Chapter 6
Stability Criteria for Delay Differential Equations

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Abstract It is shown that some recent stability criteria for delay differential equations are consequences of a well-known comparison principle for delay differential inequalities. Our approach gives not only a unified proof, but it also yields stronger results.

Keywords Delay differential equation • Stability criteria • Comparison principle • Quasimonotone

6.1 Introduction

Recently, there has been a great interest in stability criteria for delay differential equations arising in applications, such as compartmental systems and neural networks. Our aim in this paper is to show that some recent stability criteria can easily be obtained from a comparison principle for differential inequalities whose right-hand side satisfies the quasimonotone condition. We emphasize that our approach gives not only a unified proof of some recent stability criteria, but, moreover, it yields stronger results.

Let $\mathbb{R}$ be the set of real numbers. For a positive integer n, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional space of real column vectors and the space of $n \times n$ matrices with real entries, respectively. Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. The *induced norm* and the *logarithmic norm* of a matrix $A \in \mathbb{R}^{n\times n}$ is defined by

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$$\|A\| = \sup_{0 \neq x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \mu(A) = \lim_{\delta \to 0+} \frac{\|I + \delta A\| - 1}{\delta},$$

respectively, where I denotes the $n \times n$ identity matrix.

A matrix $A = (a_{ij})_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$ is said to be *nonnegative* if $a_{ij} \ge 0$ for all $i, j = 1, \dots, n$ and it is called *essentially nonnegative* if $a_{ij} \ge 0$ for all $i \neq j$, $i, j = 1, \dots, n$.

Let $x = (x_1, x_2, \dots, x_n)^T, y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n$. We write $x \le y$ $(x < y)$ if $x_i \le y_i$ $(x_i < y_i)$ for $i = 1, \dots, n$. Let $\mathbb{R}^n_+$ be the cone of nonnegative vectors in $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_+ = \left\{ x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n \mid x_i \ge 0 \ \text{ for all } \ i = 1, \dots, n \right\}.$$

Haddad and Chellaboina [3] studied the nonnegative solutions of the system

$$y'(t) = Ay(t) + F(y(t - \tau)), \tag{6.1}$$

where $\tau > 0, A \in \mathbb{R}^{n \times n}$, $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is locally Lipschitz continuous and $F(0) = 0$.

They proved the following stability result (see [3, Theorem 3.2]).

Theorem 6.1. *Suppose that $A \in \mathbb{R}^{n \times n}$ is essentially nonnegative and for some $\gamma \in (0, \infty)$,*

$$F(y) \le \gamma y, \qquad y \in \mathbb{R}^n_+. \tag{6.2}$$

Assume also that there exist $p, q \in \mathbb{R}^n$ such that $p, q > 0$ and

$$(A + \gamma I)^T p + q = 0, \tag{6.3}$$

where T denotes the transpose. Then the zero equilibrium of (6.1) is asymptotically stable with respect to nonnegative initial data.

S. Mohamad and K. Gopalsamy [5] studied the system of delay differential equations

$$z_i'(t) = -\alpha_i z_i(t) + \sum_{j=1}^{n} \beta_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} \gamma_{ij} f_j(z_j(t - \tau_{ij})) + I_i, \quad i = 1, \dots, n, \tag{6.4}$$

where $f_i : \mathbb{R} \to \mathbb{R}, \alpha_i > 0, \beta_{ij}, \gamma_{ij}, I_i \in \mathbb{R}$, and $\tau_{ij} \ge 0$ for $i, j = 1, \dots, n$.

By the method of Lyapunov functions they proved the following theorem (see [5, Theorem 2.1]).

Theorem 6.2. *Suppose that there exist constants $K_i, k_i \in (0, \infty)$, $i = 1, \ldots, n$, such that the following conditions hold:*

$$|f_i(x)| \leq K_i, \quad x \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{6.5}$$

$$|f_i(x) - f_i(y)| \leq k_i |x - y|, \quad x, y \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{6.6}$$

$$\alpha_i > k_i \textstyle\sum_{j=1}^{n} (|\beta_{ij}| + |\gamma_{ij}|), \quad i = 1, \ldots, n. \tag{6.7}$$

Then (6.4) has a unique equilibrium which is globally exponentially stable.

