A Class of Mixed Variational Problems with Applications in Contact Mechanics

Mircea Sofonea

Abstract We provide an existence result in the study of a new class of mixed variational problems. The problems are formulated on unbounded interval of time and involve history-dependent operators. The proof is based on generalized saddle point theory and various estimates, combined with fixed point arguments. Then, we consider a new mathematical model which describes the frictionless contact between a viscoelastic body and an obstacle. The process is quasistatic and the contact is modelled with a version of the normal compliance condition with unilateral constraint, which describes both the hardness and the softness of the foundation. We list the assumption on the data, derive a variational formulation of the problem, then we use our abstract result to prove its weak solvability.

1 Introduction

Mixed variational problems provide an useful framework in which a large number of problems involving unilateral constraints can be casted, analyzed, and solved numerically. For this reason, they are used both in Numerical Analysis, Optimization, Solid Mechanics and Fluid Mechanics, as well. The literature in the field was growing rapidly in the last decades. Existence and uniqueness results in the study of stationary mixed variational problems with Lagrange multipliers, together with various applications in Solid Mechanics, can be found in [3–5, 8, 10] and the references therein. Reference concerning the analysis of mixed variational problems associated with contact problems include [6, 7, 9], for instance.

The aim of this paper is twofold. The first one is to study the solvability of a new mixed variational problem involving Lagrange multipliers. The second one is to show how our abstract result can be used in the analysis of a mathematical model arising in Contact Mechanics. The paper is structured as follows. In Sect. 2 we introduce the mixed variational problem then we state and prove our main existence

M. Sofonea (✉)
Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France
e-mail: sofonea@univ-perp.fr

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result, Theorem 2.1. In Sect. 3, we describe our mathematical model of contact, list the assumption on the data and derive its variational formulation. Then we use Theorem 2.1 to prove the weak solvability of the model.

We end this short introductory section with some notation. Everywhere in this paper we use r^+ for the positive part of r, $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of non negative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$. Notation (x, y) will represent an element of the product of the sets X and Y, denoted $X \times Y$. Given a normed space $(X, \|\cdot\|_X)$ we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values on X. Also, for a subset $K \subset X$ we use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. Finally, if Y is a normed space and $\mathscr{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$, then $\mathscr{R}\eta(t)$ represents the value of the function $\mathscr{R}\eta$ at the point t, i.e. $\mathscr{R}\eta(t) = (\mathscr{R}\eta)(t)$.

2 An Abstract Existence Result

Let $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ be two real Hilbert spaces and let $(Z, \|\cdot\|_Z)$ be a real normed space. We consider two operators $A : X \to X$, $\mathscr{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Z)$, a functional $\varphi : Z \times X \to \mathbb{R}$, a bilinear form $b : X \times Y \to \mathbb{R}$, a function $f : \mathbb{R}_+ \to X$, an element h of X and a set $\Lambda \subset Y$. We are interested in the problem of finding two functions $u : \mathbb{R}_+ \to X$ and $\lambda : \mathbb{R}_+ \to \Lambda$ such that, for each $t \in \mathbb{R}_+$, the following inequalities hold:

$$\begin{aligned} &(Au(t), v - u(t))_X + \varphi(\mathscr{R}u(t), v)_X - \varphi(\mathscr{R}u(t), u(t)) \\ &\qquad + b(v - u(t), \lambda(t)) \geq (f(t), v - u(t))_X \quad \forall\, v \in X, \end{aligned} \tag{1}$$

$$b(u(t), \mu - \lambda(t)) \leq b(h, \mu - \lambda(t)) \qquad \forall\, \mu \in \Lambda. \tag{2}$$

In the study of this problem we consider the following assumptions.

