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Recent Advances in Delay Differential and Difference Equations

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Recent Advances in Delay Differential and Difference Equations

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Preface

We would like to recommend the readers the 12 research papers on differential and difference equations in this volume. Differential and difference equations are to be understood in the broad sense. They include discrete and continuous dynamical systems, stochastic differential and difference equations, and numerical simulations of the solutions and applications.

The papers are related to the research presented by the corresponding authors at the “International Conference on Delay Differential and Difference Equations and Applications, July 15–19, 2013, Balatonfüred, Hungary” organized by the Department of Mathematics, Faculty of Information Technology of the University of Pannonia, Veszprém, Hungary. This conference was dedicated to the 70th birthday of our colleague, Professor István Győri. He has been working at the University of Pannonia, Hungary, as a full-professor of mathematics since 1993. He was the head of the Department of Mathematics between 1993 and 2009, and in the period 1995–1998 he served as the president of the university.

István Győri has published more than 160 scientific papers on differential and difference equations including his monograph on oscillation theory written jointly with Professor Gerry Ladas (Oscillation Theory for Delay Differential Equations with Applications, Oxford University Press, Oxford, 1991). In addition, he has more than 50 papers in medical and informatics applications. His open problems and papers have motivated further research, and more than 2,800 citations to his papers can be counted in the literature. He published papers together with more than 80 coauthors, and he has been a supervisor of several Ph.D. dissertations.

István Győri acts as a member of the editorial boards of more than ten international scientific journals, and he has been an invited lecturer and a member of the scientific and organizing committees of numerous international conferences. Since 2004 he has been a member of the boards of directors of the International Society of Difference Equations.

Finally, we remark that each paper in this volume has been carefully reviewed. We express our sincere thanks to the referees for their service to help our editorial task.

Veszprém, Hungary
March 2014

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Acronyms

BVP	Boundary value problem
CRN	Chemical reaction network
FDE	Functional differential equation
LMI	Linear matrix inequality
LODE	Linear ordinary differential equation
LV	Lotka-Volterra
MAL-CRN	Chemical reaction network with mass action law
NFDE	Neutral functional differential equation
ODE	Ordinary differential equation
PD	Proportional derivative
PDE	Partial differential equation
QM	Quasi-monomial
QP	Quasi-polynomial
SDE	Stochastic differential equation

Chapter 1
On Necessary and Sufficient Conditions for Preserving Convergence Rates to Equilibrium in Deterministically and Stochastically Perturbed Differential Equations with Regularly Varying Nonlinearity

John A.D. Appleby and Denis D. Patterson

Abstract This paper develops necessary and sufficient conditions for the preservation of asymptotic convergence rates of deterministically and stochastically perturbed ordinary differential equations with regularly varying nonlinearity close to their equilibrium. Sharp conditions are also established which preserve the asymptotic behaviour of the derivative of the underlying unperturbed equation. Finally, necessary and sufficient conditions are established which enable finite difference approximations to the derivative in the stochastic equation to preserve the asymptotic behaviour of the derivative of the unperturbed equation, even though the solution of the stochastic equation is nowhere differentiable, almost surely.

Keywords Differential equations • Stochastic differential equations • Asymptotic stability • Global asymptotic stability • State-independent diffusion • Fading perturbation • Regular variation

1.1 Introduction

In this paper we classify the rates of convergence to a limit of the solutions of scalar ordinary and stochastic differential equations of the forms

$$x'(t) = -f(x(t)) + g(t), \quad t > 0; \quad x(0) = \xi, \tag{1.1}$$

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and

$$\mathrm{d}X(t) = -f(X(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t), \quad t \geq 0, \tag{1.2}$$

where B is a one-dimensional standard Brownian motion. The asymptotic behaviour of the derivative in the case of (1.1), and the scaled increment $(X(t+h) - X(t))\,/\,h$ for fixed $h > 0$, in the case of (1.2), is also classified.

We assume that the unperturbed equation

$$y'(t) = -f(y(t)), \quad t > 0; \quad y(0) = \zeta \tag{1.3}$$

has a unique globally stable equilibrium (which we set to be at zero). This is characterised by the condition

$$xf(x) > 0 \quad \text{for } x \neq 0, \quad f(0) = 0. \tag{1.4}$$

In order to ensure that (1.3), (1.1) and (1.2) have continuous solutions, we assume

$$f \in C(\mathbb{R};\mathbb{R}), \quad g \in C([0,\infty);\mathbb{R}), \quad \sigma \in C([0,\infty);\mathbb{R}). \tag{1.5}$$

The condition (1.4) ensures that any solution of (1.1) or (1.2) is *global*, i.e., that

$$\tau_D := \inf\{t > 0 : x(t) \notin (-\infty,\infty)\} = +\infty,$$
$$\tau_S := \inf\{t > 0 : X(t) \notin (-\infty,\infty)\} = +\infty, \quad \text{a.s.}$$

We also ensure that there is exactly one continuous solution of both (1.1) and (1.3) by assuming

$$f \text{ is locally Lipschitz continuous on } \mathbb{R}. \tag{1.6}$$

This condition ensures the existence of a unique continuous adapted process which obeys (1.2).

In (1.3), (1.1) and (1.2), we assume that $f(x)$ does *not* have linear leading order behaviour as $x \to 0$; moreover, we do not ask that f forces solutions of (1.3) to hit zero in finite time. Since f is continuous, we are free to define

$$F(x) = \int_x^1 \frac{1}{f(u)}\,\mathrm{d}u, \quad x > 0, \tag{1.7}$$

and avoid solutions of (1.3) hitting zero in finite time forces

$$\lim_{x\to 0^+} F(x) = +\infty. \tag{1.8}$$

We notice that $F : (0,\infty) \to \mathbb{R}$ is a strictly decreasing function, so it has an inverse F^{-1}. Clearly, (1.8) implies that

$$\lim_{t\to\infty} F^{-1}(t) = 0.$$

The significance of the functions F and F^{-1} is that they enable us to determine the rate of convergence of solutions of (1.3) to zero, because $F(y(t)) - F(\zeta) = t$ for $t \geq 0$ or $y(t) = F^{-1}(t + F(\zeta))$ for $t \geq 0$. It is then of interest to ask whether solutions of (1.1) or of (1.2) will still converge to zero as $t \to \infty$ and to determine conditions (on g and σ) under which the rate of decay of the solution of the underlying unperturbed equation (1.3) is preserved by the solutions of (1.1) and (1.2).

In order to do this with reasonable generality we find it convenient and natural to assume that the function f is regularly varying at zero. We recall that a measurable function $f : (0,\infty) \to (0,\infty)$ with $f(x) > 0$ for $x > 0$ is said to be regularly varying at 0 with index $\beta \in \mathbb{R}$ if

$$\lim_{x\to 0+} \frac{f(\lambda x)}{f(x)} = \lambda^{\beta}, \quad \text{for all } \lambda > 0.$$

In the case that f is regularly varying at zero with index $\beta > 1$, the function $t \mapsto F^{-1}(t)$ is regularly varying at infinity with index $-1/(\beta-1)$. A measurable function $h : [0,\infty) \to [0,\infty)$ with $h(t) > 0$ for $t \geq 0$ is said to be regularly varying at infinity with index $\alpha \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{h(\lambda t)}{h(t)} = \lambda^{\alpha}, \quad \text{for all } \lambda > 0.$$

We use the notations $f \in \mathrm{RV}_0(\beta)$ and $h \in \mathrm{RV}_\infty(\alpha)$. Many useful properties of regularly varying functions, including those employed here, are recorded in Bingham et al. [14].

The main results of the paper give (essentially) necessary and sufficient conditions under which the asymptotic rate of decay of solutions of the perturbed equations is inherited from those of (1.3). We consider first the deterministic equation (1.1). Suppose that f is regularly varying at zero with index $\beta > 1$ and is asymptotic at zero to an odd function. Suppose further that g is continuous and that it is known that $x(t) \to 0$ as $t \to \infty$. Then the following statements are equivalent:

(a) The functions f and g obey

$$\lim_{t\to\infty} \int_0^t g(s)\,\mathrm{d}s \text{ exists}, \quad \lim_{t\to\infty} \frac{\int_t^\infty g(s)\,\mathrm{d}s}{F^{-1}(t)} = 0.$$

(b) There is $\lambda \in \{-1, 0, 1\}$ such that

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = \lambda.$$

The cases $\lambda = \pm 1$ reproduce the asymptotic behaviour of the solution y of (1.3) according to whether the initial condition is positive or negative. The case $\lambda = 0$ means that solutions of the perturbed equation decay more rapidly to zero than those of the unperturbed equation. We believe that this behaviour is rare, but it can arise for special perturbations. It is notable that this result does not require sign or pointwise conditions on the rate of decay of g; indeed, it can be shown that g need not be absolutely integrable, a strictly weaker condition than the first part of condition (a). Furthermore, one can have that $\limsup_{t\to\infty} |g(t)|/\Gamma(t) = 1$ for arbitrarily rapidly growing Γ, while solutions still obey condition (b). The asymptotic oddness of f is assumed so as to ensure that convergence rates from both sides of the equilibrium are the same.

Once the above result has been established, it is straightforward to characterise conditions under which the solution of (1.1) and its derivative inherit the asymptotic behaviour of those of (1.3). In that case, under the same hypotheses as above, we prove that the following statements are equivalent:

(c) The functions f and g obey

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = 0.$$

(d) There is $\lambda \in \{-1, 0, 1\}$ such that

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = \lambda, \quad \lim_{t\to\infty} \frac{x'(t)}{f(F^{-1}(t))} = -\lambda.$$

We notice that solutions of (1.3) with positive initial condition obey (d) with $\lambda = 1$, while those with negative initial condition obey (d) with $\lambda = -1$. The condition (c), in the case of positive g and positive initial condition ξ, was employed in Appleby and Patterson [4] to establish condition (a) (with $\lambda = 1$). However, condition (a) shows that such a pointwise condition is merely sufficient, rather than necessary, to preserve the asymptotic behaviour of solutions of (1.3).

Corresponding results apply to the stochastic equation (1.2). Once again we assume that f is in $\mathrm{RV}_0(\beta)$ for $\beta > 1$ and further suppose that f is asymptotic to an odd function at zero. We note first that if $\sigma \notin L^2([0,\infty);\mathbb{R})$, then

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \text{ exists and is finite}\right] = 0.$$

This corresponds to the necessity of the first part of condition (a) to preserve the rate of decay of solutions of (1.3) in the deterministic case. In the case when $\sigma \in L^2(0,\infty)$, we have a sharp characterisation of situations under which the solution of (1.2) inherits the decay rate of solutions of (1.3). Define for sufficiently large $t > 0$ the function $\Sigma : [T,\infty) \to (0,\infty)$ by

$$\Sigma^2(t) = 2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right), \quad t \geq T,$$

and we suppose that

$$\mu := \lim_{t\to\infty} \frac{\Sigma(t)}{F^{-1}(t)} \in [0,\infty].$$

Then $\mu \in (0,\infty]$ implies that

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \text{ exists and is finite}\right] = 0$$

while $\mu = 0$ implies that there is a $\mathscr{F}^B(\infty)$-measurable random variable λ such that $\mathbb{P}[\lambda \in \{-1,0,1\}] = 1$ and

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = \lambda\right] = 1.$$

A result which is less explicit than the above, but parallel to the main result for (1.1), is the following equivalence:

(e) σ and f obey

$$\sigma \in L^2(0,\infty), \quad \lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0, \quad \text{a.s.}$$

(f) There is a $\mathscr{F}^B(\infty)$-measurable random variable λ such that $\mathbb{P}[\lambda \in \{-1,0,1\}] = 1$ and

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = \lambda\right] = 1.$$

The second condition in (e) can be replaced to some extent by a deterministic condition. If we suppose that $\varsigma(t) = \int_t^\infty \sigma^2(s)\,\mathrm{d}s$ is decreasing, and the function δ is defined for large enough t by $\delta(t) = tF^{-1}(\varsigma^{-1}(1/t))$, then

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in (-\infty,\infty)\right] > 0$$

implies $t \mapsto \delta^2(t)/t \to \infty$ as $t \to \infty$. This leads us to consider the case when $t \mapsto \delta^2(t)/t$ is increasing. If this is so, then the following are equivalent:

(g) $\sigma \in L^2([0,\infty);\mathbb{R})$ and

$$\int_1^\infty \frac{1}{t} \exp\left(-\epsilon^2 \frac{\delta^2(t)}{t}\right) \mathrm{d}t < +\infty, \quad \text{for all } \epsilon > 0;$$

(h)

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in (-\infty, \infty)\right] > 0;$$

(i)

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in \{-1, 0, 1\}\right] = 1.$$

Lastly, we establish a result analogous to the preservation of the asymptotic behaviour of (1.3) by the solution and derivative of (1.1). We suppose Ψ is the complementary standard normal distribution function, i.e.,

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Then the following statements are equivalent:

(j) For every $\epsilon > 0$,

$$S_f(\epsilon, h) = \sum_{n=1}^{\infty} \Psi\left(\frac{\epsilon}{\frac{\sqrt{\int_{nh}^{(n+1)h} \sigma^2(s)\,\mathrm{d}s}}{f(F^{-1}(nh))}}\right) < +\infty.$$

(k) There is a $\mathscr{F}^B(\infty)$-measurable random variable λ such that $\mathbb{P}[\lambda \in \{-1, 0, 1\}] = 1$ and

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = \lambda, \quad \text{a.s.}$$

and for each $h > 0$

$$\lim_{t\to\infty} \frac{\frac{X(t+h)-X(t)}{h}}{f(F^{-1}(t))} = -\lambda, \quad \text{a.s.}$$

It should be noted that this last result has a rather unexpected quality: remember first that provided $\sigma^2(t) > 0$ for all $t \geq 0$, the sample paths of X are differentiable nowhere with probability one. Therefore, we would not expect a finite difference approximation to the derivative of X (which does not exist!) to have smooth asymptotic behaviour. But in fact, that is precisely what this last result predicts: if we take $h > 0$ as small as we like and fixed, then provided that the noise σ decays rapidly enough, the sample path *observed regularly but not continuously* will appear asymptotically differentiable.

We conjecture in fact that if for any $h_1 > 0$ we have $S_f(\epsilon, h_1) < +\infty$, then for any $h > 0$ we have $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$, so the dependence on the "step size" h is not as important as might be guessed from first sight.

These asymptotic results are proven by constructing appropriate upper and lower solutions to the differential equation (1.1) as in Appleby and Buckwar [1]. In this paper, we prove a new result in which the solutions of (1.1) and (1.2) are related to that of an "internally perturbed" ordinary differential equation of the form $z'(t) = -f(z(t) + \gamma(t))$. The benefit gained from the added difficulty involved in bringing the perturbation inside the argument of the mean-reverting term is that the function γ will typically have good pointwise behaviour (obeyingfor example $\gamma(t) \to 0$ or $\gamma(t)/F^{-1}(t) \to 0$ as $t \to \infty$), while the original forcing functions g or σ in (1.1) and (1.2) may not have nice pointwise bounds. By means of this reformulation of the problem, we are able to determine a very fine characterisation of the desired asymptotic results. We also speculate that this approach may be very successful for dealing with highly nonlinear equations of the type (1.1) or (1.2) in which $f'(x) \to \infty$ as $x \to 0$ with f being in $\mathrm{RV}_0(1)$.

The paper is a continuation of work by the authors on the deterministic equation (1.1), which only covers the case when g is positive, and obeys pointwise asymptotic bounds, but which considered the asymptotic behaviour with respect to "large" perturbations. The principal achievement of that paper was therefore to give a complete description of "positively" perturbed equations in which the g had regular asymptotic behaviour. The goal here, by contrast, is to determine necessary and sufficient conditions for the preservation of convergence rates in the presence of more irregular perturbations: ones which on average may be small, but can possess "large spikes"; perturbations which may oscillate (perhaps rapidly) between being positive and negative; and also stochastic perturbations of Itô type, which, as well as adding uncertainty, remove smoothness and natural monotonicity in the perturbation size. Despite these new complications, however, we are able to capture the key asymptotic features of the solutions. The general question of sharp conditions under which stability of perturbed equations is preserved is examined by Strauss and Yorke in [25, 26]. For other literature in this direction on limiting equations, consult the references in [2].

We mention some other connections of this work to research in the stochastic literature. The paper builds directly on work of the first author with Mackey [3] which considers the asymptotic behaviour of (1.2) with $f(x) \sim a|x|^{\beta}\mathrm{sgn}(x)$ as $x \to 0$. In that work, it is shown for sufficiently rapidly decaying noise intensity that the solution inherits the decay rate of the underlying unperturbed ODE (1.3). However, completely sharp necessary conditions for the preservation of this rate were not found in this earlier work, and the analysis was confined to the case of polynomial f. Moreover, some information about the asymptotic behaviour of the derivative of f close to zero was needed, and this has now been eliminated. Finally, results concerning the finite difference approximation to the “derivative” of X were not presented in that work. Inspiration for the summation condition used to prove this result comes from the papers by Appleby et al. [12, 13] which deal with the convergence to zero (but not the rate of convergence) of linear and nonlinear stochastic differential equations of the form (1.2), and scrutiny of the proofs will show how similar arguments have been used to assist in determining the rate of decay, especially by means of an auxiliary affine SDE whose asymptotic behaviour can be determined by an essentially direct computation. Use of “asymptotic oddness” of mean-reverting functions in SDEs of the form (1.2) in order to symmetrise the dynamics can be seen in Appleby et al. [10], while a useful technical lemma concerning the asymptotic behaviour of the family of random variables $(\int_t^\infty \sigma(s)\,\mathrm{d}B(s))_{t\geq 0}$ in the case when σ is in $L^2([0,\infty);\mathbb{R})$ comes from another work of Appleby et al. [9]. The asymptotic behaviour of discretisations of SDEs of the form (1.2) is studied in [8], and some illustrative simulations of our results are given at the end of the paper.

There is a nice literature on power-like dynamics in solutions of SDEs, and we invite the reader to consult those by Mao [21, 22], Liu and Mao [19, 20] and in Liu [18] which deal with highly nonautonomous equations as well as those of Zhang and Tsoi [27, 28] and Appleby et al. [6] which are concerned with autonomous nonlinear equations.

The role of regular variation in the asymptotic analysis of the asymptotic behaviour of differential equations is a very active area. An important monograph summarising themes in the research up to the year 2000 is Maric [23]. Another important strand of research on the exact asymptotic behaviour of nonautonomous ordinary differential equations (of first and higher order) in which the equations have regularly varying coefficients has been developed. For recent contributions, see for example the work of Evtukhov and co-workers (e.g. Evtukhov and Samoilenko [15]) and Kozḿa [17], as well as the references in these papers. These papers tend to be concerned with nonautonomous features which are *multipliers* of the regularly varying state-dependent terms, in contrast to the presence of the nonautonomous term g in (1.1), which might be thought of as *additive*. Despite this extensive literature and active research concerning regular variation and asymptotic behaviour of ordinary differential equations, and despite the fact that our analysis deals with first-order equations only, it would appear that the results presented in this work are new. Finally, the first author’s interest in the theory of regularly varying functions has been much influenced by working with I. Győri and D. Reynolds on the related class of *subexponential functions* in considering slower than exponential convergence in solutions of convolution Volterra integral, integro-differential and difference equations [5, 7].

The paper is organised as follows: a short section follows with notation. Section 1.3 outlines a result concerning the asymptotic behaviour of an equation of the form $x'(t) = -f(x(t) + \gamma(t))$ which turns out to be of great importance in establishing results for the solutions of perturbed equations. Its proof is involved and deferred to Sect. 1.8. Section 1.4 states results concerning the deterministic equation (1.1); Sect. 1.5 is devoted to the stochastic equation (1.2). Section 1.6 considers some ramifications of the results and presents examples, including those which demonstrate that the perturbations g and σ can have arbitrarily large extreme growth rates, but that solutions of the perturbed equations still inherit the dynamics of the underlying ODE (1.3). Section 1.7 shows the results of some simulations of (1.2). Section 1.9 contains the proofs deferred from Sect. 1.4. Section 1.10 presents most of the proofs concerning (1.2) postponed from Sect. 1.5. One proof from Sect. 1.5 is granted its own section: Sect. 1.11 presents a result which characterises conditions under which the SDE preserves the asymptotic behaviour of the solution and derivative of (1.3). Section 1.12, which presents proofs of results stated in Sect. 1.6, concludes the paper.

1.2 Preliminaries

In this section we introduce some common notation and list known properties of regular, slow and rapidly varying functions. We also discuss the hypotheses used in the paper and then lay out and discuss the main results of the paper.

1.2.1 Notation and Properties of Regularly Varying Functions

Throughout the paper, the set of real numbers is denoted by $\mathbb{R}$. We let $C(I;J)$ stand for the space of continuous functions which map I onto J, where I and J are typically intervals in $\mathbb{R}$. Similarly, the space of differentiable functions with continuous derivative mapping I onto J is denoted by $C^1(I;J)$. If h and j are real-valued functions defined on $(0,\infty)$ and $\lim_{t\to\infty} h(t)/j(t) = 1$, we sometimes use the standard asymptotic notation $h(t) \sim j(t)$ as $t \to \infty$. We denote the space of (absolutely) integrable functions $h : [0,\infty) \to \mathbb{R}$, which obey $\int_0^\infty |h(t)|\,\mathrm{d}t < +\infty$ by $L^1([0,\infty);\mathbb{R})$, and the space of square integrable functions $h : [0,\infty) \to \mathbb{R}$ which obey $\int_0^\infty |h(t)|^2\,\mathrm{d}t < +\infty$ by $L^2([0,\infty);\mathbb{R})$.

Throughout the paper, when we work with stochastic equations, we assume that we are working on a complete filtered probability space:

$$(\Omega, \mathscr{F}, (\mathscr{F}(t))_{t\geq 0}, \mathbb{P}).$$

The abbreviation *a.s.* stands for *almost surely*. $B = \{B(t); t \geq 0\}$ is a standard Brownian motion adapted to $(\mathscr{F}(t))_{t\geq 0}$, and in fact, as we choose deterministic initial conditions for the solutions of the stochastic equations studied, there is no loss in setting the filtration to be the one naturally generated by B:

$$\mathscr{F}(t) = \mathscr{F}^B(t) = \sigma\{B(s); 0 \leq s \leq t\}, \quad t \geq 0.$$

In our analysis, we consider the stochastic differential equation (1.2) with deterministic initial condition ξ. For simplicity, we assume throughout that f is locally Lipschitz continuous and obeys $xf(x) > 0$ for $x \neq 0$.

At various points, properties of regularly varying functions are employed. We ask the reader to consult the monograph [14] for these results. Alternatively, our recent preprint [4] which concerns the asymptotic behaviour of (1.1) with g positive incorporates a self-contained section devoted to all relevant properties of regular variation used in these works.

1.3 Asymptotic Behaviour for Ordinary Differential Equations with Internal Perturbations

1.3.1 Main Result and Discussion

In this section, we deduce the asymptotic behaviour of the ordinary differential equation

$$x'(t) = -f(x(t) + \gamma(t)), \quad t > 0; \quad x(0) = \xi. \tag{1.9}$$

We demonstrate that when the "internal" perturbation γ decays to zero so rapidly that

$$\lim_{t\to\infty} \frac{\gamma(t)}{F^{-1}(t)} = 0, \tag{1.10}$$

and the solution of (1.9) tends to zero as $t \to \infty$, the asymptotic behaviour of (1.3) is preserved.

Theorem 1.1. *Let γ be continuous and x be the continuous solution of* (1.9). *Suppose that*

$$\text{There exists } \phi \text{ such that } \lim_{x\to 0} \frac{f(x)}{\phi(x)} = 1, \quad \phi \text{ is odd on } \mathbb{R} \tag{1.11}$$

and

$$f \in RV_0(\beta), \quad \beta > 0, \quad \lim_{x\to 0+} \frac{f(x)}{x} = 0, \quad f(0) = 0 \tag{1.12}$$

with $\beta > 1$, γ obeys (1.10) *and that $\lim_{t\to\infty} x(t) = 0$. Then*

$$\lim_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = 0 \text{ or } 1.$$

This proof is perhaps of independent interest, as it addresses the situation where the autonomous differential equation (1.3) is perturbed inside the argument of f (as opposed to the more commonly studied external perturbation seen in (1.1), for example). However, it transpires that studying (1.9) and employing Theorem 1.1 gives very useful information about the solution of both equations (1.1) and (1.2) and allows them to be analysed in a form which greatly facilitates the proof of necessary and sufficient conditions for preserving the asymptotic behaviour of (1.3).

1.3.2 Application of Theorem 1.1 to (1.1) and (1.2)

We now explore how Theorem 1.1 can be applied to determine sufficient conditions for certain asymptotic decay in (1.1) and (1.2). Consider first the solution x of (1.1) which we suppose obeys $x(t) \to 0$ as $t \to \infty$. Introduce the function $u(t) = \int_0^t g(s)\,ds$ and assume that it tends to a finite limit as $t \to \infty$, which we call $u(\infty)$. We are therefore free to define $\gamma(t) = u(t) - u(\infty)$ for $t \geq 0$. Clearly, γ is continuous and obeys $\gamma(t) \to 0$ as $t \to \infty$. Of course, $u'(t) = g(t)$. Consider now $z(t) = x(t) - u(t) + u(\infty) = x(t) - \gamma(t)$ for $t \geq 0$. Then z is in $C^1((0,\infty);\mathbb{R})$ and we have that $z(t) \to 0$ as $t \to \infty$. Then $z(0) = \xi + \int_0^\infty g(s)\,ds =: \xi'$ and

$$z'(t) = x'(t) - u'(t) = -f(x(t)) = -f(z(t) + \gamma(t)), \quad t \geq 0.$$

Therefore, we see that if $\gamma(t) = \int_t^\infty g(s)\,ds$ obeys (1.10), we can apply Theorem 1.1 to z to obtain $z(t)/F^{-1}(t) \to \lambda \in \{0, \pm 1\}$ as $t \to \infty$. Then, as γ obeys (1.10), we have that $x(t) = z(t) + \gamma(t)$ obeys $x(t)/F^{-1}(t) \to \lambda \in \{0, \pm 1\}$ as $t \to \infty$. Therefore, we have established the following result.

Theorem 1.2. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let g be a continuous function such that*

$$\lim_{t\to\infty} \int_0^t g(s)\,ds \textit{ exists and is finite}, \quad \lim_{t\to\infty} \frac{\int_t^\infty g(s)\,ds}{F^{-1}(t)} = 0. \tag{1.13}$$

If the continuous solution x of (1.1) obeys $\lim_{t\to\infty} x(t) = 0$, then

$$\textit{There exists a } \lambda \in \{0, \pm 1\} = 1 \textit{ such that } \lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = \lambda. \tag{1.14}$$

We will see shortly that in the conditions $u(t)$ tends to a finite limit and $(u(\infty) - u(t))/F^{-1}(t) \to 0$ are not only sufficient to ensure the appropriate asymptotic behaviour, but are also necessary.

We remark in the case that $g(t)$ is ultimately of one sign that the hypothesis $\lim_{t\to\infty} x(t) = 0$ is unnecessary, because the first part of (1.13) implies that g is integrable, which suffices to prove under the other hypotheses that $\lim_{t\to\infty} x(t) = 0$.

We show also how Theorem 1.1 can assist in determining the asymptotic behaviour of (1.2). We suppose that $\sigma \in L^2(0,\infty)$. In this case, by the martingale convergence theorem, it follows that the process

$$U(t) = \int_0^t \sigma(s)\,\mathrm{d}B(s)$$

tends to a finite limit almost surely: we call this limit $U(\infty)$. Suppose that this occurs on the (a.s.) event Ω_1; moreover, $t \mapsto U(t,\omega)$ can be taken to be continuous on this event. Let Ω_2 be the a.s. event on which there is a well-defined continuous adapted process X which solves (1.2). Since $\sigma \in L^2(0,\infty)$, it is well known that $X(t) \to 0$ as $t \to \infty$ a.s., and denote by Ω_3 the almost sure event on which this convergence occurs. Now set $\Omega_4 = \Omega_1 \cap \Omega_2 \cap \Omega_3$, which is also a.s. Consider $V(t) = X(t) - U(t)$ for $t \geq 0$, which is well defined on Ω_4. Then V obeys

$$V(t) = \xi - \int_0^t f(X(s))\,\mathrm{d}s, \quad t \geq 0.$$

Since f is continuous and $t \mapsto X(t)$ is continuous on Ω_4, it follows that for each fixed outcome $\omega \in \Omega_4$ we have that $t \mapsto V(t,\omega)$ is in $C^1((0,\infty);\mathbb{R})$ and in fact

$$V'(t,\omega) = -f(X(t,\omega)), \quad t \geq 0; \quad V(0,\omega) = \xi.$$

Now, for each $\omega \in \Omega_4$, define $\gamma(t,\omega) = U(\infty,\omega) - U(t,\omega)$. Then, it follows that $\gamma(t,\omega) \to 0$ as $t \to \infty$ and that $t \mapsto \gamma(t,\omega)$ is continuous. Finally, for $\omega \in \Omega_4$, define $Z(t,\omega) = X(t,\omega) - U(t,\omega) + U(\infty,\omega) = X(t,\omega) + \gamma(t,\omega)$ for $t \geq 0$. Then $Z(t,\omega) \to 0$ as $t \to \infty$. Furthermore, because we can view $Z(t,\omega) = V(t,\omega) + U(\infty,\omega)$, we have that $t \mapsto Z(t,\omega)$ is in $C^1((0,\infty);\mathbb{R})$ and moreover

$$Z'(t,\omega) = V'(t,\omega) = -f(X(t,\omega)) = -f(Z(t,\omega) + \gamma(t,\omega)), \quad t \geq 0;$$
$$Z(0,\omega) = \xi + \left(\int_0^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega) =: \xi'(\omega).$$

Once again, we see that $t \mapsto Z(t,\omega)$ and $t \mapsto \gamma(t,\omega)$ obey all conditions of Theorem 1.1, provided that

$$\int_t^\infty \sigma(s)\,\mathrm{d}B(s) = U(\infty) - U(t)$$

obeys

$$\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0, \quad \text{a.s.} \tag{1.15}$$

Suppose that this last limit is true on the a.s. event Ω_5, and let $\Omega_6 = \Omega_5 \cap \Omega_4$. In that case, we have that $Z(t,\omega)/F^{-1}(t) \to \lambda(\omega)$ as $t \to \infty$, where $\lambda(\omega) \in \{0, \pm 1\}$ for each $\omega \in \Omega_6$. Also, since $\gamma(t,\omega)/F^{-1}(t) \to 0$ as $t \to \infty$ for all $\omega \in \Omega_6$, we have that $X(t,\omega)/F^{-1}(t) \to \lambda(\omega)$ as $t \to \infty$ for every $\omega \in \Omega_6$. Since Ω_6 is an almost sure event, we have that $X(t)/F^{-1}(t) \to \lambda$ as $t \to \infty$ a.s., where λ must be a $\mathscr{F}^B(\infty)$-measurable random variable for which $\mathbb{P}[\lambda \in \{0, \pm 1\}] = 1$. Accordingly, we see that the following result has been established.

Theorem 1.3. *Suppose that f is continuous and obeys* (1.11) *and* (1.12) *for $\beta > 1$. Let σ be continuous, in $L^2([0,\infty);\mathbb{R})$ and obey* (1.15). *Then*

There exists an $\mathscr{F}^B(\infty)$-measurable random variable λ such that

$$\mathbb{P}[\lambda \in \{0, \pm 1\}] = 1 \textit{ and } \mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = \lambda\right] = 1. \tag{1.16}$$

Once again, we return later to establish that if the solution of (1.2) obeys (1.3), then it must be the case that $\sigma \in L^2(0,\infty)$ and obeys (1.15).

1.4 Main Results for Perturbed ODE

In this section, we list the main results of the paper. We start with analysis for the deterministic equation (1.1) and then consider the stochastic equation (1.2). In each case, we show that the sufficient conditions under which the perturbed equations inherit the asymptotic behaviour of (1.3) are also necessary. We also present results which concern the asymptotic behaviour of the derivative or increment of solutions of the perturbed equation.

A converse of Theorem 1.2 requires that (1.14) implies (1.13). We prove first that (1.14) implies that

$$\lim_{t\to\infty} \int_0^t g(s)\,\mathrm{d}s \text{ exists and is finite.} \tag{1.17}$$

Notice that this is a strictly weaker condition than requiring that g be absolutely integrable.

Theorem 1.4. *Suppose that f is continuous and obeys* (1.11) *and* (1.12) *for $\beta > 1$. Let g be continuous. If the continuous solution x of* (1.1) *obeys* (1.14)*, then g obeys* (1.17).

Given that (1.17) [which is the first part of (1.13)] holds when x obeys (1.14), we can define

$$\int_t^\infty g(s)\,\mathrm{d}s := \lim_{T\to\infty} \int_0^T g(s)\,\mathrm{d}s - \int_0^t g(s)\,\mathrm{d}s, \quad t \geq 0.$$

We now show that if x obeys (1.14), $t \mapsto \int_t^\infty g(s)\,\mathrm{d}s$ must obey both parts of (1.13).

Theorem 1.5. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let g be continuous. If the continuous solution x of (1.1) obeys (1.14), then g obeys (1.13).*

Combining the results of Theorem 1.2 and 1.5, we arrive at the following result.

Theorem 1.6. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let g be continuous. If the continuous solution x of (1.1) obeys $\lim_{t\to\infty} x(t) = 0$, then the following are equivalent:*

(a) The function g obeys (1.13).
(b) x obeys (1.14).

We next consider the situation where the solution and derivative of (1.1) both inherit their asymptotic behaviour from the solution of (1.3).

Theorem 1.7. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let g be continuous. If the continuous solution x of (1.1) obeys $\lim_{t\to\infty} x(t) = 0$, then the following are equivalent:*

(a) The function g obeys

$$\lim_{t\to\infty} \frac{g(t)}{f(F^{-1}(t))} = 0. \tag{1.18}$$

(b) There exists $\lambda \in \{-1, 0, 1\}$ such that

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = \lambda, \quad \lim_{t\to\infty} \frac{x'(t)}{f(F^{-1}(t))} = -\lambda. \tag{1.19}$$

Proof. The proof is easy and we present it here. We show first that (a) implies (b). Equation (1.18) implies that g is in $L^1([0,\infty);\mathbb{R})$ so the first part of (1.13) holds. By L'Hôpital's rule, (1.18) implies the second part of (1.13). Therefore the first part of the limit in (1.19) is true, by Theorem 1.2. In the proof of Theorem 1.5 it was shown that

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = \lambda$$

for $\lambda = \pm 1, 0$ implies

$$\lim_{t\to\infty} \frac{\varphi(x(t))}{\varphi(F^{-1}(t))} = \lambda,$$

where φ is the function asymptotic to f which is introduced in Lemma 1.10. Since $f(x)/\varphi(x) \to 1$ as $x \to 0$ and $x(t) \to 0$ as $t \to \infty$, it follows that

$$\lim_{t\to\infty} \frac{f(x(t))}{f(F^{-1}(t))} = \lambda.$$

Taking limits in (1.1), the last limit and (1.18) yield the second part of (1.19), as claimed.

To prove that (b) implies (a), simply rearrange (1.1) to get

$$\frac{g(t)}{f(F^{-1}(t))} = \frac{x'(t)}{f(F^{-1}(t))} + \frac{f(x(t))}{f(F^{-1}(t))}. \tag{1.20}$$

By hypothesis, (1.19) holds, so by the argument above, the second term on the right-hand side of (1.20) obeys

$$\lim_{t\to\infty} \frac{\varphi(x(t))}{\varphi(F^{-1}(t))} = \lambda.$$

The first term on the right-hand side of (1.20) has limit $-\lambda$ by hypothesis, so inserting these limits into (1.20) yields (1.18), as required. □

The pointwise condition (1.18) was used in [4] to obtain the first part of (1.19); it is a sharp condition, in the sense that if the limit in (1.18) exists and is non-zero, the asymptotic behaviour in the first part of (1.19) does not hold. However, in the case that the limit does not exist, it can still be the case that the first part of (1.19) holds, as the condition (1.13) [which is of course implied by (1.18)] can be true even when (1.18) is violated. However, Theorem 1.7 reveals the true significance of the condition (1.18): it is the critical size of the perturbation g that is allowed in order for the solution of (1.1) to be sufficiently well behaved asymptotically that it inherits the appropriate rate of decay by virtue of the fact that its derivative is well behaved. Naturally, such a stipulation places greater restrictions on the pointwise behaviour of g, by virtue of the form of (1.1).

It is tacit in this last statement that the second part of (1.19) drives the behaviour of x: in fact, it is easily seen by L'Hôpital's rule that the second part (1.19) implies the first.

We finish this section by noting in the case when g is positive and the initial condition is positive (so that $x(t) > 0$ for all $t \geq 0$), the limit in (1.14) is unity.

Theorem 1.8. *Suppose that f is continuous and is in $RV_0(\beta)$ for $\beta > 1$. Suppose further that g is continuous and positive. If x is the continuous solution (1.1) with $x(0) = \xi > 0$, then the following are equivalent:*

(a) The function g obeys (1.13).

(b) x obeys

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

Proof. Since $x(t) > 0$ for all $t \geq 0$, we do not need to assume that f is asymptotically odd, as in other results in this section. If (a) holds, then as g is positive, we have that $g \in L^1([0,\infty);(0,\infty))$. Therefore, we have that $x(t) \to 0$ as $t \to \infty$. Therefore, by Theorem 1.2 and the fact that x is positive, we have that

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 0 \text{ or } 1.$$

On the other hand, define $z'(t) = -f(z(t))$ for $t \geq 0$ and $z(0) = x(0)/2 > 0$. Then $x(t) > z(t)$ for all $t \geq 0$. Integration yields that $F(z(t))/t \to 1$ as $t \to \infty$, and as F^{-1} is regularly varying, it follows that $z(t)/F^{-1}(t) \to 1$ as $t \to \infty$. Therefore we have that

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \geq 1,$$

and this forces x to obey (b). If (b) holds, we have that $x(t) \to 0$ as $t \to \infty$, and since all other hypotheses of Theorem 1.5 are true, we have that (a) holds. □

Finally, we prove a result concerning the global stability of solutions of (1.1), a hypothesis which we require in many of the above results. It is to be noted that to achieve this, no additional conditions are required of f close to the equilibrium of (1.3). Instead, we require some asymptotic control of f at infinity. The condition we use is

$$\liminf_{x\to+\infty} |f(x)| > 0, \quad \liminf_{x\to-\infty} |f(x)| > 0, \tag{1.21}$$

and this was employed in [10] to cover the case of equations of the form (1.1) in which the perturbation g obeys $g(t) \to 0$ as $t \to \infty$. We remark that examples in [2, 10] show that if this condition is violated, it can happen that solutions of (1.1) tend to $\pm\infty$ as $t \to \infty$. Hence, we can see that this condition is not excessively restrictive. We are of course free to postulate alternative sufficient conditions under which $x(t) \to 0$ as $t \to \infty$, so as to allow the use of the above theorems; however, we prefer to separate hypotheses which ensure convergence from those explicitly required to preserve exact rates of decay.

As usual, in the following theorem, we assume without explicitly saying that f obeys (1.4). We assume in this result that f is locally Lipschitz continuous, as this enables us to simplify the argument: however, we conjecture that with more effort it is possible to require only that f is continuous. In this case, we may not have uniqueness of continuous solutions, but all solutions x should obey $x(t) \to 0$ as $t \to \infty$.

Theorem 1.9. *Let f be locally Lipschitz continuous and suppose* (1.17) *holds. Suppose further that* (1.21) *holds. Then the unique, continuous solution x of* (1.1) *obeys $x(t) \to 0$ as $t \to \infty$.*

1.5 Main Results for SDEs

We now present the main results for solutions of (1.2). We start by considering the results directly for X and later consider the asymptotic behaviour of the scaled increments of X.

1.5.1 Asymptotic Decay Rates of Solutions of (1.2)

We have already shown in Theorem 1.3 that

$$\sigma \in L^2([0,\infty);\mathbb{R}), \qquad \lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0, \quad \text{a.s.} \tag{1.22}$$

imply (1.3). We now establish the converse to this result, along the lines used to prove the converse of Theorem 1.2 in the deterministic case. Firstly, we prove that (1.3) implies the first condition in (1.22), namely $\sigma \in L^2([0,\infty);\mathbb{R})$.

Lemma 1.1. *Suppose that f is continuous and obeys* (1.11) *and* (1.12) *for $\beta > 1$. Let σ be continuous. If the continuous adapted process X which obeys* (1.2) *also satisfies* (1.3)*, then $\sigma \in L^2([0,\infty);\mathbb{R})$.*

Proof. Writing (1.2) in integral form and rearranging yields

$$\int_0^t \sigma(s)\,\mathrm{d}B(s) = X(t) - X(0) + \int_0^t f(X(s))\,\mathrm{d}s, \quad t \ge 0. \tag{1.23}$$

We have been granted as a hypothesis that X obeys (1.3): this implies that $X(t) \to 0$ as $t \to \infty$ a.s. Therefore, if it can be shown that the last term on the right-hand side of (1.23) tends to a finite limit a.s. as $t \to \infty$, we have that

$$\lim_{t\to\infty} \int_0^t \sigma(s)\,\mathrm{d}B(s) \text{ exists and is finite, a.s.}$$

By virtue of the martingale convergence theorem, this forces σ to be in $L^2([0,\infty);\mathbb{R})$. Hence it is enough to prove that

$$\lim_{t\to\infty} \int_0^t f(X(s))\,\mathrm{d}s \text{ exists and is finite a.s.}$$

under the hypothesis (1.3). However, this can be established by employing pathwise (i.e., to each ω in the almost sure event for which (1.3) holds and for which $\lambda(\omega) = 0, 1$, or -1) the argument used to prove the convergence of the integral

$$\int_0^t f(x(s))\,\mathrm{d}s$$

in the proof of Theorem 1.4, under the hypothesis that the function x obeys (1.14). $\square$

Next we show that the second part of (1.22) is necessary if we are to have that the solution X of (1.2) obeys (1.3). Since $\sigma \in L^2([0,\infty);\mathbb{R})$, we have that

$$\lim_{T\to\infty}\int_0^T \sigma(s)\,\mathrm{d}B(s) \text{ exists and is finite a.s.}$$

Therefore, for each outcome ω in an a.s. event Ω_0, we can define

$$\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega) := \lim_{T\to\infty}\left(\int_0^T \sigma(s)\,\mathrm{d}B(s)\right)(\omega) - \left(\int_0^t \sigma(s)\,\mathrm{d}B(s)\right)(\omega), \quad t \ge 0. \quad (1.24)$$

It is more useful to view this as an (uncountably) infinite family of $\mathscr{F}^B(\infty)$-measurable random variables that are well defined on Ω_0 rather than as a conventional process. One reason for this is that $I(t) := \int_t^\infty \sigma(s)\,\mathrm{d}B(s)$ is not $\mathscr{F}^B(t)$-measurable, as it depends on the Brownian motion B after time t: therefore, I is not adapted to the filtration $(\mathscr{F}^B(t))_{t\ge 0}$ on which the solutions of the SDE evolves. However, this is not a limitation, as our arguments can be applied path by path, and therefore can be viewed as essentially deterministic, once the outcome ω has been selected in an a.s. event on which desirable asymptotic properties hold.

Theorem 1.10. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. If the continuous adapted process X which obeys (1.2) also satisfies*

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X(t,\omega)}{F^{-1}(t)} = \lambda(\omega) \in (-\infty,\infty)\right] = 1,$$

then X obeys (1.3) and σ obeys (1.22).

In order to prove this result, and another result later in this section, the following result is needed concerning the asymptotic behaviour of $\int_t^\infty \sigma(s)\,\mathrm{d}B(s)$ when $\sigma \in L^2([0,\infty);\mathbb{R})$.

Lemma 1.2. *Suppose σ is a continuous function such that $\sigma \in L^2([0,\infty);\mathbb{R})$ and*

$$\int_t^\infty \sigma^2(s)\,\mathrm{d}s > 0 \textit{ for all } t \ge 0, \quad (1.25)$$

then

$$\limsup_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{\sqrt{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}} = 1, \quad a.s.$$

and

$$\liminf_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{\sqrt{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}} = -1, \quad a.s.$$

The proof of this lemma can be found in [9]. The condition (1.25) is important; if it did not hold, however, the dynamics of the SDE (1.2) would collapse to those of (1.3). To see this, notice that if (1.25) does not hold, then there exists a deterministic $T > 0$ such that $\int_t^\infty \sigma^2(s)\,\mathrm{d}s = 0$ for all $t \geq T$. Since σ^2 is nonnegative and continuous, this implies that $\sigma(t) = 0$ a.s. for $t \in [T, \infty)$ and therefore that

$$\int_T^t \sigma(s)\,\mathrm{d}B(s) = 0 \quad \text{for all } t \in [T, \infty) \text{ a.s.}$$

Therefore, for $t \geq T$, (1.2) reads

$$X(t) = X(T) - \int_T^t f(X(s))\,\mathrm{d}s,$$

so $X'(t) = -f(X(t))$ for $t \geq T$ a.s. with "initial condition" $X(T)$ being a random variable. Clearly, we have that

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = \operatorname{sgn}(X(T)), \quad \text{a.s.}$$

so in this case, we have (1.3). We therefore tacitly assume that (1.25) holds in the future, because otherwise the stochastic equation (1.2) is simply an equation of the form (1.3) with a random initial condition.

In the case that (1.25) holds, we see that the function

$$\Sigma(t) = \sqrt{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}$$

is positive for all t sufficiently large and that $\Sigma(t) \to 0$ as $t \to \infty$. Therefore, by Lemma 1.2,

$$\liminf_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{\Sigma(t)} = -1, \quad \limsup_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{\Sigma(t)} = 1, \quad \text{a.s}$$

and because $F^{-1}(t) > 0$ for all $t \geq 0$ and $F^{-1}(t) \to 0$ as $t \to \infty$, we have

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} \text{ exists and is finite and non-zero}\right] = 0. \tag{1.26}$$

Combining the results of Theorems 1.3 and 1.10 we obtain the following result.

Theorem 1.11. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. Suppose that X is the continuous adapted process X which obeys (1.2). Then the following are equivalent:*

(a) $\sigma \in L^2([0,\infty);\mathbb{R})$ *and*

$$\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0, \quad \text{a.s.}$$

(b)

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in (-\infty,\infty)\right] = 1;$$

(c)

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in \{-1,0,1\}\right] = 1.$$

We have shown that (a) implies (c) in Theorem 1.3); (c) clearly implies (b); and by Theorem 1.10, (b) implies (a).

The second condition in (1.22) is difficult to check a priori. Instead, we may use Lemma 1.2 to arrive at a more direct theorem, contingent on the following additional assumption on σ. There exists $\mu \in [0,\infty]$ such that

$$\mu^2 := \lim_{t\to\infty} \frac{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}{F^{-1}(t)^2}. \tag{1.27}$$

We state this result now.

Theorem 1.12. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. Suppose that X is the continuous adapted process which obeys (1.2).*

(a) Suppose $\sigma \notin L^2([0,\infty);\mathbb{R})$. *Then*

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in (-\infty,\infty)\right] = 0.$$

(b) Suppose $\sigma \in L^2([0,\infty);\mathbb{R})$.

(i) If μ *defined by* (1.27) *is zero, then*

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in \{-1,0,1\}\right] = 1.$$

(ii) If μ *defined by* (1.27) *is in* $(0,\infty]$, *then*

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in (-\infty,\infty)\right] = 0.$$

In the next subsection, we give another result which replaces the second condition in (1.22) with a different deterministic condition. The theorem appears somewhat more comprehensive than Theorem (1.12), in that it gives necessary and sufficient conditions for preserving the rate of decay of solutions. It is however more complicated to apply than Theorem (1.12) and makes some additional monotonicity demands.

1.5.2 *Characterisation of Preserved Decay Rate in Terms of an Upper Class Condition*

In this subsection, B is always a standard Brownian motion defined on $(\Omega,\mathscr{F},\mathbb{P})$. Our result follows from a number of lemmas. The first places restrictions on the rate of growth of deterministic functions which majorise Brownian motion with positive probability. Notice that we make no regularity or monotonicity restrictions on the majorising function δ: the reason for this will become apparent presently.

Lemma 1.3. *Let* $\delta \in C([0,\infty);(0,\infty))$ *and suppose that*

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{B(t)}{\delta(t)} = 0\right] > 0.$$

Then $\delta^2(t)/t \to \infty$ *as* $t \to \infty$.

Proof. Before starting the proof proper, we introduce some useful random sequences and facts. Let $t_n \nearrow \infty$ be any increasing deterministic sequence with $t_n \to \infty$ as $n \to \infty$. Define $G_n = B(t_n)/\delta(t_n)$ for $n \geq 1$. Then G_n is normally distributed with zero mean and variance $v_n^2 := t_n/\delta^2(t_n)$. We sometimes find it convenient to work with the standardised random variables $\tilde{G}_n = G_n/v_n$. If $n > m$, we have that

$$\mathrm{Corr}(G_n,G_m) = \mathrm{Corr}(\tilde{G}_n,\tilde{G}_m) = \sqrt{\frac{t_m}{t_n}},$$

where $\mathrm{Corr}(X, Y)$ denotes the correlation of the random variables X and Y. If we define $\alpha \in (0, 1]$ as the probability in the statement of the lemma, then clearly $\mathbb{P}[\lim_{n\to\infty} G_n = 0] \geq \alpha$.

We suppose next, in contrary to the conclusion of the lemma, that

$$\limsup_{t\to\infty} \frac{t}{\delta^2(t)} =: L \in (0, \infty].$$

Our proof now consists of showing that sequences t_n (which must be deterministic ones) on which this limsup is achieved will contradict the fact that $\mathbb{P}[\lim_{n\to\infty} G_n = 0] \geq \alpha > 0$.

Let t_n be an increasing sequence on which the limsup is achieved; in other words

$$\lim_{n\to\infty} \frac{t_n}{\delta^2(t_n)} = L.$$

Therefore $\mathrm{Var}[G_n] \to L$ as $n \to \infty$. Select a subsequence (t_{n_j}) of t_n such that $t_{n_{j+1}} > t_{n_j}/4$ and define $G_j^* = \tilde{G}_{n_j}$. Then we have for any $k > 0$ that $t_{n_{j+k}} > t_{n_j}/4^k$. Hence

$$0 < \mathrm{Corr}(G_j^*, G_{j+k}^*) = \sqrt{\frac{t_{n_j}}{t_{n_{j+k}}}} < \frac{1}{2^k}.$$

Since G_j^* is a standard normal random variable for each j, the above correlation bound implies

$$\limsup_{j\to\infty} \frac{G_j^*}{\sqrt{2\log j}} = 1, \quad \text{a.s.}$$

(for a proof of this fact, see [11], for example). Therefore

$$\limsup_{j\to\infty} \frac{G_{n_j}}{\sqrt{\mathrm{Var}[G_{n_j}]}} = +\infty, \quad \text{a.s.}$$

Since $\mathrm{Var}[G_{n_j}] \to L \in (0, \infty]$ as $j \to \infty$, it follows that $\limsup_{j\to\infty} G_{n_j} = +\infty$ a.s. Hence, as n_j is subsequence of the integers, we have $\limsup_{n\to\infty} G_n = +\infty$ a.s. But this contradicts the supposition that $\lim_{n\to\infty} G_n = 0$ with positive probability. Therefore, we must have $\limsup_{t\to\infty} t/\delta^2(t) = 0$, or $\lim_{t\to\infty} t/\delta^2(t) = 0$, proving the claim. □

Define

$$\varsigma(t) = \int_t^\infty \sigma^2(s)\,\mathrm{d}s. \tag{1.28}$$

If ς is decreasing, there exists $T' > 0$ such that we may define

$$\delta(t) = tF^{-1}(\varsigma^{-1}(t)), \quad t \geq T'. \tag{1.29}$$

The assumption that ς is decreasing is relatively mild and quite natural: it arises in the important case where $\sigma(t) \neq 0$ for all $t \geq T'$, in which case the equation is always authentically stochastic (more technically, the diffusion coefficient is non-degenerate) for sufficiently large time.

We now give a result which essentially says that $\int_t^\infty \sigma(s)\,\mathrm{d}B(s)/F^{-1}(t)$ is asymptotically negligible as $t \to \infty$ whenever $B(t)/\delta(t)$ is, where δ is defined by (1.29). This is clear progress, as it is easier to determine the asymptotic behaviour of B directly rather than that the delicate family of random variables $\int_t^\infty \sigma(s)\,\mathrm{d}B(s)$.

Lemma 1.4. *Let σ be continuous and in $L^2([0,\infty);\mathbb{R})$. Suppose ς in (1.28) is decreasing. If δ is defined in (1.29), then*

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0\right] = \mathbb{P}\left[\lim_{t\to\infty} \frac{B(t)}{\delta(t)} = 0\right].$$

Proof. Writing $M(t) = \int_t^\infty \sigma(s)\,\mathrm{d}B(s)$, the martingale time change theorem asserts the existence of another standard Brownian motion $\tilde{B}$ such that $M(t) = \tilde{B}(\langle M\rangle(t))$ for all $t \geq 0$. Notice also that M has an a.s. limit at infinity. Define $T := \int_0^\infty \sigma^2(s)\,\mathrm{d}s = \langle M\rangle(\infty) < +\infty$. Then $\varsigma(t) = T - \langle M\rangle(t)$ for all $t \geq 0$. Since $\int_t^\infty \sigma(s)\,\mathrm{d}B(s) = M(\infty) - M(t)$, we have

$$\int_t^\infty \sigma(s)\,\mathrm{d}B(s) = M(\infty) - M(t) = \tilde{B}(T) - \tilde{B}(\langle M\rangle(t)) = \tilde{B}(T) - \tilde{B}(T - \varsigma(t)).$$

Since $\varsigma(t)$ is decreasing to 0 as $t \to \infty$, we have

$$\begin{aligned}
&\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0\right] \\
&= \mathbb{P}\left[\lim_{t\to\infty} \frac{\tilde{B}(T) - \tilde{B}(T-\varsigma(t))}{F^{-1}(t)} = 0\right] \\
&= \mathbb{P}\left[\lim_{\tau\downarrow 0} \frac{\tilde{B}(T) - \tilde{B}(T-\tau)}{F^{-1}(\varsigma^{-1}(\tau))} = 0\right],
\end{aligned}$$

where we made the substitution $\tau = \varsigma(t)$ at the last step. Now, notice that $B_2(\tau) = \tilde{B}(T-\tau) - \tilde{B}(\tau)$ for $\tau \in [0, T]$ is a standard Brownian motion, so therefore

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0\right] = \mathbb{P}\left[\lim_{\tau\downarrow 0} \frac{B_2(\tau)}{F^{-1}(\varsigma^{-1}(\tau))} = 0\right].$$

Since $B_3(t) = tB_2(1/t)$ for $t > 0$ and $B_3(0) = 0$ is also standard Brownian motion, we make the substitution $\tau = 1/t$ and use the definition of δ in (1.29) to get

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0\right] = \mathbb{P}\left[\lim_{t\to\infty} \frac{B_3(t)}{\delta(t)} = 0\right].$$

Since B_3 has the same distribution as B, the claim is proven. $\square$

The first conclusion of the next result can be proven using the argument from Lemma 1.1. The second conclusion can be proven in an almost identical manner to Theorem 1.10. The proof is therefore omitted.

Lemma 1.5. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. Suppose that X is the continuous adapted process X which obeys (1.2). If*

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \in (-\infty,\infty)\right] > 0,$$

then $\sigma \in L^2(0,\infty)$ and

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} = 0\right] > 0.$$

Our next result can be deduced from the Kolmogorov–Erdős characterisation of the law of the iterated logarithm for standard Brownian motion. We state this now for completeness.

Lemma 1.6. *Let $\psi \in C([0,\infty);(0,\infty))$ be increasing with $\lim_{t\to\infty}\psi(t) = +\infty$.*

(a) If

$$\int_1^\infty \frac{1}{t}\psi(t)\mathrm{e}^{-\frac{1}{2}\psi^2(t)}\,\mathrm{d}t < +\infty,$$

then

$$\mathbb{P}\left[\frac{|B(t)|}{\sqrt{t}\cdot\psi(t)} < 1 \text{ i.o. as } t\to\infty\right] = 1.$$

(b) If

$$\int_1^\infty \frac{1}{t}\psi(t)\mathrm{e}^{-\frac{1}{2}\psi^2(t)}\,\mathrm{d}t = +\infty,$$

then

$$\mathbb{P}\left[\frac{|B(t)|}{\sqrt{t}\cdot\psi(t)} > 1 \text{ i.o. as } t \to \infty\right] = 1.$$

We have used here the standard abbreviation i.o. to refer to events that are realised infinitely often. For a proof of Lemma 1.6 see for example Itô and McKean [16]. Our corollary to Lemma 1.6 follows. We notice in its statement that a factor of $s(t) := \delta(t)/\sqrt{t}$, which appears in the Kolmogorov–Erdős characterisation, is omitted in Lemma 1.7 because the dependence of the parameter ϵ enables this subdominant term s to be subsumed into the more rapidly decaying exponential term.

Lemma 1.7. *Suppose that $\delta \in C([1,\infty);(0,\infty))$ is such that $t \mapsto \delta^2(t)/t$ is increasing with $\delta^2(t)/t \to \infty$ as $t \to \infty$. Then the following are equivalent:*

(a)

$$\int_1^\infty \frac{1}{t}\exp\left(-\epsilon^2\frac{\delta^2(t)}{t}\right)\,\mathrm{d}t < +\infty, \quad \textit{for all } \epsilon > 0;$$

(b)

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{B(t)}{\delta(t)} = 0\right] = 1.$$

Proof. Define $\phi(t) = \delta(t)/\sqrt{t}$ for $t \geq 1$. Then ϕ is increasing with $\phi(t) \to \infty$ as $t \to \infty$. Let $\epsilon > 0$. Then, if we define $\phi_\epsilon(t) = \epsilon\phi(t)$ for $t \geq 1$, we see that ϕ_ϵ is increasing and $\phi_\epsilon(t) \to \infty$ as $t \to \infty$.

Suppose that (a) holds. It is equivalent to

$$\int_1^\infty \frac{1}{t}\exp\left(-\frac{1}{2}\phi_\epsilon^2(t)\right)\,\mathrm{d}t < +\infty$$

for every $\epsilon > 0$. Since this holds for every $\epsilon > 0$ and $x = o(\mathrm{e}^{x^2/2})$ as $x \to \infty$ this is equivalent to

$$\int_1^\infty \frac{1}{t}\phi_\eta(t)\exp\left(-\frac{1}{2}\phi_\eta^2(t)\right)\,\mathrm{d}t < +\infty$$

for every $\eta > 0$. Therefore, by part (a) of Lemma 1.6 with $\psi = \phi_\eta$, we have

$$\mathbb{P}\left[\frac{|B(t)|}{\sqrt{t}\cdot\phi_\eta(t)} < 1 \text{ i.o. as } t \to \infty\right] = 1, \quad \text{for each } \eta > 0.$$

Call the almost sure event on which this statement holds Ω_η. Then

$$\limsup_{t\to\infty} \frac{|B(t)|}{\sqrt{t}\cdot\phi(t)} < \eta, \quad \text{a.s. on } \Omega_\eta.$$

Consider finally $\Omega^* = \cap_{\eta\in\mathbb{Q}\cap(0,1)}\Omega_\eta$. Then Ω^* is almost sure and since $\phi(t) = \delta(t)/\sqrt{t}$ we have

$$\limsup_{t\to\infty} \frac{|B(t)|}{\delta(t)} = 0, \quad \text{a.s. on } \Omega^*.$$

This shows that (a) implies (b).

Conversely, suppose that (b) holds. Then it follows for each $\eta > 0$ that there is an a.s. event Ω_η such that

$$\limsup_{t\to\infty} \frac{|B(t)|}{\sqrt{t}\cdot\phi(t)} < \eta, \quad \text{a.s. on } \Omega_\eta.$$

In other words

$$\mathbb{P}\left[\frac{|B(t)|}{\sqrt{t}\cdot\phi_\eta(t)} < 1 \text{ i.o. as } t\to\infty\right] = 1, \quad \text{for each } \eta > 0. \tag{1.30}$$

Now suppose, in contradiction to (a), that there exists $\eta' > 0$ such that

$$\int_1^\infty \frac{1}{t}\phi_{\eta'}(t)\exp\left(-\frac{1}{2}\phi_{\eta'}^2(t)\right)\,\mathrm{d}t = +\infty.$$

This implies that

$$\int_1^\infty \frac{1}{t}\exp\left(-\frac{1}{2}\phi_{\eta'}^2(t)\right)\,\mathrm{d}t = +\infty.$$

By part (b) of Lemma 1.6 with $\psi = \phi_{\eta'}$ it now follows that

$$\mathbb{P}\left[\frac{|B(t)|}{\sqrt{t}\cdot\phi_{\eta'}(t)} > 1 \text{ i.o. as } t\to\infty\right] = 1.$$

But this contradicts (1.30), and so it must follow that (a) is true. □

We make a remark and then prove our main result. Suppose that ς is increasing. Notice from Lemma 1.5 combined with Lemma 1.4 that

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)} \in (-\infty,\infty)\right] > 0$$

implies

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{B(t)}{\delta(t)}=0\right]>0.$$

Taking this in conjunction with Lemma 1.3 we see that $t\mapsto\delta^2(t)/t$ must tend to infinity as $t\to\infty$. Hence, if we want to preserve any of the main features of the decay rate of the underlying deterministic equation, even with positive probability, we must demand that $\delta^2(t)/t\to\infty$ as $t\to\infty$. Strengthening this to ask that the limit is reached *monotonically*, we may give a deterministic characterisation of the preservation of the rate of decay of the solution of $y'(t)=-f(y(t))$ in the solution of (1.2).

Theorem 1.13. *Suppose that f is continuous and obeys* (1.11) *and* (1.12) *for $\beta>1$. Let σ be continuous. Suppose that X is the continuous adapted process X which obeys* (1.2)*. Suppose finally that ς defined in* (1.28) *is decreasing and that δ is the function defined in* (1.29)*.*

(i) If

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}\in(-\infty,\infty)\right]>0,$$

then $t\mapsto\delta^2(t)/t\to\infty$ as $t\to\infty$.

(ii) If moreover $t\mapsto\delta^2(t)/t$ is increasing, then the following are equivalent:

(a) $\sigma\in L^2([0,\infty);\mathbb{R})$ and

$$\int_1^\infty\frac{1}{t}\exp\left(-\epsilon^2\frac{\delta^2(t)}{t}\right)\mathrm{d}t<+\infty,\quad \textit{for all } \epsilon>0;$$

(b)

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}\in(-\infty,\infty)\right]>0;$$

(c)

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}\in\{-1,0,1\}\right]=1.$$

Proof. We have proved (i) in the discussion above. Now we prove (ii). Suppose that (c) holds. Then clearly (b) is true. This implies that $\sigma\in L^2([0,\infty);\mathbb{R})$. By Lemma 1.5 we have that

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)}=0\right]>0.$$

Then by Lemma 1.4 we have

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)}=0\right]=\mathbb{P}\left[\lim_{t\to\infty}\frac{B(t)}{\delta(t)}=0\right]>0. \tag{1.31}$$

Now suppose that there exists $\epsilon_0 > 0$ such that

$$\int_1^\infty \frac{1}{t}\exp\left(-\epsilon_0^2\frac{\delta^2(t)}{t}\right)\,\mathrm{d}t=+\infty. \tag{1.32}$$

Recall that part of the proof that (b) implies (a) in Lemma 1.7 shows that

$$\int_1^\infty \frac{1}{t}\exp\left(-\frac{1}{2}\eta^2\frac{\delta^2(t)}{t}\right)\,\mathrm{d}t=+\infty$$

implies

$$\mathbb{P}\left[\frac{|B(t)|}{\delta(t)}>\eta \text{ i.o. as } t\to\infty\right]=1.$$

Since $t\mapsto \delta^2(t)/t$ is increasing, it therefore follows from this remark and (1.32) that

$$\mathbb{P}\left[\limsup_{t\to\infty}\frac{|B(t)|}{\delta(t)}>\sqrt{2}\epsilon_0\right]=1.$$

Hence

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{B(t)}{\delta(t)}=0\right]=0,$$

in contradiction to (1.31). Hence it must follow that

$$\int_1^\infty \frac{1}{t}\exp\left(-\epsilon^2\frac{\delta^2(t)}{t}\right)\,\mathrm{d}t<+\infty,\quad \text{for all } \epsilon>0,$$

which proves (a). It remains to show that (a) implies (c). Since (a) holds, by Lemma 1.7, it follows that

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{B(t)}{\delta(t)}=0\right]=1.$$

Therefore Lemma 1.4 and the monotonicity of ς give

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)}=0\right]=1.$$

Finally, by Theorem 1.3 it follows that X obeys (c). □

1.5.3 The Scaled Increments of X

We now turn to the situation in which the solution of (1.2) possesses asymptotic behaviour similar to that of (1.3) by virtue of the scaled h-increment

$$\frac{X(t+h)-X(t)}{h}$$

possessing good asymptotic behaviour. In fact, we will give necessary and sufficient conditions for the solution of (1.2) to obey

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}=\lambda,\quad \lim_{t\to\infty}\frac{\frac{X(t+h)-X(t)}{h}}{f(F^{-1}(t))}=-\lambda,\quad \lambda\in\{-1,0,1\}\right]=1 \quad (1.33)$$

for each $h>0$. It turns out if we let Ψ be the complementary normal distribution function and define

$$S_f(\epsilon,h):=\sum_{n=0}^{\infty}\Psi\left(\frac{\epsilon}{\theta(n)}\right),\quad \theta^2(n):=\frac{\int_{nh}^{(n+1)h}\sigma^2(s)ds}{(f\circ F^{-1})^2(nh)}, \quad (1.34)$$

then (1.33) holds if and only if for a fixed $h>0$, we have $S_f(\epsilon,h)<+\infty$ for all $\epsilon>0$. We first establish the sufficiency of the finiteness of $S_f(\epsilon,h)$.

Theorem 1.14. *Suppose that f is locally Lipschitz continuous and obeys (1.11) and (1.12) for $\beta>1$. Let σ be continuous. Let X be the continuous adapted solution of (1.2). Let $h>0$ and define $S_f(\epsilon,h)$ as in (1.34). If $S_f(\epsilon,h)<+\infty$ for all $\epsilon>0$, then*

There exists a $\mathscr{F}^B(\infty)$-measurable random variable λ such that

$$\lambda\in\{-1,0,1\}\ a.s.,\quad \lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}=\lambda,\ a.s.,\quad \lim_{t\to\infty}\frac{\frac{X(t+h)-X(t)}{h}}{(f\circ F^{-1})(t)}=-\lambda,\ a.s. \quad (1.35)$$

Once this result has been secured, we see that it admits a converse.

Theorem 1.15. *Suppose that f is locally Lipschitz continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. Let X be the continuous adapted solution of (1.2). Let $h > 0$ and define $S_f(\epsilon, h)$ as in (1.34). Suppose that there exists an event A with $\mathbb{P}[A] > 0$ such that*

$$A = \left\{\omega : \lim_{t\to\infty} \frac{X(t,\omega)}{F^{-1}(t)} = \lambda(\omega), \right. \\ \left. \lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f \circ F^{-1})(t)} = -\lambda(\omega),\ \lambda(\omega) \in \{-1, 0, 1\}\right\}. \quad (1.36)$$

Then $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$.

We now consolidate the last two theorems to demonstrate the necessary and sufficient conditions under which the increments of the process as well as the process itself enjoy the same convergence rate to zero as the derivative of the solution of (1.3), as well as the solution itself.

Theorem 1.16. *Suppose that f is locally Lipschitz continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. Let X be the continuous adapted solution of (1.2). Let $h > 0$ and define $S_f(\epsilon, h)$ as in (1.34). Then the following are equivalent:*

(a) $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$.
(b) There exists an event A with $\mathbb{P}[A] > 0$ such that

$$A = \left\{\omega : \lim_{t\to\infty} \frac{X(t,\omega)}{F^{-1}(t)} = \lambda(\omega), \right. \\ \left. \lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f \circ F^{-1})(t)} = -\lambda(\omega),\ \lambda(\omega) \in \{-1, 0, 1\}\right\}.$$

(c) There exists an event A with $\mathbb{P}[A] = 1$ such that

$$A = \left\{\omega : \lim_{t\to\infty} \frac{X(t,\omega)}{F^{-1}(t)} = \lambda(\omega), \right. \\ \left. \lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f \circ F^{-1})(t)} = -\lambda(\omega),\ \lambda(\omega) \in \{-1, 0, 1\}\right\}.$$

The proof that the statements are equivalent is now easy: part (a) implies part (c) by Theorem 1.14; part (c) trivially implies part (b); and part (b) implies (a) by Theorem 1.15.

1.6 Examples

We present in this section some examples to illustrate the scope of the results.

We note that all limits are possible in Theorem (1.2). To see this, simply take $x(0) > 0$ and $g(t) > 0$ for $t \geq 0$; then if $\int_t^\infty g(s)\,\mathrm{d}s/F^{-1}(t) \to 0$ as $t \to \infty$ we have that $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$ by Theorem 1.8. If $x(0) < 0$, $g(t) < 0$ for all $t \geq 0$, we can prove similarly that $x(t)/F^{-1}(t) \to -1$ as $t \to \infty$.

To show that a zero limit can obtain (and indeed that x can decay to zero arbitrarily rapidly), suppose that $d(0) = 1$ and that $d \in C^1((0,\infty);(0,\infty))$ with $d(t) \to 0$ as $t \to \infty$. Suppose that d decays to zero faster than any exponential function by assuming that $d'(t)/d(t) \to -\infty$ as $t \to \infty$. Then

$$\lim_{t\to\infty} \frac{d'(t)}{f(F^{-1}(t))} = 0.$$

Hence $d(t)/F^{-1}(t) \to 0$ as $t \to \infty$. Let $\xi > 0$ and define $g(t) = \xi d'(t) + f(\xi d(t))$ for $t \geq 0$. Then $x(t) = \xi d(t)$ for $t \geq 0$ is the solution of (1.1), and we have that

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 0.$$

Moreover, as $f(x)/x \to 0$ as $x \to 0$, we have that $|f(\xi d(t))| < d(t)$ for t sufficiently large. Thus $f(\xi d(t))/d'(t) \to 0$ as $t \to \infty$. Hence $g(t)/d'(t) \to \xi$ as $t \to \infty$. Therefore, as $d' \in L^1([0,\infty);\mathbb{R}$ with $d(t) = \int_t^\infty -d'(s)\,\mathrm{d}s$, we have that

$$\lim_{t\to\infty} \frac{\int_t^\infty g(s)\,\mathrm{d}s}{F^{-1}(t)} = 0,$$

as predicted by Theorem 1.5.

We start with a lemma which can be used to show that g can obey

$$\lim_{t\to\infty} \int_0^t g(s)\,\mathrm{d}s \text{ that exists and is finite,}$$

without being absolutely integrable and that this gives rise to less conservative stability conditions. In fact, we will use the lemma to demonstrate that there are perturbations g whose extremes can grow arbitrarily fast as $t \to \infty$ and which change sign infinitely often, but which nevertheless satisfy (1.13).

Lemma 1.8. *Let $k \in C^1((0,\infty);(0,\infty))$ be such that $k \notin L^1([0,\infty);(0,\infty))$. Suppose also that*

$$\lim_{t\to\infty} \sup_{0\leq s\leq T} \left|\frac{k(t+s)}{k(t)} - 1\right| = 0, \quad \lim_{t\to\infty} \sup_{0\leq s\leq T} \left|\frac{k'(t+s)}{k'(t)} - 1\right| = 0.$$

Define $k_0(t) = k(t)\sin(t)$ *for* $t \geq 0$. *Then*

(i) $\lim_{t\to\infty} \int_0^t k_0(s)\,\mathrm{d}s$ *is finite, but* $\lim_{t\to\infty} \int_0^t |k_0(s)|\,\mathrm{d}s = +\infty$*;*
(ii)

$$\limsup_{t\to\infty} \frac{\left|\int_t^\infty k_0(s)\,\mathrm{d}s\right|}{k(t)} = 1, \quad \liminf_{t\to\infty} \frac{\left|\int_t^\infty k_0(s)\,\mathrm{d}s\right|}{k(t)} = 0.$$

We note that if k is a positive, non-integrable function with $-k'$ regularly varying, then all the above conditions hold.

