

Algorithms Parameterized by Vertex Cover and Modular Width, through Potential Maximal Cliques*

Fedor V. Fomin¹, Mathieu Liedloff², Pedro Montealegre², and Ioan Todinca²

¹ Department of Informatics, University of Bergen, N-5020 Bergen, Norway
fomin@ii.uib.no

² Univ. Orléans, INSA Centre Val de Loire, LIFO EA 4022, BP 6759, F-45067
Orléans Cedex 2, France
{mathieu.liedloff,ioan.todinca,pedro.montealegre}@univ-orleans.fr

Abstract. In this paper we give upper bounds on the number of *minimal separators* and *potential maximal cliques of graphs* w.r.t. two graph parameters, namely *vertex cover* (vc) and *modular width* (mw). We prove that for any graph, the number of minimal separators is $\mathcal{O}^*(3^{vc})$ and $\mathcal{O}^*(1.6181^{mw})$, the number of potential maximal cliques is $\mathcal{O}^*(4^{vc})$ and $\mathcal{O}^*(1.7347^{mw})$, and these objects can be listed within the same running times. (The \mathcal{O}^* notation suppresses polynomial factors in the size of the input.) Combined with known results [3,12], we deduce that a large family of problems, e.g., TREEWIDTH, MINIMUM FILL-IN, LONGEST INDUCED PATH, FEEDBACK VERTEX SET and many others, can be solved in time $\mathcal{O}^*(4^{vc})$ or $\mathcal{O}^*(1.7347^{mw})$.

1 Introduction

The *vertex cover* of a graph G , denoted by $vc(G)$, is the minimum number of vertices that cover all edges of the graph. The *modular width* $mw(G)$ can be defined as the maximum degree of a prime node in the modular decomposition of G (see [20] and Section 4 for definitions). The main results of this paper are of combinatorial nature: we show that the number of *minimal separators* and the number of *potential maximal cliques* of a graph (see Section 2 and also [3] for definitions) are upper bounded by a function in each of these parameters. More specifically, we prove the number of minimal separators is at most 3^{vc} and $\mathcal{O}^*(1.6181^{mw})$, and the number of potential maximal cliques is $\mathcal{O}^*(4^{vc})$ and $\mathcal{O}^*(1.7347^{mw})$, and these objects can be listed within the same running time bounds. Recall that the \mathcal{O}^* notation suppresses polynomial factors in the size of the input, i.e., $\mathcal{O}^*(f(k))$ should be read as $f(k) \cdot n^{\mathcal{O}(1)}$ where n is the number of vertices of the input graph. Minimal separators and potential maximal cliques have been used for solving several classical optimization problems, e.g., TREEWIDTH, MINIMUM FILL-IN [10], LONGEST INDUCED PATH, FEEDBACK

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VERTEX SET or INDEPENDENT CYCLE PACKING [12]. Pipelined with our combinatorial bounds, we obtain a series of algorithmic consequences in the area of FPT algorithms parameterized by the vertex cover and the modular width of the input graph. In particular, the problems mentioned above can be solved in time $\mathcal{O}^*(4^{vc})$ and $\mathcal{O}^*(1.7347^{mw})$. These results are complementary in the sense that graphs with small vertex cover are sparse, while graphs with small modular width may be dense.

Vertex cover and modular width are strongly related to treewidth (tw) and cliquewidth (cw) parameters, since for any graph G we have $tw(G) \leq vc(G)$ and $cw(G) \leq mw(G) + 2$. The celebrated theorem of Courcelle [6] states that all problems expressible in Counting Monadic Second Order Logic (CMSO₂) can be solved in time $f(tw) \cdot n$ for some function f depending on the problem. A similar result for cliquewidth [7] shows that all CMSO₁ problems can be solved in time $f(cw) \cdot n$, if the clique-decomposition is also given as part of the input. (See the full version [11] for definitions of different types of logic. Informally, CMSO₂ allows logic formulae with quantifiers over vertices, edges, edge sets and vertex sets, and counting modulo constants. The CMSO₁ formulae are more restricted, we are not allowed to quantify over edge sets.)

Typically function f is a tower of exponentials, and the height of the tower depends on the formula. Moreover Frick and Grohe [15] proved that this dependency on treewidth or cliquewidth cannot be significantly improved in general. Lampis [18] shows that the running time for CMSO₂ problems can be improved $2^{2^{\mathcal{O}(vc)}} \cdot n$ when parametrized by vertex cover, but he also shows that this cannot be improved to $\mathcal{O}^*(2^{2^{\mathcal{O}(vc)}})$ (under the exponential time hypothesis). We are not aware of similar improvements for parameter modular width, but we refer to [16] for discussions on problems parameterized by modular width.

