On Interpreting Three-Way Decisions through Two-Way Decisions

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Abstract. Three-way decisions for classification consist of the actions of acceptance, rejection and non-commitment (i.e., neither acceptance nor rejection) in deciding whether an object is in a class. A difficulty with three-way decisions is that one must consider costs of three actions simultaneously. On the other hand, for two-way decisions, one simply considers costs of two actions. The main objective of this paper is to take advantage of the simplicity of two-way decisions by interpreting three-way decisions as a combination of a pair of two-way decision models. One consists of acceptance and non-acceptance and the other consists of rejection and non-rejection. The non-commitment of the three-way decision model is viewed as non-acceptance and non-rejection of the pair of two-way decision models.

1 Introduction

In concept learning, concept formation and classification, one typically uses a strategy of binary, two-way decisions. That is, an object is either accepted or rejected as being an instance of a concept or a class. One makes a decision with minimum errors or costs. An advantage of a two-way decision strategy is its simplicity. One only needs to consider two actions. On the other hand, when one is forced to make either an acceptance or a rejection decision, it is impossible to arrive at both a low level of incorrect acceptance error and a low level of incorrect rejection error at the same time [1]. To avoid this difficulty, three-way decisions are widely used in many fields and disciplines [2–11] as an alternative effective strategy. In contrast to two-way decisions, a third option called non-commitment, namely, neither acceptance nor rejection, is added. Three-way decisions enable us to reduce incorrect acceptance error and incorrect rejection error simultaneously at the expense of non-commitment for some objects.

In earlier formulations of three-way decisions with rough sets [12], three actions are considered and compared simultaneously. The consideration of six types of costs of three actions makes three-way decisions more complicated than two-way decisions, as the latter only need to consider four types of costs. A recent study by Yao [13] on three-way decisions based on two evaluations suggests that three-way decisions can in fact be interpreted through two-way decisions. In this

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paper, we revisit the two-evaluation-based three-way decision model and explicitly show that three-way decisions can be formulated as a combination of a pair of two-way decision models, namely, an acceptance model and a rejection model. To achieve this goal, we slightly modify the interpretation of a two-way decision model. In the standard interpretation of a two-way decision model, acceptance and rejection are dual actions. That is, failing to accept is the same as rejecting, and vice versa. However, in our acceptance model, we have acceptance and non-acceptance decisions. Failing to accept is non-acceptance, rather than rejecting. Similarly, in the rejection model, we have rejection and non-rejection. Failing to rejection is non-rejection, rather than accepting. By combining the two models together, we have three actions of acceptance, rejection, and non-commitment, where non-commitment is interpreted as non-acceptance and non-rejection.

The rest of the paper are organized as follows. Section 2 gives a brief review of two-evaluation-based three-way decisions proposed by Yao [13]. Section 3 proposes an interpretation of three-way decisions based on a pair of two-way decisions. Section 4 calculates the acceptance and rejection thresholds using a pair of two-way decision models.

2 An Overview of Three-Way Decisions

The purpose of three-way decisions is to divide objects in an universe U, according to a criteria, into three pair-wise disjoint regions, namely, positive, negative and boundary regions, denoted by POS, NEG and BND. A criteria can be considered to be a set of conditions. Due to a lack of information, the true state of satisfiability of the criteria usually is unknown. The degree to which the criteria is satisfied must be estimated according to the partial information. We use a pair of evaluation functions to estimate whether the criteria is satisfied or not.

Definition 1. *Suppose U is a finite non-empty set of objects. Based on a pair of posets $(L_a, \preceq_a)$ and $(L_r, \preceq_r)$, where $\preceq_a$ and $\preceq_r$ are two partial orderings, an acceptance evaluation and a rejection evaluation can be defined as a pair of mappings: $v_a : U \longrightarrow L_a$ and $v_r : U \longrightarrow L_r$. For each object $x \in U$, $v_a(x) \in L_a$ and $v_r(x) \in L_r$ represent the acceptance-evaluation value and rejection-evaluation value, respectively.*

For two objects $x, y \in U$, $v_a(x) \preceq_a v_a(y)$ indicates that x is not more acceptable than y. Similarly, $v_r(x) \preceq_a v_r(y)$ indicates that x is not more rejectable than y. Notions, such as benefits, risks, costs or confidence, can be used to interpret partial orderings $\preceq_a$ and $\preceq_r$.