We can use this lemma to demonstrate that there are perturbations g whose extremes can grow arbitrarily fast as $t \to \infty$, and which change sign infinitely often, but which nevertheless satisfy (1.13).

Theorem 1.17. *Suppose that* f *is continuous and obeys* (1.11) *and* (1.12) *for* $\beta > 1$. *Let* Γ *be a function that obeys*

$$\Gamma \in C([0,\infty);(0,\infty)), \quad \lim_{t\to\infty} \Gamma(t) = +\infty.$$

Then there exists a function g *which obeys* (1.13)*, changes sign infinitely often and satisfies*

$$\limsup_{t\to\infty} \frac{|g(t)|}{\Gamma(t)} = 1,$$

and hence the continuous solution x *of* (1.1) *obeys* (1.14)*.*

The proof is deferred to Sect. 1.12. We notice that the function g constructed to verify this theorem does not merely oscillate, but does so with increasing frequency as $t \to \infty$. In fact, as $t \to \infty$, the number of sign changes of g in the interval $[t, t+1]$ tends to infinity. This rapid "self-cancellation" in g is what accounts for the good asymptotic behaviour of $\int_t^\infty g(s)\,\mathrm{d}s$. Furthermore, for an appreciable proportion of the time as $t \to \infty$, we have that $|g(t)| > \Gamma(t)/2$, so the periods of extreme behaviour of g are common.

In the case when g is a positive function which has rapidly growing extremes, we cannot rely on such fortuitous self-cancellation to preserve the asymptotic behaviour of the solution of (1.3) in (1.1). Instead, we show that while arbitrarily rapidly growing perturbations g can still preserve the rate of decay, such frequent extreme behaviour should be limited to relatively short intervals of time. In other words, short "spikes" in g are still admissible.

In order to demonstrate this, we start by establishing the following lemma. As often, the proof is deferred to the end.

Lemma 1.9. *Suppose* k_s *is a positive,* $C^1(0,\infty)$ *function with*

$$\lim_{t\to\infty} \int_t^\infty k_s(s) ds = 0$$

and $\lim_{t\to\infty} k_s(t) = 0$. *Suppose also that* $\Gamma \in C^1(0,\infty)$ *is increasing, with* $\Gamma(t) \nearrow \infty$. *Define* $\Gamma_+(t) = \Gamma(t) + \bar{k}_s + 1$, $t \geq 0$, *where* $\bar{k}_s = \sup_{t\geq 0} k_s(t)$ *and also define the sequence* $\{w_j\}_{j=0}^{\infty}$ *by*

$$w_j := \frac{1}{2} \wedge \frac{\int_{j+1}^{j+2} k_s(u)du}{\Gamma_+(j+1)}. \tag{1.37}$$

Suppose $a > 0$ *and* $b > 0$ *and consider the function*

$$h_s(x,a,b) := \begin{cases} b\left(1 - 3(\frac{x-a}{a})^2 - 2(\frac{x-a}{a})^3\right), \; x \in [0,a], \\ h(2a-x,a,b), \; x \in (a,2a]. \end{cases}$$

Then the function defined for $t \in [n, n+1]$, *for all* $n \geq 0$, *by*

$$k(t) := \begin{cases} k_s(t) + h(t-n, \frac{w_n}{2}, \Gamma_+(t) - k_s(t)), \; t \in [n, n+w_n), \\ k_s(t), \; t \in [n+w_n, n+1]. \end{cases} \tag{1.38}$$

is $C^1(0,\infty)$ *and obeys*

$$\lim_{t\to\infty} \frac{\int_t^\infty k(s)ds}{\int_t^\infty k_s(s)ds} = 1. \tag{1.39}$$

Furthermore, we note that

$$\limsup_{t\to\infty} \frac{k(t)}{\Gamma(t)} = 1. \tag{1.40}$$

Armed with this result we can now prove the following theorem.

Theorem 1.18. *Suppose that* f *is continuous and obeys* (1.11) *and* (1.12) *for* $\beta > 1$. *Let* Γ *be a function that obeys*

$$\Gamma \in C^1([0,\infty);(0,\infty)), \quad \lim_{t\to\infty} \Gamma(t) = +\infty.$$

Then there exists a function $g \in C^1((0,\infty);(0,\infty))$ *which obeys* (1.13) *and satisfies*

$$\limsup_{t\to\infty} \frac{|g(t)|}{\Gamma(t)} = 1,$$

and hence the continuous solution x *of* (1.1) *obeys* (1.14).

Proof. Let $k_s(t) = (\varphi \circ \Phi^{-1})(t)/(1+t)$ for $t \geq 0$. Notice that $k_s \in C^1((0,\infty);(0,\infty))$ tends to zero. In fact $k_s(t)/(f \circ F^{-1})(t) \to 0$ as $t \to \infty$, so by L'Hôpital's rule we get

$$\lim_{t\to\infty} \frac{\int_t^\infty k_s(u)\,\mathrm{d}u}{F^{-1}(t)} = 0.$$

Given this function k_s, by Lemma 1.9, there is a positive and C^1 function k defined by (1.38) which additionally satisfies

$$\lim_{t\to\infty} \frac{\int_t^\infty k(u)\,\mathrm{d}u}{\int_t^\infty k_s(u)\,\mathrm{d}u} = 1, \quad \limsup_{t\to\infty} \frac{k(t)}{\Gamma(t)} = 1.$$

Now let $g(t) = k(t)$ for all $t \geq 0$, so that g obeys (1.13) and

$$\limsup_{t\to\infty} g(t)/\Gamma(t) = 1,$$

as required, and hence Theorem 1.2 applies to the solution x of (1.1), as claimed. □

We can prove a similar result for the stochastic differential equation.

Theorem 1.19. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let Γ be a function that obeys*

$$\Gamma \in C^1([0,\infty);(0,\infty)), \quad \lim_{t\to\infty} \Gamma(t) = +\infty.$$

Then there exists a function $\sigma^2 \in C^1((0,\infty);(0,\infty))$ which obeys (1.27) with $\mu = 0$ and satisfies

$$\limsup_{t\to\infty} \frac{\sigma^2(t)}{\Gamma(t)} = 1,$$

and hence the continuous adapted process X which obeys (1.2) satisfies the conclusions of Theorem 1.3.

Proof. Let $\gamma = 1/(\beta - 1) + 1/2$. Let $k_s(t) = (1+t)^{-2\gamma-\epsilon}$ for $t \geq 0$. Notice that $k_s \in C^1((0,\infty);(0,\infty))$ tends to zero and $k_s \in L^1([0,\infty);\mathbb{R})$. We have that $\int_t^\infty k_s(u)\,\mathrm{d}u \in \mathrm{RV}_\infty(-2\gamma + 1 - \epsilon)$. Hence

$$t \mapsto \int_t^\infty k_s(u)\,\mathrm{d}u \log\log\left(\frac{1}{\int_t^\infty k_s(u)\,\mathrm{d}u}\right) \in \mathrm{RV}_\infty(-2\gamma + 1 - \epsilon).$$

Given this function k_s, by Lemma 1.9 there is a positive and C^1 function k defined by (1.38) which additionally satisfies

$$\lim_{t\to\infty} \frac{\int_t^\infty k(u)\,\mathrm{d}u}{\int_t^\infty k_s(u)\,\mathrm{d}u} = 1, \quad \limsup_{t\to\infty} \frac{k(t)}{\Gamma(t)} = 1.$$

Now let $\sigma^2(t) = k(t)$ for all $t \geq 0$. Then we have that

$$t \mapsto \int_t^\infty \sigma^2(u)\,\mathrm{d}u \log\log\left(\frac{1}{\int_t^\infty \sigma^2(u)\,\mathrm{d}u}\right) \in \mathrm{RV}_\infty(-2\gamma + 1 - \epsilon)$$

Since $F^{-1} \in \mathrm{RV}_\infty(-1/(\beta-1))$, we have that (1.27) holds with $\mu = 0$, because $2\gamma - 1 + \epsilon > 2/(\beta - 1)$. Moreover, $\limsup_{t\to\infty} \sigma^2(t)/\Gamma(t) = 1$, as required, and hence Theorem 1.3 applies to the solution X of (1.2), as claimed. □

In many cases it is straightforward to determine the asymptotic behaviour of differential equations directly, because the asymptotic behaviour of F^{-1} can be determined. The following result gives an easily checked and sufficient condition on f under which the asymptotic behaviour of F^{-1} can be read off.

Proposition 1.1. *Suppose that $f \in RV_0(\beta)$ is continuous and $\beta > 1$. Define*

$$\ell(x) = \left(\frac{f(x)}{x^\beta}\right)^{-1/(\beta-1)}$$

and assume that

$$\lim_{x\to 0} \frac{\ell(x\ell(x))}{\ell(x)} = 1. \tag{1.41}$$

If F is defined by (1.7),

$$F(x) \sim \frac{1}{\beta - 1}\frac{x}{f(x)}, \quad \text{as } x \to 0^+,$$

and

$$F^{-1}(t) \sim \left(\frac{1}{\beta-1}\right)^{1/(\beta-1)} t^{-1/(\beta-1)}\ell(t^{-1/(\beta-1)}), \quad \text{as } t \to \infty. \tag{1.42}$$

Proof. The proof is not hard and introduces useful notation for the rest of this section, so we give it here. Define $l(x) = f(x)/x^\beta$. Then $\ell(x) = l(x)^{-1/(\beta-1)}$. Since $f \in \mathrm{RV}_0(\beta)$ for $\beta > 1$ it follows that l and ℓ are both in $\mathrm{RV}_0(0)$. The asymptotic behaviour of F is well known. Since $f(x)/x^\beta = l(x)$, we have that $1/l(x) = \ell(x)^{\beta-1}$ as $x \to 0^+$, and so it is true that

$$F(x) \sim \frac{1}{\beta-1} x^{1-\beta}\ell(x)^{\beta-1}, \quad \text{as } x \to 0^+.$$

Define

$$G(t) = \left(\frac{1}{\beta - 1}\right)^{1/(\beta-1)} t^{-1/(\beta-1)} \ell(t^{-1/(\beta-1)}), \quad t \geq 1.$$

If we can show that $\lim_{t\to\infty} F(G(t))/t = 1$, then as $F^{-1} \in \mathrm{RV}_\infty(-1/(\beta-1))$, it follows that $G(t)/F^{-1}(t) \to 1$ as $t \to \infty$, which proves the claim.

Clearly, as $\ell \in \mathrm{RV}_0(0)$, we have that $G \in \mathrm{RV}_\infty(-1/(\beta-1))$ and thus $G(t) \to 0$ as $t \to \infty$. Hence, as $t \to \infty$, the asymptotic behaviour of F at 0 and the definition of G give

$$F(G(t)) \sim \frac{1}{\beta - 1} G(t)^{1-\beta} \ell(G(t))^{\beta-1} \sim t \left(\frac{\ell(G(t))}{\ell(t^{-1/(\beta-1)})}\right)^{\beta-1}. \tag{1.43}$$

Since $\ell \in \mathrm{RV}_0(0)$, we have that

$$\lim_{t\to\infty} \frac{\ell(G(t))}{\ell(t^{-1/(\beta-1)}\ell(t^{-1/(\beta-1)}))} = 1.$$

Therefore

$$\lim_{t\to\infty} \frac{\ell(G(t))}{\ell(t^{-1/(\beta-1)})}$$
$$= \lim_{t\to\infty} \frac{\ell(G(t))}{\ell(t^{-1/(\beta-1)}\ell(t^{-1/(\beta-1)}))} \cdot \frac{\ell(t^{-1/(\beta-1)}\ell(t^{-1/(\beta-1)}))}{\ell(t^{-1/(\beta-1)})} = 1,$$

because the second limit is unity, by (1.41). Returning to (1.43), we see that $F(G(t))/t \to 1$ as $t \to \infty$, as we required. □

Once a regularly varying function f has been given, ℓ is determined. It happens that many regularly varying functions f enjoy the property (1.41). We give the details now for a parameterised family of such functions.

Example 1.1. Suppose for instance that $\beta > 1$ and β_1 and β_2 are real and f obeys

$$\lim_{x\to 0^+} \frac{f(x)}{a|x|^\beta \log^{\beta_1}(1/|x|)\{\log\log(1/|x|)\}^{\beta_2}\mathrm{sgn}(x)} = 1.$$

Then, in the terminology above, we may take

$$l(x) = a \log^{\beta_1}(1/x) \log\log(1/x)^{\beta_2}$$

for $x > 0$ sufficiently small. Then

$$\ell(x) = l(x)^{-1/(\beta-1)} = a^{-1/(\beta-1)} \log^{-\beta_1/(\beta-1)}(1/x)\{\log\log(1/x)\}^{-\beta_2/(\beta-1)}.$$

Hence $x\ell(x)$ is in $\text{RV}_0(1)$ and so $\log(1/(x\ell(x)))/\log(1/x) \to 1$ as $x \to 0^+$. This implies $\log\log(1/(x\ell(x)))/\log\log(1/x) \to 1$ as $x \to 0^+$. Armed with these limits and the definition of ℓ we get

$$\begin{aligned}
&\lim_{x\to 0^+} \frac{\ell(x\ell(x))}{\ell(x)} \\
&= \lim_{x\to 0^+} \frac{a^{-1/(\beta-1)} \log^{-\beta_1/(\beta-1)}(1/(x\ell(x)))\{\log\log(1/(x\ell(x)))\}^{-\beta_2/(\beta-1)}}{a^{-1/(\beta-1)} \log^{-\beta_1/(\beta-1)}(1/x)\log\log(1/x)^{-\beta_2/(\beta-1)}} \\
&= 1,
\end{aligned}$$

and so (1.41) holds. Therefore, by Proposition 1.1, we have that

$$F^{-1}(t) \sim \left(\frac{1}{a(\beta-1)}\right)^{1/(\beta-1)} t^{-1/(\beta-1)} \left(\frac{1}{\beta-1}\log t\right)^{-\beta_1/(\beta-1)} \times (\log\log t)^{-\beta_2/(\beta-1)}, \quad \text{as } t \to \infty.$$

Suppose now that g is continuous such that $\int_t^\infty g(s)\,\mathrm{d}s > 0$ and obeys

$$\lim_{t\to\infty} \frac{\int_t^\infty g(s)\,\mathrm{d}s}{t^{-1/(\beta-1)}\,(\log t)^{-\beta_1/(\beta-1)}\,(\log\log t)^{-\beta_2/(\beta-1)}} =: \mu_D \in [-\infty, \infty].$$

If we suppose that f has the above asymptotic behaviour at zero, is locally Lipschitz continuous on $\mathbb{R}$ and obeys (1.21), then there is a unique continuous solution of (1.1) which obeys $x(t) \to 0$ as $t \to \infty$. Furthermore, if $\mu_D = 0$, then

$$\lim_{t\to\infty} \frac{x(t)}{t^{-1/(\beta-1)}\left(\frac{1}{\beta-1}\log t\right)^{-\beta_1/(\beta-1)}\,(\log\log t)^{-\beta_2/(\beta-1)}} \in \{-1, 0, 1\}.$$

If, on the other hand, $\mu_D \neq 0$, then the above limit may not exist and cannot be 0 or ± 1.

Suppose that f has the same properties [but not necessarily (1.21)], and consider instead the solution of the stochastic equation (1.2) where $\sigma \in L^2(;0,\infty);\mathbb{R})$ is a continuous function for which

$$\lim_{t\to\infty} \frac{\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(1/\int_t^\infty \sigma^2(s)\,\mathrm{d}s\right)}{t^{-2/(\beta-1)}\,(\log t)^{-2\beta_1/(\beta-1)}\,(\log\log t)^{-2\beta_2/(\beta-1)}} =: \mu_S \in [0, \infty].$$

Then the unique continuous adapted process X which obeys (1.2) obeys $X(t) \to 0$ as $t \to \infty$ a.s. Furthermore, if $\mu_S = 0$, we have that

$$\frac{X(t)}{t^{-1/(\beta-1)}\left(\frac{1}{\beta-1}\log t\right)^{-\beta_1/(\beta-1)}\,(\log\log t)^{-\beta_2/(\beta-1)}} \in \{-1, 0, 1\}, \quad \text{a.s.,}$$

while if $\mu_S \neq 0$, we have that $X(t)/F^{-1}(t)$ tends to a limit with zero probability.

Example 1.2. We have seen that when $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$ and *some* $h > 0$, then $X(t)/F^{-1}(t)$ has limit in $\{-1, 0, 1\}$ a.s. and that $-(X(t+h) - X(t))/h)/(f \circ F^{-1})(t)$ also has the same limit a.s. Therefore, we see that preservation of the asymptotic behaviour of the finite difference approximation to the derivative of (1.3) in the solution of (1.2) requires a condition on σ which is not weaker than that required to preserve solely the asymptotic behaviour of the solution of (1.3).

In the following example, we explore two aspects of these asymptotic results. First, it is our conjecture that if there is *some* $h' > 0$ for which $S_f(\epsilon, h') < +\infty$ for all $\epsilon > 0$, then it is the case that $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$ and *all* $h > 0$. Therefore, if the asymptotic behaviour of the finite difference approximation to the derivative of (1.3) is preserved for any step-size $h' > 0$, it will be preserved for any fixed time step $h > 0$. Second, we see that the condition under which the finite difference approximation of the derivative is preserved is *strictly* stronger than that needed to preserve the asymptotic behaviour of the underlying unperturbed deterministic equation (1.3).

Let us take for definiteness the simple case when $f(x) \sim a|x|^\beta \mathrm{sgn}(x)$ as $x \to 0$ for some $a > 0$ and $\beta > 1$. Suppose also that $\sigma(t) \sim ct^{-\gamma}$ as $t \to \infty$ for $c \neq 0$ and $\gamma > 0$. If $\gamma \leq 1/2$, we have that $\sigma \notin L^2([0,\infty); \mathbb{R})$, so the solution of (1.2) cannot inherit the decay properties of the solution of (1.3).

Therefore, we let $\gamma > 1/2$. We note that elementary considerations, or Proposition 1.1, enable us to show that

$$F^{-1}(t) \sim \left(\frac{1}{a(\beta-1)}\right)^{1/\beta-1} t^{-1/(\beta-1)}, \quad \text{as } t \to \infty$$

and of course $F^{-1} \in \mathrm{RV}_\infty(-1/(\beta-1))$. Clearly $\sigma^2 \in \mathrm{RV}_\infty(-2\gamma)$, and so $t \mapsto \int_t^\infty \sigma^2(s)\,\mathrm{d}s \in \mathrm{RV}_\infty(-2\gamma+1)$. On account of the logarithmic factor, we see that for $\gamma > (\beta+1)/(2(\beta-1))$ we have that

$$\lim_{t\to\infty} \frac{X(t)}{\left(\frac{1}{a(\beta-1)}\right)^{1/\beta-1} t^{-1/(\beta-1)}} \in \{-1, 0, 1\}, \quad \text{a.s.}$$

while for $\gamma \leq (\beta+1)/(2(\beta-1))$ the limit on the left-hand side exists with probability zero.

Considering $S_f(\varepsilon, h)$, we need to find the asymptotic behaviour of

$$\frac{\int_{nh}^{(n+1)h} \sigma^2(s)\,\mathrm{d}s}{(f \circ F^{-1})^2(nh)}.$$

The numerator scales like $K_1(h)n^{-2\gamma}$ as $n \to \infty$; since $(f \circ F^{-1})(t) \sim K_2 t^{-\beta/(\beta-1)}$ as $t \to \infty$, the denominator behaves according to $(f \circ F^{-1})(nh) \sim K_2(h)^2 n^{-2\beta/(\beta-1)}$ as $n \to \infty$. Hence

$$\left(\frac{\int_{nh}^{(n+1)h} \sigma^2(s)\,\mathrm{d}s}{(f \circ F^{-1})^2(nh)}\right)^{1/2} \sim K_3(h) n^{-\gamma+\beta/(\beta-1)}$$

as $n \to \infty$. Therefore, if $\gamma > \beta/(\beta-1)$, it follows that $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$ and all $h > 0$: thus in this case, $(X(t+h) - X(t)/h)/t^{-\beta/(\beta-1)}$ tends to a (known) constant limit with probability one. If, however, $\gamma \le \beta/(\beta-1)$, then for every $h > 0$, $S_f(\epsilon, h) = +\infty$ for all $\epsilon > 0$, and so $(X(t+h) - X(t)/h)/t^{-\beta/(\beta-1)}$ tends to a finite limit with probability zero.

Since for $\beta > 1$ it is always the case that $(\beta+1)/(2(\beta-1)) < \beta/(\beta-1)$, we see that there exists γ for which the asymptotic behaviour of the approximation to the derivative does not behave like that of the underlying deterministic equation, while the asymptotic behaviour of X itself does.

1.7 Simulations

In order to graphically illustrate our results for stochastic equation, in this section we plot graphs derived from a simulation of a single sample path from the SDE

$$\mathrm{d}X(t) = -\mathrm{sgn}(X(t))|X(t)|^{\beta}\,\mathrm{d}t + (1+t)^{-\gamma}\mathrm{d}B(t),\ t \ge 0, \tag{1.44}$$

where B is a standard Brownian motion. This is an example of the class of equation examined in Example 1.2.

In the first instance, we look at the case where $\beta = 3$ and $\gamma = 2.5$. With these parameters we expect to see both ODE-like asymptotics in the solution and the finite difference of approximation of the (non-existent) derivative of X behaving like those of the ODE as $t \to \infty$. This equation was discretised using the standard explicit Euler–Maruyama method. Convergent solutions of such a numerical scheme are known to possess some asymptotic properties in common with the underlying SDE, as established in [8], and therefore such simulations should capture faithfully the asymptotic behaviour of the SDE. However, a proof of that this is the case remains to date open: we hope to address this situation for both explicit and implicit Euler-type schemes in a later work.

It should be remarked that the limit $F(X(t))/t \to 1$ as $t \to \infty$ is observed. Simulations seem to confirm that the limits $X(t)/F^{-1}(t) \to \pm 1$ both occur with positive probability but that the limit $X(t)/F^{-1}(t) \to 0$ seems to happen with probability zero (Figs. 1.1 and 1.2).

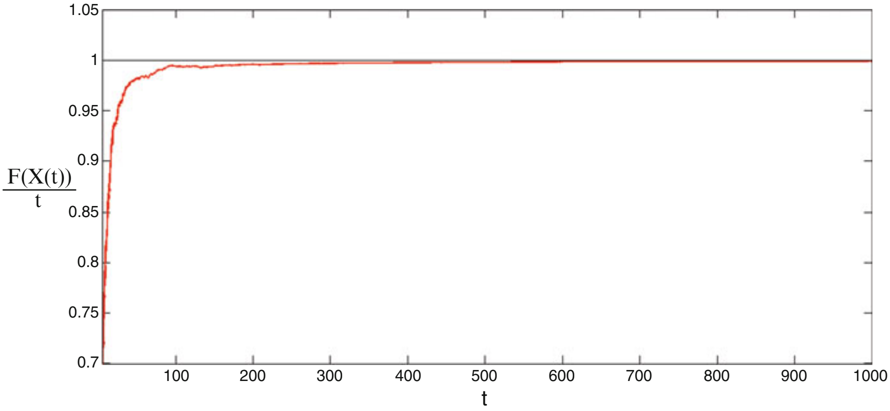

Fig. 1.1 We observe that the asymptotic regime of the solution quickly settles down to that of the corresponding ODE

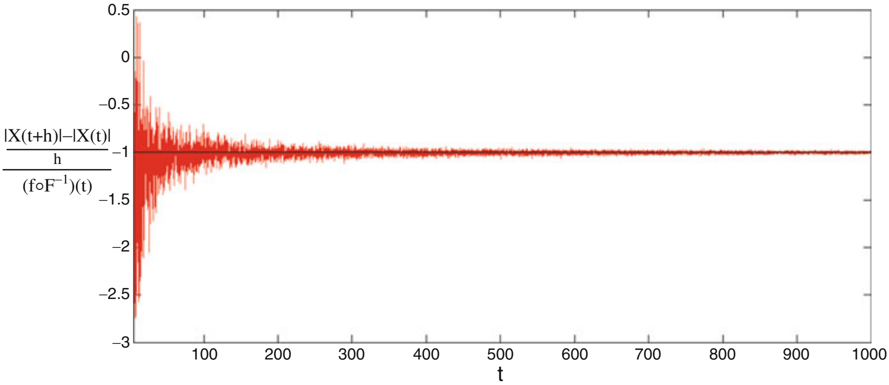

Fig. 1.2 Once time becomes large enough the approximation to the derivative is well behaved

Below we have two more graphs derived from a single path of (1.44), but in this case the parameters are $\beta = 3$ and $\gamma = 1.5$. Hence we expect to see the ODE asymptotics preserved but we do not expect to retain the nice asymptotic behaviour of the derivative of the underlying ODE being preserved. The plots confirm this hypothesis (Figs. 1.3 and 1.4). All graphs presented thus far have been with initial condition $X(0) = 1$. We now show two graphs derived from a path of (1.44) with $\beta = 3$ and $\gamma = 2.5$, as before, but with initial condition $X(0) = 0$. This helps us to demonstrate some novel behaviour of the scaled finite differences, in particular, the appearance of transient phases which considerably slow convergence to the expected limiting value (Figs. 1.5 and 1.6).

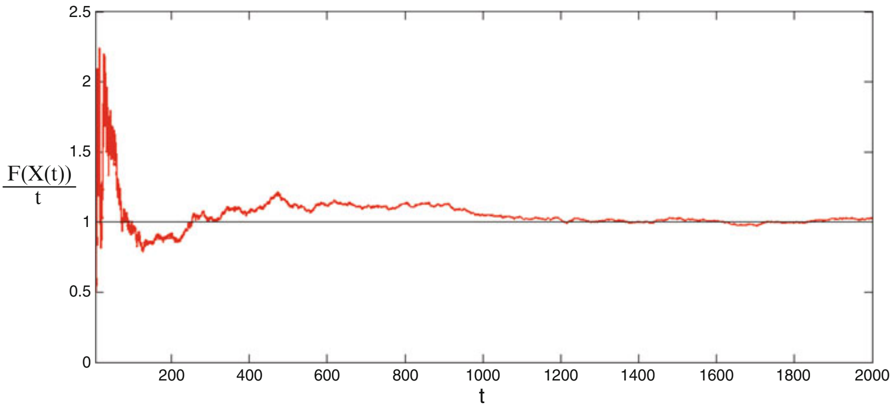

Fig. 1.3 As before the solution settles down to asymptotic regime of the corresponding ODE

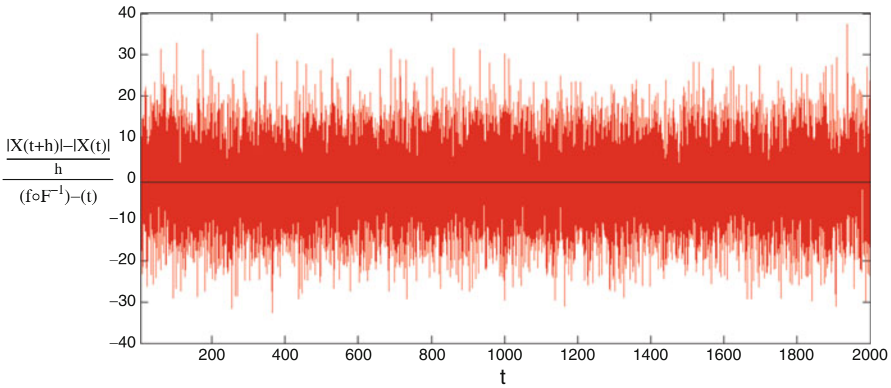

Fig. 1.4 It is clear that the increased volume of "noise" present has caused us to lose the limiting behaviour of the scaled difference in this instance

1.8 Proof of Theorem 1.1

1.8.1 Idea and Outline of the Proof

Theorem 1.1 is the key underlying result of this paper, and its proof relies on careful asymptotic analysis and a number of interlinked intermediate results. Accordingly, we take a moment to summarise the structure of the proof, which consists of a number of steps. First, we establish that f being asymptotically odd and regularly varying implies that f is asymptotic to a regularly varying, increasing and C^1 function φ that is odd: in other words, f is asymptotic to a function with improved regularity properties. Then, we show three things: first, that $t \mapsto |x(t)|$ can be written in terms of the solution of a differential inequality which depends solely on

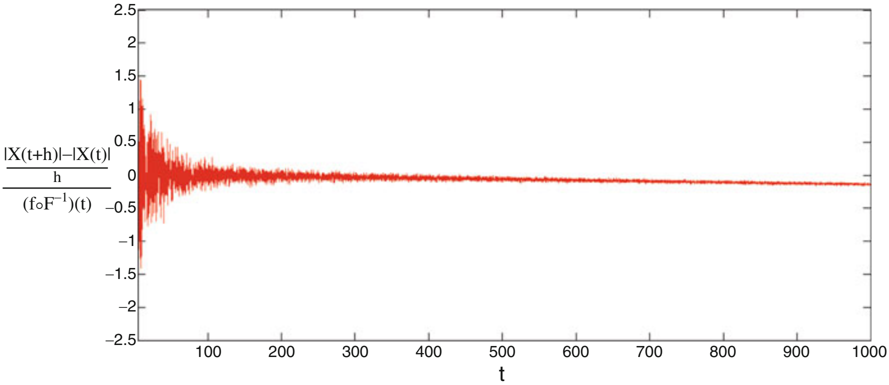

Fig. 1.5 We note here how the scaled differences go through a surprisingly long transient period in which they are very close to zero, almost up to time 1,000. However, upon careful inspection, we can see that the graph is in fact beginning to lift away from zero

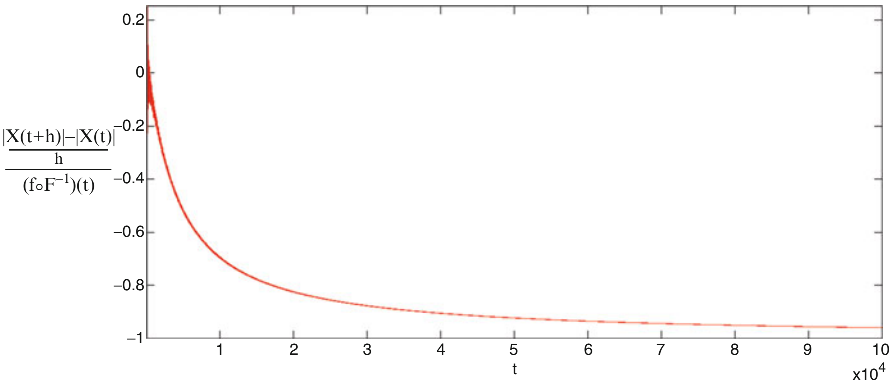

Fig. 1.6 In this case we show the full path out to time 100,000 and it is clear that the above graph was indeed a transient phase, as claimed, and that the expected limit does eventually prevail

φ modulo some small parameter which deals with the asymptotic behaviour of f; second, that $x(t) \to 0$ as $t \to \infty$; and third, that $\lim_{t\to\infty} \gamma(t)/F^{-1}(t) \to 0$ as $t \to \infty$.

The rest of the proof involves a successive "ratcheting" of the asymptotic results: the last two steps of the proof in particular rely on constructing functions that are guaranteed to majorise and minorise x for sufficiently large t. The majorisation relies on a comparison principle based on the differential inequality derived for $t \mapsto |x(t)|$; the minorisation also relies on a comparison argument, but on this occasion the original ODE (1.1) is employed to make the comparison argument work. In particular, we prove the result through the following steps:

Steps 1: $\liminf_{t\to\infty} |x(t)|/F^{-1}(t) = 0$ or 1.
Steps 2: $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 0$ or $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) \in [1,\infty)$.

Steps 3: If $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) > 0$, then $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 1$.
Steps 4: If $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 1$, then $\liminf_{t\to\infty} |x(t)|/F^{-1}(t) = 1$.

Of course, it can be seen that Steps 3 and 4 together imply that the limit of $t \mapsto x(t)/F^{-1}(t)$ must exist and be 0, -1, or 1, which is the desired result.

The sequence of steps mimics those used to determine the asymptotic behaviour in [3] for stochastic differential equations and in [8] for stochastic difference equations in the special case that $f(x)$ is asymptotic to $a|x|^{\beta}\mathrm{sgn}(x)$ as $x \to 0$ for $a > 0$. The proofs of Steps 1, 3, and 4 differ from those in both papers, although comparison arguments are employed. The proof of Step 2 is essentially identical to that used in both papers.

The rest of this section is devoted to the statement and proof of technical results.

1.8.2 Statement and Proofs of Technical Results

Lemma 1.10. *Suppose that f obeys (1.11) and (1.12). Then there exists a function φ such that*

$$\varphi \text{ is increasing, in } C^1(\mathbb{R}), \text{ is odd and } \varphi \in RV_0(\beta), \tag{1.45}$$

and

$$\lim_{x\to 0} \frac{f(x)}{\varphi(x)} = 1. \tag{1.46}$$

Moreover, if $\beta > 1$ and we define

$$\Phi(x) = \int_x^1 \frac{1}{\varphi(u)}\,\mathrm{d}u, \quad x > 0, \tag{1.47}$$

we have that

$$\lim_{x\to 0+} \frac{\Phi(x)}{F(x)} = 1, \quad \lim_{t\to\infty} \frac{\Phi^{-1}(t)}{F^{-1}(t)} = 1. \tag{1.48}$$

Proof. Recall that $f \in \mathrm{RV}_0(\beta)$ implies that there exists $\phi_+ \in C^1$ such that

$$\lim_{x\to 0+} \frac{f(x)}{\phi_+(x)} = 1, \quad \lim_{x\to 0+} \frac{x\,\phi_+'(x)}{\phi_+(x)} = \beta > 0.$$

Thus there exists $\delta > 0$ such that ϕ_+ is increasing and C^1 on $(0, \delta)$. We can extend ϕ_+ to all of $(0, \infty)$ in such a manner that ϕ_+ is increasing and C^1 on all of $(0, \infty)$. Define

$$\varphi(x) = \begin{cases} \phi_+(x), & x > 0, \\ 0, & x = 0, \\ -\phi_+(x), & x < 0. \end{cases}$$

Then φ is increasing and odd on $\mathbb{R}$. Moreover, we have

$$\varphi'(0^+) = \lim_{x\to 0^+} \frac{\varphi(x) - \varphi(0)}{x} = \lim_{x\to 0^+} \frac{\phi_+(x)}{x} = 0,$$

since $f \sim \phi_+$. Similarly,

$$\varphi'(0^-) = \lim_{x\to 0^-} \frac{\varphi(x) - \varphi(0)}{x} = \lim_{x\to 0^-} \frac{-\phi_+(-x)}{x} = \lim_{x\to 0^+} \frac{\phi_+(-x)}{-x} = 0.$$

Hence, as φ is in $C^1(0,\infty)$ and $C^1(-\infty,0)$, we conclude that $\varphi \in C^1(\mathbb{R})$. Finally,

$$\lim_{x\to 0^+} \frac{f(x)}{\varphi(x)} = \lim_{x\to 0^+} \frac{f(x)}{\phi_+(x)} = 1.$$

Similarly, we have

$$\begin{aligned}\lim_{x\to 0^-} \frac{f(x)}{\varphi(x)} &= \lim_{x\to 0^-} \frac{f(x)}{\varphi(x)} \frac{\varphi(x)}{-\phi_+(-x)} = \lim_{x\to 0^-} \frac{f(x)}{\varphi(x)} \frac{-\varphi(-x)}{-\phi_+(-x)} \\ &= \lim_{x\to 0} \frac{f(x)}{\varphi(x)} \frac{\varphi(-x)}{f(-x)} \frac{f(-x)}{\phi_+(-x)} = 1,\end{aligned}$$

as required. For $\beta > 1$, the asymptotic behaviour of Φ defined in (1.48) is a consequence of the regular variation of f and the fact that f is asymptotic to φ. □

Although x is continuously differentiable, $t \mapsto |x(t)|$ will not be differentiable if x assumes zero values. Since this cannot be ruled out, we derive a differential inequality (in terms of Dini derivatives) for $t \mapsto |x(t)|$. Accordingly, we use in the next proof the notation

$$D_+u(t) = \limsup_{h\to 0, h>0} \frac{u(t+h) - u(t)}{h}$$

for the appropriate Dini derivative.

Lemma 1.11. *Suppose that f satisfies (1.11) and (1.12) with $\beta > 1$. Suppose that γ is continuous and x is the unique continuous solution of*

$$x'(t) = -f(x(t) + \gamma(t)), \quad t \geq 0, \tag{1.49}$$

such that

$$\lim_{t\to\infty} x(t) = 0. \tag{1.50}$$

Suppose also that γ *and* F *obey* (1.10)*. If* φ *is the function in* (1.45) *which satisfies (1.46), and* Φ *is defined by* (1.47)*, then for every* $\epsilon \in (0,1)$ *there exist* $T_1(\epsilon) > 0$ *and* $T(\epsilon) > 0$ *such that*

$$|\gamma(t)| < \epsilon\Phi^{-1}(t), \quad t \geq T_1(\epsilon), \tag{1.51}$$

and

$$D_+|x(t)| \leq -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \geq T(\epsilon), \tag{1.52}$$

where

$$\varphi_\epsilon(x) := \min\left((1+\epsilon)\varphi(x),\ (1-\epsilon)\varphi(x)\right).$$

Proof. Fix $t > 0$ and suppose that $x(t) > 0$. Then as $x \in C^1$, there exists a h_1 small enough so that $x(t+h) > 0$ for all $0 < h < h_1$. Thus for $h < h_1$

$$\frac{|x(t+h)| - |x(t)|}{h} = \frac{x(t+h) - x(t)}{h} = \frac{1}{h}\int_t^{t+h} -f(x(s)+\gamma(s))ds.$$

Thus we obtain $D_+|x(t)| = -f(x(t)+\gamma(t)) = -f(|x(t)|+\gamma(t))$. If $x(t) < 0$, then there exists a $h_2 > 0$ such that $x(t+h) < 0$ for all $0 < h < h_2$. Similarly, we can write

$$\frac{|x(t+h)| - |x(t)|}{h} = -\frac{x(t+h) - x(t)}{h} = -\frac{1}{h}\int_t^{t+h} -f(x(s)+\gamma(s))ds.$$

Hence $D_+|x(t)| = f(x(t)+\gamma(t)) = f(-|x(t)|+\gamma(t))$. Finally, if $x(t) = 0$, for $h > 0$ we have

$$\frac{|x(t+h)| - |x(t)|}{h} = \left|\frac{x(t+h)}{h}\right| = \left|\frac{x(t+h) - x(t)}{h}\right|.$$

Thus $D_+|x(t)| = |x'(t)| = |-f(x(t)+\gamma(t))| = |f(x(t)+\gamma(t))|$. Therefore we have

$$D_+|x(t)| = -f(|x(t)|+\gamma(t)),\ x(t) > 0. \tag{1.53}$$

$$D_+|x(t)| = f(-|x(t)|+\gamma(t)),\ x(t) < 0. \tag{1.54}$$

$$D_+|x(t)| = |f(x(t)+\gamma(t))|,\ x(t) = 0. \tag{1.55}$$

Next, by Lemma 1.10, there exists a function φ satisfying (1.45) and (1.46). Since $\gamma(t)/F^{-1}(t) \to 0$ as $t \to \infty$, we have that $\gamma(t)/\Phi^{-1}(t) \to 0$ as $t \to \infty$. Hence, for every $\epsilon > 0$, there exists $T_1(\epsilon) > 0$ such that $|\gamma(t)| < \epsilon\Phi^{-1}(t)$ for all $t \geq T_1(\epsilon)$, as claimed.

By (1.46), for all $\epsilon > 0$ there is $x_1(\epsilon) > 0$ such that

$$1 - \epsilon < \frac{f(x)}{\varphi(x)} < 1 + \epsilon, \quad |x| < x_1(\epsilon), x \neq 0.$$

Therefore, as $f(0) = \varphi(0) = 0$ and $x\varphi(x) > 0$ for all $x \neq 0$, this implies that

$$\begin{aligned} -(1+\epsilon)\varphi(x) \leq -f(x) \leq -(1-\epsilon)\varphi(x), &\quad 0 \leq x < x_1(\epsilon), \\ -(1-\epsilon)\varphi(x) < -f(x) < -(1+\epsilon)\varphi(x), &\quad -x_1(\epsilon) < x < 0. \end{aligned}$$

Since $x(t) \to 0$ as $t \to \infty$ and $\gamma(t) \to 0$ as $t \to \infty$, there is a $T^*(\epsilon)$ large enough such that for all $\epsilon > 0$ we have $|x(t)| + |\gamma(t)| < x_1(\epsilon)$. Set $T(\epsilon) = 1 + \max(T^*(\epsilon), T_1(\epsilon))$. We now deduce that the differential inequality (1.52) holds for $t \geq T(\epsilon)$ by considering separately the cases when $x(t)$ is positive, negative and zero.

1. If $x(t) > 0$, we have from (1.53) that $D_+|x(t)| = -f(|x(t)| + \gamma(t))$. Therefore, for $t \geq T(\epsilon)$, the argument of $-f$ has modulus less than $x_1(\epsilon)$. Hence, if $|x(t)| + \gamma(t) \geq 0$, we have $-f(|x(t)| + \gamma(t)) \leq -(1-\epsilon)\varphi(|x(t)| + \gamma(t)) \leq 0$. Now $-\epsilon\Phi^{-1}(t) < \gamma(t)$, so $|x(t)| - \epsilon\Phi^{-1}(t) < |x(t)| + \gamma(t)$. Since φ is increasing, we have

   $$-(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)) > -(1-\epsilon)\varphi(|x(t)| + \gamma(t)).$$

   Hence

   $$\begin{aligned} D_+|x(t)| = -f(|x(t)| + \gamma(t)) &\leq -(1-\epsilon)\varphi(|x(t)| + \gamma(t)) \\ &< -(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)). \end{aligned}$$

   Suppose on the other hand that $|x(t)| + \gamma(t) < 0$. Since $t > T(\epsilon)$ we have that $-x_1(\epsilon) < |x(t)| + \gamma(t) < 0$. Then $-f(|x(t)| + \gamma(t)) < -(1+\epsilon)\varphi(|x(t)| + \gamma(t))$ and it is moreover the case that $0 > \gamma(t) > -\epsilon\Phi^{-1}(t)$. Hence $|x(t)| - \epsilon\Phi^{-1}(t) < |x(t)| + \gamma(t)$. Since φ is increasing, we have

   $$-(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)) > -(1+\epsilon)\varphi(|x(t)| + \gamma(t)).$$

   Hence

   $$\begin{aligned} D_+|x(t)| = -f(|x(t)| + \gamma(t)) &< -(1+\epsilon)\varphi(|x(t)| + \gamma(t)) \\ &< -(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)). \end{aligned}$$

Therefore, when $x(t) > 0$, by using the fact that $a < b$ and $a < c$ imply $a < \max(b, c)$, we have

$$\begin{aligned} D_+|x(t)| &< \max(-(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)), -(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t))) \\ &= -\min((1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)), (1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t))) \\ &= -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \end{aligned}$$

where we have used the definition of φ_ϵ at the last step. Hence

$$D_+|x(t)| <= -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \geq T(\epsilon), \quad x(t) > 0. \tag{1.56}$$

2. If $x(t) < 0$, so $|x(t)| = -x(t) > 0$. First we note that for $t \geq T(\epsilon)$ that $D_+|x(t)| = f(-|x(t)| + \gamma(t))$. Suppose first that $-|x(t)| + \gamma(t) \geq 0$. Then, we have that $\gamma(t) \geq 0$. Hence $\gamma(t) > -\epsilon\Phi^{-1}(t)$. Therefore, as $x_1(\epsilon) > -|x(t)| + \gamma(t) \geq 0$, $-f(-|x(t)| + \gamma(t)) \geq -(1+\epsilon)\varphi(-|x(t)| + \gamma(t))$. Hence, as φ is odd, we get

$$\begin{aligned} D_+|x(t)| = f(-|x(t)| + \gamma(t)) &\leq (1+\epsilon)\varphi(-|x(t)| + \gamma(t)) \\ &= -(1+\epsilon)\varphi(|x(t)| - \gamma(t)). \end{aligned}$$

Next as φ is increasing, $-(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)) > -(1+\epsilon)\varphi(|x(t)| - \gamma(t))$, so

$$D_+|x(t)| \leq -(1+\epsilon)\varphi(|x(t)| - \gamma(t)) < -(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)).$$

Suppose next that $-|x(t)| + \gamma(t) < 0$. Then $-x_1(\epsilon) < -|x(t)| + \gamma(t) < 0$, and we have that $-f(-|x(t)| + \gamma(t)) > -(1-\epsilon)\varphi(-|x(t)| + \gamma(t))$. Hence as φ is odd we get

$$\begin{aligned} D_+|x(t)| = f(-|x(t)| + \gamma(t)) &< (1-\epsilon)\varphi(-|x(t)| + \gamma(t)) \\ &= -(1-\epsilon)\varphi(|x(t)| - \gamma(t)). \end{aligned}$$

Now, as φ is increasing, it follows that we have $-(1-\epsilon)\varphi(|x(t)| - \gamma(t)) < -(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t))$, so

$$D_+|x(t)| < -(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)).$$

Therefore, regardless of the sign of $-|x(t)| + \gamma(t)$, we have that

$$D_+|x(t)| < \max(-(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)), -(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)))$$

and so by the definition of φ_ϵ we get

$$D_+|x(t)| < -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \ge T(\epsilon), \quad x(t) < 0. \tag{1.57}$$

3. When $x(t) = 0$ we have

$$D_+|x(t)| = |f(\gamma(t))| \le (1+\epsilon)|\varphi(\gamma(t))|, \quad t \ge T(\epsilon).$$

Therefore, as φ is odd and increasing, we have

$$\begin{aligned}|\varphi(\gamma(t))| &= \varphi(|\gamma(t)|) \le \varphi(\epsilon\Phi^{-1}(t)) \\ &= -\varphi(-\epsilon\Phi^{-1}(t)) = -\varphi(|x(t)| - \epsilon\Phi^{-1}(t)).\end{aligned}$$

Hence for $t \ge T(\epsilon)$ we obtain

$$\begin{aligned}D_+|x(t)| &\le -(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)) \\ &\le \max(-(1+\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t)), -(1-\epsilon)\varphi(|x(t)| - \epsilon\Phi^{-1}(t))) \\ &= -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)))\end{aligned}$$

from the definition of φ_ϵ. Hence

$$D_+|x(t)| \le -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \ge T(\epsilon), \quad x(t) = 0. \tag{1.58}$$

Combining (1.56), (1.57) and (1.58) we have that for all $t \ge T(\epsilon)$

$$D_+|x(t)| \le -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \ge T(\epsilon), \tag{1.59}$$

as required.

□

Lemma 1.12. *Suppose that f satisfies (1.11) and (1.12) with $\beta > 1$. Suppose that γ is continuous and x is the unique continuous solution of (1.49) which satisfies (1.50). Suppose also that γ and F obey (1.10). Then*

$$\liminf_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = 0 \text{ or } 1.$$

Proof. Either $\liminf_{t\to\infty} |x(t)|/F^{-1}(t) = 0$ or $\liminf_{t\to\infty} |x(t)|/F^{-1}(t) \in (0,\infty]$. Suppose that

$$\liminf_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = M \in (0,\infty), \quad M \ne 1.$$

Then there exists $T_0 > 0$ such that $|x(t)| > \frac{M}{2}F^{-1}(t)$ for all $t \geq T_0$. Hence it follows that $\lim_{t\to\infty}(x(t)+\gamma(t))/x(t) = 1$, since $\gamma(t)/F^{-1}(t) \to 0$ as $t \to \infty$. Thus, as φ is asymptotic to f, we have

$$\lim_{t\to\infty} \frac{f(x(t)+\gamma(t))}{\varphi(x(t))} = 1.$$

Hence $\lim_{t\to\infty} x'(t)/\varphi(x(t)) = -1$, and integrating yields

$$\lim_{t\to\infty} \Phi(|x(t)|)/t = 1,$$

which implies that $\lim_{t\to\infty} |x(t)|/\Phi^{-1}(t) = 1$. Hence $\lim_{t\to\infty} |x(t)|/F^{-1}(t) = 1$. Since by supposition $\liminf_{t\to\infty} |x(t)|/F^{-1}(t) = M \neq 1$, we have a contradiction. Therefore, if the liminf is finite and non-zero, it must be unity. We now rule out the possibility that

$$\liminf_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = +\infty$$

Suppose this holds. Then there is $T_0 > 0$ such that for all $t \geq T_0$, $|x(t)| > 2F^{-1}(t)$. Arguing as above, we prove once again that this leads to

$$\lim_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = 1,$$

which contradicts our supposition. Therefore, we must have that the liminf is either zero or unity, as all other possibilities have been eliminated. □

Lemma 1.13. *Suppose that f satisfies (1.11) and (1.12) with $\beta > 1$. Suppose that γ is continuous and x is the unique continuous solution of (1.49) which satisfies (1.50). Suppose also that γ and F obey (1.10). Then*

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = 0 \text{ or } \limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} \in [1,\infty].$$

Proof. Applying Lemma 1.10 to f we know there exists a φ satisfying (1.45) and (1.46) with $\beta > 1$. Thus

$$\frac{\varphi((\lambda+\epsilon)x)}{\varphi(x)} < (\lambda+\epsilon)^{\beta}(1+\epsilon), \quad |x| < x_0(\epsilon).$$

Suppose that

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = \lambda \in (0,\infty).$$

Then, for every $\epsilon > 0$, there is $T'(\epsilon) > 0$ such that $|x(t)| < (\lambda + \frac{\epsilon}{2})F^{-1}(t)$ for all $t \geq T'(\epsilon)$. By (1.10), we also have that there is $T''(\epsilon) > 0$ and $T^* > 0$ such that $|\gamma(t)| < \frac{\epsilon}{2}F^{-1}(t)$ for all $t \geq T''(\epsilon)$ and $F^{-1}(t) < x_0(\epsilon)$ for all $t \geq T^*$. Define $T'''(\epsilon) = \max(T'(\epsilon), T''(\epsilon))$, which implies $|x(t) + \gamma(t)| < (\lambda + \epsilon)F^{-1}(t)$ for all $t \geq T'''(\epsilon)$. Hence, as φ is odd and increasing, for $t \geq T'''(\epsilon)$, we have

$$\begin{aligned}|f(x(t) + \gamma(t))| &< (1 + \epsilon)\varphi(|x(t) + \gamma(t)|) < (1 + \epsilon)\varphi((\lambda + \epsilon)F^{-1}(t)) \\ &< (1 + \epsilon)^2(\lambda + \epsilon)^\beta(\varphi \circ F^{-1})(t).\end{aligned}$$

Therefore, for $t \geq T'''(\epsilon)$,

$$\left|\int_t^\infty f(x(s) + \gamma(s))\, ds\right| \leq (1 + \epsilon)^2(\lambda + \epsilon)^\beta \int_t^\infty (\varphi \circ F^{-1})(s)\, ds.$$

By (1.46), for every $\epsilon \in (0, 1)$ there exists $T^*(\epsilon)$ such that

$$\varphi(F^{-1}(t)) < \frac{f(F^{-1}(t))}{1 - \epsilon}, \quad t \geq T^*(\epsilon).$$

This allows us to write, for $t \geq \max(T'''(\epsilon), T^*(\epsilon))$, the inequality

$$\begin{aligned}\left|\int_t^\infty f(x(s) + \gamma(s))\, ds\right| &\leq \frac{(1 + \epsilon)^2(\lambda + \epsilon)^\beta}{1 - \epsilon} \int_t^\infty (f \circ F^{-1})(s)\, ds \\ &= \frac{(1 + \epsilon)^2(\lambda + \epsilon)^\beta}{(1 - \epsilon)} F^{-1}(t).\end{aligned}$$

Since $x(t) = \int_t^\infty f(x(s) + \gamma(s))\, ds$ we have, for $t \geq \max(T'''(\epsilon), T^*(\epsilon))$,

$$\frac{|x(t)|}{F^{-1}(t)} = \frac{|\int_t^\infty f(x(s) + \gamma(s))ds|}{F^{-1}(t)} \leq \frac{(1 + \epsilon)^2(\lambda + \epsilon)^\beta}{(1 - \epsilon)}.$$

Hence, taking the lim sup yields

$$\lambda \leq \frac{(1 + \epsilon)^2(\lambda + \epsilon)^\beta}{(1 - \epsilon)}.$$

Letting $\epsilon \to 0^+$ gives us $\lambda \leq \lambda^\beta$ or $\lambda^{\beta-1} \geq 1$. Hence, $\lambda \geq 1$, as required. □

Lemma 1.14. *Suppose that f satisfies (1.11) and (1.12) with $\beta > 1$. Suppose that γ is continuous and x is the unique continuous solution of (1.49) which satisfies (1.50). Suppose also that γ and F obey (1.10). If*

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} > 0,$$

then

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = 1.$$

Proof. From Lemma 1.13, if

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} > 0,$$

we have that

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} \geq 1.$$

In the case when $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 1$, we are done. We assume therefore that $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) > 1$. From Lemma 1.12 we have that either

$$\liminf_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} = 0 \text{ or } 1.$$

If this prevails, for every $\epsilon \in (0,1)$ sufficiently small, there exists $t_n(\epsilon) \nearrow \infty$ such that $|x(t_n)| = (1 + \frac{2\beta}{\beta-1}\epsilon)\Phi^{-1}(t_n)$. Let $\eta = 3\beta/(\beta-1)$ and define the function

$$h(\epsilon) := (1 + (\eta-1)\epsilon)^{\beta}(1-\epsilon)^{\beta} - (1+\eta\epsilon), \quad \epsilon \in [0,1).$$

Note that $h(0) = 0$ and $h'(0) > 0$. Thus there exists $x_1(\beta) > 0$ such that $h(\epsilon) > 0$ for all $\epsilon < x_1(\beta)$. Let $\lambda(\epsilon) = 1 + \frac{3\beta}{\beta-1}\epsilon = 1 + \eta\epsilon$. Therefore

$$\lambda - \epsilon = 1 + \left(\frac{3\beta}{\beta-1} - 1\right)\epsilon = 1 + \left(\frac{2\beta+1}{\beta-1}\right)\epsilon > 1.$$

Furthermore, for $\epsilon < x_1(\beta)$, we have

$$\begin{aligned}&(\lambda(\epsilon)-\epsilon)^{\beta}(1-\epsilon)^{\beta-1} - \frac{\lambda}{1-\epsilon} \\ &= (1+(\eta-1)\epsilon)^{\beta}(1-\epsilon)^{\beta-1} - \frac{1+\eta\epsilon}{1-\epsilon} = \frac{1}{1-\epsilon}h(\epsilon) > 0.\end{aligned}$$

Since $\varphi \in \mathrm{RV}_0(\beta)$ we have that

$$\frac{\varphi((1+\frac{2\beta+1}{\beta-1}\epsilon)x)}{\varphi(x)} > \left(1 + \frac{2\beta+1}{\beta-1}\epsilon\right)^{\beta}(1-\epsilon)^{\beta}, \quad x < x_2(\epsilon).$$

Since $\Phi^{-1}(t) \to 0$ as $t \to \infty$, there exists $T_2(\epsilon)$ such that $\Phi^{-1}(t) < x_2(\epsilon)$ for all $t > T_2(\epsilon)$ and as γ obeys (1.10), we have that there is $T_1(\epsilon)$ such that $|\gamma(t)| < \epsilon\Phi^{-1}(t)$ for all $t > T_1(\epsilon)$. Also, by Lemma 1.11, there exists $T(\epsilon) > 0$ such that we have

$$D_+|x(t)| \leq -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \geq T(\epsilon).$$

Let $T^*(\epsilon) := \inf\{t_n(\epsilon) : t_n > T_1(\epsilon) \vee T_2(\epsilon) \vee T(\epsilon)\}$. Define $x_+(t) = \lambda(\epsilon)\Phi^{-1}(t)$ for $t \geq T^*(\epsilon)$. Therefore $x_+'(t) = -\lambda(\epsilon)(\varphi \circ \Phi^{-1})(t)$ for all $t \geq T^*(\epsilon)$. Using the regular variation of φ and the fact that $\epsilon < x_1(\beta)$, for $t \geq T^*(\epsilon)$, we have

$$\frac{\varphi((\lambda(\epsilon) - \epsilon)\Phi^{-1}(t))}{\varphi(\Phi^{-1}(t))} > (\lambda(\epsilon) - \epsilon)^\beta (1 - \epsilon)^{\beta - 1} > \frac{\lambda}{1 - \epsilon}.$$

Thus

$$\begin{aligned} -\lambda(\epsilon)\varphi(\Phi^{-1}(t)) &> -(1 - \epsilon)\varphi((\lambda(\epsilon) - \epsilon)\Phi^{-1}(t)) \\ &= -(1 - \epsilon)\varphi(x_+(t) - \epsilon\Phi^{-1}(t)). \end{aligned}$$

Therefore we have

$$x_+'(t) > -(1 - \epsilon)\varphi(x_+(t) - \epsilon\Phi^{-1}(t)), \quad t \geq T^*(\epsilon).$$

Furthermore, we have

$$\begin{aligned} x_+(T^*) = \lambda(\epsilon)\Phi^{-1}(T^*) &= \left(1 + \frac{3\beta}{\beta - 1}\epsilon\right)\Phi^{-1}(T^*) > \left(1 + \frac{2\beta}{\beta - 1}\epsilon\right)\Phi^{-1}(T^*) \\ &= |x(T^*)|. \end{aligned}$$

Hence

$$\begin{aligned} &x_+'(t) > -(1 - \epsilon)\varphi(x_+(t) - \epsilon\Phi^{-1}(t)), \quad t \geq T^*(\epsilon) \\ &x_+(T^*) > |x(T^*)|. \end{aligned} \tag{1.60}$$

Also, by Lemma 1.11, we have

$$D_+|x(t)| \leq -\varphi_\epsilon(|x(t)| - \epsilon\Phi^{-1}(t)), \quad t \geq T^*(\epsilon). \tag{1.61}$$

Suppose there is a minimal $t' > T^*(\epsilon)$ such that $|x(t')| = x_+(t') = \lambda(\epsilon)\Phi^{-1}(t')$. Then $|x(t')| - \epsilon\Phi^{-1}(t') = x_+(t') - \epsilon\Phi^{-1}(t') = (\lambda - \epsilon)\Phi^{-1}(t') > 0$. Hence

$$\begin{aligned}
&\varphi_\epsilon(|x(t')| - \epsilon\Phi^{-1}(t')) \\
&= \min\{(1+\epsilon)\varphi(|x(t')| - \epsilon\Phi^{-1}(t')), (1-\epsilon)\varphi(|x(t')| - \epsilon\Phi^{-1}(t'))\} \\
&= (1-\epsilon)\varphi(|x(t')| - \epsilon\Phi^{-1}(t')).
\end{aligned}$$

Hence by (1.60) and (1.61) we get

$$\begin{aligned}
D_+|x(t)| &\leq -(1-\epsilon)\varphi(|x(t')| - \epsilon\Phi^{-1}(t')) \\
&= -(1-\epsilon)\varphi(x_+(t') - \epsilon\Phi^{-1}(t')) < x_+'(t').
\end{aligned}$$

The minimality of t' implies that $D_+|x(t')| \geq x_+'(t')$, which gives a contradiction. Therefore we must have $|x(t)| < x_+(t)$ for all $t \geq T^*(\epsilon)$. Hence,

$$|x(t)| < x_+(t) = \lambda(\epsilon)\Phi^{-1}(t) = \left(1 + \frac{3\beta}{\beta-1}\epsilon\right)\Phi^{-1}(t), \quad t \geq T^*(\epsilon).$$

Thus

$$\limsup_{t\to\infty} \frac{|x(t)|}{\Phi^{-1}(t)} \leq 1 + \frac{3\beta}{\beta-1}\epsilon,$$

so by letting $\epsilon \to 0^+$ and using the fact that $\Phi^{-1} \sim F^{-1}$, we get

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} \leq 1.$$

This contradicts the supposition that $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) > 1$, and so we must have $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 1$ or $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 0$, as claimed. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Define

$$\eta = \frac{3\beta}{\beta-1}, \quad \lambda(\epsilon) = 1 - \eta\epsilon, \quad 0 < \epsilon < \frac{1}{\eta+1} < \frac{1}{\eta}.$$

Then $\lambda(\epsilon) \in (0,1)$ and $\epsilon < \lambda(\epsilon)$. Define $h(\epsilon) := (1-\eta\epsilon) - (1-\eta\epsilon+\epsilon)^\beta(1+\epsilon)^\beta$. We note that $h(0) = 0$ and $h'(0) > 0$. Hence there exists $\epsilon' = \epsilon'(\beta) > 0$ such that $h(\epsilon) > 0$ for all $\epsilon < \epsilon'(\beta) < 1$. Therefore

$$(1-\eta\epsilon) - (1-\eta\epsilon+\epsilon)^\beta(1+\epsilon)^\beta > 0, \quad \epsilon < \epsilon',$$

or

$$\lambda(\epsilon) - (\lambda(\epsilon) + \epsilon)^{\beta}(1+\epsilon)^{\beta} > 0, \quad \epsilon < \epsilon'.$$

This implies

$$\lambda(\epsilon) > (\lambda(\epsilon) + \epsilon)^{\beta}(1+\epsilon)^{\beta}, \quad \epsilon < \epsilon'.$$

For every $\epsilon \in (0,1)$, there is $x_1(\epsilon) > 0$ such that $f(x) < (1+\epsilon)\varphi(x)$ for $x < x_1(\epsilon)$ and

$$\frac{\varphi((\lambda+\epsilon)x)}{f(x)} < (\lambda+\epsilon)^{\beta}(1+\epsilon)^{\beta-1}, \quad x < x_1(\epsilon).$$

Since $F^{-1}(t) \to 0$ as $t \to \infty$, for every $\epsilon > 0$ there is $T_1(\epsilon) > 0$ such that $t > T_1(\epsilon)$ implies $F^{-1}(t) < x_1(\epsilon)$ and $F^{-1}(t) < x_2(\epsilon)/(\lambda+\epsilon)$. Also, as γ obeys (1.10) for every $\epsilon > 0$, there exists $T_2(\epsilon) > 0$ such that for $t > T_2(\epsilon)$, we have $|\gamma(t)| < \epsilon F^{-1}(t)$. Since $\limsup_{t\to\infty} |x(t)|/F^{-1}(t) = 1$, we have that either

$$\text{(I)} \quad \limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1 \quad \text{or} \quad \text{(II)} \quad \limsup_{t\to\infty} \frac{-x(t)}{F^{-1}(t)} = 1.$$

We consider case (I) first. If it holds, there exists $t_n \nearrow \infty$ such that $x(t_n) > (1-\frac{\eta}{2}\epsilon)F^{-1}(t_n)$. Let $T(\epsilon) = \inf\{t_n(\epsilon) : t_n(\epsilon) > T_1 \vee T_2\}$ and define $x_-(t) = \lambda(\epsilon)F^{-1}(t)$ for all $t \geq T(\epsilon)$. Then $x_-(T) = x_-(t_n) = \lambda(\epsilon)F^{-1}(t_n)$ and we have

$$x(T) = x(t_n) > (1-\frac{\eta}{2}\epsilon)F^{-1}(t_n) > (1-\eta\epsilon)F^{-1}(t_n) = x_-(t_n) = x_-(T).$$

Hence $x(T) > x_-(T)$. Now for $t \geq T(\epsilon)$, $x_-(t) + \gamma(t) < (\lambda(\epsilon)+\epsilon)F^{-1}(t)$, which implies

$$\begin{aligned} f(x_-(t)+\gamma(t)) &< (1+\epsilon)\varphi((\lambda(\epsilon)+\epsilon)F^{-1}(t)) \\ &< (1+\epsilon)(\lambda+\epsilon)^{\beta}(1+\epsilon)^{\beta-1} f(F^{-1}(t)) < \lambda(\epsilon)(f\circ F^{-1})(t), \end{aligned}$$

since $F^{-1}(t) < x_1(\epsilon)$, $(\lambda+\epsilon)F^{-1}(t) < x_2(\epsilon)$ and $\epsilon < \epsilon'$. Thus

$$-f(x_-(t)+\gamma(t)) > -\lambda(\epsilon)(f\circ F^{-1})(t) = -x_-'(t), \quad t \geq T(\epsilon).$$

Therefore $x_-'(t) < -f(x_-(t)+\gamma(t))$ for $t \geq T(\epsilon)$ and $x_-(T) < |x(T)|$. Now suppose there exists $t' > T$ such that $x(t') = x_-(t')$. Then $x'(t') \leq x_-'(t')$. Hence

$$\begin{aligned} x_-'(t') &< -f(x_-(t')+\gamma(t')) = -f(x(t')+\gamma(t')) = x'(t') \\ &\leq x_-'(t'), \end{aligned}$$

which gives a contradiction. Hence $x_-(t) < x(t), t \geq T(\epsilon)$. Thus

$$x(t) > x_-(t) = \lambda(\epsilon)F^{-1}(t) = \left(1 - \frac{3\beta}{\beta - 1}\epsilon\right) F^{-1}(t), \quad t \geq T(\epsilon).$$

Therefore we have that

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \geq 1 - \frac{3\beta}{\beta - 1}\epsilon,$$

so by letting $\epsilon \to 0^+$ we get

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \geq 1.$$

Thus, if $\limsup_{t\to\infty} x(t)/F^{-1}(t) = 1$, we have $\liminf_{t\to\infty} x(t)/F^{-1}(t) \geq 1$. Therefore we have

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1 \text{ implies } \lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

In case (II), if we have that

$$\limsup_{t\to\infty} \frac{-x(t)}{F^{-1}(t)} = 1,$$

then let $z(t) = -x(t)$ and follow the same argument as before. In this case we let

$$z_-(t) = \lambda(\epsilon)F^{-1}(t), \text{ for all } t > T^*(\epsilon)$$

and similarly we arrive at $z(t) > z_-(t)$ for all $t > T^*(\epsilon)$. Translating this back to a statement about $x(t)$ we obtain

$$\limsup_{t\to\infty} \frac{-x(t)}{F^{-1}(t)} = 1 \text{ implies } \lim_{t\to\infty} \frac{-x(t)}{F^{-1}(t)} = 1,$$

as required. □

1.9 Proofs from Sect. 1.4

Proof of Theorem 1.4

We start by making uniform asymptotic estimates of the terms involving x in the integrated form of (1.1), namely

$$x(t) = x(0) + \int_0^t f(x(s))\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}s. \tag{1.62}$$

This entails making a pointwise estimate of $f(x(t))$. If it can be shown that the function $t \mapsto \int_0^t f(x(s))\,\mathrm{d}s$ tends to a finite limit as $t \to \infty$, the result is secured, because the hypothesis (1.14) implies that $x(t) \to 0$ as $t \to \infty$ and therefore that g obeys (1.17).

By Lemma 1.10, there is a function φ such that

$$\frac{1}{2} < \frac{f(x)}{\varphi(x)} < \frac{3}{2}, \ |x| < x_1,$$

for some $x_1 > 0$, where φ is increasing, odd and $\varphi \in RV_0(\beta)$. Since $\varphi \in RV_0(\beta)$ we also have that

$$\frac{\varphi(x)}{\varphi(\frac{x}{L+1})} < 2(|\lambda|+1)^\beta, \ \text{for } |x| < x_2.$$

For some $x_2 > 0$. Thus $|f(x)| < \frac{3}{2}\varphi(|x|)$ for all $|x| < x_1$. Since $x(t) \to 0$ as $t \to \infty$, $|x(t)| < x_1$ for all $t \geq T_1$. Since x obeys (1.14) and $F^{-1}(t) \to 0$ as $t \to \infty$, we have that there exist $T_2 > 0$ and $T_3 > 0$ such that $|x(t)| < (|\lambda|+1)F^{-1}(t)$ for $t \geq T_2$ and $(|\lambda|+1)F^{-1}(t) < x_1$, for $t \geq T_3$. Hence, for $t \geq T := 1 + T_1 \vee T_2 \vee T_3$, we have $|f(x(t))| \leq 2\varphi(|x(t)|) \leq 2\varphi((|\lambda|+1)F^{-1}(t))$. Now we estimate the integral involving $f(x(t))$. For $t \geq T$ we have

$$\begin{aligned}\left|\int_T^t f(x(s))\,\mathrm{d}s\right| &\leq \int_T^t \frac{3}{2}\varphi((|\lambda|+1)F^{-1}(s))\,\mathrm{d}s \\ &= \frac{3}{2(|\lambda|+1)} \int_{F^{-1}(t)}^{F^{-1}(T)} \frac{\varphi(u)}{\varphi(\frac{u}{|\lambda|+1})} \cdot \frac{\varphi(\frac{u}{|\lambda|+1})}{f(\frac{u}{|\lambda|+1})}\,\mathrm{d}u. \end{aligned} \tag{1.63}$$

Now $(|\lambda|+1)F^{-1}(T) \leq (|\lambda|+1)F^{-1}(T_3) < x_1$ so if $0 < u \leq F^{-1}(T)$, then

$$\frac{u}{|\lambda|+1} \leq \frac{F^{-1}(T)}{|\lambda|+1} < \frac{x_1}{(|\lambda|+1)^2} < x_1.$$

Hence

$$\frac{\varphi(\frac{u}{|\lambda|+1})}{f(\frac{u}{|\lambda|+1})} < 2, \ \text{for } u \leq F^{-1}(T). \tag{1.64}$$

Next $T > T_3$, so $F^{-1}(T) < F^{-1}(T_3)$ so $(|\lambda|+1)F^{-1}(T) < (|\lambda|+1)F^{-1}(T_3) < x_2$. Hence $u \leq F^{-1}(T)$ implies $u < x_2$. Thus

$$\frac{\varphi(u)}{\varphi(\frac{u}{|\lambda|+1})} < 2(|\lambda|+1)^{\beta}, \text{ for } u \leq F^{-1}(T). \tag{1.65}$$

If we insert equations (1.64) and (1.65) into (1.63) we obtain the following inequalities, for $t \geq T$:

$$\begin{aligned}\left|\int_T^t f(x(s))\,\mathrm{d}s\right| &\leq \frac{3}{2(|\lambda|+1)}\int_{F^{-1}(t)}^{F^{-1}(T)} 2\cdot 2(|\lambda|+1)^{\beta}\,\mathrm{d}u\\ &\leq 6(|\lambda|+1)^{\beta-1}F^{-1}(T),\end{aligned}$$

which is finite. Since T is finite, $\lim_{t\to\infty}\int_0^t f(x(s))\,\mathrm{d}s$ is finite, and so g obeys (1.17), as required.

Proof of Theorem 1.5

By Theorem 1.4, we have that $\lim_{t\to\infty}\int_0^t g(s)\,\mathrm{d}s$ exists and is finite. By (1.14), it follows that $\lim_{t\to\infty} x(t) = 0$. Also, by Theorem 1.4 it follows that

$$\lim_{t\to\infty}\int_0^t f(x(s))\,\mathrm{d}s$$

is finite, so $\int_t^\infty f(x(s))\,\mathrm{d}s$ is well defined for all $t \geq 0$. Hence we have

$$\int_0^\infty g(s)\,\mathrm{d}s = \int_0^\infty f(x(s))\,\mathrm{d}s - x(0), \qquad \int_0^t g(s)\,\mathrm{d}s = \int_0^t f(x(s))\,\mathrm{d}s + x(t) - x(0).$$

Therefore we have

$$\frac{1}{F^{-1}(t)}\int_t^\infty g(s)\,\mathrm{d}s = \frac{1}{F^{-1}(t)}\int_t^\infty f(x(s))\,\mathrm{d}s - \frac{x(t)}{F^{-1}(t)}, \quad t \geq 0. \tag{1.66}$$

We now analyse the asymptotic behaviour of the right-hand side of (1.66) to prove the second part of (1.13). Under (1.14), we have either

$$\text{(i)}\ \lim_{t\to\infty}\frac{x(t)}{F^{-1}(t)} = 0 \ \text{ or}$$

$$\text{(ii)}\ \lim_{t\to\infty}\frac{x(t)}{F^{-1}(t)} = \pm 1.$$

By L'Hôpital's rule, and recalling the properties of the function φ introduced in Lemma 1.10, we may consider

$$\lim_{t\to\infty} \frac{\int_t^\infty f(x(s))\,\mathrm{d}s}{F^{-1}(t)} = \lim_{t\to\infty} \frac{-f(x(t))}{-f(F^{-1}(t))} = \lim_{t\to\infty} \frac{f(x(t))}{f(F^{-1}(t))} = \lim_{t\to\infty} \frac{\varphi(x(t))}{\varphi(F^{-1}(t))},$$

provided that the limit on the right-hand side exists. We now show that it does when (1.14) prevails. In case (i), as $\varphi \in RV_0(\beta)$, φ is odd and $|x(t)|/F^{-1}(t) \to 0$ as $t \to \infty$ we have

$$\lim_{t\to\infty} \frac{|\varphi(x(t))|}{\varphi(F^{-1}(t))} = \lim_{t\to\infty} \frac{\varphi(|x(t)|)}{\varphi(F^{-1}(t))} = 0.$$

Thus

$$\lim_{t\to\infty} \frac{\int_t^\infty f(x(s))\,\mathrm{d}s}{F^{-1}(t)} = 0,$$

so by taking limits on both sides of (1.66), we have the second part of (1.13), as required.

