Trees and Co-trees with Bounded Degrees in Planar 3-connected Graphs*

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Abstract. This paper considers the conjecture by Grünbaum that every planar 3-connected graph has a spanning tree T such that both T and its co-tree have maximum degree at most 3. Here, the *co-tree* of T is the spanning tree of the dual obtained by taking the duals of the non-tree edges. While Grünbaum's conjecture remains open, we show that every planar 3-connected graph has a spanning tree T such that both T and its co-tree have maximum degree at most 5. It can be found in linear time.

Keywords: Planar graph, canonical ordering, spanning tree, maximum degree.

1 Introduction

In 1966, Barnette showed that every planar 3-connected graph has a spanning tree with maximum degree at most 3 [2]. (In the following, a *k-tree* denotes a tree with maximum degree at most k.) Since the dual of a 3-connected planar graph is also 3-connected, the dual graph G^* also has a spanning 3-tree. In 1970, Grünbaum [11] conjectured that there are spanning 3-trees in the graph and its dual that are simultaneous in the sense of being tree and co-tree. For any spanning tree T in a planar graph, define the *co-tree* to be the subgraph of the dual graph formed by taking the dual edges of the edges in $G - T$. Since cuts in planar graphs correspond to union of cycles in the dual graph, it is easy to see that the co-tree is a spanning tree of G^*. Grünbaum conjecture is hence the following:

Conjecture 1. [11] Every planar 3-connected graph has a spanning 3-tree for which the co-tree is a spanning 3-tree of the dual graph.

This conjecture was still open in 2007 [12], and to our knowledge remains open today. This paper proves a slightly weaker statement: Every planar 3-connected graph has a spanning 5-tree for which the co-tree is a spanning 5-tree of the dual graph.

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Our approach is to read this spanning 5-tree from the *canonical ordering*, a decomposition that exists for all 3-connected planar graphs [14] and that has properties useful for many algorithms for graph drawing (see e.g. [6,14,16]) and other applications (see e.g. [13]). This will be formally defined in Section 2. There are readily available implementations for finding a canonical ordering (see for example [4,7]), and getting our tree from the canonical ordering is nearly trivial, so our trees not only can be found in linear time, but it would be very easy to implement the algorithm.

The canonical ordering is useful for Barnette's theorem as well. Barnette's proof [2] is constructive, but the algorithm that can be derived from the proof likely has quadratic run-time (he did not analyze it). With a slightly more structured proof and suitable data structures, it is possible to find the 3-tree in linear time [18]. But in fact, the 3-tree can be directly read from the canonical ordering. This was mentioned by Chrobak and Kant in their technical report [5], but no details were given as to why the degree-bound holds, and they did not include the result in their journal version [6]. We provide these details in Section 3, somewhat as a warm-up and because the key lemma will be needed later. Then we prove the weakened version of Grünbaum's conjecture in Section 4.

2 Background

Assume that $G = (V, E)$ is a planar graph, i.e., it can be drawn in the plane without crossing. Also assume that G is 3-connected, i.e., for any two vertices $\{u, v\}$ the graph resulting from deleting u and v is still connected. By Whitney's theorem a 3-connected planar graph G has a unique *combinatorial embedding*, i.e., in any planar drawing of G the circular clockwise order of edges around each vertex v is the same, up to reversal of all these orders. Given a planar drawing Γ, a *face* is a maximal connected region of $\mathbb{R}^2 - \Gamma$. The unbounded face is called the *outer-face*, all other faces are *interior faces*.

Define the *dual graph* G^* as follows. For every face f in G, add a vertex f^* to G^*. If e is an edge of G with incident faces f_ℓ and f_r, then add edge $e^* := (f_\ell^*, f_r^*)$ to G^*; e^* is called the *dual edge* of e.

De Fraysseix, Pach and Pollack [9] were the first to introduce a canonical ordering for triangulated planar graphs. Kant [14] generalized the canonical ordering to all 3-connected planar graphs.

