Games on Graphs

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Abstract. Positional Games is a branch of Combinatorics which focuses on a variety of two player games, ranging from well-known games such as Tic-Tac-Toe and Hex, to purely abstract games played on graphs. The field has experienced quite a growth in recent years, with more than a few applications in related areas.

We aim to introduce the basic notions, approaches and tools, as well as to survey the recent developments, open problems and promising research directions, keeping the main focus on the games played on graphs.

Keywords: positional game, Maker-Breaker, Avoider-Enforcer, probabilistic intuition.

1 Introduction

Positional games are a class of combinatorial two-player games of perfect information, with no chance moves and with players moving sequentially. These properties already distinguish this area of research from its popular relative, Game Theory, which has its roots in Economics. Some of the more prominent positional games include Tic-Tac-Toe, Hex, Bridg-It and the Shannon's switching game.

The basic structure of a positional game is fairly simple. Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. The set X is called the "board", and the members of $\mathcal{F}$ are referred to as the "winning sets". In the positional game $(X, \mathcal{F})$, two players take turns in claiming previously unclaimed elements of X, until all the elements are claimed. In a more general setting, given positive integers a and b, in the *biased* $(a : b)$ *game* the first player claims a elements per move and the second player claims b elements per move. If $a = b = 1$, the game is called *unbiased*.

As for determining the winner, there are several standard sets of rules, and here we mention three.

– In a *strong game*, the player who is first to claim all elements of a winning set is the winner, and if all elements of the board are claimed and no player has won, the game is a draw. A strategy stealing argument ensures that the

* Partly supported by Ministry of Science and Technological Development, Republic of Serbia, and Provincial Secretariat for Science, Province of Vojvodina.

N. Hernandez et al. (Eds.): ICCS 2014, LNAI 8577, pp. 31–36, 2014.
DOI: 10.1007/978-3-319-08389-6_4,

first player can achieve at least a draw, so the two possible outcomes of a game (if played by two perfect players) are: a first player's win, and a draw. Tic-Tac-Toe (a.k.a. Noughts and Crosses, or Xs and Os) is an example of a strong game – the board consists of nine elements (usually drawn as a 3-by-3 grid), with eight winning sets. As most kids would readily tell you, this game is a draw.

Even though these games are quite easy to introduce, they turn out to be notoriously hard to analyze, and hence there are very few results in that area.

- A *Maker-Breaker game* features two players, Maker and Breaker. Maker wins if he claims all elements of a winning set (not necessarily first). Breaker wins otherwise, i.e., if all the elements of the board are claimed and Maker has not won. Hence, one of the players always wins – a draw is not possible. As it turns out, the widely popular game Hex is a Maker-Breaker game, a fact that requires a proof, see [5].
- Finally, in an *Avoider-Enforcer game* players are called Avoider and Enforcer. Here, Enforcer wins if at any point of the game Avoider claims all elements of a winning set. Avoider wins otherwise, i.e., if he manages to avoid claiming a whole winning set to the end of the game. Due to the nature of the game, the winning sets in Avoider-Enforcer games are sometimes referred to as the losing sets.

In what follows we deal with the positional games played on graphs. That means that the board of the game is the *edge set* of a graph, usually the complete graph on n vertices. The winning sets typically are all representatives of a standard graph-theoretic structure. We introduce a few games that stand out when it comes to importance and attention received in the recent years.

The research in this area was initiated by Lehman [13], who studied the *connectivity game*, a generalization of the well-known Shannon switching game, where the winning sets are the edge sets of all spanning trees of the base graph. We denote the connectivity game played on the complete graph by $(E(K_n), \mathcal{C})$. Another important game is the *Hamiltonicity game* $(E(K_n), \mathcal{H})$, where $\mathcal{H}$ consists of the edge sets of all Hamiltonian cycles of K_n. In the *clique game* the winning sets are the edge sets of all the k-cliques, for a fixed integer $k \geq 3$. We denote this game with $(E(K_n), \mathcal{K}_k)$. Note that in this game the size of the winning sets is fixed and does not depend on n, which distinguishes it from the connectivity game and the Hamiltonicity game. A simple Ramsey argument coupled with the strategy stealing argument (see [1] for details) ensures Maker's win if n is large.

Numerous topics on positional games are covered in the monograph of Beck [1]. The new book [10] gives a gentle introduction to the subject, along with a view to the recent developments.

2 Maker-Breaker Games

It is not hard to verify that the connectivity game and the k-clique game are Maker's wins when n is large enough. Showing the same for the Hamiltonicity game requires a one-page argument [4]. This, however, is not the end of the story. A standard approach to even out the odds is introduced by Chvátal and Erdős in [4], giving Breaker more power with the help of a bias.