For an object, a decision is made if its evaluation value reaches a certain level. We introduce the notion of *designated values* as sets of values for acceptance and rejection.

Definition 2. *Suppose L_a^+ is the set of designated values for acceptance, satisfying condition $\emptyset \neq L_a^+ \subseteq L_a$, and L_r^- is the set of designated values for rejection,* *satisfying condition* $\emptyset \neq L_r^- \subseteq L_r$*. The positive, negative and boundary regions of three-way decisions are defined by:*

$$\begin{aligned}
\mathrm{POS}_{(L_a^+,L_r^-)}(v_a,v_r) &= \{x \in U \mid v_a(x) \in L_a^+ \wedge v_r(x) \notin L_r^-\},\\
\mathrm{NEG}_{(L_a^+,L_r^-)}(v_a,v_r) &= \{x \in U \mid v_a(x) \notin L_a^+ \wedge v_r(x) \in L_r^-\},\\
\mathrm{BND}_{(L_a^+,L_r^-)}(v_a,v_r) &= (\mathrm{POS}_{(L_a^+,L_r^-)}(v_a,v_r) \cup \mathrm{NEG}_{(L_a^+,L_r^-)}(v_a,v_r))^c\\
&= \{x \in U \mid (v_a(x) \notin L_a^+ \wedge v_r(x) \notin L_r^-) \vee\\
&\qquad (v_a(x) \in L_a^+ \wedge v_r(x) \in L_r^-)\}, \qquad (1)
\end{aligned}$$

where $(\cdot)^c$ *denotes the complement of a set.*

The three decision regions are pair-wise disjoint and some of them may be empty. They do not necessarily form a partition of the universe.

3 Three-Way Decisions as a Combination of a Pair of Two-Way Decisions

Two models of three-way decisions are interpreted based on two-way decisions. A general model uses two evaluations and a specific model uses a single evaluation.

3.1 Two-Evaluation-Based Model

Let $v_a : U \longrightarrow L_a$ denote the acceptance-evaluation function, $(\mathrm{A}, \bar{\mathrm{A}})$ denote the two-way decisions for acceptance, and $L_a^+ \subseteq L_a$ denote the designated values for acceptance. The two decision regions of an acceptance model, i.e., the $(\mathrm{A}, \bar{\mathrm{A}})$-model, are defined by:

$$\begin{aligned}
\mathrm{POS}_{L_a^+}(v_a) &= \{x \in U \mid v_a(x) \in L_a^+\},\\
\mathrm{NPOS}_{L_a^+}(v_a) &= (\mathrm{POS}_{L_a^+}(v_a))^c = \{x \in U \mid v_a(x) \notin L_a^+\}, \qquad (2)
\end{aligned}$$

where $\mathrm{POS}_{L_a^+}(v_a)$ is called the acceptance region and $\mathrm{NPOS}_{L_a^+}(v_a)$ is called the non-acceptance region. For an object $x \in U$, we can make two-way decisions:

(A) If $v_a(x) \in L_a^+$, then take an acceptance action, *i.e.*, $x \in \mathrm{POS}_{L_a^+}(v_a)$;

($\bar{\mathrm{A}}$) If $v_a(x) \notin L_a^+$, then take a non-acceptance action, *i.e.*, $x \in \mathrm{NPOS}_{L_a^+}(v_a)$.

The acceptance rule (A) classifies objects into an acceptance region. The non-acceptance rule ($\bar{\mathrm{A}}$) classifies objects into the non-acceptance region. The two regions are disjoint and their union is the universe U.

