A Contact Problem with Normal Compliance, Finite Penetration and Nonmonotone Slip Dependent Friction

Ahmad Ramadan, Mikäel Barboteu, Krzysztof Bartosz, and Piotr Kalita

Abstract In this work, we consider a static frictional contact problem between a linearly elastic body and an obstacle, the so-called foundation. This contact is described by a normal compliance condition of such a type that the penetration is restricted with unilateral constraint. The friction is modeled with a nonmonotone law. In order to approximate the contact conditions, we consider a regularized problem wherein the contact is modeled by a standard normal compliance condition without finite penetration. Next, we present a convergence result between the solution of the regularized problem and the original problem. Finally, we provide a numerical validation of this convergence result. To this end we introduce a discrete scheme for the numerical approximation of the frictional contact problems.

1 Introduction

The aim of this paper is to study frictional contact problems in which the contact is modeled with normal compliance of such a type that the penetration is restricted with unilateral constraint. In a physical point of view, this penetration can be assimilated to the flattening of the asperities on the contact interface. Furthermore, the friction is modeled with a nonmonotone law in which the friction bound depends on the tangential displacement, the penetration, and the size of the asperities. The behavior of the material is modeled with a linear elastic constitutive law. In the present paper we consider two frictional contact problems. The first problem is characterized by normal compliance in which the penetration is restricted by unilateral constraint and the second problem represents a regularization of the first problem by considering penetrations without restriction.

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Our interest in this paper is to present the convergence of the solution of the regularized problem with nonmonotone friction to the solution of the original problem with normal compliance, finite penetration, and nonmonotone friction. After, we provide numerical simulations which illustrate the mechanical behavior of the contact model and the numerical validation of the convergence result.

The rest of the paper is structured as follows. In Sect. 2 we present the classical formulation of the contact problems, the variational formulation of the problems, the existence of the weak solution of the regularized problems, and the convergence result. Finally, in Sect. 3 we present the numerical solution of the problems and we provide some numerical simulations on an academic two-dimensional example including a numerical validation of the convergence result.

2 Mechanical Problems and Variational Formulations

In this section we describe the model for the nonmonotone frictional contact with normal compliance and finite penetration as well as a family of auxiliary models used for its approximation. The physical setting is as follows. A linearly elastic body occupies an open bounded connected set $\Omega \subset \mathbb{R}^d$ ($d \le 3$ in applications) with a Lipschitz boundary Γ that is partitioned into three disjoint parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, and $\overline{\Gamma}_3$ with Γ_1, Γ_2, and Γ_3 being relatively open, and meas $(\Gamma_1) > 0$. The body is clamped on Γ_1 and thus the displacement field vanishes there. A volume force of density f_0 acts in Ω and a surface traction of density f_2 acts on Γ_2. The body is in frictional contact with an obstacle on Γ_3. We consider $H = L^2(\Omega)^d = \{ u = (u_i) \mid u_i \in L^2(\Omega) \}$, $Q = \{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \}$, $H_1 = \{ u = (u_i) \mid \varepsilon(u) \in Q \}$ et $Q_1 = \{ \sigma \in Q \mid \mathrm{Div}\, \sigma \in H \}$.

The classical formulation of the frictional contact problem considered is the following.

Problem. $\mathscr{P}_M$. Find a displacement field $u : \Omega \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \to \mathbb{S}^d$ such that

$$\sigma = \mathscr{E}\varepsilon(u) \qquad \text{in } \Omega, \tag{1}$$

$$\mathrm{Div}\, \sigma + f_0 = 0 \qquad \text{in } \Omega, \tag{2}$$

$$u = 0 \qquad \text{on } \Gamma_1, \tag{3}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2, \tag{4}$$

$$\sigma_\nu + p(u_\nu) \le 0,\ u_\nu - g \le 0,\ (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \qquad \text{on } \Gamma_3, \tag{5}$$

$$\left.\begin{array}{ll} |\sigma_\tau| \le N(u_\nu)\mu(|u_\tau|) & \text{if} \quad u_\tau = 0, \\ -\sigma_\tau = N(u_\nu)\mu(|u_\tau|)\frac{u_\tau}{|u_\tau|} & \text{if} \quad u_\tau \neq 0, \end{array}\right. \qquad \text{on } \Gamma_3. \tag{6}$$