Most of our algorithmic applications concern a restricted, though still large subset of CMSO₂ problems, but we guarantee algorithms that are single exponential in the vertex cover: $\mathcal{O}^*(4^{vc})$ and in the modular width: $\mathcal{O}^*(1.7347^{mw})$. We point out that our result for modular width extends the result of [13,12], who show a similar bound of $\mathcal{O}^*(1.7347^n)$ for the number of potential maximal cliques and for the running times for these problems, but parameterized by the number of vertices of the input graph.

We use the following generic problem proposed by [12], that encompasses many classical optimization problems. Fix an integer $t \geq 0$ and a CMSO₂ formula φ . Consider the problem of finding, in the input graph G , an induced subgraph $G[F]$ together with a vertex subset $X \subseteq F$, such that the treewidth of $G[F]$ is at most t , the graph $G[F]$ together with the vertex subset X satisfy formula φ , and X is of maximum size under this conditions. This optimization problem is called MAX INDUCED SUBGRAPH OF $tw \leq t$ SATISFYING φ :

$$\begin{array}{ll}
 \text{Max} & |X| \\
 \text{subject to} & \text{There is a set } F \subseteq V \text{ such that } X \subseteq F; \\
 & \text{The treewidth of } G[F] \text{ is at most } t; \\
 & (G[F], X) \models \varphi.
 \end{array} \tag{1}$$

Note that our formula φ has a free variable corresponding to the vertex subset X . For several examples, in formula φ the vertex set X is actually equal to F . E.g., even when φ only states that $X = F$, for $t = 0$ we obtain the MAXIMUM INDEPENDENT SET PROBLEM, and for $t = 1$ we obtain the MAXIMUM INDUCED FOREST. If $t = 1$ and φ states that $X = F$ and $G[F]$ is a path we obtain the LONGEST INDUCED PATH problem. Still under the assumption that $X = F$, we can express the problem of finding the largest induced subgraph $G[F]$ excluding a fixed planar graph H as a minor, or the largest induced subgraph with no cycles of length $0 \pmod l$. But X can correspond to other parameters, e.g. we can choose the formula φ such that $|X|$ is the number of connected components of $G[F]$. Based on this we can express problems like INDEPENDENT CYCLE PACKING, where the goal is to find an induced subgraph with a maximum number of components, and such that each component induces a cycle.

The result of [12] states that problem MAX INDUCED SUBGRAPH OF $\text{tw} \leq t$ SATISFYING φ can be solved in a running time of the type $\# \text{pmc} \cdot n^{t+4} \cdot f(\varphi, t)$ where $\# \text{pmc}$ is the number of potential maximal cliques of the graph, assuming that the set of all potential maximal cliques is also part of the input. Thanks to our combinatorial bounds we deduce that the problem MAX INDUCED SUBGRAPH OF $\text{tw} \leq t$ SATISFYING φ can be solved in time $\mathcal{O}(4^{\text{vc}} n^{t+c})$ and $\mathcal{O}(1.7347^{\text{mw}} n^{t+c})$, for some small constant c .

There are several other graph parameters that can be computed in time $\mathcal{O}^*(\# \text{pmc})$ if the input graph is given together with the set of its potential maximal cliques. E.g., TREEWIDTH, MINIMUM FILL-IN [10], their weighted versions [1,17] and several problems related to phylogeny [17], or TREELength [19]. Pipelined with our main combinatorial result, we deduce that all these problems can be solved in time $\mathcal{O}^*(4^{\text{vc}})$ or $\mathcal{O}^*(1.7347^{\text{mw}})$. Recently Chapelle et al. [5] provided an algorithm solving TREEWIDTH and PATHWIDTH in $\mathcal{O}^*(3^{\text{vc}})$, but those completely different techniques do not seem to work for MINIMUM FILL-IN or TREELength. The interested reader may also refer., e.g., to [8,9] for more (layout) problems parameterized by vertex cover.

