Some Recent Developments in Finding Systematically Conservation Laws and Nonlocal Symmetries for Partial Differential Equations

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Abstract This chapter presents recent developments in finding systematically conservation laws and nonlocal symmetries for partial differential equations. There is a review of local symmetries, including Lie's algorithm to find local symmetries in evolutionary form and their applications. The Direct Method for finding local conservation laws is reviewed and its relationship to and extension of Noether's theorem are discussed. Moreover, it is shown how symmetries, including discrete symmetries may yield additional conservation laws from known conservation laws. Systematic procedures are presented to seek nonlocally related PDE systems for a given PDE system with two independent variables. In particular, these procedures include the use of conservation laws, point symmetries, and subsystems (including subsystems arising after appropriate invertible transformations of variables) to obtain trees of equivalent nonlocally related PDE systems. In turn, it is shown how the calculation of point symmetries of such nonlocally related systems leads to the discovery of nonlocal symmetries for a given PDE system. The situation of systematically constructing useful nonlocally related systems in multidimensions is considered. Many illustrative examples are provided.

1 Introduction

This chapter is concerned with recent developments in finding conservation laws (CLs) and nonlocal symmetries for partial differential equations (PDEs). It focuses on recent research of the authors and some of the first author's collaborators, including Stephen Anco, Alexei Cheviakov, Temuer Chaolu, Jean-François Ganghoffer, Nataliya

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Ivanova, Sukeyuki Kumei, Ian Lisle, Alex Ma, Greg Reid, Vladimir Shtelen and Thomas Wolf. Much of the material in this chapter appears in more detail in [1, 2].

In the latter part of the 19th century, Sophus Lie initiated his studies on continuous groups of transformations (Lie groups of transformations) in order to put order to, and thereby extend systematically, the hodgepodge of heuristic techniques for solving ordinary differential equations (ODEs). In particular, Lie showed the following.

- The problem of finding a Lie group of point transformations leaving invariant a differential equation (*point symmetry* of a differential equation) is systematic and reduces to solving a related linear system of determining equations for the coefficients (infinitesimals) of its *infinitesimal generator*.
- A point symmetry of an ODE leads to reducing systematically the order of an ODE (irrespective of any imposed initial conditions).
- A point symmetry of a PDE leads to finding systematically special solutions called *invariant* (*similarity*) *solutions*.
- A point symmetry of a differential equation generates a one-parameter family of solutions from any known solution of the differential equation that is not an invariant solution.

However there were limitations to the applicability of Lie's work.

- There were a restricted number of applications for point symmetries, especially for PDE systems.
- Few differential equations have point symmetries.
- For PDE systems having point symmetries, the invariant solutions arising from point symmetries normally yield only a small submanifold of the solution manifold of the PDE system and hence few posed boundary value problems can be solved.
- There was the computational difficulty of finding point symmetries.

Since the end of the 19th century there have been significant extensions of Lie's work on symmetries of PDEs to extend its range of applicability.

- Further applications of point symmetries have been found to include linearizations, other mappings and solutions of boundary value problems. In particular, knowledge of the point symmetries of a nonlinear PDE system (contact symmetries in the case of a scalar PDE), allows one to determine whether the system can be mapped invertibly to a linear system and yields a procedure to find such a mapping when one exists [2–4]. Knowledge of the point symmetries of a linear PDE system with variable coefficients allows one to determine whether the system can be mapped invertibly to a linear system with constant coefficients and yields a procedure to find such a mapping when one exists [2, 3].
- Extensions of the spaces of symmetries of a given PDE system to include local symmetries (higher-order symmetries) as well as nonlocal symmetries [2, 5–8].
- Extension of the applications of symmetries to include variational symmetries that yield conservation laws for variational systems [2, 8].

- Extension of variational symmetries to more general multipliers and resulting conservation laws for essentially any given PDE system [2, 8–11].
- The discovery of further solutions that arise from the extension of Lie's method to the "nonclassical method" as well as other generalizations [2, 12, 13].
- The development of symbolic computation software to solve efficiently the (overdetermined) linear system of symmetry and/or multiplier determining equations as well as related calculations for solving the nonlinear systems of determining equations arising when one uses the nonclassical method [14–18].

1.1 What is a Symmetry of a PDE System and How to Find One?

A symmetry (discrete or continuous) of a PDE system is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE system to a solution of the same system. In particular, continuous symmetries of a PDE system are continuous deformations of its solutions to solutions of the same PDE system. Hence continuous symmetries are defined topologically and not restricted to just point or local symmetries. Thus, in principle, any nontrivial PDE has symmetries. The problem is to find and use the symmetries of a given PDE system. Practically, to find symmetries of a given PDE system, one considers transformations, acting locally on the variables of some finite-dimensional space, which leave invariant the solution manifold of the PDE system and its differential consequences. However, these variables do not have to be restricted to just the independent and dependent variables of the given PDE system.

Higher-order symmetries (*local symmetries*) arise when the solutions of the linear determining equations for infinitesimals are allowed to depend on a finite number of derivatives of dependent variables of the PDE system.

- Infinitesimals for a point symmetry in evolutionary form allow at most linear dependence on first derivatives of dependent variables of a PDE system.
- Infinitesimals for a contact symmetry in evolutionary form (only exists for a scalar PDE) allow arbitrary dependence on at most first derivatives of the dependent variable of a scalar PDE.

In making the extension from point and contact symmetries to higher-order symmetries, it is essential to realize that the linear determining equations for local symmetries are the linearized system (*Fréchet derivative*) of the given PDE system that holds for all of its solutions. Globally, point and contact symmetries act on finite-dimensional spaces whereas higher-order symmetries act on infinite-dimensional spaces consisting of the dependent and independent variables of a given PDE system as well as all of their derivatives. Well-known integrable equations of mathematical physics such as the Korteweg-de-Vries equation have an infinite number of higher-order symmetries [19].

Another extension is to consider solutions of the determining equations where infinitesimals have an ad-hoc dependence on nonlocal variables such as integrals of the dependent variables [20–23]. For some PDEs, such *nonlocal symmetries* can be found formally through *recursion operators* that depend on inverse differentiation. Integrable equations such as the sine-Gordon and cubic Schrödinger equations have an infinite number of such nonlocal symmetries.

1.2 Conservation Laws

In her celebrated 1918 paper [5], Emmy Noether showed that if a DE system admits a variational principle, then any local transformation group leaving invariant the action integral for its Lagrangian density, i.e., a *variational symmetry*, yields a *local conservation law*. Conversely, any local CL of a variational DE system arises from a variational symmetry, and hence there is a direct correspondence between local CLs and variational symmetries (Noether's theorem).

However there are limitations in the use of Noether's theorem.

- Its application is restricted to variational systems. In particular, a given DE system, *as written*, is variational if and only if its linearized system is self-adjoint.
- One has the difficulty of finding local symmetries of the action integral. In general, not all local symmetries of a variational DE system are variational symmetries.
- The use of Noether's theorem to find local conservation laws is coordinate-dependent.

The *Direct Method* for finding CLs allows one to find local CLs more generally for a given DE system. A CL of a given DE system is a divergence expression that vanishes on all solutions of the DE system. Local CLs arise from scalar products formed by linear combinations of *local CL multipliers* (factors that are functions of independent and dependent variables and their derivatives) multiplying each DE in the system. This scalar product is annihilated by the Euler operators associated with each of its dependent variables without restricting these variables in the scalar product to solutions of the system of DEs, i.e., the dependent variables are replaced by arbitrary functions of the independent variables.

If a given DE system, *as written*, is variational, then local CL multipliers correspond to variational symmetries. In the variational situation, using the Direct Method, local CL multipliers satisfy a linear system of determining equations that includes the linearizing system of the given DE system augmented by additional determining equations that taken together correspond to the action integral being invariant under the associated variational symmetry.

More generally, in using the Direct Method for any given DE system, the local CL multipliers are the solutions of an easily found linear determining system that includes the adjoint system of the linearizing DE system [1, 2, 9–11].

For any set of local CL multipliers, usually one can directly find the fluxes and density of the corresponding local CL and, if this proves difficult, there is an integral formula that yields them without the need of a specific functional (Lagrangian) even in the case when the given DE system is variational [9–11].

One can compare the number of local symmetries and the number of local CLs of a given DE system. When a DE system is variational, i.e., its linearized system is self-adjoint, then local CLs arise from a subset of its local symmetries and the number of linearly independent local CLs cannot exceed the number of higher-order symmetries. In general, this will not be the case when a system is not variational. Here a given DE system can have more local conservation laws than local symmetries as well as vice versa.

For any given DE system, a transformation group (continuous or discrete) that leaves it invariant yields an explicit formula that maps a CL to a CL of the same system, whether or not the given system is variational. If the transformation group is a one-parameter Lie group of point (or contact) transformations, then in terms of a parameter expansion a given CL can map into more than one additional CL for the given DE system [2, 24].

1.3 Nonlocally Related Systems and Nonlocal Symmetries

Systematic procedures have been found to seek nonlocal symmetries of a given PDE system through applying Lie's algorithm to nonlocally related systems. In particular, to apply symmetry methods to PDE systems, one needs to work in some specific coordinate frame in order to perform calculations. A procedure to find symmetries that are nonlocal and yet are local in some related coordinate frame involves embedding a given PDE system in another PDE system obtained by adjoining nonlocal variables in such a way that the resulting nonlocally related PDE system is equivalent to the given system. Consequently, any local symmetry of the nonlocally related system yields a symmetry of the given system (The converse also holds). A local symmetry of the nonlocally related system, with the corresponding infinitesimals for the variables of the given PDE system having an essential dependence on nonlocal variables, yields a nonlocal symmetry of the given PDE system.

There are two known systematic ways to find such an embedding.

- Each local CL of a given PDE system yields a nonlocally related system. For each local CL, one can introduce a potential variable(s). Here the nonlocally related system is the given PDE system augmented by a corresponding potential system [2, 25–27].
- Each point symmetry of a given PDE system yields a nonlocally related system. Here, as a first step, the given PDE system naturally yields a locally related PDE system (intermediate system) arising from the canonical coordinates of the point symmetry. In turn, the intermediate system has a natural CL which yields a nonlocally related system (inverse potential system) for the given PDE system [28, 29]. The intermediate system plays the role of a potential system for the inverse potential system.

If a local symmetry of such a nonlocally related system has an essential dependence on nonlocal variables when projected to the given system, then it yields a nonlocal symmetry of the given PDE system. It turns out that many PDE systems have such systematically constructed nonlocal symmetries. Furthermore, one can find additional nonlocal symmetries of a given PDE system through seeking local symmetries of an equivalent subsystem of the given system or one of its constructed nonlocally related systems provided that such a subsystem is nonlocally related to the given PDE system.

There are many applications of nonlocally related systems.

- Invariant solutions of nonlocally related systems (arising from CLs or point symmetries) can yield further solutions of a given PDE system.
- Since a point symmetry-based or CL-based nonlocal symmetry is a local symmetry of a constructed nonlocally related system, it generates a one-parameter family of solutions from any known solution (that is not an invariant solution) of such a nonlocally related system. In turn, this yields a one-parameter family of solutions from any known solution of the given PDE system.
- Local CLs of such nonlocally related systems can yield nonlocal CLs of a given PDE system if their local CL multipliers have an essential dependence on nonlocal variables.

Still wider classes of nonlocally related systems can be constructed systematically for a given PDE system. One can further extend embeddings through the effective use of local CLs to systematically construct trees of nonlocally related but equivalent PDE systems. If a given PDE system has n local CLs, then each CL yields potentials and corresponding potential systems. From the n local CLs, one can directly construct up to $2^n - 1$ independent nonlocally related systems of PDEs by considering corresponding potential systems individually (n singlets), in pairs ($n(n-1)/2$ couplets), ..., taken all together (one n-plet). Any of these systems could lead to the discovery of new nonlocal symmetries and/or nonlocal CLs of the given PDE system or any of the other nonlocally related systems. Such nonlocal CLs could yield further nonlocally related systems, etc. Furthermore, subsystems of such nonlocally related systems could yield further nonlocally related systems. Correspondingly, a tree of nonlocally related, and equivalent, systems is constructed for a given PDE system [2, 30, 31].

The situation in the case of multidimensional PDE systems (i.e., with at least three independent variables) is especially interesting. Here one can show that nonlocal symmetries and nonlocal CLs arising from the CL-based approach cannot arise from potential systems unless they are augmented by gauge constraints [2, 32, 33].

There exist many applications of such systematically constructed nonlocally related systems that further extend the use of symmetry methods for PDE systems.

- Through such constructions, one can systematically relate Eulerian and Lagrangian coordinate descriptions of gas dynamics and nonlinear elasticity. In particular, for the Eulerian coordinate description, a subsystem of the potential system arising from conservation of mass, naturally yields the corresponding description in Lagrangian coordinates [2, 30, 31, 34, 35].
- For a given class of PDEs with constitutive functions, one finds trees of nonlocally related systems yielding symmetries and CLs with respect to various forms of its constitutive functions.

- One can systematically seek noninvertible mappings of nonlinear PDE systems to linear PDE systems. Consequently, further nonlinear PDE systems can be mapped into equivalent linear PDE systems beyond those obtained through invertible mappings [2, 27, 36].
- One can systematically extend the class of linear PDE systems with variable coefficients that can be mapped into equivalent linear PDE systems with constant coefficients through inclusion of noninvertible mappings [2, 37, 38].

The rest of this chapter is organized as follows. In Sect. 2, we review local symmetries, Lie's algorithm to find local symmetries in evolutionary form, applications of local symmetries and as examples consider the heat equation and the Kortweg-de Vries equation. In Sect. 3, we consider the construction of conservation laws, introduce the Direct Method and its relationship to Noether's theorem, and show how symmetries could yield additional CLs from known CLs. As examples, we consider nonlinear telegraph equations, the Korteweg-de Vries equation, the Klein-Gordon equation, and nonlinear wave equations. In Sect. 4, we present systematic procedures to seek nonlocally related systems and nonlocal symmetries of a given PDE system with two independent variables. We introduce conservation law and point symmetry based methods as well as the use of subsystems to obtain trees of equivalent nonlocally related PDE systems. As examples, we focus on nonlinear wave equations, nonlinear telegraph equations, planar gas dynamics equations, and nonlinear reaction diffusion equations. In Sect. 5, we consider the situation of nonlocality in multidimensions. We show that if one directly applies the CL-based method to a single CL, then it is necessary to append a gauge constraint relating potential variables of the resulting vector potential system when seeking nonlocal symmetries. Some open problems are discussed.

2 Local Symmetries

Lie's algorithm for seeking point symmetries can be extended to seek more general local symmetries admitted by PDE systems. In the extension of Lie's algorithm, one uses differential consequences of the given PDE system, i.e., invariance of a given PDE system is understood to include its differential consequences. Here it is important to consider the infinitesimal generators for point symmetries in their *evolutionary form* where the independent variables are themselves invariant and the action of a group of point transformations is strictly an action on the dependent variables of the PDE system, so that solutions are *directly mapped into other solutions* under the group action. This allows one to readily extend Lie's algorithm to seek *contact symmetries* (only existing for scalar PDEs) where now the components of infinitesimal generators for dependent variables can depend at most on the first derivatives of the dependent variable of a given scalar PDE (if this dependence is at most linear on the first derivatives, then a contact symmetry is a point symmetry).

A contact symmetry is equivalent to a point transformation acting on the space of the given independent variables, the dependent variable and its first derivatives and, through this, can be naturally extended to a point transformation acting on the space of the given independent variables, the dependent variable and its derivatives to any finite order greater than one.

Lie's algorithm can be still further extended by allowing the infinitesimal generators in evolutionary form to depend on derivatives of dependent variables to any finite order. This allows one to calculate symmetries that are called *higher-order symmetries*. In the scalar case, contact symmetries are first-order symmetries. Otherwise, higher-order symmetries are not equivalent to point transformations acting on a finite-dimensional space including the independent variables, the dependent variables and their derivatives to some finite order. Higher-order symmetries are local symmetries in the sense that the components of the dependent variables in their infinitesimal generators depend at most on a finite number of derivatives of a given PDE system's dependent variables so that their calculation only depends on the local behaviour of solutions of a given PDE system.

Local symmetries include point symmetries, contact symmetries and higher-order symmetries. Local symmetries are uniquely determined when infinitesimal generators are represented in evolutionary form.

Sophus Lie considered contact symmetries. Emmy Noether introduced the notion of higher-order symmetries in her celebrated paper on conservation laws [5]. The well-known infinite sequences of conservation laws of the Korteweg-de Vries (KdV) and sine-Gordon equations are directly related to admitted infinite sequences of local symmetries obtained through the use of recursion operators [19].

Consider a given scalar PDE of order k

$$R(x, t, u, \partial u, \ldots, \partial^k u) = 0 \tag{1}$$

with independent variables (x, t) and dependent variable $u(x, t)$; $\partial^j u$ denotes the jth order partial derivatives of $u(x, t)$ appearing in the PDE (1). In *evolutionary form*, the *local symmetries of order* p of a PDE (1), in terms of their infinitesimal generators

$$\eta(x, t, u, \partial u, \ldots, \partial^p u)\frac{\partial}{\partial u}$$

are the solutions $\eta(x, t, u, \partial u, \ldots, \partial^p u)$ of its linearized system (*Fréchet derivative*)

$$\left[\frac{\partial R}{\partial u}\eta + \frac{\partial R}{\partial u_x}\mathrm{D}_x\eta + \frac{\partial R}{\partial u_t}\mathrm{D}_t\eta + \frac{\partial^2 R}{\partial {u_x}^2}(\mathrm{D}_x)^2\eta + \cdots\right]_{\substack{R=0,\\ \mathrm{D}_x R=0,\\ \mathrm{D}_t R=0,\\ \vdots}} = 0$$

in terms of *total derivative operators*

$$\mathrm{D}_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots$$
$$\mathrm{D}_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + \cdots$$

and holding for all solutions $u = \theta(x, t)$ of the PDE (1) and its differential consequences.