Condition (5) was first introduced in [1] and it was used in various papers, see [2] and the references therein where p is the compliance function. Condition (6) was introduced in [3] and $N(u_\nu)\mu(|u_\tau|)$ represents the magnitude of the limiting friction traction at which slip begins. In this case, the friction coefficient μ depends on the tangential displacement $|u_\tau|$ and the magnitude of the friction bound depends also on the penetrations and the size of the asperities via the function N defined by

$$N(x,\eta) = \begin{cases} 0 \ \text{ for } \ \eta \le 0, \\ S\frac{\eta}{g(x)} \ \text{ for } \ \eta \in (0, g(x)), \\ S \ \text{ for } \ \eta \ge g(x). \end{cases} \tag{7}$$

In the above formula the value $S \ge 0$ is a given value. Next we define the approximate problems corresponding to Problem $\mathscr{P}_M$. Let $n \in \mathbb{N}$.

Problem. $\mathscr{P}_M^n$. Find a displacement field $u^n : \Omega \to \mathbb{R}^d$ and a stress field $\sigma^n : \Omega \to \mathbb{S}^d$ such that (1)–(4) and (6) hold for $u = u^n$ and $\sigma = \sigma^n$, and

$$-\sigma_\nu^n \in \begin{cases} \{p(u_\nu^n)\} & \text{if} \quad u_\nu^n < g, \\ [p(g), p(g) + nc_2] & \text{if} \quad u_\nu^n = g, \\ \{p(g) + nc_2 + nc_3(u_\nu^n - g)\} & \text{if} \quad u_\nu^n > g, \end{cases} \qquad \text{on } \Gamma_3. \tag{8}$$

In (8) c_2 and c_3 are arbitrary nonnegative constants such that $c_2 + c_3 > 0$.

Proceeding in a standard way, we obtain the following variational formulations of Problems $\mathscr{P}_M$ and $\mathscr{P}_M^n$. We consider $V = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_1\}$, $K = \{v \in V,\, v_\nu \le g(x) \ \text{ on } \ \Gamma_3\}$, $B : V \to V^*$ tel que $\langle Bu, v\rangle = (\mathscr{E}\varepsilon(u), \varepsilon(v))_Q$ and f the element of V' such that $\langle f, v\rangle = \int_\Omega f_0 \cdot v\,dx + \int_{\Gamma_2} f_2 \cdot v\,d\,\Gamma$.

Problem. $\mathscr{P}_V$. Find the displacement field $u \in K$ and the friction density $\sigma_\tau \in L^2(\Gamma_3)^d$ such that for all $v \in K$ we have

$$\langle Bu - f, v - u\rangle + \int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu)\,d\,\Gamma \ge \int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - u_\tau)\,d\,\Gamma, \tag{9}$$

$$\textit{with} \quad -\sigma_\tau \in N(u_\nu)\partial j_\tau(u_\tau) \ \textit{ a.e. on } \Gamma_3 \tag{10}$$

where function $j_\tau : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$j_\tau(\xi) = \int_0^{|\xi|} \mu(t)\,dt, \tag{11}$$

then under some assumptions, see [3] we can prove that the conditions (6) are equivalent to the subdifferential inclusion (10).

