{G}$ be a graph, and let $\mathcal{K}$ be a spanning subgraph.

(i) $\mathcal{L}$ is called a $\mathcal{K}$-*cutset* if $\mathcal{L}$ is a cutset of $\mathcal{G}$ and a spanning subgraph of $\mathcal{K}$.
(ii) l is called a $\mathcal{K}$-*loop* if l is a loop and all branches of l are contained in $\mathcal{K}$.

Lemma 4.10 *Let $\mathcal{G}$ be a connected graph with n branches and m nodes, no self-loops, an incidence matrix $A \in \mathbb{R}^{m-1,n}$, and a loop matrix $B \in \mathbb{R}^{n-m+1,n}$. Further, let $\mathcal{K}$ be a spanning subgraph. Assume that the branches of $\mathcal{G}$ are sorted so that*

$$A = \begin{bmatrix} A_{\mathcal{K}} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{\mathcal{K}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix}.$$

(a) *The following three assertions are equivalent:*

   (i) *$\mathcal{G}$ does not contain $\mathcal{K}$-cutsets;*
   (ii) $\ker A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} = \{0\}$;
   (iii) $\ker B_{\mathcal{K}} = \{0\}$.

(b) *The following three assertions are equivalent:*

   (i) *$\mathcal{G}$ does not contain $\mathcal{K}$-loops;*
   (ii) $\ker A_{\mathcal{K}} = \{0\}$;
   (iii) $\ker B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} = \{0\}$.

Proof

(a) The equivalence of (i) and (ii) follows from Theorem 4.4 (b). To show that (ii) implies (iii), assume that $B_{\mathcal{K}} x = 0$. Then

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \in \ker \begin{bmatrix} B_{\mathcal{K}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix} = \operatorname{im} \begin{bmatrix} A_{\mathcal{K}}^{\mathrm{T}} \\ A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} \end{bmatrix},$$

that is, there exists $y \in \mathbb{R}^{m-1}$ such that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{bmatrix} A_{\mathcal{K}}^{\mathrm{T}} \\ A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} \end{bmatrix} y.$$

In particular, we have $A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} y = 0$, whence, by assumption (ii), $y = 0$. Thus, $x = A_{\mathcal{K}}^{\mathrm{T}} y = 0$.

To prove that (iii) is sufficient for (ii), we can perform the same argumentation by interchanging the roles of $A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}}$ and $B_{\mathcal{K}}$.

(b) The equivalence of (i) and (ii) follows from Theorem 4.4 (d). The equivalence of (ii) and (iii) can be proven analogously to part (a) (by interchanging the roles of $\mathcal{K}$ and $\mathcal{G} - \mathcal{K}$ and of the loop and incidence matrices). □

The subsequent two auxiliary results are concerned with properties of subgraphs of subgraphs and gives some equivalent characterizations in terms of properties of their incidence and loop matrices.

Lemma 4.11 *Let $\mathcal{G}$ be a connected graph with n branches and m nodes, no self-loops, an incidence matrix $A \in \mathbb{R}^{n-1,m}$, and a loop matrix $B \in \mathbb{R}^{n-m+1,n}$. Further,* *let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted so that*

$$A = \begin{bmatrix} A_{\mathcal{L}} & A_{\mathcal{K}-\mathcal{L}} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{\mathcal{L}} & B_{\mathcal{K}-\mathcal{L}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix},$$

and define

$$A_{\mathcal{K}} = \begin{bmatrix} A_{\mathcal{L}} & A_{\mathcal{K}-\mathcal{L}} \end{bmatrix}, \qquad B_{\mathcal{K}} = \begin{bmatrix} B_{\mathcal{L}} & B_{\mathcal{K}-\mathcal{L}} \end{bmatrix},$$
$$A_{\mathcal{G}-\mathcal{L}} = \begin{bmatrix} A_{\mathcal{K}-\mathcal{L}} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}, \qquad B_{\mathcal{G}-\mathcal{L}} = \begin{bmatrix} B_{\mathcal{K}-\mathcal{L}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix}.$$

Then the following four assertions are equivalent:

(i) *$\mathcal{G}$ does not contain $\mathcal{K}$-loops except for $\mathcal{L}$-loops*;

(ii)

$$\ker A_{\mathcal{K}} = \ker A_{\mathcal{L}} \times \{0\}.$$

(iii) *For a matrix $Z_{\mathcal{L}}$ with* $\operatorname{im} Z_{\mathcal{L}} = \ker A_{\mathcal{L}}^{\mathrm{T}}$,

$$\ker Z_{\mathcal{L}}^{\mathrm{T}} A_{\mathcal{K}-\mathcal{L}} = \{0\}.$$

(iv)

$$\ker B_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}} = \ker B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}}.$$

(v) *For a matrix $Y_{\mathcal{G}-\mathcal{K}}$ with* $\operatorname{im} Y_{\mathcal{G}-\mathcal{K}} = \ker B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}}$,

$$Y_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} B_{\mathcal{G}-\mathcal{K}} = 0.$$

Proof To show that (i) implies (ii), let $\tilde{B}_{\mathcal{K}}$ be a loop matrix of the graph $\mathcal{K}$ (note that, in general, $\tilde{B}_{\mathcal{K}}$ and $B_{\mathcal{K}}$ do not coincide). The assumption that all $\mathcal{K}$-loops are actually $\mathcal{L}$-loops implies that $\tilde{B}_{\mathcal{K}}$ is structured as

$$\tilde{B}_{\mathcal{K}} = \begin{bmatrix} \tilde{B}_{\mathcal{L}} & 0 \end{bmatrix}.$$

Since $\operatorname{im} \tilde{B}_{\mathcal{K}} = \ker A_{\mathcal{K}}$, we have $\ker A_{\mathcal{K}} = \operatorname{im} \tilde{B}_{\mathcal{L}}^{\mathrm{T}} \times \{0\}$. This further implies that $\operatorname{im} \tilde{B}_{\mathcal{L}}^{\mathrm{T}} = \ker A_{\mathcal{L}}$ or, in other words, (b) holds.

Now we show that (ii) is sufficient for (i). Let l be a loop in $\mathcal{K}$. Assume that $\mathcal{K}$ has n_K branches and $\mathcal{L}$ has n_L branches. Define the vector $b_l = [b_{l1}, \ldots, b_{ln_K}] \in \mathbb{R}^{1,m} \setminus \{0\}$ with

$$b_{lk} = \begin{cases} 1 & \text{if branch } k \text{ belongs to } l \text{ and has the same orientation,} \\ -1 & \text{if branch } k \text{ belongs to } l \text{ and has the contrary orientation,} \\ 0 & \text{otherwise.} \end{cases}$$

Then (ii) gives rise to $b_{ln_L+1} = \cdots = b_{n_K} = 0$, whence the branches of $\mathcal{K} - \mathcal{L}$ are not involved in l, that is, l is actually an $\mathcal{L}$-loop.

Aiming to show that (iii) holds, assume (ii). Let $x \in \ker Z_{\mathcal{L}}^{\mathrm{T}} A_{\mathcal{K}-\mathcal{L}}$. Then

$$A_{\mathcal{K}-\mathcal{L}} x \in \ker Z_{\mathcal{L}}^{\mathrm{T}} = (\operatorname{im} Z_{\mathcal{L}})^{\perp} = \left(\ker A_{\mathcal{L}}^{\mathrm{T}}\right)^{\perp} = \operatorname{im} A_{\mathcal{L}}.$$

Thus, there exists a real vector y such that

$$A_{\mathcal{K}-\mathcal{L}} x = A_{\mathcal{L}} y.$$

This gives rise to

$$\begin{pmatrix} -y \\ x \end{pmatrix} \in \ker \begin{bmatrix} A_{\mathcal{L}} \\ A_{\mathcal{K}-\mathcal{L}} \end{bmatrix} = \ker A_{\mathcal{K}} = \ker A_{\mathcal{L}} \times \{0\},$$

and, consequently, x vanishes.

For the converse implication, it suffices to show that (c) implies $\ker A_{\mathcal{K}} \subset \ker A_{\mathcal{L}} \times \{0\}$ (the reverse inclusion holds in any case). Assume that

$$\begin{pmatrix} y \\ x \end{pmatrix} \in \ker A_{\mathcal{K}},$$

that is, $A_{\mathcal{L}} y + A_{\mathcal{K}-\mathcal{L}} x = 0$. Multiplying this equation from the left by $Z_{\mathcal{L}}^{\mathrm{T}}$, we obtain $x \in \ker Z_{\mathcal{L}}^{\mathrm{T}} A_{\mathcal{K}-\mathcal{L}} = \{0\}$, that is, $x = 0$ and $A_{\mathcal{L}} y = 0$. Hence,

$$\begin{pmatrix} y \\ x \end{pmatrix} \in \ker A_{\mathcal{L}} \times \{0\}.$$

The following proof concerns the sufficiency of (ii) for (iv): It suffices to show that (ii) implies

$$\ker B_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}} \subset B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}}$$

since the converse inclusion holds in any case. Assume that $B_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}} x = 0$. Then

$$B^{\mathrm{T}} x = \begin{pmatrix} B_{\mathcal{L}}^{\mathrm{T}} x \\ B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} x \\ 0 \end{pmatrix} \in \ker A_{\mathcal{K}} = \ker A_{\mathcal{L}} \times \{0\},$$

whence $B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} x$.

Conversely, assume that (iv) holds and let

$$\begin{pmatrix} y \\ x \end{pmatrix} \in \ker A_{\mathcal{K}}.$$

Then

$$\begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \in \ker A = \operatorname{im} B^{\mathrm{T}} = \operatorname{im} \begin{bmatrix} B_{\mathcal{L}}^{\mathrm{T}} \\ B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} \\ B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} \end{bmatrix},$$

that is, there exists a real vector z such that $y = B_{\mathcal{L}}^{\mathrm{T}} z$, $x = B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} z$ and $B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} z = 0$. The latter implies that $x = B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} z = 0$, that is, (b) holds.

It remains to show that (iv) and (v) are equivalent. Assume that (iv) holds. Then

$$\ker B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} \subset \ker B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} = \operatorname{im} Y_{\mathcal{K}-\mathcal{L}},$$

whence

$$Y_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} B_{\mathcal{G}-\mathcal{K}} = \left(B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} Y_{\mathcal{K}-\mathcal{L}}\right)^{\mathrm{T}} = 0.$$

Finally, assume that $Y_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} B_{\mathcal{G}-\mathcal{K}} = 0$ and let $B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} x = 0$. Then $x \in \operatorname{im} Y_{\mathcal{K}-\mathcal{L}}$, that is, there exists a real vector y such that $x = Y_{\mathcal{K}-\mathcal{L}} y$. This implies

$$B_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}} x = \begin{pmatrix} B_{\mathcal{L}}^{\mathrm{T}} x \\ B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} x \end{pmatrix} = \begin{pmatrix} B_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} Y_{\mathcal{K}-\mathcal{L}} y \\ B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} Y_{\mathcal{K}-\mathcal{L}} y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So far, we have shown that $Y_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} B_{\mathcal{G}-\mathcal{K}} = 0$ implies $\ker B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} \subset \ker B_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}}$. Since the other inclusion holds in any case ($B_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}}$ is a submatrix of $B_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}}$), the overall result has been proven. □

Lemma 4.12 *Let $\mathcal{G}$ be a connected graph with n branches and m nodes, no self-loops, an incidence matrix $A \in \mathbb{R}^{m-1,n}$, and a loop matrix $B \in \mathbb{R}^{n-m+1,n}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{L}$. Assume that the branches of $\mathcal{G}$ are sorted so that*

$$A = \begin{bmatrix} A_{\mathcal{L}} & A_{\mathcal{K}-\mathcal{L}} & A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{\mathcal{L}} & B_{\mathcal{K}-\mathcal{L}} & B_{\mathcal{G}-\mathcal{K}} \end{bmatrix}.$$

Then the following four assertions are equivalent:

(i) *$\mathcal{G}$ does not contain $\mathcal{K}$-cutsets except for $\mathcal{L}$-cutsets;*
(ii) *The initial and terminal nodes of each branch of $\mathcal{K}-\mathcal{L}$ are connected by a path in $\mathcal{G}-\mathcal{K}$.*
(iii)

$$\ker A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}} = \ker A_{\mathcal{G}-\mathcal{L}}^{\mathrm{T}}.$$

(iv) *For a matrix $Z_{\mathcal{G}-\mathcal{K}}$ with $\operatorname{im} Z_{\mathcal{G}-\mathcal{K}} = \ker A_{\mathcal{G}-\mathcal{K}}^{\mathrm{T}}$,*

$$Z_{\mathcal{K}-\mathcal{L}}^{\mathrm{T}} A_{\mathcal{G}-\mathcal{K}} = 0.$$

(v)

$$\ker B_{\mathcal{K}} = \ker B_{\mathcal{L}} \times \{0\}.$$

(vi) *For a matrix $Y_{\mathcal{L}}$ with $\operatorname{im} Y_{\mathcal{L}} = \ker B_{\mathcal{L}}^{\mathrm{T}}$,*

$$\ker Y_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{K}-\mathcal{L}} = \{0\}.$$

Proof By interchanging the roles of loop and incidence matrices, the proof of equivalence of the assertions (c)–(f) is totally analogous to the proof of equivalence of (ii)–(v) in Lemma 4.11. Hence, it suffices to show that (i), (ii), and (iii) are equivalent:

First, we show that (i) implies (iii): As a first observation, note that since $A_{\mathcal{K}-\mathcal{L}}$ is a submatrix of $A_{\mathcal{K}}$, (iii) is equivalent to $\operatorname{im} A_{\mathcal{K}-\mathcal{L}} \subset \operatorname{im} A_{\mathcal{G}-\mathcal{K}}$. Now seeking for a contradiction, assume that (iii) is not fulfilled. Then, by the preliminary consideration, there exists a column vector a_1 of $A_{\mathcal{K}-\mathcal{L}}$ with $a_1 \notin \operatorname{im} A_{\mathcal{G}-\mathcal{K}}$. Now, for k as large as possible, successively construct column vectors $\tilde{a}_1, \dots, \tilde{a}_k$ of $A_{\mathcal{K}}$ with the property that

$$a_1 \notin \operatorname{im} A_{\mathcal{G}-\mathcal{K}} + \operatorname{span}\{\tilde{a}_1, \dots, \tilde{a}_i\} \quad \text{for all } i \in \{1, \dots, k\}. \tag{29}$$

Let $a_2, \dots, a_j$ be the set of column vectors of $A_{\mathcal{K}}$ that have not been chosen by the previous procedure. Since the overall incidence matrix A has full row rank, the construction of $\tilde{a}_1, \dots, \tilde{a}_k$ leads to

$$A_{\mathcal{G}-\mathcal{K}} + \operatorname{span}\{\tilde{a}_1, \dots, \tilde{a}_k, a_i\} = \mathbb{R}^{n-1} \quad \text{for all } i \in \{1, \dots, j\}. \tag{30}$$

Now construct the spanning graph $\mathcal{C}$ by taking the branches $a_1, \dots, a_j$. Due to (29), $\mathcal{G}-\mathcal{C}$ is disconnected. Furthermore, $\mathcal{C}$ contains a branch of $\mathcal{K}-\mathcal{L}$, namely the one corresponding to the column vector a_1. Since, furthermore, (30) implies that the addition of any branch of $\mathcal{C}$ to $\mathcal{G}-\mathcal{C}$ results is a connected graph, we have constructed a cutset in $\mathcal{K}$ that contains branches of $\mathcal{K}-\mathcal{L}$.

The next step is to show that (iii) is sufficient for (ii): Assume that the nodes are sorted by connected components in $\mathcal{G}-\mathcal{K}$, that is,

$$A_{\mathcal{G}-\mathcal{K}} = \operatorname{diag}(A_{\mathcal{G}-\mathcal{K},1}, \dots, A_{\mathcal{G}-\mathcal{K},n}). \tag{31}$$

Then the matrices $A_{\mathcal{G}-\mathcal{K},i}$ $i = 1, \dots, n$, are all-node incidence matrices of the connected components (except for the component i_g connected to the grounding node; then $A_{\mathcal{G}-\mathcal{K},i_g}$ is an incidence matrix). Seeking for a contradiction, assume that e is a branch in $\mathcal{K}-\mathcal{L}$ whose incidence nodes are not connected by a path in $\mathcal{G}-\mathcal{K}$. Then a_k has not more than two nonzero entries, and one of the following two cases holds:

(a) If e is connected to the grounding node, then a_k is the multiple of a unit vector corresponding to a position not belonging to the grounded component, whence $a_k \notin A_{\mathcal{G}-\mathcal{K}}$.
(b) If e connects two nongrounded nodes, then a_k has two nonzero entries, which are located at rows corresponding to two different matrices $A_{\mathcal{G}-\mathcal{K},i}$ and $A_{\mathcal{G}-\mathcal{K},j}$ in $A_{\mathcal{G}-\mathcal{K}}$. This again implies $a_k \notin A_{\mathcal{G}-\mathcal{K}}$. This is again a contradiction to (iii).

For the overall statement, it suffices to prove that (ii) implies (i). Let $\mathcal{C}$ be a cutset of $\mathcal{G}$ that is contained in $\mathcal{K}$ and assume that e is a branch of $\mathcal{C}$ that is contained in $\mathcal{K}-\mathcal{L}$. Since there exists some path in $\mathcal{G}-\mathcal{K}$ that connects the incidence nodes of e, the addition of e to $\mathcal{G}-\mathcal{C}$ (which is a supergraph of $\mathcal{G}-\mathcal{K}$) does not connect two different connected components. The resulting graph is therefore still disconnected, which is a contradiction to $\mathcal{C}$ being a cutset of $\mathcal{G}$. □

2.4.4 Notes and References

(i) The representation of the Kirchhoff laws by means of incidence and loop matrices is also called *nodal analysis* and *mesh analysis*, respectively [16, 19, 32].

(ii) The part in Proposition 4.10 about incidence matrices and subgraphs has also been shown in [22]; the parts in Lemmas 4.11 and 4.12 about incidence matrices and subgraphs have also been shown in [22]. The parts on loop matrices is novel.

(iii) The correspondence between subgraph properties and linear algebraic properties of the corresponding incidence and loop matrices is an interesting feature. It can be seen from (20) that the kernel of a transposed incidence matrix can be computed by a determination of the connected components of a graph. As well, we can infer from (23) and the preceding argumentation that loop matrices can be determined by a simple determination of a tree. Conversely, the computation of the kernel of an incidence matrix leads to the determination of the loops in a (sub)graph. It is further shown in [9, 28] that a matrix $Z_{\mathcal{L}}^{\mathrm{T}} A_{\mathcal{K}-\mathcal{L}}$ (see Lemma 4.11) has an interpretation as an incidence matrix of the graph, which is constructed from $\mathcal{K} - \mathcal{L}$ by merging those nodes that are connected by a path in $\mathcal{L}$. The determination of its nullspace thus again leads a graph theoretical problem.

Note that to determine nullspaces, graph computations are by far preferable to linear algebraic method. Efficient algorithms for the aforementioned problems can be found in [18]. Note that the aforementioned graph theoretical features have been used in [20, 21] to analyze special properties of circuit models.

2.5 Circuit Components: Sources, Resistances, Capacitances, Inductances

We have seen in the previous section that, for a connected electrical circuit with n branches and m nodes, the Kirchhoff laws lead to $n = (m - 1) + (n - m + 1)$ linearly independent algebraic equations for the voltages and currents. Since, altogether, voltages and currents are $2n$ variables, mathematical intuition gives rise to the fact that n further relations are missing to completely describe the circuit. The behavior of a circuit does, indeed, not only depend of interconnectivity, the so-called *network topology*, but also on the type of electrical components located on the branches. These can, for instance, be sources, resistances, capacitances, and inductances. These will either (such as in case of a source) prescribe the voltage or the current, or they form a relation between voltage and current of a certain branch. In this section, we will collect these relations for the aforementioned components.

Fig. 7 Symbol of a voltage source

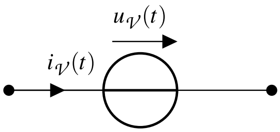


Fig. 8 Symbol of a current source

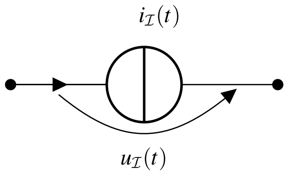
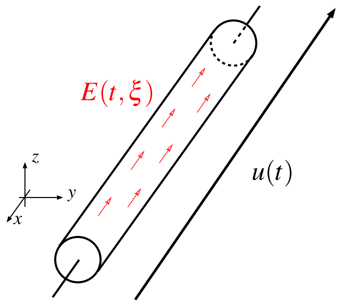


Fig. 9 Model of a resistance



2.5.1 Sources

Sources describe physical interaction of an electrical circuit with the environment. Voltage sources are elements where the voltage $u_{\mathcal{V}}(\cdot) : I \to \mathbb{R}$ is prescribed. In current sources, the current $i_{\mathcal{I}}(\cdot) : I \to \mathbb{R}$ is given beforehand. The symbols of voltage and current sources are presented in Figs. 7 and 8.

We will see in Sect. 2.6 that the physical variables $i_{\mathcal{V}}(\cdot), u_{\mathcal{I}}(\cdot) : I \to \mathbb{R}$ (and therefore also energy flow through sources) are determined by the overall electrical circuit. Some further assumptions on the prescribed functions $u_{\mathcal{V}}(\cdot), i_{\mathcal{I}}(\cdot) : I \to \mathbb{R}$ (such as, e.g., smoothness) will also depend on the connectivity of the overall circuit; this will as well be a subject of Sect. 2.6.

2.5.2 Resistances

We make the following ansatz for a resistance: Consider a conductor material in the cylindric spatial domain (see Fig. 9)

$$\Omega = [0,\ell] \times \left\{(\xi_y,\xi_z) : \xi_y^2 + \xi_z^2 \le r^2\right\} \subset \mathbb{R}^3 \tag{32}$$

with length ℓ and radius r.