A local symmetry of order p, $\eta(x, t, u, \partial u, \ldots, \partial^p u)\frac{\partial}{\partial u}$ (including its natural extension to action on derivatives), maps *any* solution $u = \theta(x, t)$ of PDE (1) (that is not an invariant solution of PDE (1)) into a one-parameter (ε) family of solutions of PDE (1) given by the expression

$$u = \left(\mathrm{e}^{\varepsilon\left(\eta \frac{\partial}{\partial u} + (\mathrm{D}_x \eta) \frac{\partial}{\partial u_x} + (\mathrm{D}_t \eta) \frac{\partial}{\partial u_t} + \cdots\right)} u \right) \Big|_{u=\theta(x,t)}$$

and is equivalent to the transformation

$$\begin{aligned} x^* &= x \\ t^* &= t \\ u^* &= \mathrm{e}^{\varepsilon\left(\eta \frac{\partial}{\partial u} + (\mathrm{D}_x \eta) \frac{\partial}{\partial u_x} + (\mathrm{D}_t \eta) \frac{\partial}{\partial u_t} + \cdots\right)} u \\ &= u + \varepsilon \eta(x, t, u, \partial u, \ldots, \partial^p u) + O(\varepsilon^2). \end{aligned}$$

If $p = 1$, then the first order symmetry is equivalent to the *contact symmetry*

$$\begin{aligned} x^* &= x + \varepsilon \frac{\partial \eta}{\partial u_x} + O(\varepsilon^2) \\ t^* &= t + \varepsilon \frac{\partial \eta}{\partial u_t} + O(\varepsilon^2) \\ u^* &= u + \varepsilon \left(u_x \frac{\partial \eta}{\partial u_x} + u_t \frac{\partial \eta}{\partial u_t} - \eta \right) + O(\varepsilon^2) \\ u_x^* &= u_x + \varepsilon \left(-u_x \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial x} \right) + O(\varepsilon^2) \\ u_t^* &= u_t + \varepsilon \left(-u_t \frac{\partial \eta}{\partial u} - \frac{\partial \eta}{\partial t} \right) + O(\varepsilon^2). \end{aligned}$$

If a first order symmetry has an infinitesimal of the form

$$\eta(x, t, u, \partial u) = \xi(x, t, u) u_x + \tau(x, t, u) u_t - \omega(x, t, u)$$

then it is equivalent to the *point symmetry*

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2)$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2)$$
$$u^* = u + \varepsilon\omega(x,t,u) + O(\varepsilon^2).$$

2.1 Example 1: The Heat Equation

The heat equation

$$R = u_t - u_{xx} = 0$$

has the point symmetries [12, 13]

$$\mathrm{X}_1 = u_x\frac{\partial}{\partial u}, \qquad \mathrm{X}_2 = u_t\frac{\partial}{\partial u}, \qquad \mathrm{X}_3 = (xu_x + 2tu_t)\frac{\partial}{\partial u}$$
$$\mathrm{X}_4 = (xtu_x + t^2u_t + [\tfrac{1}{4}x^2 + \tfrac{1}{2}t]u)\frac{\partial}{\partial u}$$
$$\mathrm{X}_5 = (tu_x + \tfrac{1}{2}xu)\frac{\partial}{\partial u}, \quad \mathrm{X}_6 = u\frac{\partial}{\partial u}.$$

2.2 Example 2: The Korteweg-de Vries Equation

The Korteweg-de Vries (KdV) equation

$$R = u_t + uu_x + u_{xxx} = 0$$

has an infinite sequence of higher-order symmetries given by

$$(\mathbf{R}^n)u_x, \qquad n = 0, 1, 2, \ldots$$

in terms of the recursion operator [19]

$$\mathbf{R} = (\mathrm{D}_x)^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x(\mathrm{D}_x)^{-1}.$$

Specifically, one obtains corresponding nonlocal symmetries

$$u_x\frac{\partial}{\partial u}, \quad (uu_x + u_{xxx})\frac{\partial}{\partial u}$$
$$(\tfrac{5}{6}u^2u_x + 4u_xu_{xx} + \tfrac{5}{3}uu_{xxx} + u_{xxxxx})\frac{\partial}{\partial u}, \ldots .$$

For a given PDE system, local symmetries can be used to determine

- specific invariant solutions.
- a one-parameter family of solutions from "any" known solution.
- whether it can be linearized by an invertible transformation and find the linearization when it exists [3, 4, 21].
- whether an inverse scattering transform exists.
- whether a given linear PDE with variable coefficients can be invertibly mapped into a linear PDE with constant coefficients and find such a mapping when it exists [39, 40].

3 Construction of Conservation Laws

In this section, we consider the problem of finding the *local conservation laws* for a given PDE system. In particular, we present the Direct Method for the construction of CLs. In the Direct Method one first derives the determining equations yielding the multipliers (*local CL multipliers*). Following this, one finds the fluxes and densities of corresponding local CLs. It is shown that a subset of the determining equations for local CL multipliers includes the adjoint equations of the determining equations yielding the local symmetries (in evolutionary form) of a given PDE system. The self-adjoint case is especially interesting since here the given PDE system is variational and thus the local CL multipliers are also local symmetries (the converse is false) of the given PDE system. A comparison is made with the classical Noether theorem. Further connections between symmetries and CLs are presented. In particular, it is shown how a symmetry of a PDE system maps a known CL to a CL of the same PDE system. In the case of a local symmetry it is shown that a parameter expansion could yield more than one new CL from a known CL.

3.1 Uses of Conservation Laws

Conservation laws can yield constants of motion for any posed boundary value problem for a given PDE system. For this reason, for global convergence of an approximation scheme, it is important to preserve CLs, at least those CLs considered to be of importance for a particular posed boundary value problem.

From knowledge of the local CL multipliers for a given nonlinear PDE system, one can determine whether it can be mapped invertibly to a linear PDE system and set up the equations to find such a mapping when one exists [2].

In Sect. 4, it will be shown how one can use local CLs to find nonlocally related systems for a given PDE system. In turn, invariant solutions arising from local symmetries of such a nonlocally related system could yield further solutions of the given PDE system beyond those obtained as invariant solutions arising from local symmetry reductions. Moreover, the computation of local CLs of a nonlocally related system could yield nonlocal CLs of a given PDE system and to noninvertible linearizations of nonlinear PDE systems.

3.2 Direct Method for Construction of Conservation Laws

Consider a given system $\mathbf{R}\{x; u\}$ of N PDEs of order k with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{2}$$

A *local conservation law* of the PDE system (2) is an expression

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_1 \Phi^1[u] + \cdots + \mathrm{D}_n \Phi^n[u] = 0 \tag{3}$$

holding for any solution of the PDE system (2). In (3), the operators D_i, $i = 1, \ldots, n$ are total derivative operators.

Definition 1 A PDE system $\mathbf{R}\{x; u\}$ (2) is *totally non-degenerate* if (2) and its differential consequences have maximal rank and are locally solvable.

The proof of the following theorem appears in [11].

Theorem 1 *Suppose* $\mathbf{R}\{x; u\}$ *(2) is a totally non-degenerate PDE system. Then for every nontrivial local conservation law*

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_i \Phi^i(x, u, \partial u, \ldots, \partial^r u) = 0$$

of (2), there exists a set of multipliers, called local conservation law multipliers*,*

$$\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U), \qquad \sigma = 1, \ldots, N$$

such that

$$\mathrm{D}_i \Phi^i[U] \equiv \Lambda_\sigma[U] R^\sigma[U]$$

holds for arbitrary $U(x)$.

Definition 2 The *Euler operator* with respect to U^j is the operator

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - \mathrm{D}_i \frac{\partial}{\partial U^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_s} \frac{\partial}{\partial U^j_{i_1 \cdots i_s}} + \cdots .$$

The proofs of the following two theorems follow from direct computations.

Theorem 2 *For* any divergence expression $\mathrm{D}_i \Phi^i[U]$*, one has*

$$\mathrm{E}_{U^j}(\mathrm{D}_i \Phi^i[U]) \equiv 0, \qquad j = 1, \ldots, m.$$

Theorem 3 *Let* $F[U] = F(x, U, \partial U, \ldots, \partial^s U)$. *Then*

$$\mathrm{E}_{U^j} F[U] \equiv 0, \qquad j = 1, \ldots, m$$

holds for arbitrary $U(x)$ *if and only if*

$$F[U] \equiv \mathrm{D}_i \Psi^i(x, U, \partial U, \ldots, \partial^{s-1} U)$$

for some set of functions $\{\Psi^i(x, U, \partial U, \ldots, \partial^{s-1} U)\}$.

The next theorem follows directly from Theorems 2 and 3.

Theorem 4 *A set of local multipliers* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}$ *yields a divergence expression for PDE system (2) if and only if*

$$\mathrm{E}_{U^j}(\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U) R^\sigma(x, U, \partial U, \ldots, \partial^k U)) \equiv 0, \qquad j = 1, \ldots, m \tag{4}$$

holds for arbitrary $U(x)$.

3.2.1 Summary of Direct Method to Find Local CLs

The Direct Method to find local CLs for a given PDE system (2) can be summarized as follows. Further details can be found in [2, 10, 11].

1. Seek multipliers of the form $\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)$ with derivatives $\partial^l U$ to some specified order l.
2. Obtain and solve the determining Eq. (4) to find the multipliers of local conservation laws.
3. For each set of multipliers, find the corresponding fluxes $\Phi^i[U] = \Phi^i(x, U, \partial U, \ldots, \partial^r U)$ satisfying the identity

$$\Lambda_\sigma[U] R^\sigma[U] \equiv \mathrm{D}_i \Phi^i[U]. \tag{5}$$

4. Consequently, one obtains the local CL

$$\mathrm{D}_i \Phi^i[u] = \mathrm{D}_i \Phi^i(x, u, \partial u, \ldots, \partial^r u) = 0$$

with fluxes $\Phi^i[u]$ holding for any solution of the PDE system (2).

The fluxes $\Phi^i[U] = \Phi^i(x, U, \partial U, \ldots, \partial^r U)$ in (5) can be found in the following ways:

- Directly manipulate the left-hand side of (5) to obtain the right-hand side divergence form.

- Treat the fluxes as unknowns in expression (5). Expand the right-hand side to set up a linear set of PDEs for the fluxes. Solve this linear set of PDEs.
- If one is unable to perform either of the first two ways successfully, then one can formally obtain the fluxes through use of an integral (homotopy) formula that appears in [11].

Example 1 Nonlinear Telegraph System

Consider the nonlinear telegraph system

$$\begin{aligned} R_1[u, v] &= v_t - (u^2 + 1)u_x - u = 0 \\ R_2[u, v] &= u_t - v_x = 0. \end{aligned} \tag{6}$$

We seek local CL multipliers of the form

$$\Lambda_1 = \xi[U, V] = \xi(x, t, U, V), \qquad \Lambda_2 = \varphi[U, V] = \varphi(x, t, U, V) \tag{7}$$

for the nonlinear telegraph system (6). In terms of the Euler operators

$$\mathrm{E}_U = \frac{\partial}{\partial U} - \mathrm{D}_x \frac{\partial}{\partial U_x} - \mathrm{D}_t \frac{\partial}{\partial U_t}, \quad \mathrm{E}_V = \frac{\partial}{\partial V} - \mathrm{D}_x \frac{\partial}{\partial V_x} - \mathrm{D}_t \frac{\partial}{\partial V_t}$$

the multipliers (7) yield a local CL of the nonlinear telegraph system (6) if and only if the determining equations

$$\begin{aligned} \mathrm{E}_U(\xi[U, V]R_1[U, V] + \varphi[U, V]R_2[U, V]) \equiv 0 \\ \mathrm{E}_V(\xi[U, V]R_1[U, V] + \varphi[U, V]R_2[U, V]) \equiv 0 \end{aligned} \tag{8}$$

hold for arbitrary differentiable functions $U(x, t)$, $V(x, t)$. It is straightforward to show that the Eq. (8) hold if and only if

$$\begin{aligned} \varphi_V - \xi_U &= 0 \\ \varphi_U - (U^2 + 1)\xi_V &= 0 \\ \varphi_x - \xi_t - U\xi_V &= 0 \\ (U^2 + 1)\xi_x - \varphi_t - U\xi_U - \xi &= 0. \end{aligned} \tag{9}$$

The five linearly independent solutions [41] of the linear determining system (9) are given by

$$\begin{aligned} &(\xi_1, \varphi_1) = (0, 1), \qquad (\xi_2, \varphi_2) = (t, x - \tfrac{1}{2}t^2), \qquad (\xi_3, \varphi_3) = (1, -t) \\ &(\xi_4, \varphi_4) = (\mathrm{e}^{x+\frac{1}{2}U^2+V}, U\mathrm{e}^{x+\frac{1}{2}U^2+V}), \quad (\xi_5, \varphi_5) = (\mathrm{e}^{x+\frac{1}{2}U^2-V}, -U\mathrm{e}^{x+\frac{1}{2}U^2-V}). \end{aligned}$$

Correspondingly, through manipulation, one obtains the following five local conservation laws [41]

$$
\begin{aligned}
\mathrm{D}_t u + \mathrm{D}_x[-v] &= 0\\
\mathrm{D}_t[(x - \tfrac{1}{2}t^2)u + tv] + \mathrm{D}_x[(\tfrac{1}{2}t^2 - x)v - t(\tfrac{1}{3}u^3 + u)] &= 0\\
\mathrm{D}_t[v - tu] + \mathrm{D}_x[tv - (\tfrac{1}{3}u^3 + u)] &= 0\\
\mathrm{D}_t[\mathrm{e}^{x+\frac{1}{2}u^2+v}] + \mathrm{D}_x[-u\mathrm{e}^{x+\frac{1}{2}u^2+v}] &= 0\\
\mathrm{D}_t[\mathrm{e}^{x+\frac{1}{2}u^2-v}] + \mathrm{D}_x[u\mathrm{e}^{x+\frac{1}{2}u^2-v}] &= 0.
\end{aligned}
$$

Example 2 KdV Equation

As a second example, consider again the KdV equation [10]

$$R[u] = u_t + uu_x + u_{xxx} = 0. \tag{10}$$

It is convenient to also write (10) as

$$u_t = g[u] = -(uu_x + u_{xxx}). \tag{11}$$

Due to the evolutionary form of the KdV equation (10), it follows that all local CL multipliers are of the form $\Lambda[U] = \Lambda(t, x, U, \partial_x U, \ldots, \partial_x^l U)$, $l = 1, 2, \ldots$. Then $\mathrm{E}_U(\Lambda[U](U_t + UU_x + U_{xxx})) \equiv 0$ if and only if

$$
\begin{aligned}
&-\mathrm{D}_t\Lambda - U\mathrm{D}_x\Lambda - \mathrm{D}_x^3\Lambda + (U_t + UU_x + U_{xxx})\Lambda_U\\
&-\mathrm{D}_x((U_t + UU_x + U_{xxx})\Lambda_{\partial_x U}) + \cdots\\
&+(-1)^l\mathrm{D}_x^l((U_t + UU_x + U_{xxx})\Lambda_{\partial_x^l U}) \equiv 0.
\end{aligned} \tag{12}
$$

Note that the linear determining Eq. (12) is of the form

$$\alpha_1 + \alpha_2 U_t + \alpha_3\partial_x U_t + \cdots + \alpha_{l+2}\partial_x^l U_t \equiv 0 \tag{13}$$

where in Eq. (13) each coefficient α_i depends at most on t, x, U and x-derivatives of U. Since $U(x, t)$ is an arbitrary function in Eq. (13), it follows that each of the terms $U_t, \partial_x U_t, \ldots, \partial_x^l U_t$ must be treated as independent variables in (13). Hence $\alpha_i = 0$, $i = 1, \ldots, l + 2$. Thus Eq. (13) splits into an overdetermined linear system of $l + 2$ determining equations for the local multipliers $\Lambda(t, x, U, \partial_x U, \ldots, \partial_x^l U)$, given by

$$\tilde{\mathrm{D}}_t\Lambda + U\mathrm{D}_x\Lambda + \mathrm{D}_x^3\Lambda = 0 \tag{14}$$

$$\sum_{k=1}^{l} (-\mathrm{D}_x)^k \Lambda_{\partial_x^k U} = 0 \tag{15}$$

$$(1-(-1)^q)\Lambda_{\partial_x^q U} + \sum_{k=q+1}^{l} \frac{k!}{q!(k-q)!}(-\mathrm{D}_x)^{k-q}\Lambda_{\partial_x^k U} = 0, \quad q = 1, \ldots, l-1 \tag{16}$$

$$(1-(-1)^l)\Lambda_{\partial_x^l U} = 0 \tag{17}$$

where $\tilde{\mathrm{D}}_t = \frac{\partial}{\partial t} + g[U]\frac{\partial}{\partial U} + (g[U])_x\frac{\partial}{\partial U_x} + \cdots$ is the total derivative operator restricted to the KdV equation, with $g[U] = -(UU_x + U_{xxx})$.

Now we seek local CL multipliers of the form $\Lambda[U] = \Lambda(x, t, U)$. Then the determining Eqs. (15)–(17) are satisfied and the determining Eq. (14) becomes

$$\begin{aligned}(\Lambda_t + U\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxU}U_x + 3\Lambda_{xUU}U_x^2 \\ + \Lambda_{UUU}U_x^3 + 3\Lambda_{xU}U_{xx} + 3\Lambda_{UU}U_xU_{xx} = 0.\end{aligned} \tag{18}$$

Equation (18) holds for arbitrary values of x, t, U, U_x and U_{xx}. Hence Eq. (18) splits into six equations. Their solution yields the three local CL multipliers $\Lambda_1 = 1$, $\Lambda_2 = U$, $\Lambda_3 = tU - x$. In turn, after simple manipulations, these three multipliers yield the divergence expressions

$$\begin{aligned}
U_t + UU_x + U_{xxx} &\equiv \mathrm{D}_t U + \mathrm{D}_x(\tfrac{1}{2}U^2 + U_{xx}) \\
U(U_t + UU_x + U_{xxx}) &\equiv \mathrm{D}_t(\tfrac{1}{2}U^2) + \mathrm{D}_x(\tfrac{1}{3}U^3 + UU_{xx} - \tfrac{1}{2}U_x^2) \\
(tU - x)(U_t + UU_x + U_{xxx}) &\equiv \mathrm{D}_t(\tfrac{1}{2}tU^2 - xU) \\
&\quad + \mathrm{D}_x(-\tfrac{1}{2}xU^2 + tUU_{xx} - \tfrac{1}{2}tU_x^2 - xU_{xx} + U_x).
\end{aligned}$$

Thus the corresponding local conservation laws for the KdV Eq. (10) are given by

$$\begin{aligned}
\mathrm{D}_t u + \mathrm{D}_x(\tfrac{1}{2}u^2 + u_{xx}) &= 0 \\
\mathrm{D}_t(\tfrac{1}{2}u^2) + \mathrm{D}_x(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2) &= 0 \\
\mathrm{D}_t(\tfrac{1}{2}tu^2 - xu) + \mathrm{D}_x(-\tfrac{1}{2}xu^2 + tuu_{xx} - \tfrac{1}{2}tu_x^2 - xu_{xx} + u_x) &= 0.
\end{aligned}$$

One can show that there is only one additional local CL multiplier of the form $\Lambda[U] = \Lambda(x, t, U, U_x, U_{xx})$, given by

$$\Lambda_4 = U_{xx} + \tfrac{1}{2}U^2.$$

Moreover, one can show that in terms of the recursion operator

$$\mathbf{R}^*[U] = \mathrm{D}_x^2 + \tfrac{1}{3}U + \tfrac{1}{3}\mathrm{D}_x^{-1} \circ U \circ \mathrm{D}_x$$

the KdV equation has an infinite sequence of local CL multipliers given by

$$\Lambda_{2n} = (\mathbf{R}^*[U])^n U, \qquad n = 1, 2, \ldots .$$

General Expression Relating Local CL Multipliers and Solutions of Adjoint Equations.