2 Minimal Separators and Potential Maximal Clique

Let $G = (V, E)$ be an undirected, simple graph. We denote by n its number of vertices and by m its number of edges. The *neighborhood* of a vertex v is $N(v) = \{u \in V : \{u, v\} \in E\}$. We say that a vertex x *sees* a vertex subset S (or vice-versa) if $N(x)$ intersects S . For a vertex set $S \subseteq V$ we denote by $N(S)$ the set $\bigcup_{v \in S} N(v) \setminus S$. We write $N[S]$ (resp. $N[x]$) for $N(S) \cup S$ (resp. $N(x) \cup \{x\}$). Also $G[S]$ denotes the subgraph of G induced by S , and $G - S$ is the graph $G[V \setminus S]$.

A *connected component* of graph G is the vertex set of a maximal induced connected subgraph of G . Consider a vertex subset S of graph G . Given two vertices u and v , we say that S is a u, v -separator if u and v are in different connected components of $G - S$. Moreover, if S is inclusion-minimal among all u, v -separators, we say that S is a *minimal u, v -separator*. A vertex subset S is

called a *minimal separator* of G if S is a u, v -minimal separator for some pair of vertices u and v .

Let C be a component of $G - S$. If $N(C) = S$, we say that C is a *full component* associated to S .

Proposition 1 (folklore). *A vertex subset S of G is a minimal separator if and only if $G - S$ has at least two full components associated to S . Moreover, S is a minimal x, y -separator if and only if x and y are in different full components associated to S .*

A graph H is *chordal* or *triangulated* if every cycle with four or more vertices has a chord, i.e., an edge between two non-consecutive vertices of the cycle. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $H = (V, E')$ such that $E \subseteq E'$. Graph H is a *minimal triangulation* of G if for every edge set E'' with $E \subseteq E'' \subset E'$, the graph $F = (V, E'')$ is not chordal.

A set of vertices $\Omega \subseteq V$ of a graph G is called a *potential maximal clique* if there is a minimal triangulation H of G such that Ω is a maximal clique of H .

The following statement due to Bouchitté and Todinca [3] provides a characterization of potential maximal cliques, and in particular allows to test in polynomial time if a vertex subset Ω is a potential maximal clique of G :

Proposition 2 ([3]). *Let $\Omega \subseteq V$ be a set of vertices of the graph $G = (V, E)$ and $\{C_1, \dots, C_p\}$ be the set of connected components of $G - \Omega$. We denote $\mathcal{S}(\Omega) = \{S_1, S_2, \dots, S_p\}$, where $S_i = N(C_i)$ for all $i \in \{1, \dots, p\}$. Then Ω is a potential maximal clique of G if and only if*

1. *each $S_i \in \mathcal{S}(\Omega)$ is strictly contained in Ω ;*
2. *the graph on the vertex set Ω obtained from $G[\Omega]$ by completing each $S_i \in \mathcal{S}(\Omega)$ into a clique is a complete graph.*

Moreover, if Ω is a potential maximal clique, then $\mathcal{S}(\Omega)$ is the set of minimal separators of G contained in Ω .

Another way of stating the second condition is that for any pair of vertices $u, v \in \Omega$, if they are not adjacent in G then there is a component C of $G - \Omega$ seeing both x and y .

To illustrate Proposition 2, consider, e.g., the cube graph depicted in Figure 2. The set $\Omega_1 = \{a, e, g, c, h\}$ is a potential maximal clique and the minimal separators contained in Ω_1 are $\{a, e, g, c\}$ and $\{a, h, c\}$. Another potential maximal clique of the cube graph is $\Omega_2 = \{a, c, f, h\}$ containing the minimal separators $\{a, c, f\}$, $\{a, c, h\}$, $\{a, f, h\}$ and $\{c, f, h\}$.

Based on Propositions 1 and 2, one can easily deduce:

Corollary 1 (see e.g., [3]). *There is an $O(m)$ time algorithm testing if a given vertex subset S is a minimal separator of G , and $O(nm)$ time algorithm testing if a given vertex subset Ω is a potential maximal clique of G .*

We also need the following observation.