For $\xi_x \in [0,\ell]$, we define the cross-sectional area by

$$\mathcal{A}_{\xi_x} = \{\xi_x\} \times \left\{(\xi_y,\xi_z) : \xi_y^2 + \xi_z^2 \le r^2\right\}. \tag{33}$$

To deduce the relation between the resistive voltage and current from Maxwell's equations, we make the following assumptions.

Assumption 5.1 (The electromagnetic field inside resistances)

(a) The electromagnetic field inside the conductor material is stationary, that is,

$$\frac{\partial}{\partial t} D \equiv \frac{\partial}{\partial t} B \equiv 0.$$

(b) Ω does not contain any electric charges.
(c) For all $\xi_x \in [0, \ell]$, the voltage between two arbitrary points of $\mathcal{A}_{\xi_x}$ vanishes.
(d) The conductance function $g : \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$ has the following properties:

  (i) g is continuously differentiable.
  (ii) g is homogeneous, that is, $g(E, \xi_1) = g(E, \xi_2)$ for all $E \in \mathbb{R}^3$ and $\xi_1, \xi_2 \in \Omega$.
  (iii) g is strictly incremental, that is, $(E_1 - E_2)^{\mathrm{T}} g(E_1 - E_2, \xi) > 0$ for all distinct $E_1, E_2 \in \mathbb{R}^3$ and $\xi \in \Omega$.
  (iv) g is isotropic, that is, $g(E, \xi)$ and E are linearly dependent for all $E \in \mathbb{R}^3$ and $\xi \in \Omega$.

Using the definition of the voltage (10), property (c) implies that the electric field intensity is directed according to the conductor, that is, $E(t, \xi) = e(t, \xi) \cdot e_x$, where e_x is the canonical unit vector in the x-direction, and $e(\cdot, \cdot)$ is some scalar-valued function. Homogeneity and isotropy, smoothness, and the incrementation property of the conductance function then imply that

$$j(t, \xi) = g\big(E(t, \xi), \xi\big) = g_x\big(e(t, \xi)\big) \cdot e_x$$

for some strictly increasing and differentiable function $g_x : \mathbb{R} \to \mathbb{R}$ with $g_x(0) = 0$. Further, by using (9) we can infer from the stationarity of the electromagnetic field that the field of electric current density is divergence-free, that is, $\operatorname{div} j(\cdot, \cdot) \equiv 0$. Consequently, $g_x(e(t, \xi))$ is spatially constant. The strict monotonicity of g_x then implies that $e(t, \xi)$ is spatially constant, whence we can set up

$$E(t, \xi) = e(t) \cdot e_x$$

for some scalar-valued function e only depending on time t (see Fig. 12).

Consider now the straight path $\mathcal{S}$ between $(0, 0, 0)$ and $(\ell, 0, 0)$. The normal of this path fulfills $n(\xi) = e_x$ for all $\xi \in \mathcal{S}$. As a consequence, the voltage reads

$$\begin{aligned} u(t) &= \int_{\mathcal{S}} \nu^{\mathrm{T}}(\xi) \cdot E(t, \xi)\, ds(\xi) \\ &= \int_{\mathcal{S}} e_x^{\mathrm{T}} \cdot e(t) \cdot e_x\, ds(\xi) \end{aligned}$$

$$= \int_{\mathcal{S}} e(t)\, ds(\xi)$$
$$= \int_0^{\ell} e(t)\, d\xi = \ell e(t). \tag{34}$$

Consider the cross-sectional area $\mathcal{A}_0$ (compare (33)). The normal of $\mathcal{A}_0$ fulfills $n(\xi) = e_x$ for all $\xi \in \mathcal{A}_0$. Then obtain for the voltage $u(t)$ between the ends of the conductor and the current $i(t)$ through the conductor that

$$\begin{aligned} i(t) &= \iint_{\mathcal{A}_0} n^{\mathrm{T}}(\xi) j(t,\xi)\, dS(\xi) \\ &= \iint_{\mathcal{A}_0} n^{\mathrm{T}}(\xi) g_x\big(e(t)\big) \cdot e_x\, dS(\xi) \\ &= \iint_{\mathcal{A}_0} e_x^{\mathrm{T}} g_x\big(e(t)\big) \cdot e_x\, dS(\xi) \\ &= \iint_{\mathcal{A}_0} g_x\big(e(t)\big)\, dS(\xi) \\ &= \big(\pi r^2\big) \cdot g_x\big(e(t)\big) = \underbrace{\big(\pi r^2\big) \cdot g_x\left(\frac{u(t)}{\ell}\right)}_{=:g(u(t))}. \end{aligned}$$

As a consequence, we obtain the algebraic relation

$$i(t) = g\big(u(t)\big), \tag{35}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and differentiable function with $g(0) = 0$. The symbol of a resistance is presented in Fig. 10.

Remark 5.2 (Linear resistance) Note that in the case where the friction function is furthermore linear (i.e., $g(E(t,\xi),\xi) = c_g \cdot E(t,\xi)$), the resistance relation (35) becomes

$$i(t) = \mathcal{G} \cdot u(t), \tag{36}$$

where

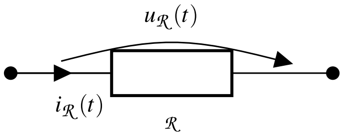
$$\mathcal{G} = \frac{\pi r^2 \cdot c_g}{\ell} > 0$$

is the so-called *conductance value* of the linear resistance.

Equivalently, we can write

$$u(t) = \mathcal{R} \cdot i(t), \tag{37}$$

Fig. 10 Symbol of a resistance



where

$$\mathcal{R} = \frac{\ell}{\pi r^2 \cdot c_g} > 0.$$

Remark 5.3 (Resistance, energy balance) The energy balance of a general resistance that is operated in the time interval $[t_0, t_f]$

$$W_r = \int_{t_0}^{t_f} u(\tau)i(\tau)\,d\tau = \int_{t_0}^{t_f} u(\tau)g\big(u(\tau)\big)\,d\tau \geq 0,$$

where the latter inequality holds since the integrand is positive. A resistance is therefore an *energy-dissipating element*, that is, it consumes energy.

Note that, in the linear case, the energy balance simplifies to

$$W_r = \mathcal{G} \cdot \int_{t_0}^{t_f} u^2(\tau)\,d\tau \geq 0.$$

2.5.3 Capacitances

We make the following ansatz for a capacitance: Consider again an electromagnetic medium in a cylindric spatial domain $\Omega \subset \mathbb{R}^3$ as in (32) with length ℓ and radius r (see also Fig. 9). To deduce the relation between capacitive voltage and current from Maxwell's equations, we make the following assumptions.

Assumption 5.4 (The electromagnetic field inside capacitances)

(a) The magnetic flux intensity inside the medium is stationary, that is,

$$\frac{\partial}{\partial t} B \equiv 0.$$

(b) The medium is a perfect isolator, that is, $j(\cdot, \xi) \equiv 0$ for all $\xi \in \Omega$.
(c) In the lateral area

$$\mathcal{A}_{\text{lat}} = [0, \ell] \times \left\{(\xi_y, \xi_z) : \xi_y^2 + \xi_z^2 = r^2\right\} \subset \partial\Omega$$

of the cylindric domain Ω, the magnetic field intensity is directed orthogonally to $\mathcal{A}_{\text{lat}}$. In other words, for all $\xi \in \mathcal{A}_{\text{lat}}$ and all times t, the positively oriented normal $n(\xi)$ and $H(t, \xi)$ are linearly dependent.

(d) There is no explicit algebraic relation between the electric current density and the electric field intensity.
(e) Ω does not contain any electric charges.
(f) For all $\xi_x \in [0, \ell]$, the voltage between two arbitrary points of $\mathcal{A}_{\xi_x}$ (compare (33)) vanishes.
(g) The function $f_e : \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$ has the following properties:

   (i) f_e is continuously differentiable.
   (ii) f_e is homogeneous, that is, $f_e(D, \xi_1) = f_e(D, \xi_2)$ for all $D \in \mathbb{R}^3$ and $\xi_1, \xi_2 \in \Omega$.
   (iii) The function $f_e(\cdot, \xi) : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible for some (and hence any) $\xi \in \Omega$.
   (iv) f_e is isotropic, that is, $f_e(D, \xi)$ and D are linearly dependent for all $D \in \mathbb{R}^3$ and $\xi \in \Omega$.

Using the definition of the voltage (10), property (c) implies that the electric field intensity is directed according to the conductor, that is, $E(t, \xi) = e(t, \xi) \cdot e_x$ for some scalar-valued function $e(\cdot, \cdot)$. Isotropy, homogeneity, and the invertibility of f_e then implies that the electrical displacement is as well directed along the conductor, whence

$$D(t, \xi) = f_e^{-1}\big(E(t, \xi), \xi\big) = q_x\big(e(t, \xi)\big) \cdot e_x$$

for some differentiable and invertible function $q_x : \mathbb{R} \to \mathbb{R}$. Further, by using that, by the absence of electric charges, the field of electric displacement is divergence-free, we obtain that it is even spatially constant. Consequently, the electric field intensity is as well spatially constant, and we can set up

$$E(t, \xi) = e(t) \cdot e_x$$

for some scalar-valued function $e(\cdot)$ only depending on time.

Using that the magnetic field is stationary, we can, as for resistances, infer that the electrical field is spatially constant, that is,

$$E(t, \xi) = e(t) \cdot e_x$$

for some scalar-valued function $e(\cdot)$ only depending on time, and we can use the argumentation in as in (34) to see that the voltage reads

$$u(t) = \ell e(t).$$

Assume that the current $i(\cdot)$ is applied to the capacitor. The current density inside Ω is additively composed of the current density induced by the applied current $j_{\mathrm{appl}}(\cdot, \cdot)$ and the current density $j_{\mathrm{ind}}(\cdot, \cdot)$ induced by the electric field. Since the medium in Ω is an isolator, the current density inside Ω vanishes. Consequently, for all times t and all $\xi \in \Omega$,

$$0 = j_{\mathrm{appl}}(t, \xi) + j_{\mathrm{ind}}(t, \xi).$$

The definition of the current yields

$$i(t) = \iint_{\mathcal{A}_0} n^{\mathrm{T}}(\xi)\, j_{\mathrm{appl}}(t,\xi)\, dS(\xi).$$

The definition of the cross-sectional area $\mathcal{A}_0$ and the lateral surface $\mathcal{A}_{\mathrm{lat}}$ yields $\partial\mathcal{A}_0 \subset \mathcal{A}_{\mathrm{lat}}$. By Maxwell's equations, Stokes theorem, stationarity of the magnetic flux intensity, and the assumption that the tangential component magnetic field intensity vanishes in the lateral surface, we obtain

$$\begin{aligned}
i(t) &= \iint_{\mathcal{A}_0} n^{\mathrm{T}}(\xi) \cdot j_{\mathrm{appl}}(t,\xi)\, dS(\xi) \\
&= -\iint_{\mathcal{A}_0} \underbrace{n^{\mathrm{T}}(\xi)}_{=e_x^{\mathrm{T}}} \cdot j_{\mathrm{ind}}(t,\xi)\, dS(\xi) \\
&= \iint_{\mathcal{A}_0} e_x^{\mathrm{T}} \cdot \frac{\partial}{\partial t} D(t,\xi) - e_x^{\mathrm{T}} \cdot \operatorname{curl} H(t,\xi)\, dS(\xi) \\
&= \frac{d}{dt} \iint_{\mathcal{A}_0} e_x^{\mathrm{T}} \cdot D(t,\xi)\, dS(\xi) - \oint_{\partial\mathcal{A}} \underbrace{\nu^{\mathrm{T}}(\xi) \cdot H(t,\xi)}_{=0}\, ds(\xi) \\
&= \frac{d}{dt} \iint_{\mathcal{A}_0} e_x^{\mathrm{T}} \cdot f_e^{-1}\big(E(t,\xi),\xi\big)\, dS(\xi) \\
&= \frac{d}{dt} \iint_{\mathcal{A}_0} e_x^{\mathrm{T}} \cdot q_x\big(e(t)\big) \cdot e_x\, dS(\xi) \\
&= \frac{d}{dt} \pi r^2 \cdot q_x\big(e(t)\big) \\
&= \frac{d}{dt} \underbrace{\pi r^2 \cdot q_x\left(\frac{u(t)}{\ell}\right)}_{=:q(u(t))}.
\end{aligned}$$

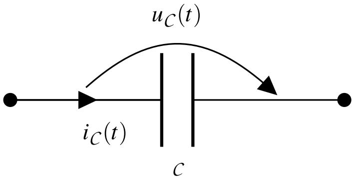
That is, we obtain the dynamic relation

$$i(t) = \frac{d}{dt} q\big(u(t)\big) \tag{38}$$

for some function $q : \mathbb{R} \to \mathbb{R}$. Note that the quantity $q(u)$ has the physical dimension of electric charge, whence $q(\cdot)$ is called a *charge function*. It is sometimes spoken about the charge $q(u(t))$ of the capacitance. Note that $q(u(t))$ is a virtual quantity. Especially, there is no direct relation between the charge of a capacitance and the electric charge (density) as introduced in Sect. 2.3. The symbol of a capacitance is presented in Fig. 11.

Remark 5.5 (Linear capacitance) Note that, in the case where the constitutive relation is furthermore linear (i.e., $f_e(D(t,\xi),\xi) = c_c \cdot D(t,\xi)$), the capacitance relation

Fig. 11 Symbol of a capacitance



(35) becomes

$$i(t) = C \cdot \dot{u}(t), \tag{39}$$

where

$$C = \frac{\pi r^2}{\ell c_c} > 0$$

is the so-called *capacitance value* of the linear capacitance.

Remark 5.6 (Capacitance, energy balance) Isotropy and homogeneity of f_e and the construction of the function q_x further implies that the electric energy density fulfills

$$\frac{\partial}{\partial D} V_e^{\mathrm{T}}\big(q_x(e) \cdot e_x, \xi\big) = f_e\big(q_x(e) \cdot e_x, \xi\big) = e \cdot e_x.$$

Hence, the function $q_x : \mathbb{R} \to \mathbb{R}$ is invertible with

$$q_x^{-1}(q) = e_x^{\mathrm{T}} \frac{\partial}{\partial D} V_e^{\mathrm{T}}(q \cdot e_x) = \frac{d}{dq} V_{e,x}(q),$$

where

$$\begin{aligned} V_{e,x} \colon \quad & \mathbb{R} \to \mathbb{R}, \\ & q \mapsto V_e(q \cdot e_x). \end{aligned}$$

In particular, this function fulfills $V_{e,x}(0) = 0$ and $V_{e,x}(q) > 0$ for all $q \in \mathbb{R} \setminus \{0\}$.

The construction of the capacitance function and assumption (3) on f_e implies that $q : \mathbb{R} \to \mathbb{R}$ is invertible with

$$q^{-1}(\cdot) = \ell \cdot q_x^{-1}\left(\frac{\cdot}{\pi r^2}\right) = \frac{d}{dq} \underbrace{l \pi r^2 V_{e,x}\left(\frac{\cdot}{\pi r^2}\right)}_{=:V_C(\cdot)}.$$

Moreover, $V_C(0) = 0$ and $V_C(q_C) > 0$ for all $q_C \in \mathbb{R} \setminus \{0\}$.

Now we consider the energy balance of a capacitance that is operated in the time interval $[t_0, t_f]$

$$\begin{aligned} W_C &= \int_{t_0}^{t_f} u(\tau) i(\tau)\, d\tau \\ &= \int_{t_0}^{t_f} q^{-1}\big(q\big(u(\tau)\big)\big) \cdot \frac{d}{d\tau} q\big(u(\tau)\big)\, d\tau \end{aligned}$$

$$\begin{aligned}
&= \int_{t_0}^{t_f} \frac{d}{dq} V_C\big(q\big(u(\tau)\big)\big) \cdot \frac{d}{d\tau} q\big(u(\tau)\big)\, d\tau \\
&= \int_{t_0}^{t_f} \frac{d}{d\tau} V_C\big(q\big(u(\tau)\big)\big)\, d\tau \\
&= V_C\big(q\big(u(\tau)\big)\big)\Big|_{\tau=t_0}^{\tau=t_f}. \qquad (40)
\end{aligned}$$

Consequently, the function V_C has the physical interpretation of an *energy storage function*. A capacitance is therefore a *reactive element*, that is, it stores energy.

Note that, in the linear case, the storage function simplifies to

$$V_C\big(q(u)\big) = \frac{1}{2} \cdot \mathcal{C}^{-1} \cdot q^2(u) = \frac{1}{2} \cdot \mathcal{C}^{-1} \cdot \big(\mathcal{C}(u)\big)^2 = \frac{1}{2} \cdot \mathcal{C} \cdot u^2,$$

whence the energy balance then reads

$$W_C = \frac{1}{2} \cdot \mathcal{C} \cdot u^2(\tau)\big|_{\tau=t_0}^{\tau=t_f}.$$

Remark 5.7 (Capacitances and differentiation rules) The previous assumptions imply that the function $q : \mathbb{R} \to \mathbb{R}$ is differentiable. By the chain rule, (38) can be rewritten as

$$i(t) = \mathcal{C}\big(u(t)\big) \cdot \dot{u}(t), \qquad (41)$$

where

$$\mathcal{C}(u_C) = \frac{d}{du_C} q(u_C).$$

Monotonicity of q further implies that $\mathcal{C}(\cdot)$ is a pointwise positive function.

By the differentiation rule for inverse functions, we obtain

$$\mathcal{C}(u_C) = \frac{d}{du_C} q(u_C) = \left(\frac{d}{dq} V_C\big(q(u_C)\big)\right)^{-1}.$$

2.5.4 Inductances

It will turn out in this part that inductances are components that store magnetic energy. We will see that there are certain analogies to capacitances if one replaces electric by accordant magnetic physical quantities. The mode of action of an inductance can be explained by a conductor loop. We further make the (simplifying) assumption that the conductor with domain Ω forms a circle that is interrupted by an isolator of width zero (see Fig. 12). Assume that the circle radius is given by r, where the radius is here defined to be the distance from the circle midpoint to any conductor midpoint. Further, let l_h be the conductor width.

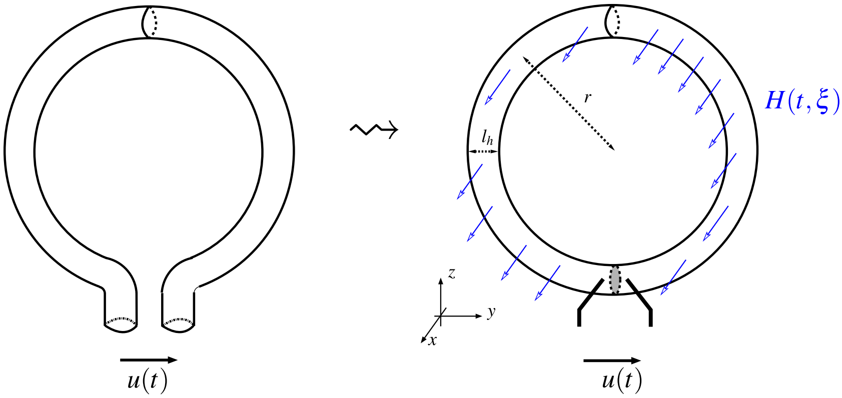


Fig. 12 Model of an inductance

To deduce the relation between inductive voltage and current from Maxwell's equations, we make the following assumptions.

Assumption 5.8 (The electromagnetic field inside capacitances)

(a) The electric displacement inside the medium Ω is stationary, that is,

$$\frac{\partial}{\partial t} D \equiv 0.$$

(b) The medium is a perfect conductor, that is, $E(\cdot, \xi) \equiv 0$ for all $\xi \in \Omega$.
(c) There is no explicit algebraic relation between the electric current density and the electric field intensity.
(d) Ω does not contain any electric charges.
(e) The function $f_m : \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$ has the following properties:

   (i) f_m is continuously differentiable.
   (ii) f_m is homogeneous, that is, $f_m(B, \xi_1) = f_m(B, \xi_2)$ for all $B \in \mathbb{R}^3$ and $\xi_1, \xi_2 \in \Omega$.
   (iii) The function $f_m(\cdot, \xi) : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible for some (and hence any) $\xi \in \Omega$.
   (iv) f_m is isotropic, that is, $f_m(B, \xi)$ and B are linearly dependent for all $B \in \mathbb{R}^3$ and $\xi \in \Omega$.

Let $\xi = \xi_x e_x + \xi_y e_y + \xi_z e_z$, and let $h_s : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$h_s(x) = 0 \quad \text{for all } x \in [0, r - l_h/2] \cup [r + l_h/2, \infty),$$

and

$$h_s(x) > 0 \quad \text{for all } x \in (r - l_h/2, r + l_h/2).$$

We make the following ansatz for the magnetic flux intensity:

$$H(t,\xi) = h_s\big(\xi_y^2 + \xi_z^2\big) \cdot h(t) \cdot e_x,$$

where $h(\cdot)$ is a scalar-valued function defined on a temporal domain in which the process evolves (see Fig. 12).