Consider a given PDE system (2). Let $R^\sigma[U] = R^\sigma(x, U, \partial U, \ldots, \partial^k U)$, $\sigma = 1, \ldots, N$, where $U(x) = (U^1(x), \ldots, U^m(x))$ is *arbitrary* and $U(x) = u(x)$ solves the PDE system (2).

In terms of m *arbitrary* functions $V(x) = (V^1(x), \ldots, V^m(x))$, the *linearizing operator* $\mathrm{L}[U]$ associated with the PDE system (2) is given by

$$\mathrm{L}_\rho^\sigma[U]V^\rho \equiv \left[\frac{\partial R^\sigma[U]}{\partial U^\rho} + \frac{\partial R^\sigma[U]}{\partial U_i^\rho}\mathrm{D}_i + \cdots + \frac{\partial R^\sigma[U]}{\partial U_{i_1\ldots i_k}^\rho}\mathrm{D}_{i_1}\cdots\mathrm{D}_{i_k}\right]V^\rho,$$
$$\sigma = 1, \ldots, N$$

and, in terms of N *arbitrary* functions $W(x) = (W_1(x), \ldots, W_N(x))$, the *adjoint operator* $\mathrm{L}^*[U]$ associated with the PDE system (2) is given by

$$\mathrm{L}^{*\sigma}_{\ \rho}[U]W_\sigma \equiv \frac{\partial R^\sigma[U]}{\partial U^\rho}W_\sigma - \mathrm{D}_i\left(\frac{\partial R^\sigma[U]}{\partial U_i^\rho}W_\sigma\right) + \cdots$$
$$+ (-1)^k \mathrm{D}_{i_1}\cdots\mathrm{D}_{i_k}\left(\frac{\partial R^\sigma[U]}{\partial U_{i_1\cdots i_k}^\rho}W_\sigma\right), \qquad \rho = 1, \ldots, m.$$

In particular, $W_\sigma \mathrm{L}_\rho^\sigma[U]V^\rho - V^\rho \mathrm{L}^{*\sigma}_{\ \rho}[U]W_\sigma$ is a divergence expression.

Let

$$W_\sigma = \Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U), \quad \sigma = 1, \ldots, N.$$

By direct calculation, in terms of Euler operators, one can show that

$$\mathrm{E}_{U^\rho}(\Lambda_\sigma[U]R^\sigma[U]) \equiv \mathrm{L}^{*\sigma}_{\ \rho}[U]\Lambda_\sigma[U] + F_\rho(R[U]) \tag{19}$$

with

$$F_\rho(R[U]) = \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho}R^\sigma[U] - \mathrm{D}_i\left(\frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho}R^\sigma[U]\right) + \cdots$$
$$+ (-1)^l \mathrm{D}_{i_1}\cdots\mathrm{D}_{i_l}\left(\frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1\cdots i_l}^\rho}R^\sigma[U]\right), \qquad \rho = 1, \ldots, m. \tag{20}$$

From (19), it follows that $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ yields a set of local CL multipliers for the PDE system (2) if and only if the right-hand side of (19) vanishes for arbitrary $U(x)$. Moreover, since the expressions (20) vanish on any solution $U(x) = u(x)$ of $\mathbf{R}\{x; u\}$ (2), it follows that every set of local CL multipliers $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ of the PDE system (2) must be a solution of its adjoint system of PDEs, which is the adjoint of its linearizing system of PDEs, when $U(x) = u(x)$ is a solution of $\mathbf{R}\{x; u\}$ (2), i.e.,

$$\mathrm{L}^{*\sigma}_{\rho}[u]\Lambda_\sigma[u] = 0, \qquad \rho = 1, \ldots, m. \tag{21}$$

The proof of the following theorem follows directly from expression (19).

Theorem 5 *Consider a given PDE system (2). A set of functions* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ *yields a set of local CL multipliers for PDE system (2) if and only if the identities*

$$\mathrm{L}^{*\sigma}_{\rho}[U]\Lambda_\sigma[U] + \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - \mathrm{D}_i\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_i} R^\sigma[U]\right) + \cdots$$
$$+ (-1)^l \mathrm{D}_{i_1} \cdots \mathrm{D}_{i_l}\left(\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho_{i_1\cdots i_l}} R^\sigma[U]\right) \equiv 0, \qquad \rho = 1, \ldots, m$$

hold for m arbitrary functions $U(x) = (U^1(x), \ldots, U^m(x))$ *in terms of the components* $\{\mathrm{L}^{*\sigma}_{\rho}[U]\}$ *of the adjoint operator of the linearizing operator (Fréchet derivative) for the given PDE system (2).*

The derivation leading to Eq. (21) can be summarized in terms of the following theorem.

Theorem 6 *Consider a given PDE system (2). Suppose one has a set of local CL multipliers* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ *for the PDE system (2). Let* $\{\mathrm{L}^{*\sigma}_{\rho}[U]\}$ *be the components of the adjoint operator of the linearizing operator (Fréchet derivative) for the PDE system (2) and let* $U(x) = u(x) = (u^1(x), \ldots, u^m(x))$ *be any solution of the PDE system (2). Then* $\mathrm{L}^{*\sigma}_{\rho}[u]\Lambda_\sigma[u] = 0$.

The Situation When the Linearizing Operator is Self-adjoint

Definition 3 Let $\mathrm{L}[U]$, with its components $\mathrm{L}^{\sigma}_{\rho}[U]$, be the linearizing operator associated with a PDE system $\mathbf{R}\{x; u\}$ (2). The adjoint operator of $\mathrm{L}[U]$ is $\mathrm{L}^*[U]$, with components $\mathrm{L}^{*\sigma}_{\rho}[U]$. $\mathrm{L}[U]$ is a *self-adjoint* operator if and only if $\mathrm{L}[U] \equiv \mathrm{L}^*[U]$, i.e., $\mathrm{L}^{\sigma}_{\rho}[U] \equiv \mathrm{L}^{*\sigma}_{\rho}[U]$, $\sigma, \rho = 1, \ldots, m$.

One can show that a given PDE system, *as written*, has a variational formulation if and only if its associated linearizing operator is self-adjoint [8, 42, 43].

If the linearizing operator associated with a given PDE system is self-adjoint, then each set of local CL multipliers yields a local symmetry of the given PDE system. In particular, one has the following theorem.

Theorem 7 *Consider a given PDE system* $\mathbf{R}\{x; u\}$ *(2) with* $N = m$*, i.e., the number of dependent variables appearing in PDE system (2) is the same as the number of equations in PDE system (2). Suppose the associated linearizing operator* $\mathrm{L}[U]$ *for PDE system (2) is self-adjoint. Let* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^m$ *be a set of local CL multipliers for (2). Let*

$$\eta^\sigma(x, u, \partial u, \ldots, \partial^l u) = \Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u), \quad \sigma = 1, \ldots, m$$

where $U(x) = u(x)$ *is any solution of the PDE system (2). Then*

$$\eta^\sigma(x, u, \partial u, \ldots, \partial^l u)\frac{\partial}{\partial u^\sigma} \tag{22}$$

is a local symmetry of $\mathbf{R}\{x; u\}$.

Proof Since the hypothesis of Theorem 6 is satisfied with $\mathrm{L}[U] = \mathrm{L}^*[U]$, from the equations of this theorem it follows that in terms of the components of the associated linearizing operator $\mathrm{L}[U]$, one has

$$\mathrm{L}_\rho^\sigma[u]\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u) = 0, \qquad \rho = 1, \ldots, m \tag{23}$$

where $u = \theta(x)$ is any solution of the given PDE system (2). But the set of Eq. (23) is the set of determining equations for a local symmetry $\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u)\frac{\partial}{\partial u^\sigma}$ of PDE system (2). Hence (22) is a local symmetry of PDE system (2). □

The converse of Theorem 7 is false. In particular, suppose $\eta^\sigma(x, u, \partial u, \ldots, \partial^l u)\frac{\partial}{\partial u^\sigma}$ is a local symmetry of a PDE system $\mathbf{R}\{x; u\}$ (2) with a self-adjoint linearizing operator $\mathrm{L}[U]$. Let $\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U) = \eta^\sigma(x, U, \partial U, \ldots, \partial^l U)$, $\sigma = 1, \ldots, m$, where $U(x) = (U^1(x), \ldots, U^m(x))$ is arbitrary. Then it does not necessarily follow that $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^m$ is a set of local CL multipliers of $\mathbf{R}\{x; u\}$. This can be seen as follows: In the self-adjoint case, the set of local symmetry determining equations is a subset of the set of local multiplier determining equations. Here *each* local symmetry yields a set of local CL multipliers if and only *each* solution of the set of local symmetry determining equations also solves the remaining set of local multiplier determining equations.

To illustrate the situation, consider the following example of a nonlinear PDE whose linearizing operator is self-adjoint but the PDE has a point symmetry that does not yield a multiplier for a local CL

$$u_{tt} - u(uu_x)_x = 0. \tag{24}$$

It is easy to see that the PDE (24) has the scaling point symmetry $x \to \alpha x$, $u \to \alpha u$, corresponding to the infinitesimal generator

$$\mathrm{X} = (u - xu_x)\frac{\partial}{\partial u}. \tag{25}$$

The self-adjoint linearizing operator associated with PDE (24) is given by

$$\mathrm{L}[U] = \mathrm{D}_t^2 - U^2\mathrm{D}_x^2 - 2UU_x\mathrm{D}_x - 2UU_{xx} - U_x^2.$$

The determining equation for the local CL multipliers $\Lambda(t, x, U, U_t, U_x)$ of the PDE (24) is an identity holding for all values of the variables t, x, U, U_t, U_x, U_{tt}, U_{tx}, U_{xx}, U_{ttt}, U_{ttx}, U_{txx}, U_{xxx}, and splits into a system of two equations consisting of

$$\tilde{\mathrm{D}}_t^2\Lambda - U^2\mathrm{D}_x^2\Lambda - 2UU_x\mathrm{D}_x\Lambda - (2UU_{xx} + U_x^2)\Lambda = 0 \tag{26}$$

and

$$2\Lambda_U + \tilde{\mathrm{D}}_t\Lambda_{U_t} - \mathrm{D}_x\Lambda_{U_t} = 0 \tag{27}$$

in terms of the "restricted" total derivative operator $\tilde{\mathrm{D}}_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + U_{tx}\frac{\partial}{\partial U_x} + g[U]\frac{\partial}{\partial U_t} + U_{txx}\frac{\partial}{\partial U_{xx}} + \mathrm{D}_t(g[U])\frac{\partial}{\partial U_{tt}}$ where $g[U] = U(UU_x)_x$.

Equation (26) is the determining equation for $\Lambda(t, x, u, u_t, u_x)\frac{\partial}{\partial u}$ to be a contact symmetry of the given PDE (24). If the contact symmetry satisfies the second determining Eq. (27) then it yields a local CL multiplier $\Lambda(t, x, U, U_t, U_x)$ of PDE (24). It is easy to check that the scaling symmetry (25) obviously satisfies the contact symmetry determining Eq. (26) but does not satisfy the second determining Eq. (27) when $u(x, t)$ is replaced by an arbitrary function $U(x, t)$. Hence the scaling symmetry (25) does not yield a local conservation law of PDE (24).

3.3 Noether's Theorem

In 1918, Emmy Noether presented her celebrated procedure (*Noether's theorem*) to find local CLs for a DE system that admits a variational principle.

When a given DE system admits a variational principle, then the extremals of the associated action functional yield the given DE system (the *Euler-Lagrange equations*). In this case, Noether showed that if a one-parameter local transformation leaves invariant the action functional (action integral), then one obtains the fluxes of a local CL through an explicit formula that involves the infinitesimals of the local transformation and the Lagrangian (Lagrangian density) of the action functional.

3.3.1 Euler-Lagrange Equations

Consider a functional $J[U]$ in terms of n independent variables $x = (x^1, \ldots, x^n)$ and m arbitrary functions $U = (U^1(x), \ldots, U^m(x))$ and their partial derivatives to order k, defined on a domain Ω

$$J[U] = \int_{\Omega} L[U]dx = \int_{\Omega} L(x, U, \partial U, \ldots, \partial^k U)dx. \tag{28}$$

In (28), the function $L[U] = L(x, U, \partial U, \ldots, \partial^k U)$ is called a *Lagrangian* and the functional $J[U]$ is called an *action integral*.

Consider an infinitesimal change $U(x) \to U(x) + \varepsilon v(x)$ where $v(x)$ is any function such that $v(x)$ and its derivatives to order $k-1$ vanish on the boundary $\partial\Omega$ of the domain Ω. The corresponding infinitesimal change (variation) in the Lagrangian $L[U]$ is given by

$$\begin{aligned}\delta L &= L(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \ldots, \partial^k U + \varepsilon \partial^k v) - L(x, U, \partial U, \ldots, \partial^k U) \\ &= \varepsilon \left(\frac{\partial L[U]}{\partial U^i} v^i + \frac{\partial L[U]}{\partial U^i_j} v^i_j + \cdots + \frac{\partial L[U]}{\partial U^i_{j_1 \cdots j_k}} v^i_{j_1 \cdots j_k} \right) + O(\varepsilon^2).\end{aligned} \tag{29}$$

Let

$$\begin{aligned}W^l[U, v] = v^i &\left(\frac{\partial L[U]}{\partial U^i_l} + \cdots + (-1)^{k-1} \mathrm{D}_{j_1} \cdots \mathrm{D}_{j_{k-1}} \frac{\partial L[U]}{\partial U^i_{l j_1 \cdots j_{k-1}}} \right) \\ &+ v^i_{j_1} \left(\frac{\partial L[U]}{\partial U^i_{j_1 l}} + \cdots + (-1)^{k-2} \mathrm{D}_{j_2} \cdots \mathrm{D}_{j_{k-1}} \frac{\partial L[U]}{\partial U^i_{j_1 l j_2 \cdots j_{k-1}}} \right) \\ &+ \cdots + v^i_{j_1 \cdots j_{k-1}} \frac{\partial L[U]}{\partial U^i_{j_1 j_2 \cdots j_{k-1} l}}.\end{aligned} \tag{30}$$

After repeatedly using integration by parts, one can show that

$$\delta L = \varepsilon (v^i \mathrm{E}_{U^i}(L[U]) + \mathrm{D}_l W^l[U, v]) + O(\varepsilon^2) \tag{31}$$

where E_{U^i} is the Euler operator with respect to U^i. The corresponding variation in the action integral $J[U]$ is given by

$$\begin{aligned}\delta J = J[U + \varepsilon v] - J[U] &= \int_{\Omega} \delta L dx \\ &= \varepsilon \int_{\Omega} (v^i \mathrm{E}_{U^i}(L[U]) + \mathrm{D}_l W^l[U, v]) dx + O(\varepsilon^2) \\ &= \varepsilon \Big(\int_{\Omega} v^i \mathrm{E}_{U^i}(L[U]) dx + \int_{\partial\Omega} W^l[U, v] n^l d\sigma \Big) + O(\varepsilon^2).\end{aligned} \tag{32}$$

Hence if $U(x) = u(x)$ extremizes the action integral $J[U]$, then the $O(\varepsilon)$ term in δJ must vanish. Thus $\int_{\Omega} v^i \mathrm{E}_{u^i}(L[u]) \mathrm{d}x = 0$ for an *arbitrary* function $v(x)$ defined on the domain Ω. Hence, if $U(x) = u(x)$ extremizes the action integral $J[U]$, then

$u(x)$ must satisfy the PDE system

$$\mathrm{E}_{u^i}(L[u]) = \frac{\partial L[u]}{\partial u^i} + \cdots + (-1)^k \mathrm{D}_{j_1} \cdots \mathrm{D}_{j_k} \frac{\partial L[u]}{\partial u^i_{j_1 \cdots j_k}} = 0, \quad i = 1, \ldots, m. \quad (33)$$

The Eq. (33) are called the *Euler-Lagrange equations* satisfied by an extremum $U(x) = u(x)$ of the action integral $J[U]$. Thus the following theorem has been proved.

Theorem 8 *If a smooth function $U(x) = u(x)$ is an extremum of an action integral (28), then $u(x)$ satisfies the Euler-Lagrange equations (33).*

3.3.2 Standard Formulation of Noether's Theorem

Definition 4 In the *standard formulation of Noether's theorem*, the action integral (28) is invariant under the one-parameter Lie group of point transformations

$$\begin{aligned} (x^*)^i &= x^i + \varepsilon\xi^i(x, U) + O(\varepsilon^2), \qquad i = 1, \ldots, n \\ (U^*)^\mu &= U^\mu + \varepsilon\eta^\mu(x, U) + O(\varepsilon^2), \qquad \mu = 1, \ldots, m \end{aligned} \quad (34)$$

with infinitesimal generator $\mathrm{X} = \xi^i(x, U)\frac{\partial}{\partial x^i} + \eta^\mu(x, U)\frac{\partial}{\partial U^\mu}$, if and only if $\int_{\Omega^*} L[U^*]dx^* = \int_\Omega L[U]dx$ where Ω^* is the image of Ω under the Lie group of point transformations (34).

The *Jacobian* of the one parameter Lie group of point transformations (34) is given by $\mathrm{J} = \det(\mathrm{D}_i(x^*)^j) = 1 + \varepsilon \mathrm{D}_i \xi^i(x, U) + O(\varepsilon^2)$. Then $\mathrm{d}x^* = \mathrm{J}\mathrm{d}x$. Moreover, $L[U^*] = \mathrm{e}^{\varepsilon \mathrm{X}} L[U]$ in terms of the infinitesimal generator X. Consequently, in the standard formulation of Noether's theorem, X is a point symmetry of $J[U]$ if and only if

$$0 = \int_\Omega (\mathrm{J}\mathrm{e}^{\varepsilon \mathrm{X}} - 1)L[U]dx = \varepsilon \int_\Omega (L[U]\mathrm{D}_i\xi^i(x, U) + \mathrm{X}^{(k)}L[U])dx + O(\varepsilon^2) \quad (35)$$

holds for arbitrary $U(x)$ where $\mathrm{X}^{(k)}$ is the k-th extension (prolongation) of the infinitesimal generator X. Hence, if X is a point symmetry of $J[U]$, then the $O(\varepsilon)$ term in (35) must vanish. Thus $L[U]\mathrm{D}_i\xi^i(x, U) + \mathrm{X}^{(k)}L[U] \equiv 0$.