$$\left\{ \begin{array}{l} \text{(a) There exists } m_A > 0 \text{ such that} \\ \qquad (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall\, u, v \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \qquad \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall\, u, v \in X. \end{array} \right. \tag{3}$$

$$\left\{ \begin{array}{l} \text{For each } n \in \mathbb{N}^* \text{ there exists } r_n \geq 0 \text{ such that} \\ \quad \|\mathscr{R}u_1(t) - \mathscr{R}u_2(t)\|_Z \leq r_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \qquad \forall\, u_1, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{array} \right. \tag{4}$$

$$\begin{cases} \text{(a) The function } \varphi(\eta,\cdot): X \to \mathbb{R} \text{ is convex} \\ \qquad \text{and Lipschitz continuous, for any } \eta \in Z. \\ \text{(b) There exists } \alpha \geq 0 \text{ such that} \\ \qquad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\ \qquad\quad \leq \alpha \, \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \quad \forall\, \eta_1,\, \eta_2 \in Z,\ \ v_1,\, v_2 \in X. \end{cases} \tag{5}$$

$$\begin{cases} \text{(a) There exists } M_b > 0 \text{ such that} \\ \qquad |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y \quad \forall\, v \in X,\ \mu \in Y. \\ \text{(b) There exists } m_b > 0 \text{ such that } \inf\limits_{\mu\in Y, \mu\neq 0_Y} \ \sup\limits_{v\in X, v\neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq m_b. \end{cases} \tag{6}$$

$$f \in C(\mathbb{R}_+; X), \qquad h \in X. \tag{7}$$

$$\Lambda \text{ is a closed convex subset of } Y \text{ that contains } 0_Y. \tag{8}$$

On these assumptions we have the following comments. First, (3) shows that A is a strongly monotone Lipschitz continuous operator. Next, following the terminology introduced in [12], (4) shows that $\mathscr{R}$ is a *history-dependent operator*. Finally, condition (6)(b) is the so-called inf-sup condition, used in the saddle point theory, see, for instance, [3–5, 8] and the references therein.

The solvability of problem (1)–(2) is given by the following result.

Theorem 2.1. *Assume* (3)–(8)*. Then, there exists a couple of functions* $(u, \lambda) : \mathbb{R}_+ \to X \times Y$*, unique in* u*, such that* (1)–(2) *hold for all* $t \in \mathbb{R}_+$*. Moreover,* $u \in C(\mathbb{R}_+; X)$.

Proof. The proof of Theorem 2.1 is carried out in several steps, that we shortly describe in what follows.

(i) In the first step we consider $g \in X$, $z \in Z$ and, using arguments similar to those used in [2], we prove that there exist a couple $(u, \lambda) \in X \times \Lambda$, unique in u, such that

$$(Au, v-u)_X + \varphi(z, v) - \varphi(z, u) + b(v-u, \lambda) \geq (g,\, v-u)_X \quad \forall\, v \in X, \tag{9}$$

$$b(u, \mu-\lambda) \leq b(k, \mu-\lambda) \qquad \forall \mu \in \Lambda. \tag{10}$$

In addition, if $(u_1,\ \lambda_1)$ and $(u_2,\ \lambda_2)$ are two solutions of the problem (9)–(10) corresponding to the data $(g_1, z_1) \in X \times Z$ and $(g_2, z_2) \in X \times Z$, respectively, then there exists $c > 0$ which depends only on A and φ such that

$$\|u_1 - u_2\|_X \leq c\,(\|g_1 - g_2\|_X + \|z_1 - z_2\|_Z).$$

(ii) In the second step we consider an element $\eta \in C(\mathbb{R}_+; X)$ and introduce the notation $y_\eta = \mathscr{R}\eta \in C(\mathbb{R}_+; Z)$. Then we use the results in step i) to prove that there exists a couple of functions $(u_\eta, \lambda_\eta) : \mathbb{R}_+ \to X \times \Lambda$, unique in the first component, such that, for each $t \in \mathbb{R}_+$, the following inequalities hold:

$$(Au_\eta(t), v - u_\eta(t))_X + \varphi(y_\eta(t), v) - \varphi(y_\eta(t), u_\eta(t)) \\ + b(v - u_\eta(t), \lambda_\eta(t)) \geq (f(t), v - u_\eta(t))_X \qquad \forall\, v \in X, \tag{11}$$

$$b(u(t), \mu - \lambda_\eta(t)) \leq b(h, \mu - \lambda_\eta(t)) \qquad \forall\, \mu \in \Lambda. \tag{12}$$