Definition 1. *[14] A* canonical ordering *of a planar graph G with a fixed combinatorial embedding and outer-face is an ordered partition $V = V_1 \cup \cdots \cup V_K$ that satisfies the following:*

- *V_1 consists of two vertices v_1 and v_2 where v_2 is the counter-clockwise neighbour of v_1 on the outer-face.*
- *V_K is a singleton $\{v_n\}$ where v_n is the clockwise neighbour of v_1 on the outer-face.*
- *For each k in $2, \ldots, K$, the graph $G[V_1 \cup \cdots \cup V_k]$ induced by $V_1 \cup \cdots \cup V_k$ is 2-connected and contains edge (v_1, v_2) and all vertices of V_k on the outer-face.*

– *For each k in $2, \ldots, K-1$ one of the two following conditions hold:*
  1. *V_k contains a single vertex z that has at least two neighbours in $V_1 \cup \cdots \cup V_{k-1}$ and at least one neighbour in $V_{k+1} \cup \cdots \cup V_K$.*
  2. *V_k contains $\ell \geq 2$ vertices that induce a path $z_1 - z_2 - \cdots - z_\ell$, enumerated in clockwise order around the outer-face of $G[V_1 \cup \cdots \cup V_k]$. Vertices z_1 and z_ℓ have exactly one neighbour each in $V_1 \cup \cdots \cup V_{k-1}$, while $z_2, \ldots, z_{\ell-1}$ have no such neighbours. Each z_i, $1 \leq i \leq \ell$ has at least one neighbour in $V_{k+1} \cup \cdots \cup V_K$.*

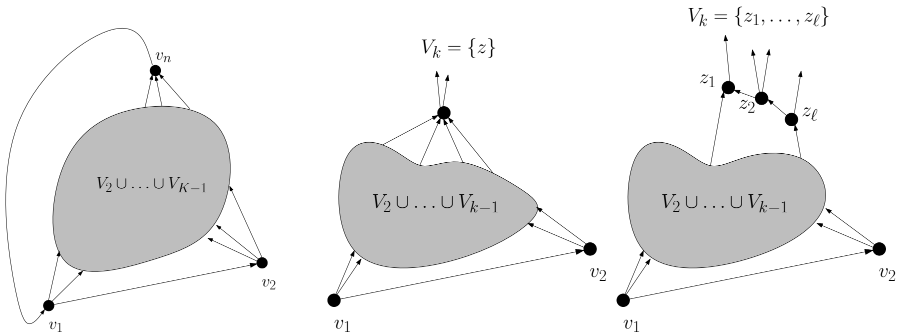

Fig. 1. The canonical ordering with its implied edge directions (defined in Section 2.1)

Figure 1 illustrates this definition. A set V_k, $k = 1, \ldots, K$ is called a *group* of the canonical ordering; a group with one vertex is a *singleton-group*, all other groups are *chain-groups*. Edges with both ends in the same group are called *intra-edges*, all others are *inter-edges*. Notice that when adding group V_k for $k \geq 2$, there exists some faces (one for a chain-group, one or more for a singleton-group) that are interior faces of $G[V_1 \cup \cdots \cup V_k]$ but were not interior faces of $G[V_1 \cup \cdots \cup V_{k-1}]$; these faces are called the *faces completed by group V_k*.

Kant [14] showed that any 3-connected planar graph has such a canonical ordering, even if the outer-face and the 2-path $v_n - v_1 - v_2$ on it to be used for the canonical ordering have been fixed. Furthermore, it can be found in linear time.