The one-parameter Lie group of point transformations (34) with infinitesimal generator X is equivalent to the one-parameter family of transformations in evolutionary form given by

$$\begin{aligned} (x^*)^i &= x^i, \qquad i = 1, \ldots, n \\ (U^*)^\mu &= U^\mu + \varepsilon[\eta^\mu(x, U) - U^\mu_i \xi^i(x, U)] + O(\varepsilon^2), \qquad \mu = 1, \ldots, m \end{aligned} \quad (36)$$

with k-th extended infinitesimal generator $\hat{\mathrm{X}}^{(k)} = \hat{\eta}^\mu[U]\frac{\partial}{\partial U^\mu} + \cdots$. Under transformation (36), $U(x) \to U(x) + \varepsilon v(x)$ has components $v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x, U) - U_i^\mu \xi^i(x, U)$. Hence $\delta L = \varepsilon \hat{\mathrm{X}}^{(k)} L[U] + O(\varepsilon^2)$. Thus

$$\int_\Omega \delta L dx = \varepsilon \int_\Omega \hat{\mathrm{X}}^{(k)} L[U] dx + O(\varepsilon^2). \tag{37}$$

Consequently, after setting $v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x, U) - U_i^\mu \xi^i(x, U)$, and comparing expressions (32) and (37), it follows that

$$\hat{\mathrm{X}}^{(k)} L[U] \equiv \hat{\eta}^\mu[U]\mathrm{E}_{U^\mu}(L[U]) + \mathrm{D}_l W^l[U, \hat{\eta}[U]]. \tag{38}$$

By direct calculation, one can show the following.

Lemma 1 *Let $F[U] = F(x, U, \partial U, \ldots, \partial^k U)$ be an arbitrary function of its arguments. Then, in terms of the extended infinitesimal generators $\mathrm{X}^{(k)}$ and $\hat{\mathrm{X}}^{(k)}$, one has the identity*

$$\mathrm{X}^{(k)} F[U] + F[U]\mathrm{D}_i \xi^i(x, U) \equiv \hat{\mathrm{X}}^{(k)} F[U] + \mathrm{D}_i(F[U]\xi^i(x, U)). \tag{39}$$

Theorem 9 Standard formulation of Noether's theorem. *Suppose a given PDE system is derivable from a variational principle, i.e., the given PDE system is a set of Euler-Lagrange equations (33) whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ with Lagrangian $L[U]$. Suppose the one-parameter Lie group of point transformations (34) with infinitesimal generator* X *leaves invariant $J[U]$. Then*

1. *The identity*

$$\hat{\eta}^\mu[U]\mathrm{E}_{U^\mu}(L[U]) \equiv -\mathrm{D}_i(\xi^i(x, U)L[U] + W^i[U, \hat{\eta}[U]]) \tag{40}$$

   holds for arbitrary functions $U(x)$, i.e., $\{\hat{\eta}[U]\}_{\mu=1}^m$ is a set of local CL multipliers of the Euler-Lagrange system (33).
2. *The local conservation law*

$$\mathrm{D}_i(\xi^i(x, u)L[u] + W^i[u, \hat{\eta}[u]]) = 0 \tag{41}$$

   holds for any solution $u = \theta(x)$ of the Euler-Lagrange system (33).

Proof Let $F[U] = L[U]$ in the identity in Lemma 1. Then the identity

$$\hat{\mathrm{X}}^{(k)} L[U] + \mathrm{D}_i(L[U]\xi^i(x, U)) \equiv 0 \tag{42}$$

holds for arbitrary functions $U(x)$. Substitution for $\hat{\mathrm{X}}^{(k)} L[U]$ in (42) through (38) yields the identity (40). If $U(x) = u(x)$ solves the Euler-Lagrange system (33), then the left-hand-side of equation (40) vanishes. This yields the local conservation law (41). □

3.3.3 Extended Formulation of Noether's Theorem

One can extend the standard formulation of Noether's theorem to find additional local conservation laws arising from invariance under higher-order transformations through a generalization of Definition 4 for the invariance of an action integral $J[U]$. Here the action integral $J[U]$ is invariant under a one-parameter family of higher-order transformations if its integrand $L[U]$ is invariant to within a divergence.

Definition 5 Let $\hat{\mathrm{X}} = \hat{\eta}^\mu(x, U, \partial U, \ldots, \partial^s U)\frac{\partial}{\partial U^\mu}$ be the infinitesimal generator of a one-parameter family of local transformations (36) in evolutionary form with infinite extension $\hat{\mathrm{X}}^{(\infty)}$. Let $\hat{\eta}^\mu[U] = \hat{\eta}^\mu(x, U, \partial U, \ldots, \partial^s U)$. Here $\hat{\mathrm{X}}$ is a local symmetry of $J[U]$ if and only if the identity

$$\hat{\mathrm{X}}^{(\infty)} L[U] \equiv \mathrm{D}_i A^i[U] \tag{43}$$

holds for some set of functions $A^i[U] = A^i(x, U, \partial U, \ldots, \partial^r U)$, $i = 1, \ldots, n$.

Theorem 10 Extended formulation of Noether's theorem. *Suppose a given PDE system is derivable from a variational principle, i.e., the given PDE system is a set of Euler-Lagrange equations (33) whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ with Lagrangian $L[U]$. Suppose $\hat{\mathrm{X}} = \hat{\eta}^\mu[U]\frac{\partial}{\partial U^\mu}$ is a local symmetry of $J[U]$. Then*

1. *The identity*
$$\hat{\eta}^\mu[U]\mathrm{E}_{U^\mu}(L[U]) \equiv \mathrm{D}_i(A^i[U] - W^i[U, \hat{\eta}[U]]) \tag{44}$$
holds for arbitrary functions $U(x)$, i.e., $\{\hat{\eta}^\mu[U]\}_{\mu=1}^m$ is a set of local CL multipliers for the Euler-Lagrange system (33).
2. *The local conservation law*
$$\mathrm{D}_i(W^i[u, \hat{\eta}[u]] - A^i[u]) = 0 \tag{45}$$
holds for any solution $u = \theta(x)$ of the Euler-Lagrange system (33).

Proof For the one-parameter family of local transformations (36) with infinitesimal generator $\hat{\mathrm{X}} = \hat{\eta}^\mu[U]\frac{\partial}{U^\mu}$, it follows that the corresponding infinitesimal change $U(x) \to U(x) + \varepsilon v(x)$ has components $v^\mu(x) = \hat{\eta}^\mu[U]$. Consequently, $\delta L = \varepsilon\hat{\mathrm{X}}^{(\infty)} L[U] + O(\varepsilon^2)$. But $\delta L = \varepsilon(\hat{\eta}^\mu[U]\mathrm{E}_{U^\mu}(L[U]) + \mathrm{D}_i(W^i[U, \hat{\eta}[U]])) + O(\varepsilon^2)$. Hence it immediately follows that the identity

$$\hat{\mathrm{X}}^{(\infty)} L[U] \equiv \hat{\eta}^\mu[U]\mathrm{E}_{U^\mu}(L[U]) + \mathrm{D}_i(W^i[U, \hat{\eta}[U]]) \tag{46}$$

holds for arbitrary functions $U(x)$. Since $\hat{\mathrm{X}} = \hat{\eta}^{\mu}[U]\frac{\partial}{U^{\mu}}$ is a local symmetry of $J[U]$, it follows that Eq. (43) holds. Substitution for $\hat{\mathrm{X}}^{(\infty)}L[U]$ in (46) through (43) yields the identity (44). If $U(x) = u(x)$ solves the Euler-Lagrange system (33), then the left-hand-side of Eq. (44) vanishes. This yields the local conservation law (45). □

The following theorem shows that any local conservation law obtained through the standard formulation of Noether's theorem can be obtained through the extended formulation of Noether's theorem.

Theorem 11 *If a local conservation law is obtained through the standard formulation of Noether's theorem, then this local conservation law can be obtained through its extended formulation.*

Proof Suppose the one-parameter Lie group of point transformations (34) with infinitesimal generator X yields a local CL of a given PDE system derivable from a variational principle with Euler-Lagrange system (33). Then the identity (42) holds. Consequently,

$$\hat{\mathrm{X}}^{(k)}L[U] = \hat{\mathrm{X}}^{(\infty)}L[U] = \mathrm{D}_i A^i[U] \tag{47}$$

where $A^i[U] = -\mathrm{D}_i(L[U]\xi^i(x, U))$. But Eq. (47) is just the condition for X to be a local symmetry of $J[U]$. Consequently, one obtains the same local conservation law from the extended formulation of Noether's theorem. □

3.3.4 Limitations of Noether's Theorem

There are several limitations in using Noether's theorem to find the local conservation laws of a given PDE system.

1. There is the difficulty of finding variational symmetries. To find the variational symmetries of a given DE system arising from a variational principle, first one determines the local symmetries $\mathrm{X} = \eta^{\sigma}[u]\frac{\partial}{\partial u^{\sigma}}$ of the Euler-Lagrange equations (33). Then for each local symmetry, one checks if X leaves invariant the Lagrangian $L[U]$ to within a divergence. Note that since all local conservation laws, obtainable by Noether's theorem, arise from local CL multipliers, one can simply use the Direct Method to check whether a local symmetry is a variational symmetry.
2. A given system of DEs is not variational as written. A given system of differential equations, as written, is variational if and only if its linearized system (Fréchet derivative) is self-adjoint. Consequently, it is necessary, but far from sufficient, that a given system of DEs, *as written*, must be of even order, have the same number of equations in the system as its number of dependent variables and be non-dissipative to directly admit a variational principle.
3. Artifices can make a given system of DEs variational that is not variational, as written. Such artifices include

- The use of multipliers. As an example, the PDE

$$u_{tt} + H'(u_x)u_{xx} + H(u_x) = 0 \tag{48}$$

as written, does not admit a variational principle since its linearized equation $\varsigma_{tt} + H'(u_x)\varsigma_{xx} + (H''(u_x) + H'(u_x))\varsigma_x = 0$ is not self-adjoint. However, the equivalent PDE $e^x[u_{tt} + H'(u_x)u_{xx} + H(u_x)] = 0$, obtained after multiplying PDE (48) by e^x, is self-adjoint!

- The use of a contact transformation. As an example, the ODE

$$y'' + 2y' + y = 0 \tag{49}$$

as written, obviously does not admit a variational principle. But the point transformation $x \to X = x$, $y \to Y = ye^x$, maps the ODE (49) to the variational ODE $Y'' = 0$. However, it is well-known that every second order ODE, written in solved form, can be mapped into $Y'' = 0$ by some contact transformation but there is no finite algorithm to find such a transformation.

- The use of a differential substitution. As an example, the KdV equation (11), as written, obviously does not admit a variational principle since it is of odd order. But the well-known differential substitution $u = v_x$ yields the equivalent transformed KdV equation $v_{xxxx} + v_x v_{xx} + v_{xt} = 0$, that is the Euler-Lagrange equation for an extremum $V(x,t) = v(x,t)$ of the action integral with Lagrangian $L[V] = \frac{1}{2}(V_{xx})^2 - \frac{1}{6}(V_x)^3 - \frac{1}{2}V_x V_t$.

4. Noether's theorem is coordinate-dependent. The use of Noether's theorem to obtain a local conservation law is coordinate-dependent since the action of a contact transformation can transform a DE having a variational principle to one that does not have one. *On the other hand it is well-known that local conservation laws are coordinate-independent in the sense that a contact transformation maps a local CL of a given DE into a local CL of the transformed DE.*
5. The artifice of a Lagrangian itself for finding the local CLs of a given DE system. One should be able to expect to directly find the local conservation laws of a given DE system without the need to find a related action integral whether or not the given DE system is variational.

3.4 Further Comments on the Direct Method to Find Local Conservation Laws vis-á-vis Noether's Theorem

The Direct Method to find local CLs addresses limitations of Noether's theorem as follows.

1. In principle, the Direct Method can be used to find local conservation laws for *any* DE system, no matter how it is written, whereas the direct application of Noether's theorem requires the linearized system of a given DE system to be self-adjoint.

Essentially, the Direct Method finds all local CLs of a given DE system. Note that Noether's theorem can only be used to find local CLs. As seen in Theorems 9 and 10, Noether's theorem is also a multiplier method.

2. In the Direct Method, no functional is required unlike the situation for Noether's theorem. Local CLs are constructed directly. In the Direct Method, local CL multipliers correspond to symmetries of a given DE system if and only if its linearization operator is self-adjoint.

Example 1 Klein-Gordon Equation

As an example to compare the use of Noether's theorem and the Direct Method to find local CLs, consider the Klein-Gordon equation

$$u_{tx} - u^n = 0, \qquad n \neq 0, 1. \tag{50}$$

The PDE (50) has the scaling point symmetry

$$x^* = \alpha^{1-n} x, \qquad t^* = t, \qquad u^* = \alpha u \tag{51}$$

with the corresponding infinitesimal generator $\mathrm{X} = (u - (1-n)xu_x)\frac{\partial}{\partial u}$. One can show that the Klein-Gordon equation (50) is variational with action functional $J[U] = \int L[U]dtdx$; $L[U] = -\frac{1}{2}U_tU_x + \frac{1}{n+1}U^{n+1}$. We now show that the point symmetry (51) of the PDE (50) does not yield a local CL of this PDE from the presented three points of view.

1. Standard formulation of Noether's theorem. Let $x^* = \alpha^{1-n}x$, $t^* = t$, $U^* = \alpha U$. Then $J[U^*] = J[\alpha U] = \int L[U^*]dt^*dx^* = \alpha^{1-n}\int L[\alpha U]dtdx$. But $L[\alpha U] = \alpha^{1+n}L[U]$. Hence $J[U^*] = \alpha^2 J[U] \neq J[U]$ for any value of $\alpha \neq 1$. Thus the point symmetry (51) of the Klein-Gordon equation (50) yields no local CL.
2. Extended formulation of Noether's theorem. Here, by direct calculation, one can show that the extended infinitesimal generator $\mathrm{X}^{(\infty)}$ of the infinitesimal generator X of the point symmetry (51) yields

$$\begin{aligned}\mathrm{X}^{(\infty)}L[U] = U^n(U - xU_x(1-n)) - \tfrac{1}{2}(U_x(U_t - xU_{xt}(1-n)\\ + U_t(U_x - xU_{xx}(1-n))).\end{aligned} \tag{52}$$

The right-hand side of the expression (52) does not yield a divergence. The best way to show this is through applying the Euler operator with respect to U to the right-hand side of (52). In particular, $\mathrm{E}_U(\mathrm{X}^{(\infty)}L[U]) \equiv 2(U_{xt} + U^n) \neq 0$. Hence the extended formulation of Noether's theorem yields no local CL.
3. Application of the Direct Method. Here $\mathrm{E}_U[(U - xU_x(1-n))(U_{tx} - U)] \neq 0$ for an arbitrary function $U(x,t)$. Hence the point symmetry (51) of the Klein-Gordon equation (50) yields no local CL multiplier and thus no local CL.

Example 2 Nonlinear Wave Equation

Now we use the nonlinear wave equation

$$u_{tt} - (c^2(u)u_x)_x = 0 \tag{53}$$

as an example to show how the Direct Method finds the fluxes for a local CL from a known local CL multiplier. In particular, one can show that $\Lambda[U] = xt$ is a local CL multiplier for the PDE (53). Then

$$xt(U_{tt} - (c^2(U)U_x)_x) = \mathrm{D}_t(T[U]) + \mathrm{D}_x(X[U]) \tag{54}$$

for some functions $T[U] = T(x, t, U, U_x, U_t)$, $X[U] = X(x, t, U, U_x, U_t)$. Consequently, the Eq. (54) becomes

$$\begin{aligned} xt(U_{tt} - 2c(U)c'(U)U_x^2 - c^2(U)U_{xx}) = T_t + T_U U_t + T_{U_t} U_{tt} + T_{U_x} U_{tx} \\ + X_x + X_U U_x + X_{U_t} U_{tx} + X_{U_x} U_{xx}. \end{aligned} \tag{55}$$

Equating to zero the coefficients of U_{xx}, U_{tt}, U_{tx}, U_x^2, U_t, U_x, and the rest of the terms in Eq. (55) straightforwardly yields the fluxes $T[U] = xtU_t - xU$, $X[U] = -xtc^2(U)U_x + t\int c^2(U)dU$.

3.5 Use of Symmetries to Seek Further Conservation Laws from a Known Conservation Law

It is now shown how any symmetry (discrete or continuous) of a given PDE system $\mathbf{R}\{x; u\}$ (2) maps any CL of (2) into a CL of (2). Usually, no additional CL of (2) is obtained.

A symmetry of a PDE system induces a symmetry that leaves invariant the linear determining system for its local CL multipliers. Hence it follows that if one determines the action of a symmetry on a set of local CL multipliers $\{\Lambda_\sigma[U]\}$ for a known local CL of $\mathbf{R}\{x; u\}$ to obtain another set of local CL multipliers $\{\hat{\Lambda}_\sigma[U]\}$, then a priori one can determine whether an additional local CL is obtained for $\mathbf{R}\{x; u\}$.

In particular, suppose the invertible point transformation

$$x = x(\tilde{x}, \tilde{u}), \qquad u = u(\tilde{x}, \tilde{u}) \tag{56}$$

with its inverse transformation given by $\tilde{x} = \tilde{x}(x, u)$, $\tilde{u} = \tilde{u}(x, u)$, is a symmetry of a PDE system (2). Then corresponding to each PDE in (2), with solutions $u(x)$ replaced by arbitrary functions $U(x)$, and $\tilde{u}(x)$ replaced by $\tilde{U}(x)$, one has

$$R^\alpha[U] = A^\alpha_\beta[\tilde{U}]R^\beta[\tilde{U}] \tag{57}$$

holding for some set of functions $\{A^\alpha_\beta[U]\}$. Consequently, by direct calculation, one can prove the following theorem. For details, see [2, 24].

Theorem 12 *Under a point transformation (56), with $u(x)$ replaced by $U(x)$ and $\tilde{u}(x)$ replaced by $\tilde{U}(x)$, in terms of any given set of functions $\{\Phi^i[U]\}$, there exists a corresponding set of functions $\{\Psi^i[\tilde{U}]\}$ such that*

$$\mathrm{J}[\tilde{U}]\mathrm{D}_i\Phi^i[U] = \tilde{\mathrm{D}}_i\Psi^i[\tilde{U}] \tag{58}$$

where the Jacobian determinant

$$\mathrm{J}[\tilde{U}] = \frac{\mathrm{D}(x^1,\ldots,x^n)}{\mathrm{D}(\tilde{x}^1,\ldots,\tilde{x}^n)} = \begin{vmatrix} \tilde{\mathrm{D}}_1x^1 & \cdots & \tilde{\mathrm{D}}_1x^n \\ & \cdots & \\ \vdots & \vdots\ \vdots & \vdots \\ \tilde{\mathrm{D}}_nx^1 & & \tilde{\mathrm{D}}_nx^n \end{vmatrix} \tag{59}$$

and

$$\Psi^{i_1}[\tilde{U}] = \pm \begin{vmatrix} \Phi^1[U] & \Phi^2[U] & \cdots & \Phi^n[U] \\ \tilde{\mathrm{D}}_{i_2}x^1 & & \cdots & \tilde{\mathrm{D}}_{i_2}x^n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\mathrm{D}}_{i_n}x^1 & & \cdots & \tilde{\mathrm{D}}_{i_n}x^n \end{vmatrix}. \tag{60}$$

By direct calculation, one can prove the following theorem with details appearing in [24].