Problem. $\mathscr{P}_V^n$. Find the displacement field $u^n \in V$, friction density $\sigma_\tau^n \in L^2(\Gamma_3)^d$ and normal stress $\sigma_\nu^n \in L^2(\Gamma_3)$ such that for all $v \in V$ we have

$$\langle Bu^n - f, v\rangle = \int_{\Gamma_3} \sigma_\tau^n \cdot v_\tau + \sigma_\nu^n v_\nu \, d\,\Gamma, \tag{12}$$

$$\textit{with} \quad -\sigma_\tau^n \in N(u_\nu^n)\partial j_\tau(u_\tau^n) \;\; \textit{a.e. on } \Gamma_3,$$

$$\textit{and} \quad -\sigma_\nu^n \in \bar{p}(u_\nu^n) + \partial j_\nu^n(u_\nu^n) \;\; \textit{a.e. on } \Gamma_3.$$

Where $j_\nu^n(x,\eta) = \begin{cases} 0 & \text{if} \quad \eta \le g(x), \\ nc_2\eta + \frac{nc_3}{2}(\eta - g(x))^2 & \text{if} \quad \eta > g(x), \end{cases}$

and $\quad \bar{p} : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ such that $\bar{p}(x,s) = \begin{cases} p(s) & \text{for} \quad s \le g(x) \\ p(g(x)) & \text{for} \quad s > g(x). \end{cases}$

Theorem 2.1. *Under some assumptions, see [3], Problem $\mathscr{P}_V^n$ has a solution for every $n \in \mathbb{N}$.*

Theorem 2.2. *Let $(u^n, \sigma_\tau^n, \sigma_\nu^n)$ be a solution of Problem $\mathscr{P}_V^n$, then under some assumptions, see [3], for a subsequence, we have $u^n \to u$ weakly in V, $\sigma_\tau^n \to \sigma_\tau$ weakly in $L^2(\Gamma_3;\mathbb{R}^d)$, where (u, σ_τ) is a solution of Problem $\mathscr{P}_V$.*

3 Numerical Solution

The numerical strategy presented in this section is based on a sequence of convex programming problems; more details can be found in [4]. We consider some materials for the discretization step. Let Ω a polyhedral domain, $\{\mathscr{T}^h\}$ a regular family of triangular finite element partitions of $\overline{\Omega}$. The space V is approximated by the finite dimensional space $V^h \subset V$ of continuous and piecewise affine functions, that is,

$$V^h = \{\, v^h \in [C(\overline{\Omega})]^d \;:\; v^h|_T \in [P_1(T)]^d \quad \forall\, T \in \mathscr{T}^h,$$
$$v^h = 0 \;\text{ at the nodes on }\; \Gamma_1 \,\},$$

where $P_1(T)$ represents the space of polynomials of degree less or equal to one in T. For the discretization of the normal contact terms, we consider the spaces $X_\nu^h = \{\, v^h_{\nu|\Gamma_3} \;:\; v^h \in V^h\,\}$; $X_\tau^h = \{\, v^h_{\tau|\Gamma_3} \;:\; v^h \in V^h\,\}$ equipped with their usual norm. Let us consider the discrete spaces of piecewise constants $Y_\nu^h \subset L^2(\Gamma_3)$ and $Y_\tau^h \subset L^2(\Gamma_3)$ related, respectively, to the discretization of the normal stress σ_ν and the friction density σ_τ. We also introduce the function $\varphi : X_\nu^h \to (-\infty, +\infty]$ and the operator $L : X_\nu^h \to Y_\nu^h$ defined by

$$\varphi(u_\nu^h) = \int_{\Gamma_3} I_{\mathbb{R}_-}(u_\nu^h - g)\, d\Gamma, \quad \forall\, u_\nu^h \in X_\nu^h,$$

$$L : X_\nu^h \to Y_\nu^h \quad \langle L u_\nu^h, v_\nu^h \rangle_{Y_\nu^h, X_\nu^h} = \int_{\Gamma_3} p(u_\nu^h) v_\nu^h \, d\Gamma \quad \forall\, u_\nu^h,\, v_\nu^h \in X_\nu^h$$

where $I_{\mathbb{R}_-}$ represents the indicator function of the set $\mathbb{R}_- = (-\infty, 0]$.