2.1 Edge Directions

Given a canonical ordering, one naturally directs inter-edges from the lower-indexed to the higher-indexed group. For proving Barnette's theorem, it will be useful to direct intra-edges as well as follows:

Definition 2. *Given a canonical ordering, enumerate the vertices as $v_1, \ldots, v_n$ as follows. Group V_1 consists of v_1 and v_2. For $2 \leq k \leq K$, let $s = |V_1| + \cdots + |V_{k-1}|$.*

- *If V_k is a singleton group $\{z\}$, then set $v_{s+1} := z$.*
- *If V_k is a chain-group $z_1, \ldots, z_\ell$, then let v_h and v_i be the neighbours of z_1 and z_ℓ in $V_1 \cup \cdots \cup V_{k-1}$, respectively. If $h < i$, then set $v_{s+j} := z_j$ for $j = 1, \ldots, \ell$, else set $v_{s+j} := z_{\ell-j+1}$ for $j = 1, \ldots, \ell$.*

Let $\mathrm{idx}(v)$ be the index of vertex v in this enumeration. Consider edges to be directed from the lower-indexed to the higher-indexed vertex, with the exception of edge (v_1, v_n), which we direct $v_n \to v_1$. These edge directions are illustrated in Figure 1, with higher-indexed vertices drawn with larger y-coordinate.

Observation 1 *(1) Every vertex has, in its clockwise order of incident edges, a non-empty interval of incoming edges followed by a non-empty interval of outgoing edges.*

(2) The edges on each of the two faces incident to (v_1, v_n) form a directed cycle.

(3) For every face not incident to (v_1, v_n), the incident edges form two directed paths.

Proof. For purposes of this proof only, consider edge (v_1, v_n) to be directed $v_1 \to v_n$. Then by properties of the canonical ordering, every vertex except v_1 has at least one incoming edge, and every vertex except v_n has at least one outgoing inter-edge. Therefore this orientation is *bi-polar*: it is acyclic with a single source v_1 and a single sink v_n. It is known [19] that property (1) holds for all vertices $\neq v_1, v_n$ in a bi-polar orientation in a planar graph. Orienting edge (v_1, v_n) as $v_n \to v_1$ also makes (1) hold at v_1 and v_n, since they then have exactly one incoming/one outgoing edge.

In the bi-polar orientation, property (3) holds for any face f [19]. Orienting edge (v_1, v_n) as $v_n \to v_1$ will not change the property unless f is incident to (v_1, v_n). If f is incident to (v_1, v_n), then v_1 (as a source) was necessarily the beginning and v_n was necessarily the end of the two directed paths. Orienting edge (v_1, v_n) as $v_n \to v_1$ therefore turns the two directed paths into one directed cycle. So (2) holds.

Define the *first* and *last* outgoing edge to be the first and last edge in the clockwise order around v that is outgoing; this is well-defined by Observation 1(1). Also define the following:

Definition 3. *For any vertex v_i, $i \geq 2$, let the* parent-edge *be the incoming edge $v_h \to v_i$ for which h is maximized.*

If $e = v \to w$ is a directed edge, then w is the *head* of e, v is the *tail* of e, and v is a *predecessor* of w. The *left face* of e is the face to the left when walking from the tail to the head, and the *right face* of e is the other face incident to e. The predecessor at the parent-edge of w is called the *parent* of w. The *predecessors of group V_k* are all vertices that are predecessors of some vertex in V_k.

2.2 Edge Labels

To read trees from the canonical ordering, it helps to assign labels to the edges incident to a vertex. They are very similar to Felsner's triorientation derived from Schnyder labellings [8] (which in turn can easily be derived from the canonical ordering [15]), but differ slightly in the handling of intra-edges and edge (v_1, v_n).

Definition 4. *Given a canonical ordering, label the edge-vertex-incidences as follows:*

- *If V_k is a singleton-group $\{z\}$ with $2 \leq k \leq K$, then the first incoming edge of z (in clockwise order) is labelled SE, the last incoming edge of z (in clockwise order) is labelled SW, and all other incoming edges of z are labelled S.*
- *If V_k is a chain-group $\{z_1, \dots, z_\ell\}$ with $2 \leq k < K$, then the incoming inter-edge of z_1 is labelled SW at z_1, the incoming inter-edge of z_ℓ is labelled SE at z_ℓ, and any intra-edge (z_i, z_{i+1}) is labelled E at z_i and W at z_{i+1}.*
- *Edge $v_1 \to v_2$ is labelled E at v_1 and W at v_2.*
- *Edge $v_n \to v_1$ is labelled S at v_1.*
- *If an inter-edge $v \to w$ is labelled SE / S / SW at w, then label it NW / N / NE at v.*

Call an edge an $\mathcal{L}$-edge (for $\mathcal{L} \in \{S, SW, W, NW, N, NE, E, SE\}$) if it is labelled $\mathcal{L}$ at one endpoint.