In case (ii) the limit

$$\lim_{t\to\infty} \frac{\int_t^\infty f(x(s))\,\mathrm{d}s}{F^{-1}(t)} = \lim_{t\to\infty} \frac{\varphi(x(t))}{\varphi(F^{-1}(t))} \tag{1.67}$$

still obtains, provided the limit on the right-hand side exists. If $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$ the limit on the right-hand side of (1.67) is 1, so (1.67) and (1.66) combine to yield the second part of (1.13), as claimed. If, on the other hand, $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$, then from (1.67) we use the fact that φ is odd to write

$$\lim_{t\to\infty} \frac{\int_t^\infty f(x(s))\,\mathrm{d}s}{F^{-1}(t)} = \lim_{t\to\infty} \frac{\varphi(x(t))}{\varphi(F^{-1}(t))} = \lim_{t\to\infty} \frac{-\varphi(-x(t))}{\varphi(F^{-1}(t))} = -1.$$

Using this, $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$ and (1.66) gives (1.13), as required.

Proof of Theorem 1.9

We define

$$\phi_+ = \liminf_{x\to+\infty} |f(x)|, \quad \phi_- = \liminf_{x\to-\infty} |f(x)|.$$

Note by (1.21) that $\phi_+, \phi_- > 0$. As before we define $u(t) = \int_0^t g(s)\,ds$ and thus (1.17) implies that $u(\infty) = \int_0^\infty g(s)\,ds$ is well defined. Hence $\gamma(t) := u(t) - u(\infty) \to 0$ as $t \to \infty$. Define $z(t) = x(t) - u(t) + u(\infty) = x(t) - \gamma(t)$ for $t \geq 0$. Therefore $z(t) \to 0$ as $t \to \infty$ if and only if $x(t) \to 0$ as $t \to \infty$. Moreover $z'(t) = x'(t) - u'(t) = -f(x(t)) = -f(z(t) + \gamma(t))$. Thus we proceed to show that z has the desired limit. We set $\phi = \min(\phi_+, \phi_-)$ and thus there exists x_1 such that

$$f(x) \geq \frac{\phi}{2} \text{ for all } x \geq x_1 > 0 \text{ and } -f(x) \geq \frac{\phi}{2} \text{ for all } x \leq -x_1.$$

Now by the Lipschitz continuity of f there exists K_1 such that

$$|f(x) - f(y)| \leq K_1|x - y| \text{ for all } x, y \text{ such that } |x|, |y| \leq x_1 + 1. \tag{1.68}$$

Choose $\delta \in (0, 1)$ to be small enough that $\phi/4 > K_1\delta$ with $\delta/2 < x_1$. Thus for $x \in (x_1 - \delta,\, x_1 + \delta)$ we have

$$-K_1\delta \leq f(x) - f(x_1) \leq K_1\delta.$$

Hence we obtain

$$f(x) \geq f(x_1) - K_1\delta \geq \frac{\phi}{2} - \frac{\phi}{4} \geq \frac{\phi}{4}.$$

Since we know that $\gamma(t) \to 0$ there exists a $T_0 > 0$ such that $|\gamma(t)| < \delta/2$ for all $t \geq T_0$.

If there exists $T_2 > T_0$ such that $z(T_2) < x_1$, it can be shown that $z(t) < x_1$ for all $t \geq T_2$. We defer the proof of this fact temporarily. Instead, we first assume to the contrary that $z(t) \geq x_1$ for all $t \geq T_0$. Therefore $z(t) + \gamma(t) \geq x_1 - \delta/2$ for $t \geq T_0$, and therefore $-z'(t) = f(z(t) + \gamma(t)) \geq \phi/4 > 0$ for $t \geq T_0$. But this implies that $z(t)$ will ultimately lie below x_1, a contradiction.

It remains to prove that if $z(T_2) < x_1$ for some $T_2 > T_0$, then $z(t) < x_1$ for all $t \geq T_2$. Suppose to the contrary that there is a minimal $T_1 > T_2$ such $z(T_1) = x_1$. Then $z'(T_1) \geq 0$. On the other hand, $f(z(T_1) + \gamma(T_1)) = f(x_1 + \gamma(T_1))$. Since $T_1 > T_0$, we have $|\gamma(T_1)| \leq \delta/2$, and so it follows that $f(x_1 + \gamma(T_1)) \geq \phi/4$. Hence $0 \leq z'(T_1) = -f(x_1 + \gamma(T_1)) \leq -\phi/4 < 0$, a contradiction.

Therefore, we have shown that there exists $T_2 > 0$ such that $z(t) < x_1$ for all $t \geq T_2$. By a similar argument, it can be shown that there is a $T_3 > 0$ such that $z(t) > -x_1$ for all $t \geq T_3$. Hence, with $T_4 = \max(T_0, T_2, T_3)$, we have that $|z(t)| \leq x_1$ for all $t \geq T_4$ and also that $|\gamma(t)| \leq \delta/2$.

It remains to show that the boundedness of z implies that it tends to zero. Write, for $t \geq 0$, $g(t) = f(z(t)) - f(z(t) + \gamma(t))$. Then g is continuous on $[0, \infty)$, by dint of the continuity of z, γ and f. Since $|\gamma(t)| \leq \delta/2 < 1/2$ for $t \geq T_4$, it follows that $|z(t) + \gamma(t)| < x_1 + 1$ and $|z(t)| < x_1 + 1$ for all $t \geq T_4$. Hence, by (1.68), we have

$|g(t)| \leq K_1|\gamma(t)$ for $t \geq T_4$. Since $\gamma(t) \to 0$ as $t \to \infty$, we have that $g(t) \to 0$ as $t \to \infty$. Moreover, by the definition of g, it follows that z obeys

$$z'(t) = -f(z(t)) + g(t), \quad t \geq 0.$$

The Lipschitz continuity of f, the properties (1.4)and (1.21) and the fact that g is continuous on $[0,\infty)$ and $g(t) \to 0$ as $t \to \infty$ mean, by a result in [10], that $z(t) \to 0$ as $t \to \infty$. This allows us to conclude that $x(t) \to 0$ as $t \to \infty$, as claimed.

1.10 Proofs from Sect. 1.5

We now prove some results from Sect. 1.5, up to but not including Theorem 1.14.

Proof of Theorem 1.10

We rearrange (1.2) and write

$$\int_0^t -f(X(s))\,\mathrm{d}s = X(t) - \xi - \int_0^t \sigma(s)\,\mathrm{d}B(s). \tag{1.69}$$

Since X obeys (1.3), it follows from Lemma 1.1 that $\sigma \in L^2(0,\infty)$. The martingale convergence theorem then implies that the last term on the right-hand side of (1.69) has a finite limit as $t \to \infty$ a.s. Moreover, X obeys (1.3) implies that $X(t) \to 0$ as $t \to \infty$ a.s., so from this it follows that all the terms on the right-hand side of (1.69) converge to 0 with probability 1. Therefore there is an event Ω_1 such that the limit as $t \to \infty$ of the left-hand side of (1.69) is well defined and we may write

$$\int_0^\infty -f(X(s))\,\mathrm{d}s = -\xi - \int_0^\infty \sigma(s)\,\mathrm{d}B(s), \text{ on } \Omega_1.$$

Taking this identity together with (1.69) on Ω_1, we can obtain

$$\int_t^\infty \sigma(s)\,\mathrm{d}B(s) = -X(t) + \int_t^\infty f(X(s))\,\mathrm{d}s. \tag{1.70}$$

Define the a.s. event on which (1.70) holds to be Ω^* and

$$A := \left\{\omega : \lim_{t\to\infty} \frac{X(t,\omega)}{F^{-1}(t)} = \lambda(\omega) \in (-\infty,\infty)\right\} \cap \Omega^*.$$

We have presumed that $\mathbb{P}[A] = 1$. We decompose $A = A_+ \cup A_- \cup A_0$ where the events $A_\cdot$ are defined by

$$A_+ = A \cap \{\omega : \lambda(\omega) > 0\}, \; A_- = A \cap \{\omega : \lambda(\omega) < 0\}, \; A_0 = A \cap \{\omega : \lambda(\omega) = 0\}.$$

Now, consider $\omega \in A$ so that $\lambda(\omega) \neq 0$; then

$$\lim_{t\to\infty} \frac{X(t,\omega)}{\lambda(\omega)F^{-1}(t)} = 1.$$

Then as φ is in $\mathrm{RV}_0(\beta)$ and $f(x)/\varphi(x) \to 1$ as $x \to 0$

$$\lim_{t\to\infty} \frac{f(X(t))}{\varphi(\lambda(\omega)F^{-1}(t))} = \lim_{t\to\infty} \frac{\varphi(X(t))}{\varphi(\lambda(\omega)F^{-1}(t))} = 1.$$

In the case when $\omega \in A_+$, since $\varphi \in \mathrm{RV}_0(\beta)$, we have that

$$\lim_{t\to\infty} \frac{f(X(t,\omega))}{\varphi(F^{-1}(t))} = \lambda(\omega)^\beta.$$

Therefore, by L'Hôpital's rule and the fact that $f(x)/\varphi(x) \to 1$ as $x \to 0$, we have

$$\lim_{t\to\infty} \frac{\int_t^\infty f(X(s,\omega))\,\mathrm{d}s}{F^{-1}(t)} = \lim_{t\to\infty} \frac{f(X(t,\omega))}{f(F^{-1}(t))} = \lambda(\omega)^\beta.$$

Rearranging (1.70) and taking limits yields for each $\omega \in A_+$

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = -\lambda(\omega) + \lambda(\omega)^\beta.$$

Now write $A_+ = A_1 \cup A_1'$ where $A_1 = A_+ \cap \{\lambda = 1\}$ and $A_1' = A_+ \cap \{\lambda \neq 1\}$. Suppose that A_1' is such that $\mathbb{P}[A_1'] > 0$. Then we have that

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} \text{ exists and is not equal to } 0\right] > 0,$$

which contradicts (1.26). Hence $\mathbb{P}[A_1'] = 0$. Thus $\mathbb{P}[A_+] = \mathbb{P}[A_1]$. Moreover, we have that

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = 0, \quad \omega \in A_1. \tag{1.71}$$

Next, we consider the case when $\omega \in A_-$, so $\lambda(\omega) < 0$. As before we have

$$\lim_{t\to\infty} \frac{f(X(t,\omega))}{\varphi(\lambda(\omega)F^{-1}(t))} = 1.$$

Using this limit and the fact that φ is odd, we have

$$\lim_{t\to\infty} \frac{f(X(t,\omega))}{\varphi(F^{-1}(t))} = \lim_{t\to\infty} \frac{\varphi(\lambda(\omega)F^{-1}(t))}{\varphi(F^{-1}(t))} = \lim_{t\to\infty} \frac{-\varphi(-\lambda(\omega)F^{-1}(t))}{\varphi(F^{-1}(t))},$$

so because $-\lambda(\omega) > 0$, the fact that $\varphi \in \mathrm{RV}_0(\beta)$ and that f and F^{-1} are asymptotic to φ and Φ^{-1}, respectively, implies that

$$\lim_{t\to\infty} \frac{\varphi(X(t,\omega))}{\varphi(\Phi^{-1}(t))} = -(-\lambda(\omega))^{\beta}.$$

Therefore, by L'Hôpital's rule and the fact that $f(x)/\varphi(x) \to 1$ as $x \to 0$, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{\int_t^\infty f(X(s,\omega))\,\mathrm{d}s}{F^{-1}(t)} &= \lim_{t\to\infty} \frac{\int_t^\infty \varphi(X(s,\omega))\,\mathrm{d}s}{\Phi^{-1}(t)} \\ &= \lim_{t\to\infty} \frac{\varphi(X(t,\omega))}{\varphi(\Phi^{-1}(t))} = -(-\lambda(\omega))^{\beta}.\end{aligned}$$

Rearranging (1.70) and taking limits yields for each $\omega \in A_-$

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = -\lambda(\omega) - (-\lambda(\omega))^{\beta}.$$

Now write $A_- = A_{-1} \cup A'_{-1}$ where $A_{-1} = A_+ \cap \{\lambda(\omega) = -1\}$ and $A'_1 = A_+ \cap \{\lambda(\omega) \neq -1\}$. Suppose that A'_{-1} is such that $\mathbb{P}[A'_{-1}] > 0$. Then we have that

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)} \text{ exists and is not equal to } 0\right] > 0,$$

which contradicts (1.26). Hence $\mathbb{P}[A'_{-1}] = 0$. Thus $\mathbb{P}[A_-] = \mathbb{P}[A_{-1}]$. Moreover, we have that

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = 0, \quad \omega \in A_{-1}. \tag{1.72}$$

Finally, we consider the situation $\omega \in A_0$, so $\lambda(\omega) = 0$. Then

$$\lim_{t\to\infty} \frac{|X(t,\omega)|}{F^{-1}(t)} = 0.$$

Therefore, as φ is odd, and in $\mathrm{RV}_0(\beta)$, we have

$$\lim_{t\to\infty}\frac{|\varphi(X(t,\omega)|}{\varphi(F^{-1}(t))}=\lim_{t\to\infty}\frac{\varphi(|X(t,\omega)|)}{\varphi(F^{-1}(t))}=0.$$

Hence

$$\lim_{t\to\infty}\frac{\varphi(X(t,\omega))}{\varphi(\Phi^{-1}(t))}=0.$$

Therefore, because by L'Hôpital's rule and the fact that $f(x)/\varphi(x)\to 1$ as $x\to 0$, we have

$$\lim_{t\to\infty}\frac{\int_t^\infty f(X(s,\omega))\,\mathrm{d}s}{F^{-1}(t)}=\lim_{t\to\infty}\frac{\int_t^\infty \varphi(X(s,\omega))\,\mathrm{d}s}{\Phi^{-1}(t)}=\lim_{t\to\infty}\frac{\varphi(X(t,\omega))}{\varphi(\Phi^{-1}(t))}=0.$$

Rearranging (1.70) and taking limits yields

$$\lim_{t\to\infty}\frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)}=0,\quad \omega\in A_0. \tag{1.73}$$

Therefore, we have shown that $1=\mathbb{P}[A]=\mathbb{P}[A_1\cup A_1'\cup A_{-1}\cup A_{-1}'\cup A_0]=\mathbb{P}[A_1\cup A_{-1}\cup A_0]$. Therefore, by (1.72)–(1.73), and this statement, we have that

$$\lim_{t\to\infty}\frac{\int_t^\infty \sigma(s)\,\mathrm{d}B(s)}{F^{-1}(t)}=0,\quad \lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}=\lambda\in\{-1,0,1\},\quad \text{a.s.,}$$

which proves (1.3) and (1.22).

Proof of Theorem 1.12

To prove part (a), we note that the event

$$A:=\{\omega\,:\,\lim_{t\to\infty}\frac{X(t,\omega)}{F^{-1}(t)}=\lambda(\omega)\in(-\infty,\infty)\}$$

is a sub-event of the event $\{\omega\,:\,\lim_{t\to\infty}X(t,\omega)=0\}$. Therefore, if we assume that $\mathbb{P}[A]>0$, it follows that $X(t)$ tends to zero with positive probability, and does so for all outcomes in A.

As in the proof of Theorem 1.10, write $A=A_+\cup A_-\cup A_0$. We can use the argument employed in Theorem 1.10 to prove that

$$\lim_{t\to\infty} \frac{f(X(t,\omega))}{\varphi(\Phi^{-1}(t))} = \lambda(\omega)^\beta, \quad \omega \in A_+,$$
$$\lim_{t\to\infty} \frac{f(X(t,\omega))}{\varphi(\Phi^{-1}(t))} = -(-\lambda(\omega))^\beta, \quad \omega \in A_-,$$
$$\lim_{t\to\infty} \frac{f(X(t,\omega))}{\varphi(\Phi^{-1}(t))} = 0, \quad \omega \in A_0.$$

Therefore, as $\varphi \circ \Phi^{-1} \in L^1([0,\infty);\mathbb{R})$, it follows that

$$\lim_{t\to\infty} \int_0^t f(X(s,\omega))\,\mathrm{d}s \text{ exists and is finite for each } \omega \in A.$$

Since $X(t,\omega) \to 0$ as $t \to \infty$ for each $\omega \in A$, it follows that every term on the right-hand side of (1.23) tends to a finite limit as $t \to \infty$, for each $\omega \in A$. Therefore, it follows that

$$\mathbb{P}\left[\lim_{t\to\infty} \int_0^t \sigma(s)\,\mathrm{d}B(s) \text{ exists and is finite}\right] > 0.$$

If $\sigma \notin L^2([0,\infty);\mathbb{R})$, we have that

$$\mathbb{P}\left[\lim_{t\to\infty} \int_0^t \sigma(s)\,\mathrm{d}B(s) \text{ exists and is finite}\right] = 0,$$

a contradiction. Hence the assumption that $\mathbb{P}[A] > 0$ must be false, proving part (a).

To prove (b), part (i), notice that $\mu = 0$ in (1.27) together with Lemma 1.2 implies

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)^2}{F^{-1}(t)^2} \\
&= \limsup_{t\to\infty} \frac{(\int_t^\infty \sigma(s)\,\mathrm{d}B(s))^2(\omega)}{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)} \cdot \frac{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s) ds}\right)}{F^{-1}(t)^2} \\
&= 1 \cdot 0 = 0, \quad \text{a.s.}
\end{aligned}$$

Therefore, by Theorem 1.3, it follows that X obeys (1.3), as required. To prove part (ii), let us again suppose that the event A defined above is of positive probability. Arguing as in the proof of Theorem 1.10, we see that on the event

$$A' := A \cap \Omega_2 := A \cap \{\omega : \left(\lim_{t\to\infty} \int_0^t \sigma(s)\,\mathrm{d}B(s)\right)(\omega) \text{ exists and is finite}\}$$

(which has the same probability as A, because $\sigma \in L^2([0,\infty);\mathbb{R})$ ensures that the second event is a.s.) we have

$$\int_t^\infty \sigma(s)\,\mathrm{d}B(s) = -X(t) + \int_t^\infty f(X(s))\,\mathrm{d}s.$$

Therefore, defining $A'_+ = A_+ \cap \Omega_2$, $A'_- = A_- \cap \Omega_2$ and $A'_0 = A_0 \cap \Omega_2$, we can argue as in Theorem 1.10 to show that

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = -\lambda(\omega) + \lambda(\omega)^\beta, \quad \omega \in A'_+,$$
$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = -\lambda(\omega) - (-\lambda(\omega))^\beta, \quad \omega \in A'_-,$$
$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} = 0, \quad \omega \in A'_0.$$

Therefore, it follows that, for all $\omega \in A'$,

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} =: \Lambda \quad \text{exists and is finite.}$$

Now, by Lemma 1.2, there is an a.s. event Ω_3 such that for all $\omega \in \Omega_3$ we have

$$\limsup_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{\sqrt{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}} = 1,$$

with the liminf being -1. Therefore, for $\omega \in A'' := A' \cap \Omega_3$, for which $\mathbb{P}[A''] = \mathbb{P}[A] > 0$, we have

$$\lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{\sqrt{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}}$$
$$= \lim_{t\to\infty} \frac{\left(\int_t^\infty \sigma(s)\,\mathrm{d}B(s)\right)(\omega)}{F^{-1}(t)} \cdot \frac{F^{-1}(t)}{\sqrt{2\int_t^\infty \sigma^2(s)\,\mathrm{d}s \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,\mathrm{d}s}\right)}} = \Lambda\frac{1}{\mu},$$

where we interpret $1/\mu = 0$ in the case when μ is infinite. But on A'' this limit does not exist, giving the required contradiction.

1.11 Proof of Theorems 1.14 and 1.15

To prove Theorem 1.14, we require preliminary asymptotic estimates on the h-increment of the Itô integral in (1.2), as well as an auxiliary stochastic process with the same diffusion coefficient as (1.2). We prove that both of these processes are small relative to $(f \circ F^{-1})(t)$ as $t \to \infty$ a.s. under the condition that $S_f(\epsilon, h)$ is finite for all $\epsilon > 0$. The proof of the converse, Theorem 1.15, is more straightforward and follows in the second subsection.

Proof of Theorem 1.14

As promised, we start with a lemma concerning the asymptotic behaviour of the h-increment of the Itô integral in (1.2).

Lemma 1.15. *Suppose that f is continuous and obeys (1.11) and (1.12) for $\beta > 1$. Let σ be continuous. Let $h > 0$ and define $S_f(\epsilon, h)$ as in (1.34). If $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$, then*

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = 0, \quad a.s.$$

Proof. Considering $\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)$ we write, for $nh \le t \le (n+1)h$,

$$\left|\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)\right| \le \left|-\int_{(n+1)h}^{t} \sigma(s)\,\mathrm{d}B(s)\right| + \left|\int_{(n+1)h}^{t+h} \sigma(s)\,\mathrm{d}B(s)\right|.$$

It follows that

$$\sup_{nh\le t\le(n+1)h} \left|\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)\right| \le \sup_{nh\le t\le(n+1)h} \left|-\int_{(n+1)h}^{t} \sigma(s)\,\mathrm{d}B(s)\right| + \sup_{nh\le t\le(n+1)h} \left|\int_{(n+1)h}^{t+h} \sigma(s)\,\mathrm{d}B(s)\right|.$$

Similarly we obtain

$$\sup_{nh\le t\le(n+1)h} \left|\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)\right| \le \left|-\int_{nh}^{(n+1)h} \sigma(s)\,\mathrm{d}B(s)\right| + \sup_{(n+1)h\le t\le(n+2)h} \left|\int_{(n+1)h}^{t} \sigma(s)\,\mathrm{d}B(s)\right|. \tag{1.74}$$

Here we note that $S_f(\epsilon, h) < +\infty$ implies that

$$\lim_{t\to\infty} \frac{\int_{nh}^{(n+1)h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = 0, \quad \text{a.s.}$$

by means of the first Borel–Cantelli lemma. Next we proceed to estimate

$$\mathbb{P}\left[Z((n+1)h) > \epsilon\right], \text{ for some } \epsilon \in (0,1),$$

where

$$Z((n+1)h) := \sup_{(n+1)h \le t \le (n+2)h} \frac{\left|\int_{(n+1)h}^{t} \sigma(s)\,\mathrm{d}B(s)\right|}{(f \circ F^{-1})((n+1)h)}, \ n \geq 1.$$

We also define the function

$$\tau(t) := \frac{\int_{(n+1)h}^{t} \sigma^2(s)ds}{(f \circ F^{-1})^2((n+1)h)}, \text{ for } t \in [(n+1)h, (n+2)h].$$

By the Martingale Time Change Theorem there exists a standard Brownian motion B_n^* such that

$$\begin{aligned}
&\mathbb{P}[Z((n+1)h) > \epsilon] \\
&= \mathbb{P}\left[\sup_{t\in[(n+1)h,(n+2)h]} \left|B_{n+1}^*(\tau(t))\right| > \epsilon\right] \\
&= \mathbb{P}\left[\sup_{u\in[0,\tau((n+2)h)]} |B_{n+1}^*(u)| > \epsilon\right] \\
&\leq \mathbb{P}\left[\sup_{u\in[0,\tau((n+2)h)]} B_{n+1}^*(u) > \epsilon\right] + \mathbb{P}\left[\sup_{u\in[0,\tau((n+2)h)]} -B_{n+1}^*(u) > \epsilon\right] \\
&= \mathbb{P}\left[|B_{n+1}^*(\tau((n+2)h))| > \epsilon\right] + \mathbb{P}\left[|B_{n+1}^{**}(\tau((n+2)h))| > \epsilon\right],
\end{aligned}$$

where $-B_{n+1}^* = B_{n+1}^{**}$ is a standard Brownian motion. Thus, as $B_{n+1}^*(\tau((n+2)h))$ is normally distributed with zero mean, we have

$$\begin{aligned}
&\mathbb{P}[Z((n+1)h) > \epsilon] \\
&\leq 2\mathbb{P}\left[|B_{n+1}^*(\tau((n+2)h))| > \epsilon\right] = 4\mathbb{P}\left[B_{n+1}^*(\tau((n+2)h)) > \epsilon\right] \\
&= 4\Psi\left(\frac{\epsilon}{\sqrt{\tau((n+2)h)}}\right) = 4\Psi\left(\frac{\epsilon}{\theta(n+1)}\right).
\end{aligned}$$

But since we assumed that $S_f(\epsilon) < +\infty$, we have

$$\sum_{n=0}^{\infty} \mathbb{P}[Z((n+1)h) > \epsilon] \leq 4 \sum_{n=0}^{\infty} \Psi\left(\frac{\epsilon}{\sqrt{\tau((n+2)h)}}\right) < +\infty.$$

We can then apply the first Borel–Cantelli Lemma to conclude that

$$\limsup_{n\to\infty} Z((n+1)h) < \epsilon \quad \text{a.s.}$$

Thus

$$\lim_{n\to\infty} \sup_{(n+1)h\leq t\leq (n+2)h} \frac{\left|\int_{(n+1)h}^{t} \sigma(s)\,\mathrm{d}B(s)\right|}{(f\circ F^{-1})((n+1)h)} = 0 \quad \text{a.s.}$$

Combining this with (1.74) we get

$$\lim_{n\to\infty} \sup_{nh\leq t\leq (n+1)h} \frac{\left|\int_{t}^{t+h} \sigma(s)\,\mathrm{d}B(s)\right|}{(f\circ F^{-1})(nh)} = 0, \quad \text{a.s.,}$$

as required. □

We next need the asymptotic behaviour of an auxiliary process which solves an affine SDE.

Lemma 1.16. *Suppose that f is continuous and obeys* (1.11) *and* (1.12) *for $\beta > 1$. Let σ be continuous. Let $h > 0$ and define $S_f(\epsilon, h)$ as in* (1.34)*. Suppose that $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$. Let Y be the unique continuous adapted process which solves*

$$\mathrm{d}Y(t) = -Y(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t),\ t \geq 0, \quad Y(0) = 0. \tag{1.75}$$

Then

$$\lim_{t\to\infty} \frac{Y(t)}{(f\circ F^{-1})(t)} = 0, \quad \textit{a.s.} \tag{1.76}$$

Proof. Define

$$V_h(n) = \int_{(n-1)h}^{nh} \mathrm{e}^{s-nh}\sigma(s)\,\mathrm{d}B(s),\ n \geq 1; \quad \tilde{V}_h(n) = \frac{V_h(n)}{(\varphi\circ\Phi^{-1})(nh)},\ n \geq 1.$$

Then $(\tilde{V}_h(n))_{n\geq 1}$ is a sequence of independent normal random variables with zero mean and variance

$$\tilde{v}_h^2(n) = \frac{1}{(\varphi \circ \Phi^{-1})(nh)^2} \int_{(n-1)h}^{nh} \mathrm{e}^{2s-2nh} \sigma^2(s)\, \mathrm{d}s, \quad n \geq 1.$$

We show first that $\tilde{V}_h(n) \to 0$ a.s. as $n \to \infty$. By the fact that $f \circ F^{-1}$ is asymptotic to $\varphi \circ \Phi^{-1}$, there is $N = N(h)$ such that for all $n \geq N(h)$ we have

$$\tilde{v}_h^2(n) \leq \frac{1}{(\varphi \circ \Phi^{-1})(nh)^2} \int_{(n-1)h}^{nh} \sigma^2(s)\, \mathrm{d}s \leq 4\theta^2(n-1).$$

Hence $\tilde{v}_h(n) \leq 2\theta(n-1)$ for $n \geq N(h)$. Also

$$\mathbb{P}[|\tilde{V}_h(n)| > \epsilon] = 2\mathbb{P}[\tilde{V}_h(n)/\tilde{v}_h(n) > \epsilon/\tilde{v}_h(n)],$$

so

$$\mathbb{P}[|\tilde{V}_h(n)| > \epsilon] = 2\Psi(\epsilon/\tilde{v}_h(n)) \leq 2\Psi(\epsilon/\tilde{v}_h(n)) \leq 2\Psi(\epsilon/2/\theta(n-1)), \;\; n \geq N(h),$$

since Ψ is decreasing, and $\epsilon/v_h(n) \geq \epsilon/(2\theta(n-1))$ for $n \geq N(h)$. Now, due to the fact that $S_f(\epsilon/2, h) < +\infty$, it follows that

$$\sum_{n=0}^{\infty} \mathbb{P}[|\tilde{V}_h(n)| > \epsilon] < +\infty$$

for every $\epsilon > 0$, and hence, by the first Borel–Cantelli lemma, it follows that $\mathbb{P}[\lim_{n\to\infty} \tilde{V}_h(n) = 0] = 1$.

Next, we note that as Y is a solution of (1.75), it obeys

$$Y(t) = \mathrm{e}^{-t} \int_0^t \mathrm{e}^s \sigma(s)\; \mathrm{d}B(s), \quad t \geq 0.$$

Notice that $Y((n+1)h) = \mathrm{e}^{-h} Y(nh) + V_h(n+1)$ for $n \geq 0$. Now we define $\tilde{Y}(t) := Y(t)/(\varphi \circ \Phi^{-1})(t)$ for $t \geq 0$ and thus we have

$$\begin{aligned}\tilde{Y}((n+1)h) &= \mathrm{e}^{-h} \frac{Y(nh)}{(\varphi \circ \Phi^{-1})((n+1)h)} + \tilde{V}_h(n+1) \\ &= \mathrm{e}^{-h} \frac{(\varphi \circ \Phi^{-1})(nh)}{(\varphi \circ \Phi^{-1})((n+1)h)} \tilde{Y}(nh) + \tilde{V}_h(n+1).\end{aligned}$$

Hence

$$\tilde{Y}((n+1)h) = a(nh)\tilde{Y}(nh) + \tilde{V}_h(n+1), \quad n \geq 0,$$

where $a(nh) := \mathrm{e}^{-h}(\varphi \circ \Phi^{-1})(nh)/(\varphi \circ \Phi^{-1})((n+1)h)$. We note that $a(nh) > 0$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} a(nh) = \mathrm{e}^{-h}$. Notice that $(1 + h/2)\mathrm{e}^{-h} < 1$ for all $h > 0$. Since $a(nh) \to \mathrm{e}^{-h}$ as $n \to \infty$, there exists $N_2(h) \in \mathbb{N}$ such that $a(nh) \leq (1 + h/2)\mathrm{e}^{-h} < 1$ for all $n \geq N_2$. Next, we may write, for all $n > N_2(h)$,

$$\begin{aligned} |\tilde{Y}((n+1)h)| &\leq a(nh)|\tilde{Y}(nh)| + |\tilde{V}_h(n+1)| \\ &\leq \mathrm{e}^{-h}(1 + h/2)|\tilde{Y}(nh)| + |\tilde{V}_h(n+1)|. \end{aligned}$$

From this inequality we define

$$\begin{aligned} \bar{Y}((n+1)h) &= \mathrm{e}^{-h}(1 + h/2)\bar{Y}(nh) + |\tilde{V}_h(n+1)|, \quad n \geq N_2(h) + 1, \\ \bar{Y}(nh) &= |\tilde{Y}(nh)| + 1, \quad n = N_2(h) + 1. \end{aligned}$$

Therefore we have that $|\tilde{Y}(nh)| < \bar{Y}(nh)$ for $n \geq N_2(h) + 1$. Since $\tilde{V}_h(n) \to 0$ as $n \to \infty$ a.s., it follows that $\bar{Y}(nh) \to 0$ as $n \to \infty$ a.s. Hence $\tilde{Y}(nh) \to 0$ as $n \to \infty$ a.s.

Next, let $t \in [nh, (n+1)h]$. Then

$$\frac{Y(t)}{(\varphi \circ \Phi^{-1})(t)} = \frac{1}{(\varphi \circ \Phi^{-1})(t)} Y(nh)\mathrm{e}^{-(t-nh)} + \frac{\mathrm{e}^{-t}}{(\varphi \circ \Phi^{-1})(t)} \int_{nh}^{t} \mathrm{e}^{s}\sigma(s)\,\mathrm{d}B(s).$$

Notice since $\varphi'(x) \to 0$ as $x \to 0$, and $(\Phi^{-1})'(t) = (\varphi \circ \Phi^{-1})(t)$ that $t \mapsto \mathrm{e}^{-t}/(\varphi \circ \Phi^{-1})(t)$ is decreasing on (T_3, ∞) for some $T_3 > 0$. Define $N_3 \in \mathbb{N}$ such that $N_3 h > T_3$. Also $t \mapsto (\varphi \circ \Phi^{-1})(t)$ is decreasing on $[0, \infty)$. Then we have for $n \geq N_3$ that $t \geq nh \geq N_3 h > T_3$, and so

$$\begin{aligned} &\sup_{t\in[nh,(n+1)h]} \frac{|Y(t)|}{(\varphi \circ \Phi^{-1})(t)} \\ &\quad \leq \sup_{t\in[nh,(n+1)h]} \frac{1}{(\varphi \circ \Phi^{-1})(t)} |Y(nh)| \\ &\qquad + \sup_{t\in[nh,(n+1)h]} \frac{\mathrm{e}^{-t}}{(\varphi \circ \Phi^{-1})(t)} \left| \int_{nh}^{t} \mathrm{e}^{s}\sigma(s)\,\mathrm{d}B(s) \right| \\ &\quad \leq \frac{|Y(nh)|}{(\varphi \circ \Phi^{-1})((n+1)h)} \\ &\qquad + \frac{\mathrm{e}^{-(n+1)h}}{(\varphi \circ \Phi^{-1})((n+1)h)} \sup_{t\in[nh,(n+1)h]} \left| \int_{nh}^{t} \mathrm{e}^{s}\sigma(s)\,\mathrm{d}B(s) \right|. \end{aligned}$$

Since $Y(nh)/\varphi\circ\Phi^{-1}(nh) \to 0$ as $n \to \infty$ a.s. and $\varphi\circ\Phi^{-1}$ is in $\mathrm{RV}_\infty(-\beta/(\beta-1))$, we have that the first term on the right-hand side has zero limit as $n \to \infty$ a.s. Therefore it remains to prove that

$$U(n+1) := \frac{\mathrm{e}^{-(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \sup_{t\in[nh,(n+1)h]} \left| \int_{nh}^{t} \mathrm{e}^{s}\sigma(s)\,\mathrm{d}B(s) \right|$$

obeys $U(n) \to 0$ as $n \to \infty$ a.s., as this will demonstrate that

$$\lim_{n\to\infty} \sup_{t\in[nh,(n+1)h]} \frac{|Y(t)|}{(\varphi\circ\Phi^{-1})(t)} = 0, \quad \text{a.s.}$$

Next, we see that with $\rho(t) = \int_{nh}^{t} \mathrm{e}^{2s}\sigma^2(s)\,\mathrm{d}s$ for $t \in [nh,(n+1)h)$, by the martingale time change theorem, there exists a standard Brownian motion B_n^* such that

$$\begin{aligned}
\mathbb{P}[U(n+1) > \epsilon] &= \mathbb{P}\left[\sup_{t\in[nh,(n+1)h]} \left| \int_{nh}^{t} \mathrm{e}^{s}\sigma(s)\,\mathrm{d}B(s) \right| > \epsilon \frac{\mathrm{e}^{(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \right] \\
&= \mathbb{P}\left[\sup_{t\in[nh,(n+1)h]} |B_n^*(\rho(t))| > \epsilon \frac{\mathrm{e}^{(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \right] \\
&= \mathbb{P}\left[\sup_{t\in[0,\rho((n+1)h)]} |B_n^*(t)| > \epsilon \frac{\mathrm{e}^{(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \right].
\end{aligned}$$

By standard arguments, we get that

$$\begin{aligned}
\mathbb{P}[U(n+1) > \epsilon] &\le 2\mathbb{P}\left[\sup_{t\in[0,\rho((n+1)h)]} B_n^*(t) > \epsilon \frac{\mathrm{e}^{(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \right] \\
&= 2\mathbb{P}\left[|B_n^*(\rho((n+1)h))| > \epsilon \frac{\mathrm{e}^{(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \right] \\
&= 4\Psi\left(\epsilon \frac{\mathrm{e}^{(n+1)h}}{(\varphi\circ\Phi^{-1})((n+1)h)} \frac{1}{\sqrt{\rho((n+1)h)}} \right).
\end{aligned}$$

Finally, we estimate the right-hand side of the above expression. Since

$$\sqrt{\rho((n+1)h)} = \left(\int_{nh}^{(n+1)h} \mathrm{e}^{2s}\sigma^2(s)\,\mathrm{d}s \right)^{1/2} \le \mathrm{e}^{(n+1)h} \left(\int_{nh}^{(n+1)h} \sigma^2(s)\,\mathrm{d}s \right)^{1/2},$$

we have

$$\epsilon \frac{e^{(n+1)h}}{(\varphi \circ \Phi^{-1})((n+1)h)} \cdot \frac{1}{\sqrt{\rho((n+1)h)}} \geq \epsilon \frac{(f \circ F^{-1})(nh)}{(\varphi \circ \Phi^{-1})((n+1)h)} \cdot \frac{1}{(f \circ F^{-1})(nh)} \left(\int_{nh}^{(n+1)h} \sigma^2(s)\,ds \right)^{-1/2}.$$

Next, the fact that $f \circ F^{-1}$ is asymptotic to $\varphi \circ \Phi^{-1}$ and that both are regularly varying functions means there is an $N_4(h) \in \mathbb{N}$ such that

$$\frac{(f \circ F^{-1})(nh)}{(\varphi \circ \Phi^{-1})((n+1)h)} \geq \frac{1}{2}, \quad n \geq N_4(h).$$

Hence, by the definition of $\theta(n)$, for $n \geq N_4(h)$, we get

$$\epsilon \frac{e^{(n+1)h}}{(\varphi \circ \Phi^{-1})((n+1)h)} \cdot \frac{1}{\sqrt{\rho((n+1)h)}} \geq \frac{\epsilon}{2} \frac{1}{\theta(n)},$$

so as Ψ is decreasing, we have for $n \geq N_4(h)$ that

$$\mathbb{P}[U(n+1) > \epsilon] \leq 4\Psi \left(\epsilon \frac{e^{(n+1)h}}{(\varphi \circ \Phi^{-1})((n+1)h)} \frac{1}{\sqrt{\rho((n+1)h)}} \right) \leq 4\Psi \left(\frac{\epsilon/2}{\theta(n)} \right).$$

Since $S_f(\epsilon/2, h) < +\infty$ for all $\epsilon > 0$, it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}[U(n+1) > \epsilon] < +\infty$$

for every $\epsilon > 0$, and therefore, by the first Borel–Cantelli lemma, it follows that $\mathbb{P}[\lim_{n\to\infty} U(n) = 0] = 1$. As noted above, this is the remaining fact that guarantees that $\lim_{t\to\infty} Y(t)/(f \circ F^{-1})(t) = 0$ a.s., as required. □

We now have all the ingredients to prove Theorem 1.14.

Proof of Theorem 1.14. Since $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$, by Lemma 1.16, we have that

$$\lim_{t\to\infty} \frac{Y(t)}{(f \circ F^{-1})(t)} = 0, \quad \text{a.s.}$$

Consider $Z(t) = X(t) - Y(t)$ for $t \geq 0$. Since $(f \circ F^{-1})(nh) \to 0$ as $n \to \infty$, it follows that $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$ this implies that

$$S(\epsilon, h) := \sum_{n=1}^{\infty} \Psi\left(\frac{\epsilon}{\sqrt{\int_{nh}^{(n+1)h} \sigma^2(s)\,\mathrm{d}s}} \right) < +\infty.$$

This has been shown in [13] to give $X(t) \to 0$ as $t \to \infty$ a.s. Therefore, we have that $Z(t) \to 0$ as $t \to \infty$ a.s. Next, we have that $Z'(t) = -f(Z(t) + Y(t)) + Y(t)$ for $t \geq 0$. Now, given that f is continuous, we have that

$$Z'(t) = -f(Z(t)) + g(t),$$

where g is continuous and is given by $g(t) = f(Z(t)) - f(Z(t) + Y(t)) + Y(t)$ for $t \geq 0$. Now, since f is locally Lipschitz continuous, it follows that there is $K_\delta > 0$ such that $|f(x) - f(y)| \leq K_\delta |x - y|$ for all $|x|, |y| \leq \delta$. Since $Z(t) \to 0$ and $Y(t) \to 0$ as $t \to \infty$, it follows that $|Z(t)| \leq \delta/2$, $|Y(t)| \leq \delta/2$ for all $t \geq T_1$. Therefore $|f(Z(t)) - f(Z(t) + Y(t))| \leq K_\delta |Y(t)|$ for $t \geq T_1$. Hence $|g(t)| \leq (1 + K_\delta)|Y(t)|$ for $t \geq T_1$. Therefore, we have that

$$\lim_{t\to\infty} \frac{g(t)}{(f \circ F^{-1})(t)} = 0, \quad \text{a.s.}$$

Thus, by Theorem 1.2, we have for each outcome in an a.s. event that

$$Z(t, \omega)/F^{-1}(t) \to \lambda(\omega) \in \{-1, 0, 1\} \quad \text{as } t \to \infty.$$

Since $f(x)/x \to 0$ as $x \to 0$, it follows that $Y(t)/F^{-1}(t) \to 0$ as $t \to \infty$ a.s., so therefore we have that

$$\lim_{t\to\infty} \frac{X(t, \omega)}{F^{-1}(t)} = \lambda(\omega) \in \{-1, 0, 1\},$$

for every outcome ω in some a.s. event. This is the first limit in (1.35). Next let

$$\Omega_0 := \left\{ \omega : \text{ a solution } X(\cdot, \omega) \text{ exists and } \lim_{t\to\infty} X(t, \omega) = 0 \right\}.$$

We further define the events

$$A_0 := \left\{ \omega : \lim_{t\to\infty} \frac{X(t, \omega)}{F^{-1}(t)} = 0 \right\} \cap \Omega_0,$$

$$A_1^+ := \left\{ \omega : \lim_{t\to\infty} \frac{X(t, \omega)}{F^{-1}(t)} = 1 \right\} \cap \Omega_0,$$

$$A_1^- := \left\{ \omega : \lim_{t\to\infty} \frac{X(t, \omega)}{F^{-1}(t)} = -1 \right\} \cap \Omega_0,$$

with $A_1 := A_1^+ \cup A_1^-$. Therefore we have that $\Omega_1 := A_0 \cup A_1$ is a.s. and since $S_f(\epsilon, h) < \infty$ for all $\epsilon > 0$, by Lemma 1.15, we have

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = 0, \quad \text{a.s.}$$

Let the event on which this limit holds be Ω_2 and let $\Omega_3 = \Omega_0 \cap \Omega_2$. For $\omega \in \Omega_3$ we have

$$\lim_{t\to\infty} \frac{\int_t^{t+h} f(X(s))ds}{(f \circ F^{-1})(t)} = \lim_{t\to\infty} \frac{\int_t^{t+h} \varphi(X(s))ds}{(\varphi \circ F^{-1})(t)},$$

where φ is odd, increasing and $\varphi \in RV_0(\beta)$. If $\omega \in A_0$, then $X(t,\omega)/F^{-1}(t) \to 0$ as $t \to \infty$. Thus

$$\lim_{t\to\infty} \frac{|\varphi(X(t,\omega))|}{\varphi(F^{-1}(t))} = \lim_{t\to\infty} \frac{\varphi(|X(t,\omega)|)}{\varphi(F^{-1}(t))} = 0.$$

Hence, as $\varphi \circ F^{-1} \in \mathrm{RV}_0(-\beta/(\beta-1))$, we have

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \varphi(X(s,\omega))ds}{\varphi(F^{-1}(t))} = 0,$$

and therefore we have

$$\lim_{t\to\infty} \frac{\int_t^{t+h} f(X(s,\omega))ds}{(f \circ F^{-1})(t)} = 0, \quad \omega \in A_0 \cap \Omega_3.$$

Hence, for $\omega \in A_0 \cap \Omega_3$, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f \circ F^{-1})(t)} &= \lim_{t\to\infty} \frac{\frac{1}{h}\int_t^{t+h} -f(X(s))\,\mathrm{d}s}{(f \circ F^{-1})(t)} + \frac{\frac{1}{h}\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} \\ &= 0 = -\lambda(\omega).\end{aligned}$$

If $\omega \in A_1^+ \cap \Omega_3$, we have $\lim_{t\to\infty} X(t,\omega)/F^{-1}(t) = 1$, so as φ is regularly varying, it follows that $\lim_{t\to\infty} \varphi(X(t,\omega))/\varphi(F^{-1}(t)) = 1$. Hence

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \varphi(X(s,\omega))\,\mathrm{d}s}{\varphi(F^{-1}(t))} = h,$$

and so

$$\lim_{t\to\infty} \frac{-\frac{1}{h}\int_t^{t+h} \varphi(X(s,\omega))\,\mathrm{d}s}{\varphi(F^{-1}(t))} = -1.$$

Therefore, for $\omega \in A_1^+ \cap \Omega_3$, we have

$$\lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f\circ F^{-1})(t)} = -1 = -\lambda(\omega).$$

Similarly, for $\omega \in A_1^- \cap \Omega_3$, we use the fact that φ is odd to get

$$\lim_{t\to\infty} \frac{\varphi(X(t,\omega))}{\varphi(F^{-1}(t))} = \lim_{t\to\infty} \frac{-\varphi(-X(t,\omega))}{\varphi(F^{-1}(t))} = -1,$$

from which we obtain, for $\omega \in A_1^- \cap \Omega_3$,

$$\lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f\circ F^{-1})(t)} = 1 = -\lambda(\omega).$$

Thus for $\omega \in A_1 \cap \Omega_3$ we have

$$\lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f\circ F^{-1})(t)} = -\lambda(\omega),$$

and so for all $\omega \in \Omega_1 \cap \Omega_3$ we have

$$\lim_{t\to\infty} \frac{\frac{X(t+h,\omega)-X(t,\omega)}{h}}{(f\circ F^{-1})(t)} = -\lambda(\omega).$$

But since Ω_1 and Ω_3 are a.s. events, we have the second limit in (1.35), as required. □

Proof of Theorem 1.15

Define $A = A_1 \cup A_{-1} \cup A_0$. If we are in the case when $\omega \in A_1$, then

$$\lim_{t\to\infty} \frac{X(t,\omega)}{F^{-1}(t)} = 1.$$

Hence $X(t,\omega) \sim F^{-1}(t)$ as $t \to \infty$ and so $f(X(t,\omega)) \sim (f \circ F^{-1})(t)$ as $t \to \infty$. Therefore we have that

$$\lim_{t\to\infty} \frac{\frac{1}{h}\int_t^{t+h} f(X(s,\omega))\,\mathrm{d}s}{(f \circ F^{-1})(t)} = 1.$$

By hypothesis we know that

$$\lim_{t\to\infty} \frac{-\frac{1}{h}\int_t^{t+h} f(X(s,\omega))\,\mathrm{d}s}{(f \circ F^{-1})(t)} + \lim_{t\to\infty} \frac{\frac{1}{h}\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = -\lambda(\omega) = -1,$$

a.s. on A_1. Thus we can conclude that

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = 0, \text{ a.s. on } A_1.$$

By the same argument (and using Lemma 1.10) we can show that on A_{-1}, we get

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = 0, \text{ a.s. on } A_{-1}.$$

A similar limit applies for A_0:

$$\lim_{t\to\infty} \frac{\int_t^{t+h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(t)} = 0, \text{ a.s. on } A_0.$$

Therefore, as the limit applies to A_1, A_{-1} and A_0, it applies to all of A, a.s., and in particular along the sequence of times nh, for $n \geq 1$:

$$\lim_{t\to\infty} \frac{\int_{nh}^{(n+1)h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(nh)} = 0, \text{ a.s. on } A. \tag{1.77}$$

Since A is an event of positive probability and the random variables

$$\tilde{Y}_n = \frac{\int_{nh}^{(n+1)h} \sigma(s)\,\mathrm{d}B(s)}{(f \circ F^{-1})(nh)}$$

are independent, the convergence in (1.77) is a.s. by the zero–one law. Moreover, since the $\tilde{Y}_n$ are independent, the Borel–Cantelli lemmas force $S_f(\epsilon, h) < +\infty$ for all $\epsilon > 0$, as claimed.

1.12 Proofs from Examples Section

Proof of Lemma 1.8

Let $j \in \mathbb{N}$ and consider

$$\int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s = \int_0^{\pi} k(u+2\pi j)\sin(u)\,\mathrm{d}u + \int_{\pi}^{2\pi} k(v+2\pi j)\sin(v)\,\mathrm{d}v.$$

Now

$$\begin{aligned}
&\int_{\pi}^{2\pi} k(v+2\pi j)\sin(v)\,\mathrm{d}v \\
&\quad = \int_0^{\pi} k(u+\pi+2\pi j)\sin(u+\pi)\,\mathrm{d}u = -\int_0^{\pi} k(u+2\pi j+\pi)\sin(u)\,\mathrm{d}u.
\end{aligned}$$

Therefore

$$\int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s = \int_0^{\pi} \{k(u+2\pi j) - k(u+2\pi j+\pi)\}\sin(u)\,\mathrm{d}u.$$

Hence

$$\begin{aligned}
&\left| \frac{1}{-k'(2j\pi)\cdot\pi} \int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s - \int_0^{\pi}\sin(u)\,\mathrm{d}u \right| \\
&\leq \int_0^{\pi} \left| \frac{\{k(u+2\pi j) - k(u+2\pi j+\pi)\}}{-k'(2j\pi)\cdot\pi} - 1 \right| |\sin(u)|\,\mathrm{d}u \\
&\leq \pi \sup_{u\in[0,\pi]} \left| \frac{\{k(u+2\pi j) - k(u+2\pi j+\pi)\}}{-k'(2j\pi)\cdot\pi} - 1 \right|.
\end{aligned}$$

By the mean value theorem, for any $u \in [0,\pi]$, there is $\xi_{j,u} \in [0,\pi]$ such that we have

$$\begin{aligned}
\left| \frac{k(u+2\pi j) - k(u+2\pi j+\pi)}{-k'(2j\pi)\cdot\pi} - 1 \right| &= \left| \frac{k'(u+2\pi j+\xi_{u,j})}{k'(2j\pi)} - 1 \right| \\
&\leq \sup_{v\in[0,2\pi]} \left| \frac{k'(v+2\pi j)}{k'(2j\pi)} - 1 \right|.
\end{aligned}$$

Since $\int_0^\pi \sin(u)\,\mathrm{d}u = 2$, we have

$$\left|\frac{1}{-k'(2j\pi)\cdot\pi}\int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s - 2\right| \\ \leq \pi \sup_{v\in[0,2\pi]}\left|\frac{k'(v+2\pi j)}{k'(2j\pi)} - 1\right|.$$

By hypothesis, we therefore have that

$$\lim_{j\to\infty}\frac{1}{-k'(2j\pi)}\int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s = 2\pi. \tag{1.78}$$

Next

$$\frac{k(2j\pi)-k(2j\pi+2\pi)}{-k'(2j\pi)} - 2\pi = \int_{2j\pi}^{2j\pi+2\pi}\left\{\frac{k'(s)}{k'(2j\pi)} - 1\right\}\,\mathrm{d}s,$$

so

$$\lim_{j\to\infty}\frac{(k(2j\pi)-k(2j\pi+2\pi))}{-2\pi k'(2j\pi)} = 1.$$

Combining this with (1.78) gives

$$\lim_{j\to\infty}\frac{1}{k(2j\pi)-k(2j\pi+2\pi)}\int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s = 1.$$

Since $k(t)\to 0$ as $t\to\infty$, by Toeplitz lemma,

$$\lim_{n\to\infty}\frac{\int_{2n\pi}^\infty k_0(s)\,\mathrm{d}s}{k(2\pi n)} = \lim_{n\to\infty}\frac{\sum_{j=n}^\infty \int_{2j\pi}^{2(j+1)\pi} k_0(s)\,\mathrm{d}s}{\sum_{j=n}^\infty \frac{1}{2}(k(2j\pi)-k(2j\pi+2\pi))} = 1. \tag{1.79}$$

Equation (1.79) demonstrates that the first part of (i) is valid. We now use it to prove part (ii). To do so, let $n(t)$ be the largest integer less than or equal to $t/(2\pi)$, i.e., $n(t) = \lfloor t/(2\pi)\rfloor$. Then

$$\frac{\int_{2\pi(n(t)+1)}^\infty k_0(s)\,\mathrm{d}s}{k(t)} = \frac{\int_{2\pi(n(t)+1)}^\infty k_0(s)\,\mathrm{d}s}{k(2\pi(n(t)+1))}\cdot\frac{k(2\pi(n(t)+1))}{k(t)} \to 1$$

as $t\to\infty$. Also

$$\lim_{t\to\infty}\left\{\frac{\int_t^{2\pi(n(t)+1)} k_0(s)\,\mathrm{d}s}{k(t)} - \int_t^{2\pi(n(t)+1)}\sin(u)\,\mathrm{d}u\right\} = 0.$$

Therefore, as $\int_t^{2\pi(n(t)+1)} \sin(u)\,\mathrm{d}u = \cos(t) - \cos(2\pi(n(t)+1)) = \cos(t) - 1$, we have

$$\lim_{t\to\infty} \left\{ \frac{\int_t^\infty k_0(s)\,\mathrm{d}s}{k(t)} - \cos(t) \right\} = 0.$$

Therefore, we see that part (ii) is true. The proof of the second part of (i)
Let $j \in \mathbb{N}$; noting that

$$\int_\pi^{2\pi} k(v + 2\pi j)|\sin(v)|\,\mathrm{d}v = \int_0^\pi k(u + 2\pi j + \pi)|\sin(u)|\,\mathrm{d}u.$$

we see that

$$\begin{aligned}\int_{2j\pi}^{2(j+1)\pi} |k_0(s)|\,\mathrm{d}s &= \int_0^\pi k(u + 2\pi j)\sin(u)\,\mathrm{d}u + \int_\pi^{2\pi} k(v + 2\pi j)|\sin(v)|\,\mathrm{d}v \\ &= 2\int_0^\pi k(u + 2\pi j)\sin(u)\,\mathrm{d}u.\end{aligned}$$

Arguing as before, we see that

$$\lim_{j\to\infty} \frac{\int_{2j\pi}^{2(j+1)\pi} |k_0(s)|\,\mathrm{d}s}{k(2\pi j)} = 2\int_0^\pi \sin(u)\,\mathrm{d}u = 4.$$

Also

$$\lim_{j\to\infty} \frac{\int_{2j\pi}^{2(j+1)\pi} k(s)\,\mathrm{d}s}{k(2\pi j)} = 2\pi.$$

Therefore

$$\lim_{j\to\infty} \frac{\int_{2j\pi}^{2(j+1)\pi} |k_0(s)|\,\mathrm{d}s}{\int_{2j\pi}^{2(j+1)\pi} k(s)\,\mathrm{d}s} = \frac{4}{2\pi}.$$

Hence, by Toeplitz lemma, we have

$$\lim_{n\to\infty} \frac{\int_0^{2n\pi} |k_0(s)|\,\mathrm{d}s}{\int_0^{2n\pi} k(s)\,\mathrm{d}s} = \lim_{n\to\infty} \frac{\sum_{j=0}^n \int_{2j\pi}^{2(j+1)\pi} |k_0(s)|\,\mathrm{d}s}{\sum_{j=0}^n \int_{2j\pi}^{2(j+1)\pi} k(s)\,\mathrm{d}s} = \frac{4}{2\pi},$$

and so $\lim_{t\to\infty} \int_0^t |k_0(s)|\,\mathrm{d}s = +\infty$, as required.

Proof of Theorem 1.17

Suppose n is an integer such that $n \geq (2\beta - 1)/(\beta - 1)$, and let

$$g(t) = \Gamma(t) \sin\left(\left\{\int_0^t \Gamma(s)\,\mathrm{d}s\right\}^n\right), \quad t \geq 0.$$

Since Γ is continuous, so is g. Moreover, the function $I(t) := \int_0^t \Gamma(s)\,\mathrm{d}s$ is in $C^1((0,\infty);(0,\infty))$ and it obeys $I(t) \to \infty$ as $t \to \infty$. Let $0 \leq t < T$. Then, using integration by substitution, we obtain

$$\int_t^T g(s)\,\mathrm{d}s = \int_{I(t)^n}^{I(T)^n} \frac{1}{n} u^{-(1-1/n)} \sin(u)\,\mathrm{d}u.$$

If we identify $k(t) = t^{-(1-1/n)}/n$, and let $k_0(t) = k(t)\sin(t)$ for $t \geq 1$, it can be seen that k obeys all the properties of Lemma 1.8 and therefore that

$$\lim_{t\to\infty} \int_1^t k_0(s)\,\mathrm{d}s =: K^*, \quad \lim_{t\to\infty} \int_1^t |k_0(s)|\,\mathrm{d}s = +\infty.$$

Since $I(T) \to \infty$ as $T \to \infty$, g is continuous on $[0, 1]$ and

$$\int_1^T g(s)\,\mathrm{d}s = \int_{I(1)^n}^{I(T)^n} k_0(u)\,\mathrm{d}u, \text{ and } \int_1^T |g(s)|\,\mathrm{d}s = \int_{I(1)^n}^{I(T)^n} |k_0(u)|\,\mathrm{d}u,$$

we have that g obeys

$$\lim_{t\to\infty} \int_0^t g(s)\,\mathrm{d}s = K^* + \int_0^1 g(s)\,\mathrm{d}s, \text{ and } \lim_{t\to\infty} \int_0^t |g(s)|\,\mathrm{d}s = +\infty.$$

Of course, the first limit implies that $\int_t^\infty g(s)\,\mathrm{d}s \to 0$ as $t \to \infty$. Clearly by construction $\limsup_{t\to\infty} |g(t)|/\Gamma(t) = 1$, and g has infinitely many changes of sign because $I(t)^n$, the argument of sin in g, tends to infinity as $t \to \infty$.

Finally, we determine the asymptotic behaviour of $\int_t^\infty g(s)\,\mathrm{d}s$ as $t \to \infty$. Since $I(t) \to \infty$ as $t \to \infty$, by Lemma 1.8, we have that

$$\limsup_{t\to\infty} \frac{\left|\int_t^\infty g(s)\,\mathrm{d}s\right|}{k(I(t)^n)} = \limsup_{t\to\infty} \frac{\left|\int_{I(t)^n}^\infty k_0(u)\,\mathrm{d}u\right|}{k(I(t)^n)} = 1.$$

Since $k(I(t)^n) = I(t)^{-(n-1)}/n$, we have

$$\limsup_{t\to\infty} \frac{\left|\int_t^\infty g(s)\,\mathrm{d}s\right|}{\frac{1}{n} I(t)^{-(n-1)}} = 1. \tag{1.80}$$

Next, as $\Gamma(t) \to \infty$ as $t \to \infty$, it follows that $I(t)/t \to \infty$ as $t \to \infty$, so there exists T_1 such that $I(t) > t$ for all $t \geq T_1$. Therefore $I(t)^{-(n-1)} < t^{-(n-1)}$ for $t \geq T_1$. Since $F^{-1} \in \mathrm{RV}_\infty(-1/(\beta-1))$, we have that

$$\lim_{t\to\infty} \frac{\log F^{-1}(t)}{\log t} = -\frac{1}{\beta - 1}.$$

Hence there is a $T_2 > 0$ such that $F^{-1}(t) > t^{-1/(\beta-1)-1/2}$ for $t \geq T_2(\epsilon)$. Now let $T_3 = \max(T_1, T_2)$. For $t \geq T_3$, we have

$$\frac{I(t)^{-(n-1)}}{F^{-1}(t)} \leq t^{-(n-1)} \cdot t^{1/(\beta-1)+1/2},$$

and as $n \geq (2\beta - 1)/(\beta - 1)$, the right-hand side of this expression tends to zero as $t \to \infty$. Combining this with (1.80) gives the second part of (1.13), as claimed. Therefore, Theorem 1.2 applies to the solution x of (1.1), as claimed.

Proof of Lemma 1.9

First we note some key properties of h_s which will be used extensively:

$$h(0, a, b) = 0, \; \frac{d}{dx} h_s(x, a, b) = 0 \text{ for } x = 0, a, b,$$
$$\frac{d}{dx} h_s(x, a, b) > 0 \text{ for } x \in (0, a) \text{ and } \frac{d}{dx} h_s(x, a, b) < 0 \text{ for } x \in (a, 2a).$$

Thus $h_s(x, a, b) \leq h_s(a, a, b) = b$ for $x \in [0, 2a]$. Now we proceed to show (1.40):

$$\begin{aligned} k(n + \frac{w_n}{2}) &= k_s(n + \frac{w_n}{2}) + h_s(\frac{w_n}{2}, \frac{w_n}{2}, \Gamma_+(n + \frac{w_n}{2}) - k_s(n + \frac{w_n}{2})) \\ &= k_s(n + \frac{w_n}{2}) + \Gamma_+(n + \frac{w_n}{2}) - k_s(n + \frac{w_n}{2}) = \Gamma_+(n + \frac{w_n}{2}), \end{aligned}$$

which is valid since $\Gamma_+(t) > g_s(t)$, $t \geq 0$. Therefore, with $t_n = n + \frac{w_n}{2}$,

$$\limsup_{t\to\infty} \frac{k(t)}{\Gamma_+(t)} \geq \limsup_{n\to\infty} \frac{k(t_n)}{\Gamma_+(t_n)} = 1.$$

Considering $t \in [n, n + w_n]$, we obtain

$$\begin{aligned} k(t) &= k_s(t) + h_s(t - n, \frac{w_n}{2}, \Gamma_+(t) - k_s(t)) \\ &\leq k_s(t) + \Gamma_+(t) - k_s(t) = \Gamma_+(t). \end{aligned}$$

For $t \in [n + w_n, n + 1]$, $k(t) = k_s(t) < \Gamma_+(t)$. Therefore $k(t) \leq \Gamma_+(t)$ for all $t \geq 0$. Thus

$$1 \leq \limsup_{t\to\infty} \frac{k(t)}{\Gamma_+(t)} \leq 1, \text{ so } \limsup_{t\to\infty} \frac{k(t)}{\Gamma(t)} = 1,$$

since $\Gamma_+(t) \sim \Gamma(t)$ as $t \to \infty$.

Given $k_s \in C^1(0,\infty)$, to show that $k \in C^1(0,\infty)$, we just need to show that it is C^1 at the points of transition. Write $h_s(x,a,b) = b\tilde{h}_s(x,a)$. Then

$$\begin{aligned}
&\frac{d}{dt} h_s(t-n, \frac{w_n}{2}, \Gamma_+(t) - k_s(t)) \\
&\quad = \frac{\partial}{\partial x} h_s(t-n, \frac{w_n}{2}, \Gamma_+(t) - k_s(t)) \\
&\quad\quad + \frac{\partial}{\partial t} h_s(t-n, \frac{w_n}{2}, \Gamma_+(t) - k_s(t)) \cdot (\Gamma_+'(t) - k_s'(t)) \\
&\quad = (\Gamma_+(t) - k_s(t)) \cdot \tilde{h}_s'(t-n, \frac{w_n}{2}) \\
&\quad\quad + \tilde{h}_s(t-n, \frac{w_n}{2})(\Gamma_+'(t) - k_s'(t)).
\end{aligned}$$

Now as $\tilde{h}_s'(0,a) = \tilde{h}_s'(a,a) = \tilde{h}_s'(2a,a) = 0$ and $\tilde{h}_s(0,a) = \tilde{h}_s(2a,a) = 0$ we have

$$k'(t) = \begin{cases} k_s'(t) + (\Gamma_+(t) - k_s(t)) \cdot \tilde{h}_s'(t-n, \frac{w_n}{2}) & \\ \quad + \tilde{h}_s(t-n, \frac{w_n}{2})(\Gamma_+'(t) - k_s'(t)), & t \in [n, n+w_n), \\ k_s'(t), \quad t \in [n + w_n, n+1]. & \end{cases}$$

Hence $\lim_{t\downarrow n} k'(t) = k'(n)$ and $\lim_{t\uparrow n+w_n} k'(t) = k'(n+w_n)$. Similarly, we have $\lim_{t\downarrow n} k(t) = k_s(n) + h_s(0, \frac{w_n}{2}, \Gamma_+(n) - k_s(n)) = k_s(n) = k(n)$ and

$$\begin{aligned}
\lim_{t\uparrow n+w_n} k(t) &= k_s(n+w_n) + h_s(w_n, \frac{w_n}{2}, \Gamma_+(n+w_n) - k_s(n+w_n)) \\
&= k_s(n+w_n) + \{\Gamma_+(n+w_n) - k_s(n+w_n)\}\tilde{h}_s(w_n, \frac{w_n}{2}) \\
&= k_s(n+w_n) = k(n+w_n).
\end{aligned}$$

Thus we have $k \in C^1(0,\infty)$, as required.

Finally we demonstrate that (1.39) holds. Suppose $t \in [n, n+w_n)$ and write

$$k(t) = k_s(t) + h_s(t-n, \frac{w_n}{2}, \Gamma_+(t) - k_s(t)) \leq \Gamma_+(t) \leq \Gamma_+(n+1).$$

Hence it can be shown that

$$\int_t^\infty k(u)\,\mathrm{d}u \le \int_t^\infty k_s(u)\,\mathrm{d}u + \sum_{j=n}^\infty w_j \Gamma_+(j+1),\ t \in [n, n+w_n).$$

Similarly, for $t \in [n+w_n, n+1]$, we have

$$\int_t^\infty k(u)\,\mathrm{d}u \le \int_t^\infty k_s(u)\,\mathrm{d}u + \sum_{j=n+1}^\infty w_j \Gamma_+(j+1).$$

Thus

$$\int_t^\infty k(u)\,\mathrm{d}u \le \int_t^\infty k_s(u)\,\mathrm{d}u + \sum_{j=n}^\infty w_j \Gamma_+(j+1),\ t \in [n, n+1].$$

Hence, for $t \in [n, n+1]$,

$$\begin{aligned}\frac{\int_t^\infty k(u)\,\mathrm{d}u}{\int_t^\infty k_s(u)\,\mathrm{d}u} &\le 1 + \frac{\sum_{j=n}^\infty w_j \Gamma_+(j+1)}{\int_t^\infty k_s(u)\,\mathrm{d}u} \le 1 + \frac{\sum_{j=n}^\infty w_j \Gamma_+(j+1)}{\int_{n+1}^\infty k_s(u)\,\mathrm{d}u} \\ &= 1 + \frac{\sum_{j=n}^\infty w_j \Gamma_+(j+1)}{\sum_{j=n}^\infty \int_{j+1}^{j+2} k_s(u)\,\mathrm{d}u}.\end{aligned}$$

Now using (1.37) we have

$$0 \le \limsup_{n\to\infty} \frac{w_n \Gamma_+(n+1)}{\int_{n+1}^{n+2} k_s(u)\,\mathrm{d}u} \le \limsup_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence

$$\lim_{n\to\infty} \frac{w_n \Gamma_+(n+1)}{\int_{n+1}^{n+2} k_s(u)\,\mathrm{d}u} = 0,$$

and by Toeplitz lemma [24]

$$\lim_{n\to\infty} \frac{\sum_{j=n}^\infty w_j \Gamma_+(j+1)}{\sum_{j=n}^\infty \int_{j+1}^{j+2} k_s(u)\,\mathrm{d}u} = 0.$$

Hence with $n(t) \in \mathbb{N}$ defined by $t \le n(t) < t+1$ we have

$$\limsup_{t\to\infty} \frac{\int_t^\infty k(u)\,\mathrm{d}u}{\int_t^\infty k_s(u)\,\mathrm{d}u} \le 1 + \limsup_{t\to\infty} \frac{\sum_{j=n(t)}^\infty w_j \Gamma_+(j+1)}{\sum_{j=n(t)}^\infty \int_{j+1}^{j+2} k_s(u)\,\mathrm{d}u} = 1.$$

Since $k(t) \geq k_s(t)$, for all $t \geq 0$,

$$\liminf_{t\to\infty} \frac{\int_t^\infty k(u)\,\mathrm{d}u}{\int_t^\infty k_s(u)\,\mathrm{d}u} \geq 1,$$

which completes the proof.

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Chapter 2
Comparison Theorems for Second-Order Functional Differential Equations

Zuzana Došlá and Mauro Marini

Abstract The effect of the deviating argument on the existence of nonoscillatory solutions for second-order differential equations with p-Laplacian is studied by means of the comparison with a half-linear equation. As a consequence, necessary and sufficient conditions for the existence of the so-called intermediate solutions are given and the coexistence with different types of nonoscillatory solutions is analyzed. Moreover, new oscillation results are established too.

Keywords Half-linear equation • Oscillation • Intermediate solution

2.1 Introduction

Consider the second-order nonlinear differential equation with the deviating argument

$$\big(a(t)|y'(t)|^{\alpha}\operatorname{sgn} y'(t)\big)' + b(t)|y(t+q)|^{\alpha}\operatorname{sgn} y(t+q) = 0, \tag{2.1}$$

and the corresponding half-linear equation

$$\big(a(t)|x'(t)|^{\alpha}\operatorname{sgn} x'(t)\big)' + b(t)|x(t)|^{\alpha}\operatorname{sgn} x(t) = 0. \tag{2.2}$$

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Throughout the paper we assume that α, q are real constants, $\alpha > 0$, and a, b are continuous functions on $[t_0, \infty)$, such that $a(t) > 0$, $b(t) \geq 0$, $\sup\{b(t) : t \geq T\} > 0$ for all $T > t_0$ and

$$I_a = \int_{t_0}^{\infty} a^{-1/\alpha}(s)\,\mathrm{d}s = \infty, \;\; I_b = \int_{t_0}^{\infty} b(s)\,\mathrm{d}s < \infty.$$

As usual, a nontrivial solution y of (2.1), which exists on some ray $[\tau, \infty)$, $\tau \geq t_0$, is said to be *nonoscillatory* if $y(t) \neq 0$ for large t and *oscillatory* otherwise. Moreover, equation (2.1) is said to be *oscillatory* if any of its solution, which is continuable up to infinity, is oscillatory. Recall that, in view of the Sturmian theory, the half-linear equation (2.2) does not admit coexistence of oscillatory and nonoscillatory solutions; see, e.g., [10]. Moreover, since any solution of (2.2) is defined for all $t \geq t_0$, for (2.2) the absence of nonoscillatory solutions is equivalent to the oscillation. Hence, equation (2.2) is said to be *nonoscillatory* if any of its solution is nonoscillatory.

Concerning the qualitative behavior of the solutions of functional differential equations, many papers are devoted to a comparison between the cases with and without deviating argument. We refer the reader to the books [1, 2, 11, 14] and references therein.

In this paper we examine the effect of the deviating argument q on the existence of possible types of nonoscillatory solutions. Our main goal here is to find necessary and sufficient conditions for the existence of the so-called intermediate solutions of (2.1); see below for the definition. This problem is very difficult, as it is pointed out in [18, Remark 1.1] and [1, page 241]. Our study is accomplished by means of a topological approach and a comparison with the half-linear case (2.2). As a consequence, the role of the deviating argument q, in studying the coexistence of possible types of nonoscillatory solutions, is illustrated.

Moreover, new conditions for the oscillation of (2.1) are established too. In particular, we prove that if $a(t) \equiv 1$, then equations (2.1) and (2.2) have the same oscillatory behavior.