Using the definition of the current (8), Maxwell's equations, property (c), and the stationarity of the electric field yields

$$\begin{aligned} i(t) &= \iint_{\{0\}\times[r-l_h/2,r+l_h/2]\times[0,l_d]} \nu^{\mathrm{T}}(\xi) \cdot j(t,\xi)\, dS(\xi) \\ &= \iint_{\{0\}\times[r-l_h/2,r+l_h/2]\times[0,l_d]} \nu^{\mathrm{T}}(\xi) \cdot \operatorname{curl} H(t,\xi)\, dS(\xi) \\ &= \iint_{\{0\}\times[r-l_h/2,r+l_h/2]\times[0,l_d]} e_x^{\mathrm{T}} \cdot 2b_s'\big(\xi_y^2 + \xi_z^2\big) \cdot e_x \cdot h(t)\, dS(\xi) \\ &= \underbrace{2 \iint_{\{0\}\times[r-l_h/2,r+l_h/2]\times[0,l_d]} b_s'\big(\xi_y^2 + \xi_z^2\big)\, dS(\xi)}_{=:c_m} \cdot h(t). \end{aligned}$$

Assume that the voltage $u(\cdot)$ is applied to the inductor. The electric field intensity inside the conductor is additively composed of the field intensity induced by the applied voltage $E_{\mathrm{appl}}(\cdot,\cdot)$ and the electric field intensity $E_{\mathrm{ind}}(\cdot,\cdot)$ induced by the magnetic field. Since the wire is a perfect conductor, the electric field intensity vanishes inside the wire. Consequently, for all times t and all $\xi \in \mathbb{R}^3$ with

$$0 \le \xi_x \le l_d \quad \text{and} \quad (r - l_h)^2 \le \xi_y^2 + \xi_z^2 \le (r + l_h)^2,$$

we have

$$0 = E_{\mathrm{appl}}(t,\xi) + E_{\mathrm{ind}}(t,\xi).$$

Let $A \subset \mathbb{R}^3$ be a circular area that is surrounded by the midline of the wire, that is,

$$A = \big\{(\xi_x,\xi_y,\xi_z) \in \mathbb{R}^3 : \xi_x = l_d/2 \text{ and } \xi_y^2 + \xi_z^2 \le r^2\big\}.$$

Isotropy, homogeneity, and the invertibility of f_m then implies that the magnetic flux is as well directed orthogonally to A, that is,

$$\begin{aligned} B(t,\xi) = f_m^{-1}\big(H(t,\xi),\xi\big) & \\ &= \psi_x\big(h_s\big(\xi_y^2 + \xi_z^2\big) \cdot h(t)\big) \cdot e_x \\ &= \psi_x\left(\frac{h_s(\xi_y^2 + \xi_z^2)}{c_m} \cdot i(t)\right) \cdot e_x \end{aligned}$$

for some differentiable function $\psi_x : \mathbb{R} \to \mathbb{R}$.

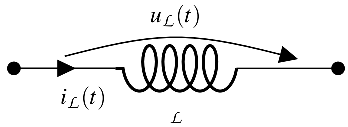


Fig. 13 Symbol of an inductance

By Maxwell's equations, Stokes theorem, the definition of the voltage, and a transformation to polar coordinates we obtain

$$\begin{aligned}
u(t) &= \oint_{\partial A} \nu^{\mathrm{T}}(\xi) \cdot E_{\mathrm{appl}}(t,\xi)\, ds(\xi) \\
&= -\oint_{\partial A} \nu^{\mathrm{T}}(\xi) \cdot E_{\mathrm{ind}}(t,\xi)\, ds(\xi) \\
&= -\iint_A \underbrace{n^{\mathrm{T}}(\xi)}_{=e_x^{\mathrm{T}}} \cdot \underbrace{\operatorname{curl} E_{\mathrm{ind}}(t,\xi)}_{=-\frac{\partial}{\partial t}B(t,\xi)}\, dS(\xi) \\
&= -\frac{d}{dt}\iint_A e_x^{\mathrm{T}} \cdot \underbrace{B(t,\xi)}_{=\psi_x(\frac{h_s(\xi_y^2+\xi_z^2)}{c_m}\cdot i(t))\cdot e_x}\, dS(\xi) \\
&= \frac{d}{dt}\iint_A \psi_x\left(\frac{h_s(\xi_y^2+\xi_z^2)}{c_m}\cdot i(t)\right) dS(\xi) \\
&= \frac{d}{dt}\underbrace{2\pi\int_{r-l_h/2}^{r+l_h/2} y\psi_x\left(\frac{h_s(y^2)}{c_m}\cdot i(t)\right) dy}_{=:\psi(i(t))}.
\end{aligned}$$

That is, we obtain the dynamic relation

$$u(t) = \frac{d}{dt}\psi\bigl(i(t)\bigr) \tag{42}$$

for some function $\psi : \mathbb{R} \to \mathbb{R}$, which is called a *magnetic flux function*. The symbol of an inductance is presented in Fig. 13.

Remark 5.9 (Linear inductance) Note that, in the case where the constitutive relation is furthermore linear (i.e., $f_m(B(t,\xi),\xi) = c_i \cdot H(t,\xi)$), the inductance relation (35) becomes

$$u(t) = \mathcal{L} \cdot \dot{i}(t), \tag{43}$$

where

$$\mathcal{L} = \frac{2\pi c_i}{c_m}\int_{r-l_h/2}^{r+l_h/2} s \cdot h_s\bigl(s^2\bigr)\, d\xi > 0$$

is the so-called *inductance value* of the linear inductance.

Remark 5.10 (Inductance, energy balance) Isotropy and homogeneity of f_m and the construction of the function ψ_x further implies that the magnetic energy density fulfills

$$\begin{aligned}
&\frac{\partial}{\partial B} V_m^{\mathrm{T}}\big(\psi_x\big(h_s\big(\xi_y^2+\xi_z^2\big)h(t)\big)\cdot e_x,\xi\big)\\
&\quad = f_m\big(\psi_x\big(h_s\big(\xi_y^2+\xi_z^2\big)\cdot h(t)\big)\cdot e_x,\xi\big) = H(t,\xi)\\
&\quad = h_s\big(\xi_y^2+\xi_z^2\big)\cdot h(t)\cdot e_x.
\end{aligned}$$

Hence, the function $\psi_x : \mathbb{R}\to\mathbb{R}$ is invertible with

$$\psi_x^{-1}(h) = e_x^{\mathrm{T}}\frac{\partial}{\partial D}V_e^{\mathrm{T}}\big((h)\cdot e_x\big) = \frac{d}{dq}V_{m,x}(h),$$

where

$$\begin{aligned}
V_{m,x}\colon\quad & \mathbb{R}\to\mathbb{R},\\
& h\mapsto V_m(h\cdot e_x).
\end{aligned}$$

In particular, this function fulfills $V_{m,x}(0)=0$ and $V_{m,x}(h)>0$ for all $h\in\mathbb{R}\setminus\{0\}$. The latter, together with the continuous differentiability of $f_m(\cdot,\xi)$ and $f_m^{-1}(\cdot,\xi)$, implies that the derivatives of both the function ψ_x^{-1} and ψ_x are positive and, furthermore, $\psi_x(0)=0$. Thus, the function $\psi : \mathbb{R}\to\mathbb{R}$ is differentiable with

$$\psi'(i) = 2\pi\int_{r-l_h/2}^{r+l_h/2} y\psi_x'\left(\frac{h_s(y^2)}{c_m}\cdot i\right)\frac{h_s(y^2)}{c_m}\,dy > 0.$$

Consequently, ψ possesses a continuously differentiable and strictly increasing inverse function $\psi^{-1} : \mathbb{R}\to\mathbb{R}$ with $\operatorname{sign}\psi^{-1}(p) = \operatorname{sign}(p)$ for all $p\in\mathbb{R}$. Now consider the function

$$\begin{aligned}
V_{\mathcal{L}}\colon\quad & \mathbb{R}\to\mathbb{R},\\
& \psi_{\mathcal{L}}\mapsto \int_0^{\psi_{\mathcal{L}}}\psi^{-1}(p)\,dp.
\end{aligned}$$

The construction of $V_{\mathcal{L}}$ implies that $V_{\mathcal{L}}(0)=0$ and $V_{\mathcal{L}}(\psi_{\mathcal{L}})>0$ for all $\psi_{\mathcal{L}}\in\mathbb{R}\setminus\{0\}$ and, furthermore,

$$\psi^{-1}(\psi_{\mathcal{L}}) = \frac{d}{d\psi_{\mathcal{L}}}V_{\mathcal{L}}(\psi_{\mathcal{L}})\quad \text{for all}\,\psi_{\mathcal{L}}\in\mathbb{R}.$$

Now we consider the energy balance of an inductance that is operated in the time interval $[t_0,t_f]$

$$\begin{aligned}
W_{\mathcal{L}} &= \int_{t_0}^{t_f} u(\tau)i(\tau)\,d\tau\\
&= \int_{t_0}^{t_f}\frac{d}{d\tau}\psi\big(i(\tau)\big)\psi^{-1}\big(\psi\big(i(\tau)\big)\big)\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t_f} \frac{d}{d\tau} \psi\big(i(\tau)\big) \frac{d}{d\psi} V_{\mathcal{L}}\big(\psi\big(i(\tau)\big)\big)\, d\tau \\
&= \int_{t_0}^{t_f} \frac{d}{d\tau} V_{\mathcal{L}}\big(\psi\big(i(\tau)\big)\big)\, d\tau \\
&= V_{\mathcal{L}}\big(\psi\big(i(\tau)\big)\big)\Big|_{\tau=t_0}^{\tau=t_f}. \qquad (44)
\end{aligned}
$$

Consequently, the function $V_{\mathcal{L}}$ has the physical interpretation of an *energy storage function*. An inductance is therefore again a reactive element.

In the linear case, the storage function simplifies to

$$V_{\mathcal{L}}\big(\psi(u)\big) = \frac{1}{2} \cdot \mathcal{L}^{-1} \cdot \psi^2(i) = \frac{1}{2} \cdot \mathcal{L}^{-1} \cdot \big(\mathcal{L}(i)\big)^2 = \frac{1}{2} \cdot \mathcal{L} \cdot i^2,$$

whence the energy balance then reads

$$W_{\mathcal{L}} = \frac{1}{2} \cdot \mathcal{L} \cdot i^2(\tau)\big|_{\tau=t_0}^{\tau=t_f}.$$

Remark 5.11 (Inductances and differentiation rules) The previous assumptions imply that the function $\psi : \mathbb{R} \to \mathbb{R}$ is differentiable. By the chain rule, (42) can be rewritten as

$$u(t) = \mathcal{L}\big(i(t)\big) \cdot \dot{i}(t), \qquad (45)$$

where

$$\mathcal{L}(u_{\mathcal{L}}) = \frac{d}{di_{\mathcal{L}}} \psi(i_{\mathcal{L}}).$$

The monotonicity of ψ further implies that the function $\mathcal{L}(\cdot)$ is pointwise positive.

By the differentiation rule for inverse functions we obtain

$$\mathcal{L}(i_{\mathcal{L}}) = \frac{d}{di_{\mathcal{L}}} \psi(i_{\mathcal{L}}) = \left(\frac{d}{d\psi} V_{\mathcal{L}}\big(\psi(i_{\mathcal{L}})\big)\right)^{-1}.$$

2.5.5 Some Notes on Diodes

Resistances, capacitances, and inductances are typical components of analogue electrical circuits. The fundamental role in electronic engineering is however taken by semiconductor devices, such as diodes and transistors (see also Notes and References). A fine modeling of such components has to be done by partial differential equations (see, e.g., [36]).

In contrast to the previous sections, we are not going to model these components on the basis of the fundamental laws of the electromagnetic field. We are rather presenting a less accurate but often reliable ansatz to the description of their behavior

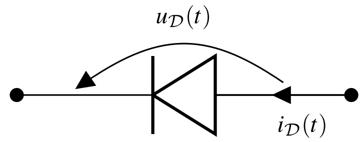


Fig. 14 Symbol of a diode

by equivalent RCL circuits. As a showcase, we are considering diodes. The symbol of a diode is presented in Fig. 14.

An ideal diode is a component that allows the current to flow in one specified direction while blocking currents with opposite sign. A mathematical lax formulation of this property is

$$i_{\mathcal{D}}(t) = g_{\mathcal{D}}\big(u_{\mathcal{D}}(t)\big) \cdot u_{\mathcal{D}}(t),$$
$$\text{where } g_{\mathcal{D}}(u) = \begin{cases} \infty & \text{if } u > 0, \\ 0 & \text{if } u \le 0. \end{cases}$$

A mathematically more precise description is given the specification of the behavior

$$\big(i_{\mathcal{D}}(t), u_{\mathcal{D}}(t)\big) \in \{0\} \times \mathbb{R}_{\le 0} \cup \mathbb{R}_{\ge 0} \times \{0\}.$$

Since the product of voltage and current of an ideal diode always vanishes, this component behaves energetically neutral.

It is clear that such a behavior is not technically realizable. It can be neverthe-less be approximated by a component consisting of a semiconductor crystal with two regions, each with a different *doping*. Such a configuration is called an *np-junction* [55].

The most simple ansatz for the modeling of a nonideal diode is by replacing it by a resistance with highly nonsymmetric conductance behavior, such as, for instance, the *Shockley diode equation* [55]

$$i_{\mathcal{D}}(t) = i_S \cdot \big(e^{\frac{u_{\mathcal{D}}(t)}{u_p}} - 1\big),$$

where $i_S > 0$ and $u_p > 0$ are material-dependent quantities. Note that the behavior of an ideal diode is the more approached, the bigger is u_p.

A refinement of this model also includes capacitive effects. This can be done by adding some (small) capacitance in parallel to the resistance model of the diode [61].

2.5.6 Notes and References

(i) In [16, 19, 32, 34, 60], component relations have also been derived. These however go with an a priori definition of capacitive charge and magnetic flux as physical quantities. In contrast to this, our approach is based on Maxwell's equations with additional assumptions.

(ii) Note that, apart from sources, resistances, and capacitances, there are various further components that occur in electrical circuits. Such components could, for instance, be *controlled sources* [22] (i.e., sources with voltage or current explicitly depending on some other physical quantity), semi-conductors [12, 36] (such as diodes and transistors), MEM devices [48, 53, 54], or transmission lines [42].

2.6 Circuit Models and Differential–Algebraic Equations

2.6.1 Circuit Equations in Compact Form

Having collected all relevant equations describing an electrical circuit, we are now ready to set up and analyze the overall model. Let a connected electrical circuit with n branches be given; let the vectors $i(t), u(t) \in \mathbb{R}^n$ be defined as in (13) and (15), that is, their components are containing voltages and current of the respective branches. We further assume that the branches are ordered by the type of component, that is,

$$i(t) = \begin{pmatrix} i_{\mathcal{R}}(t) \\ i_{\mathcal{C}}(t) \\ i_{\mathcal{L}}(t) \\ i_{\mathcal{V}}(t) \\ i_{\mathcal{I}}(t) \end{pmatrix}, \qquad u(t) = \begin{pmatrix} u_{\mathcal{R}}(t) \\ u_{\mathcal{C}}(t) \\ u_{\mathcal{L}}(t) \\ u_{\mathcal{V}}(t) \\ u_{\mathcal{I}}(t) \end{pmatrix}, \tag{46}$$

where

$$i_{\mathcal{R}}(t),\ u_{\mathcal{R}}(t) \in \mathbb{R}^{n_{\mathcal{R}}}, \qquad i_{\mathcal{C}}(t),\ u_{\mathcal{C}}(t) \in \mathbb{R}^{n_{\mathcal{C}}}, \qquad i_{\mathcal{L}}(t),\ u_{\mathcal{L}}(t) \in \mathbb{R}^{n_{\mathcal{L}}},$$
$$i_{\mathcal{V}}(t),\ u_{\mathcal{V}}(t) \in \mathbb{R}^{n_{\mathcal{V}}}, \qquad i_{\mathcal{I}}(t),\ u_{\mathcal{I}}(t) \in \mathbb{R}^{n_{\mathcal{I}}}.$$

The component relations then read, in compact form,

$$i_{\mathcal{R}}(t) = g\big(u_{\mathcal{R}}(t)\big), \qquad i_{\mathcal{C}}(t) = \frac{d}{dt}q\big(u_{\mathcal{C}}(t)\big), \qquad u_{\mathcal{L}}(t) = \frac{d}{dt}\psi\big(i_{\mathcal{L}}(t)\big)$$

for

$$g\colon \quad \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}, \qquad\qquad q\colon \quad \mathbb{R}^{n_{\mathcal{C}}} \to \mathbb{R}^{n_{\mathcal{C}}},$$
$$\begin{pmatrix} u_1 \\ \vdots \\ u_{n_{\mathcal{R}}} \end{pmatrix} \mapsto \begin{pmatrix} g_1(u_1) \\ \vdots \\ g_{n_{\mathcal{R}}}(u_{n_{\mathcal{R}}}) \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ \vdots \\ u_{n_{\mathcal{C}}} \end{pmatrix} \mapsto \begin{pmatrix} q_1(u_1) \\ \vdots \\ q_{m_{\mathcal{C}}}(u_{n_{\mathcal{C}}}) \end{pmatrix},$$
$$\psi\colon \quad \mathbb{R}^{m_{\mathcal{L}}} \to \mathbb{R}^{n_{\mathcal{L}}},$$
$$\begin{pmatrix} i_1 \\ \vdots \\ i_{n_{\mathcal{L}}} \end{pmatrix} \mapsto \begin{pmatrix} \psi_1(u_1) \\ \vdots \\ \psi_{n_{\mathcal{C}}}(i_{n_{\mathcal{C}}}) \end{pmatrix},$$

where the scalar functions $g_i, q_i, \psi_i : \mathbb{R} \to \mathbb{R}$ are respectively representing the behavior of the ith resistance, capacitance, and inductance. The assumptions of Sect. 2.5 imply that $g(0) = 0$, and for all $u \in \mathbb{R}^{m_C} \setminus \{0\}$,

$$u^{\mathrm{T}} g(u) > 0. \tag{47}$$

Further, since $q_k^{-1}(q_{Ck}) = \frac{d}{dq_{Ck}} V_{Ck}(q_{Ck})$ and $\psi_k^{-1}(\psi_{\mathcal{L}k}) = \frac{d}{d\psi_{\mathcal{L}k}} V_{\mathcal{L}k}(\psi_{\mathcal{L}k})$, the functions $q : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$ and $\psi : \mathbb{R}^{n_\mathcal{L}} \to \mathbb{R}^{n_\mathcal{L}}$ possess inverses fulfilling

$$q^{-1}(q_C) = \frac{d}{dq_C} V_C(q_C), \qquad \psi^{-1}(\psi_\mathcal{L}) = \frac{d}{d\psi_\mathcal{L}} V_\mathcal{L}(\psi_\mathcal{L}), \tag{48a}$$

where

$$V_C(q_C) = \sum_{k=1}^{n_C} V_{Ck}(q_{Ck}), \qquad V_\mathcal{L}(\psi_\mathcal{L}) = \sum_{k=1}^{n_\mathcal{L}} V_{\mathcal{L}k}(\psi_{\mathcal{L}k}). \tag{48b}$$

In particular, $V_C(0) = 0$, $V_\mathcal{L}(0) = 0$, and

$$V_C(q_C) > 0, \qquad V_\mathcal{L}(\psi_\mathcal{L}) > 0 \quad \text{for all } q_C \in \mathbb{R}^{n_C}, \psi_\mathcal{L} \in \mathbb{R}^{n_\mathcal{L}}.$$

Using the chain rule, the component relations of the reactive elements read (see Remarks 5.7 and 5.11)

$$i_C(t) = \mathcal{C}\big(u_C(t)\big) \cdot \dot{u}_C(t), \qquad u_\mathcal{L}(t) = \mathcal{L}\big(i_\mathcal{L}(t)\big) \cdot \dot{i}_C(t), \tag{49a}$$

where

$$\mathcal{C}(u_C) = \frac{d}{du_C} q(u_C), \qquad \mathcal{L}(i_\mathcal{L}) = \frac{d}{di_\mathcal{L}} \psi(i_\mathcal{L}). \tag{49b}$$

In particular, the monotonicity of the scalar charge and flux functions implies that the ranges of the functions $\mathcal{C} : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C, n_C}$ and $\mathcal{L} : \mathbb{R}^{n_\mathcal{L}} \to \mathbb{R}^{n_\mathcal{L}, n_\mathcal{L}}$ are contained in the set of diagonal and positive definite matrices.