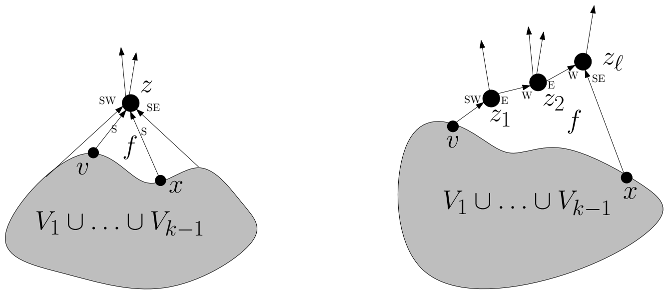

Fig. 2. The canonical ordering with its implied edge labelling. We also illustrate notations for the proof of Lemma 2.

See Figure 2 for an illustration of this labelling. The following properties are easily verified (see also [6] and [8] for similar results):

Lemma 1.
- *At each vertex there are, in clockwise order, some edges labelled S, at most one edge labelled SW, at most one edge labelled W, some edges labelled NW, at most one edge labelled N, some edges labelled NE, at most one edge labelled E, and at most one edge labelled SE.*
- *An edge is an intra-edge if and only if it is labelled E at one endpoint and W at the other.*
- *No vertex has an edge labelled W and an edge labelled SW.*
- *No vertex has an edge labelled E and an edge labelled SE.*

3 Barnette's Theorem via the Canonical Ordering

We now show that Barnette's theorem has a proof where the tree can be read directly from a canonical ordering.

Theorem 1. *Let G be a planar graph with a canonical ordering. Then the parent-edges forms a spanning tree of maximum degree 3.*

Proof. Let T be the set of parent edges. First note that each vertex $v_2, \dots, v_n$ has exactly one incoming edge in T, and there is no directed cycle since (v_1, v_n) is not a parent-edge and therefore edges are directed according to indices. So T is indeed a spanning tree. To see the bound on the maximum degree, the following lemma suffices:

Lemma 2. *Assume $v \to w$ is a parent-edge of w. Then either $v \to w$ is the first outgoing edge at v and labelled W or NW or N at v, or $v \to w$ is the last outgoing edge at v and labelled E or NE or N at v.*

Proof. $w = v_1$ is impossible since v_1 has no parent. If $w = v_2$, then its parent-edge $v_1 \to v_2$ is the last outgoing edge of v_1 and labelled W, so the claim holds. Now consider $w = v_i$ for some $i \geq 3$, which means that w belongs to some group V_k for $k \geq 2$. There are two cases:

- V_k is a chain-group $z_1 - \cdots - z_\ell$, which implies $k < K$. Assume that the chain is directed $z_1 \to \cdots \to z_\ell$; the other case is symmetric. Refer to Figure 2(right). Note that z_i is the parent of z_{i+1} for $1 \leq i < \ell$, and $z_i \to z_{i+1}$ is the last outgoing edge of z_i and labelled E, so the claim holds for $w \in \{z_2, \dots, z_\ell\}$.
  Consider $w = z_1$. The parent v of z_1 is the predecessor of V_k adjacent to z_1. Let x be the other predecessor of V_k (it is adjacent to z_ℓ). The direction of the chain implies $\text{idx}(v) > \text{idx}(x)$. Let f be the face completed by V_k and observe that it does not contain (v_1, v_n). By Observation 1(3) the boundary of f consists of two directed paths, which both end at z_ℓ. The vertex where these two paths begin cannot be v, otherwise there would be a directed path from v to x and therefore $\text{idx}(x) > \text{idx}(v)$. So v has at least one incoming edge on face f, and hence $v \to z_1$ is its last outgoing edge. Also, this edge is labelled SW at z_1, hence NE at v, as desired.
- V_k is a singleton-group $\{z\}$ with $z = w$. Refer to Figure 2(left). Let $x \to w$ be an incoming edge of w that comes before or after $v \to w$ in the clockwise order of edges at w. Such an edge must exist since w has at least two incoming edges (this holds for $w = v_n$ by 3-connectivity). Assume that the clockwise order at w contains $x \to w$ followed by $v \to w$; the other case is similar.
  Let f be the face incident to edges $v \to w$ and $x \to w$. By construction f is not incident to (v_1, v_n), and by Observation 1(3) the boundary of f consists of two directed paths, which both end at w. The vertex where these two paths begin cannot be v, otherwise there would be a directed path from v to x, hence $\text{idx}(x) > \text{idx}(v)$ contradicting the definition of parent-edge $v \to w$.