2.2 Preliminaries

For any solution y of (2.1), denote by $y^{[1]}$ its *quasiderivative*, i.e., the function

$$y^{[1]}(t) = a(t)|y'(t)|^{\alpha} \operatorname{sgn} y'(t). \tag{2.3}$$

Since $I_a = \infty$, it is easy to see that any nonoscillatory solution y of (2.1) is eventually monotone and satisfies $y(t)y^{[1]}(t) > 0$ for large t; see, e.g., [6, 18]. We denote this property by saying that y is of the class $\mathbb{M}^+$.

Thus, any nonoscillatory solution y of (2.1) is either eventually positive increasing with $y^{[1]}$ positive nonincreasing or y is eventually negative decreasing with $y^{[1]}$ negative nondecreasing. Hence, we can divide the class $\mathbb{M}^+$ of all nonoscillatory solutions of (2.1) into the subclasses:

$$\mathbb{M}^+_{\infty,\ell} = \{y \in \mathbb{M}^+ : |y(\infty)| = \infty,\ y^{[1]}(\infty) = \ell_y,\ 0 < |\ell_y| < \infty\},$$
$$\mathbb{M}^+_{\infty,0} = \{y \in \mathbb{M}^+ : |y(\infty)| = \infty,\ y^{[1]}(\infty) = 0\},$$
$$\mathbb{M}^+_{\ell,0} = \{y \in \mathbb{M}^+ : y(\infty) = \ell_y,\ y^{[1]}(\infty) = 0,\ 0 < |\ell_y| < \infty\}.$$

Following [18], solutions in $\mathbb{M}^+_{\infty,\ell}$, $\mathbb{M}^+_{\infty,0}$, $\mathbb{M}^+_{\ell,0}$ are called *dominant solutions, intermediate solutions*, and *subdominant solutions*, respectively. Indeed, if $y \in \mathbb{M}^+_{\infty,\ell}$, $w \in \mathbb{M}^+_{\infty,0}$, $z \in \mathbb{M}^+_{\ell,0}$, then we have

$$|y(t)| > |w(t)| > |z(t)| \quad \text{for large} \quad t.$$

For the half-linear equation (2.2), an important role in finding nonoscillation criteria is played by the integrals

$$J = \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_t^{\infty} b(\sigma)\,\mathrm{d}\sigma\right)^{1/\alpha} \mathrm{d}t,$$
$$K_0 = \int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{t} \frac{1}{a^{1/\alpha}(\sigma)}\,\mathrm{d}\sigma\right)^{\alpha} \mathrm{d}t,$$

see, e.g., [5, 15]. When $\alpha = 1$, in view of the Fubini theorem, we have $J = K_0$. In general, the possible cases, concerning the mutual convergence of integrals J and K_0, are the following four cases ([9, Section 3.2.1.]):

$$\begin{aligned}
&(\mathrm{C}_1): && J = \infty, && K_0 = \infty, && \alpha > 0;\\
&(\mathrm{C}_2): && J = \infty, && K_0 < \infty, && \alpha > 1;\\
&(\mathrm{C}_3): && J < \infty, && K_0 = \infty, && 0 < \alpha < 1;\\
&(\mathrm{C}_4): && J < \infty, && K_0 < \infty, && \alpha > 0.
\end{aligned} \tag{2.4}$$

For the equation (2.1) with the deviating argument q, the role of K_0 is played by

$$K_q = \int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{g_q(t)} \frac{1}{a^{1/\alpha}(\sigma)}\,\mathrm{d}\sigma\right)^{\alpha} \mathrm{d}t, \tag{2.5}$$

where $g_q(t) = \max_{t \geq t_0} \{t_0, t + q\}$.

If $q \geq 0$, then $t + q \geq t_0$ for $t \geq t_0$. If $q < 0$, then there exists $\bar{t}_0 \geq t_0$ such that $t + q \geq \bar{t}_0$. Hence, without loss of generality, we can put $\bar{t}_0 = t_0$ and so, in both cases, (2.5) becomes

$$K_q = \int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{t+q} \frac{1}{a^{1/\alpha}(\sigma)} \, \mathrm{d}\sigma \right)^{\alpha} \mathrm{d}t.$$

Clearly, we have

$$K_p \leq K_0 \leq K_q \tag{2.6}$$

for $p < 0 < q$.

Using [18, Theorem 3.1], the following result holds.

Theorem A.

(i$_1$) *Equation* (2.1) *has subdominant solutions if and only if* $J < \infty$.
(i$_2$) *Equation* (2.1) *has dominant solutions if and only if* $K_q < \infty$.

Theorem A. shows that the deviating argument q does not have influence on the existence of subdominant solutions for (2.1).

Concerning dominant solutions, the situation can be different. The following example illustrates this fact.

Example 2.1. Consider the equation ($t \geq 1$)

$$\left(\frac{1}{2t \exp(t^2)} y'(t) \right)' + \frac{1}{t^2 \exp(t^2)} y(t+q) = 0. \tag{2.7}$$

A standard calculation shows that $K_0 < \infty$ and $K_q = \infty$ for any $q > 0$. Thus, in view of Theorem A., equation (2.7) with the advanced argument does not have dominant solutions, while the corresponding linear equation possesses this type of nonoscillatory solutions. In view of the Fubini theorem, $J = K_0$ and so (2.7) has subdominant solutions for any q.

Thus, for (2.7) we have

$$\begin{aligned} q \leq 0: &\quad \mathbb{M}^+_{\infty,\ell} \neq \emptyset, \quad \mathbb{M}^+_{\ell,0} \neq \emptyset, \\ q > 0: &\quad \mathbb{M}^+_{\infty,\ell} = \emptyset, \quad \mathbb{M}^+_{\ell,0} \neq \emptyset. \end{aligned}$$

Note that an analogous example can be produced for (2.1) with the delayed argument. In this case, because $K_q \leq K_0$ for $q < 0$, there exist equations of type (2.1) with dominant solutions, while the corresponding half-linear equation does not have this type of solutions.

The discrepancy in Example 2.1 depends on the growth of a at infinity. If there exists a constant $H > 0$ such that

$$\limsup_{t \to \infty} \int_{t-H}^{t} a^{-1/\alpha}(s) \, \mathrm{d}s < \infty, \tag{2.8}$$

then the deviating argument q does not produce any discrepancy in the existence of dominant solutions, as the following result shows.

Lemma 2.1. *Assume* (2.8). *Then, we have for* $q \neq 0$

$$K_0 < \infty \quad \textit{if and only if} \quad K_q < \infty.$$

Proof. Without loss of generality, suppose $q > 0$ and let j be an integer such that $q < Hj$. Thus,

$$\int_t^{t+q} a^{-1/\alpha}(s)\,\mathrm{d}s < \int_t^{t+Hj} a^{-1/\alpha}(s)\,\mathrm{d}s = \sum_{k=1}^{j} \int_{t+H(k-1)}^{t+Hk} a^{-1/\alpha}(s)\,\mathrm{d}s. \tag{2.9}$$

In view of (2.6), it is sufficient to show

$$K_0 < \infty \;\Rightarrow\; K_q < \infty. \tag{2.10}$$

From (2.8) and (2.9), there exists $M > 0$ such that we have for any large t, say $t \geq T$,

$$\int_t^{t+q} a^{-1/\alpha}(s)\,\mathrm{d}s < M.$$

Hence, using the inequality

$$(X+Y)^\alpha \leq \sigma_\alpha (X^\alpha + Y^\alpha),$$

where

$$\sigma_\alpha = \begin{cases} 1 & \text{if } \alpha \leq 1 \\ 2^{\alpha-1} & \text{if } \alpha > 1 \end{cases},$$

we obtain

$$\int_T^\infty b(t)\left(\int_T^{t+q} \frac{1}{a^{1/\alpha}(\sigma)}\,\mathrm{d}\sigma\right)^\alpha \mathrm{d}t \leq \int_T^\infty b(t)\left(\int_T^{t} \frac{1}{a^{1/\alpha}(\sigma)}\,\mathrm{d}\sigma + M\right)^\alpha \mathrm{d}t$$
$$\leq \sigma_\alpha \int_T^\infty b(t)\left(\int_T^{t} \frac{1}{a^{1/\alpha}(\sigma)}\,\mathrm{d}\sigma\right)^\alpha \mathrm{d}t + \sigma_\alpha M^\alpha \int_T^\infty b(t)\,\mathrm{d}t$$

from which (2.10) follows, because $I_b < \infty$. □

Consequently, from Lemma 2.1 and Theorem A., we obtain the following.

Corollary 2.1. *Assume* (2.8). *Then* (2.1) *has dominant solutions if and only if* $K_0 < \infty$.

Concerning the intermediate solutions, writing (2.1) as the coupled nonlinear differential system

$$\begin{cases} u' = a^{-1/\alpha}(t)|v(t)|^{1/\alpha}\operatorname{sgn} v(t) \\ v' = -b(t)|u((t+q)|^{\alpha}\operatorname{sgn} u(t+q) \end{cases}, \tag{2.11}$$

and using [6, Theorem 3.1], we obtain the following sufficient criterion (see also [18, Theorem 1.3] when $a \equiv 1$).

Theorem B. *Equation* (2.1) *has intermediate solutions if*

$$J = \infty \quad \text{and} \quad K_q < \infty.$$

As it is claimed in [18, Remark 1.1] (see also [1, page 241]), the question to find a necessary and sufficient condition for the existence of intermediate solutions is a very difficult problem, due to the lack of good *a priori* bounds.

This problem has been completely resolved in [5] for the half-linear equation. More precisely, from Theorem 6 and Theorem 7 in [5], we obtain the following.

Theorem C. *Equation* (2.2) *has intermediate solutions if and only if any of the following conditions is satisfied:*

(i_1) *The case (C_2), defined in* (2.4)*, occurs.*
(i_2) *The case (C_3), defined in* (2.4)*, occurs.*
(i_3) *The case (C_1), defined in* (2.4)*, occurs and* (2.2) *is nonoscillatory.*

Thus, for the half-linear equation (2.2), the situation is summarized as follows:

(1) *If the case* (C_1) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} = \emptyset$, $\mathbb{M}^+_{\ell,0} = \emptyset$.
Moreover, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, *if* (2.2) *is nonoscillatory.*
(2) *If the case* (C_2) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\ell,0} = \emptyset$.
(3) *If the case* (C_3) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} = \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\ell,0} \neq \emptyset$.
(4) *If the case* (C_4) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} = \emptyset$, $\mathbb{M}^+_{\ell,0} \neq \emptyset$.

Remark 2.1. Theorems A., B., and C. have been proved by assuming the positivity of the function b. Nevertheless, it is easy to verify that these results continue to hold also in the case here considered $b(t) \geq 0$, and $\sup\{b(t) : t \geq T\} > 0$ for all $T > t_0$.

Remark 2.2. When $\alpha = 1$, as already noticed, we have either $J = K_0 < \infty$ or $J = K_0 = \infty$. Thus, in virtue of Theorem A. and Theorem C., for the linear equation

$$\big(a(t)x'(t)\big)' + b(t)x(t) = 0, \tag{2.12}$$

intermediate solutions cannot coexist with dominant solutions or subdominant solutions.

2.3 Intermediate Solutions

In this section we present our main result, which is a necessary and sufficient condition for the existence of intermediate solutions for (2.1).

Theorem 2.1. *Assume* (2.8). *Equation* (2.1) *has intermediate solutions if and only if the same occurs for* (2.2).

Clearly, if $a(t) \equiv 1$, then (2.8) is satisfied.

To prove Theorem 2.1, the following result is useful.

Lemma 2.2. *Let y be a nonoscillatory solution of* (2.2) *such that* $\lim_{t\to\infty} y^{[1]}(t) = 0$. *Then for any fixed $h > 0$*

$$\lim_{t\to\infty} [y(t) - y(t-h)] = 0.$$

Proof. Without loss of generality, suppose $y(t+q-h) > 0$, $y^{[1]}(t-h) > 0$ for $t \geq t_0$. Thus, from (2.1) $y^{[1]}$ is nonincreasing for $t \geq t_0$. Hence we have for $t \geq t_0$

$$\begin{aligned} y(t) - y(t-h) &= \int_{t-h}^{t} \left(\frac{y^{[1]}(s)}{a(s)} \right)^{1/\alpha} \mathrm{d}s \\ &\leq \left(y^{[1]}(t-h) \right)^{1/\alpha} \int_{t-h}^{t} a^{-1/\alpha}(s)\, \mathrm{d}s. \end{aligned}$$

If $h \leq H$, the assertion follows from (2.8), recalling that $\lim_{t\to\infty} y^{[1]}(t) = 0$. If $h > H$, choose $n_0 \in \mathbb{N}$ large so that $Hn_0 > h$. Then we have

$$\int_{t-h}^{t} a^{-1/\alpha}(s)\, \mathrm{d}s \leq \int_{t-n_0H}^{t} a^{-1/\alpha}(s)\, \mathrm{d}s = \sum_{i=0}^{i=n_0-1} \int_{t_i-H}^{t_i} a^{-1/\alpha}(s)\, \mathrm{d}s,$$

where $t_i = t - iH$. Hence, by using the same argument as before, the assertion again follows. □

Proof of Theorem 2.1. For the sake of simplicity, we prove the assertion in the case $q > 0$. When $q < 0$, the argument is similar, with minor changes.

Step 1. Assume that (2.2) has an intermediate solution x and suppose, without loss of generality, $x(t) > 1$, $x^{[1]}(t) > 0$ for $t \geq t_0$. Integrating (2.2) on (T, t), $T \geq t_0$, we obtain

$$x(t) - x(T) = \int_{T}^{t} \left(\frac{1}{a(s)} \left(\int_{s}^{\infty} b(r) x^{\alpha}(r))\, \mathrm{d}r \right) \right)^{1/\alpha} \mathrm{d}s. \tag{2.13}$$

Let $t_1 = t_0 + q$. Thus $t - q \geq t_0$ on $[t_1, \infty)$. Fixing $M > 1$, in virtue of Lemma 2.2, we can suppose that for large t

$$x(t) - x(t-q) \le 1 - M^{-1}. \tag{2.14}$$

For simplicity, we can assume that (2.14) is valid for any $t \ge t_1$. Consider in the Fréchet space $C[t_1,\infty)$ of all continuous functions on $[t_1,\infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_1,\infty)$, the set Ω given by

$$\Omega = \{u \in C[t_1,\infty) : x(t-q) \le u(t) \le Mx(t-q)\} \,.$$

Define in Ω the operator T as follows

$$T(u)(t) = x(t_1) + \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_s^{\infty} b(r) u^{\alpha}(r+q)\, \mathrm{d}r \right)^{1/\alpha} \mathrm{d}s. \tag{2.15}$$

Since $u \in \Omega$, we have for $t \ge t_1$

$$x(t) \le u(t+q) \le Mx(t)$$

and thus, from (2.15), we get

$$T(u)(t) \le x(t_1) + M \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_s^{\infty} b(r) x^{\alpha}(r)\, \mathrm{d}r \right)^{1/\alpha} \mathrm{d}s.$$

Hence, in view of (2.13), we obtain

$$T(u)(t) \le x(t_1) + Mx(t) - Mx(t_1). \tag{2.16}$$

Thus, from (2.14), we have

$$\begin{aligned} T(u)(t) &\le x(t_1) + Mx(t-q) + M - 1 - Mx(t_1) = \\ &= Mx(t-q) - (x(t_1) - 1)(M-1) \end{aligned}$$

or

$$T(u)(t) \le Mx(t-q),$$

because $x(t_1) > 1$ and $M > 1$. Moreover, from (2.13) and (2.15), we get also

$$T(u)(t) \ge x(t_1) + \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_s^{\infty} b(r) x^{\alpha}(r)\, \mathrm{d}r \right)^{1/\alpha} \mathrm{d}s = x(t), \tag{2.17}$$

from which we obtain $T(u)(t) \ge x(t-q)$, because x is increasing on $[t_1,\infty)$. Hence the operator T maps Ω into itself.

Let us show that $T(\Omega)$ is relatively compact, i.e., $T(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[t_1, \infty)$. Because $T(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, in view of the above estimates, we have for any $u \in \Omega$

$$\int_t^\infty b(r)x^\alpha(r))\,\mathrm{d}r \le a(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}T(u)(t)\right)^\alpha \le M\int_t^\infty b(r)x^\alpha(r))\,\mathrm{d}r\,,$$

which yields the equicontinuity of the elements in $T(\Omega)$. Now we prove the continuity of T in Ω. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence in Ω which uniformly converges on every compact interval of $[t_1, \infty)$ to $\bar{u} \in \Omega$. Because $T(\Omega)$ is relatively compact, the sequence $\{T(u_n)\}$ admits a subsequence $\{T(u_{n_j})\}$ converging, in the topology of $C[t_1, \infty)$, to $\bar{z}_u \in \overline{T(\Omega)}$. From (2.16) and (2.17) we have

$$x(t) \le T(u)(t) \le x(t_1) + M(x(t) - x(t_1)).$$

Thus, in virtue of the Lebesgue dominated convergence theorem, the sequence $\{T(u_{n_j})(t)\}$ pointwise converges to $T(\bar{u})(t)$. In view of the uniqueness of the limit, $T(\bar{u}) = \bar{z}_u$ is the only cluster point of the compact sequence $\{T(u_n)\}$, that is, the continuity of T in the topology of $C[t_1, \infty)$. Hence, by the Tychonov fixed point theorem, the operator T has a fixed point y, which, clearly, is a solution of (2.1). Moreover, since $y \in \Omega$ and x is an intermediate solution, we get $\lim_{t\to\infty} y(t) = \infty$. Moreover, from

$$x^{[1]}(t) = \int_t^\infty b(r)x^\alpha(r))\,\mathrm{d}r \le y^{[1]}(t) \le Mx^{[1]}(t)$$

we get $\lim_{t\to\infty} y^{[1]}(t) = 0$, that is, y is an intermediate solution of (2.1).

Step 2. Assume that (2.1) has an intermediate solution y and suppose $y(t) > 0$, $y^{[1]}(t) > 0$ for $t \ge t_0$. Integrating (2.1) on (T, t), $T \ge t_0$, we obtain

$$y(t) - y(T) = \int_T^t \frac{1}{a^{1/\alpha}(s)}\left(\int_s^\infty b(r)y^\alpha(r+q))\,\mathrm{d}r\right)^{1/\alpha}\mathrm{d}s. \tag{2.18}$$

In virtue of Lemma 2.2, we have for large t

$$y(t+q) - y(t) \le \frac{1}{2}. \tag{2.19}$$

For simplicity, we can assume that (2.19) is valid for any $t \ge t_0$. Set

$$M_0 = 1 + \frac{1}{y(t_0)}.$$

Thus $1 + y(t_0) = M_0 y(t_0)$ and

$$y(t_0) + \frac{1}{2} < M_0 y(t_0).$$

Fixing ρ such that

$$y(t_0) + \frac{1}{2} < \rho < M_0 y(t_0), \tag{2.20}$$

consider in the Fréchet space $C[t_0, \infty)$ of all continuous functions on $[t_0, \infty)$, the set Ω_0 given by

$$\Omega_0 = \{v \in C[t_0, \infty) : y(t+q) \le v(t) \le M_0 y(t+q)\} .$$

Define in Ω_0 the operator T_0 as follows

$$T_0(v)(t) = \rho + \int_{t_0}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_s^\infty b(r) v^\alpha(r))\,\mathrm{d}r \right)^{1/\alpha} \mathrm{d}s. \tag{2.21}$$

Thus, we obtain

$$T_0(v)(t) \le \rho + M_0 \int_{t_0}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_s^\infty b(r) y^\alpha(r+q))\,\mathrm{d}r \right)^{1/\alpha} \mathrm{d}s$$

and, in view of (2.18) and (2.20), we have

$$T_0(v)(t) \le \rho + M_0 y(t) - M_0 y(t_0) < M_0 y(t).$$

Since y is increasing for $t \ge t_0$ we obtain

$$T_0(v)(t) \le M_0 y(t+q).$$

Moreover, from (2.18) and (2.21), we get also

$$\begin{aligned} T_0(v)(t) \ge \rho + \int_{t_0}^{t} \frac{1}{a^{1/\alpha}(s)} \left(\int_s^\infty b(r) y^\alpha(r+q))\,\mathrm{d}r \right)^{1/\alpha} \mathrm{d}s = \\ = \rho + y(t) - y(t_0). \end{aligned}$$

Using (2.19) and (2.20), we obtain

$$T_0(v)(t) \ge \rho + \frac{1}{2} + y(t+q) - y(t_0) \ge y(t+q),$$

that is, the operator T_0 maps Ω_0 into itself.

The same argument to the one given in Step 1 proves that $T_0(\Omega_0)$ is relatively compact in the topology of $C[t_0,\infty)$. Hence, the assertion follows by using again the Tychonov fixed point theorem. □

From Theorem 2.1, we obtain immediately the following.

Corollary 2.2. *Assume* (2.8)*. If* (2.1) *has intermediate solutions for a fixed q, then the same occurs for any q.*

In Theorem 2.1 the assumption (2.8) cannot be eliminated, as the following example shows.

Example 2.2. Consider the equation ($t \geq 1$)

$$\left(\frac{1}{2t\exp(t^2)}y'(t)\right)' + 2\gamma t\exp(-t^2)y(t-1) = 0, \tag{2.22}$$

where γ is a positive constant. Since we have for any $H > 0$

$$\int_{t-H}^{t} \frac{1}{a(s)}\,\mathrm{d}s = \exp t^2 - \exp(t-H)^2,$$

assumption (2.8) is not satisfied and so Theorem 2.1 cannot be applied. A standard calculation gives $J = \infty$, $K_{-1} < \infty$. Thus, in view of Theorem B., equation (2.22) has intermediate solutions for $\gamma > 0$. Nevertheless, the corresponding linear equation

$$\left(\frac{1}{2t\exp(t^2)}x'(t)\right)' + 2\gamma t\exp(-t^2)x(t) = 0 \tag{2.23}$$

is oscillatory if $\gamma > \frac{1}{4}$, and so does not admit intermediate solutions. Indeed, the change of variable

$$t = \sqrt{\log(s+1)}, \quad z(s) = x(\sqrt{\log(s+1)})$$

transforms (2.23) into the Euler equation

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}z(s) + \frac{\gamma}{(s+1)^2}z(s) = 0,$$

which is oscillatory in virtue of the Hille-Kneser theorem.

2.4 Applications

Here we present some consequences of Theorem 2.1 that concern with the oscillation of (2.1) and (2.2).

Corollary 2.3. *Assume* (2.8).

(i_1) If (2.2) *is oscillatory, then* (2.1) *is oscillatory for any q.*
(i_2) If (2.1) *is oscillatory for a fixed q, then* (2.2) *is oscillatory.*

Consequently, (2.1) *is oscillatory if and only if* (2.2) *is oscillatory.*

Proof. Claim (i_1). Assume there exists $\overline{q}$ such that (2.1) has a nonoscillatory solution y for $q = \overline{q}$. If $y \in \mathbb{M}^+_{\ell,0}$, in view of Theorem A. we have $J < \infty$. Thus, again from Theorem A., the class $\mathbb{M}^+_{\ell,0}$ is nonempty also for (2.2). Hence, (2.2) is nonoscillatory, which is a contradiction. Similarly, if $y \in \mathbb{M}^+_{\infty,\ell}$, in view of Corollary 2.1, we have $K_0 < \infty$, that is, (2.2) is nonoscillatory, which is a contradiction. Finally, if y is an intermediate solution, the contradiction follows from Theorem 2.1.

Claim (i_2) can be proved in a similar way, by taking into account Corollary 2.1 and Theorem 2.1. □

The following example illustrates Corollary 2.3.

Example 2.3. Consider the equation ($t \geq 1$)

$$\left(|y'(t)|^\alpha \operatorname{sgn} y'(t)\right)' + \frac{\gamma}{t^{\alpha+1}} |y(t+q)|^\alpha \operatorname{sgn} y(t+q) = 0, \tag{2.24}$$

jointly with its corresponding half-linear equation

$$\left(|y'(t)|^\alpha \operatorname{sgn} y'(t)\right)' + \frac{\gamma}{t^{\alpha+1}} |y(t)|^\alpha \operatorname{sgn} y(t) = 0, \tag{2.25}$$

where γ is a positive constant. Equation (2.25) is the *half-linear Euler equation* and its oscillation depends on the so-called critical point μ, that is,

$$\mu = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1};$$

see, e.g., [10, Section 1.4.2]. More precisely, if $\gamma > \mu$, then (2.25) is oscillatory, while if $\gamma \leq \mu$, then (2.25) is nonoscillatory. Moreover, if (2.25) is nonoscillatory, then it has intermediate solutions; see, e.g., [7]. Clearly, assumption (2.8) is verified for (2.24). Thus, in view of Corollary 2.3, if $\gamma > \mu$, then (2.24) is oscillatory for any q. Similarly, if $\gamma \leq \mu$, in view of Theorem 2.1, equation (2.24) has intermediate solutions for any q.

Applying Theorem 2.1 and Theorem C., we obtain a sufficient criterion for the existence of intermediate solutions of (2.1), which extends Theorem B..

Corollary 2.4. *Assume* (2.8). *Equation* (2.1) *has intermediate solutions if any of the following conditions is satisfied:*

(i_1) The case (C_2)*, defined in* (2.4)*, occurs.*
(i_2) The case (C_3)*, defined in* (2.4)*, occurs.*
(i_3) The case (C_1)*, defined in* (2.4)*, occurs and* (2.2) *is nonoscillatory.*

Conversely, if $K_0 < \infty$ *and* $J < \infty$*, then* (2.1) *does not admit intermediate solutions.*

Now, we give comparison oscillation theorems for equation

$$y''(t) + b(t)y(t+q) = 0, \tag{2.26}$$

and its corresponding linear equation

$$x''(t) + b(t)x(t) = 0. \tag{2.27}$$

Since $a(t) \equiv 1$, the assumption (2.8) is satisfied for (2.26) and the following holds.

Corollary 2.5. *Equation* (2.27) *is oscillatory if and only if equation* (2.26) *is oscillatory.*

Proof. Assume that (2.27) is oscillatory. Hence

$$J = K_0 = \int_0^\infty t\, b(t)\, \mathrm{d}t = \infty. \tag{2.28}$$

Then, in virtue of Theorem A. and Corollary 2.1, for equation (2.26), we obtain $\mathbb{M}^+_{\ell,0} = \mathbb{M}^+_{\infty,\ell} = \emptyset$. Moreover, in view of Theorem 2.1, equation (2.26) has no intermediate solutions, so the assertion follows. If (2.26) is oscillatory, the argument is similar. □

Remark 2.3. Corollary 2.5 deals with equations for which $a(t) \equiv 1$. Example 2.2 shows that Corollary 2.5 can fail when the function $1/a$ is, roughly speaking, rapidly varying at infinity.

Corollary 2.6. *If* (2.27) *is nonoscillatory, then* (2.26) *has intermediate solutions if and only if*

$$\int_0^\infty t\, b(t)\, \mathrm{d}t = \infty. \tag{2.29}$$

Proof. Since (2.27) is nonoscillatory, in view of (2.28), (2.29) and Theorem A., equation (2.27) has intermediate solutions. Thus, the assertion follows from Theorem 2.1. □

The following example illustrates Corollary 2.5 and Corollary 2.6.

Example 2.4. Consider the equation ($t \geq 2$)

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\delta}{(\log t)^2}\right) y(t+q) = 0, \tag{2.30}$$

and its corresponding linear equation

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\delta}{(\log t)^2}\right) y(t) = 0, \tag{2.31}$$

where δ is a positive constant.

Equation (2.31) is the *Riemann-Weber equation* and has been considered in [7, 13, 17]. In particular, if $\delta > 1/4$, then (2.31) is oscillatory, while if $\delta \leq 1/4$, then (2.31) is nonoscillatory. Moreover, when $\delta \leq 1/4$, then any solution of (2.31) is intermediate; see, e.g., [7].

If $\delta > 1/4$, then by Corollary 2.5 equation (2.30) is oscillatory for any q. If $\delta \leq 1/4$, then, taking into account that (2.29) is valid and applying Corollary 2.6, we get that (2.30) has intermediate solutions for any q. Moreover, by Theorem A. and Corollary 2.1 every nonoscillatory solution is of this type.

2.5 The Coexistence of Nonoscillatory Solutions

When the condition (2.8) is satisfied, in virtue of Theorem A., Corollary 2.1, and Theorem 2.1, the situation for (2.1) and (2.2) is almost the same and it is summarized as follows:

Assume (2.8). *Then for* (2.1) *we have:*

(1) *If the case* (C_1) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} = \emptyset$, $\mathbb{M}^+_{\ell,0} = \emptyset$.
Moreover, $\mathbb{M}^+_{\infty,0} \neq \emptyset$ *if* (2.2) *is nonoscillatory.*
(2) *If the case* (C_2) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\ell,0} = \emptyset$.
(3) *If the case* (C_3) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} = \emptyset$, $\mathbb{M}^+_{\infty,0} \neq \emptyset$, $\mathbb{M}^+_{\ell,0} \neq \emptyset$.
(4) *If the case* (C_4) *occurs* $\Longrightarrow \mathbb{M}^+_{\infty,\ell} \neq \emptyset$, $\mathbb{M}^+_{\infty,0} = \emptyset$, $\mathbb{M}^+_{\ell,0} \neq \emptyset$.

When (2.8) is not valid, Example 2.1 shows that the deviating argument can produce some differences between (2.1) and (2.2) in the existence and coexistence of possible types of nonoscillatory solutions.

Especially, for equation (2.2) with $\alpha = 1$, that is for the linear equation (2.12), intermediate solutions cannot coexist with dominant or subdominant solutions; see Remark 2.2. However, for the corresponding equation with deviating argument, that is, for

$$\big(a(t)y'(t)\big)' + b(t)y(t+q) = 0, \tag{2.32}$$

this fact is not true. Example 2.2 illustrates that there exist equations of type (2.32) for which intermediate solutions coexist with dominant solutions. Observe that the corresponding linear equation in Example 2.2 can be oscillatory (if $\gamma > \frac{1}{4}$) or nonoscillatory with only intermediate solutions (if $0 < \gamma \le \frac{1}{4}$).

Finally, the following example shows that for (2.1) intermediate solutions can coexist simultaneously with dominant solutions and subdominant solutions, that is, roughly speaking for (2.1), also the triple coexistence of nonoscillatory solutions is possible.

Example 2.5. Consider the equation ($t > 1$)

$$\left(\frac{1}{t^2 \exp t^2} y'(t)\right)' + \frac{2}{t^2 \exp(t+1)^2} y(t+1) = 0. \tag{2.33}$$

We have

$$J \le \int_1^\infty \frac{2t^2 \exp t^2}{\exp(t+1)^2} \int_t^\infty \frac{1}{r^2}\,\mathrm{d}r\,\mathrm{d}t = 2\mathrm{e}^{-1} \int_1^\infty t \exp(-2t)\,\mathrm{d}t < \infty.$$

Hence, (2.33) has subdominant solutions. Moreover, a standard calculation gives also $K_1 < \infty$ and so (2.33) has also dominant solutions. Finally, because $y(t) = \exp(t^2)$ is its solution and $y^{[1]}(t) = 2t^{-1}$, (2.33) has also intermediate solutions.

Observe that this triple coexistence is impossible for the corresponding linear equation, as well as for the half-linear equation, as it follows from Theorem A., Theorem C., and (2.4).

Conclusion. The deviating argument q does not have influence on the existence of subdominant solutions for (2.1).

The delayed argument $q < 0$ does not have influence on the existence of dominant solutions for (2.1), but the advanced argument $q > 0$ may cause that dominant solutions do not exist even if the corresponding half-linear equation (2.2) has dominant solutions.

The delayed argument $q < 0$ may produce the existence of intermediate solutions even if the corresponding half-linear equation (2.2) is oscillatory.

When the condition (2.8) holds, then the deviating argument q does not have influence on the asymptotic and oscillatory behavior of solutions of (2.1).

2.6 Open Problems

We close this paper with two open problems.

1. In Example 2.2, the function $1/a$ is rapidly varying at infinity; see, e.g., [16]. The same occurs in Example 2.1 and Example 2.5. Now, the following question arises. In Theorem 2.1, can assumption (2.8) be weakened, for instance, by assuming that the function

$$A(t) = \left(\frac{1}{a(t)}\right)^{1/\alpha}$$

is regularly varying at infinity?

2. Assume either the case (C_4) or (C_3). Thus, from Theorem A., for (2.1) we have $\mathbb{M}^+_{\ell,0} \neq \emptyset$. Then solutions $y \in \mathbb{M}^+_{\ell,0}$, such that $y(t) > 0$ for large t, are, as $t \to \infty$, the smallest eventually positive solutions. In the half-linear case these solutions are called principal solutions; see [8, 12] for the definition and [3, 4] for some of their properties. In particular, in [8] it is proved that when $\alpha > 1$, a solution x of (2.2) is in the class $\mathbb{M}^+_{\ell,0}$ if and only if

$$Q_x := \int^{\infty} \frac{x'(t)}{x^2(t)x^{[1]}(t)}\,\mathrm{d}t = \infty.$$

When $0 < \alpha < 1$ in [3, 4], other integral characterizations of principal solutions are presented. It should be interesting to study under which assumptions these integral characterizations can be extended also to equation (2.1) with deviating argument.

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Chapter 3
Analysis of Qualitative Dynamic Properties of Positive Polynomial Systems Using Transformations

Katalin M. Hangos and Gábor Szederkényi

Abstract Two classes of positive polynomial systems, quasi-polynomial (QP) systems and reaction kinetic networks with mass action law (MAL-CRN), are considered. QP systems are general descriptors of ODEs with smooth right-hand sides; their stability properties can be checked by algebraic methods (linear matrix inequalities). On the other hand, MAL-CRN systems possess a combinatorial characterization of their structural stability properties using their reaction graph.

Dynamic equivalence and similarity transformations applied either to the variables (quasi-monomial and time-reparametrization transformations) or to the phase state space (translated X-factorable transformation) will be applied to construct a dynamically similar linear MAL-CRN model to certain given QP system models. This way one can establish sufficient structural stability conditions based on the underlying reaction graph properties for the subset of QP system models that enable such a construction.

Keywords Polynomial ODEs • Positive systems • Dynamic equivalence • Dynamic similarity • Structural stability

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3.1 Introduction

The class of positive polynomial ODEs plays an important role in describing the dynamics of physical, chemical, and ecological systems, where the positivity of the variables is dictated and ensured by the physical meaning, as certain quantities – such as concentrations, pressures, population numbers, etc. – cannot take negative values. The dynamic model of such systems may often originate from first physical, chemical, or engineering principles – such as conservation – that implies a well-defined structure to the right-hand sides of the ODE models.

The notion of positive systems builds upon the *essential nonnegativity* of a function $f = [f_1 \ \dots \ f_n]^T : [0,\infty)^n \to \mathbb{R}^n$ that holds if, for all $i = 1,\dots,n$, $f_i(x) \geq 0$ for all $x \in [0,\infty)^n$, whenever $x_i = 0$ [6]. An autonomous nonlinear system is defined on the nonnegative orthant $[0,\infty)^n = \overline{\mathbb{R}}_+^n \subset \mathscr{X}$

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \ x(0) = x_0, \tag{3.1}$$

where $f : \mathscr{X} \to \mathbb{R}^n$ is locally Lipschitz, $\mathscr{X}$ is an open subset of $\mathbb{R}^n$, and $x_0 \in \mathscr{X}$ is nonnegative (or positive) when the nonnegative (or positive) orthant is invariant for the dynamics (3.1). This property holds if and only if f is essentially nonnegative.

The subclass of quasi-polynomial systems (QP systems in short), to which the well-known Lotka-Volterra equations belong, forms a general descriptor class of dynamic systems with smooth nonlinearities in the sense that such systems can be embedded into QP form by adding new auxiliary variables to the system [4]. There exists a parameter-dependent sufficient condition for a given QP system to be globally asymptotically stable [11],which can be checked by solving a linear matrix inequality (LMI).

Deterministic kinetic systems with mass action kinetics or simply chemical reaction networks (CRNs) form a wide class of nonnegative polynomial systems that are able to produce all the important qualitative phenomena (e.g., stable/unstable equilibria, oscillations, limit cycles, multiplicity of equilibrium points, and even chaotic behavior) present in the dynamics of nonlinear processes [2]. The importance of the CRN system class with mass action law (abbreviated as MAL-CRNs) lies in the fact that strong structural (i.e., parameter-independent) stability results exist for the deficiency zero weakly reversible case [8] and recently for the detailed balanced case, when each of the chemical reactions is assumed to be reversible (see [5, 12, 13, 17] and recently [7]).

The aim of this paper is to try to establish a dynamic similarity relationship between the Lotka-Volterra form of QP systems and the linear MAL-CRNs in order to obtain structural stability conditions for the former.

3.2 Quasi-Polynomial (QP) Systems

The most general class of positive polynomial systems is the class of quasi-polynomial (QP) ones that are time-dependent autonomous ODEs (3.1) evolving in the positive orthant $\mathbb{R}^n_+$, i.e., $x(t) \succ 0$ (element-wise) for $t \geq 0$ and $x_0 \succ 0$.

3.2.1 *The ODE Form*

Two sets of variables are present in the ODE form of a QP system:

- The **state variables** $x_i, i = 1, \ldots, n$
- The **quasi-monomials (QMs)** $q_j, j = 1, \ldots, m$

We assume $m \geq n$.

With these variables the system dynamics is described by an autonomous ODE with quasi-polynomial right-hand sides defined on the positive orthant

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \left(\lambda_i + \sum_{j=1}^{m} A_{ij} q_j \right), \quad i = 1, \ldots, n, \tag{3.2}$$

that is augmented by the following algebraic equations:

$$q_j = \prod_{i=1}^{n} x_i^{B_{ji}}, \quad j = 1, \ldots, m. \tag{3.3}$$

The above equations (3.3) are the so-called quasi-monomial (QM) relationships.

The state space $\mathscr{X} \subseteq \mathbb{R}^n_+$ will also be called phase space in the paper.

3.2.1.1 Algebraic Characterization

The real vector $\lambda \in \mathbb{R}^n$ and matrices $B \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times m}$ are the parameters of a QP system model (3.2)–(3.3).

3.2.2 *Quasi-Monomial Transformation and the Lotka-Volterra Canonical Form*

The so-called quasi-monomial transformation is an equivalence transformation on the class of QP systems that allows to form equivalence classes. These classes can be represented by their member in Lotka-Volterra canonical form [3].

3.2.2.1 The Quasi-Monomial Transformation

The so-called quasi-monomial transformation or **QM-transformation** in short introduces new state variables

$$x'_j = \prod_{i=1}^{n} x_i^{\Gamma_{ji}}, \quad j = 1, \ldots, n. \tag{3.4}$$

The parameter of the QM-transformation is the real square invertible matrix $\Gamma \in \mathbb{R}^{n \times n}$.

The parameters A and B of a QP system model are transformed to $A' = \Gamma^{-1} A$ and $B' = B\Gamma$; therefore the product $M = BA$ is invariant under the QM-transformation.

3.2.2.2 Lotka-Volterra (LV) Canonical Form

Being an equivalence transformation, the QM-transformation splits the set of QP models into equivalence classes. **From any QP model** (3.2)–(3.3) with parameters (A, B, λ) of an equivalence class, the LV model form can be obtained **by QM-transformation and variable extension** such that $B' = I$ with $x' = q$. Then the transformed matrix A' becomes

$$A' = M = B \cdot A. \tag{3.5}$$

The resulting transformed ODE in LV form

$$\frac{\mathrm{d}q_l}{\mathrm{d}t} = q_l \left(\Lambda_l + \sum_{j=1}^{m} M_{ij} q_j \right), \quad l = 1, \ldots, m \tag{3.6}$$

is a homogeneous bilinear ODE that describes the dynamics in the monomial space $\mathcal{Q} \subseteq \mathbb{R}_+^m$. However, because of the variable extension and the relationship $m \geq n$, the dynamics lives in a lower n-dimensional manifold of the monomial space $\mathcal{Q}$.

Steady-State Points

The nontrivial nonnegative steady-state points of the original QP equation (3.2) can be obtained (if they exist) by solving the equation

$$0 = \lambda + A \cdot q^* \tag{3.7}$$

for q^*. It is important to note that the equilibrium point is determined in the monomial space $\mathscr{Q}$ and then it is transformed back to the state space.

Equation (3.7) is a linear under-determined equation for the vector $q^* \in \mathbb{R}^m$, but Eq. (3.3) gives $m - n$ algebraic relationships between the elements of q^*; therefore one may have a well-posed solution (even if it is not unique). As the monomial space is only a subset of $\mathbb{R}^m$ ($\mathscr{Q} \subseteq \mathbb{R}^m_+$), it may occur that no positive equilibrium point exists. Without further investigations, however, we only consider here the case when **a finite number of positive steady-state points exist** in the state space.

The Vector-Matrix Form

In order to develop a compact vector-matrix form, the following notations are introduced

$$\underline{\ln}\, q^T = [\ln q_1, \ldots, \ln q_m]^T, \quad \underline{diag}\, q = \begin{bmatrix} q_1 & 0 & \ldots & 0 & 0 \\ 0 & \cdot & q_i & \cdot & 0 \\ 0 & 0 & \ldots & 0 & q_m \end{bmatrix}. \tag{3.8}$$

Then the dynamics (3.6) of a Lotka-Volterra system with a positive steady-state point q^* can be written in the following form:

$$\frac{\mathrm{d}\,\underline{\ln}\, q}{\mathrm{d}t} = M \cdot (q - q^*) \quad \text{or} \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \underline{diag}\, q \cdot M \cdot (q - q^*). \tag{3.9}$$

3.2.3 The Time-Rescaling Transformation

The so-called time-rescaling transformation [10] maps a QP system model to another QP system model in the following way. Let us introduce a transformation vector $\Omega = [\Omega_1, \ldots, \Omega_n]^T \in \mathbb{R}^n$ that is used to "rescale" the time in a state-dependent way

$$\mathrm{d}t = \prod_{k=1}^{n} x_k^{\Omega_k} \mathrm{d}t'.$$

Then the original QP model (3.2) with parameters (A, B, λ) is transformed to the model that is also in QP form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t'} = x_i \sum_{j=1}^{m+1} [\tilde{A}]_{i,j} \prod_{k=1}^{n} x_k^{[\tilde{B}]_{j,k}}, \; i = 1, \ldots, n, \tag{3.10}$$

where the new parameters $\tilde{A} \in \mathbb{R}^{n\times(m+1)}$ and $\tilde{B} \in \mathbb{R}^{(m+1)\times n}$ are

$$\begin{aligned}
\tilde{A}_{i,j} &= A_{i,j}, \;\; i = 1,\ldots,n; \;\; j = 1,\ldots,m \\
\tilde{A}_{i,m+1} &= \lambda_i, \;\; i = 1,\ldots,n \\
\tilde{B}_{i,j} &= B_{i,j} + \Omega_j, \;\; i = 1,\ldots,m; \;\; j = 1,\ldots,n \\
\tilde{B}_{m+1,j} &= \Omega_j, \;\; j = 1,\ldots,n.
\end{aligned}$$

Note that the **number of monomials is increased by one**, and the new parameter vector $\tilde{\lambda}$ is zero in the transformed system.

It is important to note that by assuming strictly positive state variables, the time-rescaling transformation is a **similarity transformation** that leaves the equilibrium points and the stability properties unchanged [10].

3.2.4 Stability Condition for QP Systems

Thanks to the well-characterized structure of QP systems with a positive equilibrium point q^*, an easy-to-check sufficient condition for their global (asymptotic) stability exists [11].

A QP system (3.2)–(3.3) with a positive equilibrium point q^* is globally stable if the the linear matrix inequality

$$M^T C + CM \leq 0 \tag{3.11}$$

is solvable for a positive diagonal matrix C, with $M = BA$. In this case, the matrix M is called **diagonally stable**. (The stability is asymptotic, if the inequality (3.11) is strict.) Given the parameter matrix M of the system, the condition (3.11) can be checked effectively by solving a linear matrix inequality (LMI) [19].

It is important to note that the above condition is derived using the following Lyapunov function candidate:

$$V(q) = \sum_{i=1}^{m} c_i \left(q_i - q_i^* - q_i^* \ln \frac{q_i}{q_i^*} \right). \tag{3.12}$$

Unfortunately, however, the condition (3.11) is rather conservative. Therefore, one may use time rescaling of the original QP system model to find a dynamically similar QP system model such that it fulfills (3.11). This, however, requires to solve a bilinear matrix inequality (BMI) [15].

3.3 Chemical Reaction Networks with Mass Action Law

Chemical reaction networks with mass action law (MAL-CRNs in short) form an important special subclass of positive polynomial systems. Their special structure, which will be described briefly in this section, enables to apply parameter-independent robust conditions for their asymptotic stability.

3.3.1 Formal Description

Chemical reaction networks [8] are abstract versions of reaction kinetic models in chemistry and biochemistry. They are composed of **irreversible elementary reaction steps** in the form

$$\sum_{s=1}^{n} \alpha_{sj} \mathrm{A}_s \rightarrow \sum_{s=1}^{n} \beta_{sl} \mathrm{A}_s, \tag{3.13}$$

where A_s, $s = 1, \ldots, n$ are the chemical components, while α_{sj} and β_{sl} are the **stoichiometric coefficients** that are always nonnegative integers.

The linear combinations of components present on each side of a reaction step are called **complexes**, i.e., $C_j = \sum_{s=1}^{n} \alpha_{sj} \mathrm{A}_s$ $(j = 1, \ldots, m)$ form the set of complexes.

The dynamics of a MAL-CRN is described by an autonomous ODE with polynomial right-hand side on the positive orthant in the following form:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = Y \cdot A_k \cdot \varphi(x) \tag{3.14}$$

$$\varphi_j(x) = \prod_{s=1}^{n} x_s^{\alpha_{sj}}, \quad j = 1, \ldots, m, \tag{3.15}$$

where the state vector is composed of the concentrations (these are nonnegative quantities) of the components (x_s is the concentration of A_s). The nonnegative variables in the vector φ are called **(reaction) monomials**; they span the monomial space.

It should be emphasized that in contrast to the reversible reaction steps traditionally considered in the applied mathematical literature (see [5,12,13,17] and recently [7]) we assume **irreversible reactions** in (3.13). This implies that the reaction rate $r_j = k_{j,l} \varphi_j(x)$ of the reaction (3.13) depends only on the composition of the reactant complex, i.e., on the stoichiometric coefficients α_{sj}, $s = 1, \ldots, n$ but not on the coefficients β_{sj}, $s = 1, \ldots, n$.

The parameters of the model are the complex composition matrix $Y_{sj} = \alpha_{sj}$ and the reaction matrix A_k:

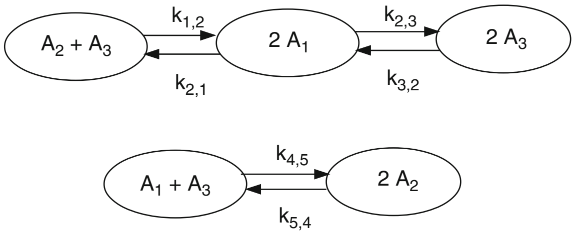

Fig. 3.1 The reaction graph of the example

$$[A_k]_{lj} = \begin{cases} -\sum_{\ell=1,\ \ell\neq j}^{m} k_{j,\ell}, & \text{if } l = j \\ k_{j,l}, & \text{if } l \neq j, \end{cases} \tag{3.16}$$

where $k_{l,j} > 0$ is the **reaction rate constant** (a positive number) of the reaction $C_l \to C_j$.

It is important to note that $A_k \in \mathbb{R}^{m\times m}$ is a **Kirchhoff matrix** with zero column sum. Therefore, A_k is rank-deficient.

The reaction structure of a MAL-CRN is described by the so-called **reaction graph**, that is, a weighted directed graph. The vertices of the reaction graph correspond to the complexes, and the edges describe reactions that connect the complexes. This means that a directed edge from the vertex C_j to C_l exists, if there is a reaction $C_j \to C_l$ in the CRN. The edge weight is the corresponding reaction rate coefficient $k_{j,l}$. Therefore, the Kirchhoff matrix A_k determines the reaction graph (Fig. 3.1).

It is important to note that the dynamics of CRNs with (generalized) mass conservation evolve in a lower dimensional subspace of the state space $\mathscr{X} \subseteq \mathbb{R}^n_+$ that is determined by the initial conditions and is called the **stoichiometric compatibility class**.

3.3.1.1 Example: A Simple Nonlinear MAL-CRN

Let us consider a simple MAL-CRN with the following three reversible reactions:

$$\mathrm{A}_2 + \mathrm{A}_3 \underset{k_{2,1}}{\overset{k_{1,2}}{\rightleftharpoons}} 2\mathrm{A}_1, \quad \mathrm{A}_1 + \mathrm{A}_3 \underset{k_{5,4}}{\overset{k_{4,5}}{\rightleftharpoons}} 2\mathrm{A}_2, \quad 2\mathrm{A}_1 \underset{k_{3,2}}{\overset{k_{2,3}}{\rightleftharpoons}} 2\mathrm{A}_3\,.$$

Because the elementary reaction steps are considered irreversible, we break down these reactions into six irreversible steps connecting five complexes (i.e., $m = 5$ with $C = \{\mathrm{A}_2 + \mathrm{A}_3,\ 2\mathrm{A}_1,\ 2\mathrm{A}_3,\ \mathrm{A}_1 + \mathrm{A}_3,\ 2\mathrm{A}_2\}$) with the following reaction monomials:

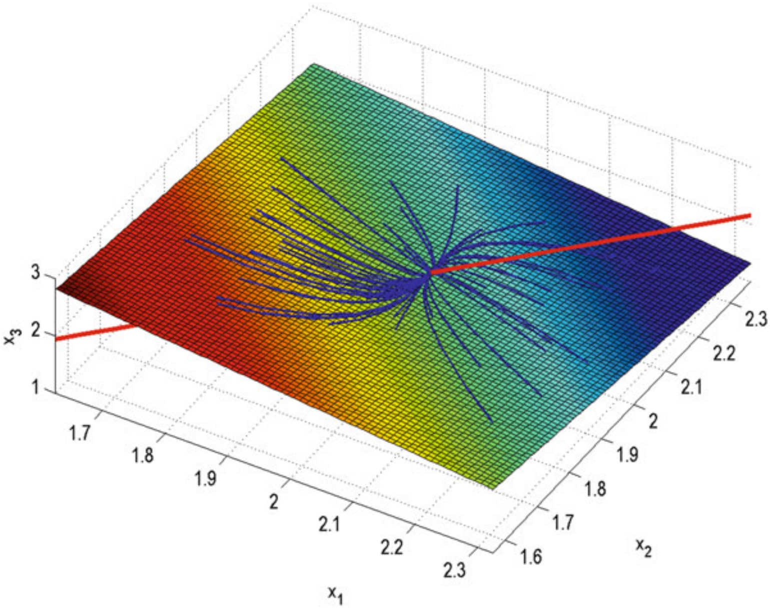

Fig. 3.2 The dynamics of the example evolving in a stoichiometric compatibility class

$$\varphi(x) = [x_2x_3,\ x_1^2,\ x_3^2,\ x_1x_3,\ x_2^2]^T.$$

The dynamic model equations are as follows:

$$\dot{x}_1 = 2k_{1,2}x_2x_3 - (2k_{2,1} + 2k_{2,3})x_1^2 + 2k_{3,2}x_3^2 - k_{4,5}x_1x_3 + k_{5,4}x_2^2$$
$$\dot{x}_2 = -k_{1,2}x_2x_3 + k_{2,1}x_1^2 + 2k_{4,5}x_1x_3 - 2k_{5,4}x_2^2$$
$$\dot{x}_3 = -k_{1,2}x_2x_3 + (k_{2,1} + 2k_{2,3})x_1^2 - 2k_{3,2}x_3^2 - k_{4,5}x_1x_3 + k_{5,4}x_2^2.$$

3.3.2 MAL-CRN Structural Stability

The **structure** of a MAL-CRN system is determined by the complex composition matrix Y and by its reaction graph (or equivalently its reaction matrix A_k) without its weights, i.e., irrespective of the actual values of the reaction rate constants (Fig. 3.2).

The structural stability of an ODE can also be defined following this idea.

Definition 3.1. An ODE $\frac{dz}{dt} = F(z, P)$ with parameters P will be called structurally stable with respect to a parameter set $\mathscr{P}$, if it is stable for every $P \in \mathscr{P}$.

3.3.2.1 MAL-CRN Structural Properties

The structural properties of a MAL-CRN model are defined based on the graph structure of the reaction graph without its edge weights and on the complex compositions.

A CRN is called **weakly reversible** if whenever there exists a directed path from C_i to C_j in its reaction graph, there exists a directed path from C_j to C_i. In graph theoretic terms, this means that all components of the reaction graph are strongly connected components. We shall use the fact known from the literature that a CRN is weakly reversible if and only if there exists a vector b with strictly positive elements in the kernel of A_k, i.e., there exists $b \in \mathbb{R}^n_+$ such that $A_k \cdot b = 0$ [9]. This also implies that the CRN has a positive equilibrium point in the monomial space.

The notion of the **deficiency** of a CRN is built on the set of **reaction vectors** that are defined as $\mathscr{R} = \{\rho^{(l,k)} = \eta^{(l)} - \eta^{(k)} \mid C_k C_l \in E \text{ in } G\}$, where $\eta^{(i)}$ denotes the ith column of Y. Then the deficiency δ is an integer number that is defined as

$$\delta = m - \ell - s \tag{3.17}$$

where m is the number of complexes and ℓ is the number of connected components in the reaction graph, while s is the dimension of the stoichiometric subspace, i.e., $s = rank(\mathscr{R})$. The zero deficiency property implies the stability of equilibria in a weakly reversible MAL-CRN system.

Deficiency Zero Theorem

The **deficiency zero theorem** [8] shows a very robust stability property of a certain class of kinetic systems. It says that deficiency zero weakly reversible networks possess well-characterizable equilibrium points, and independent of the weights of the reaction graph (i.e., that of the system parameters), they are at least locally stable with a known logarithmic Lyapunov function that is also independent of the system parameters. (According to the so-called **Global Attractor Conjecture** that was proved for the single linkage class case in [1], this stability is actually global.)

3.3.3 Linear CRN Systems

A linear MAL-CRN is characterized by the equation $Y = I$, that is, $m = n$, and the components form the complexes ($C_i = \mathrm{A}_i,\ i = 1, \ldots, m$). Then the dynamics is described by the following ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_k x \tag{3.18}$$

where A_k is the reaction matrix, that is, a Kirchhoff matrix (see Eq. (3.16)). This implies

$$[A_k]_{ii} < 0; \quad [A_k]_{ij} \geq 0,\ i \neq j; \quad \underline{1} \cdot A_k = \underline{0}, \tag{3.19}$$

where $\underline{1} = [1, \ldots, 1]$ is a row vector.

Note that the state and monomial spaces of a linear CRN coincide and the dynamics is linear in this space.

Because of the $Y = I$ equality, **a linear CRN has always zero deficiency**.

3.4 Transforming LV Models to a Linear MAL-CRN Form

Based on the notion of dynamic similarity and on model transformations we aim at constructing a dynamically similar linear MAL-CRN model to a given Lotka-Volterra model. If this is possible, then we can use the structural stability conditions of the linear MAL-CRN model to infer the structural stability of the LV model.

3.4.1 The Translated X-Factorable Transformation

Given an ODE

$$\frac{\mathrm{d}z}{\mathrm{d}t} = F(z) \tag{3.20}$$

on the positive orthant $z \in \mathbb{R}^n_+$ with $F(z) = 0$.

The **nonlinear translated X-factorable transformation** maps the above ODE into

$$\frac{\mathrm{d}z'}{\mathrm{d}t} = \underline{diag}\, z \cdot F(z - z^*), \tag{3.21}$$

where the elements of $z^* = [z_1^*, \ldots, z_n^*]^T$ are positive real numbers and $z = [z_1, \ldots, z_n]^T$.

If $F(z)$ is composed of polynomial-type functions with a finite number of singular solutions, then the above transformation can move the singular solutions into the positive orthant and leaves the geometry of the state (or phase) space unchanged within it (but not at or near the boundary) [18].

The dynamics of the solutions of Eqs. (3.20) and (3.21) are called **structurally similar**.

3.4.2 Constructing a Dynamically Similar Linear CRN Form

Let us have a QP system model in its LV form defined on the positive orthant

$$\frac{\mathrm{d}\overline{x}}{\mathrm{d}t} = \underline{diag}\,\overline{x}\,(\Lambda + M\overline{x}) \tag{3.22}$$

with a positive steady-state point $\overline{x}^*$. We want to construct a linear CRN model of the form

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = \tilde{A}_k\,\chi \tag{3.23}$$

such that the two systems are dynamically similar and $\tilde{A}_k$ is a Kirchhoff matrix.

Definition 3.2. Two ODEs are called dynamically similar if they have topologically equivalent state spaces (topologically equivalent phase spaces) [16], and the stability properties of all of their steady-state manifolds are the same.

The requirement of dynamic similarity implies that the linear CRN model will also have a positive steady-state point χ^* with the same stability property (e.g., globally asymptotically stable).

The construction will be done in two steps. First a dynamically similar linear homogeneous model will be constructed that will be transformed to a CRN model in the second step – if possible.

3.4.2.1 Dynamically Similar Homogeneous Linear System

We observe that MAL-CRN models in Eq. (3.14)–(3.15) form a homogeneous set of equations where the equilibrium point does not appear in the equations. Therefore, we augment the state vector $\overline{x}$ of the model (3.22) by a constant element; then the **homogeneous form** is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \underline{diag}\,x\left(\left[\begin{array}{c|c} M & \Lambda \\ \hline 0 & 0 \end{array}\right] x\right), \tag{3.24}$$

where $n = \overline{n} + 1$ and $n \in \mathbb{R}^n$ is the new state vector.

Next we follow the procedures described in [14] by using X-factorable transformation to associate a dynamically similar **linear ODE**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left[\begin{array}{c|c} M & \Lambda \\ \hline 0 & 0 \end{array}\right] x = \check{M}x. \tag{3.25}$$

3.4.2.2 Transforming the Linear ODE to a Potential CRN

In the second step we ensure the zero column sum property of the model by applying a linear state transformation (that is an equivalence transformation of the state spaces) using the invertible transformation matrix

$$T = \left[\begin{array}{c|c} I & 0 \\ \hline -1 \ldots -1 & 1 \end{array}\right], \quad T^{-1} = \left[\begin{array}{c|c} I & 0 \\ \hline 1 \ldots 1 & 1 \end{array}\right]. \tag{3.26}$$

We apply T to Eq. (3.25) to have $\frac{d\chi}{dt} = \tilde{A}\chi$ with $\chi = Tx$ and

$$\hat{M} = T\check{M} = \left[\begin{array}{c|c} M & \Lambda \\ \hline \underline{m} & \Lambda_n \end{array}\right], \quad \tilde{M} = \hat{M}T^{-1} = \left[\begin{array}{c|c} M + \Lambda\underline{1} & \Lambda \\ \hline \underline{m} + \Lambda_n\underline{1} & \Lambda_n \end{array}\right], \tag{3.27}$$

where $m_i = -\sum_{l=1}^{\overline{n}} [M]_{li}$ and $\Lambda_n = -\sum_{l=1}^{\overline{n}} \Lambda_l$ and the row vectors are denoted by underlining.

The column conservation property holds for both $\hat{M}$ and $\tilde{M}$.

Finally we obtain that $\tilde{M}$ corresponds to the coefficient matrix $\tilde{A}_k$ of the dynamically similar linear ODE (3.23) that can only be a CRN if $\tilde{M}$ has the required sign patterns in Eq. (3.19) besides of the column conservation property.

3.4.3 *Structural Stability Analysis*

The sufficient conditions in the deficiency zero theorem will be used to establish conditions for robust structural stability using the transformed coefficient matrix $\tilde{M}$ in (3.27).

The following properties of the original LV parameter matrices (M, Λ) are needed as sufficient conditions for the structural stability that originate from the required sign pattern property (3.19) of the CRN coefficient matrix $\tilde{A}_k$ in Eq. (3.23):

1. Nonnegativity of the parameter vector Λ, i.e.,

$$\Lambda_i > 0, \quad i = 1, \ldots, \overline{n} \tag{3.28}$$

2. The sign pattern and the strict dominant main diagonal property of M, i.e.,

$$M_{ii} < 0, \quad M_{ij} \geq 0, \quad |M_{ii}| \geq \sum_{\substack{l=1 \\ l \neq i}}^{\overline{n}} M_{li} \tag{3.29}$$

3.5 Conclusion and Future Work

A sufficient condition for structural stability is established for QP systems with a positive steady-state point by transforming it to a linear CRN. The resulting conditions are simple inequalities (3.28) and (3.28) that represent sign conditions of the LV parameter vector Λ and matrix M and the dominant main diagonal property of the latter.

Future work includes the use of time rescaling to enlarge the possibility of the LV parameters Λ and M to fulfill the above sufficient conditions, the checking of which will lead to solving an LMI.

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Chapter 4
Almost Oscillatory Solutions of Second Order Difference Equations of Neutral Type

Robert Jankowski and Ewa Schmeidel

Abstract By means of Riccati technique, we establish some new oscillation criteria for difference equations of neutral type in terms of the coefficients. The results are illustrated by examples.

Keywords Second-order difference equation • Superlinear • Sturm-Liouville difference equation • Oscillation • Riccati technique

4.1 Introduction

In this paper we consider the equations of neutral type in the form

$$\Delta\left(\left(\Delta\left(x_n + c_n x_{n-k}\right)\right)^{\gamma}\right) + q_n x_{n+1}^{\alpha} + e_n \operatorname{sgn}(x_n) = 0, \tag{4.1}$$

where k is a nonnegative integer, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$, $\alpha > \gamma \geq 1$ are the ratio of odd positive integers, and (c_n), (q_n), and (e_n) are positive sequences, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Here the signum function of a real number x is defined as usually by

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$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

By a solution of equation (4.1), we mean a real-valued sequence (x_n) defined on $\mathbb{N}_k := \{k, k+1, \ldots\}$ which satisfies (4.1) for every $n \in \mathbb{N}_k$.

Definition 4.1. Solution (x_n) of equation (4.1) is said to be oscillatory if for every integer $n_1 \in \mathbb{N}_k$, there exists $n \geq n_1$ such that $x_n x_{n+1} \leq 0$; otherwise, it is called nonoscillatory.

Definition 4.2. Solution (x_n) of equation (4.1) is said to be almost oscillatory if either (x_n) is oscillatory or (Δx_n) is oscillatory, or $x_n \to 0$ as $n \to \infty$.

In the last years the study concerning oscillatory and asymptotic behavior of solutions of difference equations, in particular second-order difference equations, has been in interest of many authors (see, e.g., papers of Medina and Pinto [11], Migda [12], Migda and Migda [13], Migda, Schmeidel and Zbąszyniak [15], Musielak and Popenda [16], Saker [17–19], Schmeidel [20], Schmeidel and Zbąszyniak [21], and Thandapani, Arul, Graef, and Spikes [22]).

Neutral difference equations were studied in many other papers, for instance, in Grace and Lalli [5] and [8], Lalli and Zhang [9], Migda and Migda [14], and Luo and Bainov [10].

For the reader's convenience, we note that the background for difference equations theory can be found in numerous well-known monographs: Agarwal [1], Agarwal, Bohner, Grace, and O'Regan [2], Agarwal and Wong [3], as well as by Elaydi [4], Kelley and Peterson [6], or Kocić and Ladas [7].

The following lemma which can be found in [23] will be used for proving the main result.

Lemma 4.1. *Set*

$$F(x) = a\, x^{\alpha-\gamma} + \frac{b}{x^{\gamma}} \textit{ for } x > 0. \tag{4.2}$$

If $a > 0, b > 0$ *and*

$$\alpha > \gamma \geq 1 \tag{4.3}$$

then $F(x)$ *attains its minimum*

$$F_{min} = \frac{\alpha a^{\frac{\gamma}{\alpha}} b^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}}}. \tag{4.4}$$

Lemma 4.2. *For all $x \geq y \geq 0$ and $\gamma \geq 1$ we have the following inequality:*

$$x^{\gamma} - y^{\gamma} \geq (x - y)^{\gamma}\ .$$

Proof. Let $x \geq y \geq 0$ and $\gamma \geq 1$. Then there exist $h \geq 0$ such that $x = y + h$ and then

$$\begin{aligned}(y+h)^{\gamma} &= (y+h)(y+h)^{\gamma-1} = y(y+h)^{\gamma-1} + h(y+h)^{\gamma-1} \\ &\geq y \cdot y^{\gamma-1} + h \cdot h^{\gamma-1} = y^{\gamma} + h^{\gamma}.\end{aligned}$$

□

4.2 Main Results

In this section, by using the Riccati substitution, we will establish some almost oscillation criteria for equation (4.1).

Theorem 4.1. *Let condition* (4.3) *be satisfied,*

$$\alpha \text{ and } \gamma \text{ be ratio of odd positive integers,} \tag{4.5}$$

(c_n) *be a real nonnegative sequence, and* (e_n), (q_n) *be real positive sequences. If there exists a positive sequence* (p_n) *such that*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \left(p_i Q_i - \frac{(\Delta p_i)^2}{4p_i} \right) = \infty, \tag{4.6}$$

where $Q_n = \min\{Q_n^*, Q_n^{**}\}$,

$$Q_n^* = \frac{\alpha q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}} (1+c_{n+1})^{\gamma}}$$

and

$$Q_n^{**} = \frac{\alpha q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}} (1+c_{n+k+1})^{\gamma}}.$$

Then any solution of equation (4.1) *is almost oscillatory.*

Proof. Let us define companion sequence (z_n) of sequence (x_n) as follows:

$$z_n = x_n + c_n x_{n-k}. \tag{4.7}$$

For the sake of contradiction, suppose that equation (4.1) has eventually positive solution (x_n) and Δx_n is eventually of one sign such that

$$\lim_{n\to\infty} x_n = l > 0. \tag{4.8}$$

Because of positivity of sequences (x_n) and (c_n) we get that the sequence (z_n) is positive, too.
Let us rewrite equation (4.1) in the following form:

$$\Delta\left((\Delta z_n)^\gamma\right) = -(q_n x_{n+1}^\alpha + e_n). \tag{4.9}$$

Hence, $\Delta\left((\Delta z_n)^\gamma\right) < 0$ for $n \ge n_2$. This implies that the sequence $((\Delta z_n)^\gamma)$ is a decreasing sequence. So, $((\Delta z_n)^\gamma)$ is also eventually of one sign as well as (Δz_n). This implies that the sequence (z_n) is monotonic. Therefore, (z_n) is eventually one sign. Because of Definition 4.2, the case that Δx_n changes the sign is also excluded. In conclusion, there are four possible cases to consider:

1. $x_n > 0,\ \Delta x_n > 0,\ z_n > 0,\ \Delta z_n > 0$
2. $x_n > 0,\ \Delta x_n < 0,\ z_n > 0,\ \Delta z_n > 0$
3. $x_n > 0,\ \Delta x_n > 0,\ z_n > 0,\ \Delta z_n < 0$
4. $x_n > 0,\ \Delta x_n < 0,\ z_n > 0,\ \Delta z_n < 0$

for enough large n, say $n \ge n_3 \ge n_2$.

Case (i). Since (x_n) is a positive increasing sequence we have $x_n > x_{n-k}$ for $n \ge n_3 + k = n_4$. By (4.7), we get $x_n > \frac{z_n}{1+c_{n+1}}$. Then, using equation (4.9), we obtain

$$\frac{\Delta\left((\Delta z_n)^\gamma\right)}{z_{n+1}^\gamma} < -\left(\frac{q_n}{(1+c_{n+1})^\alpha} z_{n+1}^{\alpha-\gamma} + \frac{e_n}{z_{n+1}^\gamma}\right). \tag{4.10}$$

Putting $a = \frac{q_n}{(1+c_{n+1})^\alpha}$, $b = e_n$ and $x = z_n$ in (4.2), we have

$$F(z_n) = \frac{q_n}{(1+c_{n+1})^\alpha} z_{n+1}^{\alpha-\gamma} + \frac{e_n}{z_{n+1}^\gamma}.$$

Hence, we rewrite inequality (4.10) as follows:

$$\frac{\Delta\left((\Delta z_n)^\gamma\right)}{z_{n+1}^\gamma} < -F(z_n).$$

By Lemma 4.1, we get

$$F(z_n) \ge \frac{\alpha q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}} (1+c_{n+1})^\gamma}.$$

Thus

$$\frac{\Delta\left((\Delta z_n)^\gamma\right)}{z_{n+1}^\gamma} < -\frac{\alpha q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}} (1+c_{n+1})^\gamma} = -Q_n^* \text{ for } n \geq n_4. \tag{4.11}$$

Let us denote

$$w_n = p_n \frac{(\Delta z_n)^\gamma}{z_{n+1}^\gamma}, \quad n \geq n_4, \tag{4.12}$$

where z_n is defined by (4.7). From the above we get $w_n > 0$, and

$$\begin{aligned}
\Delta w_n &= p_{n+1}\frac{(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} - p_n \frac{(\Delta z_n)^\gamma}{z_{n+1}^\gamma} = \\
&= p_n \frac{\Delta\,(\Delta z_n)^\gamma}{z_{n+1}^\gamma} + p_{n+1}\frac{(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} - p_n \frac{(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} \\
&\quad + p_n \frac{(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} - p_n \frac{(\Delta z_{n+1})^\gamma}{z_{n+1}^\gamma} \\
&= p_n \frac{\Delta\,(\Delta z_n)^\gamma}{z_{n+1}^\gamma} + \frac{\Delta p_n}{p_{n+1}} w_{n+1} + \frac{p_n\,(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma z_{n+1}^\gamma}\left[z_{n+1}^\gamma - z_{n+2}^\gamma\right] \\
&= p_n \frac{\Delta\,(\Delta z_n)^\gamma}{z_{n+1}^\gamma} + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^\gamma}{z_{n+1}^\gamma}.
\end{aligned}$$

From (4.11), we get the following inequality:

$$\Delta w_n < -p_n Q_n^* + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^\gamma}{z_{n+1}^\gamma}. \tag{4.13}$$

For $\Delta z_n > 0$ we have $z_{n+2} > z_{n+1}$ and $z_{n+2}^\gamma > z_{n+1}^\gamma$. Because of positivity of the sequence (z_n) for $n \geq n_4$, we obtain $\frac{1}{z_{n+2}^\gamma} < \frac{1}{z_{n+1}^\gamma}$. From (4.12) and by Lemma 4.2, we get

$$\Delta w_n < -p_n Q_n^* + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}^2} w_{n+1}^2. \tag{4.14}$$

This implies that

$$\begin{aligned}
\Delta w_n &< -p_n Q_n^* + \frac{(\Delta p_n)^2}{4p_n} - \left[\frac{\sqrt{p_n}}{p_{n+1}} w_{n+1} - \frac{1}{2\sqrt{p_n}} \Delta p_n\right]^2 \\
&< -p_n Q_n^* + \frac{(\Delta p_n)^2}{4p_n}.
\end{aligned}$$

Summing both sides of the above inequality from $i = n_4$ to $n-1$, we obtain

$$w_n - w_{n_4} < -\sum_{i=n_4}^{n-1}\left(p_i Q_i^* - \frac{(\Delta p_i)^2}{4p_i}\right).$$

The above inequality and $w_{n_4} > w_{n_4} - w_n$ yield

$$w_{n_4} > \sum_{i=n_4}^{n-1}\left(p_i Q_i^* - \frac{(\Delta p_i)^2}{4p_i}\right).$$

Letting n into infinity we obtain

$$w_{n_4} > \limsup_{n\to\infty}\sum_{i=n_4}^{n-1}\left(p_i Q_i^* - \frac{(\Delta p_i)^2}{4p_i}\right).$$

This is a contradiction with (4.6).