If an unbiased game $(X, \mathcal{F})$ is a Maker's win, we choose to play the same game with $(1 : b)$ bias, increasing b until Breaker starts winning. Formally, we want to answer the following question: What is the largest integer $b_{\mathcal{F}}$ for which Maker can win the biased $(1 : b_{\mathcal{F}})$ game? This value is called the *threshold bias* of $\mathcal{F}$. The existence of the threshold bias for every game is guaranteed by the so-called *bias monotonicity* of Maker-Breaker games, the fact that a player can only benefit from claiming additional elements at any point of the game.

For the connectivity game, it was shown by Chvátal and Erdős [4] and Gebauer and Szabó [6] that the threshold bias is $b_{\mathcal{C}} = (1 + o(1))\frac{n}{\log n}$. The result of Krivelevich [11] gives the leading term of the threshold bias for the Hamiltonicity game, $b_{\mathcal{H}} = (1+o(1))\frac{n}{\log n}$. In the k-clique game, Bednarska and Łuczak [2] found the order of the threshold bias, $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$. Determining the leading constant inside the $\Theta(.)$ remains an open problem that appears to be very challenging.

3 Avoider-Enforcer Games

Combinatorial game theory devotes a lot of attention to pairs of two-player games where the way for a player to win in one game becomes the way for him to lose in the other game – while the playing rules in both games are identical, the rule for deciding if the first player won one game is exactly the negation of the same rule for the first player in the other game. We have that setup in corresponding Maker-Breaker and Avoider-Enforcer variants of a positional game. In light of that, an Avoider-Enforcer game is said to be the *misére* version of its Maker-Breaker counterpart.

We already mentioned that in Maker-Breaker games bonus moves do not harm players, if a player is given one or more elements of the board at any point of the game he can only profit from it. Naturally, one wonders if an analogous statement holds for Avoider-Enforcer games. At first sight, it makes sense that a player trying to avoid something cannot be harmed when some of the elements he claimed are "unclaimed". But this turns out not to be true, as the following example shows.

Consider the Avoider-Enforcer $(a : b)$ game played on the board with four elements, and two disjoint winning sets of size two. It is easy to see that for $a = b = 2$ Avoider wins, for $a = 1, b = 2$ the win is Enforcer's, and finally for $a = b = 1$ Avoider is the winner again.

This feature is somewhat disturbing as, to start with, the existence of the threshold bias is not guaranteed. This prompted the authors of [9] to adjust, in a rather natural way, the game rules to ensure bias monotonicity. Under the so-called *monotone rules*, for given bias parameters a and b and a positional game $\mathcal{F}$, in a monotone $(a : b)$ Avoider-Enforcer game $\mathcal{F}$ in each turn Avoider claims *at least* a elements of the board, where Enforcer claims at *least* b elements of the board. These rules can be easily argued to be bias monotone, and thus the threshold bias becomes a well defined notion. We will refer to the original rules, where each player claims *exactly* as many elements as the respective bias suggests, as the *strict* rules. Perhaps somewhat surprisingly, monotone Avoider-Enforces games turn out to be rather different from those played under strict rules, and in quite a few cases known results about strict rules provide a rather misleading clue about the location of the threshold bias for the monotone version.

From now on, each game can be viewed under two different sets of rules – the *strict* game and the *monotone* game. Given a positional game $\mathcal{F}$, for its *strict* version we define the *lower threshold bias* $f_{\mathcal{F}}^-$ to be the largest integer such that Enforcer can win the $(1:b)$ game $\mathcal{F}$ for every $b \leq f_{\mathcal{F}}^-$, and the *upper threshold bias* $f_{\mathcal{F}}^+$ to be the smallest non-negative integer such that Avoider can win the $(1:b)$ game $\mathcal{F}$ for every $b > f_{\mathcal{F}}^+$.

If we play the game $\mathcal{F}$ under *monotone* rules, the bias monotonicity implies the existence of the unique threshold bias $f_{\mathcal{F}}^{mon}$ as the non-negative integer for which Enforcer has a winning strategy in the $(1:b)$ game if and only if $b \leq f_{\mathcal{F}}^{mon}$.

The leading term of the threshold bias for the monotone version of several well-studied positional games with spanning winning sets is given by the following two results. In [9], it was shown that for $b \geq (1+o(1))\frac{n}{\ln n}$ Avoider has a winning strategy in the monotone $(1:b)$ min-degree-1 game, the game in which his goal is to avoid touching all vertices. On the other hand, we have that for $b \leq (1-o(1))\frac{n}{\ln n}$ Enforcer has a winning strategy in the monotone $(1:b)$ Hamiltonicity game, and also in the k-connectivity game, for any fixed k, see [12].

These results give that the leading term of the threshold biases for the monotone versions of the connectivity game and the Hamiltonicity game (as well as for some other important games, like the perfect matching game, the min-degree-k game, for $k \geq 1$, the k-edge-connectivity game, for $k \geq 1$, and the k-connectivity game, for $k \geq 1$) is $(1+o(1))\,\frac{n}{\ln n}$. Indeed, each of these graph properties implies min-degree-1, and each of them is implied either by Hamiltonicity or k-connectivity. Note that for all these games we have the same threshold bias in the Maker-Breaker version of the game.