Similarly, let v_r denote the rejection-evaluation function, $(\mathrm{R}, \bar{\mathrm{R}})$ denote the two-way decisions for rejection, and $L_r^- \subseteq L_r$ denote the designated values for rejection. The two-way decision regions of a rejection model, i.e., $(\mathrm{R}, \bar{\mathrm{R}})$-model, are defined by:

$$\begin{aligned}
\mathrm{NEG}_{L_r^-}(v_r) &= \{x \in U \mid v_r(x) \in L_r^-\},\\
\mathrm{NNEG}_{L_r^-}(v_r) &= (\mathrm{NEG}_{L_r^-}(v_r))^c = \{x \in U \mid v_r(x) \notin L_r^-\}, \qquad (3)
\end{aligned}$$

where $\mathrm{NEG}_{L_r^-}(v_r)$ is called the rejection region and $\mathrm{NNEG}_{L_r^-}(v_r)$ is called the non-rejection region. For an object $x \in U$, we can make two-way decisions:

(R) If $v_r(x) \in L_r^-$, then take a rejection action, *i.e.*, $x \in \mathrm{NEG}_{L_r^-}(v_r)$;

($\bar{\mathrm{R}}$) If $v_r(x) \notin L_r^-$, then take a non-rejection action, *i.e.*, $x \in \mathrm{NNEG}_{L_r^-}(v_r)$.

The rejection rule (R) classifies objects into the rejection region. The non-rejection rule ($\bar{\mathrm{R}}$) classifies objects into the non-rejection region. The two regions are disjoint and their union is the universe U.

By combining decision rules of the pair of two-way decision models for acceptance and for rejection, we have three-way decision rules: for each object $x \in U$,

(P) If $v_a(x) \in L_a^+ \wedge v_r(x) \notin L_r^-$, then take an acceptance action, *i.e.*, $x \in \mathrm{POS}_{(L_a^+, L_r^-)}(v_a, v_r)$;

(R) If $v_r(x) \in L_r^- \wedge v_a(x) \notin L_a^+$, then take a rejection action, *i.e.*, $x \in \mathrm{NEG}_{(L_a^+, L_r^-)}(v_a, v_r)$;

(B) If $(v_a(x) \in L_a^+ \wedge v_r(x) \in L_r^-)$ or $(v_a(x) \notin L_a^+ \wedge v_r(x) \notin L_r^-)$, then take a non-commitment action, *i.e.*, $x \in \mathrm{BND}_{(L_a^+, L_r^-)}(v_a, v_r)$.

They in fact define the three regions, $\mathrm{POS}_{(L_a^+, L_r^-)}(v_a, v_r)$, $\mathrm{NEG}_{(L_a^+, L_r^-)}(v_a, v_r)$ and $\mathrm{BND}_{(L_a^+, L_r^-)}(v_a, v_r)$ of three-way decisions given in Equation (1). Table 1 shows the connection between two-way decisions and three-way decisions. An acceptance decision is interpreted as a combination of acceptance and non-rejection, i.e.,

$$\mathrm{POS}_{(L_a^+, L_r^-)}(v_a, v_r) = \mathrm{POS}_{L_a^+}(v_a) \cap \mathrm{NNEG}_{L_r^-}(v_r). \tag{4}$$

Combining rejection and non-acceptance decisions forms the rejection decision, i.e.,

$$\mathrm{NEG}_{(L_a^+, L_r^-)}(v_a, v_r) = \mathrm{NEG}_{L_r^-}(v_r) \cap \mathrm{NPOS}_{L_a^+}(v_a). \tag{5}$$

Making both an acceptance and a rejection decision is a contradiction of a pair of two-way decisions, which results in a non-commitment decision. Neither making an acceptance nor making a rejection decision leads to a different type of non-commitment decision. The union of the two sets forms the boundary region of three-way decisions:

$$\begin{aligned}\mathrm{BND}_{(L_a^+, L_r^-)}(v_a, v_r) = &(\mathrm{POS}_{L_a^+}(v_a) \cap \mathrm{NEG}_{L_r^-}(v_r)) \\ &\cup(\mathrm{NPOS}_{L_a^+}(v_a) \cap \mathrm{NNEG}_{L_r^-}(v_r)).\end{aligned} \tag{6}$$

In many applications, we tend to avoid making a contradiction during decision-makings by imposing the following condition:

$$\mathrm{POS}_{(L_a^+)}(v_a) \cap \mathrm{NEG}_{(L_r^-)}(v_r) = \emptyset. \tag{7}$$

As a result, we have $\mathrm{BND}_{(L_a^+, L_r^-)}(v_a, v_r) = \mathrm{NPOS}_{L_a^+}(v_a) \cap \mathrm{NNEG}_{L_r^-}(v_r)$, and the non-commitment decision are interpreted as neither acceptance nor rejection.