Moreover, $u_\eta \in C(\mathbb{R}_+; X)$. In addition, if (u_1, λ_1) and (u_2, λ_2) are two solutions of problem (11)–(12) corresponding to the data $\eta_1, \eta_2 \in C(\mathbb{R}_+; X)$ then, for each positive integer n, we have

$$\|u_1(t) - u_2(t)\|_X \leq \frac{\alpha r_n}{m_A} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X \, ds \qquad \forall\, t \in [0, n]. \tag{13}$$

(iii) In the next step we define the operator $\Theta : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ by equality $\Theta\eta = u_\eta$ for all $\eta \in C(\mathbb{R}_+; X)$. We use estimate (13) and a fixed point result obtained in [14] to prove that the operator Θ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

(iv) Let η^* be the unique fixed point of the operator Θ. Then, writing (11)–(12) for $\eta = \eta^*$ and using the equalities $u_{\eta^*} = \eta^*$, $y_{\eta^*} = \mathscr{R}\eta^*$, it follows that the couple $(u_{\eta^*}, \lambda_{\eta^*})$ is a solution of problem (1)–(2). Moreover, $u_{\eta^*} \in C(\mathbb{R}_+; X)$. The uniqueness of the solution in the first component follows from the uniqueness of the fixed point of the operator Θ, guaranteed by the step (iii).

□

3 A Viscoelastic Contact Model

In this section we introduce a model of frictionless contact which can be studied by using the abstract result presented in Sect. 2. The physical setting is as follows. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, with the boundary Γ partitioned into three disjoint measurable parts Γ_1, Γ_2, Γ_3, such that $meas\, \Gamma_1 > 0$. We assume that Γ is Lipschitz continuous and we denote by $\boldsymbol{\nu}$ its unit outward normal, defined almost everywhere. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in Ω, surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and, finally, we assume that the body is in contact with a deformable foundation on Γ_3. The contact is frictionless and we model it with a version of the multivalued normal compliance condition with unilateral constraint. The process is quasistatic and we study it in the time interval $\mathbb{R}_+ = [0, \infty)$. The classical formulation of the problem is the following.

Problem 1. Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in} \quad \Omega, \tag{14}$$

$$\text{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{15}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{16}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{17}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{18}$$

for all $t \in \mathbb{R}_+$, and there exists $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ which satisfies

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\Big) = 0, \\ 0 \le \xi(t) \le F\Big(\int_0^t u_\nu^+(s)\,ds\Big), \\ \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0,\ \xi(t) = F\Big(\int_0^t u_\nu^+(s)\,ds\Big) \ \text{ if } \ u_\nu(t) > 0 \end{array}\right\} \ on \ \ \Gamma_3, \tag{19}$$

for all $t \in \mathbb{R}_+$.

Here and below $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$ and, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} = (x_i)$; the indices i, j, k, l run between 1 and d and the summation convention over repeated indices is used; an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i/\partial x_j$; $\boldsymbol{\varepsilon}$ represents the deformation operator given by $\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v}))$, $\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j} + v_{j,i})$ and Div is the divergence operator, i.e. $\text{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$.

Equation (14) represents the viscoelastic constitutive law of the material, already used in a large number of works, see, for instance, the books [11, 13] and the references therein. Equation (15) is the equilibrium equation and we use it here since the process is assumed to be quasistatic. Conditions (16) and (17) are the displacement and traction boundary conditions, respectively, and condition (18) represents the frictionless condition.

We now provide some comments on condition (19) in which $g \ge 0$ is a given bound for the penetration, p represents a positive function which vanishes for a negative argument and u_ν, σ_ν represent the normal displacement and the normal stress, respectively. This condition was used in [1] in the case when F vanishes and in [15] in the case when F is given. There, various mechanical interpretations related to this condition were provided. Here we restrict ourselves to recall that condition (19) describes the following features of the contact: when there is separation between the body's surface and the foundation then the normal stress vanishes; the penetration arises only if the absolute value of the normal stress reaches the critical value F; when there is penetration the contact follows a normal compliance-type condition but only up to the bound g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g. Note that, in contrast with [15], in this paper we assume that the yield value F depends on the history of the penetration, represented by the integral term in (19); this dependence describes the hardening and the softening properties of the foundation, makes the contact problem more general, and leads to a new and interesting mathematical model.