Now we switch our attention to the games played under strict rules. For the connectivity game under strict rules we know the exact value of the lower and upper threshold bias, and they are the same, $f_{\mathcal{C}}^- = f_{\mathcal{C}}^+ = \left\lfloor \frac{n-1}{2} \right\rfloor$, see [7]. This is one of very few games on graphs for which we have completely tight bounds for the threshold bias, with equal upper and lower threshold biases. Note the substantial difference between these threshold biases and the monotone threshold bias for the connectivity game. Much less is known for the Hamiltonicity game, where we just have the lower bound $(1-o(1))\,\frac{n}{\ln n}$ for the lower threshold bias [12]. As for the bounds from above, we have only the obvious. We say that Avoider has a *trivial strategy* when Enforcer's bias is that large that the total number of edges Avoider will claim in the whole game is less than the size of the smallest losing set, so he can win no matter how he plays. It is not clear how far can we expect to get, as for example in the connectivity game a trivial Avoider's strategy turns out to be the optimal one.

As for the k-clique game, as well as for most of the other games in which the winning sets are of constant size, we are quite far from determining the leading term for any of the threshold biases, with only few non-trivial bounds currently available. This gives a whole range of very important open problems.

4 Games on the Random Board

As we have already mentioned, for many standard positional games the outcome of the unbiased Maker-Breaker game played on a (large) complete graph is an obvious Maker's win, and one way to help Breaker gain power is by increasing his bias. An alternative way is the so-called game on the random board, introduced in [16]. Informally speaking, we randomly thin out the board before the game starts, some of the winning sets disappear in that process, Maker's chances drop and Breaker gains momentum.

For a positional game $(X, \mathcal{F})$ and probability p, the *game on the random board* $(X_p, \mathcal{F}_p)$ is a probability space of games, where each $x \in X$ is included in X_p with probability p (independently), and $\mathcal{F}_p = \{W \in \mathcal{F} | \; W \subseteq X_p\}$.

Now even if an unbiased game is an easy Maker's win, as we decrease p the game gets harder for Maker and at some point he is not expected to be able to win anymore. To formalize that, we observe that "being a Maker's win in $\mathcal{F}$" is an increasing graph property. Indeed, no matter what positional game $\mathcal{F}$ we take, addition of board elements does not hurt Maker. Hence, there has to exist a threshold probability $p_{\mathcal{F}}$ for this property, and we are searching for $p_{\mathcal{F}}$ such that in the $(1:1)$ game Pr[Breaker wins $(X_p, \mathcal{F}_p)$] $\to 1$ for $p \ll p_{\mathcal{F}}$, and Pr[Maker wins $(X_p, \mathcal{F}_p)$] $\to 1$ for $p \gg p_{\mathcal{F}}$.

For games played on the edge set of the complete graph K_n, note that the board in now the edge set of the Erdős-Rényi random graph, $G(n, p)$.

The threshold probability for the connectivity game was determined to be $\frac{\log n}{n}$ in [16], and shown to be sharp. As for the Hamiltonicity game, the order of magnitude of the threshold was given in [15]. Using a different approach, it was proven in [8] that the threshold is $\frac{\log n}{n}$ and it is sharp. Finally, as a consequence of a hitting time result, Ben-Shimon et al. [3] closed this question by giving a very precise description of the low order terms of the limiting probability.

The threshold for the triangle game was determined in [16], $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$. The leading term for the threshold probability in the k-clique game for $k \geq 4$ was shown to be $n^{-\frac{2}{k+1}}$ in [14].

Probabilistic Intuition. As it turns out for many standard games on graphs $\mathcal{F}$, the outcome of the game played by "perfect" players is often similar to the game played by "random" players. In other words, the inverse of the threshold bias $b_{\mathcal{F}}$ in the Maker-Breaker game played on the complete graph is "closely related" to the probability threshold for the appearance of a member of $\mathcal{F}$ in $G(n, p)$. Now we add another related parameter to the picture – the threshold probability $p_{\mathcal{F}}$ for Maker's win when the game is played on the edge set of $G(n, p)$. As we have seen in case of the connectivity game and the Hamilton cycle game, for both of those games all three mentioned parameters are exactly equal to $\frac{\ln n}{n}$.

In the k-clique game, for $k \geq 4$, the threshold bias is $\Theta(n^{\frac{2}{k+1}})$ and the threshold probability for Maker's win is the inverse (up to the leading constant), $n^{-\frac{2}{k+1}}$, supporting the random graph intuition. But, the threshold probability for the appearance of a k-clique in $G(n,p)$ is not at the same place, it is $n^{-\frac{2}{k-1}}$. And in the triangle game there is even more disagreement, as all three parameters are different – they are, respectively, $n^{\frac{1}{2}}$, $n^{-\frac{5}{9}}$ and n^{-1}.

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