Theorem 13 *Suppose the point transformation (56) is a symmetry of $\mathbf{R}\{x;u\}$ (2) and $\{\Lambda_\sigma[U]\}$ is a set of local CL multipliers for $\mathbf{R}\{x;u\}$ with fluxes $\{\Phi^i[U]\}$. Then*

$$\hat{\Lambda}_\beta[\tilde{U}]R^\beta[\tilde{U}] = \tilde{\mathrm{D}}_i\Psi^i[\tilde{U}] \tag{61}$$

where

$$\hat{\Lambda}_\beta[\tilde{U}] = \mathrm{J}[\tilde{U}]A^\alpha_\beta[\tilde{U}]\Lambda_\alpha[U], \qquad \beta = 1,\ldots,N \tag{62}$$

with the components of the derivatives in $\{\Lambda_\alpha[U]\}$ expressed in terms of the prolongation of the point transformation (56). In Eq. (61), the functions $\Psi^i[\tilde{U}]$ are yielded by determinant (60). In Eq. (62), the functions $A^\alpha_\beta[\tilde{U}]$ are obtained through Eq. (57), and the Jacobian $\mathrm{J}[\tilde{U}]$ is yielded by the determinant (59).

After replacing $\tilde{x}^i$ by x^i, $\tilde{U}^\alpha$ by U^α, etc., in Eq. (62), one obtains the following corollary.

Corollary 1 *If $\{\Lambda_\alpha[U]\}$ is a set of local CL multipliers for the PDE system $\mathbf{R}\{x;u\}$ (2) that has the symmetry (56), then $\{\hat{\Lambda}_\beta[U]\}$ yields a set of local CL multipliers for $\mathbf{R}\{x;u\}$ where $\{\hat{\Lambda}_\beta[U]\}$ is given by (62) after replacing $\tilde{x}^i$ by x^i, $\tilde{U}^\sigma$ by U^σ, $\tilde{U}^\sigma_i$ by U^σ_i, etc. The set of local CL multipliers $\{\hat{\Lambda}_\beta[U]\}$ yields a new local CL of PDE* *system (2) if and only if this set is nontrivial on all solutions $U = u(x)$ of PDE system (2), i.e., $\hat{\Lambda}_\beta[u] \not\equiv c\Lambda_\beta[u]$, $\beta = 1, \ldots, N$, for some constant c.*

Now suppose the symmetry (56) is a one-parameter Lie group of point transformations

$$x = x(\tilde{x}, \tilde{U}; \varepsilon) = \mathrm{e}^{\varepsilon\tilde{\mathrm{X}}}\tilde{x}, \qquad U = U(\tilde{x}, \tilde{U}; \varepsilon) = \mathrm{e}^{\varepsilon\tilde{\mathrm{X}}}\tilde{U} \tag{63}$$

in terms of its infinitesimal generator (and extensions) $\tilde{\mathrm{X}} = \xi^j(\tilde{x}, \tilde{U})\frac{\partial}{\partial\tilde{x}^j} + \eta^\sigma(\tilde{x}, \tilde{U})\frac{\partial}{\partial\tilde{U}^\sigma}$.

If Eq. (61) holds, then from Eq. (58) and the Lie group properties of (63), it follows that

$$\mathrm{J}[U; \varepsilon]\mathrm{e}^{\varepsilon\mathrm{X}}(\Lambda_\sigma[U]R^\sigma[U]) = \mathrm{D}_i\Psi^i[U; \varepsilon] \tag{64}$$

in terms of the infinitesimal generator (and its extensions) $\mathrm{X} = \xi^j(x, U)\frac{\partial}{\partial x^j} + \eta^\sigma(x, U)\frac{\partial}{\partial U^\sigma}$. Then, after expanding both sides of Eq. (64) in terms of power series in ε, one obtains an expression of the form

$$\sum_p \varepsilon^p\hat{\Lambda}_\sigma[U; p]R^\sigma[U] = \sum_p \varepsilon^p\mathrm{D}_i(\tfrac{1}{p!}\tfrac{\mathrm{d}^p}{\mathrm{d}\varepsilon^p}\Psi^i[U; \varepsilon])\Big|_{\varepsilon=0}. \tag{65}$$

Corresponding to the sequence of sets of local CL multipliers $\{\hat{\Lambda}_\sigma[U; p]\}$, $p = 1, 2, \ldots$, arising in expression (65), one obtains a sequence of local CLs

$$\mathrm{D}_i(\tfrac{\mathrm{d}^p}{\mathrm{d}\varepsilon^p}\Psi^i[u; \varepsilon])\Big|_{\varepsilon=0} = 0, \qquad p = 1, 2, \ldots$$

for PDE system (2) from its known local CL $\mathrm{D}_i\Phi^i[u] = 0$.

Example 1 A Nonlinear Telegraph System

Consider the nonlinear telegraph PDE system

$$\begin{aligned} v_t + (1 - 2\mathrm{e}^{2u})u_x - \mathrm{e}^u &= 0 \\ v_x - u_t &= 0. \end{aligned} \tag{66}$$

The PDE system (66) has the set of local CL multipliers

$$\begin{aligned} \Lambda_1 = \xi &= \mathrm{e}^{-\frac{1}{2}(U+t/\sqrt{2})}\sin(\tfrac{1}{2}(V + (x + 2\mathrm{e}^U)/\sqrt{2})) \\ \Lambda_2 = \varphi &= -\mathrm{e}^{-\frac{1}{2}(U+t/\sqrt{2})}(\sqrt{2}\mathrm{e}^U\sin(\tfrac{1}{2}(V + (x + 2\mathrm{e}^U)/\sqrt{2})) \\ &\quad + \cos(\tfrac{1}{2}(V + (x + 2\mathrm{e}^U)/\sqrt{2}))) \end{aligned}$$

and corresponding fluxes

$$
\begin{aligned}
T &= -2\mathrm{e}^{-\frac{1}{2}(u+t/\sqrt{2})}\cos(\tfrac{1}{2}(v+(x+2\mathrm{e}^{u})/\sqrt{2}))\\
X &= 2\mathrm{e}^{-\frac{1}{2}(u+t/\sqrt{2})}(\sqrt{2}\mathrm{e}^{u}\cos(\tfrac{1}{2}(v+(x+2\mathrm{e}^{u})/\sqrt{2}))\\
&\quad -\sin(\tfrac{1}{2}(v+(x+2\mathrm{e}^{u})/\sqrt{2}))).
\end{aligned}
$$

The nonlinear telegraph PDE system (66) obviously has the discrete reflection symmetry $(t, x, u, v) = (-\tilde{t}, \tilde{x}, \tilde{u}, -\tilde{v})$ and the translational point symmetry $(t, x, u, v) = (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v} + \varepsilon)$. One can show that for the above local CL of PDE system (66), these symmetries yield three additional local CLs as follows.

1. Reflection symmetry applied to the above local CL.
2. Translation symmetry applied to the above local CL.
3. Reflection symmetry applied again to the local CL found in (2).

For further details, see [41].

Example 2 Another Nonlinear Telegraph System

Consider another nonlinear telegraph PDE system given by

$$
v_t - (\operatorname{sech}^2 u)u_x + \tanh u = 0, \qquad v_x - u_t = 0. \tag{67}
$$

The PDE system (67) has the set of local CL multipliers

$$
\Lambda_1 = \xi = \mathrm{e}^{x}(2x + t^2 - V^2 - 2\log(\cosh U)), \qquad \Lambda_2 = \varphi = 2\mathrm{e}^{x}(V\tanh U - t)
$$

and corresponding fluxes

$$
\begin{aligned}
T &= \mathrm{e}^{x}(2tu - \tfrac{1}{3}v^3 + v(t^2 + 2x - 2\log(\cosh u)))\\
X &= \mathrm{e}^{x}((v^2 - t^2 - 2x + 2(1 + \log(\cosh u)))\tanh u - 2(vt + u)).
\end{aligned}
$$

The nonlinear telegraph PDE system (67) has the point symmetries with infinitesimal generators given respectively by

$$
\mathrm{X}_1 = \frac{\partial}{\partial t}, \qquad \mathrm{X}_2 = v\frac{\partial}{\partial t} + \tanh u\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + t\frac{\partial}{\partial v}.
$$

One can show that for the above local CL of PDE system (67), these two point symmetries yield three additional local CLs as follows.

1. The $O(\varepsilon)$, $O(\varepsilon^2)$ terms that result from applying the translation symmetry X_1 to the above local CL yield two additional local CLs.

2. The action of the second point symmetry X_2 on the additional $O(\varepsilon)$ local CL, obtained in (1), yields a third additional CL.

For further details, see [41].

4 Nonlocally Related Systems and Nonlocal Symmetries

Often a given PDE system has no local symmetry or no local conservation law. Even if a given PDE system has a local symmetry, it may not be useful for the problem at hand. The aim is to extend existing methods for finding local symmetries and local CLs to PDE systems that are nonlocally related and equivalent to a given PDE system in order to seek nonlocal symmetries and nonlocal CLs for a given PDE system. Two systematic and natural ways will be presented to find such nonlocally related systems for a given PDE system. In particular, it will be shown that for any PDE system, each local CL as well as each point symmetry systematically yields a nonlocally related system. Further systematic extensions for seeking additional nonlocally related systems will also be presented.

4.1 Conservation Law-based Method to Obtain Nonlocally Related Systems and Nonlocal Symmetries: Subsystems

Initially, we focus on the situation of a scalar PDE with two independent variables. As will be seen, no extra complication arises for a PDE system with two independent variables. But the situation for a PDE system with three or more independent variables is more complicated as will be seen in Sect. 5.

For a local conservation law

$$\mathrm{D}_t T(x, t, u, \partial u, \ldots, \partial^r u) + \mathrm{D}_x X(x, t, u, \partial u, \ldots, \partial^r u) = 0 \tag{68}$$

of a given scalar PDE

$$R[u] = R(x, t, u, \partial u, \ldots, \partial^k u) = 0 \tag{69}$$

one can form an equivalent augmented *potential system* P given by

$$\begin{aligned} \frac{\partial v}{\partial t} &= X(x, t, u, \partial u, \ldots, \partial^r u) \\ \frac{\partial v}{\partial x} &= -T(x, t, u, \partial u, \ldots, \partial^r u) \\ R(x, t, u, \partial u&, \ldots, \partial^k u) = 0. \end{aligned} \tag{70}$$

If $(u(x,t), v(x,t))$ solves the potential system P, then $u(x,t)$ solves the given scalar PDE (69). Conversely, if $u(x,t)$ solves the given scalar PDE (69), then there exists a solution $(u(x,t), v(x,t))$ of the potential system P since the integrability condition $v_{xt} = v_{tx}$ is satisfied due to the existence of the local CL (68). But the equivalence relationship is *nonlocal* and *non-invertible* since for any solution $u(x,t)$ of the given scalar PDE (69), if $(u(x,t), v(x,t))$ solves the potential system P, then so does $(u(x,t), v(x,t)+C)$ for any constant C.

Consequently, any symmetry (CL) of the given scalar PDE (69) yields a symmetry (CL) of the equivalent potential system P. Conversely, any symmetry (CL) of the potential system P yields a symmetry (CL) of the given scalar PDE (69).

Now suppose the equivalent potential system P has a point symmetry given by an infinitesimal generator

$$\xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \omega(x,t,u,v)\frac{\partial}{\partial u} + \varphi(x,t,u,v)\frac{\partial}{\partial v}\cdot \tag{71}$$

The point symmetry (71) of the potential system P yields a *nonlocal symmetry* of the given scalar PDE (69) if and only if its infinitesimal components satisfy the relationship

$$(\xi_v)^2 + (\tau_v)^2 + (\omega_v)^2 \not\equiv 0. \tag{72}$$

Hence, through a local CL of the PDE (69), a nonlocal symmetry of (69) can be obtained from a point symmetry (71) of the nonlocally related potential system P given by the PDE system (70) if the components of the point symmetry (71) satisfy the inequality (72).

The converse is also true. In particular, suppose a scalar PDE (69) has a point symmetry given by the infinitesimal generator

$$\alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \gamma(x,t,u)\frac{\partial}{\partial u}\cdot \tag{73}$$

The point symmetry (73) of the PDE (69) yields a *nonlocal symmetry* of the potential system P if and only if the potential system P has no corresponding point symmetry of the form $\alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \gamma(x,t,u)\frac{\partial}{\partial u} + \delta(x,t,u,v)\frac{\partial}{\partial v}$ for some function $\delta(x,t,u,v)$.

Next, we show how to obtain further nonlocally related systems for a given PDE system.

4.1.1 Use of n Local CLs to Obtain up to $2^n - 1$ Nonlocally Related Systems

Suppose there are n local CL multipliers $\{\Lambda_i(x,t,U,\partial U,\ldots,\partial^q U)\}_{i=1}^n$ yielding n independent local CLs of a given scalar PDE. Let v^i be the potential variable arising from the local CL multiplier $\Lambda_i[U]$. Then one obtains n *singlet* potential systems P^i, $i = 1,\ldots,n$. Moreover, one can consider potential systems in *couplets*

$\{P^i, P^j\}_{i,j=1}^n$ with two potential variables; in *triplets* $\{P^i, P^j, P^k\}_{i,j,k=1}^n$ with three potential variables; ...; in an *n-plet* $\{P^1, \ldots, P^n\}$ with n potential variables. Consequently from n local CLs of a given scalar PDE, one obtains $2^n - 1$ *distinct potential systems*!

Moreover, starting from *any* one of these $2^n - 1$ potential systems, one can continue the process. In particular, if one of these potential systems has N "local" CLs, in principle one could obtain up to $2^N - 1$ further distinct potential systems. However, not all local CLs of these $2^n - 1$ potential systems yield additional potential systems. *In particular, one can show that if a set of local CL multipliers depends only on independent variables (x, t) then no additional potential system is obtained.* See [2, 30, 31] for further details.

Any potential system could yield additional nonlocal symmetries or additional nonlocal CLs for any other potential system or the "given" PDE. Furthermore, one of the constructed potential systems could be a "given" PDE system. A more direct way of seeing this will be presented in Sect. 5 through the symmetry-based method for obtaining nonlocally related systems.

4.1.2 Nonlocally Related Subsystems

Definition 6 Suppose one has a given PDE system $\mathbf{S}\{x, t; u^1, \ldots, u^M\}$ with the indicated M dependent variables. A *subsystem* excluding a dependent variable, say u^M, is *nonlocally related* to the given system $\mathbf{S}\{x, t; u^1, \ldots, u^M\}$ if u^M cannot be directly expressed from the equations of $\mathbf{S}\{x, t; u^1, \ldots, u^M\}$ in terms of x, t, the remaining dependent variables $u^1, \ldots, u^{M-1}$, and their derivatives.

Subsystems for consideration can arise following an interchange of one or more of the dependent and independent variables of a given system $\mathbf{S}\{x, t; u^1, \ldots, u^M\}$. Consequently, for a given PDE system, one obtains a tree of nonlocally related (but equivalent) PDE systems arising from local conservation laws and subsystems. *Each PDE system in such a tree is equivalent in the sense that the solution set for any system in the tree can be found from the solution set for any other PDE system in the tree through a connection formula.* Due to the equivalence of the solution sets and the nonlocal relationship between PDE systems in a tree, it follows that any coordinate-independent method of analysis (quantitative, analytical, numerical, perturbation, etc.) when applied to some PDE system in a tree may yield simpler computations and/or results that cannot be obtained when the method of analysis is directly applied to any particular PDE system in a tree. In particular, it is important to note that a "given" system could be any system in such a tree!!

Example 1 Nonlinear Wave Equation

Suppose a given PDE $\mathbf{U}\{x, t; u\}$ is the nonlinear wave equation

$$u_{tt} = (c^2(u)u_x)_x. \tag{74}$$

Directly, one obtains the singlet potential system (local CL multiplier is 1) $\mathbf{UV}\{x, t; u, v\}$ given by

$$v_x - u_t = 0, \qquad v_t - c^2(u)u_x = 0. \tag{75}$$

Through the invertible point transformation (hodograph transformation) $x = x(u, v)$, $t = t(u, v)$, the potential system $\mathbf{UV}\{x, t; u, v\}$ becomes the invertibly equivalent PDE system $\mathbf{XT}\{u, v; x, t\}$ given by

$$x_v - t_u = 0, \qquad x_u - c^2(u)t_v = 0. \tag{76}$$

One can show that there are only three additional local CL multipliers of the form $\Lambda(x, t, U) = xt, x, t$ for the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74) for an *arbitrary* wave speed $c(u)$. This yields three additional singlet potential systems $\mathbf{UA}\{x, t; u, a\}$, $\mathbf{UB}\{x, t; u, b\}$, and $\mathbf{UW}\{x, t; u, w\}$, respectively given by the PDE systems

$$a_x - x[tu_t - u] = 0, \qquad a_t - t[xc^2(u)u_x - \int c^2(u)du] = 0 \tag{77}$$

$$b_x - xu_t = 0, \qquad b_t - [xc^2(u)u_x - \int c^2(u)du] = 0 \tag{78}$$

and

$$w_x - [tu_t - u] = 0, \qquad w_t - tc^2(u)u_x = 0. \tag{79}$$

Nonlocally related subsystems $\mathbf{T}\{u, v; t\}$ and $\mathbf{X}\{u, v; x\}$ arise from $\mathbf{UV}\{x, t; u, v\}$ through $\mathbf{XT}\{u, v; x, t\}$ after one respectively deletes the dependent variables x and t from $\mathbf{XT}\{u, v; x, t\}$

$$t_{vv} - c^{-2}(u)t_{uu} = 0 \tag{80}$$

and

$$x_{vv} - (c^{-2}(u)x_u)_u = 0. \tag{81}$$

One can show that the symmetry classifications of the PDEs (80) and (81) are "equivalent" [25]. Hence we concentrate on $\mathbf{T}\{u, v; t\}$. Since the PDE $\mathbf{T}\{u, v; t\}$ (80) is linear and self-adjoint, it follows that any solution of $\mathbf{T}\{u, v; t\}$ yields a local CL multiplier for $\mathbf{T}\{u, v; t\}$. Four of these local CL multipliers, for an arbitrary wave speed $c(u)$, are given by $\Lambda(u, v, T) = c^2(u), uc^2(u), vc^2(u), uvc^2(u)$. These yield three additional singlet potential systems $\mathbf{TP}\{u, v; t, p\}$, $\mathbf{TQ}\{u, v; t, q\}$, $\mathbf{TR}\{u, v; t, r\}$, respectively given by

$$p_v - (ut_u - t) = 0, \qquad p_u - uc^2(u)t_v = 0 \tag{82}$$

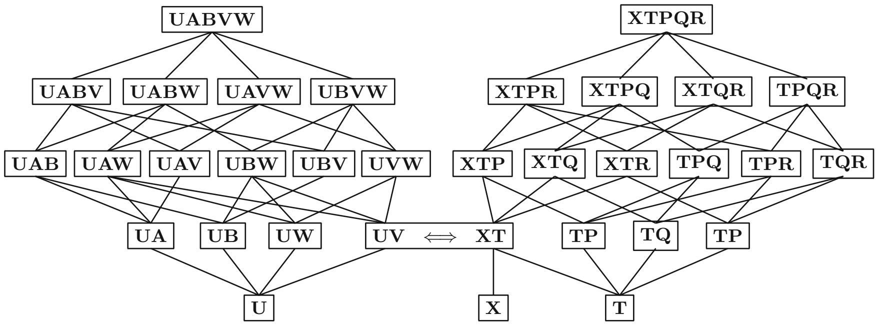

Fig. 1 A tree of nonlocally related systems for the nonlinear wave equation (74) for arbitrary wave speed $c(u)$

$$q_v - vt_u = 0, \qquad q_u + c^2(u)(t - vt_v) = 0 \tag{83}$$

and

$$r_v - v(ut_u - t) = 0, \qquad r_u - uc^2(u)(vt_v - t) = 0. \tag{84}$$

Consequently, one obtains the following (*far from exhaustive*) tree (Fig. 1) of nonlocally related systems for the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74), holding for an arbitrary wave speed $c(u)$.