Table 1. Interpretation of three-way decisions based on two-way decisions

(A, Ā)-model \ (R, R̄)-model	rejection	non-rejection
acceptance	non-commitment (contradiction)	acceptance
non-acceptance	rejection	non-commitment

3.2 Single-Evaluation-based Model

Suppose $v : U \longrightarrow L$ is an evaluation function defined on a totally ordered set $(L, \preceq)$, where $\preceq$ is a total ordering, and let $L^+ \subseteq L$ and $L^- \subseteq L$ denote the sets of designated values for acceptance and rejection, respectively. When we use v in the $(\mathrm{A}, \bar{\mathrm{A}})$-model, we define the two regions of two-way decisions for acceptance by:

$$\begin{aligned}\mathrm{POS}_{L^+}(v) &= \{x \in U \mid v(x) \in L^+\},\\ \mathrm{NPOS}_{L^+}(v) &= (\mathrm{POS}_{L^+}(v))^c = \{x \in U \mid v(x) \notin L^+\},\end{aligned} \tag{8}$$

where $\mathrm{POS}_{L^+}(v)$ is the acceptance region and $\mathrm{NPOS}_{L^+}(v)$ is the non-acceptance region. Similarly, when we use v in the $(\mathrm{R}, \bar{\mathrm{R}})$-model, we define the two regions of two-way decisions for rejection by:

$$\begin{aligned}\mathrm{NEG}_{L^-}(v) &= \{x \in U \mid v(x) \in L^-\},\\ \mathrm{NNEG}_{L^-}(v) &= (\mathrm{NEG}_{L^-}(v))^c = \{x \in U \mid v(x) \notin L^-\},\end{aligned} \tag{9}$$

where $\mathrm{NEG}_{L^-}(v)$ is the rejection region, and $\mathrm{NNEG}_{L^-}(v)$ is the non-rejection region.

As shown by Figure 1, combining the pair of two-way decisions forms three-way decision regions.

In order to ensure that the three-way decision regions are pair-wise disjoint, we assume that the following property holds:

$$\text{(t)} \quad L^+ \cap L^- = \emptyset.$$

According to condition (t), we can define three-way decision regions using a single evaluation by:

$$\begin{aligned}\mathrm{POS}_{(L^+,L^-)}(v) &= \{x \in U \mid v(x) \in L^+\},\\ \mathrm{NEG}_{(L^+,L^-)}(v) &= \{x \in U \mid v(x) \in L^-\},\\ \mathrm{BND}_{(L^+,L^-)}(v) &= \{x \in U \mid v(x) \notin L^+ \wedge v(x) \notin L^-\}.\end{aligned} \tag{10}$$

That is, a single-evaluation-based three-way decision model can be interpreted based on a pair of two-way decision models.

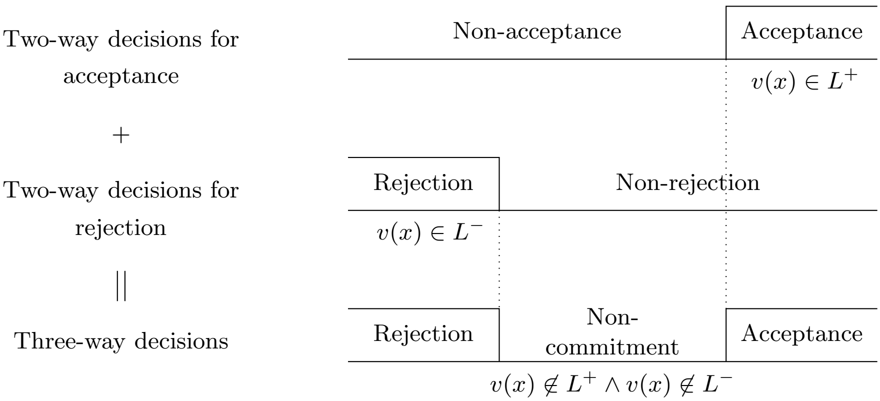

Fig. 1. Interpretation of a single-evaluation-based three-way decisions

4 Probabilistic Three-Way Decisions

In order to apply three-way decisions, we need to investigate the following fundamental issues [13]:

a) construction and interpretation of evaluation functions,
b) construction and interpretation of the sets of designated values of acceptance and rejection, respectively, and
c) analysis of the cost or risk of two-way decisions and three-way decisions, as well as their relationships.