The incidence and loop matrices can, as well, be partitioned according to the subdivision of $i(t)$ and $u(t)$ in (46), that is,

$$A = \begin{bmatrix} A_\mathcal{R} & A_C & A_\mathcal{L} & A_\mathcal{V} & A_\mathcal{I} \end{bmatrix}, \qquad B = \begin{bmatrix} B_\mathcal{R} & B_C & B_\mathcal{L} & B_\mathcal{V} & B_\mathcal{I} \end{bmatrix}.$$

Kirchhoff's laws can now be represented in two alternative ways, namely the incidence-based formulation (see (24) and (26))

$$\begin{aligned} &A_\mathcal{R} i_\mathcal{R}(t) + A_C i_C(t) + A_\mathcal{L} i_\mathcal{L}(t) + A_\mathcal{V} i_\mathcal{V}(t) + A_\mathcal{I} i_\mathcal{I}(t) = 0, \\ &u_\mathcal{R}(t) = A_\mathcal{R}^{\mathrm{T}} \phi(t), \qquad u_C(t) = A_C^{\mathrm{T}} \phi(t), \qquad u_\mathcal{L}(t) = A_\mathcal{L}^{\mathrm{T}} \phi(t), \\ &u_\mathcal{L}(t) = A_\mathcal{L}^{\mathrm{T}} \phi(t), \qquad u_\mathcal{V}(t) = A_\mathcal{V}^{\mathrm{T}} \phi(t), \qquad u_\mathcal{I}(t) = A_\mathcal{I}^{\mathrm{T}} \phi(t) \end{aligned} \tag{50}$$

or the loop-based formulation (see (25) and (27))

$$\begin{aligned}
&B_{\mathcal{R}}u_{\mathcal{R}}(t)+B_{\mathcal{C}}u_{\mathcal{C}}(t)+B_{\mathcal{L}}u_{\mathcal{L}}(t)+B_{\mathcal{V}}u_{\mathcal{V}}(t)+B_{\mathcal{I}}u_{\mathcal{I}}(t)=0,\\
&i_{\mathcal{R}}(t)=B_{\mathcal{R}}^{\mathrm{T}}\iota(t),\qquad i_{\mathcal{C}}(t)=B_{\mathcal{C}}^{\mathrm{T}}\iota(t),\qquad i_{\mathcal{L}}(t)=B_{\mathcal{L}}^{\mathrm{T}}\iota(t),\\
&i_{\mathcal{L}}(t)=B_{\mathcal{L}}^{\mathrm{T}}\iota(t),\qquad i_{\mathcal{V}}(t)=B_{\mathcal{V}}^{\mathrm{T}}\iota(t),\qquad i_{\mathcal{I}}(t)=B_{\mathcal{I}}^{\mathrm{T}}\iota(t).
\end{aligned}\tag{51}$$

Having in mind that the functions $u_{\mathcal{V}}(\cdot)$ and $i_{\mathcal{I}}(\cdot)$ are prescribed, the overall circuit is described by the resistance law $i_{\mathcal{R}}(t)=g(u_{\mathcal{R}}(t))$, the differential equations (49a) for the reactive elements, and the Kirchhoff laws either in the form (50) or (51). This altogether leads to a coupled system of equations of pure algebraic nature (such as the Kirchhoff laws and the component relations for resistances) together with a set of differential equations (such as the component relations for reactive elements). This type of systems is, in general, referred to as *differential–algebraic equations*. A more rigorous definition and some general facts on type is presented in Sect. 2.6.2. Since many of the above-formulated equations are explicit in one variable, several relations can be inserted into one another to obtain a system of smaller size. In the following, we discuss two possibilities:

(a) **Modified nodal analysis (MNA)**

We are now using the component relations together with the incidence-based formulation of the Kirchhoff laws: Based on the KCL, we eliminate the resistive and capacitive currents and voltages. Then we obtain

$$A_{\mathcal{C}}C\big(A_{\mathcal{C}}^{\mathrm{T}}\phi(t)\big)A_{\mathcal{C}}^{\mathrm{T}}\frac{d}{dt}\phi(t)+A_{\mathcal{R}}g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi(t)\big)+A_{\mathcal{L}}i_{\mathcal{L}}(t)+A_{\mathcal{V}}i_{\mathcal{V}}(t)+A_{\mathcal{I}}i_{\mathcal{I}}(t)=0.$$

Plugging the KVL for the inductive voltages into the component relation for inductances, we are led to

$$-A_{\mathcal{L}}^{\mathrm{T}}\phi(t)+\mathcal{L}\big(i_{\mathcal{L}}(t)\big)\cdot\frac{d}{dt}i_{\mathcal{L}}(t)=0.$$

Together with the KVL for the voltage sources, this gives the so-called *modified nodal analysis*

$$\begin{aligned}
&A_{\mathcal{C}}C\big(A_{\mathcal{C}}^{\mathrm{T}}\phi(t)\big)A_{\mathcal{C}}^{\mathrm{T}}\frac{d}{dt}\phi(t)+A_{\mathcal{R}}g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi(t)\big)+A_{\mathcal{L}}i_{\mathcal{L}}(t)+A_{\mathcal{V}}i_{\mathcal{V}}(t)\\
&\quad+A_{\mathcal{I}}i_{\mathcal{I}}(t)=0,\\
&-A_{\mathcal{L}}^{\mathrm{T}}\phi(t)+\mathcal{L}\big(i_{\mathcal{L}}(t)\big)\frac{d}{dt}i_{\mathcal{L}}(t)=0,\\
&-A_{\mathcal{V}}^{\mathrm{T}}\phi(t)+u_{\mathcal{V}}(t)=0.
\end{aligned}\tag{52}$$

The unknown variables of this system are the functions for node potentials, inductive currents, and currents of voltage sources. The remaining physical variables (such as the voltages and the resistive and capacitive currents) can be algebraically reconstructed from the solutions of the above system.

(b) **Modified loop analysis (MLA)**

Additionally assuming that the characteristic functions g_k of all resistances are strictly monotonic and surjective, the conductance function possesses some continuous and strictly monotonic inverse function $r : \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}$. This function as well fulfills $r(0) = 0$ and

$$i_{\mathcal{R}} \cdot r(i_{\mathcal{R}}) > 0 \quad \text{for all } i_{\mathcal{R}} \in \mathbb{R}^{n_{\mathcal{R}}} \setminus \{0\}.$$

Now using the component relations together with the loop-based formulation of the Kirchhoff laws, we obtain from the KVL, the component relations for resistances and inductances, and the KCL for resistive and inductive currents that

$$B_{\mathcal{L}}\mathcal{L}\big(B_{\mathcal{L}}^{\mathrm{T}}\iota(t)\big)B_{\mathcal{L}}^{\mathrm{T}}\frac{d}{dt}\iota(t) + B_{\mathcal{R}}r\big(B_{\mathcal{R}}^{\mathrm{T}}\iota(t)\big) + B_{\mathcal{C}}u_{\mathcal{C}}(t) + B_{\mathcal{I}}u_{\mathcal{I}}(t) + B_{\mathcal{V}}u_{\mathcal{V}}(t) = 0.$$

Moreover, the KCL, together with the component relation for capacitances, reads

$$-B_{\mathcal{C}}^{\mathrm{T}}\iota(t) + \mathcal{C}\big(u_{\mathcal{C}}(t)\big) \cdot \frac{d}{dt}u_{\mathcal{C}}(t) = 0.$$

Using these two relations together with the KVL for the voltage sources, we are led to the *modified loop analysis*

$$\begin{aligned}
&B_{\mathcal{L}}\mathcal{L}\big(B_{\mathcal{L}}^{\mathrm{T}}\iota(t)\big)B_{\mathcal{L}}^{\mathrm{T}}\frac{d}{dt}\iota(t) + B_{\mathcal{R}}r\big(B_{\mathcal{R}}^{\mathrm{T}}\iota(t)\big) + B_{\mathcal{C}}u_{\mathcal{C}}(t) + B_{\mathcal{I}}u_{\mathcal{I}}(t)\\
&\quad + B_{\mathcal{V}}u_{\mathcal{V}}(t) = 0,\\
&-B_{\mathcal{C}}^{\mathrm{T}}\iota(t) + \mathcal{C}\big(u_{\mathcal{C}}(t)\big)\frac{d}{dt}u_{\mathcal{C}}(t) = 0,\\
&-B_{\mathcal{I}}^{\mathrm{T}}\iota(t) + i_{\mathcal{I}}(t) = 0.
\end{aligned} \tag{53}$$

The unknown variables of this system are the functions for loop currents, capacitive voltages, and voltages of current sources.

2.6.2 *Differential–Algebraic Equations, General Facts*

Modified nodal analysis and modified loop analysis are systems of equations with a vector-valued function in one indeterminate as unknown. Some of these equations contain the derivative of certain components of the to-be-solved function, whereas other equations are of purely algebraic nature. Such systems are called *differential–algebraic equations*. A rigorous definition and some basics of this type are presented in the following.

Definition 6.1 (Differential–algebraic equation, solution) Let $U, V \subset \mathbb{R}^n$ be open sets, let $I = [t_0, t_f)$ be an interval for some $t_f \in (t_0, \infty]$. Let $\mathcal{F} : U \times V \times I \to \mathbb{R}^k$ be a function. Then an equation of the form

$$\mathcal{F}\big(\dot{x}(t), x(t), t\big) = 0 \tag{54}$$

is called a *differential–algebraic equation (DAE)*. A function $x(\cdot) : [t_0, \omega) \to V$ is said to be a *solution* of the DAE (54) if it is differentiable with $\dot{x}(t)$ for all $t \in [t_0, \omega)$ and (54) is pointwise fulfilled for all $t \in [t_0, \omega)$.

A vector $x_0 \in V$ is called a *consistent initial value* if (54) has a solution with $x(t_0) = x_0$.

Remark 6.2

(i) If $\mathcal{F} : U \times V \times I \to \mathbb{R}^k$ is of the form $\mathcal{F}(\dot{x}, x, t) = \dot{x} - f(x, t)$, then (54) reduces to an ordinary differential equation (ODE). In this case, the assumption of continuity of $f : V \times I$ gives rise to the consistency of any initial value. If, moreover, f is locally Lipschitz continuous with respect to x (that is, for all $(x, t) \in V \times I$, there exist a neighborhood $\mathcal{U}$ and $L > 0$ such that $\|f(x_1, \tau) - f(x_2, \tau)\| \leq \|x_1 - x_2\|$ for all $(x_1, \tau), (x_2, \tau) \in \mathcal{U}$), then any initial condition determines the local solution uniquely [8, §7.3]. The local Lipschitz continuity is, for instance, fulfilled if f is continuously differentiable.

(ii) If $\mathcal{F}(\cdot, \cdot, \cdot)$ is differentiable and $\frac{d}{d\dot{x}}\mathcal{F}(\dot{x}_0, x_0, t_0)$ is an invertible matrix at some $(\dot{x}_0, x_0, t_0) \in U \times V \times I$, then the implicit function theorem [59, Sect. 17.8] implies that the differential–algebraic equation (54) is locally equivalent to an ODE.

Since theory of ODEs is well understood, it is—at least from a theoretical point of view—desirable to lead back a differential–algebraic equation to an ODE in a certain way. This is done in what follows.

Definition 6.3 (Derivative array, differentiation index) Let $U, V \subset \mathbb{R}^n$ be open sets, let $I = [t_0, t_f)$ be an interval for some $t_f \in (t_0, \infty]$. Let $l \in \mathbb{N}$, $\mathcal{F} : U \times V \times I \to \mathbb{R}^k$, and let a differential–algebraic equation (54) be given. Then the *μth derivative array of* (54) is given by the first μ formal derivatives of (54) with respect to time, that is,

$$\mathcal{F}_\mu\big(x^{(\mu+1)}(t), x^{(\mu)}(t), \ldots, \dot{x}(t), x(t), t\big) = \begin{pmatrix} \mathcal{F}(\dot{x}(t), x(t), t) \\ \frac{d}{dt}\mathcal{F}(\dot{x}(t), x(t), t) \\ \vdots \\ \frac{d^\mu}{dt^\mu}\mathcal{F}(\dot{x}(t), x(t), t) \end{pmatrix} = 0. \tag{55}$$

The differential–algebraic equation (54) is said to have *a differentiation index* $\mu \in \mathbb{N}$ if for all $(x, t) \in V \times I$, there exists a unique $\dot{x} \in V$ such that there exist $\ddot{x}, \ldots, x^{(\mu+1)} \in U$ such that $\mathcal{F}_\mu(x^{(\mu+1)}, x^{(\mu)}, \ldots, \dot{x}, x(t), t) = 0$. In this case, there exists a function $f : V \times I \to V$ with $(x, t) \mapsto \dot{x}$ for t, x, and $\dot{x}$ with the above properties. The ODE

$$\dot{x}(t) = f\big(x(t), t\big) \tag{56}$$

is said to be an *inherent ordinary differential equation of* (54).

Remark 6.4

(i) By the chain rule, we have

$$\begin{aligned}0 &= \frac{d}{dt}\mathcal{F}\big(\dot{x}(t), x(t), t\big)\\ &= \frac{\partial}{\partial \dot{x}}\mathcal{F}\big(\dot{x}(t), x(t), t\big)\cdot \ddot{x}(t) + \frac{\partial}{\partial x}\mathcal{F}\big(\dot{x}(t), x(t), t\big)\cdot \dot{x}(t)\\ &\quad + \frac{\partial}{\partial t}\mathcal{F}\big(\dot{x}(t), x(t), t\big).\end{aligned}$$

A further successive application of the chain and product rules leads to a derivative array of higher order.

(ii) Since the inherent ODE is obtained by differentiation of the differential–algebraic equation, any solution of (54) solves (56) as well.

(iii) The inherent ODE is obtained by picking equations of the μth derivative array that are explicit for the components of $\dot{x}$. In particular, the equations in $\mathcal{F}_\mu(x^{(\mu+1)}(t), x^{(\mu)}(t), \ldots, \dot{x}(t), x(t), t) = 0$ that contain higher derivatives of x can be abolished. For instance, a so-called *semiexplicit differential–algebraic equation*, that is, a DAE of the form

$$0 = \begin{pmatrix} \dot{x}_1(t) - f_1(x_1(t), x_2(t), t) \\ f_2(x_1(t), x_2(t), t) \end{pmatrix} \tag{57}$$

may be transformed to its inherent ODE by only differentiating the equation $f_2(x_1(t), x_2(t), t) = 0$. This yields

$$\begin{aligned}0 &= \frac{\partial}{\partial x_1} f_2\big(x_1(t), x_2(t), t\big)\dot{x}_1(t) + \frac{\partial}{\partial x_2} f_2\big(x_1(t), x_2(t), t\big)\dot{x}_2(t)\\ &= \frac{\partial}{\partial x_1} f_2\big(x_1(t), x_2(t), t\big) f_1\big(x_1(t), x_2(t), t\big) + \frac{\partial}{\partial x_2} f_2\big(x_1(t), x_2(t), t\big)\dot{x}_2(t).\end{aligned} \tag{58}$$

If $\frac{\partial}{\partial x_2} f_2(x_1(t), x_2(t), t)$ is invertible, then the system is of differentiation index $\mu = 1$, and the inherent ODE reads

$$\begin{aligned}&\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}\\ &\quad = \begin{pmatrix} f_1(x_1(t), x_2(t), t) \\ -(\frac{\partial}{\partial x_2} f_2(x_1(t), x_2(t), t))^{-1} \frac{\partial}{\partial x_1} f_2(x_1(t), x_2(t), t) f_1(x_1(t), x_2(t), t) \end{pmatrix}.\end{aligned} \tag{59}$$

In this case, $(x_1(\cdot), x_2(\cdot))$ solves the differential–algebraic equation (57) if and only if it solves the inherent ODE (59) and the initial value (x_{10}, x_{20}) fulfills the *algebraic constraint* $f_2(x_{10}, x_{20}, t_0) = 0$.

In case of singular $\frac{\partial}{\partial x_2} f_2(x_1(t), x_2(t), t)$, some further differentiations are necessary to obtain the inherent ODE. A semiexplicit form may then be obtained by applying a state space transformation $\bar{x}(t) = T(x(t), t)$ for some differentiable mapping $T : V \times I \to \bar{V}$ with the property that $T(\cdot, t) : V \times \bar{V}$ is bijective for all $t \in I$ and, additionally, by applying some suitable mapping $W : \mathbb{R}^k \times I \times I \to \mathbb{R}^k$ to the differential–algebraic equation that consists of $\dot{x}_1(t) - f_1(x_1(t), x_2(t), t)$ and the differentiated algebraic constraint. The algebraic constraint obtained in this way is referred to as a *hidden algebraic constraint*. This procedure is repeated until no hidden algebraic constraint is obtained anymore. In this case, the solution set of the differential–algebraic equation (57) equals the solution set of its inherent ODE with the additional property that the initial value fulfills all algebraic and hidden algebraic constraints.

The remaining part of this subsection is devoted to a differential–algebraic equation of special structure comprising both MNA and MLA, namely

$$\begin{aligned} 0 &= E\alpha\big(E^{\mathrm{T}}x_1(t)\big)E^{\mathrm{T}}\dot{x}_1(t) + A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t), \\ 0 &= \beta\big(x_2(t)\big)\dot{x}_2(t) - B_2^{\mathrm{T}}x_1(t), \\ 0 &= -B_3^{\mathrm{T}}x_1(t) + f_3(t), \end{aligned} \tag{60}$$

with the following properties.

Assumption 6.5 (Matrices and functions in the DAE (60)) Given are matrices $E \in \mathbb{R}^{n_1,m_1}$, $A \in \mathbb{R}^{n_1,m_2}$, $B_2 \in \mathbb{R}^{n_1,n_2}$, $B_3 \in \mathbb{R}^{n_1,n_3}$ and continuously differentiable functions $\alpha : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1,m_1}$, $\beta : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2,n_2}$, and $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ such that

(a) $\operatorname{rank}[\,E\,,\ A\,,\ B_2\,,\ B_3\,] = n_1$;
(b) $\operatorname{rank} B_3 = n_3$;
(c) $\alpha(z_1) > 0$, $\beta(z_2) > 0$ for all $z_1 \in \mathbb{R}^{m_1}$, $z_2 \in \mathbb{R}^{m_2}$;
(d) $\rho'(z) + (\rho')^{\mathrm{T}}(z) > 0$ for all $z \in \mathbb{R}^{n_2}$.

Next we analyze the differentiation index of differential–algebraic equations of type (60).

Theorem 6.6 *Let a differential–algebraic equation* (60) *be given and assume that matrices* $E \in \mathbb{R}^{n_1,m_1}$, $A \in \mathbb{R}^{n_1,m_2}$, $B_2 \in \mathbb{R}^{n_1,n_2}$, $B_3 \in \mathbb{R}^{n_1,n_3}$ *and functions* $\alpha : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1,m_1}$, $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2,m_2}$, $\beta : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2,n_2}$ *have the properties as in Assumptions* 6.5. *Then, for the differentiation index* μ *of* (60), *we have*

(a) $\mu = 0$ *if and only if* $n_3 = 0$ *and* $\operatorname{rank} E = n_1$.
(b) $\mu = 1$ *if and only if it is not zero and*

$$\operatorname{rank}[E, A, B_3] = n_1 \text{ and } \ker\big[E^{\mathrm{T}}, B_3\big] = \ker E^{\mathrm{T}} \times \{0\}. \tag{61}$$

(c) $\mu = 2$ *if and only if* $\mu \notin \{0, 1\}$.

We need the following auxiliary results for the proof of the above statement.

Lemma 6.7 *Let* $A \in \mathbb{R}^{n_1,m}$, $B \in \mathbb{R}^{n_1,n_2}$, $C \in \mathbb{R}^{m,m}$ *with* $C + C^{\mathrm{T}} > 0$. *Then for*

$$M = \begin{bmatrix} ACA^{\mathrm{T}} & B \\ -B^{\mathrm{T}} & 0 \end{bmatrix},$$

we have

$$\ker M = \ker[A,\ B]^{\mathrm{T}} \times \ker B. \tag{62}$$

In particular, M is invertible if and only if $\ker A \cap \ker B^{\mathrm{T}} = \{0\}$ *and* $\ker B = \{0\}$.

Proof The inclusion "$\subset$" in (62) is trivial. To show that the converse subset relation holds as well, assume that $x \in \ker M$ and partition

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

according to the block structure of M. Then we obtain

$$0 = x^{\mathrm{T}} M x = \frac{1}{2} x_1^{\mathrm{T}} A\big(C + C^{\mathrm{T}}\big) A^{\mathrm{T}} x_1 = 0,$$

whence, by

$$C + C^{\mathrm{T}} > 0$$

we have $A^{\mathrm{T}} x_1 = 0$. The equation $Mx = 0$ then implies that $Bx_2 = 0$ and $B^{\mathrm{T}} x_1 = 0$. □

Note that, by setting $n_2 = 0$ in Lemma 6.7, we obtain $\ker ACA^{\mathrm{T}} = \ker A^{\mathrm{T}}$.

Lemma 6.8 *Let matrices* $E \in \mathbb{R}^{n_1,m_1}$, $A \in \mathbb{R}^{n_1,m_2}$, $B_2 \in \mathbb{R}^{n_1,n_2}$, $B_3 \in \mathbb{R}^{n_1,n_3}$ *and functions* $\alpha : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1,m_1}$, $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2,m_2}$, $\beta : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2,n_2}$ *with the properties as in Assumptions 6.5 be given. Further, let*

$$\begin{aligned} &W \in \mathbb{R}^{n_1,p}, && \mathcal{W} \in \mathbb{R}^{n_1,\widetilde{p}}, \\ &W_1 \in \mathbb{R}^{p,p_1}, && \mathcal{W} \in \mathbb{R}^{p,\widetilde{p}_1}, \\ &W_2 \in \mathbb{R}^{n_3,p_2}, && \mathcal{W}_2 \in \mathbb{R}^{n_3,\widetilde{p}_2} \end{aligned} \tag{63a}$$

be matrices with full column rank and

$$\begin{aligned} \operatorname{im} W &= \ker E^{\mathrm{T}}, & \operatorname{im} \mathcal{W} &= \operatorname{im} E, \\ \operatorname{im} W_1 &= \ker[A, B_3]^{\mathrm{T}} W, & \operatorname{im} \mathcal{W}_1 &= \operatorname{im} W^{\mathrm{T}}[A, B_3], \\ \operatorname{im} W_2 &= \ker W^{\mathrm{T}} B_3, & \operatorname{im} \mathcal{W}_2 &= \operatorname{im} B_3^{\mathrm{T}} W. \end{aligned} \tag{63b}$$

Then we have:

(a) *The matrices* $[W,\ \mathcal{W}]$, $[W_1,\ \mathcal{W}_1]$, *and* $[W_2, \mathcal{W}_2]$ *are invertible*;
(b) $\ker E^{\mathrm{T}}\mathcal{W} = \{0\}$;
(c) $\ker W^{\mathrm{T}} B_3 = \{0\}$ *if and only if* $\ker[E^{\mathrm{T}}, B_3] = \ker E^{\mathrm{T}} \times \{0\}$;
(d) $W W_1$ *has full column rank*, *and* $\operatorname{im} W W_1 = \ker[E, A, B_3]^{\mathrm{T}}$;
(e) $\ker \mathcal{W}_1^{\mathrm{T}} Z^{\mathrm{T}} B_3 \mathcal{W}_2 = \{0\}$;
(f) $\ker[A, B_3\mathcal{W}_2]^{\mathrm{T}} W\mathcal{W}_1 = \{0\}$;
(g) $\ker B_2^{\mathrm{T}} W W_1 = \{0\}$;
(h) $\ker \mathcal{W}^{\mathrm{T}} B_3 W_2 = \{0\}$.