Case (ii). Since (x_n) is a positive decreasing sequence we have $x_{n-k} > x_n$ for large n. Then, there exists $n_5 \in \mathbb{N}$ such that $z_n < (1+c_n)x_{n-k}$ for $n \geq n_5$. Thus $x_n > \frac{z_{n+k}}{1+c_{n+k}}$. From properties of the sequence (z_n) we have $z_{n+k} > z_n$. Hence, we get $x_n > \frac{z_n}{1+c_{n+k}}$. Then, using equation (4.9), we obtain

$$\frac{\Delta\left((\Delta z_n)^\gamma\right)}{z_{n+1}^\gamma} < -\left(\frac{q_n}{(1+c_{n+k+1})^\alpha} z_{n+1}^{\alpha-\gamma} + \frac{e_n}{z_{n+1}^\gamma}\right). \tag{4.15}$$

Take $a = \frac{q_n}{(1+c_{n+k+1})^\alpha}$, $b = e_n$ and $x = z_n$ in (4.2). Analogously as in Case (i), we obtain

$$w_{n_5} > \limsup_{n\to\infty}\sum_{i=n_5}^{n-1}\left(p_i Q_i^{**} - \frac{(\Delta p_i)^2}{4p_i}\right),$$

where Q_n^{**} is defined in Theorem 4.1. This is a contradiction with (4.6).

Case (iii) and (iv). From (4.9), $\Delta\left((\Delta z_n)^\gamma\right) < 0$. This implies that $(\Delta z_{n+1})^\gamma < (\Delta z_n)^\gamma$ for $n \geq n_2$. Hence, $\Delta z_{n+1} < \Delta z_n$ and $\Delta^2 z_n < 0$. By Kneser theorem (see [1]), we know that if $\Delta^2 z_n < 0$ and $\Delta z_n < 0$ then $z_n < 0$. It is a contradiction with positivity of the sequence (z_n).

Finally, for $x_n < 0$ for all $n \geq n_3$, we use the transformation $y_n = -x_n$. Then the sequence (y_n) is eventually a positive solution of the equation

$$\Delta\left((\Delta(y_n + c_n y_{n-k}))^\gamma\right) + q_n y_{n+1}^\alpha + e_n = 0.$$

Now, the proof is completed. ◻

Example 4.1. Let us consider the difference equation

$$\Delta^2\left(x_n + \frac{1}{n}x_{n-2k}\right) + \left(5 + \frac{4n^2+8n+2}{n(n+1)(n+2)}\right)x_{n+1}^3 + \operatorname{sgn}(x_n) = 0.$$

Here $c_n = \frac{1}{n}$, $\gamma = 1$, $q_n = 5 + \frac{4n^2+8n+2}{n(n+1)(n+2)}$, $\alpha = 3$, and $e_n = 1$. For $p_n = 1$, all assumptions of Theorem 4.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. In fact, sequence $x_n = (-1)^n$ is one of such solutions. Here (x_n) is oscillatory.

Example 4.2. Let us consider equation (4.1) where $\alpha = 5$, $\gamma = 3$,

$$c_n = \frac{1}{n},$$
$$e_n = n,$$
$$q_n = n.$$

For $p_n = 1$, all assumptions of Theorem 4.1 are satisfied. Hence, any solution of the above equation is almost oscillatory.

Example 4.3. Let us consider equation (4.1) where $\alpha = 3$, $\gamma = 1$,

$$c_n = \frac{1}{n-2k},$$
$$e_n = 1,$$
$$q_n = \frac{4n+9}{(n+1)^3}.$$

For $p_n = 1$, all assumptions of Theorem 4.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. In fact, sequence $x_n = (-1)^n n$ is one of such solutions. Here, (x_n) is oscillatory.

Example 4.4. Let us consider equation (4.1) where $\alpha = \frac{5}{3}$, $\gamma = 1$,

$$c_n = 0,$$
$$e_n = \frac{1}{n(n+1)(n+2)},$$
$$q_n = \frac{(n+1)^{\frac{5}{3}}\left(4n^2+8n+3\right)}{n(n+1)(n+2)}.$$

For $p_n = 1$, all assumptions of Theorem 4.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. In fact, sequence $x_n = \frac{(-1)^n}{n}$ is one of such solutions. Here, (x_n) tends to zero.

The next example shows that convergence of series in (4.6) may be changed depending on using Q_n^* or Q_n^{**}. It means that convergence of series in (4.6) may be changed according to the terms of sequence (c_n), which is used in the sum.

Example 4.5. Let us consider equation (4.1) where $\alpha = 3$, $\gamma = 1$, $k = 1$, $e_n \equiv 1$,

$$c_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}, \quad q_n = \begin{cases} \frac{1}{n^3} & \text{if } n \text{ is even} \\ \frac{1}{n^6} & \text{if } n \text{ is odd} \end{cases}.$$

For example, for $p_i = 1$, we get

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \left(p_i Q_i^* - \frac{(\Delta p_i)^2}{4p_i} \right) = \sum_{i=1}^{\infty} Q_i^* < \infty,$$

whereas

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \left(p_i Q_i^{**} - \frac{(\Delta p_i)^2}{4p_i} \right) = \sum_{i=1}^{\infty} Q_i^{**} = \infty.$$

Notice that in this case Theorem 4.1 is not applicable.

In Theorem 4.1 we do not assume monotonicity of the sequence (c_n). Notice that monotonicity of sequences (x_n) and (z_n) does not imply monotonicity of the sequence (c_n). Such situation is illustrated by the following example.

Example 4.6. Let $x_n = n$ and $c_n = 1 - \frac{(-1)^n}{n}$. Sequence $z_n = 2n - k - (-1)^n(1 - \frac{k}{n})$, which is defined by (4.7), is eventually increasing as well as sequence (x_n). But the sequence (c_n) is oscillatory.

Corollary 4.1. *Let condition (4.3) be satisfied. Assume also that condition (4.5) is held, (c_n) is a real nonnegative bounded sequence, and (e_n), (q_n) are real positive sequences. If*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}} = \infty, \tag{4.16}$$

then any solution of the equation (4.1) is almost oscillatory.

Proof. Take $p_n = 1$. This result follows directly from Theorem 4.1. □

Putting $c_n \equiv c$, where c is a constant and $\gamma = 1$ in the equation (4.1), we get equation in the form

$$\Delta^2 (x_n + c x_{n-k}) + q_n x_{n+1}^{\alpha} + e_n \operatorname{sgn}(x_n) = 0. \tag{4.17}$$

Hence, Theorem 4.1 is reduced to the following corollary.

Corollary 4.2. *Let condition (4.3) be satisfied. Assume also that α is a ratio of odd positive integers, $c \geq 0$, and (e_n) and (q_n) are real positive sequences. If there exists a positive sequence (p_n) such that*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \left(p_i q_i^{\frac{1}{\alpha}} e_i^{1-\frac{1}{\alpha}} - \frac{(\Delta p_i)^2}{4sp_i} \right) = \infty,$$

where

$$s = \frac{\alpha}{(\alpha-1)^{1-\frac{1}{\alpha}}(1+c)},$$

then any solution of equation (4.17) is almost oscillatory.

As a special case of Corollary 4.2, for $c = 0$, we obtain difference equation of perturbed Sturm-Liouville type

$$\Delta^2 x_n + q_n x_{n+1}^{\alpha} + e_n \operatorname{sgn}(x_n) = 0. \tag{4.18}$$

For equation (4.18), Corollary 4.2 holds with $s = \frac{\alpha \sqrt[\alpha]{\alpha-1}}{\alpha-1}$.

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Chapter 5
Uniform Weak Disconjugacy and Principal Solutions for Linear Hamiltonian Systems

Russell Johnson, Sylvia Novo, Carmen Núñez, and Rafael Obaya

Abstract The paper analyzes the property of (uniform) weak disconjugacy for nonautonomous linear Hamiltonian systems, showing that it is a convenient replacement for the more restrictive property of disconjugacy. In particular, its occurrence ensures the existence of principal solutions. The analysis of the properties of these solutions provides ample information about the dynamics induced by the Hamiltonian system on the Lagrange bundle.

Keywords Nonautonomous linear Hamiltonian systems • Uniform weak disconjugacy • Principal solutions

5.1 Introduction and Preliminaries

The analysis of nonautonomous linear Hamiltonian systems with the disconjugacy property, which is closely related to their oscillation properties and which has applications in the calculus of variations, is an extended and classical branch of the study of linear differential systems. The basic facts concerning this property are discussed in Hartman [6], Coppel [1], and Reid [17]. One of the most interesting characteristics of systems of this type is the existence of principal solutions.

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These solutions constitute an extension to the nonuniformly hyperbolic case of the Lagrange planes associated, in the case of exponential dichotomy, to the bounded solutions at $+\infty$ and $-\infty$.

The analysis of disconjugacy was continued in the works [9, 10] of Johnson et al. The use of the methods of the modern theory of nonautonomous differential systems allows the authors to study the dynamical and ergodic properties of the principal solutions and to go much deeper into the close relation between principal solutions, Lyapunov indices, and exponential dichotomy than had been done before.

A less restrictive condition, called weak disconjugacy, was introduced and analyzed later by Fabbri et al. in [5]. It turns out to be often but not always equivalent to the classical disconjugacy property. The main advantage of weak disconjugacy is that it can be guaranteed under a much weakened version of the condition of identical normality, which is usually assumed when studying the classical disconjugacy property. From the first moment, it was clear that the concept of weak disconjugacy was closely related to the oscillatory properties of the system analyzed (or, more precisely, to the absence of oscillation) and that it was suited to optimize the hypotheses of certain results based on the properties of the rotation number.

But in fact the interest of weak disconjugacy goes beyond this first analysis. Under an additional condition of uniformity (which is still often weaker than that of disconjugacy), this property also ensures the existence of principal solutions and makes it possible to extend the analysis of the relation of these solutions with the Lyapunov indices and with the occurrence of exponential dichotomy.

During the last years, we have been using the concept of weak disconjugacy for various purposes. For instance, the authors of [2] describe mild conditions under which the lack of oscillation of a linear Hamiltonian system is equivalent to its weak disconjugacy, as well as stronger conditions which ensure the existence of principal solutions for a given system. In [12], the close relation between principal solutions and exponential dichotomy is analyzed in detail, then as a consequence it is shown that the Yakubovich frequency theorem (in its nonautonomous form as developed in [4]) can be applied to a wide range of optimization problems. This analysis relies on the strong connection between the uniform weak disconjugacy and the controllability properties of some systems constructed from the initial one, and clearly this relation has independent interest. And in [11], dedicated to the analysis of dissipative linear-quadratic control systems, the properties of the weakly disconjugate systems allow us to relax the conditions on *strict* dissipativity.

In spite of these applications, a systematic description of uniform weak disconjugacy and its main consequences has not been published yet. This is the goal of the present paper: we define and characterize the uniform weak disconjugacy concept (Sect. 5.2); we analyze the connections between disconjugacy, uniform weak disconjugacy, weak disconjugacy, and nonoscillation (Sect. 5.3); we show that uniform weak disconjugacy guarantees the existence of uniform principal solutions (also in Sect. 5.2); and we analyze further the dynamical consequences of their existence (Sect. 5.4). Each section begins with a more detailed description of its contents, not repeated here. The paper, with ideas based on those of Coppel [1], is reasonably self-contained. It shows that the theory of weak disconjugacy is a satisfactory generalization of that of disconjugacy. And it furnishes an appropriate reference to understand the scope of the results of [2, 12] and [11], as well as to go deeper into the analysis leading to those results.

In the rest of this section, we explain the general setting to which our results apply and give the basic notions and properties required later.

Let $H_0 \colon \mathbb{R} \to \mathfrak{sp}(n,\mathbb{R})$ be a $2n \times 2n$ matrix-valued function taking values in the Lie algebra of infinitesimally symplectic matrices: $JH_0 + H_0^T J = 0_{2n}$, where $J = \left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$ and the superscript T represents the transpose matrix. That is, $H_0 = \left[\begin{smallmatrix} H_{01} & H_{03} \\ H_{02} & -H_{01}^T \end{smallmatrix}\right]$ for symmetric $n \times n$ matrices H_{02} and H_{03}. Assuming that H_0 is bounded and uniformly continuous, we define Ω as the *hull* of H_0, i.e., as the closure in the compact-open topology of $\{H_t \mid t \in \mathbb{R}\}$, where $H_t(s) = H(t+s)$. Then Ω is a compact metric space, and the map $\mathbb{R}\times\Omega \to \Omega$ sending ω to $\omega{\cdot}t = \omega_t$, with $\omega_t(s) = \omega(t+s)$, defines a real continuous flow. Now we define $H \colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$, $\omega \mapsto \omega_0$, which is a continuous operator, and consider the family of linear Hamiltonian systems $\{\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z} \mid \omega \in \Omega\}$. Since $H(\omega{\cdot}t) = \omega_t(0) = \omega(t)$, the system $\mathbf{z}' = H_0(t)\,\mathbf{z}$ is included in this family: it corresponds to $\omega = H_0 \in \Omega$.

This procedure shows us the way to obtain families of linear Hamiltonian systems of so-called Bebutov type. The procedure also motivates the introduction of a somewhat more general setup: given a real continuous flow $\sigma \colon \mathbb{R} \times \Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$ on a compact metric space Ω and a continuous map $H \colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$, we consider the family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z} = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) \\ H_2(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \qquad \omega \in \Omega\,. \tag{5.1}$$

Note that this setting is like the one described in the previous paragraph if and only if there exists ω with σ-orbit dense in Ω. In this paper we will work in the more general setting just described. This is highly convenient, since most of the results that we obtain refer to *all* the systems of the family. That is, they are also valid for a particular one in the case that Ω is constructed as its hull. In addition, we will establish conditions on a single system ensuring that the whole family satisfies the hypotheses that we will assume for our analysis: this is done in Proposition 5.5.

Before describing some basic facts concerning (5.1), we recall that a real Lagrange plane $l \subset \mathbb{R}^{2n}$ is an n-dimensional linear space such that $\mathbf{z}^T J\,\mathbf{w} = 0$ for any pair of vectors $\mathbf{z}, \mathbf{w}$ in l. The set of all the real Lagrange planes constitute a compact manifold, $\mathscr{L}_{\mathbb{R}}$. By writing a basis of $l \in \mathscr{L}_{\mathbb{R}}$ in columns, we obtain a $2n \times n$ matrix $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ (for $n \times n$ real matrices L_1 and L_2) *representing* l, which we write as $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. Note that $L_2^T L_1 = L_1^T L_2$ and that the matrices $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} L_3 \\ L_4 \end{smallmatrix}\right]$ represent the same real Lagrange plane if and only if there exists a nonsingular real matrix P such that $L_1 = L_3 P$ and $L_2 = L_4 P$. The set

$$\mathscr{D} = \left\{ l \in \mathscr{L}_{\mathbb{R}} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \right\} \subset \mathscr{L}_{\mathbb{R}}\,, \tag{5.2}$$

is open and dense in $\mathscr{L}_\mathbb{R}$. Note that $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ belongs to $\mathscr{D}$ if and only if $\det L_1 \neq 0$, in which case $l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ for the real symmetric matrix $M = L_2\, L_1^{-1}$. This matrix M is the unique one *parameterizing* l *in* $\mathscr{D}$. In addition, the sequence $\left(l_j \equiv \left[\begin{smallmatrix} I_n \\ M_j \end{smallmatrix}\right]\right)$ of elements of $\mathscr{D}$ converges to an element $l \in \mathscr{D}$ in the topology of $\mathscr{L}_\mathbb{R}$ if and only if $l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ and (M_j) converges to M.

By abusing slightly language, we say that a real matrix L representing a Lagrange plane *belongs to* $\mathscr{L}_\mathbb{R}$ and that a matrix-valued function with this property *takes values in* $\mathscr{L}_\mathbb{R}$. If L represents $l \in \mathscr{D}$, then we say that L *belongs to* $\mathscr{D}$, and a matrix-valued function with this property *takes values in* $\mathscr{D}$.

Returning to the Hamiltonian family (5.1), let $U(t,\omega)$ be the fundamental matrix solution $U(t,\omega)$ given by $U(0,\omega) = I_{2n}$. Then $\mathbf{z}(t) = U(t,\omega)\,\mathbf{z}_0$ is the solution of the system corresponding to ω with initial data $\mathbf{z}(0) = \mathbf{z}_0$. By uniqueness of solutions, $U(t+s,\omega) = U(t,\omega{\cdot}s)\,U(s,\omega)$. This fact ensures that the map

$$\tau_r\colon \mathbb{R}\times\Omega\times\mathbb{R}^{2n} \to \Omega\times\mathbb{R}^{2n}\,, \quad (t,\omega,\mathbf{z}) \mapsto (\omega{\cdot}t, U(t,\omega)\,\mathbf{z})$$

defines a real continuous flow on the product space $\Omega\times\mathbb{R}^{2n}$, which is of *skew-product type*: it preserves the flow on Ω, which is the *base* of the bundle $\Omega\times\mathbb{R}^{2n}$.

Recall that a real $2n\times 2n$ matrix function V is symplectic if $V^T J V = J$. Let $V(t,\omega)$ be a matrix solution of the system (5.1). It is immediate to prove that $V^T(t,\omega) J V(t,\omega) = V^T(0,\omega) J V(0,\omega)$. Consequently, $V(t,\omega)$ takes values in the set of symplectic matrices if and only if $V(0,\omega)$ is symplectic. This is what happens with the fundamental matrix solution $U(t,\omega)$, since $U(0,\omega) = I_{2n}$. An easy consequence is that the n-dimensional linear space $U(t,\omega){\cdot}l = \{U(t,\omega)\,\mathbf{z} \mid \mathbf{z}\in l\}$ belongs to $\mathscr{L}_\mathbb{R}$ if l does. Therefore, the map

$$\tau\colon \mathbb{R}\times\Omega\times\mathscr{L}_\mathbb{R} \to \Omega\times\mathscr{L}_\mathbb{R}\,, \quad (t,\omega,l) \mapsto (\omega{\cdot}t, U(t,\omega){\cdot}l)$$

defines a real continuous skew-product flow on the space $\Omega\times\mathscr{L}_\mathbb{R}$.

The flows τ_r and τ are globally defined. This is not the case for the skew-product flow τ_s defined on $\Omega\times\mathbb{S}_n(\mathbb{R})$ (where $\mathbb{S}_n(\mathbb{R})$ is the set of real symmetric $n\times n$ matrices) by the solutions of the family of Riccati equations

$$M' = -M\,H_3(\omega{\cdot}t)\,M - M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t)\,; \tag{5.3}$$

that is,

$$\tau_s\colon \mathbb{R}\times\Omega\times\mathbb{S}_n(\mathbb{R}) \to \Omega\times\mathbb{S}_n(\mathbb{R})\,, \quad (t,\omega,M_0) \mapsto (\omega{\cdot}t, M(t,\omega,M_0))\,,$$

where $M(t,\omega,M_0)$ is the maximal solution of (5.3) with initial data $M(0,\omega,M_0) = M_0$. It is easy to check that $M(t,\omega,M_0)$ is defined as long as $U(t,\omega)\left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1(t,\omega,M_0) \\ L_2(t,\omega,M_0) \end{smallmatrix}\right]$ takes values in $\mathscr{D}$: in fact, $M(t,\omega,M_0) = L_2(t,\omega,M_0)L_1^{-1}(t,\omega,M_0)$ for these values of t. Note also that the classical results of ordinary differential equations ensure that if $M(t,\omega,M_0)$ is norm-bounded on an interval $[a,b]$, then it can be continued outside the interval.

All these facts will also be repeatedly used in the next pages. Throughout the paper, $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^m$ for any $m \in \mathbb{N}$, as well as the associated matrix norm. We will write $A \geq 0$ (or $A > 0$) for a symmetric positive semidefinite (or definite) matrix A and $A \geq B$ (or $A > B$) if $A - B \geq 0$ (or $A - B > 0$). And the symbols $\leq$ and $<$ have the obvious meaning.

Remark 5.1. The Euclidean matrix norm (any matrix norm, as a matter of fact) is semimonotone. That is, there exist $c > 0$ such that if $0 \leq A \leq B$ for two symmetric matrices A and B, then $\|A\| \leq c\|B\|$. If follows easily that if $A \leq C \leq B$ for three matrix-valued functions on a certain domain, and A and B are globally norm-bounded on that domain, then so is C.

Three more results, fundamental in our proofs, complete the section. The first one appears in Lidskiĭ [14]. As usual, W^* represents the conjugate transpose of W.

Lemma 5.1. *Let $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$ be a symplectic matrix. Then the matrix $W_V = (V_1 - \mathrm{i}V_3)^{-1}(V_1 + \mathrm{i}V_3)$ is well defined. In addition,*

(i) *$W_V^T = W_V$ and $W_V^* W_V = I_n$. In particular, all the eigenvalues of W_V lie on the unit circle of $\mathbb{C}$.*
(ii) *$W_V\,\mathbf{z} = \mathbf{z}$ if and only if $V_3\,\mathbf{z} = \mathbf{0}$. In particular, 1 is an eigenvalue of W_V if and only if* $\det V_3 = 0$.

Now we fix $\omega \in \Omega$, represent by $V(t,\omega) = \left[\begin{smallmatrix} V_1(t) & V_3(t) \\ V_2(t) & V_4(t) \end{smallmatrix}\right]$ any real symplectic matrix solution of the corresponding system (5.1), and define

$$W_V(t) = (V_1(t) - \mathrm{i}V_3(t))^{-1}(V_1(t) + \mathrm{i}V_3(t))\,. \tag{5.4}$$

Theorem II.5.2 of [13] and Lemma 5.1(i) ensure the existence of continuous functions $\rho_1, \ldots, \rho_n : \mathbb{R} \to \mathbb{C}$ with $|\rho_j(t)| = 1$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$, such that the set of eigenvalues of $W_V(t)$, repeated according to their multiplicities, coincides with the unordered n-uple $\{\rho_1(t), \ldots, \rho_n(t)\}$. Let $\varphi_1, \ldots, \varphi_n : \mathbb{R} \to \mathbb{R}$ be continuous argument functions: $\rho_j(t) = e^{\mathrm{i}\varphi_j(t)}$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$. A sketch of the proof of the next result, essentially due to Lidskiĭ [14], is given in Proposition 2.4 of [2].

Theorem 5.1. *Suppose that $H_3(\omega{\cdot}t) \geq 0$ for each $t \in \mathbb{R}$. With the above notation, the continuous function $\varphi_j : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $j = 1, \ldots, n$.*

5.2 Uniform Weak Disconjugacy and Principal Solutions

Let (Ω, σ) be a real continuous flow on a compact metric space, and write $\omega{\cdot}t = \sigma(t,\omega)$. We establish in this section conditions ensuring the so-called uniform weak disconjugacy of the family of linear systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{5.5}$$

where $H\colon \Omega \to \mathfrak{sp}(n,\mathbb{R})$ is continuous. We also derive from the previous characterization the existence of the principal solutions, whose properties will be analyzed in Sect. 5.4.

Let us write $H = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right]$ and represent by $U(t,\omega) = \left[\begin{smallmatrix} U_1(t,\omega) & U_3(t,\omega) \\ U_2(t,\omega) & U_4(t,\omega) \end{smallmatrix}\right]$ the fundamental matrix solution of (5.5) with $U(0,\omega) = I_{2n}$. The main properties of this matrix and of the flows τ_r, τ and τ_s, induced by (5.5) on $\Omega \times \mathbb{R}^{2n}$, $\Omega \times \mathcal{L}_{\mathbb{R}}$ and $\Omega \times \mathbb{S}_n(\mathbb{R})$, have been recalled in Sect. 5.1. In particular, the flow τ on $\Omega \times \mathcal{L}_{\mathbb{R}}$ is given by $\tau(t,\omega,l) = (\omega{\cdot}t, U(t,\omega){\cdot}l)$, where $U(t,\omega){\cdot}l = \{U(t,\omega)\,\mathbf{z} \mid \mathbf{z} \in l\}$.

Definition 5.1. The linear Hamiltonian system (5.5) corresponding to a point $\omega \in \Omega$ is *disconjugate on* $\mathbb{R}$ if, for any nonzero solution $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$, the vector $\mathbf{z}_1(t)$ vanishes at most once on $\mathbb{R}$.

Definition 5.2. The linear Hamiltonian system (5.5) corresponding to a point $\omega \in \Omega$ is *weakly disconjugate on* $[0,\infty)$ (resp. *on* $(-\infty,0]$) if there exists $t_0 = t_0(\omega) > 0$ such that, for any nonzero solution $\mathbf{z}(t) = \left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ with $\mathbf{z}_1(0) = \mathbf{0}$, there holds $\mathbf{z}_1(t) \neq \mathbf{0}$ for all $t \geq t_0$ (resp. for all $t \leq -t_0$).

In the next definition, Arg denotes any argument on $\mathrm{Sp}(n,\mathbb{R})$ equivalent to $\mathrm{Arg}_3\, V = \arg\det(V_1 + \mathrm{i}V_3)$ (where $V = \left[\begin{smallmatrix} V_1 & V_3 \\ V_2 & V_4 \end{smallmatrix}\right]$). See [18] and [20] for the precise definitions of arguments and of equivalence between them, which in particular ensures that the next concept is independent of the choices of Arg and $V(t,\omega)$.

Definition 5.3. The linear Hamiltonian system (5.5) corresponding to a point $\omega \in \Omega$ is said to be *nonoscillatory at* $+\infty$ (resp. *at* $-\infty$) if $\mathrm{Arg}\, V(t,\omega)$ is a bounded function on $[0,\infty)$ (resp. on $(-\infty,0]$), where $V(t,\omega)$ is any symplectic fundamental matrix solution and a continuous branch of the argument is taken along the curve.

Definition 5.4. The family (5.5) of linear Hamiltonian systems is *uniformly weakly disconjugate on* $[0,\infty)$ (resp. *on* $(-\infty,0]$) if each one of its systems is weakly disconjugate on $[0,\infty)$ (resp. on $(-\infty,0]$) and in addition the time t_0 of Definition 5.2 can be taken to be the same for all $\omega \in \Omega$.

Definition 5.5. A $2n \times n$ matrix solution $L(t,\omega) = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right]$ of the system (5.5) corresponding to ω is *principal on* $[t_1,\infty)$ (resp. *on* $(-\infty,t_1]$) if it takes values in $\mathcal{D}$ for any $t \geq t_1$ (resp. for $t \leq t_1$) and there exists

$$\lim_{t\to\infty} \left(\int_{t_1}^{t} L_1^{-1}(s,\omega)\, H_3(\omega{\cdot}s)\, (L_1^T)^{-1}(s,\omega)\, ds \right)^{-1} = 0_n \tag{5.6}$$

(resp. the same holds for the limit as $t \to -\infty$).

A principal solution on $[0,\infty)$ (resp. on $(-\infty,0]$) is called a *uniform principal solution at* ∞ (resp. *at* $-\infty$) if it takes values in $\mathcal{D}$ for any $t \in \mathbb{R}$.

Remark 5.2.

1. Obviously a disconjugate system is weakly disconjugate, which justifies the choice of the name for this behavior: just take any $t_0 > 0$ in Definition 5.2.

For the same reason, if all the systems of the family are disconjugate, then the family is uniformly weak disconjugate.

2. Note that, since $U(t,\omega)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}U_3(t,\omega)\,\mathbf{z}_2\\ U_4(t,\omega)\,\mathbf{z}_2\end{smallmatrix}\right]$, the weak disconjugacy on $[0,\infty)$ (resp. on $(-\infty,0]$) of the system (5.5) is equivalent to the existence of $t_0 = t_0(\omega) > 0$ such that $\det U_3(t,\omega) \neq 0$ for any $\omega \in \Omega$ if $t \geq t_0$ (resp. $t \leq -t_0$); and the uniform weak disconjugacy holds if and only if a t_0 common for all $\omega \in \Omega$ exists.

The next three conditions will play a fundamental role in the whole paper:

D1. The $n \times n$ matrix-valued function H_3 is positive semidefinite on Ω.

D2. For any $\omega \in \Omega$ and any nonzero solution $\mathbf{z}(t,\omega) = \left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ of the system (5.5) with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on $[t_1,\infty)$ for any $t_1 \in \mathbb{R}$.

D3. For any $\omega \in \Omega$ there exists a $2n \times n$ matrix solution $G(t,\omega) = \left[\begin{smallmatrix}G_1(t,\omega)\\ G_2(t,\omega)\end{smallmatrix}\right]$ of (5.5) taking values in $\mathscr{D}$ for any $t \in \mathbb{R}$. In other words, for any $\omega \in \Omega$, there exists $l_\omega \in \mathscr{L}_\mathbb{R}$ such that $U(t,\omega){\cdot}l_\omega \in \mathscr{D}$ for any $t \in \mathbb{R}$.

From now on the notation $\mathrm{D1}_\omega$ will be used to represent the property $H_3(\omega{\cdot}t) \geq 0$ for any $t \in \mathbb{R}$; and $\mathrm{D2}_\omega$ and $\mathrm{D3}_\omega$ will represent the properties stated in D2 and D3 just for the system given by ω.

The goal of this section is to prove the next characterization, whose scope will be analyzed in Sect. 5.3.

Theorem 5.2. *Suppose that* D1 *holds. The following properties are equivalent:*

(1) *the family* (5.5) *is uniformly weakly disconjugate on* $[0,\infty)$*;*
(2) *the family* (5.5) *is uniformly weakly disconjugate on* $(-\infty,0]$*;*
(3) *conditions* D2 *and* D3 *hold.*

In this case, each one of the systems of the family admits uniform principal solutions at $+\infty$ and $-\infty$ which are unique as matrix-valued functions taking values in $\mathscr{L}_\mathbb{R}$ and determine τ-invariant sets $\{(\omega, l^\pm(\omega)) \mid \omega \in \Omega\} \subset \Omega \times \mathscr{D}$.

The theorem is an immediate consequence of Theorems 5.3(i) and 5.4, stated below. One of its conclusions is that we can simply talk about uniform weak disconjugacy of the family. We will do that from Sect. 5.3 on. The proofs of these theorems require several previous results, which will also play a role in the next section. Note that condition D1, almost permanently assumed, is not required for the first results.

Proposition 5.1.

(i) *Condition* D2 *holds if and only if there exist $\delta > 0$ and $t_0 > 0$ with*

$$\int_0^{t_0} \|H_3(\omega{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{x}\|^2 \mathrm{d}t \geq \delta\|\mathbf{x}\|^2 \tag{5.7}$$

for any $\omega \in \Omega$ and $\mathbf{x} \in \mathbb{R}^n$, where $U_{H_1}(t,\omega)$ is the fundamental matrix solution of $\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x}$ with $U_{H_1}(0,\omega) = I_n$.

(ii) *Suppose that* D2 *holds, and let t_0 be the time provided by* (i). *Then, none of the systems of the family* (5.5) *admits a solution taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on an interval of length t_0.*

(iii) *Condition* D2 *is equivalent to*

D2′. *For any $\omega \in \Omega$ and any nonzero solution $\mathbf{z}(t,\omega) = \left[\begin{smallmatrix}\mathbf{z}_1(t,\omega)\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ of the system* (5.5) *with $\mathbf{z}_1(0,\omega) = \mathbf{0}$, the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on $(-\infty, t_1]$ for any $t_1 \in \mathbb{R}$.*

Proof.

(i) If $\mathrm{D2}_{\omega_1}$ does not hold, there is a nonzero solution of the system (5.5) for ω_1 on $[t_1,\infty)$, $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2(t,\omega_1)\end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2(t+t_1,\omega_1)\end{smallmatrix}\right]$ solves (5.5) for $\omega = \omega_1{\cdot}t_1$ on $[0,\infty)$. Thus, $\mathbf{0} = H_3(\omega{\cdot}t)\,\mathbf{z}_2(t,\omega)$ and $\mathbf{z}_2'(t,\omega) = -H_1^T(\omega{\cdot}t)\,\mathbf{z}_2(t,\omega)$ for $t \ge 0$ so that $\mathbf{z}_2(t,\omega) = (U_{H_1}^T)^{-1}(t,\omega)\,\mathbf{z}_2(0,\omega)$, and (5.7) does not hold for $\mathbf{x} = \mathbf{z}_2(0,\omega)$.

Conversely, if (5.7) does not hold, then the compactness of Ω and of the unit sphere in $\mathbb{R}^n$ ensures that

$$\int_0^m \|H_3(\omega_m{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_m)\,\mathbf{x}_m\|^2 \mathrm{d}t = 0$$

for each $m \in \mathbb{N}$, for a suitable point $(\omega_m,\mathbf{x}_m) \in \Omega \times \mathbb{R}^n$ with $\|\mathbf{x}_m\| = 1$. A convergent subsequence of $((\omega_m,\mathbf{x}_m))$ provides $(\omega_0,\mathbf{x}_0)$ with $\|\mathbf{x}_0\| = 1$ such that $\int_0^\infty \|H_3(\omega_0{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}_0\|^2\mathrm{d}t = 0$. Then the function $\mathbf{z}(t,\omega_0) = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega_0)\end{smallmatrix}\right]$ given by $\mathbf{z}_2(t,\omega_0) = (U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}_0$ is a nonzero solution of the system (5.5) corresponding to ω_0 on $[0,\infty)$, since $H_3(\omega_0{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}_0 = \mathbf{0}$ for each $t \ge 0$. This fact precludes D2.

(ii) Suppose for contradiction the existence of a solution $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ of the system (5.5) corresponding to ω on $[a, a+t_0]$. Then $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t+a)\end{smallmatrix}\right] = U(t,\omega{\cdot}a)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(a)\end{smallmatrix}\right]$ for each $t \in [0,t_0]$ and, as above, $\mathbf{0} = H_3((\omega{\cdot}a){\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega{\cdot}a)\,\mathbf{z}_2(a)$ for each $t \in [0,t_0]$, which contradicts (5.7) for $\mathbf{x} = \mathbf{z}_2(a)$ and $\omega{\cdot}a$.

(iii) It follows immediately from (ii) that D2 ensures D2′. Conversely, condition D2′ can be taken as starting point to prove the analogue of (i), which will then ensure (ii) and hence D2.

□

It turns out that condition (5.7) is equivalent to the uniform null controllability of the family of control systems

$$\mathbf{x}' = H_1(\omega{\cdot}t)\,\mathbf{x} + H_3(\omega{\cdot}t)\,\mathbf{u} \tag{5.8}$$

and that this condition is in turn ensured by an a priori weaker one: each minimal subset of Ω contains a point ω such that the corresponding system (5.8) is null controllable. The interested reader can find in [8] the details of the proof of this assertion.

Remark 5.3.

1. The weak disconjugacy on $[0,\infty)$ (resp. $(-\infty,0]$) of the system given by ω ensures condition D2$_\omega$ (resp. D2$'_\omega$). The weak disconjugacy on $[0,\infty)$ of all the systems precludes the existence of a nonzero solution of (5.5) taking the form $\mathbf{z}(t,\omega) = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ on any positive half line $[a,\infty)$, since $\mathbf{w}(t) = \mathbf{z}(t+a,\omega)$ solves $\mathbf{z}' = H((\omega{\cdot}a){\cdot}t))\,\mathbf{z}$ with $\mathbf{w}(0) = \mathbf{z}(a,\omega)$. In particular, D2 holds. Similar relations hold for the weak disconjugacy on $(-\infty,0]$ and condition D2$'$.
2. As deduced from Propositions 2.6 and 2.8 of [2], if D1$_\omega$ holds, then the system (5.5) corresponding to ω is weakly disconjugate on $[0,\infty)$ (resp. on $(-\infty,0]$) if and only if it is nonoscillatory at $+\infty$ (resp. at $-\infty$) and D2$_\omega$ (resp. D2$'_\omega$) holds.
3. It can be shown (see [19]) that if there exists a real Lagrange plane $l \equiv \left[\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right]$ such that the $2n\times n$ matrix-valued solution $\left[\begin{smallmatrix}L_1(t,\omega)\\ L_2(t,\omega)\end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right]$ of (5.5) satisfies $\det L_1(t,\omega)\neq 0$ for every t in a positive (resp. negative) half line, then the system corresponding to ω is nonoscillatory at $+\infty$ (resp. at $-\infty$).

The next result will not be required until the next section. However, it refers just to conditions D2 and D2$'$, so it seems appropriate to include it at this point.

Lemma 5.2.

(i) *Suppose that* D2$_{\omega_1}$ *holds for a point* ω_1 *in the omega-limit set of* ω_0. *Then* D2$_{\omega_0\cdot t}$ *holds for all* $t\in\mathbb{R}$.
(ii) *Suppose that* D2$'_{\omega_1}$ *holds for a point* ω_1 *in the alpha-limit set of* ω_0. *Then* D2$'_{\omega_0\cdot t}$ *holds for all* $t\in\mathbb{R}$.
(iii) *If* Ω *is minimal, then* D2$_{\omega_0}$ *(resp.* D2$'_{\omega_0}$*) holds for a point* $\omega_0\in\Omega$ *if and only if* D2 *(resp.* D2$'$*) holds.*

Proof. In order to prove (i), note that the omega-limit sets of ω_0 and $\omega_0{\cdot}t$ agree for any $t\in\mathbb{R}$, so that it is enough to prove that D2$_{\omega_0}$ holds. Suppose for contradiction the existence of a $\mathbf{z}_2\neq\mathbf{0}$ such that $U(t,\omega_0)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on $[t_1,\infty)$. Take a sequence $(s_m)\uparrow\infty$ with $\omega_1 = \lim_{m\to\infty}\omega_0{\cdot}s_m$, and choose a suitable subsequence (s_j) such that there exists the limit $\mathbf{w}_2\neq\mathbf{0}$ of $(\mathbf{z}_2(s_j)/\|\mathbf{z}_2(s_j)\|)$. It follows that $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2\end{smallmatrix}\right] = \lim_{j\to\infty}U(s_j,\omega_0)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2/\|\mathbf{z}_2(s_j)\|\end{smallmatrix}\right]$, and consequently $U(t,\omega_1)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2\end{smallmatrix}\right] = \lim_{j\to\infty}U(t,\omega_0{\cdot}s_j)\,U(s_j,\omega_0)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2/\|\mathbf{z}_2(s_j)\|\end{smallmatrix}\right] = \lim_{j\to\infty}U(t+s_j,\omega_0)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2/\|\mathbf{z}_2(s_j)\|\end{smallmatrix}\right] = \lim_{j\to\infty}\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t+s_j)/\|\mathbf{z}_2(s_j)\|\end{smallmatrix}\right]$ for any $t\in\mathbb{R}$, which provides a solution taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{w}_2(t)\end{smallmatrix}\right]$ on $\mathbb{R}$ for the system for ω_1. In particular, D2$_{\omega_1}$ does not hold, which yields the sought-for contradiction. The proof of (ii) is identical, and the properties stated in (iii) are easy consequences of the previous ones. $\square$

Remark 5.4. Given a point $\omega\in\Omega$, let $G(t,\omega) = \left[\begin{smallmatrix}G_1(t,\omega)\\ G_2(t,\omega)\end{smallmatrix}\right]$ be a $2n\times n$ matrix solution of the corresponding system (5.5). Suppose that $G(t,\omega)$ takes values in $\mathscr{D}$ for every t in an interval $\mathscr{I}$. Take $a\in\mathscr{I}$ and define

$$I_G(a,t,\omega) = \int_a^t G_1^{-1}(s,\omega)\, H_3(\omega{\cdot}s)\, (G_1^T)^{-1}(s,\omega)\, \mathrm{d}s \tag{5.9}$$

for $t \in \mathscr{I}$. It is easy to check that

$$\begin{bmatrix} G_1(t,\omega) & G_1(t,\omega)\, I_G(a,t,\omega) \\ G_2(t,\omega) & G_2(t,\omega)\, I_G(a,t,\omega) + (G_1^T)^{-1}(t,\omega) \end{bmatrix} \tag{5.10}$$

is a fundamental matrix solution of (5.5). Consequently, a $2n \times n$ matrix-valued function $\tilde{G}(t,\omega)$ solves (5.5) on $\mathscr{I}$ if and only if it takes the form

$$\begin{bmatrix} \tilde{G}_1(t,\omega) \\ \tilde{G}_2(t,\omega) \end{bmatrix} = \begin{bmatrix} G_1(t,\omega)\,(P(\omega) + I_G(a,t,\omega)\, Q(\omega)) \\ G_2(t,\omega)\,(P(\omega) + I_G(a,t,\omega)\, Q(\omega)) + (G_1^T)^{-1}(t,\omega)\, Q(\omega) \end{bmatrix} \tag{5.11}$$

for arbitrary real $n \times n$ matrices $P(\omega)$ and $Q(\omega)$. Moreover, as proved by Coppel in [1] (Proposition 3 of Chap. 2), if $\tilde{G}(t,\omega)$ belongs to $\mathscr{D}$ for $t \in \mathscr{I}$, then $P(\omega)$ is nonsingular and

$$I_{\tilde{G}}(a,t,\omega) = (P(\omega) + I_G(a,t,\omega)\, Q(\omega))^{-1} I_G(a,t,\omega)\, (P^T)^{-1}(\omega) \tag{5.12}$$

for $t \in I$, with $I_{\tilde{G}}$ defined from $\tilde{G}_1$ as I_G from G_1 in (5.9). Finally, if $\mathscr{I}$ contains a half line $[a,\infty)$ and there exists $\lim_{t\to\infty}(I_{\tilde{G}}(a,t,\omega))^{-1} = 0_n$, then

$$\lim_{t\to\infty} (I_{\tilde{G}}(a,t,\omega) + C)^{-1} = \lim_{t\to\infty} (I_{\tilde{G}}(a,t,\omega))^{-1}(I_n + C(I_{\tilde{G}}(a,t,\omega))^{-1})^{-1} = 0_n$$

for any constant matrix C, and hence there exists $\lim_{t\to\infty}(I_{\tilde{G}}(b,t,\omega))^{-1} = 0_n$ whenever $[b,\infty) \subseteq \mathscr{I}$. A similar result, taking limits at $-\infty$, holds if $\mathscr{I}$ contains a half line $(-\infty,a]$. These last properties are specially relevant when talking about uniform principal solutions on positive or negative half lines: if this is the case, any t_1 in (5.6) provides the same limit, so that a uniform principal solution at $+\infty$ or at $-\infty$ is a principal solution on $[t_1,\infty)$ or on $(-\infty,t_1]$ for any $t_1 \in \mathbb{R}$.

Lemma 5.3. *Suppose that* D1 *and* D2 *hold, and let t_0 be the positive time of* Proposition 5.1(i). *Let $G(t,\omega) = \left[\begin{smallmatrix} G_1(t,\omega) \\ G_2(t,\omega) \end{smallmatrix}\right]$ be a $2n \times n$ matrix solution of* (5.5) *taking values in $\mathscr{D}$ for every $t \geq t_1$ and $\omega \in \Omega$. Then, for any $\omega \in \Omega$, the symmetric matrix $I_G(a,t,\omega)$ defined by* (5.9) *for $t_1 \leq a < t$ is positive definite if $t - a \geq t_0$.*

Proof. It follows from (5.10) that

$$\mathbf{z}(t,\omega) = \begin{bmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{bmatrix} = \begin{bmatrix} G_1(t,\omega)\, I_G(a,t,\omega)\, \mathbf{x}_0 \\ G_2(t,\omega)\, I_G(a,t,\omega)\, \mathbf{x}_0 + (G_1^T)^{-1}(t,\omega)\, \mathbf{x}_0 \end{bmatrix}$$

solves (5.5) for any $\mathbf{x}_0 \in \mathbb{R}^n$. Take $t_2 \geq a + t_0$ and suppose for contradiction that there exist a vector $\mathbf{x}_0 \neq \mathbf{0}$ and a point $\omega_0 \in \Omega$ such that $\mathbf{x}_0^T\, I_G(a,t_2,\omega_0)\, \mathbf{x}_0 = \mathbf{0}$, which due to D1 implies that $\mathbf{x}_0^T\, I_G(a,t,\omega_0)\, \mathbf{x}_0 = \mathbf{0}$ for all $t \in [a,t_2]$. Hence $\mathbf{z}_1(t,\omega_0) = \mathbf{0}$ for all $t \in [a,t_2]$, which contradicts Proposition 5.1(ii). □

Now we can prove the theorems providing the information stated in Theorem 5.2. The second statement of Theorem 5.3 will be required in Sect. 5.4.

Theorem 5.3. *Suppose that* D1*,* D2*, and* D3 *hold. Then,*

(i) *the family* (5.5) *is uniformly weakly disconjugate on* $[0,\infty)$ *and on* $(-\infty,0]$*;*
(ii) *for each* $\omega \in \Omega$ *and* $l \in \mathscr{L}_{\mathbb{R}}$*, there exists* $s_{\omega,l}$ *such that* $U(t,\omega){\cdot}l \in \mathscr{D}$ *whenever* $|t| > s_{\omega,l}$*.*

Proof.

(i) We suppose without restriction that the matrix solution $\left[\begin{smallmatrix} G_1(t,\omega) \\ G_2(t,\omega)\end{smallmatrix}\right]$ of condition D3 is normalized to $G_1(0,\omega) = I_n$ for any $\omega \in \Omega$. Once this is done, it follows from (5.11) and from $\left[\begin{smallmatrix} U_3(0,\omega) \\ U_4(0,\omega)\end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_n \\ I_n\end{smallmatrix}\right]$ that

$$
\begin{aligned}
U_3(t,\omega) &= G_1(t,\omega)\, I_G(0,t,\omega) \\
U_4(t,\omega) &= G_2(t,\omega)\, I_G(0,t,\omega) + (G_1^T)^{-1}(t,\omega)
\end{aligned}
$$

for each $t \in \mathbb{R}$ and $\omega \in \Omega$, with $I_G(0,t,\omega)$ defined by (5.9). Lemma 5.3 ensures that $I_G(0,t,\omega)$ (and hence $U_3(t,\omega)$) is nonsingular whenever $|t| \ge t_0$, with t_0 provided by Proposition 5.1(i). As seen in Remark 5.2.2, this property is equivalent to the uniform weak disconjugacy at $+\infty$ and $-\infty$.

(ii) Fix $(\omega,l) \in \Omega \times \mathscr{L}_{\mathbb{R}}$, represent $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2\end{smallmatrix}\right]$, and choose any $\left[\begin{smallmatrix} L_3 \\ L_4\end{smallmatrix}\right] \equiv l_1 \in \mathscr{L}_{\mathbb{R}}$ such that $\left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2\end{smallmatrix}\right] \in \mathrm{Sp}(n,\mathbb{R})$ (for instance, $L_3 = L_2\,R^{-1}$ and $L_4 = -L_1\,R^{-1}$ for $R = L_1^T L_1 + L_2^T L_2$). Then $V(t,\omega) = U(t,\omega)\left[\begin{smallmatrix} L_3 & L_1 \\ L_4 & L_2\end{smallmatrix}\right] = \left[\begin{smallmatrix} L_3(t,\omega) & L_1(t,\omega) \\ L_4(t,\omega) & L_2(t,\omega)\end{smallmatrix}\right]$ is a symplectic matrix solution of (5.5). According to Remark 5.3.2, the uniform weak disconjugacy of the family on both half lines ensures that each one of the systems of the family (5.5) is nonoscillatory at $+\infty$ and at $-\infty$. This means that $\operatorname{Arg}_3 V(t,\omega)$ is bounded as $t \to \pm\infty$ for each $\omega \in \Omega$. On the other hand, according to Proposition 5.1(ii), the vector $\mathbf{z}_1(t,\omega)$ does not vanish identically on any positive or negative half line for any nonzero solution $\left[\begin{smallmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ of any of the systems. Under these conditions, and since D1 holds, it is possible to repeat step by step the arguments of Proposition 2.8 of [2] in order to show the existence of s_1 (depending on $(\omega,l) \in \Omega \times \mathscr{L}_{\mathbb{R}}$) with $\det L_1(t,\omega) \neq 0$ for $|t| > s_1$.

□

Remark 5.5. Suppose that $0 < A \le B$ for two symmetric $n \times n$ matrix-valued functions. Then $I_n \le A^{-1/2}BA^{-1/2}$ and hence $I_n \le B^{1/2}A^{-1}B^{1/2}$, since both right-hand terms have the same eigenvalues. Therefore, $0 < B^{-1} \le A^{-1}$. Clearly, there is an analogous result if the inequality is strict.

Theorem 5.4. *Suppose that* D1 *holds. Then the family* (5.5) *is uniformly weakly disconjugate on* $[0,\infty)$ *if and only if it is uniformly weakly disconjugate on* $(-\infty,0]$*. If this is the case, then the system* (5.5) *possesses uniform principal solutions at*

$\pm\infty$ *for each* $\omega \in \Omega$, $\begin{bmatrix} L_1^{\pm}(t,\omega) \\ L_2^{\pm}(t,\omega) \end{bmatrix}$, *and conditions* D2 *and* D3 *hold. In addition, the principal solutions are unique as matrix-valued functions taking values in* $\mathscr{L}_{\mathbb{R}}$. *Finally, if* $\tilde{l}^{\pm}(\omega)$ *are the real Lagrange planes represented by* $\begin{bmatrix} L_1^{\pm}(0,\omega) \\ L_2^{\pm}(0,\omega) \end{bmatrix}$, *then* $\tilde{l}^{\pm}(\omega{\cdot}t) = U(t,\omega){\cdot}\tilde{l}^{\pm}(\omega)$ *for any* $t \in \mathbb{R}$ *and* $\omega \in \Omega$.

Proof. The proof is carried out according to the following sketch. Assuming first the uniform weak disconjugacy on $[0,\infty)$, one proves the existence of a uniform principal solution at $+\infty$ with the stated properties. Consequently, it follows from Remark 5.3.1 and Definition 5.5 that the family (5.5) satisfies D2 and D3. Therefore, Theorem 5.3(i) ensures the uniform weak disconjugacy on $(-\infty,0]$. Some indications about how to adapt the first steps in order to ensure the existence of principal solution at $-\infty$ complete the proof. This method of proof can be repeated taking the uniform weak disconjugacy on $(-\infty,0]$ as starting point.

We assume from now on the uniform weak disconjugacy on $[0,\infty)$. Recall that then D2 holds, as explained in Remark 5.3.1. Let $t_0 > 0$ satisfy $\det U_3(t,\omega) \neq 0$ for $t \geq t_0$ and $\omega \in \Omega$ (see Remark 5.2.2) and condition in Lemma 5.3. Let us consider the $2n \times n$ matrix-valued function $G(t,\omega) = \begin{bmatrix} U_3(t,\omega) \\ U_4(t,\omega) \end{bmatrix}$, which represents a real Lagrange plane for any $t \in \mathbb{R}$, solves (5.5), and takes values in $\mathscr{D}$ for $t \geq t_0$. We represent $I(t,\omega) = I_G(t_0,t,\omega)$, this last matrix given by (5.9), and note that, by D1, $I(t,\omega)$ is nondecreasing in t. Lemma 5.3 ensures that $I(t,\omega)$ is positive definite for each $t \geq 2t_0$. Hence, $(I(t,\omega))^{-1}$ is positive definite for these values of t and nonincreasing in t (see Remark 5.5). Therefore, there exists the limit

$$J_+(\omega) = \lim_{t\to\infty} (I(t,\omega))^{-1}\,.$$

The next goal is to prove that $I_n - I(t,\omega)\,J_+(\omega)$ is nonsingular if $t \geq t_0$. Consider first the case $t \geq 2t_0$. By Lemma 5.3, $0 < I(t,\omega) < I(t+t_0,\omega)$, so that $J_+(\omega) < (I(t,\omega))^{-1}$ for $t \geq 2t_0$. Hence the matrix $I_n - I(t,\omega)\,J_+(\omega)$, whose eigenvalues agree with those of $I^{1/2}(t,\omega)\,\big((I(t,\omega))^{-1} - J_+(\omega)\big)\,I^{1/2}(t,\omega) > 0$, is nonsingular for each $t \geq 2t_0$. Now take $t \in [t_0,2t_0]$ and $s \geq 2t_0$, and note that the eigenvalues of the matrices $I_n - I(t,\omega)\,I^{-1}(s,\omega)$ and $I_n - I^{-1/2}(s,\omega)\,I(t,\omega)\,I^{-1/2}(s,\omega)$ agree. Taking limits as $s \to \infty$ one sees that the set of eigenvalues of the matrix $I_n - I(t,\omega)\,J_+(\omega)$ agrees with that of $I_n - J_+^{1/2}(\omega)\,I(t,\omega)\,J_+^{1/2}(\omega)$ (see, e.g., Theorem II.5.1 of [13]). Thus, the assertion is proved once it has been checked that the eigenvalues of this last matrix are strictly positive if $t_0 \leq t \leq 2t_0$, which in turn follows from

$$I_n - J_+^{1/2}(\omega)\,I(t,\omega)\,J_+^{1/2}(\omega) \geq I_n - J_+^{1/2}(\omega)\,I(2t_0,\omega)\,J_+^{1/2}(\omega):$$

the eigenvalues of the matrix in the right-hand term agree with those of the matrix $I_n - I(2t_0,\omega)\,J_+(\omega)$, which, as already seen, are strictly positive.

According to Remark 5.4, the $2n \times n$ matrix-valued function $L^+(t,\omega)$ given by

$$\begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix} = \begin{bmatrix} U_3(t,\omega)\,(I_n - I(t,\omega)\,J_+(\omega)) \\ U_4(t,\omega)\,(I_n - I(t,\omega)\,J_+(\omega)) - (U_3^T)^{-1}(t,\omega)\,J_+(\omega) \end{bmatrix} \tag{5.13}$$

solves (5.5) in $[t_0,\infty)$ and takes values in $\mathscr{L}_\mathbb{R}$. It has been just checked that in fact it takes values in $\mathscr{D}$ for $t \ge t_0$. Hence, by (5.12), if $I_{L^+}(t_0,t,\omega)$ is defined from L^+ by (5.9), then $(I_{L^+}(t_0,t,\omega))^{-1} = (I(t,\omega))^{-1} - J_+(\omega)$ if $t \ge 2t_0$, so that

$$\lim_{t\to\infty} (I_{L^+}(t_0,t,\omega))^{-1} = 0_n \tag{5.14}$$

The same symbol $\left[\begin{smallmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{smallmatrix}\right]$ will denote the extension of the solution given on $[t_0,\infty)$ to the whole real line. Suppose now that $t \to \left[\begin{smallmatrix} \bar L_1(t,\omega) \\ \bar L_2(t,\omega) \end{smallmatrix}\right]$ is any $2n \times n$ matrix solution of (5.5) which takes values in $\mathscr{D}$ for $t \in [t_0,\infty)$ and satisfies $\lim_{t\to\infty}(I_{\bar L}(t_0,t,\omega))^{-1} = 0_n$. By Remark 5.4, $\left[\begin{smallmatrix} \bar L_1(t,\omega) \\ \bar L_2(t,\omega) \end{smallmatrix}\right]$ can be defined from $\left[\begin{smallmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{smallmatrix}\right]$ for $t \in [t_0,\infty)$ by expression (5.11) for suitable functions $\bar P(\omega)$ and $\bar Q(\omega)$. The matrix $\bar P(\omega)$ is invertible and, by (5.12),

$$\begin{aligned} 0_n &= \lim_{t\to\infty} (I_{\bar L}(t_0,t,\omega))^{-1} \\ &= \lim_{t\to\infty} \bar P^T(\omega)\,((I_{L^+}(t_0,t,\omega))^{-1}\bar P(\omega) + \bar Q(\omega)) = \bar P^T(\omega)\,\bar Q(\omega)\,, \end{aligned}$$

so that $\bar Q(\omega) = 0_n$. Hence $\left[\begin{smallmatrix} \bar L_1(t,\omega) \\ \bar L_2(t,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1^+(t,\omega)\,\bar P(\omega) \\ L_2^+(t,\omega)\,\bar P(\omega) \end{smallmatrix}\right]$ for each $t \ge t_0$. By uniqueness of solutions, the same equality holds for any $t \in \mathbb{R}$. That is, in terms of the matrix representation of Lagrange planes,

$$\begin{bmatrix} \bar L_1(t,\omega) \\ \bar L_2(t,\omega) \end{bmatrix} \equiv \begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix} \tag{5.15}$$

for any $t \in \mathbb{R}$.

The next goal is to check that

$$L^+(t+r,\omega) \equiv L^+(t,\omega{\cdot}r) \qquad \text{for any } \omega \in \Omega,\ r \in \mathbb{R} \text{ and } t \in \mathbb{R}\,; \tag{5.16}$$

i.e., they represent the same Lagrange plane. Note that $t \mapsto L^+(t+r,\omega)$ and $t \mapsto L^+(t,\omega{\cdot}r)$ solve the system corresponding to $\omega{\cdot}r$. Assume first that $r \ge 0$, so that $L^+(t+r,\omega)$ belongs to $\mathscr{D}$ for any $t \ge t_0$. Then

$$\begin{aligned} &\lim_{t\to\infty} \left(\int_{t_0}^{t} (L_1^+)^{-1}(s+r,\omega)\, H_3((\omega{\cdot}r){\cdot}s)\, ((L_1^+)^T)^{-1}(s+r,\omega)\, \mathrm{d}s \right)^{-1} \\ &\quad = \lim_{t\to\infty} \left(\int_{t_0+r}^{t+r} (L_1^+)^{-1}(s,\omega)\, H_3(\omega{\cdot}s)\, ((L_1^+)^T)^{-1}(s,\omega)\, \mathrm{d}s \right)^{-1} = 0_n\,, \end{aligned}$$

as deduced from (5.14) and the last assertion of Remark 5.4. Hence, (5.15) implies (5.16) for $r \geq 0$. This in turn implies that, for $r \geq 0$,

$$L^+(t, \omega{\cdot}(-r)) \equiv L^+(t - r + r, \omega{\cdot}(-r)) \equiv L^+(t - r, \omega),$$

which completes the proof of (5.16). Hence, $L^+(t, \omega) \equiv L^+(t_0, \omega{\cdot}(t - t_0))$ belongs to $\mathscr{D}$ for any $t \in \mathbb{R}$ and any $\omega \in \Omega$.

The assertions concerning the uniform principal solution at $+\infty$ can be now explained. First, $L^+(t, \omega)$ always takes values in $\mathscr{D}$, so that relation (5.14) and a new application of the last assertion of Remark 5.4, ensure that $L^+(t, \omega)$ is a uniform principal solution at $+\infty$. Second, relation (5.15) shows that it is unique when considered as a function taking values in $\mathscr{L}_{\mathbb{R}}$. And third, (5.16) yields $U(r, \omega)L^+(0, \omega) = L^+(r, \omega) \equiv L^+(0, \omega{\cdot}r)$, so that if we call $l^+(\omega) \equiv \begin{bmatrix} L_1^+(0,\omega) \\ L_2^+(0,\omega) \end{bmatrix}$, then $U(r, \omega){\cdot}l^+(\omega) = l^+(\omega{\cdot}r)$.

As said at the beginning of the proof, the uniform weak disconjugacy on $(-\infty, 0]$ holds. To deal now with the existence, uniqueness, and invariance of the principal solution at $-\infty$, take $t_0 > 0$ satisfying Lemma 5.3 and $\det U_3(t, \omega) \neq 0$ for $t \leq -t_0$, call as before $G(t, \omega) = \begin{bmatrix} U_3(t,\omega) \\ U_4(t,\omega) \end{bmatrix}$, and define $\tilde{I}(t, \omega) = I_G(-t_0, t, \omega)$ for $t \leq -t_0$. This last matrix is negative definite for $t \leq -2t_0$ and decreases as t decreases, so that $(\tilde{I}(t, \omega))^{-1}$ is negative definite and increases as $t \to -\infty$. Hence, there exists $J_-(\omega) = \lim_{t\to-\infty}(\tilde{I}(t, \omega))^{-1}$. Changing I to $\tilde{I}$ and J_+ to J_- in (5.13) provides the definition of $\begin{bmatrix} L_1^-(t,\omega) \\ L_2^-(t,\omega) \end{bmatrix}$, which will now play the role before played by $\begin{bmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{bmatrix}$. The rest of the proof is identical with that dealing with the principal solution at $+\infty$. □

The proof of Theorem 5.2 is hence complete. More information concerning the existence of (perhaps nonuniform) principal solutions for the systems of the family (5.5) under less restrictive hypothesis will be given at the end of the next section.

5.3 Disconjugacy, Uniform Weak Disconjugacy, and Weak Disconjugacy

We consider three different sets of hypotheses for the family (5.5):

A. all the systems of the family are disconjugate;
B. the family is uniformly weakly disconjugate (on $(-\infty, 0]$ and on $[0, \infty)$);
C. all the systems of the family are weakly disconjugate on $(-\infty, 0]$ or on $[0, \infty)$.

Then,

– A implies B and B implies C: see Remark 5.2.1;
– even when D1 holds, B does not imply A: see Example 5.1 below;
– even when $H_3 > 0$, C does not imply B (or A): see Example 5.2 below.

To analyze the situations in which these properties (or at least two of them) hold simultaneously is the first goal of this section. Some properties concerning nonoscillation, interesting by themselves, will be used in the analysis. The second goal, closely related, is to establish conditions on a single system ensuring the uniform weak disconjugacy of the family of systems defined on its hull. The characterization of uniform weak disconjugacy provided by Theorem 5.2 will be fundamental in all the results.

As stated in the introduction, the great advantage of uniform weak disconjugacy, as compared with the classical notion of disconjugacy, is that it holds under a much weaker version of the condition of identical normality, not required for B to hold (see Example 5.1). Recall that the system (5.5) corresponding to ω is *identically normal* on $\mathbb{R}$ if, for any nonzero solution $\left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$, the vector $\mathbf{z}_1(t)$ does not vanish identically on any nondegenerate interval. So that it is clear that condition D2 is weaker than the simultaneous identical normality on $\mathbb{R}$ of all the systems of the family. Clearly, a disconjugate system is identically normal. Something more can be said in the case that D1 holds: the next result is proved in Chap. 2.1 of [1]. We point out that Theorem 5.2 can be understood as its extension to our less restrictive setting.

Proposition 5.2. *Suppose that $H_3(\omega{\cdot}t) \geq 0$ for a point $\omega \in \Omega$ and every $t \in \mathbb{R}$. Then the corresponding system (5.5) is disconjugate on $\mathbb{R}$ if and only it is identically normal on $\mathbb{R}$ and it admits a $2n \times n$ matrix solution $\left[\begin{smallmatrix} G_1(t) \\ G_2(t) \end{smallmatrix}\right]$ taking values in $\mathscr{D}$ for any $t \in \mathbb{R}$. In this case, there exist uniform principal solutions at $+\infty$ and $-\infty$, which are unique as functions taking values in $\mathscr{L}_{\mathbb{R}}$.*

Remark 5.6. It is almost immediate that, if $H_3 > 0$, all the systems of the family are identically normal: if $\mathbf{z}_1(t) = \mathbf{0}$ for a nonzero solution, then $\mathbf{z}_1'(t) = H_3(\omega{\cdot}t)\,\mathbf{z}_2(t) \neq \mathbf{0}$. So that D1 and D2 hold. Thus, Theorem 5.2 and Proposition 5.2 ensure that, if $H_3 > 0$, properties A and B are equivalent (and they are also equivalent to the fact that conditions D2 and D3 hold). (Incidentally, note that this is the situation when the family of Hamiltonian systems (5.5) comes from a family of Schrödinger equations $-\mathbf{x}'' + G(\omega{\cdot}t)\,\mathbf{x} = \mathbf{0}$ by taking $\mathbf{z} = \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix}\right]$, since in this case $H = \left[\begin{smallmatrix} 0_n & I_n \\ G & 0_n \end{smallmatrix}\right]$.) The conclusion is that the main contribution of the theory of uniform weak disconjugacy concerns the situations in which $H_3 \geq 0$ but it is not positive definite.

Under condition D1, Theorem 5.6 below, whose proof is a consequence of the next theorem, describes a situation at which B and C hold or not simultaneously: this happens when the base flow has a dense semiorbit. To understand its scope, recall that the existence of positive and negative semiorbits which are dense in Ω holds when the base flow is minimal, since Ω must be contained in the omega-limit set of any orbit, or in the (more general) case of existence of a σ-ergodic measure with total support Ω, as proved in Proposition 1.10 of Johnson and Nerurkar [7].

Theorem 5.5. *Let $\mathscr{O}$ and $\mathscr{A}$ be the omega-limit set and alpha-limit set of $\omega_0 \in \Omega$. Then,*

(i) *if the system (5.5) corresponding to ω_0 is nonoscillatory at $+\infty$, then all the systems corresponding to elements of $\{\omega_0{\cdot}t \mid t \in \mathbb{R}\} \cup \mathscr{O}$ are nonoscillatory at $+\infty$, and those corresponding to $\mathscr{O}$ are nonoscillatory at $-\infty$;*
(ii) *if the system (5.5) corresponding to ω_0 is nonoscillatory at $-\infty$, then all the systems corresponding to elements of $\{\omega_0{\cdot}t \mid t \in \mathbb{R}\} \cup \mathscr{A}$ are nonoscillatory at $-\infty$, and those corresponding to $\mathscr{A}$ are nonoscillatory at $+\infty$;*
(iii) *if $H_3(\omega_0{\cdot}t) \geq 0$ for any $t \geq 0$ and all the systems (5.5) corresponding to elements of $\{\omega_0\} \cup \mathscr{O}$ are weakly disconjugate on $[0,\infty)$, then the family restricted to $\mathscr{O}$ is uniformly weakly disconjugate;*
(iv) *if $H_3(\omega_0{\cdot}t) \geq 0$ for any $t \leq 0$ and all the systems (5.5) corresponding to elements of $\{\omega_0\} \cup \mathscr{A}$ are weakly disconjugate on $(-\infty,0]$, then the family restricted to $\mathscr{A}$ is uniformly weakly disconjugate.*

Proof.

(i) Let $V(t,\omega) = \left[\begin{smallmatrix} V_1(t,\omega) & V_3(t,\omega) \\ V_2(t,\omega) & V_4(t,\omega)\end{smallmatrix}\right]$ be a symplectic matrix solution of (5.5). Note that for each $s \in \mathbb{R}$ the system (5.5) corresponding to $\omega_0{\cdot}s$ is nonoscillatory at $+\infty$ because $V(t+s,\omega_0)$ is a fundamental matrix solution for this system. Now define $\operatorname{Arg}_1 V(t,\omega) = \arg\det(V_1(t,\omega) - \mathrm{i}V_2(t,\omega))$, which is an argument on $\mathrm{Sp}(n,\mathbb{R})$ equivalent to Arg_3 (see [20]). As in Proposition 2.2 of [16]

$$\int_0^t \operatorname{Tr} Q(\tau(s,\omega,l))\,\mathrm{d}s = \operatorname{Arg}_1 V(t,\omega) - \operatorname{Arg}_1 V(0,\omega)\,,$$

where $l \equiv \left[\begin{smallmatrix} V_1(0,\omega) \\ V_2(0,\omega)\end{smallmatrix}\right]$ and $\operatorname{Tr} Q(\omega,\tilde{l}) = \operatorname{tr}\left(\left[\Phi_1^T\ \Phi_2^T\right] JH(\omega)\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}\right)$ for any representation $\left[\begin{smallmatrix}\Phi_1\\ \Phi_2\end{smallmatrix}\right]$ of $\tilde{l}$ with $\Phi_1 + \mathrm{i}\Phi_2$ unitary. That is, the nonoscillation at $+\infty$ (resp. at $-\infty$) of the system corresponding to ω is equivalent to the existence of $l_\omega \in \mathscr{L}_{\mathbb{R}}$ and $c_{\omega,l_\omega} > 0$ such that $\left|\int_0^t \operatorname{Tr} Q(\tau(s,\omega,l_\omega))\,\mathrm{d}s\right| \leq c_{\omega,l_\omega}$ for any $t \geq 0$ (resp. for any $t \leq 0$). Suppose that this is the case for the point ω_0 and $t \geq 0$. Then

$$\begin{aligned}\left|\int_0^t \operatorname{Tr} Q(\tau(s+r,\omega_0,l_{\omega_0}))\,\mathrm{d}s\right| &= \left|\int_r^{t+r} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,\mathrm{d}s\right| \\ &\leq \left|\int_0^r \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,\mathrm{d}s\right| + \left|\int_0^{t+r} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,\mathrm{d}s\right| \leq 2c_{\omega_0,l_{\omega_0}}\end{aligned}$$

for any $r \geq 0$. Now, given $\omega_1 \in \mathscr{O}$, look for a sequence $(t_m) \uparrow \infty$ such that there exists $(\omega_1,l_1) = \lim_{m\to\infty} \tau(t_m,\omega_0,l_{\omega_0})$. Then, if $t \geq 0$,

$$\left|\int_0^t \operatorname{Tr} Q(\tau(s,\omega_1,l_1))\,\mathrm{d}s\right| = \lim_{m\to\infty}\left|\int_0^t \operatorname{Tr} Q(\tau(s+t_m,\omega_0,l_{\omega_0}))\,\mathrm{d}s\right| \leq 2c_{\omega_0,l_{\omega_0}}\,,$$

and hence the system corresponding to ω_1 is nonoscillatory at $+\infty$. Analogously,

$$\left|\int_{-t}^{0} \operatorname{Tr} Q(\tau(s,\omega_1,l_1))\,\mathrm{d}s\right| = \lim_{m\to\infty}\left|\int_{t_m-t}^{t_m} \operatorname{Tr} Q(\tau(s,\omega_0,l_{\omega_0}))\,\mathrm{d}s\right| \le 2c_{\omega_0,l_{\omega_0}}$$

which shows the nonoscillation at $-\infty$ and completes the proof of (i).

(iii) The assumption $H_3(\omega_0{\cdot}t) \ge 0$ ensures condition D1 on $\tilde{\mathscr{O}} = \{\omega_0{\cdot}t \mid t \ge 0\} \cup \mathscr{O}$. In addition, the weak disconjugacy hypothesis guarantees D2$_\omega$ for any $\omega \in \mathscr{O}$ (see Remark 5.3.1). Moreover, Remark 5.3.2, assertion (i), and Lemma 5.2(i) guarantee that all the systems corresponding to points $\omega \in \tilde{\mathscr{O}}$ are nonoscillatory at $+\infty$ and satisfy D2$_\omega$ and hence that all of them are weakly disconjugate on $[0,\infty)$.

By Remark 5.2.2, the weak disconjugacy of (5.5) on $[0,\infty)$ for each $\omega \in \tilde{\mathscr{O}}$ provides $t_\omega \ge 0$ with $\det U_3(t,\omega) \ne 0$ for each $t > t_\omega$. Choosing t_ω as the smallest one with this property, we have $\det U_3(t_\omega,\omega) = 0$. Take $r > t_\omega$ and consider

$$\begin{bmatrix} Z_1(t,\omega) \\ Z_2(t,\omega) \end{bmatrix} = \begin{bmatrix} U_3(t-r,\omega{\cdot}r) \\ U_4(t-r,\omega{\cdot}r) \end{bmatrix},$$

which is a matrix solution of (5.5) taking values in $\mathscr{L}_{\mathbb{R}}$ and satisfying $\left[\begin{smallmatrix} Z_1(r,\omega) \\ Z_2(r,\omega) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$. Remark 5.4 yields

$$\begin{aligned} Z_1(t,\omega) &= U_3(t,\omega)\left(\int_r^t U_3^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(U_3^T)^{-1}(s,\omega)\,\mathrm{d}s\right)U_3^T(r,\omega)\,, \\ Z_2(t,\omega) &= U_4(t,\omega)\left(\int_r^t U_3^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(U_3^T)^{-1}(s,\omega)\,\mathrm{d}s\right)U_3^T(r,\omega) \\ &\quad + (U_3^T)^{-1}(t,\omega)\,U_3^T(r,\omega) \end{aligned}$$

for each $t \ge r$. Assume for now that there exists $t_0 > 0$, common for any $\omega \in \tilde{\mathscr{O}}$, such that $\det Z_1(t,\omega) \ne 0$ for each $t \ge r + t_0$ or, equivalently, such that

$$\int_r^t U_3^{-1}(s,\omega)\,H_3(\omega{\cdot}s)\,(U_3^T)^{-1}(s,\omega)\,\mathrm{d}s > 0 \tag{5.17}$$

for each $t \ge r + t_0$ and any $\omega \in \tilde{\mathscr{O}}$. Then $\det U_3(t,\omega{\cdot}r) = \det Z_1(t+r,\omega) \ne 0$ for each $t \ge t_0$, which implies that $t_{\omega\cdot r} < t_0$ if $r > t_\omega$. This property will be fundamental to complete the proof. In order to show the existence of this t_0, note that, since $\tilde{\mathscr{O}}$ is compact, the arguments of Proposition 5.1(i) can be repeated to obtain $t_0 > 0$ and $\delta > 0$ such that (5.7) holds for any $\omega \in \tilde{\mathscr{O}}$ and $\mathbf{x} \in \mathbb{R}^n$. Then, reasoning as in Proposition 5.1(ii), one shows that none of the systems (5.5) corresponding to elements of the positively invariant set $\tilde{\mathscr{O}}$ admits a solution taking the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ in an interval of length $[r, r+t_0]$ for any $r \geq 0$. And, in turn, this property allows us to repeat the proof of Lemma 5.3 in order to check (5.17).