We discuss these issues in the context of probabilistic three-way decisions, which is a generalization of decision-theoretic rough sets [12] and has received much attention recently [14–16].

4.1 Main Results of Probabilistic Three-Way Classifications

Suppose U is a universe of objects. An object in U is represented by a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ in a d-dimensional space, where x_i, $1 \leq i \leq d$, is the object's value on the i-th attribute. Using the terminology of rough set theory, a vector $\boldsymbol{x}$ is in fact the representation of the set of objects with the same description. In the rest of this paper, we simply consider U as a set of vectors. Let C denote a concept or a class of interest, C^c denote its complement, and $Pr(C|\boldsymbol{x})$ denote the conditional probability that an object is in C given that the object is described by $\boldsymbol{x}$. The main task of probabilistic classifications is to decide, according to $Pr(C|\boldsymbol{x})$, whether an object with description $\boldsymbol{x}$ is an instance of C.

The conditional probability $Pr(C|\boldsymbol{x})$ is considered to be an evaluation function for building a three-way decision model. The unit interval $[0, 1]$ is the set of evaluation status values, that is, $L = [0, 1]$. Given a pair of thresholds (α, β)

with $0 \leq \beta < \alpha \leq 1$, we construct the sets of designated values for acceptance and rejection as follows:

$$L^+ = \{a \in [0,1] \mid a \succeq \alpha\}, \qquad L^- = \{b \in [0,1] \mid b \preceq \beta\}. \tag{11}$$

By inserting L^+ and L^- into Equation (10), we immediately obtain the main results of probabilistic three-way classifications [17]:

$$\begin{aligned} \mathrm{POS}_{(\alpha,\cdot)}(C) &= \{\boldsymbol{x} \in U \mid Pr(C|\boldsymbol{x}) \geq \alpha\}, \\ \mathrm{NEG}_{(\cdot,\beta)}(C) &= \{\boldsymbol{x} \in U \mid Pr(C|\boldsymbol{x}) \leq \beta\}, \\ \mathrm{BND}_{(\alpha,\beta)}(C) &= \{\boldsymbol{x} \in U \mid \beta < Pr(C|\boldsymbol{x}) < \alpha\}. \end{aligned} \tag{12}$$

The threshold α is called the acceptance threshold and β is called the rejection threshold. A crucial issue is how to interpret and determine the pair of thresholds [1]. We present a solution by using a pair of two-way decision models based on Bayesian decision theory [18].

4.2 Calculating the Acceptance Threshold

To build a two-way decision model, i.e., $(\mathrm{A}, \bar{\mathrm{A}})$-model, we need to introduce an acceptance threshold $0 < \alpha \leq 1$. The two regions of probabilistic two-way decisions for acceptance are given by:

$$\begin{aligned} \mathrm{POS}_{\alpha}(C) &= \{\boldsymbol{x} \in U \mid Pr(C|\boldsymbol{x}) \geq \alpha\}, \\ \mathrm{NPOS}_{\alpha}(C) &= \{\boldsymbol{x} \in U \mid Pr(C|\boldsymbol{x}) < \alpha\}. \end{aligned} \tag{13}$$

Based on Bayesian decision theory [18], we calculate the optimal acceptance threshold α.

Let $\Omega = \{C, C^c\}$ denote the set of states and $Actions_A = \{a_A, a_{\bar{A}}\}$ denote the set of two decision actions, namely, an acceptance action a_A and a non-acceptance action $a_{\bar{A}}$. We assume each action is associated with certain cost, loss or risk. Such a loss function is given by a 2×2 matrix:

Action	$\boldsymbol{x} \in C$ (Positive instance)	$\boldsymbol{x} \in C^c$ (Negative instance)
a_A	$\lambda_{AP} = \lambda(a_A\|C)$	$\lambda_{AN} = \lambda(a_A\|C^c)$
$a_{\bar{A}}$	$\lambda_{\bar{A}P} = \lambda(a_{\bar{A}}\|C)$	$\lambda_{\bar{A}N} = \lambda(a_{\bar{A}}\|C^c)$