Proof

(a) The statement for $[\,W\,,\ \mathcal{W}\,]$ follows by the fact that both W and $\mathcal{W}$ have full column rank together with

$$\operatorname{im} W = \ker E^{\mathrm{T}} = (\operatorname{im} E)^{\perp} = (\operatorname{im} \mathcal{W})^{\perp}.$$

The invertibility of the matrices $[W_1,\ \mathcal{W}_1]$ and $[W_2,\ \mathcal{W}_2]$ follows by the same arguments.
(b) Let $x \in \ker E^{\mathrm{T}}\mathcal{W}$. Then, by the definition of W and $\mathcal{W}$, $\mathcal{W}x \in \ker E^{\mathrm{T}}$ and $\mathcal{W}x \in \operatorname{im} \mathcal{W} = \operatorname{im} E = (\ker E^{\mathrm{T}})^{\perp}$, and thus $\mathcal{W}x = 0$. Since $\mathcal{W}$ has full column rank, we have $x = 0$.
(c) Assume that $\ker W^{\mathrm{T}} B_3 = \{0\}$, and let $x_1 \in \mathbb{R}^{n_1}$, $x_3 \in \mathbb{R}^{n_3}$ with

$$\begin{bmatrix} E^{\mathrm{T}} & B_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = 0.$$

Multiplication of this equation from the left by W^{T} leads to $W^{\mathrm{T}} B_3 x_3 = 0$, and thus $x_3 = 0$.

To prove the converse direction, assume that $W^{\mathrm{T}} B_3 x_3 = 0$. Then

$$B_3 x_3 \in \ker W^{\mathrm{T}} = (\operatorname{im} W)^{\perp} = \left(\ker E^{\mathrm{T}}\right)^{\perp} = \operatorname{im} E.$$

Hence, there exists $x_1 \in \mathbb{R}^{m_1}$ such that $E x_1 = B_3 x_3$, that is,

$$\begin{pmatrix} -x_1 \\ x_3 \end{pmatrix} \in \ker \begin{bmatrix} E & B_3 \end{bmatrix} = \ker E \times \{0\},$$

whence $x_3 = 0$.
(d) The matrix $W W_1$ has full column rank as a product of matrices with full column rank.

The inclusion $\operatorname{im} W W_1 \subset \ker[E, A, B_3]^{\mathrm{T}}$ follows from

$$\begin{bmatrix} E^{\mathrm{T}} \\ A^{\mathrm{T}} \\ B_3^{\mathrm{T}} \end{bmatrix} W W_1 = \begin{bmatrix} (E^{\mathrm{T}} W) W_1 \\ \left(\left[\begin{smallmatrix} A^{\mathrm{T}} \\ B_3^{\mathrm{T}} \end{smallmatrix}\right] W\right) W_1 \end{bmatrix} = 0.$$

To prove $\operatorname{im} W W_1 \supset \ker[E,\ A,\ B_3]^{\mathrm{T}}$, assume that $x \in \ker[E,\ A,\ B_3]^{\mathrm{T}}$. Since, in particular, $x \in \ker E^{\mathrm{T}}$, there exists $y \in \mathbb{R}^{p}$ with $x = W y$, and thus

$$\begin{bmatrix} A^{\mathrm{T}} \\ B_3^{\mathrm{T}} \end{bmatrix} W y = 0.$$

By the definition of W_2, there exists $y \in \mathbb{R}^{p_2}$ with $y = W_2 z$, and thus

$$x = W W_2 z \in \operatorname{im} W W_2.$$

(c) Assume that $z \in \mathbb{R}^{p_2}$ with $\mathcal{W}_1^{\mathrm{T}} W^{\mathrm{T}} B_3 \mathcal{W}_2 z = 0$. Then

$$\begin{aligned} W^{\mathrm{T}} B_3 \mathcal{W}_2 z \in \ker \mathcal{W}_1^{\mathrm{T}} &= (\operatorname{im} \mathcal{W}_1)^{\perp} \\ &= \left(\operatorname{im} W^{\mathrm{T}}[A,\ B_3]\right)^{\perp} \\ &= \ker[A,\ B_3]^{\mathrm{T}} W \subset \ker B_3^{\mathrm{T}} W = \left(\operatorname{im} W^{\mathrm{T}} B_3\right)^{\perp}, \end{aligned}$$

whence

$$W^{\mathrm{T}} B_3 \mathcal{W}_2 z \in \left(\operatorname{im} W^{\mathrm{T}} B_3\right)^{\perp} \cap \operatorname{im} W^{\mathrm{T}} B_3 = \{0\}.$$

This implies $W^{\mathrm{T}} B_3 \mathcal{W}_2 z = 0$, and thus

$$\mathcal{W}_2 z \in \ker W^{\mathrm{T}} B_3 = \operatorname{im} W_2 = (\operatorname{im} W_2)^{\perp}.$$

Therefore, we have $\mathcal{W}_2 z \in \operatorname{im} W_2 \cap \operatorname{im} \mathcal{W}_2 = \{0\}$. The property of $\mathcal{W}_2$ having full column rank then implies $z = 0$.

(f) Let $z \in \ker(A^{\mathrm{T}} W) \cap \ker B_3^{\mathrm{T}} W$. Since $Wz \in \ker E$ by the definition of W, we have

$$W z \in \ker \begin{bmatrix} E^{\mathrm{T}} \\ A^{\mathrm{T}} \\ B_3^{\mathrm{T}} \end{bmatrix} = \{0\},$$

whence $z = 0$.

(g) Let $z \in \ker B_2^{\mathrm{T}} W W_1$. Then $W W_1 \in \ker B_2^{\mathrm{T}}$, and, by assertion d),

$$W W_1 z \in \ker[E, A, B_2]^{\mathrm{T}}.$$

By the assumption that $[E,\ A,\ B_2,\ B_3]$ has full row rank we now obtain that $W W_1 z = 0$. By the property of $W W_1$ having full column rank (see (d)) we may infer that $z = 0$.

(h) Assume that $z \ker \mathcal{W}^{\mathrm{T}} B_3 W_2$. Then $W_2 z \in \ker \mathcal{W}^{\mathrm{T}} B_3$, and $W_2 z \in \ker W^{\mathrm{T}} B_3$ by the definition of W_2. Thus, we have

$$W_2 z \in \ker[W, \mathcal{W}]^{\mathrm{T}} B_3,$$

and, by the invertibility of $[W,\ \mathcal{W}]$ (see (a)), we can conclude that

$$W_2 z \in \ker B_3 = \{0\}.$$

The property of Z_2 having full column rank then gives rise to $z = 0$. □

Now we prove Theorem 6.6.

Proof of Theorem 6.6

(a) First assume that E has full row rank and $n_3 = 0$. Then by Lemma 6.7 we see that the matrix $E\alpha(E^{\mathrm{T}}x_1)E^{\mathrm{T}}$ is invertible for all $x_1 \in \mathbb{R}^{n_1}$. Since, furthermore, the last equation in (60) is trivial, the differential–algebraic equation (60) is already equivalent to the ordinary differential equation

$$\begin{aligned}\dot{x}_1(t) &= -\big(E\alpha\big(E^{\mathrm{T}}x_1(t)\big)E^{\mathrm{T}}\big)^{-1}\big(A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) \\ &\qquad + B_3x_3(t) + f_1(t)\big), \\ \dot{x}_2(t) &= \beta\big(x_2(t)\big)^{-1}B_2^{\mathrm{T}}x_1(t).\end{aligned} \tag{64}$$

Consequently, the differentiation index of (60) is zero in this case.

To prove the converse statement, assume that $\ker E^{\mathrm{T}} \neq \{0\}$ or $n_3 > 0$. The first statement implies that no derivatives of the components of $x_1(t)$ that are in the kernel of E^{T} occur, whereas the latter assumption implies that (60) does not contain any derivatives of x_3 (which is now a vector with at least one component). Hence, some differentiations of the equations in (60) are needed to obtain an ordinary differential equation, and the differentiation index of (60) is consequently larger than zero.

(b) Here (and in part (c)) we will make use of the (trivial) fact that, for invertible matrices W and T of suitable size, the differentiation indices of the DAEs $\mathcal{F}(\dot{x}(t), x(t), t) = 0$ and $W\mathcal{F}(T\dot{z}(t), Tz(t), t) = 0$ coincide.

Let $W \in \mathbb{R}^{n_1,p}$ and $\mathcal{W} \in \mathbb{R}^{n_1,\widetilde{p}}$ be matrices of full column rank with the properties as in (63a), (63b). Using Lemma 6.8, we see that there exists a unique decomposition

$$x_1(t) = Wx_{11}(t) + \mathcal{W}x_{12}(t).$$

By a multiplication of the first equation in (60) respectively from the left by W^{T} and $\mathcal{W}^{\mathrm{T}}$, we can make use of the initial statement to see that the index of (60) coincides with the index of the differential–algebraic equation

$$\begin{aligned}0 &= \mathcal{W}^{\mathrm{T}}E\alpha\big(E^{\mathrm{T}}\mathcal{W}^{\mathrm{T}}x_{12}(t)\big)E^{\mathrm{T}}\mathcal{W}\dot{x}_{12}(t) + \mathcal{W}^{\mathrm{T}}A\rho\big(A^{\mathrm{T}}Wx_{11}(t) + A^{\mathrm{T}}\mathcal{W}x_{12}(t)\big) \\ &\qquad + \mathcal{W}^{\mathrm{T}}B_2x_2(t) + \mathcal{W}^{\mathrm{T}}B_3x_3(t) + \mathcal{W}^{\mathrm{T}}f_1(t),\end{aligned} \tag{65a}$$

$$0 = \beta\big(x_2(t)\big)\dot{x}_2(t) - B_2^{\mathrm{T}}Wx_{11}(t) - B_2^{\mathrm{T}}\mathcal{W}x_{12}(t), \tag{65b}$$

$$\begin{aligned}0 &= W^{\mathrm{T}}A\rho\big(A^{\mathrm{T}}Wx_{11}(t) + A^{\mathrm{T}}\mathcal{W}x_{12}(t)\big) \\ &\qquad + W^{\mathrm{T}}B_2x_2(t) + W^{\mathrm{T}}B_3x_3(t) + W^{\mathrm{T}}f_1(t),\end{aligned} \tag{65c}$$

$$0 = -B_3^{\mathrm{T}}Wx_{11}(t) + B_3^{\mathrm{T}}\mathcal{W}x_{12}(t) + f_3(t). \tag{65d}$$

Now we show that, under the assumptions that the index of the differential–algebraic equation (65a)–(65d) is nonzero and the rank conditions in (61) hold, the index of the DAE (65a)–(65d) equals one:

Using Lemma 6.7, we see that Eqs. (65a) and (65b) can be solved for $\dot{x}_{12}(t)$ and $\dot{x}_2(t)$, that is,

$$\dot{x}_{12}(t) = -\big(\mathcal{W}^{\mathrm{T}} E\alpha\big(E^{\mathrm{T}}\mathcal{W}^{\mathrm{T}} x_{12}(t)\big)E^{\mathrm{T}}\mathcal{W}\big)^{-1}\mathcal{W}^{\mathrm{T}}\big(A\rho\big(A^{\mathrm{T}} W x_{11}(t) + A^{\mathrm{T}}\mathcal{W}x_{12}(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big), \tag{66a}$$

$$\dot{x}_2(t) = \beta\big(x_2(t)\big)^{-1} B_2^{\mathrm{T}}\big(W x_{11}(t) + \mathcal{W}x_{12}(t)\big). \tag{66b}$$

For convenience and better overview, we will further use the following abbreviations:

$$\begin{aligned}
&\rho\big(A^{\mathrm{T}} W x_{11}(t) + A^{\mathrm{T}}\mathcal{W}x_{12}(t)\big) \rightsquigarrow \rho,\\
&\rho'\big(A^{\mathrm{T}} W x_{11}(t) + A^{\mathrm{T}}\mathcal{W}x_{12}(t)\big) \rightsquigarrow \rho',\\
&\alpha\big(E^{\mathrm{T}}\mathcal{W}^{\mathrm{T}} x_{12}(t)\big) \rightsquigarrow \alpha,\\
&\beta\big(x_2(t)\big) \rightsquigarrow \beta.
\end{aligned}$$

The first-order derivative array $\mathcal{F}_1(x^{(2)}(t), \dot{x}(t), x(t), t)$ of the DAE (60) further contains the time derivatives of (65c) and (65d), which can, in compact form and by making further use of (66a), (66b), be written as

$$\begin{aligned}
&\underbrace{\begin{bmatrix} W^{\mathrm{T}} A\rho' A^{\mathrm{T}} W & W^{\mathrm{T}} B_3 \\ -B_3^{\mathrm{T}} W & 0 \end{bmatrix}}_{=:M}\begin{pmatrix} \dot{x}_{11}(t) \\ \dot{x}_3(t) \end{pmatrix}\\
&= -\begin{pmatrix} W^{\mathrm{T}} A\rho' A^{\mathrm{T}}\mathcal{W}\dot{x}_{12}(t) + W^{\mathrm{T}} B_2\dot{x}_2(t) + W^{\mathrm{T}}\dot{f}_2(t) \\ B_3^{\mathrm{T}}\mathcal{W}\dot{x}_{12}(t) + \dot{f}_3(t) \end{pmatrix}\\
&= \begin{pmatrix} W^{\mathrm{T}} A\rho' A^{\mathrm{T}}\mathcal{W}(\mathcal{W}^{\mathrm{T}} E\alpha E^{\mathrm{T}}\mathcal{W})^{-1}\mathcal{W}^{\mathrm{T}}(A\rho + B_2x_2(t) + B_3x_3(t) + f_1(t)), \\ B_3^{\mathrm{T}}\mathcal{W}(\mathcal{W}^{\mathrm{T}} E\alpha E^{\mathrm{T}}\mathcal{W})^{-1}\mathcal{W}^{\mathrm{T}}(A\rho + B_2x_2(t) + B_3x_3(t) + f_1(t)) + \dot{f}_3(t) \end{pmatrix}\\
&\quad - \begin{pmatrix} W^{\mathrm{T}} B_2\beta^{-1}B_2^{\mathrm{T}}(W x_{11}(t) + \mathcal{W}x_{12}(t)) + \mathcal{W}^{\mathrm{T}}\dot{f}_2(t) \\ 0 \end{pmatrix}.
\end{aligned} \tag{67}$$

Since, by assumption, there holds (61), we obtain from Lemma 6.8 (c) and (d) that

$$\ker W^{\mathrm{T}} B_3 = \{0\} \quad \text{and} \quad \ker[A, B_3]^{\mathrm{T}} W = \{0\}.$$

Then by using of $\rho' + \rho'^{\mathrm{T}} > 0$ we may infer from Lemma 6.7 that M is invertible. As a consequence, $\dot{x}_{11}(t)$ and $\dot{x}_3(t)$ can be expressed by suitable functions depending on $x_{12}(t)$, $x_2(t)$, and t. This implies that the index of the differential–algebraic equation equals one.

Now we show that conditions (61) are also necessary for the index of the differential–algebraic equation (60) not to exceed one:

Consider the first-order derivative array $\mathcal{F}_1(x^{(2)}(t), \dot{x}(t), x(t), t)$ of the DAE (60). Aiming to construct an ordinary differential equation (56) for

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

from $\mathcal{F}_1(x^{(2)}(t), \dot{x}(t), x(t), t)$, it can be seen that the derivatives of Eqs. (66a) and (66b) cannot be used to form the inherent ODE (the derivatives of these equations explicitly contain the second derivatives of $x_{12}(t)$ and $x_2(t)$). As a consequence, the inherent ODE is formed by Eqs. (66a), (66b) and (67). Aiming to seek for a contradiction, assume that one of the conditions in (61) is violated:

In case of $\operatorname{rank}[E, A, B_3] < n_1$, Lemma 6.8 (d) implies that

$$\ker[E, B_3]^{\mathrm{T}} W \neq \{0\}.$$

Now consider matrices W_1, $\mathcal{W}_1$ of full column rank with the properties as in (63a), (63b). By Lemma 6.8 (a) there exists a unique decomposition

$$x_{11}(t) = W_1 x_{111}(t) + \mathcal{W}_1 x_{112}(t).$$

Then the right-hand side of Eq. (67) reads

$$\begin{bmatrix} W^{\mathrm{T}} A \rho' A^{\mathrm{T}} W \mathcal{W}_1 & 0 & W^{\mathrm{T}} B_3 \\ -B_3^{\mathrm{T}} W W_1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_{111}(t) \\ \dot{x}_{112}(t) \\ \dot{x}_3(t) \end{pmatrix}.$$

Consequently, it is not possible to use the first-order derivative array to express $\dot{x}_{112}(t)$ as a function of $x(t)$. This is a contradiction to the index of the differential–algebraic equation (60) being at most one.

In case of $\ker[E^{\mathrm{T}}, B_3] \neq \ker E^{\mathrm{T}} \times \{0\}$, by Lemma 6.8 (c) we have $\ker(W^{\mathrm{T}} B_3) \neq \{0\}$. Consider matrices W_2, $\mathcal{W}_2$ of full column rank with the properties as in (63a), (63b). By Lemma 6.8 a) there exists a unique decomposition

$$x_3(t) = W_2 x_{31}(t) + \mathcal{W}_2 x_{32}(t).$$

Then the right-hand side of Eq. (67) reads

$$\begin{bmatrix} W^{\mathrm{T}} A \rho' A^{\mathrm{T}} W & W^{\mathrm{T}} B_3 \mathcal{W}_2 & 0 \\ -B_3^{\mathrm{T}} W & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_{11}(t) \\ \dot{x}_{31}(t) \\ \dot{x}_{32}(t) \end{pmatrix}.$$

Consequently, it is not possible to use the first-order derivative array to express $\dot{x}_{32}(t)$ as a function of $x(t)$. This is a contradiction to the index of the differential–algebraic equation (60) being at most one.

(c) To complete the proof, we have to show that the inherent ODE can be constructed from the second-order derivative array $\mathcal{F}_2(x^{(3)}(t), x^{(2)}(t), \dot{x}(t), x(t), t)$ of the DAE (60). With the matrices W, $\mathcal{W}$, W_1, $\mathcal{W}_1$, W_2, $\mathcal{W}_2$ and the corresponding decompositions, a multiplication of (67) from the left with

$$\begin{bmatrix} \mathcal{W}_1^{\mathrm{T}} & 0 \\ 0 & \mathcal{W}_2^{\mathrm{T}} \end{bmatrix}$$

leads to

$$\underbrace{\begin{bmatrix} \mathcal{W}_1^{\mathrm{T}} W^{\mathrm{T}} A\rho' A^{\mathrm{T}} W \mathcal{W}_1 & \mathcal{W}_1^{\mathrm{T}} W^{\mathrm{T}} B_3 \mathcal{W}_2 \\ -\mathcal{W}_2^{\mathrm{T}} B_3^{\mathrm{T}} W \mathcal{W}_1 & 0 \end{bmatrix}}_{=:M_1} \begin{pmatrix} \dot{x}_{112}(t) \\ \dot{x}_{32}(t) \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{W}_1^{\mathrm{T}} W^{\mathrm{T}} A\rho' A^{\mathrm{T}} \mathcal{W}(\mathcal{W}^{\mathrm{T}} E\alpha E^{\mathrm{T}} \mathcal{W})^{-1} \mathcal{W}^{\mathrm{T}} (A\rho + B_2 x_2(t) + B_3 x_3(t) + f_1(t)) \\ \mathcal{W}_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W}(\mathcal{W}^{\mathrm{T}} E\alpha E^{\mathrm{T}} \mathcal{W})^{-1} \mathcal{W}^{\mathrm{T}} (A\rho + B_2 x_2(t) + B_3 x_3(t) + f_1(t)) + \mathcal{W}_2^{\mathrm{T}} \dot{f}_3(t) \end{pmatrix}$$
$$- \begin{pmatrix} \mathcal{W}_1^{\mathrm{T}} \mathcal{W}^{\mathrm{T}} B_2 \beta^{-1} B_2^{\mathrm{T}} (W x_{11}(t) + \mathcal{W} x_{12}(t)) + \mathcal{W}_1^{\mathrm{T}} \mathcal{W}^{\mathrm{T}} \dot{f}_2(t) \\ 0 \end{pmatrix}. \tag{68}$$

By Lemma 6.8 (e) and (f) we have

$$\ker \mathcal{W}_1^{\mathrm{T}} W^{\mathrm{T}} B_3 \mathcal{W}_2 = \{0\} \quad \text{and} \quad \ker [A, B_3 \mathcal{W}_2]^{\mathrm{T}} = \{0\}.$$

Lemma 6.7 then implies that M_1 is invertible, and, consequently, the vectors $\dot{x}_{112}(t)$ and $\dot{x}_{32}(t)$ are expressible by suitable functions of $x_{111}(t)$, $x_{112}(t)$, $x_2(t)$, $x_{31}(t)$, $x_{32}(t)$, and t. It remains to show that the second-order derivative array might also be used to express $\dot{x}_{111}(t)$ and $\dot{x}_{31}(t)$ as functions of $x_{111}(t)$, $x_{112}(t)$, $x_2(t)$, $x_{31}(t)$, $x_{32}(t)$, and t: A multiplication of (67) from the left by

$$\begin{bmatrix} W_1^{\mathrm{T}} & 0 \\ 0 & W_2^{\mathrm{T}} \end{bmatrix}$$

yields, by making use of $W_1^{\mathrm{T}} W^{\mathrm{T}} A = 0$, that

$$\begin{aligned} 0 &= W_1^{\mathrm{T}} W^{\mathrm{T}} B_2 \beta^{-1} B_2^{\mathrm{T}} \big(W W_1 x_{111}(t) + W \mathcal{W}_1 x_{112}(t) + \mathcal{W} x_{12}(t) \big) \\ &\quad + W_1^{\mathrm{T}} W^{\mathrm{T}} \dot{f}_2(t), \end{aligned} \tag{69a}$$
$$\begin{aligned} 0 &= W_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W} \big(\mathcal{W}^{\mathrm{T}} E\alpha E^{\mathrm{T}} \mathcal{W} \big)^{-1} \mathcal{W}^{\mathrm{T}} \\ &\quad \cdot \big(A\rho + B_2 x_2(t) + B_3 W_2 x_{31}(t) + B_3 \mathcal{W}_2 x_{32}(t) + f_1(t) \big) + W_2^{\mathrm{T}} \dot{f}_3(t). \end{aligned} \tag{69b}$$

The second-order derivative array of (60) contains the derivative of these equations. Differentiating (69a) with respect to time, we obtain

$$
\begin{aligned}
&W_1^{\mathrm{T}} W^{\mathrm{T}} B_2 \beta^{-1} B_2^{\mathrm{T}} W_1 W \dot{x}_{111}(t) \\
&\quad = -W_1^{\mathrm{T}} W^{\mathrm{T}} B_2 \beta^{-1} B_2^{\mathrm{T}} \big(W \mathcal{W}_1 \dot{x}_{112}(t) + \mathcal{W} \dot{x}_{12}(t) \big) \\
&\qquad - W_1^{\mathrm{T}} W^{\mathrm{T}} B_2 \frac{d}{dt} \big(\beta^{-1} \big) B_2^{\mathrm{T}} \big(W \mathcal{W}_1 x_{112}(t) + \mathcal{W} x_{12}(t) \big) - W_1^{\mathrm{T}} \mathcal{W}^{\mathrm{T}} \ddot{f}_2(t).
\end{aligned} \tag{70}
$$

Using Lemma 6.8 (g) and Lemma 6.7, we see that the matrix

$$
W_1^{\mathrm{T}} W^{\mathrm{T}} B_2 \beta^{-1} B_2^{\mathrm{T}} W W_1 \in \mathbb{R}^{p_1, p_1}
$$

is invertible. By using the quotient and chain rule it can be inferred that $\frac{d}{dt}(\beta^{-1})$ is expressible by a suitable function depending on $x_2(t)$ and $\dot{x}_2(t)$. Consequently, the derivative of $x_{111}(t)$ can be expressed as a function depending on $x_{112}(t)$, $x_{12}(t)$, $x_2(t)$, their derivatives, and t. Since, on the other hand, $\dot{x}_{112}(t)$, $\dot{x}_{12}(t)$, and $\dot{x}_2(t)$ already have representations as functions depending on $x_{111}(t)$, $x_{112}(t)$, $x_{12}(t)$, $x_2(t)$, $x_{31}(t)$, $x_{32}(t)$, and t, this is true for $\dot{x}_{112}(t)$ as well.