The normal compliance condition with finite penetration (5) leads to the following discrete subdifferential inclusion

$$-\sigma_\nu^h \in \partial\varphi(\Pi_\nu^h u_\nu^h) + L\Pi_\nu^h u_\nu^h \quad \text{in} \quad Y_\nu^h.$$

The friction condition (6) leads to the following discrete subdifferential inclusion

$$-\sigma_\tau^h \in N(|\Pi_\nu^h u_\nu^h|)\mu(|\Pi_\tau^h u_\tau^h|)\partial|\Pi_\tau^h u_\tau^h| \quad \text{in} \quad Y_\tau^h,$$

where $\Pi_\nu^h : X_\tau^h \to Y_\nu^h$ and $\Pi_\tau^h : X_\tau^h \to Y_\tau^h$ represent, respectively, the boundary interpolation operators from X_ν^h to Y_ν^h and from X_τ^h to Y_τ^h (see [5]).

The numerical solution of the nonsmooth nonconvex variational problem $\mathscr{P}_V$ is based on the following iterative algorithm.

Let $\epsilon > 0$ and $u^{(0)}$ be given.

Then, for $k = 0, 1 \ldots,$

Problem $\mathscr{P}_{\mathrm{V}_\mathrm{C}^\mathrm{h}}$. *Find a displacement field* $u^{h,(k+1)} \in V^h$,

a contact stress $\sigma_\nu^{h,(k+1)} \in Y_\nu^h$ *and a friction stress field* $\sigma_\tau^{h,(k+1)} \in Y_\tau^h$

such that, for $\forall\, v^h \in V^h$

$$\langle B u^{h,(k+1)} - f, v^h \rangle = \int_{\Gamma_3} \sigma_\nu^{h,(k+1)} v_\nu^h \, d\,\Gamma + \int_{\Gamma_3} \sigma_\tau^{h,(k+1)} \cdot v_\tau^h \, d\,\Gamma$$

$$\textit{with } \; -\sigma_\nu^{h,(k+1)} \in \partial\varphi(\Pi_\nu^h u_\nu^{h,(k+1)}) + L\Pi_\nu^h u^{h,(k+1)} \textit{ on } \Gamma_3$$

$$\textit{and } \; -\sigma_\tau^{h,(k+1)} \in N(|\Pi_\nu^h u_\nu^{h,(k)}|)\mu(|\Pi_h u_\tau^{h,(k)}|)\partial|\Pi_\tau^h u_\tau^{h,(k+1)}| \textit{ on } \Gamma_3$$

until $\|u^{h,(k+1)} - u^{h,(k)}\| \leq \epsilon \|u^{h,(k)}\|$

and $\|\sigma^{h,(k+1)} - \sigma^{h,(k)}\|_{L^2(\Gamma_3)^d} \leq \epsilon \|\sigma^{h,(k)}\|_{L^2(\Gamma_3)^d}$

Here, k represents the index of the iterative procedure. In Problem $\mathscr{P}_{V_C^h}$ the discrete stress σ^h on the contact boundary Γ_3 can be viewed as a Lagrange stress multiplier. This numerical strategy leads to the solution of a nonsmooth convex problem $\mathscr{P}_{V_C^h}$ at each iteration k. For the numerical treatment of the nonsmooth convex Problem $\mathscr{P}_{V_C^h}$ we use the penalized method for the normal compliance contact term combined with the augmented Lagrangean approach for the unilateral condition and Coulomb friction law. For details concerning this numerical treatment see [3].

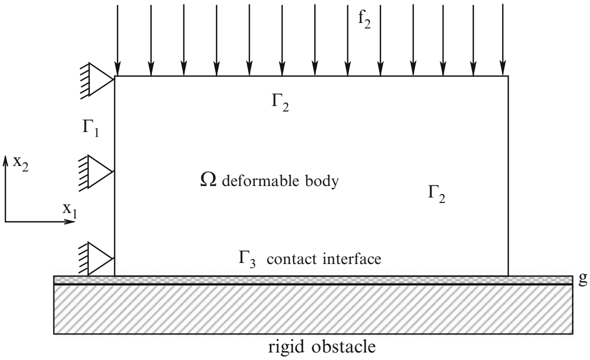

Fig. 1 Initial configuration of the two-dimensional example

Numerical Example. We consider the physical setting depicted in Fig. 1. There, $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ with $L_1,\ L_2 > 0$ and