Proposition 3 ([3]). *Let Ω be a potential maximal clique of G and let $S \subset \Omega$ be a minimal separator. Then $\Omega \setminus S$ is contained in a unique component C of $G - S$, and moreover C is a full component associated to S .*

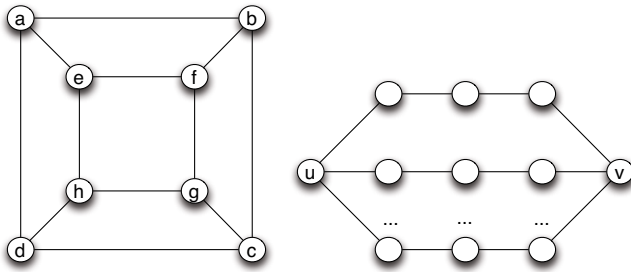


Fig. 1. Cube graph (left) and watermelon graph (right)

3 Relations to Vertex Cover

A vertex subset W is a *vertex cover* of G if each edge has at least one endpoint in W . Note that if W is a vertex cover, that $V \setminus W$ induces an *independent set* in G , i.e. $G - W$ contains no edges. We denote by $vc(G)$ the size of a minimum vertex cover of G . The parameter $vc(G)$ is called the *vertex cover number* or simply (by a slight abuse of language) the *vertex cover* of G . There is a well-known (folklore) branching algorithm computing the vertex cover of the input graph in time $\mathcal{O}^*(2^{vc})$.

Let us show that any graph G has at most $3^{vc(G)}$ minimal separators.

Lemma 1. *Let $G = (V, E)$ be a graph, W be a vertex cover and $S \subseteq V$ be a minimal separator of G . Consider a three-partition (D_1, S, D_2) of V such that both D_1 and D_2 are formed by a union of components of $G - S$, and both D_1 and D_2 contain some full component associated to S . Denote $D_1^W = D_1 \cap W$ and $D_2^W = D_2 \cap W$.*

Then $S \setminus W = \{x \in V \setminus W \mid N(x) \text{ intersects both } D_1^W \text{ and } D_2^W\}$.

Proof. Let $C_1 \subseteq D_1$ and $C_2 \subseteq D_2$ be two full components associated to S . Let $x \in S \setminus W$. Vertex x must have neighbors both in C_1 and C_2 , hence both in D_1 and D_2 . Since $x \notin W$ and W is a vertex cover, we have $N(x) \subseteq W$. Consequently x has neighbors both in D_1^W and D_2^W .

Conversely, let $x \in V \setminus W$ s.t. $N(x)$ intersects both D_1^W and D_2^W . We prove that $x \in S$. By contradiction, assume that $x \notin S$, thus x is in some component C of $G - S$. Suppose w.l.o.g. that $C \subseteq D_1$. Since $N(x) \subseteq C \cup N(C)$, we must have $N(x) \subseteq D_1 \cup S$. Thus $N(x)$ cannot intersect D_2 —a contradiction. \square

Theorem 1. *Any graph G has at most $3^{vc(G)}$ minimal separators. Moreover the set of its minimal separators can be listed in $\mathcal{O}^*(3^{vc(G)})$ time.*

Proof. Let W be a minimum size vertex cover of G . For each three-partition (D_1^W, S^W, D_2^W) of W , let $S = S^W \cup \{x \in V \setminus W \mid N(x) \text{ intersects } D_1^W \text{ and } D_2^W\}$. According to Lemma 1, each minimal separator of G will be generated this way, by an appropriate partition (D_1^W, S^W, D_2^W) of W . Thus the number of minimal separators is at most $3^{vc(G)}$, the number of three-partitions of W .

These arguments can be easily turned into an enumeration algorithm, we simply need to compute an optimum vertex cover (recall this can be done in $\mathcal{O}^*(2^{vc(G)})$ time) then test, for each set S generated from a three-partition, if S is indeed a minimal separator. The latter takes $\mathcal{O}(m)$ time for each set S using Corollary 1. \square

Observe that the bound of Theorem 1 is tight up to a constant factor. Indeed consider the watermelon graph $W_{k,3}$ formed by k disjoint paths of three vertices plus two vertices u and v adjacent to the left, respectively right ends of the paths (see Figure 2). Note that this graph has vertex cover $k + 2$ (the minimum vertex cover contains the middle of each path and vertices u and v) and it also has 3^k minimal u, v -separators, obtained by choosing arbitrarily one of the three vertices on each of the k paths.

We now extend Theorem 1 to a similar result on potential maximal cliques. Let us distinguish a particular family of potential maximal cliques, which have *active* separators. They have a particular structure which makes them easier to handle.

Definition 1 ([4]). *Let $\Omega \subseteq V$ be a potential maximal clique of graph $G = (V, E)$, let $\{C_1, \dots, C_p\}$ be the set of connected components of $G - \Omega$ and let $S_i = N(C_i)$, for $1 \leq i \leq p$.*

Consider now the graph G^+ obtained from G by completing into a clique all minimal separators S_j , $2 \leq j \leq p$, such that $S_j \not\subseteq S_1$.