Each cell represents the loss or cost of taking an action $a \in Actions$ when the state of an object is $\omega \in \Omega$, namely, $\lambda(a|\omega)$. For example, $\lambda_{\bar{A}P} = \lambda(a_{\bar{A}}|C)$ represents the risk of taking action $a_{\bar{A}}$ given that $\boldsymbol{x} \in C$. The conditional risks of taking actions a_A and $a_{\bar{A}}$ for $\boldsymbol{x} \in U$ are given by:

$$\begin{aligned} R(a_A|\boldsymbol{x}) &= \lambda(a_A|C)Pr(C|\boldsymbol{x}) + \lambda(a_A|C^c)Pr(C^c|\boldsymbol{x}), \\ R(a_{\bar{A}}|\boldsymbol{x}) &= \lambda(a_{\bar{A}}|C)Pr(C|\boldsymbol{x}) + \lambda(a_{\bar{A}}|C^c)Pr(C^c|\boldsymbol{x}). \end{aligned} \tag{14}$$

The overall risk of the acceptance and non-acceptance decisions for all objects can be expressed by:

$$\begin{aligned} R(\alpha) &= R_{\mathrm{POS}}(\alpha) + R_{\mathrm{NPOS}}(\alpha), \\ &= \sum_{\boldsymbol{x} \in \mathrm{POS}_{\alpha}(C)} R(a_A|\boldsymbol{x}) + \sum_{\boldsymbol{x} \in \mathrm{NPOS}_{\alpha}(C)} R(a_{\bar{A}}|\boldsymbol{x}). \end{aligned} \tag{15}$$

By assuming the following two conditions,

$$\text{(c0)} \qquad \lambda_{AP} < \lambda_{\bar{A}P}, \quad \lambda_{\bar{A}N} < \lambda_{AN}$$

we obtain the optimal acceptance threshold α that minimizing R:

$$\alpha = \frac{(\lambda_{AN} - \lambda_{\bar{A}N})}{(\lambda_{AN} - \lambda_{\bar{A}N}) + (\lambda_{\bar{A}P} - \lambda_{AP})} = \left(1 + \frac{\lambda_{\bar{A}P} - \lambda_{AP}}{\lambda_{AN} - \lambda_{\bar{A}N}}\right)^{-1} \tag{16}$$

It can be verified that $0 < \alpha \leq 1$. The detailed procedure for deriving α can be found in [12] .

4.3 Calculating the Rejection Threshold

To build a two-way decision model for rejection, i.e., $(\mathrm{R}, \bar{\mathrm{R}})$-model, we need a rejection threshold $0 \leq \beta < 1$. The two regions of probabilistic two-way decisions for rejection are given by:

$$\begin{aligned} \mathrm{NEG}_{\beta}(C) &= \{\boldsymbol{x} \in U \mid Pr(C|\boldsymbol{x}) \leq \beta\}, \\ \mathrm{NNEG}_{\beta}(C) &= \{\boldsymbol{x} \in U \mid Pr(C|\boldsymbol{x}) > \beta\}. \end{aligned} \tag{17}$$

The rejection threshold β can be computed by using Bayesian decision theory.

Let $\Omega = \{C, C^c\}$ denote the set of states, and $Actions_R = \{a_R, a_{\bar{R}}\}$ denote the set of two decision actions, namely, a rejection action a_R and a non-rejection action $a_{\bar{R}}$. A loss function is given by a 2×2 matrix:

Action	$x \in C$ (Positive instance)	$x \in C^c$ (Negative instance)
a_R	$\lambda_{RP} = \lambda(a_R\|C)$	$\lambda_{RN} = \lambda(a_R\|C^c)$
$a_{\bar{R}}$	$\lambda_{\bar{R}P} = \lambda(a_{\bar{R}}\|C)$	$\lambda_{\bar{R}N} = \lambda(a_{\bar{R}}\|C^c)$