So v has at least one incoming edge on face f. hence $v \to w$ is the last outgoing edge at v. Furthermore, $v \to w$ cannot be labelled SE at w (since $x \to w$ comes clockwise before it), so it is labelled SW or S at w, hence NE or N at v as desired.

So in T, every vertex is incident to at most three edges: the parent-edge, the first outgoing edge, and the last outgoing edge. This finishes the proof of Theorem 1.

In a later paper [3], Barnette strengthened his own theorem to show that in addition one can pick one vertex and require that it has degree 1 in the spanning tree. Using the canonical ordering allows us to strengthen this result even further: All vertices on one face have degree at most 2, and two of them can be required to have degree 1.

Corollary 1. *Let G be a planar graph with vertices u, w on a face f, and assume that the graph that results from adding edge (u, w) to G is 3-connected. Then G has a spanning tree T with maximum degree 3 such that $\deg_T(u) = 1 = \deg_T(w)$, and all other vertex x on face f have $\deg_T(x) \leq 2$.*

Proof. Let $G^+ = G \cup (u, w)$ and find a canonical ordering of G^+ with $u = v_1$ and $w = v_n$. Let T be the spanning 3-tree of G^+ obtained from the parent-edges; this will satisfy all properties.

Observe that (v_1, v_n) is not a parent-edge, so T is a spanning tree of G as well. Let f_ℓ and f_r be the left and right face of $v_n \to v_1$. Both faces are completed by $V_K = \{v_n\}$. It follows that any edge on f_ℓ (except $v_n \to v_1$) is a SW-edge, because only such edges may have a not-yet-completed face on their left. Therefore for any vertex $x \neq v_n$ on f_ℓ the first outgoing edge is labelled NE and by Lemma 2 it does not belong to T. So $\deg_T(x) \leq 2$ for all $x \in f_\ell$. Similarly one shows that $\deg_T(x) \leq 2$ for all $x \in f_r$. Finally, $\deg_T(v_n) = 1$ since v_n has no outgoing parent-edges, and $\deg_T(v_1) = 1$ since all vertices other than v_2 have higher-indexed predecessors.

4 On Grünbaum's Conjecture

One can easily find an example of a graph where the 3-tree from Theorem 1 yields a co-tree with unbounded degree. So unfortunately the proof of Theorem 1 does not help to solve Grünbaum's conjecture. In this section, we show that every planar 3-connected graph G has a spanning tree T such that both T and its co-tree T^* are 5-trees. Tree T will again be read from the canonical ordering, but with a different approach. Assume throughout this section that a canonical order of G has been fixed.

A crucial insight is that a canonical ordering implies a *dual canonical ordering*, i.e., a canonical ordering of the dual graph G^*. This was shown, for example, by Badent et al. [1]. An inspection of the construction shows also that the edge labels of G and G^* relate as follows:

Theorem 2. *For any canonical ordering of a 3-connected planar graph G, there exists a canonical ordering of the dual graph G^* such that the following hold:*

- *The dual of any intra-edge of G is a S-edge in G^*.*
- *The dual of any S-edge of G is an intra-edge in G^*.*
- *The dual of any SW-edge e of G is a SE-edge in G^*, and directed from the left face of e to the right face of e.*
- *The dual of any SE-edge e of G is a SW-edge in G^*, and directed from the right face of e to the left face of e.*

Now define a subgraph of G from the labels of its edges. If a vertex has NW-edges, then let the last one (in clockwise order around v) be the *NNW-edge*. Similarly define the *NNE-edge* as the first NE-edge in clockwise order.