We say that S_1 is an active separator for Ω if Ω is not a clique in this graph G^+ . A pair of vertices $x, y \in \Omega$ that are not adjacent in G^+ is called an active pair. Note that, by Proposition 2, we must have $x, y \in S_1$.

The following statement characterizes potential maximal cliques with active separators.

Proposition 4. *Let Ω be a potential maximal clique having an active separator $S \subset \Omega$, with an active pair $x, y \in S$. Denote by C the unique component of $G - S$ containing $\Omega \setminus S$. Then $\Omega \setminus S$ is a minimal x, y -separator in the graph $G[C \cup \{x, y\}]$.*

Again on the cube graph of Figure 2, for the potential maximal clique $\Omega_1 = \{a, e, g, c, h\}$, both minimal separators are active. E.g., for the minimal separator $S = \{a, e, g, c\}$ the pair $\{e, g\}$ is active. Not all potential maximal cliques have active separators, as illustrated by the potential maximal clique $\Omega_2 = \{a, c, f, h\}$ of the same graph.

Let us first focus on potential maximal cliques having an active separator. We give a result similar to Lemma 1, showing that such a potential maximal clique can be determined by a certain partition of the vertex cover W of G .

Lemma 2. *Let $G = (V, E)$ be a graph and W be a vertex cover of G . Consider a potential maximal clique Ω of G having an active separator $S \subseteq \Omega$ and an active pair $x, y \in S$. Let C be the unique connected component of $G - S$ intersecting Ω and let D_S be the union of all other connected components of $G - S$. Denote*

by D_x the union of components of $G - \Omega$ contained in C , seeing x , by D_y the union of components of $G - \Omega$ contained in C not seeing x .

Now let $D_S^W = D_S \cap W$, $D_x^W = D_x \cap W$ and $D_y^W = D_y \cap W$.

Then one of the following holds:

1. There is a vertex $t \in \Omega$ such that $\Omega \setminus S = N(t) \cap C$.
2. There is a vertex $t \in \Omega$ such that $\Omega = N[t]$.
3. A vertex $z \notin W$ is in Ω if and only if
 - (a) z sees D_S^W and $D_x^W \cup D_y^W$, or
 - (b) z does not see D_S^W but is sees $D_x^W \cup \{x\}$, $D_y^W \cup \{y\}$ and $D_x^W \cup D_y^W$.

Proof. Note that D_x, D_y, D_S and Ω form a partition of the vertex set V .

We first prove that any vertex $z \notin W$ satisfying conditions 3a or 3b must be in Ω .

Consider first the case 3a when z sees D_S^W and $D_x^W \cup D_y^W$. So z sees D_S and C ; we can apply Lemma 1 to partition (D_S, S, C) thus $z \in S$. Consider now the case 3b when z sees $D_x^W \cup D_y^W$, $D_x \cup \{x\}$ and $D_y \cup \{y\}$ but not D_S^W . Again by Lemma 1 applied to partition (D_S, S, C) , vertex z cannot be in S . Since z has a neighbor in $D_x \cup D_y$, we have $z \in C$. Let $H = G[C \cup \{x, y\}]$ and $T = \Omega \cap C$ (thus we also have $T = \Omega \setminus S$). Recall that T is an x, y -minimal separator in H by Proposition 4. By definition of set D_x , we have that $D_x \cup \{x\}$ is exactly the component of $H - T$ containing x . Note that $D_y \cup \{y\}$ is the union of the component of $H - T$ containing y and of all other components of $H - T$ (that do not see x nor y). By applying Lemma 1 on graph H , with vertex cover $(W \cap C) \cup \{x, y\}$ and with partition $(D_x \cup \{x\}, T, D_y \cup \{y\})$ we deduce that $z \in T$.

Conversely, let $z \in \Omega \setminus W$. We must prove that either z satisfies conditions 3a or 3b, or we are in one of the first two cases of the Lemma. We distinguish the cases $z \in S$ and $z \in T$. When $z \in S$, by Lemma 1 applied to partition (D_S, S, C) , z must see D_S and C . If z sees some vertex in $C \setminus \Omega$, we are done because z sees $D_x^W \cup D_y^W$ so we are in case 3a. Assume now that $N(z) \cap C \subseteq \Omega$, we prove that actually $N(z) \cap C = T = \Omega \setminus S$, so we are in case 1. Assume there is $u \in T \setminus N(z)$. By Proposition 2, there must be a connected component D of $G - \Omega$ such that $z, u \in N(D)$. Since $u \in C$, this component D must be a subset of C , so $D \subseteq C \setminus \Omega$. Together with $z \in N(D)$, this contradicts the assumption $N(z) \cap C \subseteq \Omega$.