Differentiating (69b) with respect to t, we obtain

$$
\begin{aligned}
&W_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W} \big(\mathcal{W}^{\mathrm{T}} E \alpha E^{\mathrm{T}} \mathcal{W} \big)^{-1} \mathcal{W}^{\mathrm{T}} B_3 W_2 \dot{x}_{31} \\
&\quad = W_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W} \big(\mathcal{W}^{\mathrm{T}} E \alpha E^{\mathrm{T}} \mathcal{W} \big)^{-1} \mathcal{W}^{\mathrm{T}} \\
&\qquad \cdot \big(A \rho' A W W_1 \dot{x}_{111}(t) + A \rho' A W \mathcal{W}_1 \dot{x}_{112}(t) + A \rho' A \mathcal{W} \dot{x}_{12}(t) \\
&\qquad + B_2 \dot{x}_2(t) + B_3 W_2 \dot{x}_{31}(t) + \dot{f}_1(t) \big) \\
&\qquad + W_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W} \frac{d}{dt} \big(\mathcal{W}^{\mathrm{T}} E \alpha E^{\mathrm{T}} \mathcal{W} \big)^{-1} \mathcal{W}^{\mathrm{T}} \\
&\qquad \cdot \big(A \rho + B_2 x_2(t) + B_3 W_2 x_{31}(t) + B_3 \mathcal{W}_2 x_{32}(t) + f_1(t) \big) + W_2^{\mathrm{T}} \dot{f}_3(t).
\end{aligned}
$$

Lemma 6.8 h) and Lemma 6.7 give rise to the invertibility of the matrix

$$
W_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W} \big(\mathcal{W}^{\mathrm{T}} E \alpha E^{\mathrm{T}} \mathcal{W} \big)^{-1} \mathcal{W}^{\mathrm{T}} B_3 W_2 \in \mathbb{R}^{p_2, p_2}.
$$

Then arguing as for the derivative of Eq. (69a), we can see that $\dot{x}_{31}$ is expressible by a suitable function depending on $x_{111}(t)$, $x_{112}(t)$, $x_{12}(t)$, $x_2(t)$, $x_{31}(t)$, $x_{32}(t)$, and t.

This completes the proof. □

Remark 6.9 (Differentiation index of differential–algebraic equations)

(i) The algebraic constraints of (60) are formed by (69a), (69b). Note that (69a) is trivial (i.e., it is an empty set of equations) if rank $E = n_1$. Accordingly, the hidden constraint (69a) is trivial in the case where $n_3 = 0$.

(ii) The hidden algebraic constraints of (60) are formed by (69a), (69b). Note that (69a) is trivial if $\operatorname{rank}[E,\ A,\ B_3] = n_1$, whereas, in the case where $\ker[E^{\mathrm{T}},\ B_3] = \ker E^{\mathrm{T}} \times \{0\}$, the hidden constraint (69a) becomes trivial.

(iii) From the computations in the proof of Theorem 6.6 we see that derivatives of the "right-hand side" $f_1(\cdot)$, $f_3(\cdot)$ enter the solution of the differential–algebraic equation. The order of these derivatives equals $\mu - 1$.

We close the analysis of differential–algebraic equations of type (60) by formulating the following result on consistency of initial values.

Theorem 6.10 *Let a differential–algebraic equation* (60) *be given and assume that the matrices* $E \in \mathbb{R}^{n_1,m_1}$, $A \in \mathbb{R}^{n_1,m_2}$, $B_2 \in \mathbb{R}^{n_1,n_2}$, $B_3 \in \mathbb{R}^{n_1,n_3}$ *and functions* $\alpha : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1,m_1}$, $\rho : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2,m_2}$, $\beta : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2,n_2}$ *have the properties as in Assumptions* 6.5. *Let* W, $\mathcal{W}$, W_1, $\mathcal{W}_1$, W_2, *and* $\mathcal{W}_2$ *be matrices of full column rank with the properties as in* (63a), (63b). *Let a continuous function* $f_1 : [t_0, \infty) \to \mathbb{R}^{n_1}$ *be such that*

$$W^{\mathrm{T}} f : [t_0, \infty) \to \mathbb{R}^{p}$$

is continuously differentiable and

$$W_1^{\mathrm{T}} W^{\mathrm{T}} f : [t_0, \infty) \to \mathbb{R}^{p_2}$$

is twice continuously differentiable. Further, assume that $f_3 : [t_0, \infty) \to \mathbb{R}^{n_3}$ *is continuously differentiable and such that*

$$W_2^{\mathrm{T}} f : [t_0, \infty) \to \mathbb{R}^{p_2}$$

is twice continuously differentiable. Then the initial value

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} \tag{71}$$

is consistent if and only if

$$0 = W^{\mathrm{T}}\big(A\rho\big(A^{\mathrm{T}} x_{10}\big) + B_2 x_{20} + B_3 x_{30} + f_1(t_0)\big), \tag{72a}$$

$$0 = -B_3^{\mathrm{T}} x_{10} + f_3(t_0), \tag{72b}$$

$$0 = W_1^{\mathrm{T}} W^{\mathrm{T}} B_2 \beta(x_{20})^{-1} B_2^{\mathrm{T}} x_{10} + W_1^{\mathrm{T}} W^{\mathrm{T}} \dot{f}_1(t_0), \tag{72c}$$

$$\begin{aligned} 0 = {} & W_2^{\mathrm{T}} B_3^{\mathrm{T}} \mathcal{W}\big(\mathcal{W}^{\mathrm{T}} E \alpha\big(E^{\mathrm{T}} x_{10}\big) E^{\mathrm{T}} \mathcal{W}\big)^{-1} \mathcal{W}^{\mathrm{T}} \\ & \cdot \big(A\rho\big(A^{\mathrm{T}} x_{10}\big) + B_2 x_{20} + B_3 x_{30} + f_1(t_0)\big) + W_2^{\mathrm{T}} \dot{f}_3(t_0). \end{aligned} \tag{72d}$$

Proof First, assume that a solution of (60) evolves in the time interval $[t_0, \omega)$. The necessity of the consistency conditions (72a)–(72d) follows by the fact that, by

(65c), (65c), (69a), (69a) and the definitions of $x_{111}(t)$, $x_{112}(t)$, $x_{12}(t)$, $x_{31}(t)$, and $x_{32}(t)$, the relations

$$\begin{aligned}
0 &= W^{\mathrm{T}}\big(A\rho\big(A^{\mathrm{T}}x1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big),\\
0 &= -B_3^{\mathrm{T}}x_1(t) + f_3(t),\\
0 &= W_1^{\mathrm{T}}W^{\mathrm{T}}B_2\beta\big(x_2(t)\big)^{-1}B_2^{\mathrm{T}}x_1(t) + W_1^{\mathrm{T}}\mathcal{W}^{\mathrm{T}}\dot{f}_1(t),\\
0 &= W_2^{\mathrm{T}}B_3^{\mathrm{T}}\mathcal{W}\big(\mathcal{W}^{\mathrm{T}}E\alpha\big(E^{\mathrm{T}}x_1(t)\big)E^{\mathrm{T}}\mathcal{W}\big)^{-1}\mathcal{W}^{\mathrm{T}}\\
&\quad\cdot\big(A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big) + W_2^{\mathrm{T}}\dot{f}_3(t)
\end{aligned}$$

hold for all $t \in [t_0, \omega)$. The special case $t = t_0$ gives rise to (72a)–(72d).

To show that (72a)–(72d) is sufficient for consistency of the initialization, we prove that the inherent ODE of (72a)–(72d), together with the initial value (71) fulfilling (72a)–(72d), possesses a solution that is also a solution of the differential–algebraic equation (60):

By the construction of the inherent ODE in the proof of Theorem 6.6 we see that the right-hand side is continuously differentiable. The existence of a unique solution

$$x(\cdot) = \begin{pmatrix} x_1(\cdot)\\ x_2(\cdot)\\ x_3(\cdot)\end{pmatrix} : [t_0, \omega) \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$$

is therefore guaranteed by standard results on the existence and uniqueness of solutions of ordinary differential equations.

The inherent ODE further contains the derivative of the equations in (70) with respect to time. In other words,

$$\begin{aligned}
0 &= \frac{d}{dt}\big(W_1^{\mathrm{T}}W^{\mathrm{T}}B_2\beta\big(x_2(t)\big)^{-1}B_2^{\mathrm{T}}x_1(t) + W_1^{\mathrm{T}}\mathcal{W}^{\mathrm{T}}\dot{f}_1(t)\big),\\
0 &= \frac{d}{dt}\big(W_2^{\mathrm{T}}B_3^{\mathrm{T}}\mathcal{W}\big(\mathcal{W}^{\mathrm{T}}E\alpha\big(E^{\mathrm{T}}x_1(t)\big)E^{\mathrm{T}}\mathcal{W}\big)^{-1}\mathcal{W}^{\mathrm{T}}\\
&\quad\cdot\big(A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big) + W_2^{\mathrm{T}}\dot{f}_3(t)\big)
\end{aligned}$$

for all $t \in [t_0, \omega)$. Then we can infer from (72c) and (72d) together with (71) that

$$\begin{aligned}
0 &= W_1^{\mathrm{T}}W^{\mathrm{T}}B_2\beta\big(x_2(t)\big)^{-1}B_2^{\mathrm{T}}x_1(t) + W_1^{\mathrm{T}}\mathcal{W}^{\mathrm{T}}\dot{f}_1(t),\\
0 &= W_2^{\mathrm{T}}B_3^{\mathrm{T}}\mathcal{W}\big(\mathcal{W}^{\mathrm{T}}E\alpha\big(E^{\mathrm{T}}x_1(t)\big)E^{\mathrm{T}}\mathcal{W}\big)^{-1}\mathcal{W}^{\mathrm{T}}\\
&\quad\cdot\big(A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big) + W_2^{\mathrm{T}}\dot{f}_3(t)
\end{aligned}$$

for all $t \in [t_0, \omega)$. Since, furthermore, Eq. (68) is a part of the inherent ODE, we can conclude that the solution pointwise fulfills Eq. (67). However, the latter equation is by construction equivalent to

$$
\begin{aligned}
0 &= \frac{d}{dt}\Big(W^{\mathrm{T}}\big(A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big)\Big),\\
0 &= \frac{d}{dt}\big(-B_3^{\mathrm{T}}x_1(t) + f_3(t)\big).
\end{aligned}
$$

Analogously to the above arguments, we can infer from (72a) and (72b) together with (71) that

$$
\begin{aligned}
0 &= W^{\mathrm{T}}\big(A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big),\\
0 &= -B_3^{\mathrm{T}}x_1(t) + f_3(t)
\end{aligned}
$$

for all $t \in [t_0, \omega)$. Since these equations, together with

$$
\begin{aligned}
0 &= \mathcal{W}^{\mathrm{T}}\big(E\alpha\big(E^{\mathrm{T}}x_1(t)\big)E^{\mathrm{T}}\dot{x}_1(t) + A\rho\big(A^{\mathrm{T}}x_1(t)\big) + B_2x_2(t) + B_3x_3(t) + f_1(t)\big),\\
0 &= \beta\big(x_2(t)\big)\dot{x}_2(t) - B_2^{\mathrm{T}}x_1(t),
\end{aligned}
$$

form the differential–algebraic equation (60), the desired result is proven. □

Remark 6.11 (Relaxing Assumptions 6.5) The solution theory for differential–algebraic equations of type (60) can be extended to the case where conditions (a) and (b) in Assumptions 6.5 are not necessarily fulfilled: Consider matrices

$$
\begin{aligned}
V_1 &\in \mathbb{R}^{n_1,q_1}, & \mathcal{V}_1 &\in \mathbb{R}^{n_1,\widetilde{q}_1},\\
V_3 &\in \mathbb{R}^{n_3,q_3}, & \mathcal{V}_3 &\in \mathbb{R}^{n_3,\widetilde{q}_3}
\end{aligned}
$$

of full column rank such that

$$
\begin{aligned}
\operatorname{im} V_1 &= \ker[E, A, B_2, B_3]^{\mathrm{T}}, & \operatorname{im} \mathcal{V}_1 &= \operatorname{im}[E, A, B_2, B_3],\\
\operatorname{im} V_3 &= \ker B_3, & \operatorname{im} \mathcal{V}_3 &= \operatorname{im} B_3^{\mathrm{T}}.
\end{aligned}
$$

Then, multiplying the first equation in (60) from the left by $\mathcal{V}_1$ and the third equation in (60) from the left by $\mathcal{V}_3$, and setting

$$
x_1(t) = V_1\bar{x}_1(t) + \mathcal{V}_1\widetilde{x}_1(t), \qquad x_3(t) = V_3\bar{x}_3(t) + \mathcal{V}_3\widetilde{x}_3(t),
$$

we obtain

$$
\begin{aligned}
0 &= \mathcal{V}_1^{\mathrm{T}}E\alpha\big(E^{\mathrm{T}}\mathcal{V}_1\widetilde{x}_1(t)\big)E^{\mathrm{T}}\mathcal{V}_1\dot{\widetilde{x}}_1(t) + \mathcal{V}_1^{\mathrm{T}}A\rho\big(A^{\mathrm{T}}\widetilde{V}_1\widetilde{x}_1(t)\big) + \mathcal{V}_1^{\mathrm{T}}B_2x_2(t)\\
&\quad + \mathcal{V}_1^{\mathrm{T}}B_3\widetilde{V}_3^{\mathrm{T}}\widetilde{x}_3(t) + \mathcal{V}_1^{\mathrm{T}}f_1(t),\\
0 &= \beta\big(x_2(t)\big)\dot{x}_2(t) - B_2^{\mathrm{T}}\widetilde{V}_1\widetilde{x}_1(t),\\
0 &= -\mathcal{V}_3^{\mathrm{T}}B_3^{\mathrm{T}}\mathcal{V}_1\widetilde{x}_1(t) + \mathcal{V}_3^{\mathrm{T}}f_3(t).
\end{aligned}
\tag{73}
$$

Note that, by techniques similar as in the proof of Lemma 6.8, it can be shown that (73) is a differential–algebraic equation that fulfills the presumptions of Theorem 6.6 and Theorem 6.10.

On the other hand, multiplying the first equation from the left by V_1 and the third equation from the left by V_3, on the right-hand side, we obtain the constraints

$$V_1^{\mathrm{T}} f_1(t) = 0, \qquad V_3^{\mathrm{T}} f_3(t) = 0, \tag{74}$$

or, equivalently,

$$f_1(t) \in \operatorname{im}[E, A, B_2, B_3], \qquad f_3(t) \in \operatorname{im} B_3^{\mathrm{T}} \quad \text{for all } t \in [t_0, \infty). \tag{75}$$

Solvability of (60) therefore becomes dependent on the property of $f_1(\cdot)$ and $f_3(\cdot)$ evolving in certain subspaces. Note that the components $\bar{x}_1(t)$ and $\bar{x}_3(t)$ do not occur in any of the above equations. In case of existence of solutions, this part can be chosen arbitrarily. Consequently, a violation of (a) or (b) in Assumptions 6.5 causes the nonuniqueness of solutions.

2.6.3 Circuit Equations—Structural Considerations

Here we will apply our findings on differential–algebraic equations of type (60) to MNA and MLA equations. It will turn out that the index *structural property* of the circuit can be characterized by means of the circuit topology. The concrete behavior of the capacitance, inductance, and conductance functions does not influence the differentiation index.

In the following, we will use expressions like an "$\mathcal{LI}$-loop" for a loop in the circuit graph whose branch set consists only of branches corresponding to voltage sources and/or inductances. Likewise, by a $\mathcal{CV}$-cutset, we mean a cutset in the circuit graph whose branch set consists only of branches corresponding to current sources and/or capacitances.

The general assumptions on the electric circuits are formulated as follows.

Assumption 6.12 (Electrical circuits) Given is an electrical circuit with $n_{\mathcal{V}}$ voltage sources, $n_{\mathcal{I}}$ current sources, $n_{\mathcal{C}}$ capacitances, $n_{\mathcal{L}}$ inductances, $n_{\mathcal{R}}$ resistances, n nodes, and the following properties:

(a) there are no $\mathcal{I}$-cutsets;
(b) there are no $\mathcal{V}$-loops;
(c) the charge functions $q_1, \dots, q_{n_{\mathcal{C}}} : \mathbb{R} \to \mathbb{R}$ are continuously differentiable with $q_1'(u), \dots, q_{n_{\mathcal{C}}}'(u) > 0$ for all $u \in \mathbb{R}$;
(d) the flux functions $\psi_1, \dots, \psi_{n_{\mathcal{L}}} : \mathbb{R} \to \mathbb{R}$ are continuously differentiable with $\psi_1'(i), \dots, \psi_{n_{\mathcal{L}}}'(i) > 0$ for all $i \in \mathbb{R}$;
(e) the conductance functions $g_1, \dots, g_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}$ are continuously differentiable with $g_1'(u), \dots, g_{n_{\mathcal{R}}}'(u) > 0$ for all $u \in \mathbb{R}$;

Remark 6.13 (The assumptions on circuits) The absence of $\mathcal{V}$-loops means, in a nonmathematical manner of speaking, that there are no short circuits. Indeed, a $\mathcal{V}$-loop would cause that certain voltages of the sources cannot be chosen freely (see below).

Likewise, an $\mathcal{I}$-cutset consequences induces further algebraic constraints on the currents of the current sources.

Note that by Lemma 4.10 (b) the absence of $\mathcal{V}$-loops is equivalent to

$$\ker A_{\mathcal{V}} = \{0\}, \tag{76}$$

whereas by Lemma 4.10 (a) the absence of $\mathcal{I}$-cutsets is equivalent to

$$\ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^{\mathrm{T}} = \{0\}. \tag{77}$$

Consequently, the MNA equations are differential–algebraic equations of type (60) with the properties described in Assumptions 6.5.

Further, we can use Lemma 4.10 (b) to see that the circuit does not contain any $\mathcal{V}$-loops if and only if

$$\ker \begin{bmatrix} B_{\mathcal{L}} & B_{\mathcal{R}} & B_{\mathcal{C}} & B_{\mathcal{I}} \end{bmatrix}^{\mathrm{T}} = \{0\}. \tag{78}$$

A further use of Lemma 4.10 (a) implies that the absence of $\mathcal{I}$-cutsets is equivalent to

$$\ker B_{\mathcal{I}} = \{0\}. \tag{79}$$

If, moreover, we assume that the functions $g_1, \ldots, g_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}$ possess global inverses, which are, respectively, denoted by $r_1, \ldots, r_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}$, then the MLA equations are as well differential–algebraic equations of type (60) with the properties as described in Assumptions 6.5.

Theorem 6.14 (Index of MNA equations) *Let an electrical circuit with the properties as in Assumptions 6.12 be given. Then the differentiation index μ of the MNA equations (52) exists. In particular, we have:*

(a) *The following statements are equivalent:*

  (i) $\mu = 0$;
  (ii) $\operatorname{rank} A_{\mathcal{C}} = n - 1$ *and* $n_{\mathcal{V}} = 0$;
  (iii) *the circuit neither contains $\mathcal{RLI}$-cutsets nor voltage sources.*

(b) *The following statements are equivalent:*

  (i) $\mu = 1$;
  (ii) $\operatorname{rank}[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{V}}] = n - 1$ *and* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] = \ker A_{\mathcal{C}} \times \{0\}$;
  (iii) *the circuit neither contains $\mathcal{LI}$-cutsets nor $\mathcal{CV}$-loops except for $\mathcal{C}$-loops.*

(c) *The following statements are equivalent:*

(i) $\mu = 2$;
(ii) $\operatorname{rank}[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{V}}] < n - 1$ *or* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] \neq \ker A_{\mathcal{C}} \times \{0\}$;
(iii) *the circuit contains* $\mathcal{LI}$*-cutsets or* $\mathcal{CV}$*-loops which are no pure* $\mathcal{C}$*-loops*.