Note that statement (iii) is equivalent to the upper-boundedness of the set $\{t_\omega \mid \omega \in \mathscr{O}\}$. Suppose for contradiction the existence of a sequence $(\omega_m)_{m\in\mathbb{N}}$ in $\mathscr{O}$ with $\lim_{m\to\infty} t_{\omega_m} = \infty$. Recall that $\det U_3(t_{\omega_m}, \omega_m) = 0$ and note that there is no restriction in assuming that $t_{\omega_m} > t_0$ for every $m \in \mathbb{N}$. In addition, there exist m_0 and $t_1 \in (t_0, t_{\omega_{m_0}})$ with $\det U_3(t_1, \omega_{m_0}) \neq 0$; otherwise, one would have $\det U_3(t, \omega_m) = 0$ for each $t \in (t_0, t_{\omega_m}]$, so that the continuity of $U_3(t,\omega)$ in ω would ensure that $\det U_3(t, \tilde{\omega}) = 0$ for each $t > t_0$ for any accumulation point $\tilde{\omega} \in \mathscr{O}$ of $(\omega_m)_{m\in\mathbb{N}}$ (and there exists at least one, since $\mathscr{O}$ is compact), but this is impossible by the weak disconjugacy of the system of the family (5.5) corresponding to $\tilde{\omega}$ (see Remark 5.2.2).

According to Theorem II.5.2 of [13], it is possible to choose continuous functions $\rho_1, \ldots, \rho_n \colon \mathbb{R} \to \mathbb{C}$ such that the set of eigenvalues of $W_U(t, \omega_{m_0})$, with $W_U(t,\omega) = (U_1(t,\omega) - \mathrm{i}U_3(t,\omega))^{-1}(U_1(t,\omega) + i\,U_3(t,\omega))$, coincides with the unordered n-uple $\{\rho_1(t), \ldots, \rho_n(t)\}$, which may have repeated elements. In addition, according to Lemma 5.1(i), these functions have modulus 1. Let $\varphi_1 \ldots, \varphi_n \colon \mathbb{R} \to \mathbb{R}$ be continuous branches for their arguments: $e^{\mathrm{i}\varphi_j(t)} = \rho_j(t)$ for $j = 1, \ldots, n$ and $t \in \mathbb{R}$. According to Theorem 5.1, φ_j is nondecreasing for $j = 1, \ldots, n$. It follows from Lemma 5.1(ii) that $\det U_3(t, \omega_{m_0}) = 0$ if and only if there is $j \in \{1, \ldots, n\}$ such that $\varphi_j(t) = 2m_j\pi$ for some $m_j \in \mathbb{Z}$. Since $\det U_3(t_1, \omega_{m_0}) \neq 0$, the arguments can be chosen so that $\varphi_j(t_1) \in (-2\pi, 0)$ for $j = 1, \ldots, n$. Since $\det U_3(t_{\omega_{m_0}}, \omega_{m_0}) = 0$, there is $l \in \{1, \ldots, n\}$ and an integer $n_l \geq 0$ with $\varphi_l(t_{\omega_{m_0}}) = 2n_l\pi$. And since $\det U_3(t, \omega_{m_0}) \neq 0$ for any $t > t_{\omega_{m_0}}$, then $\varphi_l(t_2) \in (2n_l\pi, 2(n_l+1)\pi)$ for any $t_2 > t_{\omega_{m_0}}$. Fix such a value t_2.

The definition of $\mathscr{O}$ provides a sequence $(s_k) \uparrow \infty$ with $\lim_{k\to\infty} \omega_0{\cdot}s_k = \omega_{m_0}$. Theorem II.5.1 of [13] ensures that the unordered set $\mathscr{E}(t,\omega)$ of the eigenvalues of the jointly continuous matrix-valued function $W_U(t,\omega)$ varies continuously in (t,ω) in the Hausdorff sense. Therefore, by choosing k large enough, all the elements of $\mathscr{A}(t_1, \omega_0{\cdot}s_k)$ belong to $\{e^{\mathrm{i}\varphi} \mid \varphi \in (-2\pi, 0)\}$, while at least an element of $\mathscr{E}(t_2, \omega_0{\cdot}s_k)$ belongs to $\{e^{\mathrm{i}\varphi} \mid \varphi \in (2n_l\pi, 2(n_l+1)\pi)\}$. For later purposes, choose such a value of k which in addition satisfies $s_k > t_{\omega_0}$. Recall that $n_l \geq 0$. Theorem II.5.2 of [13] provides continuous functions $\bar{\rho}_1, \ldots, \bar{\rho}_n \colon \mathbb{R} \to \mathbb{C}$ such that $\mathscr{E}(t, \omega_0{\cdot}s_k) = \{\bar{\rho}_1(t), \ldots, \bar{\rho}_n(t)\}$ for $t \in [t_1, t_2]$. It follows easily the existence of $\tilde{t} \in (t_1, t_2)$ such that $1 \in \mathscr{E}(\tilde{t}, \omega_0{\cdot}s_k)$. Lemma 5.1(ii) shows that $\det U_3(\tilde{t}, \omega_0{\cdot}s_k) = 0$, so that $t_{\omega_0\cdot s_k} > t_1$. However, as checked at the beginning of the proof, $t_{\omega_0\cdot s_k} < t_0 < t_1$, since $s_k > t_{\omega_0}$. This is the sought-for contradiction, which completes the proof of(iii).

The proofs of (ii) and (iv) are analogous to those of points (i) and (iii). □

Theorem 5.6. *Suppose that* D1 *holds and that there exists a positive (resp. negative) σ-semiorbit dense in Ω. Then, all the systems* (5.5) *are weakly disconjugate on* $[0,\infty)$ *(resp. on* $(-\infty, 0]$*) if and only if the family is uniformly weakly disconjugate.*

Example 5.2 shows the optimality of this result, in the sense that, even if $H_3 > 0$ (so that D1 and D2 hold: see Remark 5.6) and, at the same time, all the systems are simultaneously weakly disconjugate both on $(-\infty, 0]$ and on $[0, \infty)$, then the existence of a dense orbit (instead of semiorbit) does not suffice to guarantee the uniform weak disconjugacy of the family.

The next result presents situations of equivalence of A, B, and C.

Proposition 5.3. *Suppose that* D1 *holds and that every system of the family* (5.5) *is identically normal. Then,*

(i) *the family* (5.5) *is uniformly weakly disconjugate if and only if all its systems are disconjugate;*
(ii) *if there exists a positive (or negative) semiorbit dense in Ω, then all the systems of* (5.5) *are weakly disconjugate on* $(-\infty, 0]$ *(or on* $(0, \infty]$*) if and only if all of them are disconjugate.*

Proof. As stated in Remark 5.2.1, disconjugacy implies uniform weak disconjugacy. The converse property of (i) follows from Theorem 5.2 and Proposition 5.2. The converse property of (ii) follows from Remark 5.3.1 and Theorem 5.6. □

Much more can be said in the case of a minimal base. Theorem 5.6 and Lemma 5.2 play a fundamental role in the proof of the next result.

Proposition 5.4. *Suppose that* D1 *holds and that Ω is minimal. Then the family* (5.5) *is uniformly weakly disconjugate if and only if there exists a point ω_0 such that the corresponding system* (5.5) *is weakly disconjugate on* $[0, \infty)$ *or on* $(-\infty, 0]$.

Proof. The "only if" assertion is trivial. Suppose that the system corresponding to a point $\omega_0 \in \Omega$ is weakly disconjugate on $[0, \infty)$. Remarks 5.2.2 and 5.3.3 ensure that it is nonoscillatory at $+\infty$, so that, by Theorem 5.5(i), all the systems of the family are. In addition, $D2_{\omega_0}$ holds (see Remark 5.3.1), so that, by Lemma 5.2, D2 holds. As explained in Remark 5.3.2, all the systems of the family are weakly disconjugate on $[0, \infty)$, and hence Theorem 5.6 proves the result. The proof is the same taking the weak disconjugacy of a system on $(-\infty, 0]$ as starting point. □

Something more can be said about the relations between properties B and C when D1 and D2 hold. These additional results are based on the properties of the rotation number of the family (5.5) associated to each σ-ergodic measure m, $\alpha(m)$: see [16] and [3]. Namely, if m-almost every system is weakly disconjugate for a σ-ergodic measure m on Ω, then $\alpha(m) = 0$. And conversely, if there exists an ergodic measure m_0 with total support with $\alpha(m_0) = 0$, then the family is uniformly weakly disconjugate. The interested reader can find in [5] the details of the proofs of some of these assertions.

In particular, the last result shows that if a particular linear Hamiltonian system is weakly disconjugate on a half line and given by a recurrent coefficient matrix $H_0(t)$ with $H_{03} \geq 0$, then the family constructed on its hull is uniformly weakly disconjugate. However, recurrence is a strong condition. The next result, which improves Proposition 3.6 of [2], establishes hypotheses substituting it and providing the same conclusion. And Proposition 5.6 combines both results to optimize the information in case of recurrence.

Proposition 5.5. *Suppose that the orbit of ω_0 is dense in Ω and that*

1. $H_3(\omega_0{\cdot}t) \geq 0$ *for all* $t \in \mathbb{R}$ *(i.e.,* $\mathrm{D1}_{\omega_0}$ *holds);*
2. *for each nonzero vector* $\mathbf{z}_2 \in \mathbb{R}^n$ *there exist numbers* $t_0 > 0$ *and* $\delta_0 > 0$ *(depending on* $\mathbf{z}_2$*) such that, if* $s \in \mathbb{R}$ *and* $\left[\begin{smallmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{smallmatrix}\right] = U(t, \omega_0{\cdot}s)\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$*, then there is* $t_s \in [0, t_0]$ *with* $\|\mathbf{z}_1(t_s)\| \geq \delta_0$*;*
3. *there exists a* $2n \times n$ *matrix solution* $G(t, \omega_0) = \left[\begin{smallmatrix} G_1(t,\omega_0) \\ G_2(t,\omega_0) \end{smallmatrix}\right]$ *of the system* (5.5) *corresponding to* ω_0 *taking values in* $\mathscr{D}$.

Then the family (5.5) *is uniformly weakly disconjugate.*

Proof. It is clear that D1 holds. To show the same for D2, as in Lemma 5.1(i), we suppose for contradiction the existence of $\omega \in \Omega$ and $\mathbf{z}_2 \neq \mathbf{0}$ such that $\left[\begin{smallmatrix} \mathbf{z}_1(t,\omega) \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right] = U(t, \omega)\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$ satisfies $\mathbf{z}_1(t, \omega) = \mathbf{0}$ for each $t \geq 0$. Let t_0 and δ_0 be the constants of hypothesis 2 for $\mathbf{z}_2$. Find a sequence (t_m) with $\omega = \lim \omega_0{\cdot}t_m$, and write $\left[\begin{smallmatrix} \mathbf{z}_{1,m}(t) \\ \mathbf{z}_{2,m}(t) \end{smallmatrix}\right] = U(t, \omega_0{\cdot}t_m)\left[\begin{smallmatrix} 0 \\ \mathbf{z}_2 \end{smallmatrix}\right]$. Then $\mathbf{z}_1(t, \omega) = \lim_{m\to\infty} \mathbf{z}_{1,m}(t)$ uniformly on $[0, t_0]$. However, for each m there is an $s_m \in [0, t_0]$ such that $\|\mathbf{z}_{1,m}(s_m)\| \geq \delta_0$, and the contradiction is easily reached.

Now represent by $\mathscr{A}$ and $\mathscr{O}$ the alpha-limit and omega-limit sets of ω_0, and note that $\Omega = \mathscr{A} \cup \{\omega_0{\cdot}t \mid t \in \mathbb{R}\} \cup \mathscr{O}$. And recall that D3 holds globally if and only if $\mathrm{D3}_\omega$ holds for all $\omega \in \Omega$.

According to Remark 5.3.3, Hypothesis 3 ensures that the system corresponding to ω_0 is nonoscillatory at $+\infty$ and at $-\infty$. By Theorem 5.5(i), all the systems corresponding to points of $\mathscr{O}$ are nonoscillatory at $+\infty$, which according to Remark 5.3.2 ensures that all of them are weakly disconjugate on $[0, \infty)$. Hence, Theorem 5.5(iii) and Theorem 5.2 ensure that $\mathrm{D3}_\omega$ holds for all $\omega \in \mathscr{O}$. Analogous arguments show that it holds for all $\omega \in \mathscr{A}$. Finally, if $s \in \mathbb{R}$, hypothesis 3 yields the solution $G(t + s, \omega_0)$ taking values in $\mathscr{D}$ of the system $\mathbf{z}' = H((\omega_0{\cdot}s){\cdot}t)\,\mathbf{z}$, so that $\mathrm{D3}_\omega$ also holds for any ω in the σ-orbit of ω_0. The proof is complete. □

Proposition 5.6. *Let Ω be minimal. Suppose that there exists $\omega_0 \in \Omega$ such that* $\mathrm{D1}_{\omega_0}$ *and* $\mathrm{D2}_{\omega_0}$ *hold and such that there exists a* $2n \times n$ *matrix solution* $G(t, \omega_0) = \left[\begin{smallmatrix} G_1(t,\omega_0) \\ G_2(t,\omega_0) \end{smallmatrix}\right]$ *of the system* (5.5) *corresponding to* ω_0 *taking values in* $\mathscr{D}$ *for any* t *in a positive or negative half line. Then the family* (5.5) *is uniformly weakly disconjugate.*

Proof. It is obvious that D1 holds, and Lemma 5.2(iii) ensures the same for D2. By Remark 5.3.3, the system corresponding to ω_0 is nonoscillatory at $+\infty$ or at $-\infty$; Remark 5.3.2 yields its weak disconjugacy on $[0, \infty)$ or on $(-\infty, 0]$; and Proposition 5.4 completes the proof. □

Our next purpose is to establish conditions on a particular system ensuring the existence of principal solutions. Note that the second hypothesis of the next result is exactly the weak disconjugacy on $[0, \infty)$ and that the third one, stronger than $\mathrm{D2}_\omega$, is rather weaker than the identical normality occurring in the case of disconjugacy (see Proposition 5.2). Example 5.2 below shows that the conclusion of the theorem is optimal, in the sense that the existence of a uniform principal solution cannot be ensured even in the identically normal case.

Theorem 5.7. *Suppose that the system corresponding to* $\omega_0 \in \Omega$ *satisfies* $\mathrm{D1}_{\omega_0}$ *and* $\det U_3(t,\omega_0) \neq 0$ *for any* $t \geq t_0$ *and that it admits no solution taking the form* $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ *on* $[t_1,\infty)$ *for any* $t_1 \geq t_0$. *Then it admits a principal solution on* $[t_0,\infty)$, *which is unique as a matrix-valued function taking values in* $\mathscr{L}_{\mathbb{R}}$.

Analogously, suppose that the system corresponding to $\omega_0 \in \Omega$ *satisfies* $\mathrm{D1}_{\omega_0}$ *and* $\det U_3(t,\omega_0) \neq 0$ *for any* $t \leq t_0$ *and that it admits no solution taking the form* $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ *on* $(-\infty,t_1]$ *for any* $t_1 \leq t_0$. *Then it admits a principal solution on* $(-\infty,t_0]$, *which is unique as a matrix-valued function taking values in* $\mathscr{L}_{\mathbb{R}}$.

Proof. As usual, the proofs of both assertions are symmetric, so that just the first one will be explained. Fix any $t_1 \geq t_0$. The arguments of Proposition 5.1(i) and (ii) provide $s(t_1) > 0$ and $\delta(t_1) > 0$ such that

$$\int_{t_1}^{t_1+s(t_1)} \|H_3(\omega_0{\cdot}t)\,(U_{H_1}^T)^{-1}(t,\omega_0)\,\mathbf{x}\|^2\mathrm{d}t \geq \delta(t_1)\,\|\mathbf{x}\|^2$$

for any $\mathbf{x} \in \mathbb{R}^n$ and show that there is no solution taking the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on $[t_1, t_1 + s(t_1)]$. The proof of Lemma 5.3 can be easily adapted to check that $I_G(t_1,t,\omega_0) > 0$ whenever $t \geq t_1 + s(t_1)$, where the maps $I_G(t_1,t,\omega_0)$ are defined for $t \geq t_1 \geq t_0$ from $G(t,\omega_0) = \left[\begin{smallmatrix}U_3(t,\omega_0)\\ U_4(t,\omega_0)\end{smallmatrix}\right]$ by (5.9). From here, the proof of Theorem 5.4 can be repeated until (5.15) is obtained, taking as starting point $I_G(t_0,t,\omega_0)$. The only point of difference is that the nonsingular character of $I_n - I_G(t_0,t,\omega_0)\,J_+(\omega_0)$ is proved first in the set $[t_0 + s(t_0),\infty)$ and then in $[t_0, t_0 + s(t_0)]$. □

Note that D1, D2, and the weak disconjugacy on $[0,\infty)$ of the system corresponding to ω_0 guarantee the hypotheses of the previous theorem and that under these conditions the family is uniformly weakly disconjugate if and only if the principal solution that it provides is uniform, as deduced from Theorem 5.2.

The next examples, previously announced, show the optimality of the results given in this section. The conclusion of the first one has already been mentioned:

– unless $H_3 > 0$, the uniform weak disconjugacy of the family is a condition less restrictive than the disconjugacy of all the systems, since it does not require the identical normality property.

In both examples, Ω is the closure of the orbit of a particular one of its elements. In the second one, $H_3 > 0$ and all the systems are identically normal and weakly disconjugate both on $(-\infty,0]$ and on $[0,\infty)$. Its conclusions, also anticipated, are:

– the weak disconjugacy of all the systems of the family guarantees neither the uniform weak disconjugacy nor the existence of uniform principal solutions, even in the case of identical normality;

– the additional conditions required in Theorem 5.6 and Proposition 5.3(ii) (existence of a dense *semi*orbit) and Proposition 5.5 (properties 2 and 3) are not superfluous.

Example 5.1. Let $a\colon \mathbb{R} \to \mathbb{R}$ be the bounded and uniformly continuous function defined by $a(t) = 0$ for $|t| \le 1$, $a(t) = |t| - 1$ for $1 \le |t| \le 2$, and $a(t) = 1$ for $|t| \ge 2$. Then $b(t) = \int_0^t a(s)\,\mathrm{d}s$ takes the value 0 on $[-1,1]$ and is strictly increasing outside that interval. Consider the two-dimensional Hamiltonian system $\mathbf{z}' = \left[\begin{smallmatrix} 0 & a(t) \\ 0 & 0 \end{smallmatrix}\right]\mathbf{z}$. It is easy to check that the hull Ω of the coefficient matrix is $\Omega = \left\{\left[\begin{smallmatrix} 0 & a_s(t) \\ 0 & 0 \end{smallmatrix}\right] \mid s \in \mathbb{R}\right\} \cup \left\{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right\}$, with $a_s(t) = a(t+s)$. We look for the solution of each one of the systems with initial data $\left[\begin{smallmatrix} 0 \\ \beta \end{smallmatrix}\right] \ne \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. For each $s \in \mathbb{R}$ one has $\left[\begin{smallmatrix} x_s(t) \\ y_s(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \beta(b(t+s)-b(s)) \\ \beta \end{smallmatrix}\right]$. Therefore, $x_s(t) \ne 0$ for $|t| > 2$. For the limiting system, one has $\left[\begin{smallmatrix} x_\infty(t) \\ y_\infty(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \beta t \\ \beta \end{smallmatrix}\right]$, and hence $x_\infty(t) \ne 0$ if $t \ne 0$. Therefore, the family is uniformly weakly disconjugate: Definition 5.4 holds for $t_0 = 2$. However, the initial system is not disconjugate: in fact $x_0(t)$ vanishes on $[-1,1]$, so that the system is not identically normal. (And the same occurs for s small enough.)

Example 5.2. Let $c\colon \mathbb{R} \to \mathbb{R}$ be a bounded and uniformly continuous function satisfying $c(t) = 1$ for $|t| \ge 3\pi$ and $c(t) = -1$ for $|t| \le 2\pi$. Then the two-dimensional linear Hamiltonian system $\mathbf{z}' = \left[\begin{smallmatrix} 0 & 1 \\ c(t) & 0 \end{smallmatrix}\right]\mathbf{z}$, with $H_3(t) = 1 > 0$ for each $t \in \mathbb{R}$, is weakly disconjugate but not disconjugate: the first component of any solution takes the form $c_1 \cos t + c_2 \sin t$ for $t \in (-2\pi, 2\pi)$, so that it vanishes at least twice; and $c_3 \mathrm{e}^t + c_4 \mathrm{e}^{-t}$ for $|t| \ge 3\pi$, so that it does not vanish for large enough $|t|$. As in the previous example, the set $\Omega = \left\{\left[\begin{smallmatrix} 0 & 1 \\ c_s(t) & 0 \end{smallmatrix}\right] \mid s \in \mathbb{R}\right\} \cup \left\{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right\}$, with $c_s(t) = c(t+s)$, is the hull of the coefficient matrix. It is easy to check that all the systems of the corresponding family (5.5) are weakly disconjugate, but only the one given by $\omega_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is disconjugate. Proposition 5.3(i) ensures that the family of systems is not uniformly weakly disconjugate: as seen in Remark 5.6, condition D1 and identical normality for every system hold, since $H_3(\omega) = 1 > 0$. In fact, this assertion can be also checked directly: as $s \to -\infty$, the "last" zero of the first component of the solution starting at $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ goes to $+\infty$.

5.4 General Properties of the Principal Functions

Theorem 5.2 establishes equivalent conditions guaranteeing the existence and uniqueness, as functions taking values in $\mathscr{L}_{\mathbb{R}}$, of the uniform principal solutions at $\pm\infty$ for each system (5.5), represented by $\left[\begin{smallmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} L_1^-(t,\omega) \\ L_2^-(t,\omega) \end{smallmatrix}\right]$. Under these conditions, we represent by $l^\pm(\omega)$ the Lagrange planes given by $\left[\begin{smallmatrix} L_1^\pm(0,\omega) \\ L_2^\pm(0,\omega) \end{smallmatrix}\right]$, which according to Definition 5.5 can be also represented by the real matrices $\left[\begin{smallmatrix} I_n \\ N^+(\omega) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I_n \\ N^-(\omega) \end{smallmatrix}\right]$. It follows from the equality $l^\pm(\omega{\cdot}t) = U(t,\omega){\cdot}l^\pm(\omega)$, established in Theorem 5.4, that

$$N^\pm(\omega{\cdot}t) = L_2^\pm(t,\omega)\,(L_1^\pm)^{-1}(t,\omega)\,. \tag{5.18}$$

Or, in other words, that the two maps $t \mapsto N^\pm(\omega{\cdot}t)$ are globally defined symmetric solutions of the Riccati equation (5.3). Note that these symmetric functions are unique, in the sense that the planes $l^\pm(\omega)$ that they parameterize in $\mathscr{D}$ are unique.

Definition 5.6. The matrix-valued functions $N^\pm\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ are the *principal functions* at $\pm\infty$.

The analysis of the general properties of the principal functions under hypotheses D1, D2, and D3 (i.e., when D1 holds and the family is uniformly weakly disconjugate) is the goal of this section. The concept and main properties of upper semicontinuous functions $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$, now given, will be fundamental. A detailed proof of Proposition 5.7 is given in Proposition 5.3 of [12].

Definition 5.7. A globally defined matrix-valued function $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ is said to be *upper semicontinuous* if it is norm-bounded and if $\omega_0 = \lim_{n\to\infty}\omega_n$ and $N_0 = \lim_{n\to\infty} N(\omega_n)$, then $N_0 \le N(\omega_0)$.

Proposition 5.7.

(i) *Any continuous function is upper semicontinuous.*
(ii) *Let $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ be upper semicontinuous. Then there exists a residual set $\Omega_N \subseteq \Omega$ of continuity points of N.*
(iii) *Let $(N_m\colon \Omega \to \mathbb{S}_n(\mathbb{R}))$ be a decreasing and uniformly norm-bounded sequence of upper semicontinuous functions, and suppose that there exists $N(\omega) = \lim_{m\to\infty} N_m(\omega)$ for every $\omega \in \Omega$. Then $N\colon \Omega \to \mathbb{S}_n(\mathbb{R})$ is upper semicontinuous.*

Proposition 5.8. *Suppose that* D1, D2, *and* D3 *hold, and let t_0 satisfy the assertions of* Remark 5.2.2 *and* Proposition 5.1(i). *Then,*

$$N^\pm(\omega) = \lim_{r\to\pm\infty} N_r(\omega)\,, \tag{5.19}$$

where N_r is the continuous symmetric matrix-valued function given by

$$N_r(\omega) = -U_3^{-1}(r,\omega)\,U_1(r,\omega) \tag{5.20}$$

for $|r| > t_0$. In addition,

$$N_{r_1}(\omega) \le N_{r_2}(\omega) \le N_{-r_2}(\omega) \le N_{-r_1}(\omega) \quad \text{for } t_0 < r_1 < r_2\,, \tag{5.21}$$

and hence

$$N_r(\omega) \le N^+(\omega) \le N^-(\omega) \le N_{-r}(\omega) \quad \text{if } r > t_0\,. \tag{5.22}$$

In particular, $\mp N^\pm$ are (norm-bounded) upper semicontinuous $n\times n$ matrix-valued functions on Ω.

Proof. Let us fix $\omega \in \Omega$ and choose uniform principal solutions $L^\pm(t,\omega)$ at $\pm\infty$ normalized to $L_1^\pm(0,\omega) = I_n$ (so that $N^\pm(\omega) = L_2^\pm(0,\omega)$). According to Remark 5.4, for each fixed $r \in \mathbb{R}$ with $|r| \ge t_0$, the $2n \times n$ matrix-valued function

$$\begin{bmatrix} L_1^r(t,\omega) \\ L_2^r(t,\omega) \end{bmatrix} = \begin{bmatrix} L_1^+(t,\omega)\,(I_n - I(t,\omega)\,(I(r,\omega))^{-1}) \\ L_2^+(t,\omega)\,(I_n - I(t,\omega)\,(I(r,\omega))^{-1}) - ((L_1^+)^T)^{-1}(t,\omega)\,(I(r,\omega))^{-1} \end{bmatrix},$$

with $I(t,\omega) = I_{L^+}(0,t,\omega)$ given by (5.9), solves (5.5). By definition of principal solution, $\lim_{r\to\infty}(I(r,\omega))^{-1} = 0_n$. Therefore, $\lim_{r\to\infty}\left[\begin{smallmatrix} L_1^r(t,\omega) \\ L_2^r(t,\omega)\end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1^+(t,\omega) \\ L_2^+(t,\omega)\end{smallmatrix}\right]$ for any $t \in \mathbb{R}$. Consequently, since $L_1^+(0,\omega) = L_1^r(0,\omega) = I_n$,

$$N^+(\omega) = \lim_{r\to\infty} N_r(\omega) \qquad \text{for } N_r(\omega) = L_2^r(0,\omega)\,.$$

From $L_1^r(r,\omega) = 0_n$ and $\left[\begin{smallmatrix} L_1^r(r,\omega) \\ L_2^r(r,\omega)\end{smallmatrix}\right] = U(r,\omega)\left[\begin{smallmatrix} L_1^r(0,\omega) \\ L_2^r(0,\omega)\end{smallmatrix}\right] = U(r,\omega)\left[\begin{smallmatrix} I_n \\ N_r(\omega)\end{smallmatrix}\right]$, we deduce that $0_n = U_1(r,\omega) + U_3(r,\omega)\,N_r(\omega)$, which shows (5.20) for $|r| > t_0$ and (5.19) for N^+. For later purposes note that

$$N_r(\omega) = N^+(\omega) - (I(r,\omega))^{-1} \tag{5.23}$$

whenever $|r| > t_0$.

Let us now define $\tilde I(t,\omega) = I_{L^-}(0,t,\omega)$ by (5.9). Repeating the above argument shows that $\left[\begin{smallmatrix} L_1^-(t,\omega) \\ L_2^-(t,\omega)\end{smallmatrix}\right] = \lim_{r\to-\infty}\left[\begin{smallmatrix} K_1^r(t,\omega) \\ K_2^r(t,\omega)\end{smallmatrix}\right]$ for each $t \in \mathbb{R}$, with

$$\begin{bmatrix} K_1^r(t,\omega) \\ K_2^r(t,\omega) \end{bmatrix} = \begin{bmatrix} L_1^-(t,\omega)\,(I_n - \tilde I(t,\omega)\,(\tilde I(r,\omega))^{-1}) \\ L_2^-(t,\omega)(I_n - \tilde I(t,\omega)(\tilde I(r,\omega))^{-1}) - ((L_1^-)^T)^{-1}(t,\omega)(\tilde I(r,\omega))^{-1} \end{bmatrix}$$

for $|r| > t_0$. Then $N^-(\omega) = \lim_{r\to-\infty}\tilde N_r(\omega)$, with $\tilde N_r(\omega) = K_2^r(0,\omega)$. As before, the equality $K_1^r(r,\omega) = 0_n$ yields $\tilde N_r(\omega) = -U_3^{-1}(r,\omega)\,U_1(r,\omega)$ for $|r| > t_0$. Hence, $\tilde N_r = N_r$ for $|r| > t_0$, so that (5.19) also holds for N^-.

Relation (5.23) provides an almost immediate proof of (5.21) and (5.22): one just has to remember that $-(I(r,\omega))^{-1}$ increases as $r \ge t_0$ increases, decreases as $r \le -t_0$ decreases, and satisfies $I^{-1}(-r,\omega) < 0_n < (I(r,\omega))^{-1}$ for $r \ge t_0$.

Therefore the functions $\mp N^\pm(\omega)$ are the limit of two decreasing sequences of continuous functions which are uniformly bounded, as deduced from (5.21) for a fixed value of r_1 and Remark 5.1. Proposition 5.7(iii) ensures that they are upper semicontinuous. The proof is complete. □

Summing up, $N^{\pm}(\omega)$ are semicontinuous functions given by pointwise limits of continuous matrix-valued functions; they are bounded solutions along the flow of the Riccati equation (5.3); and they parameterize in $\mathscr{D}$ the τ-invariant subsets $\tilde{L}^{\pm} = \big\{\big(\omega, l^{\pm}(\omega)\big) \,|\, \omega \in \Omega\big\}$ of $\Omega \times \mathscr{D} \subset \Omega \times \mathscr{L}_{\mathbb{R}}$. Each one of these sets concentrates a τ-invariant measure $\mu^{\pm}$ projecting onto a fixed ergodic measure m_0 on the base: they are defined by $\int_{\Omega\times\mathscr{L}_{\mathbb{R}}} f(\omega,l)\,d\mu^{\pm} = \int_{\Omega} f(\omega, l^{\pm}(\omega))\,dm_0$, and nothing precludes the possibility that these two measures coincide.

We continue working under hypotheses D1, D2, and D3 and define

$$
\begin{aligned}
\mathscr{J}^{+} &= \big\{(\omega,l) \in \Omega \times \mathscr{D} \,|\, l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ with } N^{+}(\omega) \le M\big\} \subset \Omega \times \mathscr{L}_{\mathbb{R}}\,,\\
\mathscr{J}^{-} &= \big\{(\omega,l) \in \Omega \times \mathscr{D} \,|\, l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ with } M \le N^{-}(\omega)\big\} \subset \Omega \times \mathscr{L}_{\mathbb{R}}\,, \qquad (5.24)\\
\mathscr{J} &= \big\{(\omega,l) \in \Omega \times \mathscr{D} \,|\, l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ with } N^{+}(\omega) \le M \le N^{-}(\omega)\big\} \subset \Omega \times \mathscr{L}_{\mathbb{R}}\,,
\end{aligned}
$$

so that $\mathscr{J} = \mathscr{J}^{+} \cap \mathscr{J}^{-}$. These three sets possess some topological and dynamical properties which contribute to describe the global dynamics induced by the family (5.5) on $\Omega \times \mathscr{L}_{\mathbb{R}}$ and on $\Omega \times \mathbb{S}_n(\mathbb{R})$. A bit more precisely, recall that, if $l \in \mathscr{D}$ and $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$, then $U(t,\omega){\cdot}l \in \mathscr{D}$ as long as the solution $M(t,\omega,M_0)$ of the Riccati equation (5.3) with $M(t,\omega,M_0) = M_0$ is defined and that these solutions define the flow τ_s on $\Omega \times \mathbb{S}_n(\mathbb{R})$. Therefore, the properties of invariance and attractivity described by Theorem 5.8 show that the principal functions N^{+} and N^{-} "delimit" the areas at which τ is globally defined as flow and as positive or negative semiflow. And Theorem 5.9 proves that any τ-invariant measure on $\Omega\times\mathscr{L}_{\mathbb{R}}$ is concentrated in $\mathscr{J}$. Some of these properties are proved in [9] for the disconjugate case under the condition $H_3 > 0$. The same proofs work here, but we include all the details for the reader's convenience.

Theorem 5.8. *Suppose that* D1, D2, *and* D3 *hold. Then,*

(i) *The sets $\mathscr{J}^{+}$, $\mathscr{J}^{-}$, and $\mathscr{J}$ are positively τ-invariant, negatively τ-invariant, and τ-invariant, respectively.*
(ii) *The set $\mathscr{J}$ is compact. In addition, if a sequence $((\omega_j, l_j))$ of points of $\mathscr{J}^{+}$ (resp. of $\mathscr{J}^{-}$) converges to a point $(\omega_0, l_0) \in \Omega \times \mathscr{D}$, then $(\omega_0, l_0) \in \mathscr{J}^{+}$ (resp. $(\omega_0, l_0) \in \mathscr{J}^{-}$).*
(iii) *Take $(\omega, l) \in \Omega \times \mathscr{L}_{\mathbb{R}}$. Then $\tau(t,\omega,l) \in \Omega \times \mathscr{D}$ for any $t \ge 0$, $t \le 0$, and $t \in \mathbb{R}$, if and only if $(\omega,l) \in \mathscr{J}^{+}$, $(\omega,l) \in \mathscr{J}^{-}$, and $(\omega,l) \in \mathscr{J}$, respectively.*
(iv) *$\mathscr{J}$ is the maximal τ-invariant subset of $\Omega \times \mathscr{D}$. Moreover, the alpha-limit set and the omega-limit set of any τ-orbit in $\Omega \times \mathscr{L}_{\mathbb{R}}$ are contained in $\mathscr{J}$. In particular, $\mathscr{J}$ contains all the minimal subsets of $\Omega \times \mathscr{L}_{\mathbb{R}}$.*

Proof.

(i) We consider the auxiliary linear equation

$$
M' = -M\,H_1(\omega{\cdot}t) - H_1^T(\omega{\cdot}t)\,M + H_2(\omega{\cdot}t) = g(\omega{\cdot}t, M)\,,
$$

whose solution with initial data M_0, represented by $M_l(t,\omega,M_0)$, is globally defined for any $\omega \in \Omega$ and any $M_0 \in \mathbb{S}_n(\mathbb{R})$. Let us take $(\omega,l) \in \mathscr{J}^+$ and represent $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$. Let $\mathscr{I}$ be the maximal interval of definition of $M(t,\omega,M_0)$. The monotonicity properties established in, e.g., Theorem 4.1 of [12], ensure that $N^+(\omega{\cdot}t) \le M(t,\omega,M_0)$ for any $t \in \mathscr{I}$. In addition, since $H_3 \ge 0$, $M'(t,\omega,M_0) \le g(\omega{\cdot}t, M(t,\omega,M_0))$ for $t \in \mathscr{I}$, so that Theorem 4.2 of [12] proves that $M(t,\omega,M_0) \le M_l(t,\omega,M_0)$ for $t \ge 0$, $t \in \mathscr{I}$. Both inequalities and Remark 5.1 show that $\|M(s,\omega,M_0)\|$ is bounded in any interval $[0,t] \subset \mathscr{I}$ and hence that $M(t,\omega,M_0)$ is defined (at least) for $t \ge 0$: assertion (i) is proved for $\mathscr{J}^+$. The proof is analogous for $\mathscr{J}^-$. And both properties imply that $\mathscr{J} = \mathscr{J}^+ \cap \mathscr{J}^-$ is τ-invariant.

(ii) We define $\mathscr{J}_r = \left\{(\omega,l) \in \Omega \times \mathscr{D} \mid l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right] \text{ and } N_r(\omega) \le M \le N_{-r}(\omega)\right\}$ for N_r given by (5.20) and note that $\mathscr{J} = \cap_{r=1}^{\infty} \mathscr{J}_r$, as can be deduced from (5.19) and (5.22). The continuity of N_r and N_{-r} ensures that each set $\mathscr{J}_r$ is compact, so that also $\mathscr{J}$ is. Now we take a sequence $((\omega_j,l_j))$ of points of $\mathscr{J}^+$ with limit $(\omega_0,l_0) \in \Omega \times \mathscr{D}$ and represent $l_j \equiv \left[\begin{smallmatrix} I_n \\ M_j \end{smallmatrix}\right]$ and $l_0 \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$. Therefore, $\lim_{j\to\infty} M_j = M_0$. By hypothesis, $M_j \ge N^+(\omega_j)$. Since the function N^+ is globally norm-bounded on Ω, we can take a subsequence $((\omega_k,l_k))$ such that there exists $\lim N^+(\omega_k) = N_0$. Hence, $M_0 \ge N_0$. The semicontinuity of $-N^+$ established in Proposition 5.8 ensures that $M_0 \ge N_0 \ge N^+(\omega_0)$, which proves the statement for $\mathscr{J}^+$. The proof is analogous for $\mathscr{J}^-$.

(iii) We must just prove the "only if" assertions, since the "if" ones follow from (i). Let us suppose that $U(t,\omega){\cdot}l \in \mathscr{D}$ for any $t \ge 0$. We write $l \equiv \left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$ and $L(t,\omega) = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right] = U(t,\omega)\left[\begin{smallmatrix} I_n \\ M_0 \end{smallmatrix}\right]$. According to Remark 5.4, if $t \ge 0$, then

$$l^+(\omega{\cdot}t) \equiv \begin{bmatrix} L_1(t,\omega)\,(P(\omega) + I_L(0,t,\omega)\,Q(\omega)) \\ L_2(t,\omega)\,(P(\omega) + I_L(0,t,\omega)\,Q(\omega)) + (L_1^T)^{-1}(t,\omega)\,Q(\omega) \end{bmatrix},$$

with $P(\omega)$ nonsingular. In particular, taking $t = 0$,

$$N^+(\omega) = M_0 + Q(\omega)\,P^{-1}(\omega)\,. \tag{5.25}$$

In addition, by (5.14) and (5.12),

$$0_n = \lim_{t\to\infty} I_{L^+}^{-1}(0,t,\omega) = P^T(\omega)\left(\lim_{t\to\infty} I_L^{-1}(0,t,\omega)\right) P(\omega) + P^T(\omega)\,Q(\omega)\,,$$

so that $Q(\omega)\,P^{-1}(\omega) = -\lim_{t\to\infty} I_L^{-1}(0,t,\omega) \le 0$. This and relation (5.25) ensure that $N^+(\omega) \le M_0$. In other words, $(\omega,l) \in \mathscr{J}^+$, as asserted. The proof of the property for $\mathscr{J}^-$ is analogous, and both of them taken together imply the assertion for $\mathscr{J}$.

(iv) By (iii), the τ-orbit of a point $(\omega, l) \notin \mathscr{J}$ is not contained in $\Omega \times \mathscr{D}$. This proves the first assertion in (iv). Now we take $(\omega_0, l_0) \in \Omega \times \mathscr{L}_{\mathbb{R}}$. By Theorem 5.3(ii), there exists $t_0 \geq 0$ such that $\tau(t, \omega_0, l_0) \in \Omega \times \mathscr{D}$ for $t \geq t_0$. Therefore, by (iii) and (i), $\tau(t, \omega_0, l_0) \in \mathscr{J}^+$ whenever $t \geq t_0$. Let (ω_1, l_1) belong to the omega-limit set of (ω_0, l_0), and write it as $(\omega_1, l_1) = \lim_{j\to\infty} \tau(t_j, \omega_0, l_0)$ for a sequence $(t_j) \uparrow \infty$. A new application of Theorem 5.3(ii) provides $t_1 \geq 0$ such that

$$\tau(t, \omega_1, l_1) \in \Omega \times \mathscr{D} \quad \text{whenever} \quad |t| \geq t_1 \,. \tag{5.26}$$

In particular, $\tau(-t_1, \omega_1, l_1) = \lim_{j\to\infty} \tau(t_j - t_1, \omega_0, l_0)$ belongs to $\Omega \times \mathscr{D}$, and by (ii), $\tau(-t_1, \omega_1, l_1) \in \mathscr{J}^+$. Again (i) implies that $\tau(-t_1 + s, \omega_1, l_1) \in \mathscr{J}^+ \subset \Omega \times \mathscr{D}$ for each $s \geq 0$, which together with (5.26) show that $\tau(t, \omega_1, l_1) \in \Omega \times \mathscr{D}$ for all $t \in \mathbb{R}$. Finally, (iii) ensures that $(\omega_1, l_1) \in \mathscr{J}$, as asserted. The proof is analogous for the alpha-limit set of (ω_0, l_0). The last assertion of (iv) is now trivial, since any minimal set is the omega-limit of each of its orbits.

□

Theorem 5.9. *Suppose that* D1*,* D2*, and* D3 *hold. Then,*

(i) *Every τ-invariant measure μ on $\Omega \times \mathscr{L}_{\mathbb{R}}$ is concentrated on $\mathscr{J}$; that is,* $\mu(\mathscr{J}) = 1$. *Thus, if m is a σ-ergodic measure on Ω, $m(\Omega_0) = 1$ for a subset $\Omega_0 \subseteq \Omega$, and $\{(\omega, l(\omega)) | \, \omega \in \Omega_0\}$ is a τ-invariant subset of $\Omega \times \mathscr{L}_{\mathbb{R}}$ with l measurable on Ω, then*

$$\Omega_1 = \big\{\omega \in \Omega_0 \,|\, (\omega, l(\omega)) \in \mathscr{J}\big\}$$

is σ-invariant with $m(\Omega_1) = 1$.

(ii) *Suppose further that there exists a subset $\Omega_0 \subseteq \Omega$ with $m_0(\Omega_0) > 0$ for a σ-ergodic measure m_0 such that the σ-orbit of ω is dense in Ω for any $\omega \in \Omega_0$. Let $\mathscr{K} \subset \Omega \times \mathscr{L}_{\mathbb{R}}$ be a τ-invariant compact subset with $\mathscr{K} = \{(\omega, l(\omega)) \,|\, \omega \in \Omega\}$ for a continuous function $l \colon \Omega \to \mathscr{L}_{\mathbb{R}}$. Then $\mathscr{K} \subset \mathscr{J}$.*

Proof.

(i) The classical result concerning the decomposition of an invariant measure into ergodic measures (see, e.g., Sect. II.6 of [15]) implies that it is sufficient to prove the assertion for a τ-ergodic measure μ, fixed in what follows. Birkhoff's ergodic theorem ensures that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_{\mathscr{J}^+}(\tau(t, \omega, l))\, \mathrm{d}t = \mu(\mathscr{J}^+) \tag{5.27}$$

for μ-almost every $(\omega, l) \in \Omega \times \mathscr{L}_{\mathbb{R}}$. Let (ω_0, l_0) be one of these points. Using Theorem 5.3(ii) and Theorem 5.8(iii) & (i), we find $t_0 \geq 0$ such that $\tau(t, \omega_0, l_0) \in \mathscr{J}^+$ whenever $t \geq t_0$. Consequently, (5.27) ensures that $\mu(\mathscr{J}^+) = 1$. Analogously, $\mu(\mathscr{J}^-) = 1$, and therefore $\mu(\mathscr{J}) = \mu(\mathscr{J}^+ \cap \mathscr{J}^-) = 1$, as stated.

We now assume that m, Ω_0, and $l\colon \Omega \to \mathcal{L}_{\mathbb{R}}$ satisfy the conditions in the last assertion of (i). Applying the previous property to the τ-ergodic measure concentrated on the τ-invariant set and projecting onto m yields $\int_\Omega \chi_{\mathcal{J}}(\omega, l(\omega))\, dm = 1$, so that $(\omega, l(\omega)) \in \mathcal{J}$ for m-a.e. $\omega \in \Omega_0$. That is, $m(\Omega_1) = 1$. The τ-invariance of $\mathcal{J}$ guarantees the σ-invariance of Ω_1.

(ii) Note that the τ-orbit of $(\omega, l(\omega))$ is dense in $\mathcal{K}$ for any $\omega \in \Omega_0$. Let Ω_1 be the subset of Ω composed of the points ω with $(\omega, l(\omega)) \in \mathcal{J}$, which according to (i) is σ-invariant and satisfies $m_0(\Omega_1) = 1$. Theorem 5.8(i) and (ii) ensures that the (dense) τ-orbit of any point $\omega \in \Omega_0 \cap \Omega_1$ is contained in the compact set $\mathcal{J}$, and hence $\mathcal{K} \subset \mathcal{J}$.

□

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Chapter 6
Stability Criteria for Delay Differential Equations

Beáta Krasznai

Abstract It is shown that some recent stability criteria for delay differential equations are consequences of a well-known comparison principle for delay differential inequalities. Our approach gives not only a unified proof, but it also yields stronger results.

Keywords Delay differential equation • Stability criteria • Comparison principle • Quasimonotone

6.1 Introduction

Recently, there has been a great interest in stability criteria for delay differential equations arising in applications, such as compartmental systems and neural networks. Our aim in this paper is to show that some recent stability criteria can easily be obtained from a comparison principle for differential inequalities whose right-hand side satisfies the quasimonotone condition. We emphasize that our approach gives not only a unified proof of some recent stability criteria, but, moreover, it yields stronger results.

Let $\mathbb{R}$ be the set of real numbers. For a positive integer n, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional space of real column vectors and the space of $n \times n$ matrices with real entries, respectively. Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. The *induced norm* and the *logarithmic norm* of a matrix $A \in \mathbb{R}^{n\times n}$ is defined by

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$$\|A\| = \sup_{0 \neq x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \mu(A) = \lim_{\delta \to 0+} \frac{\|I + \delta A\| - 1}{\delta},$$

respectively, where I denotes the $n \times n$ identity matrix.

A matrix $A = (a_{ij})_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$ is said to be *nonnegative* if $a_{ij} \ge 0$ for all $i, j = 1, \ldots, n$ and it is called *essentially nonnegative* if $a_{ij} \ge 0$ for all $i \neq j$, $i, j = 1, \ldots, n$.

Let $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. We write $x \le y$ $(x < y)$ if $x_i \le y_i$ $(x_i < y_i)$ for $i = 1, \ldots, n$. Let $\mathbb{R}^n_+$ be the cone of nonnegative vectors in $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_+ = \left\{ x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \mid x_i \ge 0 \text{ for all } i = 1, \ldots, n \right\}.$$

Haddad and Chellaboina [3] studied the nonnegative solutions of the system

$$y'(t) = Ay(t) + F(y(t - \tau)), \tag{6.1}$$

where $\tau > 0, A \in \mathbb{R}^{n \times n}$, $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is locally Lipschitz continuous and $F(0) = 0$.

They proved the following stability result (see [3, Theorem 3.2]).

Theorem 6.1. *Suppose that $A \in \mathbb{R}^{n \times n}$ is essentially nonnegative and for some $\gamma \in (0, \infty)$,*

$$F(y) \le \gamma y, \qquad y \in \mathbb{R}^n_+. \tag{6.2}$$

Assume also that there exist $p, q \in \mathbb{R}^n$ such that $p, q > 0$ and

$$(A + \gamma I)^T p + q = 0, \tag{6.3}$$

where T denotes the transpose. Then the zero equilibrium of (6.1) is asymptotically stable with respect to nonnegative initial data.

S. Mohamad and K. Gopalsamy [5] studied the system of delay differential equations

$$z_i'(t) = -\alpha_i z_i(t) + \sum_{j=1}^{n} \beta_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} \gamma_{ij} f_j(z_j(t - \tau_{ij})) + I_i, \quad i = 1, \ldots, n, \tag{6.4}$$

where $f_i : \mathbb{R} \to \mathbb{R}, \alpha_i > 0, \beta_{ij}, \gamma_{ij}, I_i \in \mathbb{R}$, and $\tau_{ij} \ge 0$ for $i, j = 1, \ldots, n$.

By the method of Lyapunov functions they proved the following theorem (see [5, Theorem 2.1]).

Theorem 6.2. *Suppose that there exist constants* $K_i, k_i \in (0, \infty)$*,* $i = 1, \ldots, n$*, such that the following conditions hold:*

$$|f_i(x)| \le K_i, \quad x \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{6.5}$$

$$|f_i(x) - f_i(y)| \le k_i |x - y|, \quad x, y \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{6.6}$$

$$\alpha_i > k_i \sum_{j=1}^n (|\beta_{ij}| + |\gamma_{ij}|), \quad i = 1, \ldots, n. \tag{6.7}$$

Then (6.4) has a unique equilibrium which is globally exponentially stable.

Consider the system of delay differential equations

$$z'(t) = Az(t) + Bg(z(t - \tau)) + J, \tag{6.8}$$

where $\tau > 0$, A, $B \in \mathbb{R}^{n \times n}$, $g : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function and $J \in \mathbb{R}^n$. L. Idels and M. Kipnis [4] proved the following theorem (see [4, Corollary 3.1]).

Theorem 6.3. *Let* $z^\star$ *be an equilibrium of (6.8). Suppose that* g *is globally Lipschitz continuous with Lipschitz constant* $k > 0$ *satisfying*

$$k\|B\| < -\mu(A), \tag{6.9}$$

where $\mu(A)$ *is the logarithmic norm of* A*. Then* $z^\star$ *is a globally attractive equilibrium of (6.8).*

In this paper, we will unify and improve all the three stability results. The proofs will be based on a known comparison theorem for quasimonotone systems formulated in Sect. 6.2. The new stability criteria are presented and proved in Sect. 6.3.

6.2 Summary of Known Results

Given $r \ge 0$, let $C = C([-r, 0]; \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with the *supremum norm,*

$$\|\varphi\| := \sup_{-r \le \theta \le 0} \|\varphi(\theta)\|, \qquad \varphi \in C.$$

Let $\phi, \psi \in C$. We write $\phi \le \psi$ and $\phi < \psi$ if the inequalities hold at each point of $[-r, 0]$.

Consider the autonomous functional equation

$$x'(t) = f(x_t), \tag{6.10}$$

where $f : \Omega \to \mathbb{R}^n$, Ω is an open subset of C, and $x_t \in C$ is defined by

$$x_t(\theta) = x(t+\theta), \qquad -r \le \theta \le 0.$$

We will assume that f is Lipschitz continuous on any compact subset of Ω. This assumption guarantees that for every $\varphi \in \Omega$, there exists a unique noncontinuable solution x of (6.10) with initial value

$$x_0 = \varphi. \tag{6.11}$$

In the sequel, the unique solution of (6.10) and (6.11) will be denoted by $x(t;\varphi)$.

For each $i = 1, \dots, n$, let f_i denote the i−th coordinate function of f so that

$$f(\varphi) = (f_1(\varphi), f_2(\varphi), \dots, f_n(\varphi))^T, \qquad \varphi \in \Omega.$$

We say that f satisfies the *quasimonotone condition* on Ω if

$$\phi, \psi \in \Omega,\ \phi \le \psi, \text{ and } \phi_i(0) = \psi_i(0) \text{ for some } i, \text{implying } f_i(\phi) \le f_i(\psi).$$

The quasimonotone condition is the analogue of the well-known *Kamke condition* for ordinary differential equations.

Our proofs will be based on the following comparison principle essentially due to Ohta [6, Theorem 3].

Proposition 6.1. *Let Ω be an open subset of C. Suppose that $f : \Omega \to \mathbb{R}^n$ is Lipschitz continuous on compact subsets of $\Omega$and f satisfies the quasimonotone condition on Ω. Let $0 < b \le \infty$. Suppose that $y : [-r, b) \to \mathbb{R}^n$ is a continuous function satisfying the differential inequality*

$$\frac{\mathrm{d}^+}{\mathrm{d}t} y(t) \le f(y_t), \qquad t \in [0, b), \tag{6.12}$$

where $\frac{\mathrm{d}^+}{\mathrm{d}t}$ denotes the right-hand derivative. Assume also

$$y_0 \le \varphi \qquad \text{for some } \varphi \in C. \tag{6.13}$$

If $x(t,\varphi)$ is the unique solution of (6.10) and (6.11), then

$$y(t) \le x(t,\varphi) \tag{6.14}$$

for all $t \in [-r, b)$ for which $x(t,\varphi)$ is defined.

We will apply the above comparison theorem to the linear system of differential inequalities

$$\frac{\mathrm{d}^+}{\mathrm{d}t} y_i(t) \le \sum_{j=1}^{n} a_{ij} y_j(t) + \sum_{j=1}^{n} b_{ij} y_j(t - \tau_{ij}), \qquad i = 1, \ldots, n, \tag{6.15}$$

where $a_{ij}, b_{ij} \in \mathbb{R}$ and $\tau_{ij} \ge 0$, $i, j = 1, \ldots, n$.

System (6.15) is a special case of (6.12) when $r = \max_{i,j=1,\ldots,n} \tau_{ij}$ and $f = (f_1, f_2, \ldots, f_n)^T$ is defined by

$$f_i(\phi) = \sum_{j=1}^{n} a_{ij} \phi_j(0) + \sum_{j=1}^{n} b_{ij} \phi_j(-\tau_{ij}) \tag{6.16}$$

for $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \in C([-r, 0]; \mathbb{R}^n)$ and $i = 1, 2, \ldots, n$. In this case Eq. (6.10) has the form

$$x_i'(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_{ij}), \qquad i = 1, 2, \ldots, n. \tag{6.17}$$

It is known [7] that if $f = (f_1, f_2, \ldots, f_n)^T$ is given by (6.16), then f satisfies the quasimonotone condition on Ω if and only if

$$A = (a_{ij})_{i,j=1,2,\ldots,n} \text{ is essentially nonnegative} \tag{6.18}$$

and

$$B = (b_{ij})_{i,j=1,2,\ldots,n} \text{ is nonnegative.} \tag{6.19}$$

According to a remarkable result due to Smith [7], for linear delay differential systems satisfying the quasimonotone condition, the exponential stability of the zero solution is equivalent to the exponential stability of the associated system of ordinary differential equations which is obtained by "ignoring the delay." More precisely, we have the following result.

Proposition 6.2. *Suppose that (6.18) and (6.19) hold so that the right-hand side of (6.17) satisfies the quasimonotone condition on C.The zero solution of (6.17) is exponentially stable if and only if the zero solution of the ordinary differential equation*

$$x' = (A + B)x \tag{6.20}$$

is exponentially stable.

Note that if (6.18) and (6.19) hold, then the coefficient matrix $M = A + B$ of (6.20) is essentially nonnegative. It is known [1] that in this case the exponential stability of the zero solution of (6.20) is equivalent to the explicit condition

$$(-1)^j \det \begin{pmatrix} m_{11} & \cdots & m_{1j} \\ \vdots & & \\ m_{j1} & & m_{jj} \end{pmatrix} > 0, \qquad j = 1, \dots, n, \tag{6.21}$$

where

$$m_{ij} = a_{ij} + b_{ij}, \qquad i, j = 1, \dots, n.$$

6.3 Stability Criteria

Theorem 6.4. *Under the hypotheses of Theorem 6.1, the zero solution of (6.1) is not only asymptotically stable, but even globally exponentially stable with respect to nonnegative initial data.*

Proof. For $\varphi \in C$, $\varphi \geq 0$, let $y(t) = y(\varphi, t)$ be the unique solution of (6.1) with initial value $y_0 = \varphi$. As shown in [3], $y(t) \geq 0$ for all $t \geq 0$. This and (6.2) imply for $t \geq 0$,

$$y'(t) = Ay(t) + F(y(t - \tau)) \leq Ay(t) + \gamma y(t - \tau).$$

Therefore $y(t)$ is a solution of the system of inequalities (6.15) where

$$b_{ij} = \begin{cases} \gamma, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \qquad i = 1, \dots, n.$$

Since A is essentially nonnegative and $B = \gamma I$ is nonnegative, the quasimonotone condition holds for the right-hand side of (6.15). By the application of Proposition 6.1, we have

$$0 \leq y(t, \varphi) \leq x(t, \varphi), \qquad t \geq -r, \tag{6.22}$$

where $x(t, \varphi)$ is the unique solution of system (6.17) with initial value φ at zero. It follows from [3, Theorem 3.1] that under condition (6.3) the zero solution of (6.17) is asymptotically and hence exponentially stable. Therefore, there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|x(t, \varphi)\| \leq M \|\varphi\| e^{-\alpha t}, \qquad t \geq 0. \tag{6.23}$$

Since the definition of the exponential stability is independent of the norm used in $\mathbb{R}^n$, we may restrict ourselves to the ℓ_1-norm. Then (6.22) and (6.23) imply

$$\|y(t,\varphi)\| \le \|x(t,\varphi)\| \le M\|\varphi\|\mathrm{e}^{-\alpha t}, \qquad t \ge 0.$$

This completes the proof. □

Theorem 6.5. *Suppose that all hypotheses of Theorem 6.2 hold except for (6.7) which is replaced with the condition*

$$(-1)^j \det\begin{pmatrix} m_{11} & \cdots & m_{1j} \\ \vdots & & \\ m_{j1} & & m_{jj} \end{pmatrix} > 0, \qquad j = 1,2,\ldots,n, \tag{6.24}$$

where

$$m_{ij} = -\alpha_i\delta_{ij} + k_j(|\beta_{ij}| + |\gamma_{ij}|), \qquad i,j = 1,2,\ldots,n, \tag{6.25}$$

and δ_{ij} is the Kronecker symbol. Then Eq. (6.4) has a unique equilibrium which is globally exponentially stable.

Remark 6.1. Condition (6.7) is equivalent to saying that the logarithmic norm of $M = (m_{ij})_{1\le i,j\le n}$ given by (6.25) induced by the l_∞-norm on $\mathbb{R}^n$ is negative (see [2, p. 41]). While this is only a sufficient condition for the stability of matrix M (see [2, p. 59]), condition (6.24) is not only sufficient, but it is also necessary for the stability of M. Thus, condition (6.24) is weaker than (6.7).

Proof. The existence of an equilibrium $z^* = (z_1^*,\ldots,z_n^*)^T$ of system (6.4) can be proved in the same manner as in the proof of [5, Theorem 2.1]. For $\varphi \in C$, let $z(t) = z(t,\varphi)$ be the unique solution of (6.4) with initial value $z_0 = \varphi$. As shown in [5], we have for $t > 0$ and $i = 1,\ldots,n$,

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}t}|z_i(t) - z_i^*| \le -\alpha_i|z_i(t) - z_i^*| + \sum_{j=1}^n |\beta_{ij}|k_j|z_j(t) - z_j^*| + \\ + \sum_{j=1}^n |\gamma_{ij}|k_j|z_j(t-\tau_{ij}) - z_j^*|,\end{aligned} \tag{6.26}$$

If we let $y_i(t) := |z_i(t) - z_i^*|$, then (6.26) can be written as

$$\frac{\mathrm{d}^+}{\mathrm{d}t}y_i(t) \le -\alpha_i y_i(t) + \sum_{j=1}^n |\beta_{ij}|k_j y_j(t) + \sum_{j=1}^n |\gamma_{ij}|k_j y_j(t-\tau_{ij}). \tag{6.27}$$

System (6.27) is a special case of (6.15) with

$$a_{ij} = -\alpha_i \delta_{ij} + k_j |\beta_{ij}|, \qquad i, j = 1, 2, \ldots, n, \tag{6.28}$$

and

$$b_{ij} = k_j |\gamma_{ij}|, \qquad i, j = 1, 2, \ldots, n. \tag{6.29}$$

Clearly, conditions (6.18) and (6.19) are satisfied. Therefore Proposition 6.1 applies and we conclude that

$$0 \leq y(t) \leq x(t, \psi), \qquad t \geq 0, \tag{6.30}$$

where $y(t) = (y_1(t), \ldots, y_n(t))^T$, and $x(t, \psi)$ is the unique solution of the system

$$x'_i(t) = -\alpha_i x_i(t) + \sum_{j=1}^{n} |\beta_{ij}| k_j x_j(t) + \sum_{j=1}^{n} |\gamma_{ij}| k_j x_j(t - \tau_{ij}), \qquad i = 1, 2, \ldots, n, \tag{6.31}$$

with the initial data

$$\psi(\theta) = |\varphi(\theta) - z^*|, \qquad \theta \in [-r, 0]. \tag{6.32}$$

As noted in Sect. 6.2, condition (6.24) implies that the zero solution of (6.31) is exponentially stable. Therefore,

$$\|x(t, \psi)\| \leq M \|\psi\| \mathrm{e}^{-\alpha t}, \qquad t \geq 0$$

for some $M \geq 1$ and $\alpha > 0$. Using the ℓ_1-norm in $\mathbb{R}^n$ again, the last inequality together with (6.30) implies for $t \geq 0$,

$$\|z(t, \varphi) - z^*\| = \|y(t)\| \leq \|x(t, \psi)\| \leq M \|\psi\| \mathrm{e}^{-\alpha t} = M \|\varphi - z^*\| \mathrm{e}^{-\alpha t}.$$

This proves the global exponential stability of the equilibrium z^*. □

Theorem 6.6. *Under the assumptions of Theorem 6.3, the equilibrium $z^\star$ of (6.8) is globally exponentially stable.*

Proof. $z(t) = z(t, \varphi)$ be the unique solution of (6.8) with initial value $z_0 = \varphi$ for $\varphi \in C([-\tau, 0], \mathbb{R}^n)$. Define

$$y(t) = z(t) - z^*, \qquad t \geq -\tau.$$

From (6.8) we get for $t \geq 0$,

$$y'(t) = Ay(t) + F(y(t - \tau)), \qquad y \in \mathbb{R}^n.$$

where

$$F(y) = B[g(y + z^*) - g(z^*)], \qquad y \in \mathbb{R}^n.$$

Using the fact that g is globally Lipschitz continuous with Lipschitz constant k, we get

$$\|F(y)\| \leq \|B\| k \|y\|, \qquad y \in \mathbb{R}^n. \tag{6.33}$$

It is known (see [2, Chap. I]) that if y is an $\mathbb{R}^n-$valued function which has a right-hand derivative u for $t = t_0$, then $\|y(t)\|$ has a right-hand derivative for $t = t_0$ which is equal to

$$\lim_{h \to 0+} \frac{\|y(t_0) + hu\| - \|y(t_0)\|}{h}.$$

Hence

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|y(t)\| = \lim_{h \to 0+} \frac{\|y(t) + hy'(t)\| - \|y(t)\|}{h}, \qquad t \geq 0. \tag{6.34}$$

For $t \geq 0$, we have

$$\|y(t) + hy'(t)\| - \|y(t)\| = \|(I + hA)y(t) + hF(y(t - \tau))\| - \|y(t)\| \leq$$

$$\leq \|I + hA\| \|y(t)\| + h\|F(y(t - \tau))\| - \|y(t)\| =$$

$$= (\|I + hA\| - 1)\|y(t)\| + h\|F(y(t - \tau))\|.$$

From this, using (6.34), we find that

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|y(t)\| \leq \mu(A)\|y(t)\| + \|F(y(t - \tau))\|, \qquad t \geq 0.$$

This, combined with (6.33), implies for $t \geq 0$,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|y(t)\| \leq \mu(A)\|y(t)\| + k\|B\|\|y(t - \tau)\|, \tag{6.35}$$

and

$$\|y(t)\| = \|\varphi(t) - z^*\|, \qquad t \in [-\tau, 0].$$

Let $x(t) = x(t, \varphi)$ be the unique solution of the linear scalar differential equation

$$x'(t) = \mu(A)x(t) + k\|B\|x(t - \tau), \tag{6.36}$$

with initial value

$$x(t) = \|\varphi(t) - z^*\|, \qquad t \in [-\tau, 0].$$

By Proposition 6.1, we have

$$\|y(t)\| \leq x(t), \qquad t \geq 0. \tag{6.37}$$

In particular, $x(t)$ is nonnegative for $t \geq 0$. Clearly, (6.36) is a special case of (6.17) with $n = 1$. Obviously, conditions (6.18) and (6.19) hold. Since condition (6.9) implies the exponential stability of the zero solution of the ordinary differential equation

$$x' = (\mu(A) + k\|B\|)x,$$

by Proposition 6.2, the zero solution of (6.36) is exponentially stable. Therefore, there exist $M \geq 1$ and $\alpha > 0$ such that for $t \geq 0$ and $\varphi \in C$,

$$\|x(t)\| \leq M\|\varphi - z^*\|e^{-\alpha t}.$$

This and (6.37) imply that for $t \geq 0$,

$$\|z(t) - z^*\| = \|y(t)\| \leq x(t) \leq M\|\varphi - z^*\|e^{-\alpha t}.$$

The proof is complete. □

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Chapter 7
Analyticity of Solutions of Differential Equations with a Threshold Delay

Tibor Krisztin

Abstract We consider the differential equation $\dot{x}(t) = f(x(t), x(t-r))$ where the delay $r = r(x(\cdot))$ is defined by the threshold condition $\int_{t-r}^{t} a(x(s), \dot{x}(s))\,\mathrm{d}s = \rho$ with a given $\rho > 0$. It is shown that if f and a are analytic functions and a is positive, then the globally defined bounded solutions are analytic.

Keywords Delay differential equation • State-dependent delay • Threshold condition • Analyticity

7.1 Introduction

We consider a differential equation of the form

$$\dot{x}(t) = f(x(t), x(t-r)), \qquad r = r(x(\cdot)), \tag{7.1}$$

where the state-dependent delay r is defined by the threshold condition

$$\int_{t-r}^{t} a(x(s), \dot{x}(s))\,\mathrm{d}s = \rho. \tag{7.2}$$

Results on existence, uniqueness, continuous dependence of solutions, linearization, and construction of local invariant manifolds can be applied to (7.1), (7.2); see, e.g., [2–4, 7, 8, 11–13].

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Our aim is to show that under certain analyticity conditions on f and a, the bounded solutions $x : \mathbb{R} \to \mathbb{R}^N$ of (7.1) and (7.2) are analytic functions. The proof uses the special structure of the threshold type delay to reduce the problem of analyticity to that of the solutions of an analytic ordinary differential equation in a suitable Banach space.

The analyticity problem of globally defined bounded solutions (e.g., periodic solutions) for (7.1) was raised in lectures at several international conferences by John Mallet-Paret and Roger Nussbaum. For equations with constant delays a typical result is as follows. If $f : \mathbb{R}^{N(M+1)} \to \mathbb{R}^N$ is analytic and $r_k \geq 0$ for $1 \leq k \leq M$ are constants, then any bounded solution $x : \mathbb{R} \to \mathbb{R}^N$ of

$$\dot{x}(t) = f(x(t), x(t - r_1), x(t - r_2), \ldots, x(t - r_M))$$

is necessarily analytic in t. This and a slightly more general version of it were given by Nussbaum [10]. The technique of [10] does not seem to work if the delays are state-dependent, for example, $r_k = r_k(x(t))$ with given analytic functions r_k. In a recent paper [9] J. Mallet-Paret and R. Nussbaum study the problem of analyticity for given time-dependent analytic delay functions $r_k(t)$. They remark in [9] that the result of the present paper (in the case when a in (7.2) depends only on $x(s)$) can be obtained by reducing the problem to equations with constant delay, i.e., where [10] is applicable.

The paper [5] assumes analyticity of periodic solutions for a class of differential equations with state-dependent delay in order to prove a global bifurcation result.

As far as we know an affirmative answer for the analyticity problem is known only for the particular cases given below in this paper. Mallet-Parret and Nussbaum [9] suspect that nonanalyticity may hold in many cases.

7.2 The Result

Let $\mathbb{K}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let D be an open subset of $\mathbb{K}^p$, $p \geq 1$ is an integer. Recall from [1] that a mapping g from D into a Banach space E over $\mathbb{K}$ is analytic if, for every $a \in D$, there is $r > 0$ such that in $\{(z_1, \ldots, z_p) \in \mathbb{K}^p : |z_k - a_k| < r, \ 1 \leq k \leq p\}$, $g(z)$ is equal to the sum of an absolutely summable power series in the p variables $z_k - a_k$, $1 \leq k \leq p$. If $\mathbb{K} = \mathbb{R}$ and $g : D(\subset \mathbb{R}^p) \to E$ is (real) analytic, then clearly g extends to be (complex) analytic in a complex neighborhood $\tilde{D} \subset \mathbb{C}^p$. If $\mathbb{K} = \mathbb{C}$ and $g : D \to E$ is continuously differentiable, then g is analytic [1].

Let $\mathbb{N}$ denote the set of nonnegative integers. If A is a subset of a normed linear space F, then $l^\infty(A)$ denotes the set of sequences $u = (u_k)_{k=0}^\infty$ in A such that $||u|| = \sup_{k \in \mathbb{N}} |u_k|$ is finite. With the norm $|| \cdot ||$, the sets $l^\infty(\mathbb{R}^N)$ and $l^\infty(\mathbb{C}^N)$ are Banach spaces.

Let $N \geq 1$ be an integer. We will use the following hypotheses.

(H1) The maps $f : U \times U \to \mathbb{C}^N$ and $a : U \times V \to \mathbb{C}$ are analytic for some open subsets $U \subset \mathbb{C}^N$ and $V \subset \mathbb{C}^N$.
(H2) The sets $\tilde{U} = U \cap \mathbb{R}^N$ and $\tilde{V} = V \cap \mathbb{R}^N$ are open subsets of $\mathbb{R}^N$, and

$$f\left(\tilde{U} \times \tilde{U}\right) \subset \tilde{V}, \quad a\left(\tilde{U} \times \tilde{V}\right) \subset (0, \infty).$$

(H3) $\rho > 0$.

A continuously differentiable mapping $x : \mathbb{R} \to \mathbb{R}^N$ will be called a globally defined bounded solution of (7.1) and (7.2) if

$$x(\mathbb{R}) \subset \hat{U}, \quad \dot{x}(\mathbb{R}) \subset \hat{V}$$

for some compact subsets $\hat{U}$ and $\hat{V}$ of $\tilde{U}$ and $\tilde{V}$, respectively, so that $f\left(\hat{U} \times \hat{U}\right) \subset \hat{V}$, and there is an $r : \mathbb{R} \to \mathbb{R}$ such that

$$\dot{x}(t) = f(x(t), x(t - r(t))), \quad \int_{t-r(t)}^{t} a(x(s), \dot{x}(s))\, \mathrm{d}s = \rho$$

hold for all $t \in \mathbb{R}$.

Now we can state our result.

Theorem 7.1. *Under hypotheses (H1), (H2), and (H3), the globally defined bounded solutions $x : \mathbb{R} \to \mathbb{R}^N$ of (7.1) and (7.2) must be analytic.*

Proof. Let $x : \mathbb{R} \to \mathbb{R}^N$ be a globally defined bounded solution of (7.1) and (7.2).