Definition 5. *Presume a canonical ordering of a planar graph G is fixed. An edge e of G is called an H-*edge *if it satisfies one of the following:*

(H1) e is an intra-edge,
(H2) e is the NNW-edge of its tail,
(H3) e is the NNE-edge of its tail,
(H4) e is the parent-edge of its head and the N-edge of its tail.

The graph formed by the H-edges of G is denoted $H(G)$.

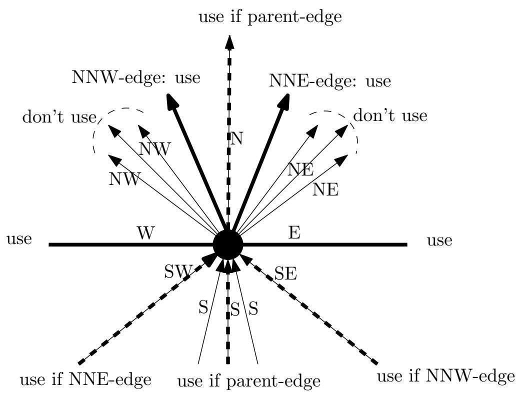

Fig. 3. Illustration of H-edges. Solid edges are H-edges; thick dashed edges may be H-edges depending on the other endpoint.

Lemma 3. *Any vertex v has at most 5 incident H-edges.*

Proof. Observe first that v has at most two incident H-edges that are outgoing inter-edges. For no such edge is added under rule (H1). Rules (H2), (H3) and (H4) add at most one such H-edge each. But if rule (H4) adds edge e, then e is the N-edge of v. By Lemma 2 it also is the first or last outgoing edge of v.

Therefore if rule (H4) applies then v has no NW-edge or no NE-edge, and so one of rules (H2) and (H3) does not apply.

Next consider the group of edges at v consisting of the intra-edges at v, and the SW-edge and SE-edge. Clearly this group has at most four edges, but actually they are only two edges by Lemma 1. So v has at most two incident H-edges in this group.

All edges at v that are neither outgoing inter-edges nor in the above group are incoming edges labelled S. Only one such edge (namely, the parent-edge of v) can be an H-edge. So v has at most 5 incident H-edges.

Let $H(G^*)$ be the graph formed by the H-edges of G^*, using the dual canonical ordering. $H(G^*)$ also has maximum degree 5. Neither $H(G)$ nor $H(G^*)$ is necessarily a tree, and it is not even obvious that they are connected. The plan is now to find a spanning tree of $H(G)$ for which the co-tree belongs to $H(G^*)$. Two lemmas are needed for this.

Lemma 4. *Let e be an edge in $G - H(G)$. Then the dual edge e^* of e belongs to $H(G^*)$.*

Proof. If e is a N-edge, then its dual is an intra-edge and hence belongs to $H(G^*)$. Edge e cannot be a NNW-edge or NNE-edge or intra-edge since it is not in $H(G)$. The remaining case is hence that e is a NW-edge of its tail v, but not the NNW-edge. (The case of a NE-edge that is not the NNE-edge is similar.) Figure 4 (left) illustrates this case.

Let e' be the clockwise next edge at v; this is also a NW-edge of v since e is not the NNW-edge. Let f be the face between e and e' at v. By Theorem 2, edge $(e')^*$ is labelled SW at f^* while e^* is labelled NE. Since e^* and e'^* are consecutive at f^*, therefore e^* is the NNE-edge of f^* and hence in $H(G^*)$.