The overall risk of the rejection and non-rejection decisions for all objects is expressed by:

$$R'(\beta) = \sum_{\boldsymbol{x} \in \mathrm{NEG}_{\beta}(C)} R(a_R|\boldsymbol{x}) + \sum_{\boldsymbol{x} \in \mathrm{NNEG}_{\beta}(C)} R(a_{\bar{R}}|\boldsymbol{x}), \tag{18}$$

where $R(a_R|\boldsymbol{x})$ and $R(a_{\bar{R}}|\boldsymbol{x})$ are conditional risks of taking actions a_R and $a_{\bar{R}}$ for $\boldsymbol{x} \in U$, respectively. By assuming the following conditions,

$$\text{(c0}'\text{)} \qquad \lambda_{\bar{R}P} < \lambda_{RP}, \quad \lambda_{RN} < \lambda_{\bar{R}N}.$$

we obtain the optimal rejection threshold β that minimizing R':

$$\beta = \frac{(\lambda_{\bar{R}N} - \lambda_{RN})}{(\lambda_{\bar{R}N} - \lambda_{RN}) + (\lambda_{RP} - \lambda_{\bar{R}P})} = \left(1 + \frac{(\lambda_{RP} - \lambda_{\bar{R}P})}{(\lambda_{\bar{R}N} - \lambda_{RN})}\right)^{-1} \tag{19}$$

It can be verified that $0 \leq \beta < 1$.

4.4 Combing Results of a Pair of Two-Way Decision Models

To build a three-way classification model, we combine a pair of two-way classification models introduced in the last two subsections. According to condition (t), we require $0 \leq \beta < \alpha \leq 1$, that is,

$$\text{(c1)} \qquad \alpha > \beta \Longleftrightarrow (1 + \frac{(\lambda_{\bar{A}P} - \lambda_{AP})}{(\lambda_{AN} - \lambda_{\bar{A}N})})^{-1} > (1 + \frac{(\lambda_{RP} - \lambda_{\bar{R}P})}{(\lambda_{\bar{R}N} - \lambda_{RN})})^{-1}.$$

Using this condition, we immediate obtain three-way probabilistic regions in Equation (12).

To establish a connection to the existing formulation of probabilistic three-way classification [12], we further assume:

$$\text{(c2)} \quad \lambda_{NP} = \lambda_{\bar{A}P} = \lambda_{\bar{R}P}, \quad \lambda_{NN} = \lambda_{\bar{A}N} = \lambda_{\bar{R}N},$$

where λ_{NP} and λ_{NN} denote the costs of decisions of non-commitment. That is, the cost of non-acceptance is the same as the cost of non-rejection, and both of them are the same as the cost of non-commitment. In this case, for three-way decisions, we have a set of three actions. Suppose $Actions = \{a_A, a_R, a_N\}$ is the set of actions for acceptance, rejection and non-commitment. The loss function is given by a 3×2 matrix:

Action	$\boldsymbol{x} \in C$ (Positive instance)	$\boldsymbol{x} \in C^c$ (Negative instance)
a_A	$\lambda_{AP} = \lambda(a_A \mid C)$	$\lambda_{AN} = \lambda(a_A \mid C^c)$
a_R	$\lambda_{RP} = \lambda(a_R \mid C)$	$\lambda_{RN} = \lambda(a_R \mid C^c)$
a_N	$\lambda_{NP} = \lambda(a_N \mid C)$	$\lambda_{NN} = \lambda(a_N \mid C^c)$

It can be proved that the pair of thresholds (α, β) computed from Equations (16) and (19) in fact minimizes the overall risk of three-way classifications:

$$R(\alpha, \beta) = R_{\text{POS}}(\alpha) + R_{\text{NEG}}(\beta) + R_{\text{BND}}(\alpha, \beta), \tag{20}$$

where the risks of the three regions are defined similarly as earlier. In this way, we obtain three-way decisions classifications by combining a pair of two-way decision models.

5 Conclusion

A pair of two-way decision models, i.e., one for acceptance and non-acceptance, the other for rejection and non-rejection, is used to derive and interpret three-way decisions. An advantage of this interpretation is the simplicity derived from a consideration of only two actions, rather than three actions, simultaneously. That is, we investigate separately two 2×2 loss matrices of two-way decisions and combine the results into a 3×2 loss matrix of three-way decisions. We compute an acceptance threshold and a rejection threshold independently in a pair of two-way decision models. By combine the two thresholds together, we obtain a three-way classification model. Our analysis clearly shows the relative independence and the connection of the two thresholds in probabilistic three-way classification.

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