Consider the system of delay differential equations

$$z'(t) = Az(t) + Bg(z(t - \tau)) + J, \tag{6.8}$$

where $\tau > 0$, $A, B \in \mathbb{R}^{n \times n}$, $g : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function and $J \in \mathbb{R}^n$. L. Idels and M. Kipnis [4] proved the following theorem (see [4, Corollary 3.1]).

Theorem 6.3. *Let $z^\star$ be an equilibrium of (6.8). Suppose that g is globally Lipschitz continuous with Lipschitz constant $k > 0$ satisfying*

$$k\|B\| < -\mu(A), \tag{6.9}$$

where $\mu(A)$ is the logarithmic norm of A. Then $z^\star$ is a globally attractive equilibrium of (6.8).

In this paper, we will unify and improve all the three stability results. The proofs will be based on a known comparison theorem for quasimonotone systems formulated in Sect. 6.2. The new stability criteria are presented and proved in Sect. 6.3.

6.2 Summary of Known Results

Given $r \geq 0$, let $C = C([-r, 0]; \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with the *supremum norm,*

$$\|\varphi\| := \sup_{-r \leq \theta \leq 0} \|\varphi(\theta)\|, \qquad \varphi \in C.$$

Let $\phi, \psi \in C$. We write $\phi \leq \psi$ and $\phi < \psi$ if the inequalities hold at each point of $[-r, 0]$.

Consider the autonomous functional equation

$$x'(t) = f(x_t), \tag{6.10}$$

where $f : \Omega \to \mathbb{R}^n$, Ω is an open subset of C, and $x_t \in C$ is defined by

$$x_t(\theta) = x(t+\theta), \qquad -r \le \theta \le 0.$$

We will assume that f is Lipschitz continuous on any compact subset of Ω. This assumption guarantees that for every $\varphi \in \Omega$, there exists a unique noncontinuable solution x of (6.10) with initial value

$$x_0 = \varphi. \tag{6.11}$$

In the sequel, the unique solution of (6.10) and (6.11) will be denoted by $x(t;\varphi)$.

For each $i = 1, \ldots, n$, let f_i denote the $i-$th coordinate function of f so that

$$f(\varphi) = (f_1(\varphi), f_2(\varphi), \ldots, f_n(\varphi))^T, \qquad \varphi \in \Omega.$$

We say that f satisfies the *quasimonotone condition* on Ω if

$$\phi, \psi \in \Omega,\ \phi \le \psi, \text{ and } \phi_i(0) = \psi_i(0) \text{ for some } i, \text{implying } f_i(\phi) \le f_i(\psi).$$

The quasimonotone condition is the analogue of the well-known *Kamke condition* for ordinary differential equations.

Our proofs will be based on the following comparison principle essentially due to Ohta [6, Theorem 3].

Proposition 6.1. *Let Ω be an open subset of C. Suppose that $f : \Omega \to \mathbb{R}^n$ is Lipschitz continuous on compact subsets of $\Omega$and f satisfies the quasimonotone condition on Ω. Let $0 < b \le \infty$. Suppose that $y : [-r, b) \to \mathbb{R}^n$ is a continuous function satisfying the differential inequality*

$$\frac{\mathrm{d}^+}{\mathrm{d}t} y(t) \le f(y_t), \qquad t \in [0, b), \tag{6.12}$$

where $\frac{\mathrm{d}^+}{\mathrm{d}t}$ denotes the right-hand derivative. Assume also

$$y_0 \le \varphi \qquad \textit{for some } \varphi \in C. \tag{6.13}$$

If $x(t,\varphi)$ is the unique solution of (6.10) and (6.11), then

$$y(t) \le x(t,\varphi) \tag{6.14}$$

for all $t \in [-r, b)$ for which $x(t,\varphi)$ is defined.

We will apply the above comparison theorem to the linear system of differential inequalities

$$\frac{\mathrm{d}^+}{\mathrm{d}t} y_i(t) \le \sum_{j=1}^{n} a_{ij} y_j(t) + \sum_{j=1}^{n} b_{ij} y_j(t - \tau_{ij}), \qquad i = 1, \ldots, n, \tag{6.15}$$

where $a_{ij}, b_{ij} \in \mathbb{R}$ and $\tau_{ij} \ge 0$, $i, j = 1, \ldots, n$.