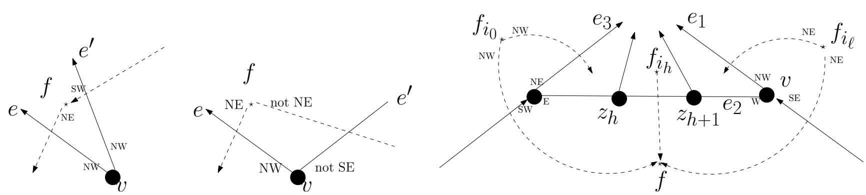

Fig. 4. For the proofs of Lemma 4 and 5. Edges in the dual are dashed.

Lemma 5. *Let C be a cycle of edges in $H(G)$. Then there exists an edge $e \in C$ such that e^* belongs to $H(G^*)$.*

Proof. There are three cases where e can be found easily; the bulk of the proof deals with the more complicated situation where none of them applies.

Case (C1): C contains a N-edge e. Then e^* is an intra-edge and belongs to $H(G^*)$ by rule (H1).

Case (C2): C contains a NW-edge e such that the clockwise next edge e' at the tail v of e is not a SE-edge. This case is illustrated in Figure 4(middle). Let f be the face between e and e'. Since e is a NW-edge, e^* is a NE-edge. Since e' is not a SE-edge, $(e')^*$ is not a NE-edge. So e^* is the NNE-edge of f^* and belongs to $H(G^*)$ by rule (H2).

Case (C3): C contains a NE-edge e such that the counter-clockwise next edge at e's tail is not a SW-edge. With a symmetric argument to (C2) one then shows that e^* is a NNW-edge and belongs to $H(G^*)$ by rule (H3).

Case (C4): None of the above cases applies. Since intra-edges form paths, cycle C must contain some inter-edges. Let e_1 be the inter-edge of C that minimizes the index of its tail v. e_1 is not a N-edge, otherwise (C1) would apply. So e_1 is either a NW-edge or a NE-edge of v. By definition of H-edges, therefore e_1 is the NNW-edge or the NNE-edge of v. Assume the former, the other case is symmetric. We will show that the situation is as in Figure 4(right).

Let e_2 be the other edge in C incident to v. Edge e_2 cannot be a N-edge at v, otherwise (C1) would apply. It also cannot be a NE-edge or E-edge at v, otherwise the clockwise edge after e_1 at v is not a SE-edge and (C2) would apply. Edge e_2 also cannot be a SE-edge or S-edge or SW-edge at v, otherwise it would be an incoming inter-edge and its tail would have a smaller index than v, contradicting the choice of e_1. Also e_2 cannot be a NW-edge at v, because the NNW-edge e_1 is the only NW-edge that is an H-edge at v. Thus edge e_2 must be an intra-edge labelled W at v.

Let $V_k = \{z_1, \ldots, z_\ell\}$ be the chain-group containing edge e_2. Notice that v has no E-edge (otherwise (C2) would apply), so $v = z_\ell$. Let a be the minimal index such that that path $z_a - z_{a+1} - \cdots - z_\ell$ is part of C. Let e_3 be the edge incident to z_a that is on C and different from (z_a, z_{a+1}). Observe that e_3 is an inter-edge, for if it were an intra-edge then its other endpoint would be z_{a-1}, contradicting the definition of a. Also observe that e_3 cannot be incoming at z_a, for otherwise the index of its tail would be smaller than all indices in V_k, and in particular smaller than the index of $v = z_\ell$; this contradicts the choice of e_1.

So e_3 is an outgoing inter-edge at z_a. If e_3 were a N-edge then (C1) would apply. If it were a NW-edge, then (due to E-edge (z_a, z_{a+1})) (C2) would apply. So e_3 is a NE-edge. Since it is an H-edge, it is the NNE-edge of z_a. Since (C3) does not apply, z_a cannot have a W-edge, which shows that $a = 1$.