The point symmetry classification for the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74) is given in [44]. The point symmetry classifications for the potential system $\mathbf{XT}\{u, v; x, t\}$ (76) (of course, it is exactly the same as that for the potential system $\mathbf{UV}\{x, t; u, v\}$ (75)) and the subsystem $\mathbf{T}\{u, v; t\}$ (80) is given in [25]. A partial point symmetry classification for the potential system $\mathbf{TP}\{u, v; t, p\}$ (82) can be adapted from results presented in [45]. The complete point symmetry classifications for the potential systems $\mathbf{UA}\{x, t; u, a\}$ (77), $\mathbf{UB}\{x, t; u, b\}$ (78), $\mathbf{UW}\{x, t; u, w\}$ (79), $\mathbf{TP}\{u, v; t, p\}$ (82), and $\mathbf{TQ}\{u, v; t, q\}$ (83) are given in [46]. Many nonlocal symmetries of the nonlinear wave equation are found from each of these nonlocally related systems in terms of specific forms of the nonlinear wave speed $c(u)$. In particular, the following additional nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74) have been found.

For the potential system $\mathbf{UB}\{x, t; u, b\}$ (78), setting $F(u) = \int c^2(u)\, du$, one finds that if $F(u)$ satisfies the ODE

$$\frac{F''(u)}{F'(u)^2} = \frac{4F(u) + 2C_1}{(F(u) + C_2)^2 + C_3}$$

in terms of arbitrary constants C_1, C_2, C_3, then the potential system $\mathbf{UB}\{x, t; u, b\}$ (78) has the point symmetry

$$X = (F(u)+C_1)x\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + \frac{(F(u)+C_2)^2+C_3}{F'(u)}\frac{\partial}{\partial u} + (2C_2 b - ({C_2}^2 + C_3)t)\frac{\partial}{\partial b}$$

that is a nonlocal symmetry of the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74).

For the potential system $\mathbf{UW}\{x, t; u, w\}$ (79), if the wave speed $c(u)$ satisfies the ODE

$$\frac{c'(u)}{c(u)} = -\frac{2u + C_1}{u^2 + C_2}$$

in terms of arbitrary constants C_1, C_2, then it has the point symmetry

$$X = w\frac{\partial}{\partial x} + (u + C_1)t\frac{\partial}{\partial t} + (u^2 + C_2)\frac{\partial}{\partial u} - C_2 x\frac{\partial}{\partial w}$$

that is a nonlocal symmetry of the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74).

The potential system $\mathbf{TP}\{u, v; t, p\}$ (82), for $c(u) = u^{-2}e^{1/u}$, has the point symmetries

$$X_1 = (pu - 2tv(u+1))\frac{\partial}{\partial t} - 2u^2 v\frac{\partial}{\partial u} + (u^2 + e^{2/u})\frac{\partial}{\partial v} + tu^{-1}e^{2/u}\frac{\partial}{\partial p}$$

$$X_2 = t(u+1)\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$$

that are both nonlocal symmetries of the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74).

For the potential system $\mathbf{TR}\{u, v; t, r\}$ (84), new nonlocal symmetries are found for $\mathbf{U}\{x, t; u\}$ (74) from the point symmetries of $\mathbf{TR}\{u, v; t, r\}$ when $c(u) = u^{-4/3}$.

For details and a table of listed nonlocal symmetries derived from the above tree of nonlocally related systems for the nonlinear wave equation $\mathbf{U}\{x, t; u\}$ (74), see [46].

Example 2 Nonlinear Telegraph Equation

Suppose a given PDE $\mathbf{U}\{x, t; u\}$ is the nonlinear telegraph (NLT) equation

$$u_{tt} - (F(u)u_x)_x - (G(u))_x = 0. \tag{85}$$

Case (a) For *arbitrary* $F(u)$, $G(u)$, one obtains two singlet potential systems $\mathbf{UV}_1\{x, t; u, v_1\}$ and $\mathbf{UV}_2\{x, t; u, v_2\}$ respectively given by the PDE systems

$$v_{1x} - u_t = 0, \qquad v_{1t} - (F(u)u_x + G(u)) = 0 \tag{86}$$

and

$$v_{2x} - (tu_t - u) = 0, \qquad v_{2t} - t(F(u)u_x + G(u)) = 0. \tag{87}$$

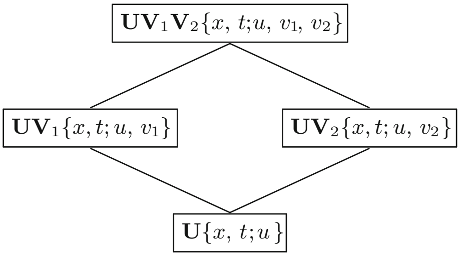

Fig. 2 Tree of nonlocally related PDE systems for the NLT equation (85) for arbitrary $F(u)$, $G(u)$

Case (b) For *arbitrary* $G(u)$, $F(u) = G'(u)$, one obtains two additional singlet potential systems $\mathbf{UB}_3\{x, t; u, b_3\}$ and $\mathbf{UB}_4\{x, t; u, b_4\}$ respectively given by the PDE systems

$$b_{3x} - e^x u_t = 0, \qquad b_{3t} - e^x F(u)u_x = 0 \tag{88}$$

and

$$b_{4x} - e^x(tu_t - u) = 0, \qquad b_{4t} - te^x F(u)u_x = 0. \tag{89}$$

Case (c) For *arbitrary* $F(u)$, $G(u) = u$, in addition to the singlet potential systems $\mathbf{UV}_1\{x, t; u, v_1\}$ (86) and $\mathbf{UV}_2\{x, t; u, v_2\}$ (87), one again obtains two further singlet potential systems $\mathbf{UC}_3\{x, t; u, c_3\}$ and $\mathbf{UC}_4\{x, t; u, c_4\}$ respectively given by the PDE systems

$$\begin{aligned} c_{3x} - ((x - \tfrac{1}{2}t^2)u_t + tu) &= 0 \\ c_{3t} - (x - \tfrac{1}{2}t^2)(F(u)u_x + u) + \int F(u)du &= 0 \end{aligned} \tag{90}$$

and

$$\begin{aligned} c_{4x} + (\tfrac{1}{6}t^3 - tx)u_t + (x - \tfrac{1}{2}t^2)u &= 0 \\ c_{4t} + (\tfrac{1}{6}t^3 - tx)(F(u)u_x + u) + t\int F(u)du &= 0. \end{aligned} \tag{91}$$

The corresponding trees of nonlocally related systems for the NLT equation are illustrated in Figs. 2 and 3.

In the cases where $F(u)$ and $G(u)$ are power law functions, see [47] for tabulations of nonlocal symmetries and nonlocal conservation laws for the NLT equation $\mathbf{U}\{x, t; u\}$ (85), arising for many of the above listed nonlocally related systems.

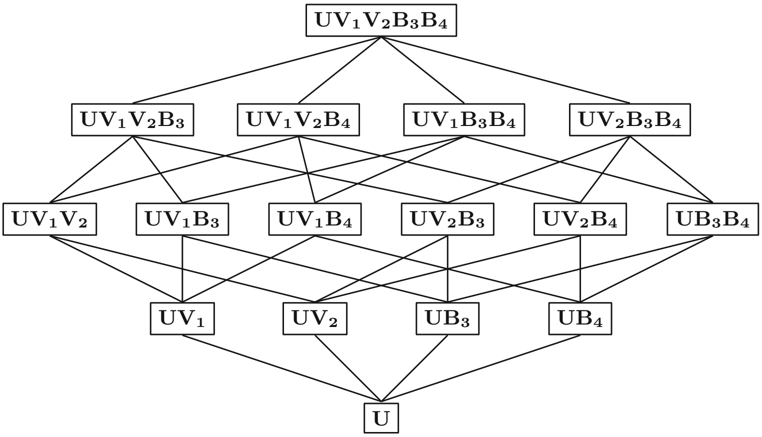

Fig. 3 Tree of nonlocally related PDE systems for the NLT equation (85) for arbitrary $G(u)$, $F(u) = G'(u)$

4.1.3 Conservation Law and Symmetry Classification Problems for the NLT Equation U$\{x, t; u\}$ and its Potential System UV$_1\{x, t; u, v_1\}$

Now we consider symmetry and conservation law classification problems for the NLT equation $\mathbf{U}\{x, t; u\}$ (85) and its potential system $\mathbf{UV}_1\{x, t; u, v_1\}$ (86). For specific $(F(u), G(u))$ pairs, the CL classification problem for $\mathbf{UV}_1\{x, t; u, v_1\}$ yields additional CLs and hence further potential systems for consideration [41].

Nonlocal Symmetries of $\mathbf{U}\{x, t; u\}$ *Arising from Point Symmetries of* $\mathbf{UV}_1\{x, t; u, v_1\}$.

The potential system $\mathbf{UV}_1\{x, t; u, v_1\}$ has a point symmetry corresponding to the infinitesimal generator

$$\mathrm{X} = \xi(x, t, u, v_1)\frac{\partial}{\partial x} + \tau(x, t, u, v_1)\frac{\partial}{\partial t} + \eta(x, t, u, v_1)\frac{\partial}{\partial u} + \varphi(x, t, u, v_1)\frac{\partial}{\partial v_1} \tag{92}$$

if and only if the coefficients of (92) satisfy the determining equations

$$\begin{aligned}
\xi_{v_1} - \tau_u &= 0\\
\eta_u - \varphi_{v_1} + \xi_x - \tau_t &= 0\\
G(u)[\eta_{v_1} + \tau_x] + \eta_t - \varphi_x &= 0\\
\xi_u - F(u)\tau_{v_1} &= 0\\
\varphi_u - G(u)\tau_u - F(u)\eta_{v_1} &= 0\\
G(u)\xi_{v_1} + \xi_t - F(u)\tau_x &= 0\\
F(u)[\varphi_{v_1} - \tau_t + \xi_x - \eta_u - 2G(u)\tau_{v_1}] - F'(u)\eta &= 0\\
G(u)[\varphi_{v_1} - \tau_t - G(u)\tau_{v_1}] - F(u)\eta_x - G'(u)\eta + \varphi_t &= 0
\end{aligned} \tag{93}$$

for *arbitrary* values of x, t, u, v_1.

The solution of the determining Eq. (93) appears in [48] and the resulting nonlocal symmetries for the NLT equation $\mathbf{U}\{x, t; u\}$ (85) are summarized by the following theorem.

Theorem 14 *A point symmetry of the potential system* $\mathbf{UV}_1\{x, t; u, v_1\}$ *(86) yields a nonlocal symmetry of the NLT equation* $\mathbf{U}\{x, t; u\}$ *(85) if and only if the pair of constitutive functions* $(F(u), G(u))$ *satisfies the first order ODE system*

$$\begin{aligned}
(c_3u + c_4)F'(u) - 2(c_1 - c_2 - G(u))F(u) &= 0\\
(c_3u + c_4)G'(u) + G^2(u) - (c_1 - 2c_2 + c_3)G(u) - c_5 &= 0
\end{aligned} \tag{94}$$

in terms of arbitrary constants $c_1, \ldots, c_5$. *For any pair* $(F(u), G(u))$ *satisfying (94), the potential system* $\mathbf{UV}_1\{x, t; u, v_1\}$ *(86) has the point symmetry (92) with*

$$\begin{aligned}
\xi &= c_1x + \int F(u)du\\
\tau &= c_2t + v_1\\
\eta &= c_3u + c_4\\
\varphi &= c_5t + (c_1 - c_2 + c_3)v_1
\end{aligned}$$

which is a (nonlocal) potential symmetry of the scalar NLT equation $\mathbf{U}\{x, t; u\}$ *(85).*

Modulo translations and scalings in u and G and scalings in F (involving 5/7 parameters), one obtains six distinct classes for $(F(u), G(u))$ for which the scalar NLT equation $\mathbf{U}\{x, t; u\}$ (85) has a potential symmetry. These classes are summarized in Table 1.

Point Symmetry Classification of the Scalar NLT Equation $\mathbf{U}\{x, t; u\}$ (85)

The NLT equation $\mathbf{U}\{x, t; u\}$ (85) has a point symmetry corresponding to the infinitesimal generator $\mathrm{X} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u}$ if and only if the determining equations

Table 1 Classification table for potential symmetries of the NLT equation (85)

Relationship	$G(u)$	$F(u)$
$F(u)=\frac{u^\beta}{\alpha}G'(u)$	$\frac{u^{2\alpha}-1}{u^{2\alpha}+1}$	$\frac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}+1)^2}$
	$\frac{u^{2\alpha}+1}{u^{2\alpha}-1}$	$-\frac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}-1)^2}$
$F(u)=\frac{u^\beta}{\alpha}G'(u)$	$\tan(\alpha\ln u)$	$u^{\beta-1}\sec^2(\alpha\ln u)$
$F(u)=u^\beta G'(u)$	$(\ln u)^{-1}$	$-u^{\beta-1}(\ln u)^{-2}$
$F(u)=e^{2\beta u}G'(u)$	$\tan u$	$e^{2\beta u}\sec^2 u$
$F(u)=e^{2\beta u}G'(u)$	$\tanh u$	$e^{2\beta u}\mathrm{sech}^2 u$
	$\coth u$	$-e^{2\beta u}\mathrm{csch}^2 u$
$F(u)=e^{2\beta u}G'(u)$	u^{-1}	$-u^{-2}e^{2\beta u}$

Table 2 Classes of $(F(u), G(u))$ yielding additional point symmetries of the scalar NLT equation $\mathbf{U}\{x,t;u\}$ (85)

$G(u)$	$F(u)$	Admitted additional point symmetries
e^u	$e^{(\alpha+1)u}$	$2\alpha x\frac{\partial}{\partial x}+(\alpha-1)t\frac{\partial}{\partial t}+2\frac{\partial}{\partial u}$
$u^{\alpha+\beta+1}$	u^α	$2\beta x\frac{\partial}{\partial x}+(\alpha+2\beta)t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}$
u^{-1}	u^{-2}	$t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u},\ e^x\frac{\partial}{\partial x}-ue^x\frac{\partial}{\partial u}$
$\ln u$	u^α	$2(\alpha+1)x\frac{\partial}{\partial x}+(\alpha+2)t\frac{\partial}{\partial t}+2u\frac{\partial}{\partial u}$
u	$e^{\alpha u}$	$2\alpha x\frac{\partial}{\partial x}+\alpha t\frac{\partial}{\partial t}+2\frac{\partial}{\partial u}$
u^{-3}	u^{-4}	$2t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u},\ t^2\frac{\partial}{\partial t}+tu\frac{\partial}{\partial u}$

$$\begin{aligned}
\xi_u=\tau_x=\tau_u=\eta_{uu}=\xi_t&=0\\
2F(u)[-\tau_t+\xi_x]-F'(u)\eta&=0\\
\eta_{tt}-F(u)\eta_{xx}-G'(u)\eta_x&=0\\
2\eta_{tu}-\tau_{tt}&=0\\
F(u)[2\eta_{xu}-\xi_{xx}]+\xi_{tt}+2F'(u)\eta_x-G'(u)[\xi_x-2\tau_t]+G''(u)\eta&=0
\end{aligned}$$

are satisfied for *arbitrary* values of x, t, and u.

For arbitrary $(F(u), G(u))$, the scalar NLT equation $\mathbf{U}\{x,t;u\}$ (85) is only invariant under translations in x and t. The classification of its point symmetries for specific forms of $(F(u), G(u))$, modulo scalings and translations in u, is presented in Table 2.

The following theorem holds. See [48] for details.

Theorem 15 *A point symmetry of the scalar NLT equation* $\mathbf{U}\{x,t;u\}$ *(85) yields a point symmetry of the NLT potential system* $\mathbf{UV}_1\{x,t;u,v_1\}$ *(86) for all cases except when* $(F(u),G(u))=(u^{-4},u^{-3})$. *In this case, its admitted point symmetry* $t^2\frac{\partial}{\partial t}+tu\frac{\partial}{\partial u}$ *yields a nonlocal symmetry of the NLT potential system* $\mathbf{UV}_1\{x,t;u,v_1\}$ *(86).*

Local Conservation Laws of the Potential System $\mathbf{UV}_1\{x,t;u,v_1\}$.