System (6.15) is a special case of (6.12) when $r = \max_{i,j=1,\ldots,n} \tau_{ij}$ and $f = (f_1, f_2, \ldots, f_n)^T$ is defined by

$$f_i(\phi) = \sum_{j=1}^{n} a_{ij} \phi_j(0) + \sum_{j=1}^{n} b_{ij} \phi_j(-\tau_{ij}) \tag{6.16}$$

for $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \in C([-r, 0]; \mathbb{R}^n)$ and $i = 1, 2, \ldots, n$. In this case Eq. (6.10) has the form

$$x_i'(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_{ij}), \qquad i = 1, 2, \ldots, n. \tag{6.17}$$

It is known [7] that if $f = (f_1, f_2, \ldots, f_n)^T$ is given by (6.16), then f satisfies the quasimonotone condition on Ω if and only if

$$A = (a_{ij})_{i,j=1,2,\ldots,n} \text{ is essentially nonnegative} \tag{6.18}$$

and

$$B = (b_{ij})_{i,j=1,2,\ldots,n} \text{ is nonnegative.} \tag{6.19}$$

According to a remarkable result due to Smith [7], for linear delay differential systems satisfying the quasimonotone condition, the exponential stability of the zero solution is equivalent to the exponential stability of the associated system of ordinary differential equations which is obtained by "ignoring the delay." More precisely, we have the following result.

Proposition 6.2. *Suppose that (6.18) and (6.19) hold so that the right-hand side of (6.17) satisfies the quasimonotone condition on C. The zero solution of (6.17) is exponentially stable if and only if the zero solution of the ordinary differential equation*

$$x' = (A + B)x \tag{6.20}$$

is exponentially stable.

Note that if (6.18) and (6.19) hold, then the coefficient matrix $M = A + B$ of (6.20) is essentially nonnegative. It is known [1] that in this case the exponential stability of the zero solution of (6.20) is equivalent to the explicit condition

$$(-1)^j \det \begin{pmatrix} m_{11} & \cdots & m_{1j} \\ \vdots & & \\ m_{j1} & & m_{jj} \end{pmatrix} > 0, \qquad j = 1, \ldots, n, \tag{6.21}$$

where

$$m_{ij} = a_{ij} + b_{ij}, \qquad i, j = 1, \ldots, n.$$

6.3 Stability Criteria

Theorem 6.4. *Under the hypotheses of Theorem 6.1, the zero solution of (6.1) is not only asymptotically stable, but even globally exponentially stable with respect to nonnegative initial data.*

Proof. For $\varphi \in C$, $\varphi \geq 0$, let $y(t) = y(\varphi, t)$ be the unique solution of (6.1) with initial value $y_0 = \varphi$. As shown in [3], $y(t) \geq 0$ for all $t \geq 0$. This and (6.2) imply for $t \geq 0$,

$$y'(t) = Ay(t) + F(y(t-\tau)) \leq Ay(t) + \gamma y(t-\tau).$$

Therefore $y(t)$ is a solution of the system of inequalities (6.15) where

$$b_{ij} = \begin{cases} \gamma, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \qquad i = 1, \ldots, n.$$

Since A is essentially nonnegative and $B = \gamma I$ is nonnegative, the quasimonotone condition holds for the right-hand side of (6.15). By the application of Proposition 6.1, we have

$$0 \leq y(t, \varphi) \leq x(t, \varphi), \qquad t \geq -r, \tag{6.22}$$

where $x(t, \varphi)$ is the unique solution of system (6.17) with initial value φ at zero. It follows from [3, Theorem 3.1] that under condition (6.3) the zero solution of (6.17) is asymptotically and hence exponentially stable. Therefore, there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|x(t, \varphi)\| \leq M \|\varphi\| \mathrm{e}^{-\alpha t}, \qquad t \geq 0. \tag{6.23}$$

Since the definition of the exponential stability is independent of the norm used in $\mathbb{R}^n$, we may restrict ourselves to the ℓ_1-norm. Then (6.22) and (6.23) imply

$$\|y(t,\varphi)\| \leq \|x(t,\varphi)\| \leq M\|\varphi\|\mathrm{e}^{-\alpha t}, \qquad t \geq 0.$$