The compactness of $\hat{U}$, $\hat{V}$ implies the existence of $a_1 > a_0 > 0$ such that

$$a\left(\hat{U} \times \hat{V}\right) \subset [a_0, a_1].$$

Clearly, $r : \mathbb{R} \to \mathbb{R}$ is unique, C^1-smooth, and

$$r(t) \in \left[\frac{\rho}{a_1}, \frac{\rho}{a_0}\right] \qquad (t \in \mathbb{R}).$$

Define the C^1-map $\eta : \mathbb{R} \to \mathbb{R}$ by $\eta(t) = t - r(t)$. Let the iterates $\eta^k : \mathbb{R} \to \mathbb{R}$ of η be given by

$$\eta^0(t) = t, \quad \eta^j(t) = \eta\left(\eta^{j-1}(t)\right) \qquad (t \in \mathbb{R},\ j \in \mathbb{N}).$$

Observe that for all $t \in \mathbb{R}$ and $j \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\eta^j(t) &= \eta'\left(\eta^{j-1}(t)\right)\frac{\mathrm{d}}{\mathrm{d}t}\eta^{j-1}(t)\\ &= \eta'\left(\eta^{j-1}(t)\right)\eta'\left(\eta^{j-2}(t)\right)\cdots\eta'(\eta(t))\,\eta'(t)\,.\end{aligned}$$

Introduce the notation $b(t) = a(x(t), \dot{x}(t))$. Differentiating the equation $\int_{\eta(t)}^{t} b(s)\,\mathrm{d}s = \rho$, we find that

$$\eta'(t) = \frac{b(t)}{b(\eta(t))}, \qquad \eta'\left(\eta^k(t)\right) = \frac{b(\eta^k(t))}{b(\eta^{k+1}(t))}.$$

Consequently,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\eta^j(t) &= \frac{b(\eta^{j-1}(t))}{b(\eta^j(t))}\frac{b(\eta^{j-2}(t))}{b(\eta^{j-1}(t))}\cdots\frac{b(\eta(t))}{b(\eta^2(t))}\frac{b(t)}{b(\eta(t))} \\
&= \frac{b(t)}{b(\eta^j(t))} \\
&= \frac{a(x(t), \dot{x}(t))}{a(x(\eta^j(t)), \dot{x}(\eta^j(t)))} \\
&= \frac{a\left(x(t), f(x(t), x(\eta(t)))\right)}{a\left(x(\eta^j(t)), f(x(\eta^j(t)), x(\eta^{j+1}(t)))\right)}.
\end{aligned}$$

Define the mapping $Y : \mathbb{R} \to l^\infty(\mathbb{R}^N)$ as follows:

$$Y(t) = (Y_0(t), Y_1(t), \ldots), \qquad Y_j(t) = x(\eta^j(t)).$$

Then, for all $t \in \mathbb{R}$ and $j \in \mathbb{N}$, we have

$$\begin{aligned}
\dot{Y}_j(t) &= \dot{x}\left(\eta^j(t)\right)\frac{\mathrm{d}}{\mathrm{d}t}\eta^j(t) \\
&= f\left(x(\eta^j(t)), x(\eta^{j+1}(t))\right)\frac{b(t)}{b(\eta^j(t))} \\
&= f\left(Y_j(t), Y_{j+1}(t)\right)\frac{a\left(Y_0(t), f(Y_0(t), Y_1(t))\right)}{a\left(Y_j(t), f(Y_j(t), Y_{j+1}(t))\right)}.
\end{aligned}$$

By using these equations and the smoothness of f, a, it follows that Y_j is C^2-smooth and there is a $K > 0$ such that $|\ddot{Y}_j(t)| \leq K$ for all $t \in \mathbb{R}$ and $j \in \mathbb{N}$. This is sufficient to guarantee that $Y : \mathbb{R} \to l^\infty(\mathbb{R}^N)$ is C^1-smooth and satisfies the differential equation

$$\dot{Y}(t) = G(Y(t))$$

in $l^\infty(\mathbb{R}^N)$ for all $t \in \mathbb{R}$, where

$$G : l^\infty(\tilde{U}) \to l^\infty(\mathbb{R}^N)$$

is given by

$$G_j(Y) = f\left(Y_j, Y_{j+1}\right) \frac{a\left(Y_0, f(Y_0, Y_1)\right)}{a\left(Y_j, f(Y_j, Y_{j+1})\right)}.$$

By conditions (H1) and (H2) there are open neighborhoods $\hat{U}_{\mathbb{C}} \subset U$ and $\hat{V}_{\mathbb{C}} \subset V$ in $\mathbb{C}$ of the sets $\hat{U}$ and $\hat{V}$, respectively, such that $f\left(\hat{U}_{\mathbb{C}} \times \hat{U}_{\mathbb{C}}\right) \subset \hat{V}_{\mathbb{C}}$, and the map $c : \left(\hat{U}_{\mathbb{C}}\right)^4 \to \mathbb{C}^N$ given by

$$c(u_0, u_1, u_2, u_3) = f\,(u_2, u_3) \frac{a\left(u_0, f(u_0, u_1)\right)}{a\left(u_2, f(u_2, u_3)\right)}$$

is analytic. Moreover, by choosing the neighborhoods $\hat{U}_{\mathbb{C}} \subset U$ and $\hat{V}_{\mathbb{C}} \subset V$ small enough, there is $L > 0$ so that for the derivatives Dc and D^2c of c the inequalities $||Dc(u)|| \le L$, $||D^2c(u)|| \le L$ hold for all $u \in \left(\hat{U}_{\mathbb{C}}\right)^4$. Hence it is easy to show that the map

$$H : l^\infty(\hat{U}_{\mathbb{C}}) \to l^\infty(\mathbb{C}^N)$$

given by

$$H_j(u) = c(u_0, u_1, u_j, u_{j+1})$$

is continuously differentiable with

$$\begin{aligned}(DH(u)v)_j =& D_1c(u_0, u_1, u_j, u_{j+1})v_0 + D_2c(u_0, u_1, u_j, u_{j+1})v_1 \\ &+ D_3c(u_0, u_1, u_j, u_{j+1})v_j + D_4c(u_0, u_1, u_j, u_{j+1})v_{j+1},\end{aligned}$$

where $u \in l^\infty(\hat{U}_{\mathbb{C}})$, $v \in l^\infty(\mathbb{C}^N)$.

Now Cauchy's existence theorem (see, e.g., [1]) gives that for any $t_0 \in \mathbb{R}$ the differential equation

$$\dot{u} = H(u)$$

with initial condition $u(t_0) = Y(t_0)$ has a unique continuously differentiable solution defined on an open ball J in $\mathbb{C}$ with center t_0. The continuous differentiability of $u : J \to l^\infty(\mathbb{C}^N)$ implies its analyticity in J [1].

Clearly, G and H coincide on $l^\infty(\tilde{U} \cap \hat{U}_{\mathbb{C}})$, and their restrictions to $l^\infty(\tilde{U} \cap \hat{U}_{\mathbb{C}})$ are C^1-smooth, considering them as mappings into $l^\infty(\mathbb{C}^N)$. Then the Cauchy problem

$$\dot{v} = G(v), \quad v(t_0) = Y(t_0)$$

has a unique continuously differentiable solution from an open interval $I \subset \mathbb{R}$ with center at t_0 into $l^\infty(\mathbb{C}^N)$. Both $Y|_I$ and $u|_{\mathbb{R}\cap J}$ are solutions. Consequently, $Y|_{I\cap J} = u|_{I\cap J}$. Therefore, the analyticity of u implies the analyticity of Y in a neighborhood of t_0. Then obviously $x(t) = Y_0(t)$ is also analytic in a neighborhood of t_0. This completes the proof. □

Remark 7.1. In the introduction of the paper [9] Mallet-Paret and Nussbaum remark that (if a in condition (7.2) depends only on $x(s)$) by introducing the new time variable

$$\tau = \int_{t_0}^{t} a(x(s))\,\mathrm{d}s,$$

and letting $y(\tau) = x(t)$, the differential equation with constant delay

$$\dot{y}(\tau) = \frac{1}{a(y(\tau))} f(y(\tau), y(\tau - \rho))$$

is obtained. For this equation Nussbaum's classic result [10] gives the analyticity of y. Reversing the change of variables by

$$t = t_0 + \int_0^{\tau} a(y(s))^{-1}\,\mathrm{d}s,$$

the analyticity of x follows.

This idea of Mallet-Paret and Nussbaum [9] can be applied to extend Theorem 7.1 to equations of the form

$$\dot{x}(t) = f(x(t), x(t - r_1), x(t - r_2), \ldots, x(t - r_M)), \qquad r_k = r_k(x(\cdot)),$$

with the threshold conditions

$$\int_{t-r_k}^{t} a(x(s), \dot{x}(s))\,\mathrm{d}s = \rho_k,$$

where f, a, ρ_k, $1 \le k \le M$ are assumed to satisfy hypotheses analogous to (H1), (H2), and (H3).

Example 7.1. The threshold condition

$$\int_{t-r}^{t} a(x(s))\,\mathrm{d}s = \rho$$

appears naturally in the modeling of infection disease transmission, the modeling of immune response systems, and the modeling of respiration, in the study of population dynamics involving structured models. See the review paper [3] and the references therein.

Example 7.2. In cutting processes [6] the equation

$$\alpha r = \rho + x(t) - x(t-r)$$

with positive α and ρ determines the time delay $r = r(x(\cdot))$ as a function of the solution x. Clearly, this equation is equivalent to the threshold condition

$$\int_{t-r}^{t} [\alpha - \dot{x}(s)]\ \mathrm{d}s = \rho,$$

and this is a particular case of (7.2) with $a(u, v) = \alpha - v$. Function a is positive provided the derivative of the solution x is sufficiently small.

Example 7.3. A nonlinear version of the above example is

$$\int_{t-r}^{t} [A(x(s)) - DB(x(s))\dot{x}(s)]\ \mathrm{d}s = \rho$$

which is equivalent to

$$\int_{t-r}^{t} A(x(s))\,\mathrm{d}s = \rho + B(x(t)) - B(x(t-r))$$

with analytic functions $A : \mathbb{R}^N \to \mathbb{R}$ and $B : \mathbb{R}^N \to \mathbb{R}$.

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Chapter 8
Application of Advanced Integrodifferential Equations in Insurance Mathematics and Process Engineering

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Abstract In this paper we consider a dual risk model with general inter-arrival time distribution and general size distribution. A special Gerber–Shiu discounted penalty function is defined and an integral equation is derived for it in the case of dependent inter-arrival times and sizes. We prove the existence and uniqueness of the solution of the integral equation in the set of bounded functions and we show that the solution tends to zero exponentially. If the density function of the inter-arrival time satisfies a linear differential equation with constant coefficients, the integral equation is transformed into an integrodifferential equation with advances in arguments and an explicit solution is given without any assumption on the size distribution. We also present a link between the Lundberg fundamental equations of the Sparre Andersen risk model and the dual risk model.

Keywords Dual risk model • Advanced integrodifferential equation • Process engineering • LODE-type distribution

8.1 Introduction

Consider the time process

$$V(t) = x - ct + \sum_{n=1}^{N(t)} Y_n, \qquad t \geq 0, \tag{8.1}$$

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where $x \geq 0$ is the initial value. $Y_n\ n = 1, 2, \ldots$ are nonnegative, independent, and identically distributed random variables with distribution function $G(y)$, density function $g(y)$, and finite expectation μ_G. Let us denote the inter-arrival times by t_i, which are also nonnegative, independent, and identically distributed random variables with distribution function $F(t)$, density function $f(t)$, and finite expectation μ_F. The time process $\{N(t) : t \geq 0\}$ denotes the number of arrivals up to time t and is defined as

$$N(t) = \begin{cases} 0, & \text{if } t_1 > t \\ \max\left\{k : \sum_{i=1}^{k} t_i \leq t\right\}, & \text{if } t_1 \leq t. \end{cases} \tag{8.2}$$

We do not assume the independence of t_i and Y_i for a fixed value of i, but we do suppose the independence of (t_i, Y_i) and (t_j, Y_j) if $i \neq j$. This model is referred to as the dependent dual risk model. The joint distribution function of (t_i, Y_i) is denoted by $H(t, y)$, the joint density function by $h(t, y)$. Obviously

$$f(t) = \int_0^{\infty} h(t, y)\,\mathrm{d}y$$

and

$$g(y) = \int_0^{\infty} h(t, y)\,\mathrm{d}t.$$

The ruin of the process $V(t)$ occurs when $V(t)$ becomes negative, i.e., the inequality $V(t) < 0$ holds. The time of ruin is the first time when the process goes below zero, namely

$$T_V(x) = \begin{cases} \inf\{t : V(t) < 0\} \\ \infty, \quad if \quad V(t) \geq 0\ \forall t \geq 0. \end{cases} \tag{8.3}$$

Thus the probability of ruin with initial surplus x equals $P(T_V(x) < \infty)$.

This model is applied in the life annuity or pension insurance where a life annuity rate or pension is paid continuously from the insurance company to the policyholder and where the random size corresponds to the death of the policyholder by what the insurance company earns an amount of money equal to "the expected annuity or pension to be paid" [8]. This model appears to be applicable also for companies such as ones in the pharmaceutical or petroleum industry where c denotes the constant rate at which expenses are paid out while the jumps are interpreted as the net present values of future incomes from inventions or discoveries [3]. Moreover, this model has also a natural interpretation in modelling buffered production systems [15].

In insurance literature, the dual risk model has also been called the negative risk sums model. Dong and Wang studied the negative risk sums model with generalized Erlang(n) time process and developed an integral equation for the probability of ruin [5]. They transformed this equation into an integrodifferential equation, and in the special case of Erlang(2) time process and exponential distribution of the random variable Y_i provided a closed form of the ruin probability. A perturbed negative risk sums model was investigated in a paper by Dong and Wang dealing with the ruin probability, where they provided an analytical solution for the Poisson-exponential case [6]. Dong generalized the model by Dong and Wang [5] for the case when the time process was described by means of two correlated risk processes, but he studied only the probability of ruin [4].

The model has been considered with modifications as well. Avanzi and his coauthors investigated the problem of optimal dividends [3]. Albrecher and his coauthors studied the dual risk process in the presence of tax payments [1], and Ng considered the dual of the compound Poisson model under a threshold dividend strategy [14].

It was an important step in the development of insurance mathematics when Gerber and Shiu introduced the Gerber–Shiu discounted penalty function [7]. Using this approach, lots of important problems could be investigated and solved (see, e.g., [9–11] and the references cited therein). In this paper we turn our attention to the Gerber–Shiu function in the dual model.

Regarding the Gerber–Shiu function, the main difference between the Sparre Andersen risk process and the dual risk model is the following. In the first model the ruin happens by jumps, while in the second one the process crosses the level zero continuously. Consequently, in the dual model, it makes no sense to use a penalty function having two variables, namely one for the surplus "immediately before ruin" and the other for the surplus "after ruin," because both values are zero. Therefore, we use the Gerber–Shiu function in the following special form.

Let the function for $m_\delta(x)$ for $0 \le x,\ 0 \le \delta$ be defined as

$$m_\delta(x) := E\left(\mathrm{e}^{-\delta T_V(x)} 1_{T_V(x)<\infty}\right), \tag{8.4}$$

where $1_{T_V(x)<\infty} = 1$ if $T_V(x) < \infty$ and $1_{T_V(x)<\infty} = 0$ otherwise.

The function $m_\delta(x)$ corresponds to the Gerber–Shiu discounted penalty function in the special case when $w \equiv 1$ [7].

The function $m_\delta(x)$ is the Laplace transform of the density function of the time of ruin; hence it can be used in deriving the density function of $T_V(x)$, its expectation, and the probability of ruin as well. In economical context, the variable δ may also be interpreted as a force of interest. In this paper we mainly investigate such a time process in which the inter-arrival time distribution satisfies a linear differential equation with constant coefficients (LODE-type distribution) without any assumptions on the size distribution. We analyze the Lundberg fundamental equation and using those results we provide an analytical solution for the Gerber–Shiu function.

The rest of the paper is organized as follows. In Sect. 8.2 we set up an integral equation satisfied by the function $m_\delta(x)$ and prove the existence and uniqueness of its solution in the set of bounded functions and we deal with the order of the convergence of $m_\delta(x)$. In Sect. 8.3 we restrict our attention to the LODE-type inter-arrival size distribution and we transform the integral equation into an integrodifferential equation. In Sect. 8.4, on the way to finding the solution in a special form, we get to the Lundberg equation and we analyze it under some assumptions. Meanwhile, we present the link between the risk and the dual risk model. Finally, in Sect. 8.5, we give an analytic formula for the solution of the integrodifferential equation in the previously investigated special cases.

8.2 The Integral Equation for $m_\delta(x)$

In this section we show that $m_\delta(x)$ satisfies an integral equation.

First we note that $m_\delta(x)$ is a bounded function. If the joint density function $h(t, y)$ is continuous, then $m_\delta(x)$ is a continuous and differentiable function of the variable δ satisfying

$$(-1)^i \left. \frac{\partial^i m_\delta(x)}{\partial \delta^i} \right|_{\delta=0} = E(T_V^i(x) 1_{T_V(x)<\infty})$$

and

$$\lim_{\delta \to 0+} m_\delta(x) = m_0(x) = E(1_{T_V(x)<\infty}) = P(T_V(x) < \infty).$$

Theorem 8.1. *Let h be a continuous density function on $[0, \infty) \times [0, \infty)$. For any $x \geq 0$ and $\delta \geq 0$, the function $m_\delta(x)$ satisfies the following integral equation*

$$m_\delta(x) = \int_0^\infty \int_0^{\frac{x}{c}} \mathrm{e}^{-\delta t} m_\delta(x + y - ct) h(t, y)\, \mathrm{d}t \mathrm{d}y + \int_{\frac{x}{c}}^\infty \mathrm{e}^{-\delta \frac{x}{c}} f(t)\, \mathrm{d}t. \quad (8.5)$$

Proof. The proof follows the arguments usually applied in renewal technique taking into account that

$$E\left(\mathrm{e}^{-\delta T_V(x)} 1_{T_V(x)<\infty} | t_1 = t, Y_1 = y\right) = \mathrm{e}^{-\frac{x}{c}\delta} \quad \text{when } t_1 > \frac{x}{c},$$

while

$$E\left(\mathrm{e}^{-\delta T_V(x)} 1_{T_V(x)<\infty} | t_1 = t, Y_1 = y\right) = \mathrm{e}^{-\delta t} m_\delta(x + y - ct) \quad \text{when } t_1 \leq \frac{x}{c}.$$

□

Note that $m_\delta(0) = 1$, which follows from Eq. (8.5).

Theorem 8.2. *Let h be a continuous density function on $[0, \infty) \times [0, \infty)$. Then for any fixed $\delta > 0$ Eq. (8.5) has a unique solution in the set of bounded and measurable functions on $[0, \infty)$ and this solution is continuous in x.*

Proof. Define the norm of a function as $\|m_\delta\| = \sup_{x \in R_0^+} |m_\delta(x)|$. The set of bounded and continuous functions on $[0, \infty)$ is a Banach space with the above norm. Let us introduce the operator K_δ mapping the set of bounded and continuous functions into the set of bounded and continuous functions by the following definition:

$$K_\delta(\varphi)(x) = \int_0^\infty \int_0^{\frac{x}{c}} e^{-\delta t} \varphi(x + y - ct) h(t, y) \, dt \, dy + \int_{\frac{x}{c}}^\infty e^{-\delta \frac{x}{c}} f(t) \, dt.$$

If φ_1 and φ_2 are bounded and continuous functions, then

$$\begin{aligned} &|K_\delta(\varphi_1)(x) - K_\delta(\varphi_2)(x)| \\ &\quad \le \int_0^\infty \int_0^{\frac{x}{c}} e^{-\delta t} |\varphi_1(x + y - ct) - \varphi_2(x + y - ct)| h(t, y) \, dt \, dy \\ &\quad \le \|\varphi_1 - \varphi_2\| C, \end{aligned}$$

where $0 < C = \int_0^\infty e^{-\delta t} f(t) \, dt < 1 \, (\delta > 0)$, hence $\|K_\delta(\varphi_1) - K_\delta(\varphi_2)\| \le C \, \|\varphi_1 - \varphi_2\|$. This means that K_δ is a contraction, and thus there exists a unique solution of the equation $\varphi = K_\delta(\varphi)$ in the set of bounded and continuous functions on $[0, \infty)$.

If φ_1 and φ_2 are different bounded and measurable but not necessarily continuous solutions of Eq. (8.5), i.e., $\varphi_1 \ne \varphi_2$, then

$$\begin{aligned} &|\varphi_1(x) - \varphi_2(x)| \\ &\quad = \int_0^\infty \int_0^{\frac{x}{c}} e^{-\delta t} \, |\varphi_1(x + y - ct) - \varphi_2(x + y - ct)| \, h(t, y) \, dt \, dy, \end{aligned}$$

so

$$\begin{aligned} \|\varphi_1 - \varphi_2\| &\le \|\varphi_1 - \varphi_2\| \int_0^\infty \int_0^\infty e^{-\delta t} h(t, y) \, dt \, dy \\ &= \|\varphi_1 - \varphi_2\| \int_0^\infty e^{-\delta t} f(t) \, dt \\ &< \|\varphi_1 - \varphi_2\| \, , \end{aligned}$$

and it is a contradiction. Consequently there are no two different bounded and measurable solutions to Eq. (8.5). Hence, for a fixed value of $\delta > 0$, the solution of Eq. (8.5) is unique in the set of bounded and measurable functions and this solution is continuous in x. □

Note that in the case of $\delta = 0$ we have $\int_0^\infty e^{-\delta t} f(t) \, dt = \int_0^\infty f(t) \, dt = 1$; hence the argument in the proof of Theorem 8.2 is not valid. Moreover, in this case, the uniqueness does not hold because the constant 1 function is a solution to Eq. (8.5) and in some cases there exists another bounded solution as well [16].

Corollary 8.1. *If the assumptions of Theorem 8.2 hold, then the function defined by Eq. (8.4) is the unique bounded solution of Eq. (8.5).*

Theorem 8.3. *If $h(t, y)$ is a continuous function, $\delta > 0$ is fixed, then $m_\delta(x) \leq e^{-\frac{\delta}{c}x}$.*

Proof. If $T_V(x) < \infty$ then $\frac{x}{c} \leq T_V(x)$, therefore

$$m_\delta(x) = E\left(e^{-\delta T_V(x)} \cdot 1_{T_V(x)<\infty}\right) \leq e^{-\delta \frac{x}{c}}.$$

□

Corollary 8.2. *For any fixed value of $\delta > 0$ $\lim\limits_{x\to\infty} m_\delta(x) = 0$ holds supposing that $h(t, y)$ is a continuous function.*

8.3 An Integrodifferential Equation for LODE-Type Inter-Arrival Time Distribution

In this section we investigate the case when the density function of the inter-arrival time satisfies a linear differential equation with constant coefficients subject to some general initial conditions. We call these random variables LODE-type random variables. Although this generalization concerning initial conditions requires more computations, we investigate this case in order to be able to handle Coxian distributions, K_n-type distributions, and LODE-type distributions with almost homogeneous initial conditions in a unified way. For this type of random variables we transform the integral equation (8.5) into an integrodifferential equation (8.6). We emphasize that in the present and following sections we assume the independence of the inter-arrival time and size, that is,

$$h(t, y) = f(t)g(y).$$

We also suppose that $g(y)$ is a continuous function.

First we summarize what is known about a density function that satisfies a linear differential equation with constant coefficients.

Let $p(z) = \sum_{i=0}^{n} a_i z^i$ with coefficients $a_n = 1, a_0 \neq 0$. Assume that the density function $f(t)$ satisfies the linear differential equation

$$p\left(\frac{d}{dt}\right)(f)(t) = \sum_{i=0}^{n} a_i f^{(i)}(t) \equiv 0$$

subject to the initial conditions

$$f^{(j)}(0) = A_j, \qquad j = 0, 1, 2, \ldots, n-1.$$

From the theory of the differential equations we know the following: if the number of different roots of the polynomial $p(z)$ is m, the different roots are denoted by ρ_i, and the multiplicity of the root ρ_i is k_i, then the solution of the linear differential equation is the density function

$$f(t) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} b_{ij} t^{j-1} e^{\rho_i t},$$

where b_{ij} are appropriate real numbers. $f(t)$ is a density function; therefore, in the case $\mathrm{Re}(\rho_i) \geq 0$ $b_{ij} = 0$ hold for all $j = 1, 2, \ldots, k_i$. After some computations we can see that $\int_0^\infty f(t)\,dt = 1$ holds iff $\sum_{i=1}^n a_i A_{i-1} = a_0$, furthermore, $\lim_{t\to\infty} f^{(j)}(t) = 0$ for $j = 0, 1, 2, 3, \ldots$.

We note that there exist such examples when the real parts of the roots of the polynomial $p(z)$ are nonnegative, but because the coefficients belonging to them are zero, these roots do not play role in $f(t)$. See, for example, $p(z) = (z+1)(z^2 - 2z + 2)$. Here the density function $f(t) = e^{-t}$ satisfies the differential equation $p\left(\frac{d}{dt}\right)(f)(t) \equiv 0$ subject to the initial conditions $A_0 = 1, A_1 = -1, A_2 = 1$, and the roots of $p(z)$ are $-1, 1+i, 1-i$. In this case $f(t)$ has the above form with coefficients $b_{11} = 1, b_{21} = 0$ and $b_{31} = 0$. But one can realize that $f(t) = e^{-t}$ satisfies the differential equation $p^*\left(\frac{d}{dt}\right)(f)(t) \equiv 0$ subject to $f(0) = 1$ as well, with $p^*(z) = z + 1$. The possibility of the polynomial reduction will be detailed later in Sect. 8.5.

Now we transform the integral equation into an integrodifferential equation.

Theorem 8.4. *Let $p(z) = \sum_{i=0}^n a_i z^i$, with $a_n = 1$, $a_0 \neq 0$, and we suppose that the density function $f(t)$ satisfies the following linear differential equation with constant coefficients*

$$p\left(\frac{d}{dt}\right)(f)(t) = \sum_{i=0}^{n} a_i f^{(i)}(t) \equiv 0 \tag{8.6}$$

subject to initial conditions

$$f^{(j)}(0) = A_j, \qquad j = 0, 1, \ldots, n-1. \tag{8.7}$$

Then, with the notation

$$q_0(z) \equiv 0,\ q_i(z) = \sum_{k=0}^{i-1} A_k z^{i-1-k} \quad i = 1, 2, \ldots, n,\ \text{and}\ q(z) = \sum_{i=0}^{n} a_i \cdot q_i(z), \tag{8.8}$$

the function $m_\delta(x)$ defined by Eq. (8.4) satisfies the following integrodifferential equation

$$p\left(\delta\mathscr{I} + c\frac{d}{dx}\right)(m_\delta)(x) = \int_0^\infty q\left(\delta\mathscr{I} + c\frac{d}{dx}\right)(m_\delta)(x+y)g(y)\,dy \tag{8.9}$$

subject to appropriate initial conditions, where $\mathscr{I}$ denotes the identity operator.

Proof. Our starting point is again Eq. (8.5). Substituting $\tau = x - ct$ and multiplying by $\mathrm{e}^{\frac{\delta x}{c}}$ we get the equation

$$c\mathrm{e}^{\frac{\delta x}{c}} m_\delta(x) = \int_0^\infty \int_0^x \mathrm{e}^{\frac{\delta\tau}{c}} f\left(\frac{x-\tau}{c}\right) m_\delta(\tau+y) g(y)\,\mathrm{d}\tau\mathrm{d}y + c\int_{\frac{x}{c}}^\infty f(t)\,\mathrm{d}t.$$

Taking the derivative of both sides with respect to x and using $f(0) = A_0$ we can conclude

$$\begin{aligned} &c\mathrm{e}^{\frac{\delta x}{c}}\left(\delta\mathscr{I} + c\frac{\mathrm{d}}{\mathrm{d}x}\right)(m_\delta)(x) \\ &\quad = c\int_0^\infty \mathrm{e}^{\frac{\delta x}{c}} A_0 m_\delta(x+y) g(y)\,\mathrm{d}y \\ &\qquad + \int_0^\infty \int_0^x \mathrm{e}^{\frac{\delta\tau}{c}} f'\left(\frac{x-\tau}{c}\right) m_\delta(\tau+y) g(y)\,\mathrm{d}\tau\mathrm{d}y - cf\left(\frac{x}{c}\right). \end{aligned}$$

Similarly, for $j = 2, \dots, n$ (using $f^{(j-1)}(0) = A_{j-1}$)

$$\begin{aligned} &c\mathrm{e}^{\frac{\delta}{c}x}\left(\delta\mathscr{I} + c\frac{\mathrm{d}}{\mathrm{d}x}\right)^j (m_\delta)(x) \\ &\quad = c\sum_{i=1}^{j-1} c\mathrm{e}^{\frac{\delta x}{c}} A_{j-i-1} \int_0^\infty \left(\delta\mathscr{I} + c\frac{\mathrm{d}}{\mathrm{d}x}\right)^i (m_\delta)(x+y) g(y)\,\mathrm{d}y \\ &\qquad + \int_0^\infty \int_0^x \mathrm{e}^{\frac{\delta\tau}{c}} f^{(j)}\left(\frac{x-\tau}{c}\right) m_\delta(\tau+y) g(y)\,\mathrm{d}\tau\mathrm{d}y - cf^{(j-1)}\left(\frac{x}{c}\right). \end{aligned}$$

Multiplying the jth equation above by a_j, summing them up, and dividing by c, we get the following equation:

$$\begin{aligned} &\mathrm{e}^{\frac{\delta}{c}x} p\left(\delta\mathscr{I} + c\frac{\mathrm{d}}{\mathrm{d}x}\right)(m_\delta)(x) \\ &\quad = \mathrm{e}^{\frac{\delta}{c}x}\int_0^\infty q\left(\delta\mathscr{I} + c\frac{\mathrm{d}}{\mathrm{d}x}\right)(m_\delta)(x+y) g(y)\,\mathrm{d}y \\ &\qquad + \frac{1}{c}\int_0^\infty \int_0^x \mathrm{e}^{\frac{\delta\tau}{c}} p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)(f)\left(\frac{x-\tau}{c}\right) m_\delta(\tau+y) g(y)\,\mathrm{d}\tau\mathrm{d}y \\ &\qquad + a_0\left(1 - F\left(\frac{x}{c}\right)\right) - \sum_{j=1}^n a_j f^{(j-1)}\left(\frac{x}{c}\right). \end{aligned} \tag{8.10}$$

As $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$, it follows that

$$\frac{1}{c}\int_0^\infty \int_0^x \mathrm{e}^{\frac{\delta\tau}{c}} p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)(f)\left(\frac{x-\tau}{c}\right) m_\delta(\tau+y) g(y)\,\mathrm{d}\tau\mathrm{d}y = 0.$$

Moreover, since $\lim_{t\to\infty} f^{(j-1)}(t) = 0$, we get

$$\begin{aligned}
a_0\left(1 - F\left(\frac{x}{c}\right)\right) - \sum_{j=1}^{n} a_j f^{(j-1)}\left(\frac{x}{c}\right)
&= a_0\left(1 - F\left(\frac{x}{c}\right)\right) + \int_{\frac{x}{c}}^{\infty} \sum_{j=1}^{n} a_j f^{(j)}(t)\,\mathrm{d}t \\
&= a_0\left(1 - F\left(\frac{x}{c}\right)\right) - a_0 \int_{\frac{x}{c}}^{\infty} f(t)\,\mathrm{d}t \\
&= 0.
\end{aligned}$$

Now dividing by $\mathrm{e}^{\frac{\delta}{c}x}$, Eq. (8.10) reduces to Eq. (8.9).

The appropriate initial conditions can be determined by substituting 0 into the equations obtained after taking derivatives. We note that if $A_k = 0$ for $k = 0, 1, 2, \ldots, n-2$ and $a_0 = A_{n-1} \neq 0$ (see [2]), then the initial conditions are $m_\delta^{(j)}(0) = \left(-\frac{\delta}{c}\right)^j$, $j = 0, 1, 2, \ldots, n-1$. □

Taking into account that the steps of the proof can be reversed, we can state that Eq. (8.9) with the appropriate initial conditions implies Eq. (8.5). Therefore, for a fixed value of $\delta > 0$, Eq. (8.9) (with appropriate initial conditions) has a unique solution in the set of bounded functions. We present this solution in the following section.

Now we present a relevant relationship between the polynomials $p(z)$, $q(z)$ defined by Eq. (8.8) and the Laplace transform of the density function $f(t)$.

Theorem 8.5. *Assume that the density function $f(t)$ satisfies $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ subject to the initial conditions $f^{(j)}(0) = A_j$, $j = 0, 1, 2, \ldots, n-1$, moreover $a_n = 1$, $a_0 \neq 0$. Denote the Laplace transform of the function f by $\tilde{f}(z) = \int_0^\infty \mathrm{e}^{-zt} f(t)\,\mathrm{d}t$. Then the equality*

$$p(z)\tilde{f}(z) = q(z) \tag{8.11}$$

holds true whenever $\mathrm{Re}(z) > \sigma$, *for some* $\sigma < 0$.

Proof. Let σ be the abscissa of the convergence of the Laplace transform of f. Since $\tilde{f}(0) = 1$, $\sigma \le 0$. Moreover, since only the roots with negative real parts play role in the representation of $f(t)$, we can see that $\sigma < 0$ holds as well.

Applying multiple partial integrations we get that $\int_0^\infty f^{(i)}(t)\mathrm{e}^{-zt}\,\mathrm{d}t = -q_i(z) + z^i\tilde{f}(z)$ for $i = 0, 1, 2, \ldots, n$.

Multiplying the above equations by a_i, respectively, and summing them up we can conclude that

$$0 = \sum_{i=0}^{n} a_i \int_0^\infty f^{(i)}(t)\mathrm{e}^{-zt}\,\mathrm{d}t = -\sum_{i=0}^{n} a_i q_i(z) + \sum_{i=0}^{n} a_i z^i \tilde{f}(z),$$

consequently, $p(z)\tilde{f}(z) = q(z)$ whenever $\mathrm{Re}(z) > \sigma$. □

Remark 8.1. The above connection can be reversed. Namely, if $p(z)\tilde{f}(z) = q(z)$ is satisfied and the degree of $q(z)$ is less than the degree of $p(z)$, then $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ holds as well. The initial conditions can be determined from the coefficient of the polynomial $q(z)$ by solving a system of linear equations. If $p(z)$ and $q(z)$ do not have common roots, this can be seen on the basis of the inverse Laplace transformation. Taking into account that the rational fraction form of the ratio $\frac{q(z)}{p(z)}$ is $\sum_{i=1}^{m}\sum_{j=1}^{n_i-1}\frac{C_{ij}}{(z-\rho_i)^j}$, the inverse Laplace transform of the terms in the sums are products of polynomials and exponential functions with exponent $\rho_i t$; hence $f(t)$ is the linear combination of them. Since exactly these functions form the fundamental system of the differential equation $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(u)(t) \equiv 0$, all their linear combinations satisfy the homogeneous differential equation $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ as well. Finally, note that if $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$, then for any polynomial of the form $p_2(z) = p_1(z)p(z) \quad p_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ is also satisfied and this solves the case when the numerator and the denominator in the ratio $\frac{q(z)}{p(z)}$ have common roots.

8.4 The Lundberg Fundamental Equation of the Model

In this section we make preparations to derive an explicit solution to Eq. (8.9) with a general density function $g(y)$. We emphasize that the polynomial $q(z)$ [defined by Eq. (8.8)] is the consequence of the initial conditions of the ordinary differential equation satisfied by the inter-arrival time distribution.

The problems arising from the possible existence of roots of $p(z)$ with nonnegative real parts make us to be cautious; hence we suppose first that all the roots of $p(z)$ have negative real parts. In the last section we show that this assumption holds in many cases, such as K_n-type distributions (including Erlang, Coxian, phase-type distributions), and also for the general polynomial $p(z)$ with almost homogeneous initial conditions used in [2] as well. Moreover, we prove that if this assumption does not hold, we can reduce the polynomial $p(z)$ to another polynomial, whose degree is less than the degree of the original one such that the roots of the reduced polynomial have negative real parts. In other words, in the general case, $f(t)$ also satisfies another linear differential equation with constant coefficients, and the number of initial conditions has to be decreased by omitting the last one or last two ones.

Our method to determine the solution of Eq. (8.9) is to find the solution in a special form. As the solution is unique in the set of bounded functions this is an appropriate method to find the unknown function $m_\delta(x)$. We will see that during the computations we get to the Lundberg fundamental equation and we have to analyze it. We prove that if $p(z)$ has exactly n roots with negative real parts, then the Lundberg equation has also exactly n roots with negative real parts without any assumption concerning the density function $g(y)$. The linear combination of the products of polynomials and exponential functions expressed by these roots of the Lundberg fundamental equation serves as $m_\delta(x)$.

Theorem 8.6. *The function $k(x) = \mathrm{e}^{vx}$ satisfies the integrodifferential equation (8.9) if and only if the exponent v satisfies the following Lundberg fundamental equation*

$$p(\delta + cv) = q(\delta + cv) \int_0^\infty \mathrm{e}^{vy} g(y)\,\mathrm{d}y. \tag{8.12}$$

Proof. Substituting e^{vx} into Eq. (8.9), taking the derivatives, and simplifying by e^{vx} we get Eq. (8.12). We note that Eq. (8.12) is called the characteristic equation of Eq. (8.9) in the theory of integrodifferential equations. □

Remark 8.2. Equation (8.12) corresponds to the Lundberg fundamental equation used in [2, Eq. (12)] and in [13, Eq. (4)]. We are interested in its roots with negative real parts. If we use the form $k(x) = \mathrm{e}^{-v^* x}$, then after some computations we can realize that Eq. (8.12) coincides with the Lundberg fundamental equation of that risk model. This connection between the risk process and the dual model is worth emphasizing. Since the following statement of the Lundberg fundamental equation is more general than the theorem proved in [12, 13] and than the statement used in [2] as well, it can be used to generalize their methods.

Theorem 8.7. *Assume that the density function $f(t)$ satisfies $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ subject to the initial conditions $f^{(j)}(0)=A_j,\ j=0,1,2,\dots,n-1,\ (a_n=1,\ a_0\neq 0)$. Suppose, moreover, that all the roots of the polynomial $p(z)$ have negative real parts. Then Eq. (8.12) has exactly n roots with negative real parts.*

Proof. If $p(z) = 0$ has exactly n roots with negative real parts, then $p(\delta + cv) = 0$ has n roots with real parts smaller than or equal to $-\frac{\delta}{c}$. We prove that Eq. (8.12) has n roots with negative real parts. To do this, we draw a closed contour on which the absolute value of the left-hand side of Eq. (8.12) is larger than the absolute value of the right-hand side of Eq. (8.12) and all the roots of the polynomial $p(\delta + cv)$ are situated inside the contour. The contour consists of the imaginary axis and a left half-circle with an appropriate large radius.

First consider the imaginary axis $\mathrm{Re}(v) = 0$. Taking the absolute value of both sides of Eq. (8.11) we can see that $|p(z)|\,|\tilde{f}(z)| = |q(z)|$ if $\sigma < \mathrm{Re}(z)$. By the nonnegativity of the function f, if $\mathrm{Re}(z) > 0$, then $|\tilde{f}(z)| < \int_0^\infty \mathrm{e}^{-\mathrm{Re}\, z\cdot t}\,|f(t)|\,\mathrm{d}t \leq \int_0^\infty f(t)\,\mathrm{d}t = 1$. Taking into account our assumption that $|p(z)| \neq 0$ whenever $\mathrm{Re}(z) > 0$, the inequality $|q(z)| < |p(z)|$ hold as well whenever $\mathrm{Re}(z) > 0$. (We note that the last step is not true if $p(z)$ has a root with nonnegative real part.) As $\mathrm{Re}(\delta + cv) > 0$ if $\mathrm{Re}(v) = 0$, it follows that $|q(\delta + cv)| < |p(\delta + cv)|$ whenever $\mathrm{Re}(v) = 0$. Moreover, if $\mathrm{Re}(v) = 0$, then $\left|\int_0^\infty \mathrm{e}^{vy} g(y)\,\mathrm{d}y\right| \leq 1$, consequently on the line $\mathrm{Re}(v) = 0$ the inequality $\left|q(\delta + cv)\int_0^\infty \mathrm{e}^{vy} g(y)\,\mathrm{d}y\right| < |p(\delta + cv)|$ holds.

Let us turn our attention to the negative half-circle. The degree of the polynomial $q(z)$ is less than the degree of the polynomial $p(z)$, consequently, $\lim_{|z|\to\infty}\left|\frac{p(z)}{q(z)}\right| = \infty$. Furthermore, $\left|\int_0^\infty \mathrm{e}^{vy} g(y)\,\mathrm{d}y\right| < 1$ if $\mathrm{Re}(v) < 0$, consequently,

$$\left|q(\delta + cv)\int_0^\infty \mathrm{e}^{vy} g(y)\,\mathrm{d}y\right| < |p(\delta + cv)|$$

holds for the points of the left half-circle $|v| = R$ with an appropriately large radius R and the set $\{z : |z| \leq R, \mathrm{Re}(z) < 0\}$ contains all the roots of the polynomial $p(\delta + cv)$ inside. Now Rouché's theorem can be applied and it implies our statement. □

Remark 8.3. If we make some slight modifications in the previous proof, we can see that the real parts of all the roots of Eq. (8.12) are smaller than or equal to $-\frac{\delta}{c}$.

8.5 An Analytical Solution of the Integrodifferential Equation

Now we give an explicit formula for the solution of the integrodifferential equation in the special case when the roots of the polynomial $p(z)$ have negative real parts. Then we show that this assumption holds in many previously investigated cases. Finally we prove that the general case (when $p(z)$ has at least one root with nonnegative real part) can be reduced to this special case.

If we know the roots of the Eq. (8.12), the solution of Eq. (8.5) can be expressed explicitly.

Theorem 8.8. *Suppose that the density function $f(t)$ satisfies $p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ with $a_n = 1$, $a_0 \neq 0$ subject to the initial conditions $f^{(j)}(0) = A_j$, $j = 0, 1, 2, \ldots, n-1$. Assume that all the roots of the polynomial $p(z)$ have negative real parts.*

Then

$$m_\delta(x) = \sum_{i=1}^{r} \sum_{j=0}^{n_i(\delta)-1} c_{ij}(\delta) x^j \mathrm{e}^{k_i(\delta)x},$$

where $c_{ij}(\delta)$ are appropriate real numbers and $k_i(\delta)$ are the roots of Eq. (8.12) with multiplicity $n_i(\delta)$ for $i = 1, 2, .., r$, where $\sum_{i=1}^{r} n_i(\delta) = n$.

Proof. Recalling Theorem 8.4, we have to find the unique bounded solution of Eq. (8.9) subject to appropriate initial conditions. According to Theorem 8.7, the Lundberg fundamental equation (characteristic equation) of the model has exactly n roots with negative real parts. Based on the theory of integrodifferential equations one can set up the fundamental system of solutions which consists of exponential functions or products of polynomials and exponential functions depending on the multiplicity of the roots. Since the fundamental solutions are linearly independent, so the initial conditions are satisfied uniquely by a linear combination of these functions. As the real parts of the roots are negative, the fundamental solutions are bounded, hence so is their linear combination. Consequently, we obtain a bounded solution of Eq. (8.9) subject to the initial conditions, which equals $m_\delta(x)$. □

We note that this result together with Remark 8.3 coincides with the exponential convergence of $m_\delta(x)$ proved for the general case in Theorem 8.3.

Now we show that all the roots of the polynomial $p(z)$ have negative real parts in many used cases.

First consider the case investigated in [2].

Special Case 1 Let $p(z) = \sum_{i=0}^{n} a_i z^i$, and $f(t)$ be a density function with

$$p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv \sum_{i=0}^{n} a_i f^{(i)}(t) \equiv 0, \quad a_n = 1, \quad a_0 \neq 0,$$

$$f^{(j)}(0) = 0, \quad j = 0, 1, \ldots, n-2, \quad f^{(n-1)}(0) = a_0.$$

Then all the roots of the polynomial $p(z)$ have negative real parts.

Proof. Recall Eq. (8.11). It is easy to see that $q(z) = a_0$. If $\mathrm{Re}(z) \geq 0$, then

$$\left|\tilde{f}(z)\right| \leq \int_0^\infty |f(t)| \cdot \mathrm{e}^{-\mathrm{Re}(z)t}\, \mathrm{d}t \leq \int_0^\infty f(t)\, \mathrm{d}t = 1.$$

This, together with $p(z)\tilde{f}(z) = a_0$, implies the inequality $|p(z)| \geq |a_0| > 0$ whenever $\mathrm{Re}(z) \geq 0$. Consequently there is no root of $p(z)$ with positive or zero real part. □

Now we turn to K_n-type distributions, which were used in [12].

Special Case 2 The inter-arrival time is K_n-type distributed, that is, the Laplace transform of its density function is of the following form:

$$\tilde{f}(z) = \frac{\prod_{i=1}^{n} \lambda_i + z \sum_{j=0}^{n-2} \beta_j z^j}{\prod_{i=1}^{n} (z + \lambda_i)}, \qquad \text{with } 0 < \lambda_i, \quad i = 1, 2, \ldots, n.$$

Applying Remark 8.1 we can see that in this case $f(t)$ is LODE type with $p(z) = \prod_{i=1}^{n}(z + \lambda_i), a_n = 1,\ a_0 = \prod_{i=1}^{n} \lambda_i \neq 0$ and all the roots of $p(z)$ are real and negative.

We also mention an important subset of the K_n-type distributions, which is formed by Coxian distributions. This set is often used because it is dense among the nonnegative distributions [17].

Special Case 3 The inter-arrival time is Coxian distributed, that is, the Laplace transform of the density function f is of the following form:

$$\tilde{f}(z) = \sum_{i=1}^{n} b_i \prod_{k=1}^{i} \frac{\lambda_k}{z + \lambda_k} \text{ with } 0 < \lambda_k,\ k = 1, 2, \ldots, n.$$

Remark 8.4. Let $\tilde{f}(z) = \frac{\lambda^2}{(\lambda+z)^2}$, i.e., the inter-arrival times, be Erlang(2) distributed, and suppose that the roots of the Lundberg fundamental equation with negative real parts are different. Let $k_2(\delta) < k_1(\delta) < 0$ denote the different roots of Eq. (8.12). Then, according to Theorem 8.8, $m_\delta(x) = c_1(\delta)\mathrm{e}^{k_1(\delta)x} + c_2(\delta)\mathrm{e}^{k_2(\delta)x}$, where the coefficients are

$$c_1(\delta) = \frac{-\frac{\delta}{c} - k_2(\delta)}{k_1(\delta) - k_2(\delta)} > 0 \quad \text{and} \quad c_2(\delta) = \frac{k_1(\delta) + \frac{\delta}{c}}{k_1(\delta) - k_2(\delta)}.$$

Consequently, $P(T_V(x) < \infty) = m_0(x) = \lim_{\delta \to 0} m_\delta(x) = c_1(0)\mathrm{e}^{k_1(0)x} + c_2(0)\mathrm{e}^{k_2(0)x}$. In the case of exponentially distributed sizes this formula coincides with the results of Dong and Wang given by (4.7) in [5].

Finally we turn to the most general case, when at least one of the roots of the polynomial $p(z) = 0$ has positive or zero real parts. This case will be reduced to the case of a polynomial with lower degree. In other words, we prove that in this case $f(t)$ satisfies a LODE defined by a polynomial of degree less than n which has only roots with negative real parts. The key is the following simple statement:

Theorem 8.9. *Let* $p(z) = \sum_{i=0}^{n} a_i z^i, a_n = 1, a_0 \neq 0, p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ *subject to the initial conditions* $f^{(j)}(0) = A_j, j = 0, 1, 2, \ldots, n-1$, *and let* $p(z_1) = 0$ *for some* z_1 *with* $\mathrm{Re}(z_1) \geq 0$. *Then the equality* $q(z_1) = 0$ *holds as well.*

Proof. Recall Eq. (8.11), namely, $p(z)\tilde{f}(z) = q(z)$ whenever $\mathrm{Re}(z) > \sigma$ and $\sigma < 0$. If we suppose $\mathrm{Re}(z_1) \geq 0$ with $p(z_1) = 0$ and $q(z_1) \neq 0$, then $\tilde{f}$ would not be regular at $z = z_1$ which contradicts the fact that $\sigma < 0$. □

Remark 8.5. Let us assume that $p(z_1) = 0$ and $\mathrm{Re}(z_1) \geq 0$. If $\mathrm{Im}(z_1) = 0$, then $p(z) = (z - z_1)p^*(z)$ and $q(z) = (z - z_1)q^*(z)$. If $\mathrm{Im}(z_1) \neq 0$, then $p(z) = (z - z_1)(z - \bar{z}_1)p^*(z)$ and $q(z) = (z - z_1)(z - \bar{z}_1)q^*(z)$. The coefficients of $p^*(z)$ and $q^*(z)$ are real numbers, the principal coefficients are 1, and the constant coefficients differ from zero. From Eq. (8.11) it can be easily seen that $p^*(z)\tilde{f}(z) = q^*(z)$. Applying Remark 8.1 we can deduce that $p^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(f)(t) \equiv 0$ subject to the initial conditions $f^{(j)}(0) = A_j$, $j = 0, 1, 2, \ldots, n-2$, or $j = 0, 1, 2, \ldots, n-3$. Consequently, the simplification by $(z - z_1)$ or by $(z - z_1)(z - \bar{z}_1)$ can be executed and continued until the roots of $p(z)$ with nonnegative real parts disappear. Hence we have reduced the problem to the case of a polynomial $p^*(z)$ which has only roots with negative real parts. In that case Theorem 8.8 gives the explicit solution.

8.6 Summary

In this paper the Gerber–Shiu function is investigated in the dual risk model. An integral equation is set up in the case of dependence between the inter-arrival time and size. We proved the existence and uniqueness of its solution and we proved that the solution tends to zero exponentially if the argument tends to infinity. In the case when the inter-arrival time density function satisfies a linear differential equation with constant coefficients subject to general initial conditions we transformed the integral equation into an integrodifferential equation. Seeking the solution in a special form we got to the Lundberg fundamental equation and we analyzed it in a special case. We have presented a link between the equations of the risk and the dual risk model. On the basis of the result of our analysis we gave the exact form of the solution. We have shown that a special assumption holds in most of cases investigated in the literature of the risk processes previously. The cases when the assumption is not satisfied were reduced to the previously solved problem. This way we have completed the determination of the Gerber–Shiu function in the case of LODE-type distribution of inter-arrival time.

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Chapter 9
Stability and Control of Systems with Propagation

Vladimir Răsvan

Abstract A natural way of introducing time delay equations is to consider boundary value problems for hyperbolic partial differential equations (PDEs) in two variables. Such problems account for the so-called propagation phenomena that may be found in several physical and engineering applications. Association of some functional (differential/integral) equations to the aforementioned boundary value problems represents a way of tackling basic theory (existence, uniqueness, data dependence, i.e. well posedness) but also some qualitative properties arising from ODEs (ordinary differential equations) such as stability, oscillations, dissipativeness, Perron condition, and others. On the other hand automatic feedback control for systems described by partial differential equations (PDEs), systems called also *with distributed parameters*, is often ensured by boundary control: the control signals appear as forcing signals at the boundaries. In control applications stability of the feedback structure is the very first requirement. In order to achieve stability, Lyapunov functionals are considered aiming to obtain simultaneously the control structure and stability of the controlled system.

Keywords Hyperbolic partial differential equations • Derivative boundary conditions • Equations with deviated arguments • Hamilton variational principle • Energy identity • Lyapunov energy functional • Feedback stabilization

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9.1 Introduction and Motivation

9.1.1 An early basic result We shall refer first to a paper of Cooke [12] that did not circulate too much. In this paper there was considered the following BVP (boundary value problem) for hyperbolic PDEs in two variables

$$
\begin{aligned}
&\frac{\partial u^+}{\partial t} + \tau^+(\lambda,t)\frac{\partial u^+}{\partial \lambda} = \Phi^+(\lambda,t)\\
&\frac{\partial u^-}{\partial t} + \tau^-(\lambda,t)\frac{\partial u^-}{\partial \lambda} = \Phi^-(\lambda,t),\ 0 \le \lambda \le 1,\ t \ge t_0,\\
&\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(0,t) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(0,t)\right] = f_0(t)\\
&\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(1,t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(1,t)\right] = f_1(t)\\
&u^\pm(x,0) = \omega^\pm(x),\ \ 0 \le x \le 1,
\end{aligned}
\tag{9.1}
$$

with $\tau^+(\lambda,t) > 0$, $\tau^-(\lambda,t) < 0$.

It was shown in that paper that by integrating the solutions of (9.1)—assumed to exist—along the characteristics, one can associate the solutions of a certain system of differential equations with deviated argument. Moreover, defining

$$
\begin{aligned}
&L^+ = \max\{1 \le j \le m | a_j^+(t) \not\equiv 0\},\ \ L^- = \max\{1 \le j \le m | b_j^-(t) \not\equiv 0\}\\
&K^+ = \max\{1 \le j \le m | b_j^+(t) \not\equiv 0\},\ \ K^- = \max\{1 \le j \le m | a_j^-(t) \not\equiv 0\},
\end{aligned}
$$

and assuming these numbers are well-defined, there is defined also

$$M = L^+ + L^- - (K^+ + K^-),$$

and the boundary conditions (with the associated equations with deviated arguments) are called of retarded, neutral, or advanced type if $M > 0$, $M = 0$, or $M < 0$ accordingly. The terminology is consistent with the standard one—see [6]—as well as with the more general classification of Kamenskii [16].

We started with this reference in order to stress that BVPs for hyperbolic PDEs in two variables represent a natural way of introducing all kinds of differential equations with deviated argument. It has to be mentioned that [12] represents some kind of generalization of the BVP occurring in applications concerning electrical circuits containing lossless transmission lines [8,9]. Some even earlier results of the same type are to be met in [21,22,32].

It might thus appear that motivations for such kind of equations are restricted to applications which are 50–70 years old. However, more recent applications arising from the control of mechanical systems, such as marine risers, overhead cranes, flexible manipulators, and oil well drill strings, display the same kind of equations [3, 4, 11, 20, 30] what sends to the observation that the study of the equations modeling such systems is still of interest.

9.1.2 Two research directions Starting from the motivating applications we shall consider two directions for developing mathematical problems. The first direction is connected to the *basic theory*—existence, uniqueness, and data dependence (well posedness). The basic theory is discussed for the associated functional equations obtained by integration along the characteristics [12]: since a one to one correspondence between the solutions of the BVP and of the functional equations is established, the results obtained for one mathematical object will be projected on the other one. Worth mentioning that an even earlier paper [1] allows to discuss more general cases of BVPs, e.g., for equations with nonuniformities [4, 10]; a more recent paper in this line is [25].

It is important to point out that this approach allows an easy extension to PDEs of such topics as stability, dissipativeness, oscillations, and Perron condition which were introduced primarily for ordinary differential equations (ODEs).

The other direction is concerned with feedback control of, what is called in engineering, systems with distributed parameters, which are boundary controlled. We consider here some applications described by hyperbolic PDEs in two variables, with dynamic boundary conditions—as in (9.1)—which contain forcing signals representing the control action. Several approaches for feedback control may be taken—model-based control, adaptive control, and sliding mode control [26]—but the stability requirement is basic. With respect to this, the *energy identity* which is well known in the theory of PDEs is able to provide a natural Lyapunov functional that can be used in designing a stabilizing feedback control. As it will appear from the sequel, the second aforementioned task (of control synthesis) may be achieved at the formal level (as engineers often do) thus obtaining the BVP describing the closed-loop system. For this system the basic theory must be constructed using the approach emphasized above; in this way the control problem is put on a sound mathematical basis.

Taking into account these considerations, the paper is organized as follows. First, the benchmark dynamics of the overhead crane is deduced in a rather complete form using the variational principle of Hamilton. Some simplified versions viewed as limit cases when some small parameters are taken 0 are considered. For one of them there is an associated system of NFDEs (neutral functional differential equations) via integration of the Riemann invariants on the characteristics, and the basic theory and the stability also are discussed for the associated equations and then projected back on the initial BVP using the representation formulae for the solutions. Next, the energy identity is established for the basic model and the stabilizing feedback is obtained in the formal way. The paper continues with the discussion of the asymptotic stability in a limit case by using once more the associated NFDEs. Since these equations do not fulfil the requirements of the standard theory for NFDEs, the analysis continues in the general case based on the approach of the maximal monotone operators. There is sketched an extension of this approach to the case of dynamic boundary conditions. The paper ends with a section of conclusions where some open problems are mentioned.

9.2 A Benchmark Dynamics: The Overhead Crane and its Mathematical Model

In automatic control there exist several problems which are considered benchmarks: various if not all approaches and methods are tested on such models. The *overhead crane* (Fig.9.1) is one of such benchmarks—see [17, 24].

In the cited references the lumped parameter model described by ODEs is considered. However, if the elastic flexibilities of the cable are taken into account, a model described by a BVP for hyperbolic PDEs with derivative boundary conditions is obtained. We stress that the model of [4] contains a sign error in one of the boundary conditions. This assertion follows if *the generalized variational Hamilton principle* is applied to deduce the mathematical model of the overhead crane as it is the case in [20] for the marine flexible riser.

9.2.1 Variational deduction of the model In order to deduce this model we write down

1. The kinetic energy of the motorized platform of mass M, the flexible cable, and the payload mass m, given by

$$E_k(t) = \frac{1}{2}\left[m\,(y_t(0,t))^2 + M\,(y_t(L,t))^2 + \int_0^L \rho(s) y_t^2(s,t)\,\mathrm{d}s\right],$$

where $y(s,t)$ is the position of the moving flexible cable at the local coordinate s (including the elastic deflection) and at the moment of time t, $\rho(s)$ being the local mass density at the local coordinate s. We shall have also

$$y(s,t) = X_p(t) + v(s,t), \quad y(L,t) \equiv X_p(t), \tag{9.2}$$

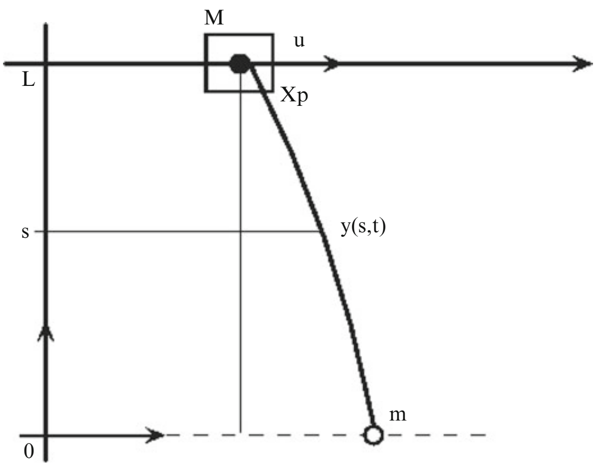

Fig. 9.1 Flexible crane mechanics—reproduced after [4]: s current coordinate on the flexible cable, $y(s,t)$ current position of the flexible cable element including local deformation, M mass of the controlled motorized platform, m payload mass, $u(t)$ the control force

where X_p is the motorized platform position and $\nu(s,t)$ is the elastic deflection of the flexible cable. Here and in the sequel the subscripts t and s denote the partial derivatives with respect to these variables.

2. The potential energy due to the strain energy of the flexible cable given by

$$E_p(t) = \frac{1}{2}\int_0^L T(s)\nu_s^2(s,t)\,\mathrm{d}s.$$

By $T(s)$ the strain force within the flexible cable was denoted. Remark that (9.2) implies $\nu_s(s,t) \equiv y_s(s,t)$.

3. The only external force acting on this mechanical system is the control force $u(t)$. Its work is given by

$$W_m(t) = u(t)X_p(t) = u(t)y(L,t).$$

The *generalized variational principle of Hamilton* stresses that, along the allowed ("legitimate") trajectories of a system, the functional

$$I = \int_{t_1}^{t_2} (E_k(t) - E_p(t) + W_m(t))\,\mathrm{d}t, \tag{9.3}$$

where $t_i, i = 1, 2$ are arbitrary, is minimal. Following the approach of the variational calculus [2] the following variations are introduced:

$$y(s,t) = \bar{y}(s,t) + \varepsilon\eta(s,t) \tag{9.4}$$

where $\bar{y}(s,t)$ corresponds to an extremal. Along (9.4) the functional (9.3) depends on ε. The necessary extremum condition is

$$I'(0) = \left.\frac{\mathrm{d}I}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0,$$

which is straightforward. We deduce

$$\int_{t_1}^{t_2} [my_t(0,t)\eta_t(0,t) + My_t(L,t)\eta_t(L,t) + u(t)\eta(L,t)]\,\mathrm{d}t +$$
$$+ \int_{t_1}^{t_2}\int_0^L [\rho(s)y_t(s,t)\eta_t(s,t) - T(s)y_s(s,t)\eta_s(s,t)]\,\mathrm{d}s\mathrm{d}t = 0.$$

Several integrations by parts and the use of the standard conditions on the Euler Lagrange variations—$\eta(s,t_i) \equiv 0, i = 1, 2$—will lead to the following equations of the overhead crane dynamics:

$$\begin{aligned} &-\rho(s)y_{tt} + (T(s)y_s)_s = 0 \\ &-my_{tt}(0,t) + T(0)y_s(0,t) = 0 \\ &-My_{tt}(L,t) - T(L)y_s(L,t) + u(t) = 0, \end{aligned} \tag{9.5}$$

which are much alike to those of [20]. We now take into account the conditions on the physics of the model given in [4]:

$$\rho(s) \equiv \rho, \;\; T(s) = mg + \rho s g, \tag{9.6}$$

that is the cable is uniform and the strain is induced by the payload and the mass of the flexible cable. Taking (9.6) into account and introducing the rated cable length coordinate $s = \sigma L$, $0 \le \sigma \le 1$, (9.5) are transformed into

$$\begin{aligned}
&\frac{L}{g}\cdot\frac{\rho L}{m} y_{tt} - \left(\left(1 + \frac{\rho L}{m}\sigma\right) y_\sigma\right)_\sigma = 0, \;\; t > 0, \;\; 0 < \sigma < 1 \\
&\frac{L}{g} y_{tt}(0,t) = y_\sigma(0,t) \\
&\frac{L}{g} y_{tt}(1,t) = -\frac{m}{M}\left(1 + \frac{\rho L}{m}\right) y_\sigma(1,t) + \frac{L}{Mg} u(t).
\end{aligned} \tag{9.7}$$

This form of the model allows introduction of the "pendulum-like" time constant $T_0 = \sqrt{L/g}$ and of the non-dimensional parameter $(\rho L)/m$. It has the form already used in [7, 27]; however, the correct sign in the third equation is in (9.7) only; in the aforementioned papers the error of [4] has migrated.

9.2.2 Small parameter analysis We shall reproduce here from [7, 27] the small parameter analysis. Let $(\rho L)/m << 1$ in order to consider negligible this ratio and equate it to 0. System (9.7) becomes

$$\begin{aligned}
&y_{\sigma\sigma} = 0, \;\; \frac{L}{g} y_{tt}(0,t) = y_\sigma(0,t) \\
&\frac{L}{g} y_{tt}(1,t) = -\frac{m}{M} y_\sigma(1,t) + \frac{L}{Mg} u(t).
\end{aligned}$$

We deduce

$$y(\sigma,t) = \phi_0(t) + \phi_1(t)\sigma, \;\; 0 \le \sigma \le 1,$$

which will thus be like an output defined along the solutions of the following system of ODEs

$$\frac{L}{g}\ddot{\phi}_0 = \phi_1, \;\; \frac{L}{g}\ddot{\phi}_1 = -\left(1 + \frac{m}{M}\right)\phi_1 + \frac{L}{Mg} u(t), \tag{9.8}$$

If we take into account the possibility of a local controller at $\sigma = 1$, i.e., on the motorized platform, then it is better to have $X_p = y(1,t) = \phi_0 + \phi_1$ as state variable; system (9.8) becomes

$$\begin{aligned}
&\frac{L}{g}\ddot{X}_p = -\frac{m}{M}(X_p - \phi_0) + \frac{L}{Mg} u(t) \\
&\frac{L}{g}\ddot{\phi}_0 = X_p - \phi_0,
\end{aligned} \tag{9.9}$$

with the distributed output

$$y(\sigma,t) = \sigma X_p(t) + (1-\sigma)\phi_0. \tag{9.10}$$

Assume now that $(\rho L)/m$ is small but not the coefficient of y_{tt} in the first equation of (9.7). This assumption signifies that regular perturbations are allowed but not singular ones. System (9.7) becomes

$$\begin{aligned}
&\frac{L}{g}\cdot\frac{\rho L}{m} y_{tt} - y_{\sigma\sigma} = 0,\ \ t>0,\ \ 0<\sigma<1\\
&\frac{L}{g} y_{tt}(0,t) = y_\sigma(0,t)\\
&\frac{L}{g} y_{tt}(1,t) = -\frac{m}{M} y_\sigma(1,t) + \frac{L}{Mg}u(t)
\end{aligned} \tag{9.11}$$

It appears that a possible singular perturbation construction will give a system of ODEs as describing the limit behavior of a BVP for hyperbolic PDE. This aspect will remain outside the structure of the present paper.

9.3 The Basic Theory for System (9.11)

We denote first

$$\frac{L}{g} = T_0^2,\ \ \frac{\rho L}{m} = \varepsilon_0,\ \ \frac{L}{Mg}u(t) = \mu(t),\ \ \frac{m}{M} = \delta_0 \tag{9.12}$$

to have

$$\begin{aligned}
&\varepsilon_0 T_0^2 y_{tt} - y_{\sigma\sigma} = 0,\ \ t>0,\ \ 0<\sigma<1\\
&T_0^2 y_{tt}(0,t) = y_\sigma(0,t)\\
&T_0^2 y_{tt}(1,t) = -\delta_0 y_\sigma(1,t) + \mu(t),
\end{aligned} \tag{9.13}$$

We follow now the approach of [12]: define the characteristic equations by

$$\frac{\mathrm{d}t}{\mathrm{d}\sigma} = \pm T_0\sqrt{\varepsilon_0}.$$

Therefore each point $(\sigma,t) \in \{(\sigma,t)|0 \le \sigma \le 1, t \ge 0\}$ is crossed by two characteristic lines:

$$t^\pm(\lambda;\sigma,t) = t \pm T_0\sqrt{\varepsilon_0}(\lambda-\sigma).$$

Introduce now the Riemann invariants, i.e., the forward wave $w^+(\sigma,t)$ and the backward wave $w^-(\sigma,t)$ defined by

$$y_t(\sigma,t) = w^+(\sigma,t) + w^-(\sigma,t),\quad y_\sigma(\sigma,t) = T_0\sqrt{\varepsilon_0}(w^-(\sigma,t) - w^+(\sigma,t)),\quad (9.14)$$

a system that can be reversed

$$w^\pm(\sigma,t) = \frac{1}{2}\left(y_t(\sigma,t) \mp \frac{1}{T_0\sqrt{\varepsilon_0}}y_\sigma(\sigma,t)\right). \quad (9.15)$$

We write (9.13) using the Riemann invariants

$$\begin{aligned}
&w_t^+ + \frac{1}{T_0\sqrt{\varepsilon_0}}w_\sigma^+ = 0,\quad w_t^- - \frac{1}{T_0\sqrt{\varepsilon_0}}w_\sigma^- = 0\\
&T_0\frac{\mathrm{d}}{\mathrm{d}t}(w^+(0,t) + w^-(0,t)) = \sqrt{\varepsilon_0}(w^-(0,t) - w^+(0,t))\\
&T_0^2\frac{\mathrm{d}}{\mathrm{d}t}(w^+(1,t) + w^-(1,t)) = -\delta_0 T_0\sqrt{\varepsilon_0}(w^-(1,t) - w^+(1,t)) + \mu(t),
\end{aligned} \quad (9.16)$$

which is of the type described in [12]. In the sequel we shall sketch that approach applied to (9.16). Let

$$\phi^+(\lambda) := w^+(\lambda, t^+(\lambda;0,t)) = w^+(\lambda, t + \lambda T_0\sqrt{\varepsilon_0})$$

$$\phi^-(\lambda) := w^-(\lambda, t^-(\lambda;1,t)) = w^-(\lambda, t + (1-\lambda)T_0\sqrt{\varepsilon_0}).$$

It is easily seen that $\mathrm{d}\phi^\pm/\mathrm{d}\lambda = 0$ hence integration along the characteristics will give

$$\begin{aligned}
&\phi^+(0) = w^+(0,t) = \phi^+(1) = w^+(1, t + T_0\sqrt{\varepsilon_0})\\
&\phi^-(1) = w^-(1,t) = \phi^-(0) = w^-(0, t + T_0\sqrt{\varepsilon_0}).
\end{aligned} \quad (9.17)$$

Denoting

$$\zeta^+(t) := w^+(1,t),\quad \zeta^-(t) := w^-(0,t), \quad (9.18)$$

the following system of equations with deviated argument is associated to (9.16)

$$\begin{aligned}
&T_0\frac{\mathrm{d}}{\mathrm{d}t}(\zeta^+(t + T_0\sqrt{\varepsilon_0}) + \zeta^-(t)) = -\sqrt{\varepsilon_0}(\zeta^+(t + T_0\sqrt{\varepsilon_0}) - \zeta^-(t))\\
&T_0^2\frac{\mathrm{d}}{\mathrm{d}t}(\zeta^-(t + T_0\sqrt{\varepsilon_0}) + \zeta^+(t)) = -\delta_0 T_0\sqrt{\varepsilon_0}(\zeta^-(t + T_0\sqrt{\varepsilon_0}) - \zeta^+(t)) + \mu(t).
\end{aligned} \quad (9.19)$$

Its initial conditions can also be constructed starting from the initial values of (9.16) given by

$$w^{\pm}(\sigma, 0) = w_0^{\pm}(\sigma), \quad 0 \le \sigma \le 1.$$

At their turn $w_0^{\pm}(\sigma)$ may be associated with the initial conditions of (9.11), based on (9.14). The construction of the initial conditions for (9.19), on the interval $0 \le t \le T_0\sqrt{\varepsilon_0}$, is realized by considering those characteristics that cross the abscissa $t = 0$, i.e., $t^+(\lambda; 1, t)$ and $t^-(\lambda; 0, t)$ for $0 \le t \le T_0\sqrt{\varepsilon_0}$. Integrating along these characteristics we find

$$\zeta_0^+(t) = w_0^+\left(1 - \frac{t}{T_0\sqrt{\varepsilon_0}}\right), \; \zeta_0^-(t) = w_0^-\left(\frac{t}{T_0\sqrt{\varepsilon_0}}\right), \quad 0 \le t \le T_0\sqrt{\varepsilon_0}. \tag{9.20}$$

We are now in position to state

Theorem D. *Consider system (9.16) with the initial conditions* $w_0^{\pm}(\sigma)$, $0 \le \sigma \le 1$ *and let* $w^{\pm}(\sigma, t)$ *be a solution of (9.16). Then* $\zeta^{\pm}(t)$, $t > T_0\sqrt{\varepsilon_0}$, *is a solution of (9.19) with the initial conditions defined by (9.20), having the degree of smoothness of* $w_0^{\pm}(\sigma)$ *and with discontinuities at* $t = kT_0\sqrt{\varepsilon_0}$, $k \in \mathbb{N}$.

Conversely, let $\zeta^{\pm}(t)$ *be a solution of (9.16) with the initial conditions given by* $\zeta_0^{\pm}(t)$ *on* $(0, T_0\sqrt{\varepsilon_0})$*, these initial conditions being smooth enough. Then the functions defined by*

$$\begin{aligned} w^+(\sigma, t) &= \begin{cases} \zeta^+(t + T_0\sqrt{\varepsilon_0}(1-\sigma)), & t - \sigma T_0\sqrt{\varepsilon_0} > 0 \\ \zeta_0^+(t + T_0\sqrt{\varepsilon_0}(1-\sigma)), & -T_0\sqrt{\varepsilon_0} < t - \sigma T_0\sqrt{\varepsilon_0} < 0, \end{cases} \\ w^-(\sigma, t) &= \begin{cases} \zeta^-(t + \sigma T_0\sqrt{\varepsilon_0}), & t + \sigma T_0\sqrt{\varepsilon_0} > T_0\sqrt{\varepsilon_0} \\ \zeta_0^-(t + \sigma T_0\sqrt{\varepsilon_0}), & 0 < t + \sigma T_0\sqrt{\varepsilon_0} < T_0\sqrt{\varepsilon_0} \end{cases} \end{aligned} \tag{9.21}$$

are solutions of (9.16) satisfying the equations except on the characteristics

$$t + T_0\sqrt{\varepsilon_0}(1-\sigma) = nT_0\sqrt{\varepsilon_0}, \quad t + \sigma T_0\sqrt{\varepsilon_0} = nT_0\sqrt{\varepsilon_0}; \quad n \in \mathbb{N},$$

with the initial conditions

$$w_0^+(\sigma) = \zeta_0^+(T_0\sqrt{\varepsilon_0}(1-\sigma)), \quad w_0^-(\sigma) = \zeta_0^-(\sigma T_0\sqrt{\varepsilon_0}); \quad 0 \le \sigma \le 1.$$

In this way all problems of the basic theory for (9.11) are projected via (9.16) on the system of differential equations with deviated argument (9.19).