We turn now to the variational formulation of Problem 1. To this end we use the notation "$\cdot$" and $\|\cdot\|$ for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, as well as the standard notation for the Lebesgue and Sobolev spaces associated with Ω and Γ. Moreover, we consider the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},$$

$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \,\}$$

which are real Hilbert spaces endowed with their canonical inner products and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. We also consider the space $S = \{\, \boldsymbol{w} = \boldsymbol{v}|_{\Gamma_3} : \boldsymbol{v} \in V \,\}$, where $\boldsymbol{v}|_{\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{v} \in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3;\mathbb{R}^d)$ where $H^{1/2}(\Gamma_3;\mathbb{R}^d)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^d$. It is known that S can be organized as a Hilbert space, in a canonical way. The dual of the space S will be denoted by D and the duality paring between D and S will be denoted by $\langle\cdot,\cdot\rangle_{\Gamma_3}$. For simplicity, we shall write $\langle\boldsymbol{\mu},\boldsymbol{v}\rangle_{\Gamma_3}$ instead of $\langle\boldsymbol{\mu},\boldsymbol{v}|_{\Gamma_3}\rangle_{\Gamma_3}$ when $\boldsymbol{\mu} \in D$ and $\boldsymbol{v} \in V$.

For a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\, \mathscr{E} = (\mathscr{E}_{ijkl}) \;:\; \mathscr{E}_{ijkl} = \mathscr{E}_{jikl} = \mathscr{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d \,\},$$

and we recall that $\mathbf{Q}_\infty$ is a real Banach space with its usual norm.

In the study of the mechanical problem (14)–(19) we assume that the viscosity operator $\mathscr{A}$ and the relaxation tensor $\mathscr{B}$ satisfy the following conditions.

$$\begin{cases} \text{(a) } \mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathscr{A}} > 0 \text{ such that} \\ \qquad \|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathscr{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m_{\mathscr{A}} > 0 \text{ such that} \\ \qquad (\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathscr{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{20}$$

$$\mathscr{B} \in C(\mathbb{R}_+; \mathbf{Q}_\infty). \tag{21}$$

The densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{22}$$

The normal compliance function p and the surface yield function F satisfy

$$\begin{cases} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p\, |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}. \\ \text{(d) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{23}$$

$$\begin{cases} \text{(a) } F : \mathbb{R}_+ \to \mathbb{R}_+. \\ \text{(b) There exists } L_F > 0 \text{ such that} \\ \qquad |F(r_1) - F(r_2)| \le L_F |r_1 - r_2| \quad \text{for all } r_1, r_2 \in \mathbb{R}_+. \end{cases} \tag{24}$$

Finally, we assume that

$$\text{there exists } \boldsymbol{\theta} \in V \text{ such that } \theta_\nu = 1 \text{ a.e. on } \Gamma_3. \tag{25}$$

Next, we define the sets $K \subset V$ and $\Lambda \subset D$, the bilinear form $b : V \times D \to \mathbb{R}$, the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ and the Lagrange multiplier $\boldsymbol{\lambda} : \mathbb{R}_+ \to \Lambda$ by equalities

$$K = \{\, \boldsymbol{v} \in V \;:\; v_\nu \le 0 \quad \text{a.e. on } \Gamma_3 \,\},$$

$$\Lambda = \{\, \boldsymbol{\mu} \in D \;:\; \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \le 0 \quad \forall\, \boldsymbol{v} \in K \,\},$$

$$b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+,$$

$$\langle \boldsymbol{\lambda}(t), \boldsymbol{w}\rangle_{\Gamma_3} = -\int_{\Gamma_3} (\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)) w_\nu \, da \quad \forall\, \boldsymbol{w} \in S,\ t \in \mathbb{R}_+.$$

Then, using standard arguments based on integration by part combined with assumption (25), we obtain the following variational formulation of Problem 1.