Proof Since the MNA equations (52) form a differential–algebraic equation of type (60) with the properties as in Assumptions 6.5, the equivalences between (i) and (ii) in (a), (b), and (c) are immediate consequences of Theorem 6.6.

The equivalence of (a) (ii) and (a) (iii) follows from the definition of $n_{\mathcal{V}}$ and the fact that, by Lemma 4.10 (a), the absence of $\mathcal{RLI}$-cutsets (which is the same as the absence of $\mathcal{RLIV}$-cutsets since the circuit does not contain any voltage sources) is equivalent to $\ker A_{\mathcal{C}}^{\mathrm{T}} = \{0\}$.

Since, by Lemma 4.10 (a),

$$\ker[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{V}}]^{\mathrm{T}} = \{0\}$$

$$\Leftrightarrow \quad \text{the circuit does not contain any } \mathcal{LI}\text{-cutsets,}$$

and, by Lemma 4.11,

$$\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] = \ker A_{\mathcal{C}} \times \{0\}$$

$$\Leftrightarrow \quad \text{the circuit does not contain any } \mathcal{CV}\text{-cutsets except for } \mathcal{C}\text{-cutsets,}$$

assertions (b) (ii) and (b) (iii) are equivalent. By the same arguments we see that (c) (ii) and (c) (iii) are equivalent as well. □

Theorem 6.15 (Index of MLA equations) *Let an electrical circuit with the properties as in Assumptions 6.12 be given. Moreover, assume that the functions*

$$g_1, \ldots, g_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}$$

possess global inverses, which are, respectively, denoted by

$$r_1, \ldots, r_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}.$$

Then the differentiation index μ *of the MLA equations* (53) *exists. In particular, we have*:

(a) *The following statements are equivalent*:

(i) $\mu = 0$;
(ii) $\operatorname{rank} B_{\mathcal{L}} = n - m + 1$ *and* $n_{\mathcal{I}} = 0$;
(iii) *the circuit contains neither* $\mathcal{CRV}$*-loops nor current sources*.

(b) *The following statements are equivalent*:

(i) $\mu = 1$;
(ii) $\operatorname{rank}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_{\mathcal{I}}] = n - m + 1$ *and* $\ker[B_{\mathcal{L}}, B_{\mathcal{I}}] = \ker B_{\mathcal{L}} \times \{0\}$;
(iv) *the circuit contains neither* $\mathcal{CV}$*-loops nor* $\mathcal{LI}$*-cutsets except for* $\mathcal{L}$*-cutsets*.

(c) *The following statements are equivalent*:

(i) $\mu = 2$;
(ii) $\operatorname{rank}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_{\mathcal{I}}] < n - m + 1$ *or* $\ker[B_{\mathcal{L}}, B_{\mathcal{I}}] \neq \ker B_{\mathcal{L}} \times \{0\}$;
(iii) *the circuit contains* $\mathcal{CV}$*-loops or* $\mathcal{LI}$*-cutsets that are not pure* $\mathcal{L}$*-loops*.

Proof The MLA equations (52) form a differential–algebraic equation of type (60) with the properties as formulated in Assumptions 6.5. Hence, the equivalences of (i) and (ii) in (a), (b), and (c) are immediate consequences of Theorem 6.6.

The equivalence of (a) (ii) and (a) (iii) follows from the definition of $n_{\mathcal{I}}$ and the fact that, by Lemma 4.10 (b), the absence of $\mathcal{CRV}$-loops (which is the same as the absence of $\mathcal{RLIV}$-cutsets since the circuit does not contain any current sources), is equivalent to $\ker B_{\mathcal{L}}^{\mathrm{T}} = \{0\}$.

By Lemma 4.12 we have

$$\begin{aligned} &\ker[B_{\mathcal{L}}, B_{\mathcal{I}}] = \ker B_{\mathcal{L}} \times \{0\} \\ &\qquad \Leftrightarrow \quad \text{the circuit does not contain any } \mathcal{LI}\text{-cutsets except for } \mathcal{L}\text{-cutsets,} \end{aligned}$$

and by Lemma 4.11 we have

$$\begin{aligned} &\ker[B_{\mathcal{L}}, B_{\mathcal{R}}, B_{\mathcal{I}}]^{\mathrm{T}} = \{0\} \\ &\qquad \Leftrightarrow \quad \text{the circuit does not contain any } \mathcal{CV}\text{-loops.} \end{aligned}$$

As a consequence, assertions (b) (ii) and (b) (iii) are equivalent. By the same arguments, we see that (c) (ii) and (c) (iii) are equivalent as well. □

Next, we aim to apply Theorem 6.10 to explicitly characterize consistency of the initial values of the MNA and MLA equations. For the result about consistent initialization of the MNA equations, we introduce the matrices of full column rank

$$\begin{aligned} Z_C &\in \mathbb{R}^{n-1, p_C}, & \mathcal{Z}_C &\in \mathbb{R}^{n-1, \widetilde{p}_C}, \\ Z_{\mathcal{RV}-C} &\in \mathbb{R}^{p_C, p_{\mathcal{RV}C}}, & \mathcal{Z}_{\mathcal{RV}-C} &\in \mathbb{R}^{p_C, \widetilde{p}_{\mathcal{RV}C}}, \\ \overline{Z}_{\mathcal{V}-C} &\in \mathbb{R}^{n_{\mathcal{V}}, \overline{p}_{\mathcal{V}-C}}, & \overline{\mathcal{Z}}_{\mathcal{V}-C} &\in \mathbb{R}^{n_{\mathcal{V}}, \overline{\widetilde{p}}_{\mathcal{V}-C}} \end{aligned} \tag{80a}$$

such that

$$\begin{aligned} \operatorname{im} Z_C &= \ker A_C^{\mathrm{T}}, & \operatorname{im} \mathcal{Z}_C &= \operatorname{im} A_C, \\ \operatorname{im} Z_{\mathcal{RV}-C} &= \ker[A_{\mathcal{R}}, A_{\mathcal{V}}]^{\mathrm{T}} Z_C, & \operatorname{im} \mathcal{Z}_{\mathcal{RV}-C} &= \operatorname{im} Z_C^{\mathrm{T}}[A_{\mathcal{R}},\ A_{\mathcal{V}}], \\ \operatorname{im} \overline{Z}_{\mathcal{V}-C} &= \ker Z_C^{\mathrm{T}} A_{\mathcal{V}}, & \operatorname{im} \overline{\mathcal{Z}}_{\mathcal{V}-C} &= \operatorname{im} A_{\mathcal{V}}^{\mathrm{T}} Z_C. \end{aligned} \tag{80b}$$

The following result (as the corresponding result on MLA equations) is an immediate consequence of Theorem 6.10.

Theorem 6.16 *Let an electrical circuit with the properties as in Assumptions 6.12 be given. Let $Z_{\mathcal{C}}$, $\mathcal{Z}_{\mathcal{C}}$, $Z_{\mathcal{RV}-\mathcal{C}}$, $\mathcal{Z}_{\mathcal{RV}-\mathcal{C}}$, $\overline{Z}_{\mathcal{V}-\mathcal{C}}$, and $\overline{\mathcal{Z}}_{\mathcal{V}-\mathcal{C}}$ be matrices of full column rank with the properties as in (80a), (80b). Let $i_{\mathcal{I}}[t_0,\infty) \to \mathbb{R}^{n_{\mathcal{I}}}$ be continuous and such that*

$$Z_{\mathcal{C}}^{\mathrm{T}} A_{\mathcal{I}} i_{\mathcal{I}} : [t_0,\infty) \to \mathbb{R}^{p_{\mathcal{C}}}$$

is continuously differentiable and

$$Z_{\mathcal{RV}-\mathcal{C}}^{\mathrm{T}} Z_{\mathcal{C}}^{\mathrm{T}} A_{\mathcal{I}} i_{\mathcal{I}} : [t_0,\infty) \to \mathbb{R}^{p_{\mathcal{RVC}}}$$

is twice continuously differentiable.

Further, assume that $u_{\mathcal{V}} : [t_0,\infty) \to \mathbb{R}^{n_{\mathcal{V}}}$ is continuously differentiable and such that

$$\overline{Z}_{\mathcal{V}-\mathcal{C}}^{\mathrm{T}} u_{\mathcal{V}} : [t_0,\infty) \to \mathbb{R}^{\overline{p}_{\mathcal{V}-\mathcal{C}}}$$

is twice continuously differentiable.

Then the initial value

$$\begin{pmatrix} \phi(t_0) \\ i_{\mathcal{L}}(t_0) \\ i_{\mathcal{V}}(t_0) \end{pmatrix} = \begin{pmatrix} \phi_0 \\ i_{\mathcal{L}0} \\ i_{\mathcal{V}0} \end{pmatrix} \tag{81}$$

is consistent if and only if

$$0 = Z_{\mathcal{C}}^{\mathrm{T}}\big(A_{\mathcal{R}} g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi_0\big) + A_{\mathcal{L}} i_{\mathcal{L}0} + A_{\mathcal{V}} i_{\mathcal{V}0} + A_{\mathcal{I}} i_{\mathcal{I}0}\big), \tag{82a}$$

$$0 = -A_{\mathcal{V}}^{\mathrm{T}}\phi_0 + u_{\mathcal{V}0}, \tag{82b}$$

$$0 = Z_{\mathcal{RV}-\mathcal{C}}^{\mathrm{T}} Z_{\mathcal{C}}^{\mathrm{T}} A_{\mathcal{L}} L(i_{\mathcal{L}0})^{-1} A_{\mathcal{L}}^{\mathrm{T}}\phi_0 + Z_{\mathcal{RV}-\mathcal{C}}^{\mathrm{T}} Z_{\mathcal{C}}^{\mathrm{T}} A_{\mathcal{I}} \dot{i}_{\mathcal{I}}(t_0), \tag{82c}$$

$$\begin{aligned} 0 = {} & \overline{Z}_{\mathcal{V}-\mathcal{C}}^{\mathrm{T}} A_{\mathcal{V}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{C}}\big(\mathcal{Z}_{\mathcal{C}}^{\mathrm{T}} A_{\mathcal{R}} g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi_0\big) A_{\mathcal{R}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{C}}\big)^{-1} \mathcal{Z}_{\mathcal{C}}^{\mathrm{T}} \\ & \cdot \big(A_{\mathcal{R}} g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi_0\big) + A_{\mathcal{L}} i_{\mathcal{L}0} + A_{\mathcal{V}} i_{\mathcal{V}0} + A_{\mathcal{I}} i_{\mathcal{I}}(t_0)\big) + \overline{Z}_{\mathcal{V}-\mathcal{C}}^{\mathrm{T}} \dot{u}_{\mathcal{V}}(t_0). \end{aligned} \tag{82d}$$

To formulate a corresponding result for the MLA, consider the matrices of full column rank

$$\begin{aligned} & Y_{\mathcal{L}} \in \mathbb{R}^{m-n+1, q_{\mathcal{L}}}, && \mathcal{Y}_{\mathcal{L}} \in \mathbb{R}^{m-n+1, \widetilde{q}_{\mathcal{L}}}, \\ & Y_{\mathcal{RI}-\mathcal{L}} \in \mathbb{R}^{q_{\mathcal{L}}, q_{\mathcal{RI}-\mathcal{L}}}, && \mathcal{Y}_{\mathcal{RI}-\mathcal{L}} \in \mathbb{R}^{q_{\mathcal{L}}, \widetilde{q}_{\mathcal{RI}-\mathcal{L}}}, \\ & \overline{Y}_{\mathcal{I}-\mathcal{L}} \in \mathbb{R}^{n_{\mathcal{I}}, \overline{p}_{\mathcal{I}-\mathcal{L}}}, && \overline{\mathcal{Z}}_{\mathcal{I}-\mathcal{L}} \in \mathbb{R}^{n_{\mathcal{I}}, \overline{\widetilde{q}}_{\mathcal{I}-\mathcal{L}}} \end{aligned} \tag{83a}$$

such that

$$\begin{aligned} & \operatorname{im} Y_{\mathcal{L}} = \ker B_{\mathcal{L}}^{\mathrm{T}}, && \operatorname{im} \mathcal{Y}_{\mathcal{L}} = \operatorname{im} B_{\mathcal{L}}, \\ & \operatorname{im} Y_{\mathcal{RV}-\mathcal{C}} = \ker[B_{\mathcal{R}}, B_{\mathcal{I}}]^{\mathrm{T}} Y_{\mathcal{L}}, && \operatorname{im} \mathcal{Y}_{\mathcal{RI}-\mathcal{L}} = \operatorname{im} Y_{\mathcal{L}}^{\mathrm{T}}[B_{\mathcal{R}}, B_{\mathcal{I}}], \\ & \operatorname{im} \overline{Y}_{\mathcal{I}-\mathcal{L}} = \ker Y_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{I}}, && \operatorname{im} \overline{\mathcal{Y}}_{\mathcal{I}-\mathcal{L}} = \operatorname{im} B_{\mathcal{I}}^{\mathrm{T}} Y_{\mathcal{L}}. \end{aligned} \tag{83b}$$

These matrices will be used to characterize consistency of the initial values of the MLA system.

Theorem 6.17 *Let an electrical circuit with the properties as in Assumptions 6.12 be given. Moreover, assume that the functions* $g_1, \dots, g_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}$ *possess global inverses, which are, respectively, denoted by* $r_1, \dots, r_{n_{\mathcal{R}}} : \mathbb{R} \to \mathbb{R}$. *Let* $Y_{\mathcal{L}}$, $\mathcal{Y}_{\mathcal{L}}$, $Y_{\mathcal{R}\mathcal{I}-\mathcal{L}}$, $\mathcal{Y}_{\mathcal{R}\mathcal{I}-\mathcal{L}}$, $\overline{Z}_{\mathcal{I}-\mathcal{L}}$, *and* $\overline{\mathcal{Z}}_{\mathcal{I}-\mathcal{L}}$ *be matrices of full column rank with the properties as in (80a), (80b). Let* $i_{\mathcal{I}} : [t_0, \infty) \to \mathbb{R}^{n_{\mathcal{I}}}$ *be continuously differentiable and such that*

$$\overline{Y}_{\mathcal{I}-\mathcal{L}}^{\mathrm{T}} i_{\mathcal{I}} : [t_0, \infty) \to \mathbb{R}^{\overline{q}_{\mathcal{I}-\mathcal{L}}}$$

is twice continuously differentiable.

Further, assume that $u_{\mathcal{V}}[t_0, \infty) \to \mathbb{R}^{n_{\mathcal{V}}}$ *is continuous and such that*

$$Z_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{V}} u_{\mathcal{V}} : [t_0, \infty) \to \mathbb{R}^{q_{\mathcal{L}}}$$

is continuously differentiable and

$$Y_{\mathcal{R}\mathcal{I}-\mathcal{L}}^{\mathrm{T}} Y_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{V}} u_{\mathcal{V}} : [t_0, \infty) \to \mathbb{R}^{q_{\mathcal{R}\mathcal{I}\mathcal{L}}}$$

is twice continuously differentiable.

Then the initial value

$$\begin{pmatrix} \iota(t_0) \\ u_{\mathcal{C}}(t_0) \\ u_{\mathcal{I}}(t_0) \end{pmatrix} = \begin{pmatrix} \iota_0 \\ u_{\mathcal{C}0} \\ u_{\mathcal{I}0} \end{pmatrix} \tag{84}$$

is consistent if and only if

$$0 = Y_{\mathcal{L}}^{\mathrm{T}} \big(B_{\mathcal{R}} r \big(B_{\mathcal{R}}^{\mathrm{T}} \iota_0 \big) + B_{\mathcal{C}} u_{\mathcal{C}0} + B_{\mathcal{I}} u_{\mathcal{I}0} + B_{\mathcal{V}} u_{\mathcal{V}0} \big), \tag{85a}$$

$$0 = -B_{\mathcal{I}}^{\mathrm{T}} \iota_0 + i_{\mathcal{I}0}, \tag{85b}$$

$$0 = Y_{\mathcal{R}\mathcal{I}-\mathcal{L}}^{\mathrm{T}} Y_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{C}} C(u_{\mathcal{C}0})^{-1} B_{\mathcal{C}}^{\mathrm{T}} \iota_0 + Y_{\mathcal{R}\mathcal{I}-\mathcal{L}}^{\mathrm{T}} Y_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{V}} \dot{u}_{\mathcal{V}}(t_0), \tag{85c}$$

$$\begin{aligned} 0 = {} & \overline{Y}_{\mathcal{I}-\mathcal{L}}^{\mathrm{T}} B_{\mathcal{I}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{L}} \big(\mathcal{Y}_{\mathcal{L}}^{\mathrm{T}} B_{\mathcal{R}} r \big(B_{\mathcal{R}}^{\mathrm{T}} \iota_0 \big) B_{\mathcal{R}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{C}} \big)^{-1} \mathcal{Y}_{\mathcal{C}}^{\mathrm{T}} \\ & \cdot \big(B_{\mathcal{R}} r \big(B_{\mathcal{R}}^{\mathrm{T}} \iota_0 \big) + B_{\mathcal{C}} u_{\mathcal{C}0} + B_{\mathcal{I}} u_{\mathcal{I}0} + B_{\mathcal{V}} u_{\mathcal{V}}(t_0) \big) + \overline{Y}_{\mathcal{I}-\mathcal{L}}^{\mathrm{T}} \dot{i}_{\mathcal{I}}(t_0). \end{aligned} \tag{85d}$$

Remark 6.18 ($\mathcal{V}$-loops and $\mathcal{I}$-cutsets) If a circuit contains $\mathcal{V}$-loops and $\mathcal{I}$-cutsets (compare Remark 6.13), we may apply the findings in Remark 6.11 to extract a differential–algebraic equation of type (60) that satisfies Assumptions 6.5. More precisely, we consider matrices of full column rank

$$\begin{aligned} &Z_{\mathcal{C}\mathcal{R}\mathcal{L}\mathcal{V}} \in \mathbb{R}^{n-1, p_{\mathcal{C}\mathcal{R}\mathcal{L}\mathcal{V}}}, \qquad && \mathcal{Z}_{\mathcal{C}\mathcal{R}\mathcal{L}\mathcal{V}} \in \mathbb{R}^{n-1, \widetilde{p}_{\mathcal{C}\mathcal{R}\mathcal{L}\mathcal{V}}}, \\ &\overline{Z}_{\mathcal{V}} \in \mathbb{R}^{n_{\mathcal{V}}, \overline{p}_{\mathcal{V}}}, && \overline{\mathcal{Z}}_{\mathcal{V}} \in \mathbb{R}^{n_{\mathcal{V}}, \widetilde{\overline{p}}_{\mathcal{V}}} \end{aligned}$$

such that

$$\operatorname{im} Z_{\mathcal{CRLV}} = \ker[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{L}}, A_{\mathcal{V}}]^{\mathrm{T}}, \qquad \operatorname{im} \mathcal{Z}_{\mathcal{CRLV}} = \operatorname{im}[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{L}}, A_{\mathcal{V}}],$$
$$\operatorname{im} \overline{Z}_{\mathcal{V}} = \ker A_{\mathcal{V}}, \qquad \operatorname{im} \overline{\mathcal{Z}}_{\mathcal{V}} = \operatorname{im} A_{\mathcal{V}}^{\mathrm{T}}.$$

Then, by making the ansatz

$$\begin{aligned}\phi(t) &= Z_{\mathcal{CRLVI}}\bar{\phi}(t) + \mathcal{Z}_{\mathcal{CRLVI}}\widetilde{\phi}(t),\\ i_{\mathcal{V}}(t) &= \overline{Z}_{\mathcal{V}}\bar{i}_{\mathcal{V}}(t) + \overline{\mathcal{Z}}_{\mathcal{V}}\widetilde{i}_{\mathcal{V}}(t),\end{aligned}$$

we see that the functions $\bar{\phi}(\cdot)$ and $\bar{u}_{\mathcal{V}}(\cdot)$ can be chosen arbitrarily, whereas the solvability of the MNA equations (52) is equivalent to

$$Z_{\mathcal{CRLV}} A_{\mathcal{I}} i_{\mathcal{I}}(\cdot) \equiv 0, \qquad \overline{Z}_{\mathcal{V}} u_{\mathcal{V}}(\cdot) \equiv 0.$$

The other components then satisfy

$$\begin{aligned}0 &= \mathcal{Z}_{\mathcal{CRLVI}}^{\mathrm{T}} A_{\mathcal{C}} C\big(A_{\mathcal{C}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{CRLVI}}\widetilde{\phi}(t)\big) A_{\mathcal{C}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{CRLVI}} \frac{d}{dt}\widetilde{\phi}(t)\\ &\quad + \mathcal{Z}_{\mathcal{CRLVI}}^{\mathrm{T}} A_{\mathcal{R}} g\big(A_{\mathcal{R}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{CRLVI}}\widetilde{\phi}(t)\big) + \mathcal{Z}_{\mathcal{CRLVI}}^{\mathrm{T}} A_{\mathcal{L}} i_{\mathcal{L}}(t)\\ &\quad + \mathcal{Z}_{\mathcal{CRLVI}}^{\mathrm{T}} A_{\mathcal{V}} \overline{\mathcal{Z}}_{\mathcal{V}} \widetilde{i}_{\mathcal{V}}(t) + \mathcal{Z}_{\mathcal{CRLVI}}^{\mathrm{T}} A_{\mathcal{I}} i_{\mathcal{I}}(t),\\ 0 &= -A_{\mathcal{L}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{CRLVI}}\widetilde{\phi}(t) + \mathcal{L}\big(i_{\mathcal{L}}(t)\big)\frac{d}{dt} i_{\mathcal{L}}(t),\\ 0 &= -\overline{\mathcal{Z}}_{\mathcal{V}}^{\mathrm{T}} A_{\mathcal{V}}^{\mathrm{T}} \mathcal{Z}_{\mathcal{CRLVI}}\widetilde{\phi}(t) + \overline{\mathcal{Z}}_{\mathcal{V}}^{\mathrm{T}} u_{\mathcal{V}}(t).\end{aligned} \tag{86}$$