$\{\Lambda_1(x,t,U,V),\ \Lambda_2(x,t,U,V)\}$ is a set of local CL multipliers for the NLT potential system $\mathbf{UV}_1\{x,t;u,v_1\}$ (86) if and only if the equations

$$\begin{aligned} &E_U\left(\Lambda_1\left(V_x-U_t\right)+\Lambda_2\left(V_t-\left(F(U)U_x+G(U)\right)\right)\right)\equiv 0\\ &E_V\left(\Lambda_1\left(V_x-U_t\right)+\Lambda_2\left(V_t-\left(F(U)U_x+G(U)\right)\right)\right)\equiv 0 \end{aligned} \tag{95}$$

hold for *arbitrary* differentiable functions $(U(x,t), (V(x,t))$. Equations (95) yield the system of determining equations

$$\begin{aligned} \frac{\partial \Lambda_2}{\partial V}-\frac{\partial \Lambda_1}{\partial U}&=0\\ \frac{\partial \Lambda_2}{\partial U}-F(U)\frac{\partial \Lambda_1}{\partial V}&=0\\ \frac{\partial \Lambda_2}{\partial x}-\frac{\partial \Lambda_1}{\partial t}-G(U)\frac{\partial \Lambda_1}{\partial V}&=0\\ F(U)\frac{\partial \Lambda_1}{\partial x}-\frac{\partial \Lambda_2}{\partial t}-\frac{\partial}{\partial U}[G(U)\Lambda_1]&=0. \end{aligned} \tag{96}$$

One can show that for any solution of (96), the fluxes for the corresponding local CLs of the potential NLT system $\mathbf{UV}_1\{x,t;u,v_1\}$ (86) are given by

$$\begin{aligned} X(x,t,u,v_1)=&-\int_a^u \Lambda_1(x,t,s,b)ds-\int_b^{v_1}\Lambda_2(x,t,u,s)ds\\ &-G(a)\int^x \Lambda_1(s,t,a,b)ds\\ T(x,t,u,v_1)=&\int_a^u \Lambda_2(x,t,s,b)ds+\int_b^{v_1}\Lambda_1(x,t,u,s)ds. \end{aligned}$$

One can show [41] that the solution of the determining system (96) reduces to the study of the system of two functions given by

$$\begin{aligned} d(U)&=G'^2F'''-3G'G''F''+[3G''^2-G'G''']F'\\ h(U)&=G'^2G^{(4)}-4G'G''G'''+3G''^3. \end{aligned}$$

Three cases arise

$$\begin{aligned} &d(U)\not\equiv 0, h(U)\equiv 0\\ &d(U)\not\equiv 0, h(U)\not\equiv 0\\ &d(U)=h(U)\equiv 0. \end{aligned}$$

The results are summarized as follows.

When $d(U)\not\equiv 0$, $h(U)\equiv 0$, the resulting local CL multipliers for the potential system $\mathbf{UV}_1\{x,t;u,v_1\}$ are indicated in Table 3.

Table 3 $d(U) \not\equiv 0, h(U) \equiv 0$

$F(U)$	$G(U)$	Local CL multipliers
Arbitrary	U	$(\Lambda_1, \Lambda_2) = (t, x - \frac{1}{2}t^2)$, $(\Lambda_1, \Lambda_2) = (1, -t)$
Arbitrary	$1/U$	$(\Lambda_1, \Lambda_2) = (U, V)$, $(\Lambda_1, \Lambda_2) = (UV, \frac{1}{2}V^2 + x + \int^U sF(s)ds)$

Table 4 $d(U) \not\equiv 0, h(U) \not\equiv 0$

Relationship	Local CL multipliers
$\gamma F - G' = \frac{\alpha}{\gamma}(G+\beta)^2$	$(\Lambda_1, \Lambda_2) = (\varphi_1, \varphi_2) = e^{\gamma x + \frac{\alpha}{\gamma}\int^U (G(s)+\beta)ds} e^{\sqrt{\alpha}(\beta t + V)}(1, \frac{\sqrt{\alpha}}{\gamma}(G(U)+\beta))$, $(\Lambda_1, \Lambda_2) = (\varphi_1, -\varphi_2) = (x, -t, U, -V)$
$\gamma F - G' = \frac{\alpha}{\gamma}$	$(\Lambda_1, \Lambda_2) = (\psi_1, \psi_2) = e^{\gamma x + \sqrt{\alpha} t}(1, \frac{\sqrt{\alpha}}{\gamma})$, $(\Lambda_1, \Lambda_2) = (\psi_1, -\psi_2)(x, -t)$
$\gamma F = G'$	$(\Lambda_1, \Lambda_2) = e^{\gamma x}(t, \frac{1}{\gamma})$, $(\Lambda_1, \Lambda_2) = e^{\gamma x}(V, \frac{1}{\gamma}G(U))$, $(\Lambda_1, \Lambda_2) = e^{\gamma x}(1, 0)$

When $d(U) \not\equiv 0$, $h(U) \not\equiv 0$, the resulting local CL multipliers for the potential system $\mathbf{UV}_1\{x, t; u, v_1\}$ are indicated in Table 4.

When $d(U) = h(U) \equiv 0$, using symmetry analysis (substitution and invariance of the ODE under a solvable three-parameter Lie group of point transformations), the ODE $h(U) = 0$ can be solved in terms of elementary functions (for $G(U)$). Then note that $F(U) = G(U) + \text{const}$ is a particular solution of the resulting linear ODE $d(U) = 0$. In turn, this leads to its general solution. Consequently, for $F(U) = \beta_1 G^2(U) + \beta_2 G(U) + \beta_3$, ${\beta_2}^2 \neq 4\beta_1\beta_3$, there are four highly nontrivial CLs when $G(U) = U, 1/U, e^U, \tanh U, \tan U$. In the case of a "perfect square" ${\beta_2}^2 = 4\beta_1\beta_3$, there are only two local CLs. For details, see [41].

The NLT potential system $\mathbf{UV}_1\{x, t; u, v_1\}$ (86) is not variational. In the case of a variational system, each set of local CL multipliers of the system must correspond to a local symmetry of the system written in evolutionary form. Hence, in the variational situation, for any pair of constitutive functions $(F(u), G(u))$, the number of sets of local CL multipliers is at most equal to the number of local symmetries. Note that for the PDE system $\mathbf{UV}_1\{x, t; u, v_1\}$ (86), for many pairs of constitutive functions $(F(u), G(u))$, the number of sets of local CL multipliers (which of course do not correspond to local symmetries) exceeds the number of local symmetries.

Example 3 Planar Gas Dynamics Equations

Suppose the given PDE system is the planar gas dynamics (PGD) equations. In the *Eulerian* description, the corresponding Euler PGD system $\mathbf{E}\{x, t; v, p, \rho\}$ is given by

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ \rho(v_t + v v_x) + p_x &= 0 \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x &= 0 \end{aligned} \tag{97}$$

where $v(x, t)$ is the velocity of the gas, $p(x, t)$ is the pressure, and $\rho(x, t)$ is the mass density of the gas. In the Eulerian system $\mathbf{E}\{x, t; v, p, \rho\}$ (97), in terms of the *entropy density* $S(p, \rho)$, the constitutive function $B(p, \rho^{-1})$ is given by $B(p, \rho^{-1}) = -\rho^2 S_\rho / S_p$.

In the *Lagrangian* description, in terms of Lagrange mass coordinates $s = t$, $y = \int_{x_0}^{x} \rho(\xi) d\xi$, the corresponding Lagrange PGD system $\mathbf{L}\{y, s; v, p, q\}$ is given by

$$\begin{aligned} q_s - v_y &= 0 \\ v_s + p_y &= 0 \\ p_s + B(p, q) v_y &= 0 \end{aligned} \tag{98}$$

with $q = 1/\rho$.

It is now shown that the potential system framework, based on using local CLs, yields a direct connection between the Euler system (97) and the Lagrange system (98). As well, as a consequence, other equivalent descriptions are derived. The Euler system $\mathbf{E}\{x, t; v, p, \rho\}$ (97) is used as the given PDE system. The first equation of the Euler system is written as a local CL, corresponding to conservation of mass. Through this equation, a potential variable $r(x, t)$ is introduced and leads to the Euler potential system $\mathbf{G}\{x, t; v, p, \rho, r\}$ given by

$$\begin{aligned} r_x - \rho &= 0 \\ r_t + \rho v &= 0 \\ \rho(v_t + v v_x) + p_x &= 0 \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x &= 0. \end{aligned} \tag{99}$$

Now consider an interchange of dependent and independent variables in $\mathbf{G}\{x, t; v, p, \rho, r\}$ with $r = y$, $t = s$ as independent variables and $x, v, p, q = 1/\rho$ as dependent variables to obtain the system $\mathbf{G}_0\{y, s; x, v, p, q\}$, invertibly equivalent to $\mathbf{G}\{x, t; v, p, \rho, r\}$ (99), given by

$$\begin{aligned} x_y - q &= 0 \\ x_s - v &= 0 \\ v_s + p_y &= 0 \\ p_s + B(p, q) v_y &= 0. \end{aligned} \tag{100}$$

A nonlocally related subsystem of $\mathbf{G}_0\{y, s; x, v, p, q\}$ (100) is obtained by excluding its dependent variable x through the integrability condition $x_{ys} = x_{sy}$. The resulting subsystem is the Lagrange system $\mathbf{L}\{y, s; v, p, q\}$ (98)!

A second CL of the Euler system $\mathbf{E}\{x, t; v, p, \rho\}$ (97) is obtained from its set of local CL multipliers $(\Lambda_1, \Lambda_2, \Lambda_3) = (V, 1, 0)$. This yields a second potential variable w. The resulting couplet system $\mathbf{W}\{x, t; v, p, \rho, r, w\}$ that includes the potential variables r and w is given by the PDE system

$$\begin{aligned} r_x - \rho &= 0 \\ r_t + \rho v &= 0 \\ w_x + r_t &= 0 \\ w_t + p + v w_x &= 0 \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x &= 0. \end{aligned} \tag{101}$$

The third equation of the couplet system $\mathbf{W}\{x, t; v, p, \rho, r, w\}$ (101), which is a local CL as written, yields a third potential variable z to yield an additional potential system $\mathbf{Z}\{x, t; v, p, \rho, r, w, z\}$ given by

$$\begin{aligned} r_x - \rho &= 0 \\ r_t + \rho v &= 0 \\ z_t - w &= 0 \\ z_x + r &= 0 \\ w_t + p + v w_x &= 0 \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x &= 0. \end{aligned} \tag{102}$$

The Lagrange system $\mathbf{L}\{y, s; v, p, q\}$ (98) has a nonlocally related subsystem obtained by excluding its dependent variable v through the integrability condition $v_{ys} = v_{sy}$. The resulting subsystem $\underline{\mathbf{L}}\{y, s; p, q\}$ is given by

$$q_{ss} + p_{yy} = 0, \qquad p_s + B(p, q) q_s = 0. \tag{103}$$

The resulting tree of nonlocally related systems, including two additional subsystems, is illustrated in Fig. 4.

Now treating the Lagrange system $\mathbf{L}\{y, s; v, p, q\}$ (98) as a given PDE system, from its three sets of local CL multipliers given by (1, 0, 0), (0, 1, 0), and $(y, s, 0)$, one can obtain the three singlet potential systems $\mathbf{LW}_1\{y, s; v, p, q, w_1\} = \mathbf{G}_0\{y, s; x, v, p, q\}$ (100), $\mathbf{LW}_2\{y, s; v, p, q, w_2\}$ and $\mathbf{LW}_3\{y, s; v, p, q, w_3\}$ respectively given by

$$\begin{aligned} w_{1y} - q &= 0 \\ w_{1s} - v &= 0 \\ v_s + p_y &= 0 \\ p_s + B(p, q) v_y &= 0 \end{aligned} \tag{104}$$

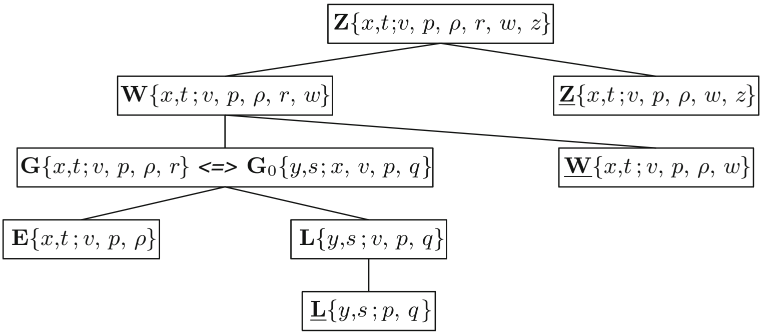

Fig. 4 Tree of nonlocally related PDE systems for PGD equations $\mathbf{E}\{x, t; v, p, \rho\}$ (97)

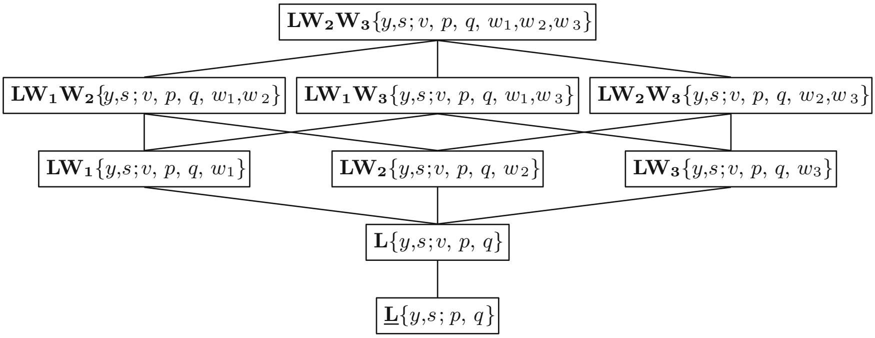

Fig. 5 Extension of tree of nonlocally related PDE systems for the Lagrange PGD system $\mathbf{L}\{y, s; v, p, q\}$ (98)

$$\begin{aligned} q_s - v_y &= 0 \\ w_{2y} - v &= 0 \\ w_{2s} + p &= 0 \\ p_s + B(p, q)v_y &= 0 \end{aligned} \tag{105}$$

and

$$\begin{aligned} w_{3y} - sv - yq &= 0 \\ w_{3s} + sp - yv &= 0 \\ v_s + p_y &= 0 \\ p_s + B(p, q)v_y &= 0. \end{aligned} \tag{106}$$

The extension of the tree illustrated in Fig. 4 is exhibited in Fig. 5.

Additional local CLs arise for the Lagrange system $\mathbf{L}\{y, s; v, p, q\}$ (98) when one considers sets of local CL multipliers of the form $\{\Lambda_i(y, s, V, P, Q)\}$, $i = 1,$

2, 3. After solving the corresponding determining equations, one can show that the resulting sets of local CL multipliers are given by

$$\begin{aligned} \Lambda_1 &= \alpha y - \beta P + B(P, Q)\mu_3 + \delta \\ \Lambda_2 &= \alpha s + \beta V + \nu \\ \Lambda_3 &= \Lambda_3(y, P, Q) \end{aligned}$$

where $\alpha, \beta, \nu, \delta$ are arbitrary constants and $\Lambda_3(y, P, Q)$ is any solution of the PDE

$$\frac{\partial \Lambda_3}{\partial Q} - \frac{\partial}{\partial P}(B(P, Q)\Lambda_3) + \beta = 0.$$

The additional local CLs that arise (for an *arbitrary* constitutive function $B(p, q)$) for the Lagrange system $\mathbf{L}\{y, s; v, p, q\}$ (98) include

- Conservation of energy $\frac{\partial}{\partial s}(\frac{1}{2}v^2 + K(p, q)) + \frac{\partial}{\partial y}(pv) = 0$ where $K(p, q)$ is any solution of the PDE $K_q - B(p, q)K_p + p = 0$.
- Conservation of entropy $\frac{\partial}{\partial s}S(p, q) = 0$ where $S(p, q)$ is any solution of the PDE $S_q - B(p, q)S_p = 0$.

In the case of a Lagrange PGD system $\mathbf{L}\{y, s; v, p, q\}$ (98), with a generalized polytropic equation of state given by

$$B(p, q) = \frac{M(p)}{q}, \qquad M''(p) \neq 0 \tag{107}$$

one can show that for local CL multipliers restricted to dependence on the independent variables (y, s), still only the three exhibited singlet potential systems (104)–(106) arise. For a generalized polytropic equation of state (107), the local symmetries arising for $\mathbf{L}\{y, s; v, p, q\}$ (98) and its resulting singlet, doublet and triplet potential systems that arise from the potential systems (104)–(106), as well as its subsystem $\underline{\mathbf{L}}\{y, s; p, q\}$ (103), are exhibited in [30].

The following remarks are noted.

- The exhibited extended trees of nonlocally related PDE systems hold for an *arbitrary* constitutive function $B(p, q)$.
- Either the Euler system $\mathbf{E}\{x, t; v, p, \rho\}$ (97) or the Lagrange system $\mathbf{L}\{y, s; v, p, q\}$ (98) can play the role of the given system in the tree.
- In a beautiful paper [49], a complete group classification with respect to the constitutive function $B(p, q)$ is given separately for the Euler and Lagrange systems but the connections between the systems are heuristic.
- To systematically construct nonlocal symmetries of the Euler and Lagrange systems, one needs to do the group classification problem for *all* PDE systems in an extended tree as well as consider other possible extended trees for specific constitutive functions followed by appropriate point symmetry analyses.

- For a Chaplygin gas given by $B(p,q) = -p/q$, one can show that the Lagrange subsystem $\underline{\mathbf{L}}\{y, s;\ p, q\}$ (103) has the point symmetry (which could not be exhibited in [49] due to its heuristic approach) $\mathrm{X} = -y^2 \frac{\partial}{\partial y} - py\frac{\partial}{\partial p} + 3yq\frac{\partial}{\partial q}$ that in turn yields a nonlocal symmetry for both the Euler and Lagrange systems.
- Further extended trees arise for the PGD equations for specific constitutive functions:

  – $B(p, 1/\rho) = \rho(1 + \mathrm{e}^p)$: Here the Euler potential system $\mathbf{G}\{x, t;\ v, p, \rho, r\}$ (99) has the family of local CLs given by $\mathrm{D}_t\left(\frac{f(r)\mathrm{e}^p}{1+\mathrm{e}^p}\right) + \mathrm{D}_x\left(\frac{f(r)v\mathrm{e}^p}{1+\mathrm{e}^p}\right) = 0$, for arbitrary $f(r)$. Such a local CL can be used to replace the fourth equation of $\mathbf{G}\{x, t;\ v, p, \rho, r\}$ (99) through introduction of a potential variable c and yields the corresponding potential system

$$\begin{aligned} r_x - \rho &= 0 \\ r_t + \rho v &= 0 \\ r_x(v_t + vv_x) + p_x &= 0 \\ c_x + \mathrm{e}^p f(r)/(1 + \mathrm{e}^p) &= 0 \\ c_t - v\mathrm{e}^p f(r)/(1 + \mathrm{e}^p) &= 0. \end{aligned}$$

  – For a Chaplygin gas given by $B(p, 1/\rho) = -p\rho$, the Euler potential system $\mathbf{G}\{x, t;\ v, p, \rho, r\}$ (99) has the family of local CLs given by $\mathrm{D}_t\left(\frac{f(r)}{p}\right) + \mathrm{D}_x\left(\frac{f(r)v}{p}\right) = 0$, for arbitrary $f(r)$. Such a local CL yields the corresponding potential system

$$\begin{aligned} r_x - \rho &= 0 \\ r_t + \rho v &= 0 \\ r_x(v_t + vv_x) + p_x &= 0 \\ d_x + f(r)/p &= 0 \\ d_t - vf(r)/p &= 0. \end{aligned} \tag{108}$$

  Here one can show that additional nonlocal symmetries arise for the Chaplygin gas Euler system $\mathbf{E}\{x, t;\ v, p, \rho\}$ (97) through the calculation of point symmetries for the potential system (108) only when $f(r) = r$, $f(r) = \text{const}$. For $f(r) = r$, the Chaplygin gas potential system (108) has the point symmetries $\mathrm{X}_{\mathbf{D}_1} = \left(-\frac{t^3}{6} + \mathrm{d}t\right)\frac{\partial}{\partial x} + \left(d - \frac{t^2}{2}\right)\frac{\partial}{\partial v} + rt\frac{\partial}{\partial p} - \frac{rt\rho}{p}\frac{\partial}{\partial p}$ and $\mathrm{X}_{\mathbf{D}_2} = \left(-\frac{t^2}{2} + d\right)\frac{\partial}{\partial x} + -t\frac{\partial}{\partial v} + r\frac{\partial}{\partial p} - \frac{r\rho}{p}\frac{\partial}{\partial p}$. The symmetry $\mathrm{X}_{\mathbf{D}_1}$ is a nonlocal symmetry for both the Euler and Lagrange systems and consequently was not able to be exhibited in [49]. On the other hand, the symmetry $\mathrm{X}_{\mathbf{D}_2}$ is a nonlocal symmetry for the Euler system but a local symmetry for the Lagrange system.