It remains to treat the case $z \in T$. Clearly $z \in C$ cannot see D_S because S separates C from D_S . We again take graph H , with vertex cover $(W \cap C) \cup \{x, y\}$, and apply Lemma 1 with partition $(D_x \cup \{x\}, T, D_y \cup \{y\})$. We deduce that z sees both $D_x^W \cup \{x\}$ and $D_y^W \cup \{y\}$. Assume that z does not see $D_x^W \cup D_y^W$. So $N(z) \cap C \setminus \Omega = \emptyset$ thus $N[z] \subseteq \Omega$. If Ω contains some vertex $u \notin N[z]$, no component of $G - \Omega$ can see both z and u (because $N(z) \subseteq \Omega$), contradicting Proposition 2. We conclude that either z sees $D_x^W \cup D_y^W$ (so satisfies condition 3b) or $\Omega = N[z]$ (thus we are in the second case of the Lemma). \square

Theorem 2. Any graph G has $\mathcal{O}^*(4^{\text{vc}(G)})$ potential maximal cliques. Moreover the set of its potential maximal cliques can be listed in $\mathcal{O}^*(4^{\text{vc}(G)})$ time.

Proof. Let us first give the upper bound and the enumeration algorithm for potential maximal cliques with active separators.

The number of potential maximal cliques with active separators satisfying the second condition of Lemma 2 is at most n , and they can all be listed in polynomial time by checking, for each vertex t , if $N[t]$ is a potential maximal clique.

For enumerating the potential maximal cliques with active separators satisfying the first condition of Lemma 2, we enumerate all minimal separators S using Theorem 1, then for each $t \in S$ and each of the at most n components C of $G - S$ we check if $S \cup (C \cap N(t))$ is a potential maximal clique. Recall that testing if a vertex set is a potential maximal clique can be done in polynomial time by Corollary 1. Thus the whole process takes $\mathcal{O}^*(3^{vc(G)})$ time, and this is also an upper bound on the number of listed objects.

It remains to enumerate the potential maximal cliques with active separators satisfying the third condition of Lemma 2. For this purpose, we “guess” the sets D_S^W, D_x^W, D_y^W as in the Lemma and then we compute Ω . More formally, for each four-partition $(D_S^W, D_x^W, D_y^W, \Omega^W)$ of W , we let $\Omega^{\overline{W}}$ be the set of vertices $z \notin W$ satisfying conditions 3a or 3b of Lemma 2, and we test using Corollary 1 if $\Omega = \Omega^W \cup \Omega^{\overline{W}}$ is indeed a potential maximal clique. By Lemma 2, this enumerates in $\mathcal{O}^*(4^{vc(G)})$ all potential maximal cliques of this type.

We have proven that G has $\mathcal{O}^*(4^{vc(G)})$ potential maximal cliques with active separators and these objects can be listed within the same running time. Due to space restrictions, the extension to all potential maximal cliques, including the ones with no active separators, is given in the full version [11]. \square

4 Relations to Modular Width

A *module* of graph $G = (V, E)$ is a set of vertices W such that, for any vertex $x \in V \setminus W$, either $W \subseteq N(x)$ or W does not intersect $N(x)$. For the reader familiar with the modular decompositions of graphs, the modular width $\text{mw}(G)$ of a graph G is the maximum size of a prime node in the modular decomposition tree. Equivalently, graph G is of modular width at most k if:

1. G has at most one vertex (the base case).
2. G is a disjoint union of graphs of modular width at most k .
3. G is a *join* of graphs of modular width at most k . I.e., G is obtained from a family of disjoint graphs of modular width at most k by taking the disjoint union and then adding all possible edges between these graphs.
4. The vertex set of G can be partitioned into $p \leq k$ modules V_1, \dots, V_p such that $G[V_i]$ is of modular width at most k , for all $i, 1 \leq i \leq p$.

The modular width of a graph can be computed in linear time, using e.g. [20]. Moreover, this algorithm outputs the algebraic expression of G corresponding to this grammar.