$$\Gamma_1 = \{0\} \times [0, L_2],\ \Gamma_2 = (\{L_1\} \times [0, L_2]) \cup ([0, L_1] \times \{L_2\}),\ \Gamma_3 = [0, L_1] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional deformable body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part $\Gamma_1 = \{0\} \times [0, L_2]$ the body is clamped and, therefore, the displacement field vanishes there. Vertical tractions act on the part $[0, L_1] \times \{L_2\}$ of the boundary and the part $\{L_1\} \times [0, L_2]$ is traction free. No body forces are assumed to act on the body during the process. The body is in frictional contact with an obstacle on the part $\Gamma_3 = [0, L_1] \times \{0\}$ of the boundary.

We model the material's behavior with a constitutive law of the form (1) in which elasticity tensor $\mathscr{E}$ satisfies

$$(\mathscr{E}\tau)_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2,$$

where E is the Young modulus, κ the Poisson ratio of the material, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The friction is modeled by a nonmonotone law (6) in which the friction bound $N(u_\nu)\mu(|u_\tau|)$ depends on the depth of the penetration u_ν and on the tangential displacement $|u_\tau|$. For the simulations, the function $N : \mathbb{R} \to \mathbb{R}^+$ given in (7) is taken. Let us also consider the following friction coefficient $\mu : \mathbb{R}^d \to \mathbb{R}$:

$$\mu(|u_\tau|) = (a - b) \cdot e^{-\alpha|u_\tau|} + b, \tag{13}$$

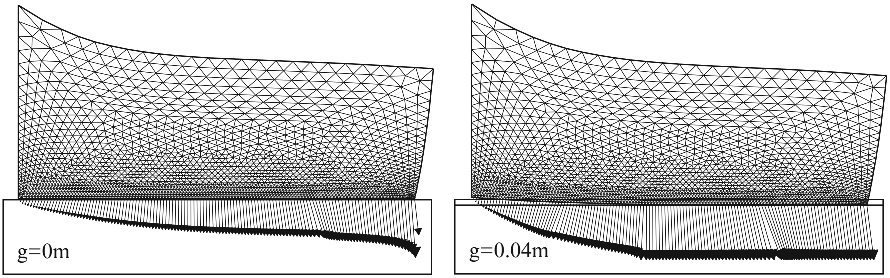

Fig. 2 Deformed meshes and frictional contact forces for $g = 0$ m and $g = 0.04$ m

with $a, b, \alpha > 0$, $a \geq b$. For the computation below we use the following data:

$$\begin{aligned}
&L_1 = 2m, \quad L_2 = 1m,\\
&E = 1000N/m^2, \quad \kappa = 0.3,\\
&f_0 = (0,0)N/m^2, \quad f_2 = \begin{cases} (0,0)\,N/m & \text{on}\{2\} \times [0,1], \\ (0,-300\,t)\,N/m & \text{on}[0,2] \times \{1\}, \end{cases}\\
&a = 1.5, \quad b = 0.5, \quad \alpha = 100, \quad S = 1N, p(u) = c_1 u_+, \quad c_1 = 100,\\
&\text{stopping criterion} : \epsilon = 10^{-6}.
\end{aligned}$$

Our results are presented in Figs. 2, 3, and 4 and are described in what follows. First, in Fig. 2, the deformed configuration as well as the frictional contact forces is plotted both in the case $g = 0$ m and $g = 0.04$ m, which represent, respectively, the case with a classical signorini unilateral contact and the case with normal compliance, finite penetration, and unilateral constraint.

In Fig. 3 we present the convergence of solution of problem $\mathscr{P}^n_{V^h}$ to the solution of problem $\mathscr{P}_{V^h}$. More precisely, we plot four deformed meshes and the associated frictional contact forces at four steps of convergence, for $n = 10,\ 100,\ 10^3,\ 10^4$. One can see that for $n = 10$ all the contact nodes are in strong penetration contact, whereas at $n = 10^4$ the contact nodes are into an admissible finite penetration, since the complete flattening of the asperities of size $g = 0.04\,m$ was reached.