Problem 2. Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to V$ and a Lagrange multiplier $\boldsymbol{\lambda} : \mathbb{R}_+ \to \Lambda$ such that

$$\begin{aligned}(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \quad (26)\\ +(p(u_\nu(t), v_\nu - u_\nu(t)))_{L^2(\Gamma_3)} + (F\Big(\int_0^t u_\nu^+(s)\, ds\Big), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)}\\ +b(\boldsymbol{v} - \boldsymbol{u}(t), \boldsymbol{\lambda}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in V,\end{aligned}$$

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\boldsymbol{\theta}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \forall\, \mu \in \Lambda, \quad (27)$$

for all $t \in \mathbb{R}_+$.

In the study of Problem 2 we have the following existence result.

Theorem 3.2. *Assume* (20)–(25). *Then, there exists a couple of functions* $(\boldsymbol{u}, \boldsymbol{\lambda}) : \mathbb{R}_+ \to V \times \Lambda$, *unique in* $\boldsymbol{u}$, *such that* (26)–(27) *hold for all* $t \in \mathbb{R}_+$. *Moreover,* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$.

Proof. We define the operators $A : V \to V$, $\mathscr{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Gamma_3))$ and the functional $\varphi : (Q \times L^2(\Gamma_3)) \times V \to \mathbb{R}$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (p(u_\nu), v_\nu)_{L^2(\Gamma_3)} \qquad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V,$$

$$\mathscr{R}\boldsymbol{u}(t) = (\int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, F\Big(\int_0^t u_\nu^+(s)\, ds\Big)) \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ t \in \mathbb{R}_+,$$

$$\varphi((\boldsymbol{\sigma}, \xi), \boldsymbol{v}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\xi^+, v_\nu^+)_{L^2(\Gamma_3)} \quad \forall\, (\boldsymbol{\sigma}, \xi) \in Q \times L^2(\Gamma_3),\ \boldsymbol{v} \in V.$$

Then it is easy to see that the couple $(\boldsymbol{u}, \boldsymbol{\lambda})$ is a solution of Problem 2 if and only if

$$\begin{aligned}(A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \varphi(\mathscr{R}\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\mathscr{R}\boldsymbol{u}(t), \boldsymbol{u}(t)) \quad (28)\\ +b(\boldsymbol{v} - \boldsymbol{u}(t), \boldsymbol{\lambda}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in V,\end{aligned}$$

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g\boldsymbol{\theta}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \forall\, \boldsymbol{\mu} \in \Lambda, \quad (29)$$

for all $t \in \mathbb{R}_+$.

We now apply Theorem 2.1 to the system (28)–(29) with $X = V$, $Y = D$, $Z = Q \times L^2(\Gamma_3)$ and $h = g\boldsymbol{\theta}$. To this end, we use assumptions (20) and (23) and the Sobolev trace theorem to see that the operator A verifies condition (3). Moreover, assumptions (21) and (24) show that the operator $\mathscr{R}$ satisfies condition (4) and, obviously, the functional φ verifies (5). Next, as showed, e.g., in [9], the bilinear form $b(\cdot, \cdot)$ is continuous and satisfies the "inf-sup" condition. We conclude from here that condition (6) holds. Also, taking into account assumption (22) it follows that $\boldsymbol{f} \in C(\mathbb{R}_+, V)$. Finally, (25) implies (7) and, obviously, condition (8) holds, too. Theorem 3.2 is now a direct consequence of Theorem 2.1. □

Let $(\boldsymbol{u}, \boldsymbol{\lambda})$ be a solution to Problem 2 and let $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ be defined by (14). Then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ is called a weak solution to Problem 1. We conclude from Theorem 3.2 that, under assumptions (20)–(25), Problem 1 has a least unique weak solution $(\boldsymbol{u}, \boldsymbol{\sigma})$. Moreover the solution satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.

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