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \dots, v_k\}$ and let $M_i = (V_i, E_i)$ be a family of pairwise disjoint graphs, for all $i, 1 \leq i \leq k$. Denote by H the graph obtained from G by replacing each vertex v_i by the module M_i . I.e.,

$H = (V_1 \cup \dots \cup V_k, E_1 \cup \dots \cup E_k \cup \{ab \mid a \in V_i, b \in V_j \text{ s.t. } v_i v_j \in E\})$. We say that graph H has been obtained from G by *expanding* each vertex v_i by the module M_i .

A vertex subset W of H is an *expansion* of vertex subset W_G of G if $W = \cup_{v_i \in W_G} V_i$. Given a vertex subset W of H , the *contraction* of W is $\{v_i \mid V_i \text{ intersects } W\}$.

We prove in Lemma 3 (resp. Lemma 4) that each minimal separator (resp. each potential maximal clique of H) actually corresponds to a minimal separator (resp. potential maximal clique) of G or to a minimal separator (resp. potential maximal clique) of one of the modules M_i . Due to space restrictions, the proofs of these statements are given [11].

Lemma 3. *Let S be a minimal separator of H . One of the following holds :*

1. S is the expansion of a minimal separator S_G of G .
2. There is $i \in \{1, \dots, k\}$ such that $S \cap V_i$ is a minimal separator of M_i and $S \setminus V_i = N_H(V_i)$.

Lemma 4. *Let Ω be a potential maximal clique of H . One of the following holds :*

1. Ω is the expansion of a potential maximal clique Ω_G of G .
2. There is some $i \in \{1, \dots, k\}$ such that $\Omega \cap V_i$ is a potential maximal clique of M_i and $\Omega \setminus V_i = N_H(V_i)$.

Lemma 3 (resp. Lemma 4) provide an injective mapping from the set of minimal separators (resp. the set of potential maximal cliques) of H to the union of the sets of minimal separators (resp. of potential maximal cliques) of G and of the graphs M_i . Therefore we have:

Corollary 2. *The number of minimal separators (resp. of potential maximal cliques) of graph H is at most the number of minimal separators (resp. of potential maximal cliques) of G plus the number of minimal separators (resp. of potential maximal cliques) of each M_i .*

The following proposition bounds the number of minimal separators and potential maximal cliques of arbitrary graphs with respect to n .

Proposition 5 ([13,14]). *Every n -vertex graph has $\mathcal{O}(1.6181^n)$ minimal separators and $\mathcal{O}(1.7347^n)$ potential maximal cliques. Moreover, these objects can be enumerated within the same running times.*

We can now prove the main result of this section.

Theorem 3. *For any graph $G = (V, E)$, the number of its minimal separators is $\mathcal{O}(n \cdot 1.6181^{\text{mw}(G)})$ and the number of its potential maximal cliques is $\mathcal{O}(n \cdot 1.7347^{\text{mw}(G)})$. Moreover, the minimal separators and the potential maximal cliques can be enumerated in $\mathcal{O}^*(1.6181^{\text{mw}(G)})$ and $\mathcal{O}^*(1.7347^{\text{mw}(G)})$ time respectively.*

Proof. Let $k = \text{mw}(G)$. By definition of modular width, there is a decomposition tree of graph G , each node corresponding to a leaf, a disjoint union, a join or

a decomposition into at most k modules. The leaves of the decomposition tree are disjoint graphs with a single vertex, thus these vertices form a partition of V . There are at most n leaves and, since each internal node is of degree at least two, there are $O(n)$ nodes in the decomposition tree. For each node N , let $G(N)$ be the graph associated to the subtree rooted in N . We prove that $G(N)$ has $O(n(N) \cdot 1.6181^k)$ minimal separators and $O(n(N) \cdot 1.7347^k)$ potential maximal cliques, where $n(N)$ is the number of nodes of the subtree rooted in N . We proceed by induction from bottom to top. The statement is clear for leaves.

Let N be an internal node N_1, N_2, \dots, N_p be its sons in the tree. Graph $G(N)$ is the expansion of some graph $G'(N)$ by replacing the i -th vertex with module $G(N_i)$. If N is a *join* node, then $G'(N)$ is a clique. When N is a *disjoint union* node, graph $G'(N)$ is an independent set, and in the last case $G'(N)$ is a graph of at most k vertices. In all cases, by Proposition 5 graph $G'(N)$ has $O(1.6181^k)$ minimal separators. Thus $G(N)$ has at most $O(1.6181^k)$ more minimal separators than all its sons taken together, which completes our proof for minimal separators.