4.2 *Symmetry-based Method to Obtain Nonlocally Related Systems and Nonlocal Symmetries*

It is now shown that any point symmetry of a given PDE system systematically yields an equivalent nonlocally related PDE system. To illustrate the situation, consider as an example the nonlinear reaction diffusion equation

$$u_t - u_{xx} = Q(u). \tag{109}$$

One can show that for any nonlinear reaction term $Q(u)$, the PDE (109) has no local conservation laws. Hence the CL-based method yields no nonlocally related systems for the PDE (109). On the other hand, note that the PDE (109) is invariant under translations in x and t.

Consider the invariance of PDE (109) under translations in x. After an interchange of the variables x and u, the PDE (109) becomes the invertibly equivalent PDE

$$x_t = \frac{x_{uu} - Q(u)x_u^3}{x_u^2}. \tag{110}$$

Accordingly, we introduce two auxiliary dependent variables $v = x_u$, $w = x_t$, and consider the *intermediate PDE system*

$$v = x_u, \qquad w = x_t, \qquad w = \frac{v_u - Q(u)v^3}{v^2}. \tag{111}$$

By its construction, the intermediate PDE system (111) is locally related to the given scalar PDE (109). Now consider the subsystem (*inverse potential system*) of the intermediate system (111) that is obtained by excluding x through the integrability condition $x_{ut} = x_{tu}$, namely

$$v_t = w_u, \qquad w = \frac{v_u - Q(u)v^3}{v^2}. \tag{112}$$

The intermediate system (111) (and hence the given PDE (109)) is nonlocally related to the inverse potential system (112). This follows from the intermediate system (111) being the potential system of the PDE system (112) with the potential variable x arising from the first equation in the inverse potential system (112), which is a local CL as written. Moreover, excluding w from the inverse potential system (112), one obtains the scalar PDE

$$v_t = \left(\frac{v_u - Q(u)v^3}{v^2}\right)_u \tag{113}$$

which is clearly nonlocally related to the given PDE (109) since the PDE (109) has no local CLs.

Hence through the example of the nonlinear reaction diffusion equation (109), one essentially sees that any point symmetry of a given PDE system naturally yields a nonlocally related system. This will be seen more explicitly as follows.

4.2.1 Construction of a Nonlocally Related System from a Point Symmetry

Consider a given PDE system

$$R^{\sigma}(x,t,u,\partial u,\ldots,\partial^{k}u)=0,\qquad \sigma=1,\ldots,N \tag{114}$$

where $u = (u^1(x,t),\ldots,u^m(x,t))$. Suppose the PDE system (114) has a point symmetry

$$\mathrm{X}=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta^{i}(x,t,u)\frac{\partial}{\partial u^{i}}\cdot \tag{115}$$

Let $X(x,t,u)$, $T(x,t,u)$, $U^1(x,t,u),\ldots,U^m(x,t,u)$ be corresponding canonical coordinates so that the point symmetry X of the PDE system transforms to $\mathrm{Y}=\frac{\partial}{\partial U^1}$, i.e., the PDE system (114) transforms invertibly to a PDE system invariant under translations in U^1 given by

$$\hat{R}^{\sigma}(X,T,\hat{U},\partial U,\ldots,\partial^{k}U)=0,\qquad \sigma=1,\ldots,N \tag{116}$$

with $\hat{U}=(U^2,\ldots,U^m)$, $U=(U^1,\ldots,U^m)$.

Now consider the *intermediate PDE system*, obtained after introducing two auxiliary dependent variables $\alpha=U^1_T$, $\beta=U^1_X$

$$\begin{aligned}
&\alpha=U^1_T\\
&\beta=U^1_X\\
&\tilde{R}^{\sigma}(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\ldots,\partial^{k-1}\alpha,\partial^{k-1}\beta,\partial^{k}\hat{U})=0,\qquad \sigma=1,\ldots,N
\end{aligned}\tag{117}$$

where $\tilde{R}^{\sigma}(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\ldots,\partial^{k-1}\alpha,\partial^{k-1}\beta,\partial^{k}\hat{U})$ is obtained from $\hat{R}^{\sigma}(X,T,\hat{U},\partial U,\ldots,\partial^{k}U)$ after making the appropriate substitutions. By construction, the intermediate system (117) is locally equivalent to the given PDE system (114). Excluding the dependent variable U^1 from the intermediate system (117), one obtains the equivalent *inverse potential system*

$$\begin{aligned}
&\alpha_X=\beta_T\\
&\tilde{R}^{\sigma}(X,T,\hat{U},\alpha,\beta,\partial\hat{U},\ldots,\partial^{k-1}\alpha,\partial^{k-1}\beta,\partial^{k}\hat{U})=0,\quad \sigma=1,\ldots,N.
\end{aligned}\tag{118}$$

The inverse potential system (118) is nonlocally related to the given PDE system (114) since the intermediate system (117) is the potential system for the inverse potential system (118) with its dependent variable U^1 playing the role of the potential variable arising from the displayed CL of the inverse potential system (118).

Consequently, the following theorem has been proved.

Theorem 16 *Any point symmetry of a PDE system (114) yields an equivalent nonlocally related PDE system (inverse potential system) given by the PDE system (118).*

This theorem can be extended to the situation of three or more independent variables. Here the resulting inverse potential system has curl-type CLs.

4.2.2 The Special Situation When the Given PDE is an Evolutionary Scalar PDE

When a given PDE system (114) is an evolutionary scalar PDE, then another related PDE system naturally arises. The situation is summarized by the following theorem whose proof is immediately obvious.

Theorem 17 *Suppose a given PDE is an evolutionary scalar PDE invariant under a point symmetry. Without loss of generality, here the given PDE can be taken to be of the form*

$$u_t = F(x, t, u_1, \ldots, u_k) \tag{119}$$

with $u_i = \frac{\partial^i u}{\partial x^i}$. *Let* $\beta = u_x$. *Then the scalar PDE*

$$\beta_t = \mathrm{D}_x F(x, t, \beta, \ldots, \beta_{k-1}) \tag{120}$$

is a locally related subsystem of the corresponding inverse potential system resulting from the invariance of the PDE (119) under translations in u.

Example Nonlinear Wave Equation

As an example, consider again the nonlinear wave equation (74) and its nonlocally related potential system (75). The invariance of the potential system (75) under translations in t and v shows that the PDE system (75) is invariant under the point symmetry with the infinitesimal generator

$$\mathrm{X} = \frac{\partial}{\partial v} - \frac{\partial}{\partial t}. \tag{121}$$

Corresponding canonical coordinates are represented by the point transformation

$$X = x, \qquad T = u, \qquad U = t + v, \qquad V = v \tag{122}$$

with the potential system (75) invariant under translations in V. The point transformation (122) maps the potential system (75) into the invertibly related PDE system

$$\begin{aligned} V_X U_T - V_T U_X - 1 = 0 \\ V_T + c^2(T)(U_X - V_X) = 0 \end{aligned} \tag{123}$$

which is invariant under translations in U and V.

From the invariance of the PDE system (123) under translations in V, one accordingly introduces auxiliary dependent variables $\alpha(X, T), \beta(X, T)$ to obtain the locally related intermediate system

$$\begin{aligned} \alpha = V_T \\ \beta = V_X \\ \beta U_T - \alpha U_X - 1 = 0 \\ \alpha + c^2(T)(U_X - \beta) = 0. \end{aligned} \tag{124}$$

Excluding V from the intermediate system (124), one obtains the inverse potential system

$$\begin{aligned} \beta_T = \alpha_X \\ \beta U_T - \alpha U_X - 1 = 0 \\ \alpha + c^2(T)(U_X - \beta) = 0. \end{aligned} \tag{125}$$

It is straightforward to exclude the dependent variables α and β from the last two equations of the inverse potential system (125) to obtain its locally related scalar PDE

$$\begin{aligned} U_{TT} + c^4(T)U_{XX} + c^2(T)[2U_{TX}U_T U_X - U_{XX}U_T^2 \\ -U_{TT}U_X^2 - 2U_{TX}] + 2c(T)c'(T)[U_X^2 U_T - U_X] = 0. \end{aligned} \tag{126}$$

In [29], it is shown that the scalar PDE (126) is nonlocally related to the scalar nonlinear wave equation (74) through comparison of the symmetry classifications of these two PDEs.

When $c(u) = u^{-2}$, one can show [28, 29] that the PDE (126) has the point symmetry $U^2 \frac{\partial}{\partial U} + TU \frac{\partial}{\partial T} - \frac{U}{T^3} \frac{\partial}{\partial X}$ that yields a previously unknown nonlocal symmetry of both the nonlinear wave equation (74) and the potential system (75).

Further details and examples of the symmetry-based method to obtain nonlocally related systems and nonlocal symmetries are presented in [28, 29].

5 Nonlocality in Multidimensions

In the multidimensional situation ($n \geq 3$ independent variables), a local conservation law for a given PDE system yields $\frac{1}{2}n(n-1)$ potential variables. It will be shown that a local symmetry of the resulting potential system *always* corresponds to a local symmetry of the given PDE system (As we have seen, this is not the situation for $n = 2$ independent variables).

In the conservation law-based approach, to obtain nonlocal symmetries of a given PDE system it is necessary to augment the potential system by a *gauge constraint*.

5.1 Divergence-type CLs and Corresponding Potential Systems

Consider a PDE system with N PDEs of order k with $n \geq 3$ independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \qquad \sigma = 1, \ldots, N. \tag{127}$$

Suppose the PDE system (127) has a divergence-type CL given by

$$\operatorname{div} \Phi[u] = \mathrm{D}_i \Phi^i[u] \equiv \mathrm{D}_i \Phi^i(x, u, \partial u, \ldots, \partial^r u) = 0. \tag{128}$$

From Poincaré's lemma, the local CL (128) yields $\frac{1}{2}n(n-1)$ potential variables $v^{jk}(x) = -v^{kj}(x)$. This leads to a set of n potential equations

$$\Phi^i[u] \equiv \mathrm{D}_j v^{ij}, \qquad i = 1, \ldots, n \tag{129}$$

equivalent to the local CL (128). The corresponding *potential system* is the union of the given PDE system (127) and the set of potential equations (129). This potential system is nonlocally related and equivalent to the given PDE system (127). In turn the potential system has the *gauge freedom* invariance given by the transformation

$$v^{ij} \to \mathrm{D}_k w^{ijk} \tag{130}$$

where the functions $w^{ijk}(x)$ are $\frac{1}{6}n(n-1)(n-2)$ *arbitrary* functions that are the components of a totally antisymmetric tensor, i.e., the constructed potential system has an infinite number of point symmetries (gauge symmetries) through the transformation (130) in terms of the infinitesimal generator

$$\mathrm{X}_{\text{gauge}} = \mathrm{D}_k w^{ijk}(x) \frac{\partial}{\partial v^{ij}}. \tag{131}$$

As it stands, the potential system is *underdetermined* due to the gauge freedom (130).

Now assume that the given PDE system (127) is *determined* in the sense that it does not have symmetries that involve *arbitrary functions* of *all* independent variables $x = (x^1, \ldots, x^n)$. In particular, suppose the potential system has a local symmetry

$$\mathrm{X} = \eta^\mu(x, u, \partial u, \ldots, \partial^P u, v, \partial v, \ldots \partial^Q v) \frac{\partial}{\partial u^\mu} + \zeta^{\alpha\beta}[u, v] \frac{\partial}{\partial v^{\alpha\beta}}. \tag{132}$$

Then the potential system has local symmetries given by the commutator $[\mathrm{X}_{\text{gauge}}, \mathrm{X}]$ that project to the symmetries

$$\left(\alpha^{ij}\frac{\partial\eta^\mu}{\partial v^{ij}}+(\mathrm{D}_{i_1}\alpha^{ij})\frac{\partial\eta^\mu}{\partial v^{ij}_{i_1}}+\cdots+(\mathrm{D}_{i_1}\cdots\mathrm{D}_{i_Q}\alpha^{ij})\frac{\partial\eta^\mu}{\partial v^{ij}_{i_1\cdots i_Q}}\right)\frac{\partial}{\partial u^\mu} \tag{133}$$

of the PDE system (127) with $\alpha^{ij}(x) = \mathrm{D}_k w^{ijk}(x)$, and $v^{ij}_{i_1\cdots i_R} = \mathrm{D}_{i_1}\cdots\mathrm{D}_{i_R}\alpha^{ij}$ denoting derivatives of v^{ij}. In the infinitesimal generator (133), $\alpha^{ij}(x)$ and each of its derivatives are arbitrary functions of $x = (x^1, \ldots, x^n)$. Since the given PDE system (127) is a *determined* system, it follows that the symmetry (133) is a symmetry of the given PDE system (127) if and only if $\frac{\partial\eta^\mu}{\partial v^{ij}} = \frac{\partial\eta^\mu}{\partial v^{ij}_{i_1}} = \cdots = \frac{\partial\eta^\mu}{\partial v^{ij}_{i_1\cdots i_Q}} \equiv 0$. Thus each local symmetry of the *underdetermined* potential system, arising from a divergence-type conservation law, yields only a local symmetry of the given *determined* PDE system (127).

Hence if a potential system arising from a divergence-type conservation law of a given PDE system (127) is to be used to seek a nonlocal symmetry of the PDE system (127) from a point symmetry of the potential system, *it is necessary to augment* the potential system with auxiliary constraint equations (*gauge constraints*) to obtain a *determined potential system*.

Definition 7 A *gauge constraint* has the property that the augmented potential system is equivalent to the given PDE system (127), i.e., every solution of the augmented potential system yields a solution of the given PDE system (127) and, conversely, every solution of the given PDE system (127) yields a solution of the augmented potential system.

Some examples of gauges (relating potential variables) include

- divergence (Coulomb) gauge
- spatial gauge
- Poincaré gauge
- Lorentz gauge (a form of divergence gauge)
- Cronstrom gauge (a form of Poincaré gauge).

For details on these gauges, see [32].

Example Wave Equation

As an example, consider the wave equation

$$u_{tt} - u_{xx} - u_{yy} = 0 \tag{134}$$

which is already a divergence-type CL. Correspondingly, one has the vector potential $v = (v^0, v^1, v^2)$ and the underdetermined potential system given by

$$\begin{aligned} u_t &= v^2_x - v^1_y \\ -u_x &= v^0_y - v^2_t \\ -u_y &= v^1_t - v^0_x. \end{aligned} \tag{135}$$

Now consider the equivalent augmented constrained system obtained by appending the Lorentz gauge

$$v^0_t - v^1_x - v^2_y = 0 \tag{136}$$

to the underdetermined potential system (135) to obtain the determined potential system

$$\begin{aligned} u_t &= v^2_x - v^1_y \\ -u_x &= v^0_y - v^2_t \\ -u_y &= v^1_t - v^0_x \\ 0 &= v^0_t - v^1_x - v^2_y. \end{aligned} \tag{137}$$

One can show [32] that the determined potential system (137) has six point symmetries that yield nonlocal symmetries as well as nonlocal CLs of the wave equation (134). One such point symmetry is given by the infinitesimal generator

$$\begin{aligned} \mathrm{X} = (yv^1 - xv^2 - tu)\frac{\partial}{\partial u} - (2tv^0 + xv^1 + yv^2)\frac{\partial}{\partial v^0} \\ - (xv^0 + 2tv^1 - yu)\frac{\partial}{\partial v^1} - (yv^0 + 2tv^2 + xu)\frac{\partial}{\partial v^2}. \end{aligned}$$

The other listed gauges yield no nonlocal symmetries from point symmetries of the corresponding determined potential systems.

5.2 Systematic Procedures to Seek Nonlocal Symmetries in Multidimensions

In the multidimensional situation ($n \geq 3$ independent variables), four systematic procedures (some with known examples) are presented to search for nonlocal symmetries of a given PDE system through seeking local symmetries of an equivalent nonlocally related PDE system.

- Potential systems arising from divergence-type conservation laws (of degree r; $1 < r \leq n-1$) augmented with gauge constraints to yield a determined potential system.
- Determined potential systems arising from curl-type conservation laws (of degree 1).
- Determined nonlocally related systems arising from admitted point symmetries. Here, each point symmetry of a given PDE system systematically yields a deter-

mined inverse potential system connected to an intermediate system through a curl-type conservation law of degree 1 [2, 50, 51].
- Determined nonlocally related subsystems.

In the case of three independent variables ($n = 3$), two types of local CLs arise.

- Degree 2 CLs (divergence-type CLs).
- Degree 1 CLs (curl-type CLs).

Potential systems arising from lower degree CLs ($r < n - 1$) essentially correspond to particular gauge constraints for underdetermined potential systems arising from divergence-type CLs.

Examples illustrating the types of nonlocal symmetries that can arise as described above appear in [50, 51].

5.3 Some Open Problems in Multidimensions

There are many open problems in seeking systematically nonlocal symmetries for multidimensional PDE systems. These include the following.

- Find examples of *nonlinear* PDE systems for which nonlocal symmetries arise as local symmetries of a potential system following from divergence-type CLs appended with gauge constraints.
- Find efficient procedures to obtain "useful" gauge constraints (eg, yielding nonlocal symmetries/nonlocal CLs) for potential systems arising from divergence-type CLs (as well as for underdetermined potential systems arising from lower-degree CLs). Can one rule out specific families of gauges for particular classes of potential systems?
- Find further examples of lower-degree CLs for PDE systems of physical importance. CLs of degree one (curl-type) are of particular interest since corresponding potential systems are determined. Examples to-date suggest that lower-degree CLs are rare and only arise when a given PDE system has a special geometrical structure. Of course, divergence-type CLs are common!
- Find examples of PDE systems of physical interest admitting point symmetries that in turn yield nonlocal symmetries of the systems.
- Find useful subsystems and useful means of obtaining subsystems (including in the two-dimensional case). Progress has been made in this direction [28, 29].
- Extend the work on obtaining nonlocally related systems to multidimensions for continuum mechanics systems such as gas dynamics equations and equations of dynamical nonlinear elasticity. A start on this has been made in [52].

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