To perform analogous manipulations to the MLA equations, consider matrices full column rank

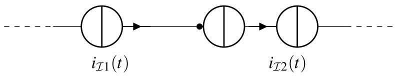
$$Y_{\mathcal{LRCI}} \in \mathbb{R}^{m-n+1, q_{\mathcal{LRCI}}}, \qquad \mathcal{Y}_{\mathcal{LRCI}} \in \mathbb{R}^{m-n+1, \widetilde{p}_{\mathcal{CRLV}}},$$
$$\overline{Y}_{\mathcal{I}} \in \mathbb{R}^{n_{\mathcal{I}}, \overline{q}_{\mathcal{I}}}, \qquad \overline{\mathcal{Y}}_{\mathcal{I}} \in \mathbb{R}^{m-n+1, \widetilde{\overline{q}}_{\mathcal{I}}}$$

such that

$$\operatorname{im} Y_{\mathcal{LRCI}} = \ker[B_{\mathcal{L}}, B_{\mathcal{R}}, B_{\mathcal{C}}, B_{\mathcal{I}}]^{\mathrm{T}}, \qquad \operatorname{im} \mathcal{Z}_{\mathcal{LRCI}} = \operatorname{im}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_{\mathcal{C}}, B_{\mathcal{I}}],$$
$$\operatorname{im} \overline{Y}_{\mathcal{I}} = \ker B_{\mathcal{I}}, \qquad \operatorname{im} \overline{\mathcal{Y}}_{\mathcal{I}} = \operatorname{im} B_{\mathcal{I}}^{\mathrm{T}}.$$

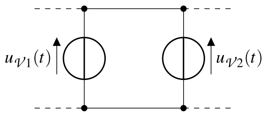
Then, by making the ansatz

$$\begin{aligned}\iota(t) &= Y_{\mathcal{LRCI}}\bar{\iota}(t) + \mathcal{Y}_{\mathcal{LRCI}}\widetilde{\iota}(t),\\ u_{\mathcal{I}}(t) &= \overline{Y}_{\mathcal{I}}\bar{u}_{\mathcal{I}}(t) + \overline{\mathcal{Y}}_{\mathcal{I}}\widetilde{u}_{\mathcal{I}}(t),\end{aligned}$$



$\Rightarrow i_{\mathcal{I}1}(t) = i_{\mathcal{I}2}(t)$

Fig. 15 Serial interconnection of current sources



$\Rightarrow u_{\mathcal{V}1}(t) = u_{\mathcal{V}2}(t)$

Fig. 16 Parallel interconnection of voltage sources

we see that the functions $\tilde{\iota}(\cdot)$ and $\bar{i}_{\mathcal{I}}(\cdot)$ can be chosen arbitrarily, whereas the solvability of the MLA equations (53) is equivalent to

$$Y_{\mathcal{LRCI}} B_{\mathcal{V}} u_{\mathcal{V}}(\cdot) \equiv 0, \qquad \overline{Y}_{\mathcal{I}} i_{\mathcal{I}}(\cdot) \equiv 0.$$

The other components then satisfy

$$\begin{aligned}
0 &= \mathcal{Y}_{\mathcal{LRCI}}^{\mathrm{T}} B_{\mathcal{L}} \mathcal{L}\big(B_{\mathcal{L}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{LRCI}} \tilde{\iota}(t)\big) B_{\mathcal{L}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{LRCI}} \frac{d}{dt} \tilde{\iota}(t) \\
&\quad + \mathcal{Y}_{\mathcal{LRCI}}^{\mathrm{T}} B_{\mathcal{R}} r\big(B_{\mathcal{R}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{LRCI}} \tilde{\iota}(t)\big) + \mathcal{Y}_{\mathcal{LRCI}}^{\mathrm{T}} B_{\mathcal{C}} u_{\mathcal{C}}(t) \\
&\quad + \mathcal{Y}_{\mathcal{LRCI}}^{\mathrm{T}} B_{\mathcal{I}} \overline{\mathcal{Y}}_{\mathcal{I}}^{\mathrm{T}} \tilde{u}_{\mathcal{I}}(t) + \mathcal{Y}_{\mathcal{LRCI}}^{\mathrm{T}} B_{\mathcal{V}} u_{\mathcal{V}}(t), && (87) \\
0 &= -B_{\mathcal{C}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{LRCI}} \tilde{\iota}(t) + \mathcal{C}\big(u_{\mathcal{C}}(t)\big) \frac{d}{dt} u_{\mathcal{C}}(t), \\
0 &= -\overline{\mathcal{Y}}_{\mathcal{I}}^{\mathrm{T}} B_{\mathcal{I}}^{\mathrm{T}} \mathcal{Y}_{\mathcal{LRCI}} \tilde{\iota}(t) + \overline{\mathcal{Y}}_{\mathcal{I}}^{\mathrm{T}} i_{\mathcal{I}}(t).
\end{aligned}$$

Note that both ansatzes have the practical interpretation that for each $\mathcal{V}$-loop, one voltage is constrained (for instance, by the equation $\overline{Z}_{\mathcal{V}} u_{\mathcal{V}}(\cdot) \equiv 0$ or equivalently by $Y_{\mathcal{LRCI}} B_{\mathcal{V}} u_{\mathcal{V}}(\cdot) \equiv 0$), and one current can be chosen arbitrarily.

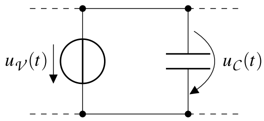
An according interpretation can be made for $\mathcal{I}$-cutsets: In each $\mathcal{I}$-cutset, one current is constrained (for instance, by the equation $Z_{\mathcal{CRLV}} A_{\mathcal{I}} i_{\mathcal{I}}(\cdot) \equiv 0$ or equivalently by $\overline{Y}_{\mathcal{I}} i_{\mathcal{I}}(\cdot) \equiv 0$), and one voltage can be chosen arbitrarily.

To illustrate this by means of an example, the configuration in Fig. 15 causes $i_{\mathcal{I}1}(\cdot) = i_{\mathcal{I}2}(\cdot)$, whereas the reduced MLA equations (87) contain $u_{\mathcal{I}1}(\cdot) + u_{\mathcal{I}2}(\cdot)$ as a component of $\tilde{u}_{\mathcal{I}}(\cdot)$. Likewise, the configuration in Fig. 16 causes $u_{\mathcal{V}1}(\cdot) = u_{\mathcal{V}2}(\cdot)$, whereas the reduced MNA equations (86) contain $i_{\mathcal{V}1}(\cdot) + i_{\mathcal{V}2}(\cdot)$ as a component of $\tilde{\iota}_{\mathcal{V}}(\cdot)$.

Remark 6.19 (Index one conditions in MNA and MLA)

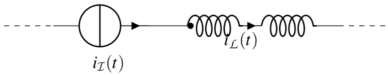
(i) The property that $\mathcal{LV}$-loops and $\mathcal{LI}$-loops cause higher index is quite intuitive from a physical perspective: In a $\mathcal{CV}$-loop, the capacitive currents are prescribed by the derivatives of the voltages of the voltage sources (see Fig. 17). In an $\mathcal{LI}$-cutset, the inductive voltages are prescribed by the derivatives of the currents of the current sources (see Fig. 18).

Fig. 17 Parallel interconnection of a voltage source and a capacitance



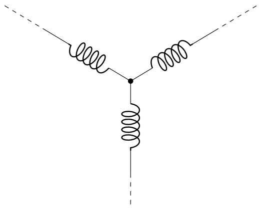
$\Rightarrow i_{\mathcal{C}0} = \mathcal{C} \cdot \dot{u}_{\mathcal{V}}(t_0)$

Fig. 18 Serial interconnection of a current source and an inductance



$\Rightarrow u_{\mathcal{L}0} = \mathcal{L} \cdot \dot{i}_{\mathcal{I}}(t_0)$

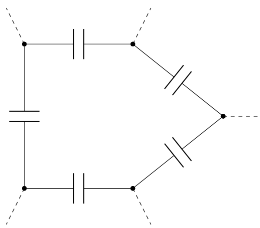
Fig. 19 $\mathcal{L}$-cutset



$\Rightarrow$ MNA system has index two

$\not\Rightarrow$ MLA system has index two

Fig. 20 $\mathcal{C}$-loop



$\Rightarrow$ MLA system has index two

$\not\Rightarrow$ MNA system has index two

(ii) An interesting feature is that $\mathcal{LI}$-cutsets (including pure $\mathcal{L}$-cutsets, see Fig. 19) cause that the MNA system has differentiation index two, whereas the corresponding index two condition for the MLA system is the existence of $\mathcal{LI}$-cutsets without pure $\mathcal{L}$-cutsets.

For $\mathcal{CV}$-loops, situation becomes, roughly speaking, vice versa: $\mathcal{CV}$-loops (including pure $\mathcal{C}$-loops, see Fig. 20) cause that the MLA system has differentiation index two, whereas the corresponding index two condition for the MNA system is the existence of $\mathcal{CV}$-loops without pure $\mathcal{C}$-loops.

Remark 6.20 (Consistency conditions for MNA and MLA equations) Note that, for an electrical circuit that contains neither $\mathcal{V}$-loops nor $\mathcal{L}$-cutsets, the following holds for the consistency conditions (82a)–(82d) and (85a)–(85d):

(i) Equation (82a) becomes trivial (that is, it contains no equations) if and only if the circuit does not contain any $\mathcal{RLIV}$-cutsets.
(ii) Equation (82b) becomes trivial if and only if the circuit does not contain any voltage sources.
(iii) Equation (82c) becomes trivial if and only if the circuit does not contain any $\mathcal{LI}$-cutsets.

(iv) Equation (82d) becomes trivial if and only if the circuit does not contain any $\mathcal{CV}$-loops except for pure $\mathcal{C}$-loops.
(v) Equation (85a) becomes trivial if and only if the circuit does not contain any $\mathcal{RCIV}$-loops.
(vi) Equation (85b) becomes trivial if and only if the circuit does not contain any current sources.
(vii) Equation (85c) becomes trivial if and only if the circuit does not contain any $\mathcal{CV}$-loops.
(viii) Equation (85d) becomes trivial if and only if the circuit does not contain any $\mathcal{LI}$-cutsets except for pure $\mathcal{L}$-cutsets.

We finally glance at the energy exchange of electrical circuits: Consider again the MNA equations

$$\begin{aligned}
&A_{\mathcal{C}}\frac{d}{dt}q\big(A_{\mathcal{C}}^{\mathrm{T}}\phi(t)\big)+A_{\mathcal{R}}g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi(t)\big)+A_{\mathcal{L}}i_{\mathcal{L}}(t)+A_{\mathcal{V}}i_{\mathcal{V}}(t)+A_{\mathcal{I}}i_{\mathcal{I}}(t)=0,\\
&-A_{\mathcal{L}}^{\mathrm{T}}\phi(t)+\frac{d}{dt}\psi\big(i_{\mathcal{L}}(t)\big)=0, \qquad (88)\\
&-A_{\mathcal{V}}^{\mathrm{T}}\phi(t)+u_{\mathcal{V}}(t)=0.
\end{aligned}$$

A multiplication of the first equation from the left by $\phi^{\mathrm{T}}(t)$, of the second equation from the left by $i_{\mathcal{L}}^{\mathrm{T}}(t)$, and of the third equation from the left by $i_{\mathcal{I}}^{\mathrm{T}}(t)$ and then a summation and according integration of these equations yields

$$\begin{aligned}
0=&\int_{t_0}^{t_f}\phi^{\mathrm{T}}(t)\bigg(A_{\mathcal{C}}\frac{d}{dt}q\big(A_{\mathcal{C}}^{\mathrm{T}}\phi(t)\big)+A_{\mathcal{R}}g\big(A_{\mathcal{R}}^{\mathrm{T}}\phi(t)\big)\\
&+A_{\mathcal{L}}i_{\mathcal{L}}(t)+A_{\mathcal{V}}i_{\mathcal{V}}(t)+A_{\mathcal{I}}i_{\mathcal{I}}(t)\bigg)\,dt\\
&+\int_{t_0}^{t_f}i_{\mathcal{L}}^{\mathrm{T}}(t)\bigg(-A_{\mathcal{L}}^{\mathrm{T}}\phi(t)+\frac{d}{dt}\psi(i_{\mathcal{L}})(t)\bigg)\,dt\\
&+\int_{t_0}^{t_f}i_{\mathcal{V}}^{\mathrm{T}}(t)\big(-A_{\mathcal{V}}^{\mathrm{T}}\phi(t)+u_{\mathcal{V}}(t)\big)\,dt.
\end{aligned}$$

Due to $\phi^{\mathrm{T}}(t)A_{\mathcal{L}}i_{\mathcal{L}}(t)=i_{\mathcal{L}}(t)A_{\mathcal{L}}^{\mathrm{T}}\phi(t)$, $\phi^{\mathrm{T}}(t)A_{\mathcal{V}}i_{\mathcal{V}}(t)=i_{\mathcal{V}}(t)A_{\mathcal{V}}^{\mathrm{T}}\phi(t)$, this equation simplifies to

$$\begin{aligned}
0=&\int_{t_0}^{t_f}\underbrace{\phi^{\mathrm{T}}(t)A_{\mathcal{C}}}_{=u_{\mathcal{C}}^{\mathrm{T}}(t)}\frac{d}{dt}q\big(\underbrace{A_{\mathcal{C}}^{\mathrm{T}}\phi(t)}_{=u_{\mathcal{C}}(t)}\big)+\underbrace{\phi^{\mathrm{T}}(t)A_{\mathcal{R}}}_{=u_{\mathcal{R}}^{\mathrm{T}}(t)}g\big(\underbrace{A_{\mathcal{R}}^{\mathrm{T}}\phi(t)}_{=u_{\mathcal{R}}(t)}\big)+\underbrace{\phi^{\mathrm{T}}(t)A_{\mathcal{I}}}_{=u_{\mathcal{I}}^{\mathrm{T}}(t)}i_{\mathcal{I}}(t)\,dt\\
&+\int_{t_0}^{t_f}i_{\mathcal{L}}^{\mathrm{T}}(t)\frac{d}{dt}\psi(i_{\mathcal{L}})(t)\,dt+\int_{t_0}^{t_f}i_{\mathcal{V}}^{\mathrm{T}}(t)u_{\mathcal{V}}(t)\,dt
\end{aligned}$$

$$= \int_{t_0}^{t_f} u_C^{\mathrm{T}}(t) \frac{d}{dt} q\big(u_C(t)\big)\, dt + \int_{t_0}^{t_f} i_{\mathcal{L}}^{\mathrm{T}}(t) \frac{d}{dt} \psi(i_{\mathcal{L}})(t)\, dt + \int_{t_0}^{t_f} u_{\mathcal{R}}^{\mathrm{T}}(t) g\big(u_{\mathcal{R}}(t)\big)\, dt$$
$$+ \int_{t_0}^{t_f} u_{\mathcal{I}}^{\mathrm{T}}(t) i_{\mathcal{I}}(t)\, dt + \int_{t_0}^{t_f} i_{\mathcal{V}}^{\mathrm{T}}(t) u_{\mathcal{V}}(t)\, dt.$$

Using the nonnegativity of $u_{\mathcal{R}}^{\mathrm{T}}(t) g(u_{\mathcal{R}}(t))$ (see (47)) and, furthermore, the representations (40), (44), and (48a) for capacitive and inductive energy, we obtain

$$\begin{aligned}
&V_C\big(q\big(u_C(t)\big)\big)\big|_{t=t_0}^{t=t_f} + V_{\mathcal{L}}\big(\psi\big(i_{\mathcal{L}}(t)\big)\big)\big|_{t=t_0}^{t=t_f} \\
&\quad \le V_C\big(q\big(u_C(t)\big)\big)\big|_{t=t_0}^{t=t_f} + V_{\mathcal{L}}\big(\psi\big(i_{\mathcal{L}}(t)\big)\big)\big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} u_{\mathcal{R}}^{\mathrm{T}}(t) g\big(u_{\mathcal{R}}(t)\big)\, dt \\
&\quad = -\int_{t_0}^{t_f} u_{\mathcal{I}}^{\mathrm{T}}(t) i_{\mathcal{I}}(t)\, dt - \int_{t_0}^{t_f} i_{\mathcal{V}}^{\mathrm{T}}(t) u_{\mathcal{V}}(t)\, dt, \qquad (89)
\end{aligned}$$

where $V_C : \mathbb{R}^{n_C} \to \mathbb{R}$ and $V_{\mathcal{L}} : \mathbb{R}^{n_{\mathcal{L}}} \to \mathbb{R}$ are the storage functions for capacitive and, respectively, inductive energy. Since, the integral of the product between voltage and current represents the energy consumptions of a specific element, relation (89) represents an energy balance of a circuit: The energy gain at capacitances and inductances is less than or equal to the energy provided by the voltage and current sources. Note that the above deviations can alternatively done on the basis of the modified loop analysis.

The difference between consumed and stored energy is given by

$$\int_{t_0}^{t_f} u_{\mathcal{R}}^{\mathrm{T}}(t) g\big(u_{\mathcal{R}}(t)\big)\, dt,$$

which is nothing but the energy lost at the resistances. Note that, for circuits without resistances (the so-called *LC resonators*), the balance (89) becomes an equation. In particular, the sum of capacitive and inductive energies remains constant if the sources are turned off.

Remark 6.21 (Analogies between Maxwell's and circuit equations) The energy balance (89) can be regarded as a lumped version of the corresponding property of Maxwell's equations; see (5a), (5b). Note that this is not the only parallelism between circuits and electromagnetic fields: For instance, Tellegen's law has a field version and a circuit version; see (12) and (28).

It seems to be an interesting task to work out these and further analogies between electromagnetic fields and electric circuits. This would, for instance, enable to interpret spatial discretizations of Maxwell's equations as electrical circuits to gain more insight.

2.6.4 Notes and References

(i) The applicability of differential–algebraic equations is not limited to electrical circuit theory: The probably most important application field outside circuit theory is in mechanical engineering [56]. The power of DAEs in (extramathematical) application has led to differential–algebraic equations becoming an own research field inside applied and pure mathematics and is the subject of several textbooks and monographs [13, 27, 33, 35, 47].

By understanding the notion *index* as a measure for the "deviation of a DAE from an ODE," various index concepts have been developed that modify and generalize the differentiation index. To mention only a few, there is, in alphabetical order, the *geometric index* [41], the *perturbation index* [25], the *strangeness index* [33] and the *tractability index* [35].

(ii) The seminal work on circuit modeling by modified nodal analysis has been done by Brennan, Ho, and Ruehli in [26], see also [16, 65]. Graph modeling of circuits has however been done earlier in [19]. Modified loop analysis has been introduced for the purpose of model order reduction in [45] and can be seen as an advancement of *mesh analysis* [19, 32]. Further circuit modeling techniques can be found in [46, 49, 50].

There exist various generalizations and modifications of the aforementioned methods for circuit modeling. For instance, models for circuits including so-called *MEM devices* has been considered in [48, 53]. The incorporation of spatially distributed components (i.e., devices that are modeled by partial differential equations) leads to so-called *partial differential–algebraic equations* (PDAEs). Such PDAE models of circuits with transmission lines (these are modeled be the *Telegraph equations*) have been considered and analyzed in [42]. Incorporation of semiconductor models (by *drift diffusion equations*) has been done in [12].

(iii) The characterization of index properties by means of the circuit topology is not new: Index determination by means of the circuit topology has been done in [22–24, 29, 38, 39, 58]. The first rigorous proof for the MNA system has been presented by Estévez Schwarz and Tischendorf in [22]. In this work, the result is even shown for circuits that contain, under some additional assumption on their connectivity, controlled sources.

Not only the index but also stability properties can be characterized by means of the circuit topology. By energy considerations (such as in Sec. 2.6.3) it can be shown that RLC circuits are stable. However, they are not necessarily asymptotically stable. Sufficient criteria for asymptotical stability by means of the circuit topology are presented by Riaza and Tischendorf in [51, 52]. These conditions are generalized to circuits containing MEM devices in [54] and to circuits containing transmission lines in [42].

The general ideas of the topological characterizations of asymptotic stability have been used in [10, 11] to analyze the asymptotic stability of the so-called *zero dynamics* for linear circuits. This allows the application of the *funnel controller*, a closed-loop control method of striking simplicity.

(iv) A further area in circuit theory is the so-called *network synthesis*. That is, from a desired input-output behavior, it is sought for a circuit whose impedance behavior matches the desired one. Network synthesis is a quite traditional area and is originated by Cauer [14], who discovered that, in the linear and time-invariant case, exactly those behaviors are realizable that are representable by a *positive real* transfer function [15]. After the discovery of the *positive real lemma* by Anderson, some further synthesis methods have been developed [2–6, 67], which are based on the positive real lemma and argumentations in the time domain. A numerical approach to network synthesis is presented in [43].

(v) An interesting physical and mathematical feature of RLC circuits is that they do not produce energy by themselves. ODE systems that provide energy balances such as (89) are called *port-Hamiltonian* (also *passive*) and are treated from a systems theoretic perspective by van der Schaft [62]. Port-Hamiltonian systems on graphs have recently be analyzed in [64], and DAE system with energy balances in [63]. Note that energy considerations play a fundamental role in model order reduction by passivity-preserving balanced truncation of electrical circuits [44].

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