This completes the proof. □

Theorem 6.5. *Suppose that all hypotheses of Theorem 6.2 hold except for (6.7) which is replaced with the condition*

$$(-1)^j \det \begin{pmatrix} m_{11} & \cdots & m_{1j} \\ \vdots & & \\ m_{j1} & & m_{jj} \end{pmatrix} > 0, \qquad j = 1, 2, \ldots, n, \tag{6.24}$$

where

$$m_{ij} = -\alpha_i \delta_{ij} + k_j(|\beta_{ij}| + |\gamma_{ij}|), \qquad i, j = 1, 2, \ldots, n, \tag{6.25}$$

and δ_{ij} is the Kronecker symbol. Then Eq. (6.4) has a unique equilibrium which is globally exponentially stable.

Remark 6.1. Condition (6.7) is equivalent to saying that the logarithmic norm of $M = (m_{ij})_{1 \leq i,j \leq n}$ given by (6.25) induced by the l_∞-norm on $\mathbb{R}^n$ is negative (see [2, p. 41]). While this is only a sufficient condition for the stability of matrix M (see [2, p. 59]), condition (6.24) is not only sufficient, but it is also necessary for the stability of M. Thus, condition (6.24) is weaker than (6.7).

Proof. The existence of an equilibrium $z^* = (z_1^*, \ldots, z_n^*)^T$ of system (6.4) can be proved in the same manner as in the proof of [5, Theorem 2.1]. For $\varphi \in C$, let $z(t) = z(t,\varphi)$ be the unique solution of (6.4) with initial value $z_0 = \varphi$. As shown in [5], we have for $t > 0$ and $i = 1, \ldots, n$,

$$\begin{aligned} \frac{\mathrm{d}^+}{\mathrm{d}t}|z_i(t) - z_i^*| \leq -\alpha_i|z_i(t) - z_i^*| + \sum_{j=1}^n |\beta_{ij}|k_j|z_j(t) - z_j^*| + \\ + \sum_{j=1}^n |\gamma_{ij}|k_j|z_j(t-\tau_{ij}) - z_j^*|, \end{aligned} \tag{6.26}$$

If we let $y_i(t) := |z_i(t) - z_i^*|$, then (6.26) can be written as

$$\frac{\mathrm{d}^+}{\mathrm{d}t} y_i(t) \leq -\alpha_i y_i(t) + \sum_{j=1}^n |\beta_{ij}|k_j y_j(t) + \sum_{j=1}^n |\gamma_{ij}|k_j y_j(t-\tau_{ij}). \tag{6.27}$$

System (6.27) is a special case of (6.15) with

$$a_{ij} = -\alpha_i \delta_{ij} + k_j |\beta_{ij}|, \qquad i, j = 1, 2, \dots, n, \tag{6.28}$$

and

$$b_{ij} = k_j |\gamma_{ij}|, \qquad i, j = 1, 2, \dots, n. \tag{6.29}$$

Clearly, conditions (6.18) and (6.19) are satisfied. Therefore Proposition 6.1 applies and we conclude that

$$0 \le y(t) \le x(t, \psi), \qquad t \ge 0, \tag{6.30}$$

where $y(t) = (y_1(t), \dots, y_n(t))^T$, and $x(t, \psi)$ is the unique solution of the system

$$x'_i(t) = -\alpha_i x_i(t) + \sum_{j=1}^{n} |\beta_{ij}| k_j x_j(t) + \sum_{j=1}^{n} |\gamma_{ij}| k_j x_j(t - \tau_{ij}), \qquad i = 1, 2, \dots, n, \tag{6.31}$$

with the initial data

$$\psi(\theta) = |\varphi(\theta) - z^*|, \qquad \theta \in [-r, 0]. \tag{6.32}$$