For the numerical convergence we denote by (u^h_n, σ^h_n) and (u^h, σ^h) the discrete solution of the contact problems $\mathscr{P}^n_{V^h}$ and $\mathscr{P}_{V^h}$, respectively. The numerical estimations of the difference

$$\|u^h_n - u^h\|_V + \|\sigma^h_n - \sigma^h\|_Q,$$

for various values of the parameter n, are presented in Fig. 4. It results from here that this difference converges to zero when n tends toward infinity, which represents a numerical validation of the theoretical convergence result obtained in Theorem 2.2.

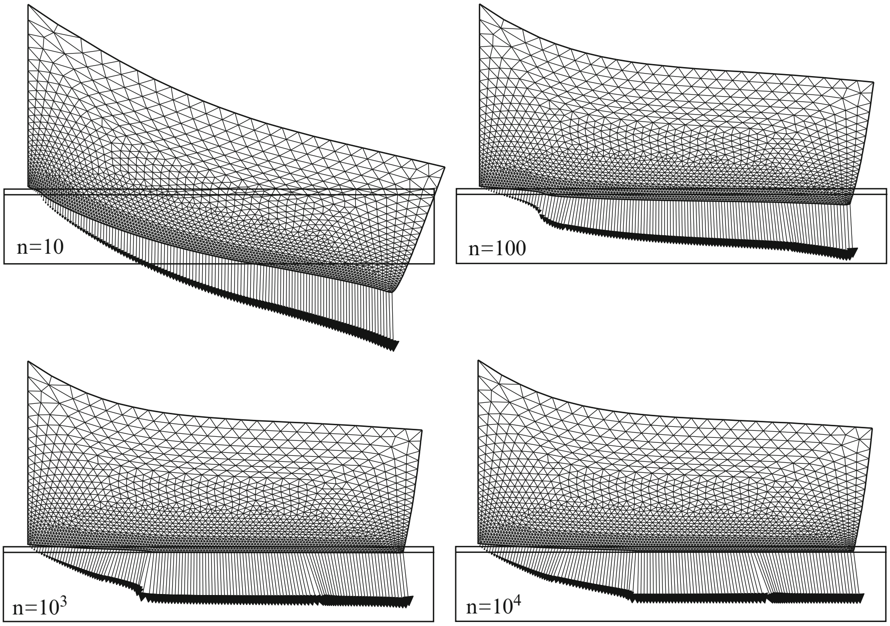

Fig. 3 Deformed meshes and frictional contact forces for $n = 10$, $n = 100$, $n = 10^3$, and $n = 10^4$

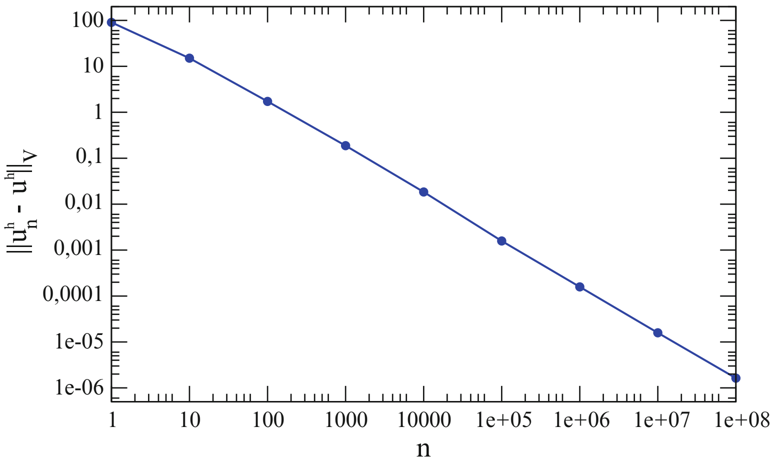

Fig. 4 Numerical validation of the convergence result in Theorem 2.2

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