Concerning potential maximal cliques, when $G'(N)$ is a clique it has exactly one potential maximal clique, and when $G'(N)$ is of size at most k it has $O(1.7347^k)$ potential maximal cliques. We must be more careful in the case when $G'(N)$ is an independent set (i.e., N is a disjoint union node), since in this case it has p potential maximal cliques, one for each vertex, and p can be as large as n . Consider a potential maximal clique Ω of $G(N)$ corresponding to an expansion of vertices of $G'(N)$ (see Lemma 4). It follows that this potential maximal clique is exactly the vertex set of some $G(N_i)$, for a child N_i of N . By construction this vertex set is disconnected from the rest of $G(N)$, and by Proposition 2 the only possibility is that this vertex set induces a clique in $G(N)$. But in this case Ω is also a potential maximal clique of $G(N_i)$. This proves that, when N is of type disjoint union, $G(N)$ has no more potential maximal cliques than the sum of the numbers of potential maximal cliques of all its sons. Hence the whole graph G has $O(n \cdot 1.7347^k)$ potential maximal cliques. All arguments are constructive and can be turned into enumeration algorithms for these objects. □

5 Applications

The *treewidth* of graph $G = (V, E)$, denoted $\text{tw}(G)$, is the minimum number k such that G has a triangulation $H = (V, E')$ of clique size at most $k + 1$. The *minimum fill in* of G is the minimum size of F , over all (minimal) triangulations $H = (V, E \cup F)$ of G . The *treelength* of G is the minimum k such that there exists a minimal triangulation H , with the property that any two vertices adjacent in H are at distance at most k in graph G .

Proposition 6. *Let Π_G denote the set of potential maximal cliques of graph G . The following problems are solvable in $O^*(|\Pi_G|)$ time, when Π_G is given in the input : (WEIGHTED) TREewidth [10,2], (WEIGHTED) MINIMUM FILL-IN [10,17], TREELength [19].*

Recall the MAX INDUCED SUBGRAPH OF $\text{tw} \leq t$ SATISFYING φ problem where, for a fixed integer t and a fixed CMSO₂ formula φ , the goal is to find a pair of vertex subsets $X \subseteq F \subseteq V$ such that $\text{tw}(G[F]) \leq t$, $(G[F], X)$ models φ and X is of maximum size.

Proposition 7 ([12]). *For any fixed integer $t > 0$ and any fixed CMSO₂ formula φ , problem MAX INDUCED SUBGRAPH OF $\text{tw} \leq t$ SATISFYING φ is solvable in $\mathcal{O}(|\Pi_G| \cdot n^{t+4})$ time, when Π_G is given in the input.*

Problem MAX INDUCED SUBGRAPH OF $\text{tw} \leq t$ SATISFYING φ generalizes many classical problems, for example MAXIMUM INDUCED FOREST, LONGEST INDUCED PATH, MAXIMUM INDUCED MATCHING, INDEPENDENT CYCLE PACKING, k -IN-A-PATH, k -IN-A-TREE, MAXIMUM INDUCED SUBGRAPH WITH A FORBIDDEN PLANAR MINOR. More examples of particular cases are given in the full version [11], see also [12]. From Theorems 2 and 3, we deduce:

Theorem 4. *Problems MAX INDUCED SUBGRAPH OF $\text{tw} \leq t$ SATISFYING φ , (WEIGHTED) TREewidth, (WEIGHTED) MINIMUM FILL-IN and TREELength can be solved in time $\mathcal{O}^*(4^{vc})$ and in time $\mathcal{O}^*(1.7347^{\text{mw}})$.*

6 Conclusion

We have provided single exponential upper bounds for the number of minimal separators and the number of potential maximal cliques of graphs, with respect to parameters vertex cover and modular width.

A natural question is whether these results can be extended to other natural graph parameters. We point out that for parameters like clique-width or maximum leaf spanning tree, one cannot obtain upper bounds of type $\mathcal{O}^*(f(k))$ for any function f . A counterexample is provided by the graph $W_{p,q}$, formed by p disjoint paths of q vertices plus two vertices u and v seeing the left, respectively right ends of the paths (similar to the watermelon graph of Figure 2). Indeed this graph has a maximum leaf spanning tree with p leaves and a cliquewidth of no more than $2p + 1$, but it has roughly $(n/p)^p$ minimal u, v -separators.

Finally, we point out that our bounds on the number of potential maximal cliques w.r.t. vertex cover and to modular width do not seem to be tight. Any improvement on these bounds, together with faster enumeration algorithms for the potential maximal cliques, will immediately provide improved algorithms for the problems mentioned in Section 5.

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