Let f be the face completed by the chain-group V_k, and let $f^*_{i_0}, \ldots, f^*_{i_\ell}$ be the predecessors of f^* in the dual canonical order. By the correspondence of edge-label of Theorem 2, f_{i_0} shares the SW-edge of z_1 with f, face f_{i_h} (for $1 \leq h < \ell$) shares (z_i, z_{i+1}) with f, and f_{i_ℓ} shares the SE-edge of z_ℓ with f.

Let $f^*_{i_p} \to f^*$ be the parent-edge of f^* in the dual canonical ordering. Observe that $p \neq 0$. For edge $(f^*_{i_0}, f^*)$ is a NW-edge at $f^*_{i_0}$, as is e^*_3. Thus $(f^*_{i_0}, f^*)$ is not the first outgoing edge at $f^*_{i_0}$, and by Lemma 2 hence not a parent-edge. Likewise one shows $p \neq \ell$. So $1 \leq p < \ell$ and the parent-edge of f^* is a N-edge. By rule (H4) the parent-edge of f^* is in $H(G^*)$. Setting $e = (z_p, z_{p+1})$ yields the result.

4.1 Putting It All Together

Theorem 3. *Every planar 3-connected graph G has a spanning tree T such that both T and its co-tree have maximum degree at most 5. T can be found in linear time.*

Proof. First observe that $H(G)$ is connected. For if it were disconnected, then there would exist a non-trivial cut with all cut-edges in $G - H(G)$. By Lemma 4 the duals of the cut-edges belong to $H(G^*)$. Since cuts in a planar graph correspond to unions of cycles in the dual, hence the duals of the cut-edges contain a non-empty cycle C of edges in $H(G^*)$. By Lemma 5 one edge of C has its dual in $H(G)$, contradicting the definition of the cut.

Let H_0 be all those edges in $H(G)$ for which the dual edge does not belong to $H(G^*)$. By Lemma 5 H_0 contains no cycle, so it is a forest. Assign a weight of 0 to all edges in H_0, a weight of 1 to all edges in $H(G) - H_0$, and a weight of ∞ to all edges in $G - H(G)$. Then compute a minimum spanning tree T of G. Since H_0 is a forest, all its edges are in T. Since $H(G)$ is connected, no edge in $G - H(G)$ belongs to T. So T is a subgraph of $H(G)$ and has maximum degree at most 5. All edges in the co-tree T^* of T are duals of edges that are in $G - H_0$, and by definition of H_0 and Lemma 4 these edges belong to $H(G^*)$. So T^* is a subgraph of $H(G^*)$ and has maximum degree at most 5.

It remains to analyze the time complexity. One can compute a canonical ordering in linear time, and from it, obtain the dual canonical ordering and the edge-sets $H(G)$ and $H(G^*)$ in linear time. The bottleneck is hence the computation of the minimum spanning tree. But there are only 3 different weights, and using a bucket-structure, rather than a priority queue, in Prim's algorithm, we can find the next vertex to add to the tree in constant time. Hence the minimum spanning tree can be found in linear time.

5 Conclusion

In this paper, we showed that every planar 3-connected graph has a spanning tree of maximum degree 5 such that the co-tree also has a spanning tree of maximum degree 5. This is a first step towards proving Grünbaum's conjecture.

Barnette's theorem has as easy consequence that every planar 3-connected graph has a *3-walk*: a walk that visits every vertex at most 3 times. But in fact, one can show a stronger statement: Every planar 3-connected graph has a 2-walk [10]. The results in the paper imply similar results: every planar 3-connected graph has a walk that alternates between faces and incident vertices and visits every vertex and every face at least once and at most 5 times. (Here by "visit v" we mean that the walk alternates between v and incident faces, and similarly for "visiting f".) An interesting open problem is, as a first step towards Grünbaum's conjecture, to try to reduce this "5" to a smaller number.

A second open problem concerns generalizations to other surfaces. Barnette's theorem generalizes to 3-connected graphs on the projective plane, torus or the Klein bottle [3]; see also a recent survey [17] for many related results. For what

k can one find a spanning k-tree in, say, a toroidal 3-connected graph such that the duals of the non-tree edges form a graph of maximum degree at most k?

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