As noted in Sect. 6.2, condition (6.24) implies that the zero solution of (6.31) is exponentially stable. Therefore,

$$\|x(t, \psi)\| \le M \|\psi\| \mathrm{e}^{-\alpha t}, \qquad t \ge 0$$

for some $M \ge 1$ and $\alpha > 0$. Using the ℓ_1-norm in $\mathbb{R}^n$ again, the last inequality together with (6.30) implies for $t \ge 0$,

$$\|z(t, \varphi) - z^*\| = \|y(t)\| \le \|x(t, \psi)\| \le M \|\psi\| \mathrm{e}^{-\alpha t} = M \|\varphi - z^*\| \mathrm{e}^{-\alpha t}.$$

This proves the global exponential stability of the equilibrium z^*. □

Theorem 6.6. *Under the assumptions of Theorem 6.3, the equilibrium $z^\star$ of (6.8) is globally exponentially stable.*

Proof. $z(t) = z(t, \varphi)$ be the unique solution of (6.8) with initial value $z_0 = \varphi$ for $\varphi \in C([-\tau, 0], \mathbb{R}^n)$. Define

$$y(t) = z(t) - z^*, \qquad t \ge -\tau.$$

From (6.8) we get for $t \ge 0$,

$$y'(t) = Ay(t) + F(y(t - \tau)), \qquad y \in \mathbb{R}^n.$$

where

$$F(y) = B[g(y + z^*) - g(z^*)], \qquad y \in \mathbb{R}^n.$$

Using the fact that g is globally Lipschitz continuous with Lipschitz constant k, we get

$$\|F(y)\| \leq \|B\|k\|y\|, \qquad y \in \mathbb{R}^n. \tag{6.33}$$

It is known (see [2, Chap. I]) that if y is an $\mathbb{R}^n-$valued function which has a right-hand derivative u for $t = t_0$, then $\|y(t)\|$ has a right-hand derivative for $t = t_0$ which is equal to

$$\lim_{h \to 0+} \frac{\|y(t_0) + hu\| - \|y(t_0)\|}{h}.$$

Hence

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|y(t)\| = \lim_{h \to 0+} \frac{\|y(t) + hy'(t)\| - \|y(t)\|}{h}, \qquad t \geq 0. \tag{6.34}$$

For $t \geq 0$, we have

$$\|y(t) + hy'(t)\| - \|y(t)\| = \|(I + hA)y(t) + hF(y(t-\tau))\| - \|y(t)\| \leq$$

$$\leq \|I + hA\|\|y(t)\| + h\|F(y(t-\tau))\| - \|y(t)\| =$$

$$= (\|I + hA\| - 1)\|y(t)\| + h\|F(y(t-\tau))\|.$$

From this, using (6.34), we find that

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|y(t)\| \leq \mu(A)\|y(t)\| + \|F(y(t-\tau))\|, \qquad t \geq 0.$$

This, combined with (6.33), implies for $t \geq 0$,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|y(t)\| \leq \mu(A)\|y(t)\| + k\|B\|\|y(t-\tau)\|, \tag{6.35}$$

and

$$\|y(t)\| = \|\varphi(t) - z^*\|, \qquad t \in [-\tau, 0].$$

Let $x(t) = x(t, \varphi)$ be the unique solution of the linear scalar differential equation

$$x'(t) = \mu(A)x(t) + k\|B\|x(t - \tau), \tag{6.36}$$

with initial value

$$x(t) = \|\varphi(t) - z^*\|, \qquad t \in [-\tau, 0].$$

By Proposition 6.1, we have

$$\|y(t)\| \leq x(t), \qquad t \geq 0. \tag{6.37}$$

In particular, $x(t)$ is nonnegative for $t \geq 0$. Clearly, (6.36) is a special case of (6.17) with $n = 1$. Obviously, conditions (6.18) and (6.19) hold. Since condition (6.9) implies the exponential stability of the zero solution of the ordinary differential equation

$$x' = (\mu(A) + k\|B\|)x,$$

by Proposition 6.2, the zero solution of (6.36) is exponentially stable. Therefore, there exist $M \geq 1$ and $\alpha > 0$ such that for $t \geq 0$ and $\varphi \in C$,

$$\|x(t)\| \leq M\|\varphi - z^*\|\mathrm{e}^{-\alpha t}.$$

This and (6.37) imply that for $t \geq 0$,

$$\|z(t) - z^*\| = \|y(t)\| \leq x(t) \leq M\|\varphi - z^*\|\mathrm{e}^{-\alpha t}.$$

The proof is complete. □

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