Introducing new notations

$$\eta^{\pm}(t) := \zeta^{\pm}(t + T_0\sqrt{\varepsilon_0}), \tag{9.22}$$

system (9.19) gets a more standard form

$$T_0\frac{\mathrm{d}}{\mathrm{d}t}(\eta^+(t)+\eta^-(t-T_0\sqrt{\varepsilon_0})) = -\sqrt{\varepsilon_0}(\eta^+(t)-\eta^-(t-T_0\sqrt{\varepsilon_0}))$$
$$T_0^2\frac{\mathrm{d}}{\mathrm{d}t}(\eta^-(t)+\eta^+(t-T_0\sqrt{\varepsilon_0})) = -\delta_0 T_0\sqrt{\varepsilon_0}(\eta^-(t)-\eta^+(t-T_0\sqrt{\varepsilon_0}))+\mu(t), \tag{9.23}$$

with the initial conditions

$$\eta_0^+(t) = w_0^+\left(-\frac{t}{T_0\sqrt{\varepsilon_0}}\right),\ \eta_0^-(t) = w_0^-\left(1+\frac{t}{T_0\sqrt{\varepsilon_0}}\right),\ \ -T_0\sqrt{\varepsilon_0}\le t\le 0. \tag{9.24}$$

We shall recall here some qualitative problems that may be considered for (9.23) and (9.24): stability, stabilization, forced oscillations, and dissipativeness in the sense of Levinson, Perron condition, and others. Through the representation formulae (9.21) these problems may be projected back on system (9.16) and, further, on (9.13), i.e., on (9.11).

We end this section by some remarks concerning system (9.7): the space-varying coefficient does not allow to separate the Riemann invariants as in the examined case. Consequently, the associated functional differential equations are more complicated—see [1, 25]; moreover in those papers the associated equations are not written explicitly. However, since they account about the so-called propagation with distortions, to obtain these functional differential equations is still an urgent task [10, 28].

9.4 The Energy Identity and the Feedback Stabilization

We turn back to system (9.7), leave aside the existence and uniqueness problem and assume existence of sufficiently smooth solutions provided the initial conditions are such [these initial conditions are suggested by (9.15)]. Multiply the first line of (9.7) by $y_t(\sigma,t)$ and integrate from 0 to 1 with respect to σ. After some integration by parts and taking into account the boundary conditions, the following *energy identity* is obtained

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{L}{g}\left[(y_t(0,t))^2+\frac{M}{m}(y_t(1,t))^2\right]-\frac{L}{g}\cdot\frac{1}{m}u(t)y_t(1,t)+$$
$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left[\frac{\rho L}{m}\cdot\frac{L}{g}y_t^2(\sigma,t)+\left(1+\frac{\rho L}{m}\sigma\right)y_\sigma^2(\sigma,t)\right]\mathrm{d}\sigma\equiv 0. \tag{9.25}$$

This identity will give a hint concerning the Lyapunov functional to use in order to stabilize the overhead crane around a steady state. Physically, the steady state is given by the crane stopped at some position $\bar{X}_p$ and with the flexible cable "frozen" in some position prescribed by the steady state of (9.7).

Since $y(1,t) \equiv X_p(t)$, $\bar{y}(1) = \bar{X}_p$ and we can compute the steady state considering $u(t) = \bar{u}$, where $\bar{u}$ is a constant. All variables will be time invariant subject to

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(1 + \frac{\rho L}{m}\sigma\right)\frac{\mathrm{d}\bar{y}}{\mathrm{d}\sigma}\right] \equiv 0$$

$$\left.\frac{\mathrm{d}\bar{y}}{\mathrm{d}\sigma}\right|_{\sigma=0} = 0, \quad \frac{m}{M}\left(1 + \frac{\rho L}{m}\right)\left.\frac{\mathrm{d}\bar{y}}{\mathrm{d}\sigma}\right|_{\sigma=1} = \frac{L}{Mg}\bar{u}.$$

From the first equation we obtain

$$\left(1 + \frac{\rho L}{m}\sigma\right)\frac{\mathrm{d}\bar{y}}{\mathrm{d}\sigma} \equiv c_1$$

with c_1 an arbitrary constant. From the condition at $\sigma = 0$ it follows that $c_1 = 0$. We deduce the first derivative also identically 0; hence $\bar{y}(\sigma)$ must be constant and equal to $\bar{X}_p$. Moreover, the control force must be 0 at rest what is consistent with the laws of mechanics.

From (9.25) the following Lyapunov functional

$$\mathscr{V}(\phi(\cdot), \psi(\cdot), Y, X, Z) = \frac{1}{2}\left\{\int_0^1 \left[\frac{\rho L}{m}\cdot\frac{L}{g}\phi^2(\sigma) + \left(1 + \frac{\rho L}{m}\sigma\right)\psi^2(\sigma)\right]\mathrm{d}\sigma + \right.$$
$$\left. + \frac{L}{g}\left(Y^2 + \frac{M}{m}Z^2\right) + a(X - \bar{X}_p)^2\right\}, \tag{9.26}$$

with $a > 0$, an arbitrary constant parameter is inferred. This functional is at least nonnegative along the smooth solutions of (9.7). According to (9.25) the derivative along these solutions will be

$$\frac{\mathrm{d}\mathscr{V}^\star(t)}{\mathrm{d}t} = \left[a(y(1,t) - \bar{X}_p) + \frac{L}{g}\cdot\frac{1}{m}u(t)\right]y_t(1,t). \tag{9.27}$$

By choosing $u(t)$ from

$$a(y(1,t) - \bar{X}_p) + \frac{L}{g}\cdot\frac{1}{m}u(t) = -f(y_t(1,t)),$$

that is

$$u(t) = -\frac{mg}{L}[a(y(1,t) - \bar{X}_p) + f(y_t(1,t))], \tag{9.28}$$

where $f(\lambda)\lambda > 0$, $f(0) = 0$, i.e., it is a standard sector-restricted nonlinearity, it follows that indeed $\bar{u} = 0$.

In fact (9.28) defines a nonlinear PD (proportional derivative) controller which ensures positioning to $\bar{y}(1) = \bar{X}_p$ and introduces a nonlinear damping at the boundary $\sigma = 1$ where the control signal acts.

In this way the following closed-loop nonlinear system is obtained

$$\begin{aligned}
&\frac{L}{g}\cdot\frac{\rho L}{m}y_{tt} - \left(\left(1+\frac{\rho L}{m}\sigma\right)y_\sigma\right)_\sigma = 0,\ \ t>0,\ \ 0<\sigma<1\\
&\frac{L}{g}y_{tt}(0,t) - y_\sigma(0,t) = 0\\
&\frac{L}{g}y_{tt}(1,t) + \frac{m}{M}\left[f(y_t(1,t)) + a(y(1,t)-\bar{X}_p) + \left(1+\frac{\rho L}{m}\right)y_\sigma(1,t)\right] = 0.
\end{aligned} \tag{9.29}$$

In the considered state space the Lyapunov functional (9.26) is positive definite. Its derivative along the solutions of (9.29) is negative semi-definite. This ensures Lyapunov stability and global boundedness of the smooth solutions of (9.29) in the sense of the norm defined by the Lyapunov functional itself.

In order to prove asymptotic stability, it is necessary to dispose of a Barbašin–Krasovskii–La Salle theorem for dynamical systems generated by BVPs like (9.29). And this sends again to the basic theory.

9.5 Asymptotic Stability in a Limit Case

9.5.1 Description of the "second" limit case We shall consider here the "second limit case" when $(\rho L)/m$ is "small," but the product$(L/g)((\rho L)/m)$ is not, i.e., only regular perturbations have to be considered. The open-loop case is described by (9.11). Consequently the closed-loop case which is deduced from (9.29) will be described by

$$\begin{aligned}
&\frac{L}{g}\cdot\frac{\rho L}{m}y_{tt} - y_{\sigma\sigma} = 0,\ \ t>0,\ \ 0<\sigma<1\\
&\frac{L}{g}y_{tt}(0,t) - y_\sigma(0,t) = 0\\
&\frac{L}{g}y_{tt}(1,t) + \frac{m}{M}[f(y_t(1,t)) + a(y(1,t)-\bar{X}_p) + y_\sigma(1,t)] = 0.
\end{aligned} \tag{9.30}$$

If notations (9.12) are used, then (9.30) reads

$$\begin{aligned}
&\varepsilon_0 T_0^2 y_{tt} - y_{\sigma\sigma} = 0\\
&T_0^2 y_{tt}(0,t) - y_\sigma(0,t) = 0\\
&T_0^2 y_{tt}(1,t) + \delta_0(f(\dot{y}(1,t)) + a(y(1,t)-\bar{X}_p) + y_\sigma(1,t)) = 0.
\end{aligned} \tag{9.31}$$

All considerations concerned with the integration of the Riemann invariants, made at Sect. 9.3, remain valid. We have nevertheless to introduce the variable $X_p := y(1,t)$ since (9.14) shows that $y(\sigma,t)$ cannot be expressed using $w^{\pm}(\sigma,t)$ without integration with respect to t. It follows that (9.31) will take the form

$$\begin{aligned}
&\varepsilon_0 T_0^2 y_{tt} - y_{\sigma\sigma} = 0\\
&T_0^2 y_{tt}(0,t) - y_\sigma(0,t) = 0\\
&\dot{X}_p - y_t(1,t) = 0\\
&T_0^2 y_{tt}(1,t) + \delta_0(f(y_t(1,t)) + a(X_p - \bar{X}_p) + y_\sigma(1,t)) = 0.
\end{aligned}\tag{9.32}$$

We may now write down (9.32) using the Riemann invariants

$$\begin{aligned}
&w_t^+ + \frac{1}{T_0\sqrt{\varepsilon_0}} w_\sigma^+ = 0, \;\; w_t^- - \frac{1}{T_0\sqrt{\varepsilon_0}} w_\sigma^- = 0\\
&T_0 \frac{\mathrm{d}}{\mathrm{d}t}(w^+(0,t) + w^-(0,t)) - \sqrt{\varepsilon_0}(w^-(0,t) - w^+(0,t)) = 0\\
&\dot{X}_p - (w^+(1,t) + w^-(1,t)) = 0\\
&T_0^2 \ddot{X}_p + \delta_0(f(\dot{X}_p) + a(X_p - \bar{X}_p) + T_0\sqrt{\varepsilon_0}(w^-(1,t) - w^+(1,t)) = 0.
\end{aligned}$$

We take now into account (9.17), (9.18), and (9.22) to obtain the system

$$\begin{aligned}
&T_0 \frac{\mathrm{d}}{\mathrm{d}t}(\eta^+(t) + \eta^-(t - T_0\sqrt{\varepsilon_0})) + \sqrt{\varepsilon_0}(\eta^+(t) - \eta^-(t - T_0\sqrt{\varepsilon_0})) = 0\\
&\dot{X}_p - (\eta^-(t) + \eta^+(t - T_0\sqrt{\varepsilon_0})) = 0\\
&T_0^2 \ddot{X}_p + \delta_0(f(\dot{X}_p) + a(X_p - \bar{X}_p) + T_0\sqrt{\varepsilon_0}(\eta^-(1,t) - \eta^+(t - T_0\sqrt{\varepsilon_0})) = 0
\end{aligned}\tag{9.33}$$

From (9.33) the following standard system of NFDEs is obtained:

$$\begin{aligned}
&T_0 \frac{\mathrm{d}}{\mathrm{d}t}(\eta^+(t) + \eta^-(t - T_0\sqrt{\varepsilon_0})) = -\sqrt{\varepsilon_0}(\eta^+(t) - \eta^-(t - T_0\sqrt{\varepsilon_0}))\\
&T_0^2 \frac{\mathrm{d}}{\mathrm{d}t}(\eta^-(t) + \eta^+(t - T_0\sqrt{\varepsilon_0})) + \delta_0(T_0\sqrt{\varepsilon_0}(\eta^-(t) - \eta^+(t - T_0\sqrt{\varepsilon_0})) +\\
&+ a(X_p - \bar{X}_p) + f(\eta^-(t) + \eta^+(t - T_0\sqrt{\varepsilon_0})) = 0\\
&\dot{X}_p = \eta^-(t) + \eta^+(t - T_0\sqrt{\varepsilon_0}).
\end{aligned}\tag{9.34}$$

The representation formulae (9.21) will be necessary also. We rewrite them as follows:

$$w^+(\sigma,t) = \eta^+(t - \sigma T_0\sqrt{\varepsilon_0}), \;\; w^-(\sigma,t) = \eta^-(t + (\sigma - 1)T_0\sqrt{\varepsilon_0}), \tag{9.35}$$

hence

$$
\begin{aligned}
y_t(\sigma,t) &= \eta^+(t-\sigma T_0\sqrt{\varepsilon_0}) + \eta^-(t+(\sigma-1)T_0\sqrt{\varepsilon_0}) \\
y_\sigma(\sigma,t) &= T_0\sqrt{\varepsilon_0}(\eta^-(t+(\sigma-1)T_0\sqrt{\varepsilon_0}) - \eta^+(t-\sigma T_0\sqrt{\varepsilon_0})).
\end{aligned}
\tag{9.36}
$$

At their turn formulae (9.35) allow the use of the Lyapunov functional (9.26) suggested by (9.25). Using (9.35) the Lyapunov functional becomes

$$
\begin{aligned}
\mathrm{V}(\eta^+(\cdot),\eta^-(\cdot),X_p) = {} & \frac{1}{2}T_0^2[(\eta^+(0)+\eta^-(-T_0\sqrt{\varepsilon_0}))^2 + \\
& + (1/\delta_0)(\eta^-(0)+\eta^+(-T_0\sqrt{\varepsilon_0}))^2] + \\
& + \frac{1}{2}T_0\sqrt{\varepsilon_0}\int_{-T_0\sqrt{\varepsilon_0}}^0 (\eta^+(\theta)+\eta^-(-T_0\sqrt{\varepsilon_0}-\theta))^2\,\mathrm{d}\theta + \\
& + \frac{1}{2}\int_{-T_0\sqrt{\varepsilon_0}}^0 (\eta^+(\theta)-\eta^-(-T_0\sqrt{\varepsilon_0}-\theta))^2\,\mathrm{d}\theta + \frac{1}{2}a(X_p-\bar{X}_p)^2
\end{aligned}
\tag{9.37}
$$

written as a functional on the state space. Taking into account (9.27) and the control choice (9.28), the derivative functional associated to (9.37) by differentiating it along the solutions of (9.34) has the form

$$
\mathscr{W}(\eta^+(\cdot),\eta^-(\cdot)) = -(\eta^-(0)+\eta^+(-T_0\sqrt{\varepsilon_0}))f(\eta^-(0)+\eta^+(-T_0\sqrt{\varepsilon_0})) \le 0.
$$

According to the Barbašin–Krasovskii–La Salle invariance principle, the solutions of (9.34)—which are bounded in the sense of the norm defined by (9.37)—will approach the largest positive invariant set contained in the set where $\mathscr{W}(\eta^+(\cdot),\eta^-(\cdot))$ vanishes. Taking into account the properties of $f(\cdot)$ we shall have

$$
\eta^-(t)+\eta^+(t-T_0\sqrt{\varepsilon_0}) \equiv 0, \quad X_p \equiv \text{const}, \quad f(\eta^-(t)+\eta^+(t-T_0\sqrt{\varepsilon_0})) \equiv 0
$$

on the invariant set. It follows that on this set η^- and η^+ are constant and they can be computed from

$$
\bar{\eta}^- + \bar{\eta}^+ = 0, \quad \bar{\eta}^- - \bar{\eta}^+ = \frac{a}{T_0\sqrt{\varepsilon_0}}(X_p-\bar{X}_p).
$$

On the other hand the first equation of (9.35) will give $\bar{\eta}^+ - \bar{\eta}^- = 0$. Thus it follows that the invariant set contains only the equilibrium $(0,0,\bar{X}_p)$ which is approached asymptotically by all solutions of (9.35). Taking into account the representation formulae (9.36) we obtain

$$
\lim_{t\to\infty} y_t(\sigma,t) = \lim_{t\to\infty} y_\sigma(\sigma,t) = 0,
$$

uniformly on $0 \le \sigma \le 1$. It follows that $y(\sigma,t) \to$ const, uniformly on $0 \le \sigma \le 1$ hence $y(\sigma,t) \to \bar{X}_p$.

Global asymptotic stability is thus obtained *provided the invariance principle holds* for system (9.34). This last aspect requires an additional discussion.

9.5.2 The difference operator From a standard reference [19] it is known that for NFDEs having the general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{D}x_t = f(x_t),$$

the invariance principle is proved for the case when the difference operator $\mathscr{D}$ is stable, i.e., its spectrum is in the open left half plane of $\mathbb{C}$. Worth mentioning that most results dealing with NFDEs are obtained for the case of the stable $\mathscr{D}$ operator. Let us consider system (9.34). Its $\mathscr{D}$ operator reads as follows:

$$\mathscr{D}\begin{pmatrix}\phi^+\\ \phi^-\\ X_p\end{pmatrix}(\cdot) = \begin{pmatrix}\phi^+(0)\\ \phi^-(0)\\ X_p\end{pmatrix} - \begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi^+(-T_0\sqrt{\varepsilon_0})\\ \phi^-(-T_0\sqrt{\varepsilon_0})\\ X_p\end{pmatrix}.$$

The matrix D which defines the delayed terms has its eigenvalues $\{0, \pm 1\}$; hence there are two chains of eigenvalues of the difference operator on the imaginary axis. The theorems of [19] cannot be applied here. It is shown however in [31] that the essential requirement for proving an invariant principle is to obtain precompactness of the positive orbits of the associated dynamical system in the Banach space. It is interesting to remark that while this precompactness has been discussed in the case of PDEs, the most cited reference being [13] (which underlies the analysis of [4]), all analysis concerning NFDEs still relies on the strong stability of the difference operator. This assumption is however liable to criticism. Two arguments may be provided here. Firstly, there exist some papers, among which we cite [33], where it is shown that NFDEs with stable $\mathscr{D}$-operator are in some sense reducible to retarded FDEs being thus a *trivial case* of neutral FDEs. On the other hand we have seen throughout the references of the present paper that neutral FDEs occur in a natural way in the context of the BVPs for hyperbolic PDEs. But, as experience shows, when mechanical systems are concerned (containing strings or beams), the associated difference operator is not stable but in the critical case. *Relaxation of the stability assumption for the difference operator* is thus *an urgent task*.

With respect to this it is interesting to cite the results of [23]. In this monograph the stability properties of NFDEs are also connected with the properties of the nonlinear difference operator $\mathscr{D} : \mathbb{R} \times \mathscr{C}(-\tau, 0; \mathbb{R}^n)$

$$\mathscr{D}(t, \phi) = \phi(0) - G(t, \phi(\cdot))$$

more precisely, of the inequality

$$|\mathscr{D}(t, y_t)| \le f(t),$$

where $f : \mathbb{R} \mapsto \mathbb{R}_+$ is continuous. The condition upon G that is shown to be useful in asymptotic stability is a contractive Lipschitz condition

$$|G(t,\phi) - G(t,\psi)| \leq \alpha\|\phi - \psi\|,\ \ 0 < \alpha < 1;\ \ t \geq 0.$$

It is not clear if this condition is less restrictive than stability in the linear case (which is in any case globally Lipschitz).

Two are the conclusions of this discussion: to analyze the conditions of [23] in connection with the standard stability ones and to seek other ways of obtaining pre-compactness of the positive orbits. One may start from the most cited reference [13] whose results apply to PDEs too [4]. We shall consider this case in the next section.

9.6 On the Basic Theory and Asymptotic Stability for the Closed-Loop System

Consider the closed-loop system (9.29) with the notation (9.12)

$$\begin{aligned}
&\varepsilon_0 T_0^2 y_{tt} - ((1+\varepsilon_0\sigma)y_\sigma)_\sigma = 0,\ \ t > 0,\ \ 0 < \sigma < 1\\
&T_0^2 y_{tt}(0,t) - y_\sigma(0,t) = 0 \\
&T_0^2 y_{tt}(1,t) + \delta_0(f(y_t(1,t)) + a(y(1,t) - \bar{X}_p) + (1+\varepsilon_0)y_\sigma(1,t)) = 0,
\end{aligned} \tag{9.38}$$

with the associated Lyapunov functional (9.26) written accordingly

$$\begin{aligned}
\mathscr{V}(\phi(\cdot),\psi(\cdot),Y,X,Z) = \frac{1}{2}\Big\{\int_0^1 \left[\varepsilon_0 T_0^2\phi^2(\sigma) + (1+\varepsilon_0\sigma)\psi^2(\sigma)\right] \mathrm{d}\sigma\ + \\
+\ T_0^2(Y^2 + (1/\delta_0)Z^2) + a(X - \bar{X}_p)^2\Big\}\,.
\end{aligned} \tag{9.39}$$

In order to further simplify the notations we rate the time variable $\tau = t/T_0$. With a slight abuse of notation we denote $y(\sigma,\tau) := y(\sigma,T_0\tau)$ and $g(\eta) := f(\eta/T_0)$. System (9.38) may be written as follows:

$$\begin{aligned}
&\varepsilon_0 y_{\tau\tau} - ((1+\varepsilon_0\sigma)y_\sigma)_\sigma = 0,\ \ \tau > 0,\ \ 0 < \sigma < 1\\
&y_{\tau\tau}(0,\tau) - y_\sigma(0,\tau) = 0 \\
&y_{\tau\tau}(1,t) + \delta_0(g(y_\tau(1,\tau)) + a(y(1,\tau) - \bar{X}_p) + (1+\varepsilon_0)y_\sigma(1,\tau) = 0.
\end{aligned} \tag{9.40}$$

For this system the energy identity reads as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(y_\tau(0,\tau))^2 + \frac{1}{\delta_0}(y_\tau(1,\tau))^2 + a(y(1,\tau - \bar{X}_p))^2\right] +$$

$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 \left[\varepsilon_0 y_\tau^2(\sigma,\tau) + (1+\varepsilon_0\sigma)y_\sigma^2(\sigma,\tau)\right]\mathrm{d}\sigma + g(y_\tau(1,\tau))y_\tau(1,\tau) \equiv 0.$$

9.6.1 Basic theory via maximal monotone operators Introducing the new variables

$$v(\sigma,\tau) = y_\tau(\sigma,\tau),\;\; w(\sigma,\tau) = y_\sigma(\sigma,\tau);\;\; X_p = y(1,\tau),$$

system (9.40) takes the form

$$\begin{aligned}
&v_\tau - \frac{1}{\varepsilon_0}((1+\varepsilon_0\sigma)w)_\sigma = 0,\;\; \tau > 0,\;\; 0 < \sigma < 1\\
&w_\tau - v_\sigma = 0\\
&v_\tau(0,\tau) - w(0,\tau) = 0\\
&\dot{X}_p - v(1,\tau) = 0\\
&v_\tau(1,\tau) + \delta_0(a(X_p - \bar{X}_p) + g(v(1,\tau)) + (1+\varepsilon_0)w(1,\tau)) = 0,
\end{aligned} \tag{9.41}$$

with the energy identity

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(v(0,\tau))^2 + \frac{1}{\delta_0}(v(1,\tau))^2 + a(X_p - \bar{X}_p)^2\right] +\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 \left[\varepsilon_0 v^2(\sigma,\tau) + (1+\varepsilon_0\sigma)w^2(\sigma,\tau)\right]\mathrm{d}\sigma + g(v(1,\tau))v(1,\tau) \equiv 0.
\end{aligned} \tag{9.42}$$

Define the following state vector

$$z = col(\phi(\cdot),\; \psi(\cdot),\; v_0,\; \xi_p,\; v_p),$$

with ϕ and ψ being functions defined on [0, 1]. This space is organized as a Hilbert space

$$\begin{aligned}
\mathscr{H} = \{&(\phi(\cdot), \psi(\cdot), v_0, \xi_p, v_p) \in L^2(0,1)\times L^2(0,1)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\\
&\mid v_0 = \phi(0), v_p = \phi(1)\}
\end{aligned}$$

with the scalar product induced by the energy identity

$$\langle(\phi(\cdot),\ \psi(\cdot),\ v_0,\ \xi_p,\ v_p),(\hat{\phi}(\cdot),\ \hat{\psi}(\cdot),\ \hat{v}_0,\ \hat{\xi}_p,\ \hat{v}_p)\rangle =$$
$$= \int_0^1 (\varepsilon_0\phi(\sigma)\hat{\phi}(\sigma) + (1+\varepsilon_0\sigma)\psi(\sigma)\hat{\psi}(\sigma))\,\mathrm{d}\sigma + v_0\hat{v}_0 + a\xi_p\hat{\xi}_p + \frac{1}{\delta_0}v_p\hat{v}_p \tag{9.43}$$

(this explains why $\mathscr{H}$ is called *energy space*).

Along the solutions of (9.41) the state vector $z(\tau)$ is defined by

$$z(\tau) = col(v(\cdot,\tau),\ w(\cdot,\tau),\ v(0,\tau),\ X_p(\tau) - \bar{X}_p,\ v(1,\tau)),$$

and (9.41) is written as

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \mathscr{A}z + \mathscr{F}(z), \tag{9.44}$$

where $\mathscr{A}$ is an unbounded differential linear operator defined as

$$\mathscr{A}z := \begin{pmatrix} -(1/\varepsilon_0)\frac{\mathrm{d}}{\mathrm{d}\sigma}((1+\varepsilon_0\sigma)\psi(\sigma)) \\ -\frac{\mathrm{d}}{\mathrm{d}\sigma}\phi(\sigma) \\ -\psi(0) \\ -v_p \\ \delta_0(1+\varepsilon_0)\psi(1) + \delta_0 a\xi_p \end{pmatrix},$$

on

$$\mathrm{Dom}\mathscr{A} = \{(\phi(\cdot),\ \psi(\cdot),\ v_0,\ \xi_p,\ v_p) \in H^1(0,1)\times H^1(0,1)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$$
$$|v_0 = \phi(0), v_p = \phi(1)\},$$

while the nonlinear operator $\mathscr{F}$ is defined by

$$\mathscr{F}(z) := col(0,\ 0,\ 0,\ 0,\ g(v_p)). \tag{9.45}$$

By adapting the construction of [4] it can be proved that $\mathscr{A}$ is maximal monotone provided the BVP

$$\begin{aligned}
&\phi'(\sigma) - \psi(\sigma) = -\psi^0(\sigma)\\
&(1+\varepsilon_0\sigma)\psi'(\sigma) - \varepsilon_0\phi(\sigma) + \varepsilon_0\psi(\sigma) = -\varepsilon_0\phi^0(\sigma)\\
&\phi(0) - \psi(0) = v_0^0\\
&(1+a\delta_0)\phi(1) + \delta_0(1+\varepsilon_0)\psi(1) = -a\delta_0\xi_p^0 + v_p^0
\end{aligned}$$

has a solution for any $z^0 = col(\phi^0(\cdot),\ \psi^0(\cdot),\ v_0^0,\ \xi_p^0,\ v_p^0) \in \mathscr{H}$—see [5]. Also if g is nondecreasing then $\mathscr{F}$ is maximal monotone. Then, following the standard approach the following results can be obtained:

(i) The canonical embedding from Dom$\mathscr{A}$, equipped with the graph norm, into $\mathscr{H}$ is *compact*.
(ii) For any initial data $z_0 \in$ Dom$\mathscr{A}$ system (9.44) has a unique strong solution $z(\tau) \in$ Dom$\mathscr{A}$.
(iii) For any initial data $z_0 \in \mathscr{H}$ system (9.44) has a unique weak solution $z(\tau) \in \mathscr{H}$ defined by $z(\tau) = S(\tau)z_0$ where $\{S(\tau)\}_{t\geq 0}$ is the semigroup of nonlinear contractions on $\mathscr{H}$ generated by the operator $\mathscr{A} + \mathscr{F}$.

9.6.2 Asymptotic stability We are now in position to obtain asymptotic stability of the equilibrium $(0, 0, 0, \bar{X}_p, 0)$. As known from the previous sections of the paper, we use the Lyapunov functional induced by the energy identity (9.42). Since

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathscr{V}(z(\tau)) = -g(v(1,\tau))v(1,\tau) \leq 0,$$

the solutions are bounded on the space $\mathscr{H}$, in particular on Dom$\mathscr{A} \subset \mathscr{H}$, where the strong solution exists and is unique. The boundedness is in the sense of the norm (9.43) induced by the Lyapunov functional hence by the energy identity. Therefore the ω-limit set in Dom$\mathscr{A}$ is nonempty and invariant with respect to $S(\tau)$. Also from the aforementioned properties the trajectory $z(\tau) = S(\tau)z_0$, $\tau \geq 0$, is precompact in $\mathscr{H}$. Let $\omega_0 \in \Omega(z_0)$ be some element in the ω-limit set of $z(\tau, z_0)$ and consider the orbit $z(\tau, \omega_0)$ which is enclosed in $\Omega(z_0)$. From the invariance principle we deduce

$$\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\tau}(z(\tau,\omega_0)) = 0,$$

hence along this trajectory we have $g(v_p(\tau))v_p(\tau) = 0$ for all $\tau \geq 0$. We deduce that $v_p(\tau) \equiv 0$, from the properties of the function $g(\cdot)$ which is nondecreasing, sector restricted and $g(0) = 0$. Here $v_p(\tau) = v(1,\tau) = 0$. This gives also $\dot{v}_p(\tau) = 0$.

It follows that $z(\tau, \omega_0)$ (or what is left of it) satisfies the system

$$\begin{aligned}
&\varepsilon_0 v_\tau - ((1+\varepsilon_0\sigma)w)_\sigma = 0, \quad \tau > 0, \quad 0 < \sigma < 1\\
&w_\tau - v_\sigma = 0\\
&\dot{v}(0,\tau) - w(0,\tau) = 0\\
&a(X_p - \bar{X}_p) + \delta_0(1+\varepsilon_0)w(1,\tau) \equiv 0.
\end{aligned} \tag{9.46}$$

Since $X_p \equiv$ const, it follows that $w(1,\tau) \equiv$ const. It is not difficult to obtain, using one of the methods of variable separation (Fourier analysis or the Laplace transform), that $v(\sigma,\tau) = w(\sigma,\tau) \equiv 0$ is the only solution of (9.46). This will give $X_p = \bar{X}_p$; hence the equilibrium $(0,\ 0,\ 0,\ \bar{X}_p,\ 0)$ is the only element of $\Omega(z_0)$ with $z_0 \in \mathscr{H}$ arbitrary. The asymptotic stability is proved.

9.7 Some Conclusions and Open Problems

The first important conclusion of the paper is that a sound deduction of the model, based on a variational approach, can offer a better basis for qualitative studies. We have considered here one of the models occurring in the control of mechanical systems; this model arises from the assimilation of the Euler–Bernoulli beam as a string. The result of this variational deduction is a BVP for the string equation, with dynamic boundary conditions.

The basic and qualitative analysis for such BVPs can be based on the association of some FDEs by integrating the Riemann invariants along the characteristics. Usually these FDEs are of neutral type; in the so-called lossless or distortionless propagation these equations allow construction of the entire qualitative theory (existence, uniqueness, data dependence, stability) which afterwards is projected back on the solutions of the BVP via the representation formulae. It has been observed however in the paper (Sect. 9.5) that the standard requirement in the theory of NFDEs—the stability of the difference operator—does not hold in the case of the systems arising from mechanics. On the other hand, as shown in Sect. 9.6, it is possible to apply the theory of the maximal monotone operators to the BVP with dynamic boundary conditions.

This theory has to be further developed, at the same time with the analysis of the FDEs occurring in more complicated cases [1, 25]. The Barbašin–Krasovskii–La Salle invariance principle for such equations of neutral type with weakly stable difference operator is waiting to be proved.

Another open direction of research is to consider the same physical structures that have motivated the present paper (flexible robot arm, overhead crane, drill string) but with other kinds of beams (Euler–Bernoulli, Rayleigh, Timoshenko, etc.)—see [29].

And last but not least, existence of a basic theory is important for computational issues also; it may contribute to the proof of convergence for some methods that are used in solution approximation. For instance the approximation of the delays by ODEs has given a basis for the method of lines for hyperbolic PDE [18].This method is again in the attention of the researchers being implemented using approaches of the artificial intelligence [14, 15].

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Chapter 10
Discrete Itô Formula for Delay Stochastic Difference Equations with Multiple Noises

Alexandra Rodkina

Abstract For stochastic difference equation with multiple noises, finite delays and a parameter h we prove a variant of discrete Itô formula. Then we apply the formula to derive conditions which provide either $\mathbf{P}\{\lim_{n\to\infty} x_n = 0\} = 1$ or $\mathbf{P}\{\liminf_{n\to\infty} |x_n| > 0\} = 1$, where x_n is a solution of the equation with sufficiently small parameter h.

Keywords Stochastic difference equation with multiple noises and finite delays • Discrete Itô formula • Asymptotic stability and instability • Martingale convergence theorems

10.1 Introduction

Consider stochastic difference equations with multiple noises and finite delays:

$$x_{n+1} = x_n \left(1 + hf\left(x_{n-K}^n\right) + \sqrt{h} \sum_{i=1}^{m} g^{[i]}\left(x_{n-K}^n\right) \xi_{n+1}^{[i]} \right), \quad n \in \mathbb{N}, \tag{10.1}$$

$$x_s = \beta_s, \quad s = 0, -1, \ldots, -K. \tag{10.2}$$

Here $\beta_0, \beta_{-1}, \ldots, \beta_{-K} \in \mathbb{R}$ are given numbers,

$$x_{n-K}^n := (x_n, x_{n-1}, \ldots, x_{n-K}) \in \mathbb{R}^{K+1}, \quad n \in \mathbb{N},$$

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$K, m \in \mathbb{N}$, $f, g^{[i]} : \mathbb{R}^{K+1} \to \mathbb{R}$ are nonrandom, continuous and uniformly bounded functions, and $\xi_{n+1}^{[i]}$ are independent, normally distributed noises, $i = 1, 2, \ldots, m$.

Equation (10.1) can be thought of as a Euler–Maruyama discretization of the delay stochastic Itô differential equation, where h is a step of discretization (see, e.g., [1, 2]). However, in this note, we consider Eq. (10.1) just as a stochastic difference equation having a parameter h which can be made as small as necessary.

We obtain the variant of the discrete Itô formula for Eq. (10.1) which is similar to the formula first introduced by Appleby et al. [4]. The main purpose of the discrete Itô formula is to mimic the classical Itô formula for continues processes when we deal with the discrete process described by the equation with small parameter h, similar to Eq. (10.1). In [4] the discrete Itô formula was applied to derive conditions of a.s. asymptotic stability and instability for the partial case of Eq. (10.1), which does not have delays, $K = 0$, and contains only one noise, $m = 1$. Later the method was applied to a special system in [6], to the scalar equation obtained after Itô–Milshtein discretization in [8], and to the system of non-linear equations with the diagonal noises in [7]. For each result the specific variant of the discrete Itô formula has been derived.

One of the advantages of the discrete Itô formula is the fact that discrete Itô formula allows to implement the idea of "stabilization by noise" (see, e.g., [1–4]). In other words, the noise coefficients are included into the stability conditions in such a way that the introduction of the noise terms into the originally unstable deterministic equation sometimes makes stochastic equation stable. Obtained conditions are quite sharp and in some cases are close to necessary and sufficient conditions.

A new variant of the discrete Itô formula is proved in Sect. 10.3. For the proof we use several results on the convergence of nonnegative martingales which we formulate in Sect. 10.2. Even though this variant of the discrete Itô formula generalizes the result from [4] to the delay multi-noise equation, in the proof we essentially use that fact that the noises are normally distributed (which is not the case for [4]). However we believe that it would be possible to extend this result for the case of noises with quickly decaying tails.

The structure of the functions f and g in Eq. (10.1) seems to be relevant to the population dynamics of a single species. In this context stability and instability results for Eq. (10.1) give information concerning the extinction or permanence of the population.

One of the obstacles in the application of the discrete Itô formula is the assumption of the boundedness of functions f and g, which is important in our analysis. To get around this assumption is planned for the future work.

In Sect. 10.4 we derive stability result; in Sect. 10.5 we derive instability result for Eq. (10.1). A simple example is given in Sect. 10.6.

10.2 Preliminaries

Assumption 10.1. *Random variables $\xi_n^{(i)}$, $i = 1, \dots, m$, $n \in \mathbb{N}$, are mutually independent. For each $i = 1, \dots, m$, $\left(\xi_n^{(i)}\right)_{n\in\mathbb{N}}$ is a sequence of normally $\mathcal{N}(0,1)$ distributed random variables.*

Let $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\in\mathbb{N}}, \mathbf{P})$ be a complete, filtered probability space, where the filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$ is naturally generated by $\left\{\xi_n^{(i)},\ i = 1, 2, \dots, m,\ n \in \mathbb{N}\right\}$:

$$\mathscr{F}_n = \sigma\left\{\xi_j^{[i]} : i = 1, 2, \dots, m,\ j = 0, 1, \dots, n\right\},$$

where sequence $\{\xi_n\}_{n\in\mathbb{N}}$ satisfies Assumption 10.1.

Among all sequences $(w_n)_{n\in\mathbb{N}}$ of random variables we distinguish those for which w_n is $\mathscr{F}_n$-measurable for all $n \in \mathbb{N}$. We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbf{P}$ throughout the text.

A stochastic sequence $\{M_n\}_{n\in\mathbb{N}}$ is an $\mathscr{F}_n$-martingale, if $\mathbf{E}|M_n| < \infty$ and $\mathbf{E}[M_n \big| \mathscr{F}_{n-1}] = M_{n-1}$, a.s., for all $n \in \mathbb{N}$.

A stochastic sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is an $\mathscr{F}_n$-martingale difference, if $\mathbf{E}|\eta_n| < \infty$ and $\mathbf{E}\left[\eta_n \big| \mathscr{F}_{n-1}\right] = 0$, a.s., for all $n \in \mathbb{N}$.

Next two results are variants of limit theorems for martingales. Lemma 10.1 can be found, e.g., in [9]; Lemma 10.2 was proved in [1].

Lemma 10.1. *If $(M_n)_{n\in\mathbb{N}}$ is a nonnegative martingale, then $\lim_{n\to\infty} M_n$ exists with probability* 1.

Lemma 10.2. *Let $(w_n)_{n\in\mathbb{N}}$ be a nonnegative $\mathscr{F}_n$-measurable process, $\mathbf{E}|w_n| < \infty\ \forall n \in \mathbb{N}$, and*

$$w_{n+1} \le w_n + a_n - b_n + \varsigma_{n+1}, \quad n = 0, 1, 2, \dots,$$

where $(\varsigma_n)_{n\in\mathbb{N}}$ is an $\mathscr{F}_n$-martingale difference, $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are nonnegative $\mathscr{F}_n$-measurable processes and $\mathbf{E}|a_n|, \mathbf{E}|b_n| < \infty \forall n \in \mathbb{N}$. Then

$$\left\{\omega : \sum_{n=1}^{\infty} a_n < \infty\right\} \subseteq \left\{\omega : \sum_{n=1}^{\infty} b_n < \infty\right\} \bigcap \{w \to\}.$$

By $\{w \to\}$ we denote the set of all $\omega \in \Omega$ for which $\lim_{n\to\infty} w_n(\omega)$ exists and is finite.

The next lemma is proved in [8] (see also, e.g., Shiryaev [9] and Williams [10]).

Lemma 10.3. *Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of nonnegative random variables defined on $(\Omega, \mathscr{F}, \mathbf{P})$, adapted to the filtration $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$, and such that $\mathbf{E}Y_i < \infty$ and $\mathbf{E}[Y_i|\mathscr{F}_{i-1}] = 1$. Then the sequence $\{M_n\}_{n\in\mathbb{N}}$ given by*

$$M_n = \prod_{i=1}^{n} Y_i, \quad n \in \mathbb{N},$$

is an $\mathscr{F}_n$-martingale.

A detailed discussion of stochastic concepts, notation and results may be found in, e.g., Shiryaev [9].

For $\bar{y} = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$ we denote $\|\bar{y}\|^2 = \sum_{i=1}^m |y_i|^2$. Also, $\bar{0} := (0, \dots, 0) \in \mathbb{R}^{K+1}$.

Denote by $\mathscr{O}$ a function $\mathscr{O} : [0, \infty) \to [0, \infty)$ such that for some $H_0 > 0$ and all $t \in [0, \infty)$ we have

$$\mathscr{O}(t) \le H_0 t. \tag{10.3}$$

Assumption 10.2. *Let $f, g^{[i]} : \mathbb{R}^{K+1} \to \mathbb{R}$ be nonrandom, continuous and uniformly bounded functions, i.e., there exists a constant $H > 0$ such that, for all $\bar{y} \in \mathbb{R}^m$ and $i = 1, 2, \dots, m$,*

$$|f(\bar{y})| \le H, \quad |g^{[i]}(\bar{y})| \le H. \tag{10.4}$$

Let also $\|g(\bar{y})\| = 0$ imply that $\bar{y} \equiv \bar{0}$.

10.3 Itô Formula

In this section we derive a variant of the Itô formula specifically for Eq. (10.1).

Lemma 10.4. *Let Assumption 10.1 hold. Let f_n and $g_n^{[i]}$, $i = 1, \dots, m$, be $\mathscr{F}_n$-measurable uniformly bounded random variables and let u_{n+1} be defined as*

$$u_{n+1} := 1 + hf_n + \sqrt{h} \sum_{i=1}^{m} g_n^{[i]} \xi_{n+1}^{[i]}, \quad n \in \mathbb{N}.$$

Assume that for function $\phi : \mathbb{R} \to \mathbb{R}$ and for $\delta \in (0, 1)$ there exists a function $\phi_\delta : \mathbb{R} \to \mathbb{R}$ such that

(i) ϕ_δ is three times differentiable and the third derivative is uniformly bounded on $\mathbb{R}$;
(ii) $\phi(u) - \phi_\delta(u) = 0$ when $u \in (-\infty, -\delta) \cup (\delta, \infty)$;
(iii) $\int_{|u|<\delta} |\phi(u) - \phi_\delta(u)|\, \mathrm{d}u < \infty$.

Then there exists h_0 such that, for all $h \le h_0$ and for each $n \in \mathbb{N}$,

$$\mathbf{E}\left[\phi(u_{n+1}) \middle| \mathscr{F}_n\right] = \phi(1) + \phi'(1)hf_n + \frac{1}{2}\phi''(1)h\|g_n\|^2 + \mathscr{O}(h^{3/2})\big[|f_n| + \|g_n\|^2\big], \tag{10.5}$$

where function $\mathscr{O}$, defined by Eq. (10.3), does not depend on n.

Remark 10.1. Discrete Itô formula in Lemma 10.4 is similar to the formula first introduced by Appleby et al. [4]. The proof follows the same steps: first we show that Eq. (10.5) holds with ϕ_δ in place of ϕ, and then we estimate

$$\Delta := \mathbf{E}\left[|\phi(u_{n+1}) - \phi_\delta(u_{n+1})| \middle| \mathscr{F}_n \right]. \tag{10.6}$$

The proof of the first step is almost identical to the one in [4]. However, the estimation of Eq. (10.6) demands to develop a specific approach, which essentially uses the normal density.

Proof. Fix $n \in \mathbb{N}$ and recall that $\xi_{n+1}^{[i]}$ are independent from the solution x_n to Eq. (10.1). So, for simplicity, we can drop conditional part in the expectation in Eq. (10.6) and treat f_n and $g_n^{[i]}$, $i = 1, \dots, m$, as constants.

For the sake of brevity write f instead of f_n, $g^{[i]}$ instead of $g_n^{[i]}$, and $\xi^{[i]}$ instead of $\xi_{n+1}^{[i]}$. Denote

$$u := 1 + hf + \sqrt{h} \sum_{i=1}^{m} g^{[i]} \xi^{[i]}, \quad v := hf + \sqrt{h} \sum_{i=1}^{m} g^{[i]} \xi^{[i]}. \tag{10.7}$$

Expanding $\phi_\delta(u)$ by the Taylor formula and applying the mathematical expectation we obtain

$$\mathbf{E}\phi_\delta(u) = \phi_\delta(1) + \phi_\delta'(1)\mathbf{E}v + \frac{\phi_\delta''(1)}{2}\mathbf{E}v^2 + \frac{\phi_\delta'''(\theta)}{6}\mathbf{E}v^3,$$

where θ is situated between 1 and $1 + v$. Applying Assumption 10.1 and boundedness of f and $g^{[i]}$ we arrive at

$$\left| \frac{\phi_\delta'''(\theta)}{6}\mathbf{E}v^3 \right| \le K_1 \mathbf{E} \left| hf + \sqrt{h} \sum_{i=1}^{m} g^{[i]} \xi^{[i]} \right|^3 \le K_2 h^{3/2} \left[|f| + \|g\|^2 \right],$$

where $K_2 > 0$ does not depend on n. Note also that

$$\phi_\delta(1) = \phi(1), \quad \phi_\delta'(1) = \phi'(1), \quad \phi_\delta''(1) = \phi''(1), \quad \mathbf{E}v = hf, \quad \mathbf{E}v^2 = h\|g\|^2.$$

Now, for Δ defined as in Eq. (10.6), we show that

$$\Delta \le h^{3/2} \|g\|^2.$$

Recall that

$$\Delta = \frac{1}{(\sqrt{2\pi})^m} \int \cdots \int_{|u| \le \delta} |\phi(u) - \phi_\delta(u)| \prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{2}} \, \mathrm{d}\xi^{[i]}. \tag{10.8}$$

Let $i_0 \in \{1, 2, \dots, m\}$ be such a number that

$$|g^{[i_0]}| = \max_{i=1,2,\dots,m} |g^{[i]}|.$$

Without loss of generality we can assume that $i_0 = 1$. Then

$$\left[\sum_{i=1}^{m} g^{[i]}\xi^{[i]}\right]^2 \le m\sum_{i=1}^{m}\left[g^{[i]}\xi^{[i]}\right]^2 \le m\left|g^{[1]}\right|^2\sum_{i=1}^{m}\left[\xi^{[i]}\right]^2. \tag{10.9}$$

Choose $h_1 > 0$ so small that for $|u| \le \delta$ and $h \le h_1$, we have

$$1 - u - hf > \frac{1}{2}.$$

Using Eqs. (10.7) and (10.9) we get, for $h \le h_1$,

$$\begin{aligned}\exp\left\{-\frac{1}{4}\sum_{i=1}^{m}\left[\xi^{[i]}\right]^2\right\} &\le \exp\left\{-\frac{1}{4mh\left|g^{[1]}\right|^2}\left[\sqrt{h}\sum_{i=1}^{m} g^{[i]}\xi^{[i]}\right]^2\right\}\\ = \exp\left\{-\frac{1}{4mh\left|g^{[1]}\right|^2}\left[1-u-hf\right]^2\right\} &\le \exp\left\{-\frac{1}{16mh\left|g^{[1]}\right|^2}\right\}. \end{aligned}\tag{10.10}$$

Now we estimate the product inside the integral in Eq. (10.8):

$$\prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{2}} = \prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}} \prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}} \le \mathrm{e}^{-\frac{1}{16mh\left|g^{[1]}\right|^2}} \prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}}.$$

Recall that for some $C_1, C_2 > 0$ and all $i = 1, 2, \dots, m$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}}\,\mathrm{d}\xi^{[i]} \le C_1, \quad \mathrm{e}^{-x} \le C_2 x^{-2}, \quad \text{and} \quad \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}} \le 1. \tag{10.11}$$

From Eqs. (10.8) and (10.11) we obtain that

$$\begin{aligned}\Delta &\le C(m)\mathrm{e}^{-\frac{1}{16mh\left|g^{[1]}\right|^2}} \int\cdots\int_{|u|\le\delta} |\phi(u) - \phi_\delta(u)| \prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}}\,\mathrm{d}\xi^{[i]}\\ &\le C_1(m)h^2\left|g^{[1]}\right|^4 \int\cdots\int_{|u|\le\delta} |\phi(u) - \phi_\delta(u)| \prod_{i=1}^{m} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}}\,\mathrm{d}\xi^{[i]}, \end{aligned}\tag{10.12}$$

where constants $C(m), C_1(m) > 0$ do not depend on n.

For u defined as in Eq. (10.7), we change the variables

$$\left(\xi^{[1]}, \xi^{[2]}, \ldots, \xi^{[m]}\right) \to \left(u, \xi^{[2]}, \ldots, \xi^{[m]}\right)$$

and denote by J the corresponding Jacobian. After straightforward calculations we get

$$|J| = \frac{1}{\sqrt{h}|g^{[1]}|}.$$

Therefore Eqs. (10.11) and (10.12) and boundedness of $\left|g^{[1]}\right|$ imply that

$$\begin{aligned}
\Delta &\le C_1(m)h^2 \left|g^{[1]}\right|^4 \frac{1}{\sqrt{h}|g^{[1]}|} \int_{-\delta}^{\delta} |\phi(u) - \phi_\delta(u)| \, \mathrm{d}u \prod_{i=2}^{m} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\left[\xi^{[i]}\right]^2}{4}} d\xi^{[i]} \\
&\le C_2(m)h^{3/2} \left|g^{[1]}\right|^3 \int_{-\delta}^{\delta} |\phi(u) - \phi_\delta(u)| \, \mathrm{d}u \le C_3(m)h^{3/2} \left|g^{[1]}\right|^2 \\
&\le C_3(m)h^{3/2} \sum_{i=1}^{m} \left|g^{[i]}\right|^2 = \mathscr{O}(h^{3/2}) \left\|g\right\|^2,
\end{aligned}$$

which completes the proof. □

Remark 10.2. Conditions of Lemma 10.4 are fulfilled for $\phi(x) = |x|^\alpha$ and $\phi(x) = |x|^{-\alpha}$, with any $\alpha \in (0,1)$. In the first case we can take $\phi_\delta(x) \equiv \phi(x)$. In the second, we can smoothly extend $\phi(x) = |x|^{-\alpha}$ from $(-\infty, -\delta) \cup (\delta, \infty)$ onto $(-\delta, \delta)$ in such a way that the obtained function $\phi_\delta(x)$ will satisfy conditions (i)–(iii) of Lemma 10.4.

Remark 10.3. Discrete Itô formula from [4] was developed for the partial case of Eq. (10.1), when $m = 1$ and there are no delays, i.e.,

$$x_{n+1} = x_n \left(1 + hf\left(x_n\right) + \sqrt{h} g\left(x_n\right) \xi_{n+1}\right), \tag{10.13}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are bounded, ξ_n are mutually independent random variables, and the density p_n of ξ_n satisfies the property

$$x^3 p_n(x) \to 0, \quad \text{when} \quad |x| \to \infty \quad \text{uniformly in } n. \tag{10.14}$$

If, in Eq. (10.1), $g^{[i]}\left(x_{n-K}^n\right) \equiv a_i$, for some $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, and all $n \in \mathbb{N}$, we may define

$$\xi_n := \frac{\sum_{i=1}^m a_i \xi_n^{[i]}}{\sqrt{\sum_{i=1}^m a_i^2}}, \quad g(x_n) := \sqrt{\sum_{i=1}^m a_i^2}. \tag{10.15}$$

Note that in Eq. (10.15) function g is bounded and random variables $\xi_n \sim \mathcal{N}(0,1)$ are mutually independent and, therefore, satisfy condition (10.14). So we can transform Eq. (10.1) to the form (10.13).

However, in general, the situation is much more complicated. Therefore we found more convenient to develop a new variant of the discrete Itô formula in order to establish stability results for Eq. (10.1).

10.4 Stability

Assume that for some $\beta \in (0,1)$

$$\sup_{\bar{y}\in\mathbb{R}^{K+1}\setminus\{\bar{0}\}} \frac{2f(\bar{y})}{\|g(\bar{y})\|^2} \le \beta. \tag{10.16}$$

Theorem 10.3. *Let Assumptions 10.1and 10.2 and condition (10.16) hold. Let x_n be a solution to Eq. (10.1) with an arbitrary initial value $\beta \in \mathbb{R}^{K+1}$. Then there exists $h_0 > 0$ such that for all $h \le h_0$ we have*

$$\mathbf{P}\left\{\lim_{n\to\infty} x_n = 0\right\} = 1.$$

Proof. Fix some $n \in \mathbb{N}$ and set, for simplicity,

$$f_n := f(x^n_{n-K}),\; g_n := g(x^n_{n-K}) = \left(g^{[1]}(x^n_{n-K}),\dots,g^{[m]}(x^n_{n-K})\right), \tag{10.17}$$

$$u_{n+1} := 1 + hf(x^n_{n-K}) + \sqrt{h}\sum_{i=1}^{m} g^{[i]}(x^n_{n-K})\xi^{[i]}_{n+1}. \tag{10.18}$$

Note that f_n and g_n are $\mathscr{F}_n$-measurable while u_{n+1} is $\mathscr{F}_{n+1}$-measurable.

Assume that $h_1 > 0$ is so small that Lemma 10.4 holds with

$$\phi(x) = |x|^\alpha, \quad \alpha < \frac{1-\beta}{2},$$

Apply Lemma 10.4 and get

$$\begin{aligned}
&\mathbf{E}\left(|u_{n+1}|^\alpha \middle| \mathscr{F}_n\right)\\
&\quad = 1 + \alpha h f_n - h\frac{\alpha(1-\alpha)}{2}\|g_n\|^2 + \mathscr{O}(h^{3/2})\left[|f_n| + \|g_n\|^2\right]\\
&\quad = 1 + \alpha h f_n\left[1 + \mathscr{O}(h^{1/2})\right] - h\frac{\alpha(1-\alpha)}{2}\|g_n\|^2\left[1 + \mathscr{O}(h^{1/2})\right]
\end{aligned} \tag{10.19}$$

$$= 1 + \frac{\alpha h \|g_n\|^2}{2} \left(\frac{2f_n \left[1 + \mathcal{O}(h^{1/2})\right]}{\|g\|^2} - (1-\alpha)\left[1 + \mathcal{O}(h^{1/2})\right] \right) \tag{10.20}$$

$$= 1 + \frac{\alpha h \|g_n\|^2}{2} \left(\frac{2f_n}{\|g_n\|^2} - 1 + \alpha + \mathcal{O}(h^{1/2}) \left[\frac{2f_n}{\|g_n\|^2} - 1 + \alpha \right] \right). \tag{10.21}$$

By choice of α, we have

$$\frac{2f_n}{\|g_n\|^2} - 1 + \alpha < \beta - 1 + \alpha \le -\frac{1-\beta}{2}.$$

Take $h_0 \le h_1$ so small that for $h \le h_0$

$$|\mathcal{O}(h^{1/2})||\beta + 1 + \alpha| < \frac{1-\beta}{4}, \quad \text{and} \quad 1 + \mathcal{O}(h^{1/2}) > \frac{1}{2}.$$

Then, for $\frac{2f_n}{\|g_n\|^2} \in [0, \beta)$, we have

$$\frac{2f_n}{\|g_n\|^2} - 1 + \alpha + \mathcal{O}(h^{1/2}) \left[\frac{2f_n}{\|g_n\|^2} - 1 + \alpha \right] \le -\frac{1-\beta}{4},$$

while for $\frac{2f_n}{\|g_n\|^2} < 0$, we have

$$\frac{2f_n \left[1 + \mathcal{O}(h^{1/2})\right]}{\|g\|^2} - (1-\alpha)\left[1 + \mathcal{O}(h^{1/2})\right] \le \frac{f_n}{\|g_n\|^2} - \frac{1-\alpha}{2} \le -\frac{1-\alpha}{2}.$$

Let

$$\tau := \min\left\{ \frac{\alpha(1-\alpha)}{4}, \ \frac{(1-\beta)\alpha}{8} \right\}.$$

Then, in both cases, we obtain from Eqs. (10.20) and (10.21) that, for $h \le h_0$,

$$\mathbf{E}\left(|u_{n+1}|^\alpha \middle| \mathcal{F}_n \right) \le 1 - \tau h \|g_n\|^2.$$

Now, we transform Eq. (10.1) to the form suitable for application of Lemma 10.2. Taking modulus of both sides of Eq. (10.1) and raising them to the degree $\alpha \in \left(0, \frac{1-\beta}{2}\right)$ we arrive at

$$|x_{n+1}|^\alpha = |x_n|^\alpha \, |u_{n+1}|^\alpha . \tag{10.22}$$

Denoting

$$w_n := |x_n|^\alpha, \quad \Phi_n := \mathbf{E}\left[|u_{n+1}|^\alpha \middle| \mathcal{F}_n \right] - 1, \quad \varsigma_{n+1} := w_n \left(|u_{n+1}|^\alpha - \Phi_n - 1 \right),$$

we write Eq. (10.22) as

$$w_{n+1} = w_n + w_n \Phi_n + \varsigma_{n+1}.$$

Note that $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a martingale difference. So, by Lemma 10.2 applied with $a_n = 0$ and $b_n = -w_n\Phi_n$, we conclude that

$$\mathbf{P}\left[\left\{\omega : -\sum_{n=1}^{\infty} w_n\Phi_n < \infty\right\} \bigcap \{w \to\}\right] = 1. \tag{10.23}$$

This, in particular, means that w_n almost surely converges to some random value w_∞. So $|x_n| \to w_\infty^{1/\alpha}$, a.s. We want to show that $\mathbf{P}\{w_\infty = 0\} = 1$.

Assume the contrary: there exists $y > 0$ such that $P\{\Omega'\} > 0$ where

$$\Omega' = \{\omega : w_\infty^{1/\alpha} \in (y/2, 3y/2)\}.$$

Then for each $\omega \in \Omega'$ there exists $N(\omega)$ such that for all $n \geq N(\omega)$, we have

$$|x_n(\omega)| \geq y/2,$$

which implies that, for all $n \geq N(\omega) + K$,

$$\|x_n^{n-K}(\omega)\| = \sqrt{\sum_{i=1}^{K} |x_{n-i}(\omega)|^2} \geq \frac{y}{2}\sqrt{K+1}.$$

So, by continuity of g, for each $\omega \in \Omega'$, there exists $\kappa(\omega) > 0$ such that, for all $n \geq N(\omega)$, we have

$$\|g(x_n^{n-K})\| \geq \kappa(\omega).$$

Therefore we have on Ω':

$$\Phi_n(\omega) \leq -\tau h\kappa(\omega),$$

which makes Eq. (10.23) impossible, since for $\omega \in \Omega'$

$$-\sum_{i=1}^{n} w_i\Phi_i \geq -\sum_{i=N(\omega)}^{n} w_i\Phi_i \geq \sum_{i=N(\omega)}^{n} \tau h|x_i|^\alpha \|g(x_i^{i-K})\|^2$$
$$\geq \sum_{i=N(\omega)}^{n} \frac{\tau h y^\alpha}{2^\alpha}\kappa(\omega) \to \infty,$$

as $n \to \infty$. □

10.5 Instability

In this subsection we assume that for some $\mu > 1$

$$\inf_{\bar{y}\in\mathbb{R}^{K+1}\setminus\{\bar{0}\}} \frac{2f(\bar{y})}{\|g(\bar{y})\|^2} \geq \mu. \tag{10.24}$$

Theorem 10.4. *Let Assumptions 10.1and 10.2 and condition (10.24) hold. Let x_n be a solution to Eq. (10.1) with an arbitrary initial value $\beta \in \mathbb{R}^{K+1}$. Then there exists $h_0 > 0$ such that for all $h \leq h_0$, we have*

$$\mathbf{P}\left\{\liminf_{n\to\infty} |x_n| > 0\right\} = 1.$$

Proof. Let f_n, g_n, and u_{n+1} be defined as in Eq. (10.17), and let $h_1 > 0$ be so small that we can apply Lemma 10.4 with

$$\phi(x) = |x|^{-\alpha}, \quad \alpha < \frac{\mu - 1}{2}.$$

Then,

$$\begin{aligned}
&\mathbf{E}\left(|u_{n+1}|^{-\alpha}\middle|\mathscr{F}_n\right)\\
&\quad= 1 - \alpha h f_n + h\frac{\alpha(1+\alpha)}{2}\|g_n\|^2 + \mathscr{O}(h^{3/2})\left[|f_n| + \|g_n\|^2\right]\\
&\quad= 1 - \alpha h f_n\left[1 + \mathscr{O}(h^{1/2})\right] + h\frac{\alpha(1+\alpha)}{2}\|g_n\|^2\left[1 + \mathscr{O}(h^{1/2})\right]\\
&\quad= 1 - \frac{\alpha h\|g_n\|^2}{2}\left(\frac{2f_n\left[1 + \mathscr{O}(h^{1/2})\right]}{\|g\|^2} - (1+\alpha)\left[1 + \mathscr{O}(h^{1/2})\right]\right).
\end{aligned}$$

Choose

$$\varepsilon \in \left(0,\ \frac{\mu - 1 - \alpha}{2(\mu + 1 + \alpha)}\right)$$

and $h_0 < h_1$ so small that, for all $h \leq h_0$,

$$\left|\mathscr{O}(h^{1/2})\right| \leq \varepsilon.$$

Then

$$\begin{aligned}
\frac{2f_n\left[1 + \mathscr{O}(h^{1/2})\right]}{\|g\|^2} - (1+\alpha)\left[1 + \mathscr{O}(h^{1/2})\right] &\geq \mu(1-\varepsilon) - (1+\alpha)(1+\varepsilon)\\
&> \frac{\mu - 1 - \alpha}{2}.
\end{aligned}$$

So,

$$\mathbf{E}\left(|u_{n+1}|^{-\alpha}\middle|\mathscr{F}_n\right) \le 1 - \frac{\alpha h(\mu - 1 - \alpha)\|g_n\|^2}{4} \le 1. \tag{10.25}$$

From Eq. (10.1) we obtain that

$$\begin{aligned} |x_n|^\alpha &= |x_0|^\alpha \prod_{i=0}^{n-1} |u_{i+1}|^\alpha \\ &= |x_0|^\alpha \prod_{i=0}^{n-1} \frac{1}{\frac{|u_{i+1}|^{-\alpha}}{\mathbf{E}\left(|u_{i+1}|^{-\alpha}\middle|\mathscr{F}_i\right)}} \times \frac{1}{\prod_{i=0}^{n-1}\mathbf{E}\left(|u_{i+1}|^{-\alpha}\middle|\mathscr{F}_i\right)}. \end{aligned} \tag{10.26}$$

Define, for all $n \in \mathbb{N}$,

$$M_n = \prod_{i=0}^{n-1} \frac{|u_{i+1}|^{-\alpha}}{\mathbf{E}\left(|u_{i+1}|^{-\alpha}\middle|\mathscr{F}_i\right)}.$$

By Lemma 10.3, the process $\{M_n\}_{n\in\mathbb{N}}$ is a nonnegative martingale, and by Lemma 10.1, it converges to a.s. finite nonnegative random variable q. This, along with Eqs. (10.25) and (10.26), proves that a.s., $|x_n(\omega)|^\alpha \ge \frac{1}{q(\omega)}|x_0|^\alpha > 0$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$ a.s., which implies that

$$\mathbf{P}\left\{\liminf_{n\to\infty} |x_n| > 0\right\} = 1.$$

□

Remark 10.4. Conditions of stability and instability developed in Theorems 10.3 and 10.4 connect appropriately to those for the stochastic differential delay equations of the form

$$\mathrm{d}X_t = X_t f\left(X_t^{t-\tau}\right)\mathrm{d}t + \sum_{i=1}^{m} X_t g^{[i]}\left(X_t^{t-\tau}\right)\mathrm{d}W_t^{[i]}, \quad t > 0, \tag{10.27}$$

$$X(s) = \varphi(s), \quad s \in [-\tau, 0]. \tag{10.28}$$

Here $\tau > 0$, $\varphi : \mathbb{R} \to \mathbb{R}$ is initial continuous, $X_t^{t-\tau}(s) = X(t+s)$ for $s \in [-\tau, 0]$, and all $t > 0$, $f : C[-\tau, 0] \to \mathbb{R}$, $g^{[i]} : C[-\tau, 0] \to \mathbb{R}$, $i = 1, 2, \ldots, m$, $W^{[i]}$ are independent Wiener processes. Corresponding results have been proved in [5] (see also [3]) for the systems of stochastic Itô equations of type (10.27) and (10.28).

In particular, for solution X_t of Eqs. (10.27) and (10.28), we can conclude from [3, 5] that

$$\sup_{u \in C[-\tau,0]\setminus\bar{0}} \frac{2f(u)}{\|g(u)\|^2} < 1 \Longrightarrow \mathbf{P}\left\{\lim_{t\to\infty} X_t = 0\right\} = 1,$$

$$\inf_{u \in C[-\tau,0]\setminus\bar{0}} \frac{2f(u)}{\|g(u)\|^2} > 1 \Longrightarrow \mathbf{P}\left\{\liminf_{t\to\infty} X_t > 0\right\} = 1.$$

10.6 Example

Assume that in Eq. (10.1) we have

$$K = 1, \quad m = 2, \qquad f(x_n, x_{n-1}) = \frac{a_0^2|x_n|^2 + a_2^2|x_{n-1}|^2}{x_n^2 + x_{n-1}^2},$$

$$g_1(x_n, x_{n-1}) = \frac{\lambda a_0|x_n|}{\sqrt{x_n^2 + x_{n-1}^2}}, \qquad g_2(x_n, x_{n-1}) = \frac{\lambda a_1|x_{n-1}|}{\sqrt{x_n^2 + x_{n-1}^2}}.$$

Here $a_0, a_1 \neq 0$, $\lambda > 0$ are arbitrary numbers.

Then

$$\frac{2f(x_n, x_{n-1})}{g_1^2(x_n, x_{n-1}) + g_1^2(x_n, x_{n-1})} = \frac{2}{\lambda}.$$

So formulae (10.16) and (10.24) hold with $\beta = \frac{2}{\lambda}$ and $\mu = \frac{2}{\lambda}$, respectively. Applying Theorems 10.3 and 10.4, we conclude that $\lambda > 2$ implies that $\mathbf{P}\{\lim_{n\to\infty} x_n = 0\} = 1$ while $\lambda < 2$ implies that $\mathbf{P}\{\liminf_{n\to\infty} |x_n| > 0\} = 1$.

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Chapter 11
On Semilinear Hyperbolic Functional Equations with State-Dependent Delays

László Simon

Abstract We consider second-order semilinear hyperbolic functional differential equations where the lower-order terms contain functional dependence and state-dependent delay on the unknown function. Existence of solutions for $t \in (0, T)$, $t \in (0, \infty)$ and some qualitative properties of the solutions in $(0, \infty)$ are shown. Further, examples are considered.

Keywords Semilinear hyperbolic equation • Functional differential equation • State-dependent delay • Qualitative properties

11.1 Introduction

We shall consider weak solutions of initial-boundary value problems of the form

$$\begin{aligned} u''(t) + \tilde{Q}(u(t)) + \varphi(x)h'(u(t)) + H(t, x; u, u([\gamma_1(u)](t))) \\ + G(t, x; u, u([\gamma_2(u)](t), u')) = F, \quad t > 0, \quad x \in \Omega, \qquad (11.1) \end{aligned}$$

$$u(0) = u_0, \quad u'(0) = u_1, \qquad (11.2)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notations $u(t) = u(t, x)$, $u' = D_t u$, $u'' = D_t^2 u$, $\tilde{Q}$ may be, e.g., a linear second-order symmetric elliptic differential operator in the variable x; h is a C^1 function having certain polynomial growth,

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$$u([\gamma_j(u)](t)) = u([\gamma_j(u)](t), x), \quad \text{where} \quad 0 \leq [\gamma_j(u)](t) \leq t.$$

Further, H and G contain nonlinear functional (nonlocal) dependence on u, $u(\gamma_j(u))$, and u', with some polynomial growth.

Semilinear hyperbolic functional equations were considered, e.g., in [3–5] with certain nonlocal terms, generally in the form of particular integral operators applied to the unknown function. First-order quasilinear evolution equations with nonlocal terms were considered, e.g., in [9, 12] and second-order quasilinear evolution equations with nonlocal terms were considered in [8], by using the theory of monotone-type operators (see [2, 7, 13]).

This paper was motivated by the work [7] of Lions where the same equation was considered in the particular case $\tilde{Q} = -\Delta$, $\varphi = 1$, $h'(\eta) = \eta|\eta|^\lambda$, $H = 0$, $G = 0$ (semilinear hyperbolic differential equation). Further, it is based on the papers [10, 11]. In [11] the particular case was studied when the terms H and G did not contain terms $u(\gamma_j(u))$. Equation (11.1) will be reduced to the equation considered in [11]. In [10] certain particular nonlinear second-order evolution equations were considered with state-dependent delays.

In Sect. 11.2 the existence of weak solutions will be proved for $t \in (0, T)$, in Sect. 11.3 examples will be shown, and in Sect. 11.4 we shall prove the existence and certain properties of solutions for $t \in (0, \infty)$.

11.2 Existence in $(0, T)$

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0, T) \times \Omega$. Denote by $W^{1,2}(\Omega)$ the Sobolev space of real-valued functions with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^2 + |u|^2 \right) \mathrm{d}x \right]^{1/2}.$$

Further, let $V \subset W^{1,2}(\Omega)$ be a closed linear subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$ [the closure of $C_0^\infty(\Omega)$], $V^\star$ be the dual space of V, $H = L^2(\Omega)$, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$, the scalar product in H will be denoted by $(\cdot, \cdot)$. Denote by $L^2(0, T; V)$ the Banach space of the set of measurable functions $u : (0, T) \to V$ with the norm

$$\|u\|_{L^2(0,T;V)} = \left[\int_0^T \|u(t)\|_V^2 \, \mathrm{d}t \right]^{1/2}$$

and $L^\infty(0, T; V)$, $L^\infty(0, T; H)$ the set of measurable functions $u : (0, T) \to V$, $u : (0, T) \to H$, respectively, with the $L^\infty(0, T)$ norm of the functions $t \mapsto \|u(t)\|_V$, $t \mapsto \|u(t)\|_H$, respectively.

Now we formulate the assumptions on the functions in (11.1).

(A_1) $\tilde{Q} : V \to V^\star$ is a linear continuous operator such that

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \langle \tilde{Q}\tilde{v}, \tilde{u}\rangle, \quad \langle \tilde{Q}\tilde{u}, \tilde{u}\rangle \geq c_0 \|\tilde{u}\|_V^2$$

for all $\tilde{u}, \tilde{v} \in V$ with some constant $c_0 > 0$. Further we shall use the notation $(Qu)(t) = \tilde{Q}(u(t))$.

(A_2) $\varphi : \Omega \to \mathbb{R}$ is a measurable function satisfying

$$c_1 \leq \varphi(x) \leq c_2 \text{ for a.a. } x \in \Omega$$

with some positive constants c_1, c_2.

(A_3) $h : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$h(\eta) \geq 0, \quad |h'(\eta)| \leq \text{const}|\eta|^\lambda \text{ for } |\eta| > 1,$$

where

$$1 < \lambda \leq \lambda_0 = \frac{n}{n-2} \text{ if } n \geq 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A_3') $h : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying with some positive constants c_3, c_4

$$h(\eta) \geq 0, \quad c_3|\eta|^\lambda \leq |h'(\eta)| \leq c_4|\eta|^\lambda \text{ for } |\eta| > 1, \quad n \geq 3,$$

where $\lambda > \lambda_0 = \frac{n}{n-2}$,

$$|h'(\eta)| \leq c_4|\eta|^\lambda \quad \text{for } |\eta| > 1, \quad n = 2, \quad \text{where } 1 < \lambda < \infty.$$

(B_4) $H : Q_T \times L^2(Q_T) \times L^2(Q_T) \to \mathbb{R}$ is a function for which $(t, x) \mapsto H(t, x; u, z)$ is measurable for all fixed $u, z \in L^2(Q_T)$; H has the Volterra property, i.e. for all $t \in [0, T]$, $H(t, x; u, z)$ depends only on the restriction of u, z to $(0, t)$; the following inequality holds for all $t \in [0, T]$ and $u, z \in L^2(Q_T)$:

$$\int_0^t \int_\Omega |H(\tau, x; u, z)|^2 \,\mathrm{d}x\mathrm{d}\tau \leq \text{const} \int_0^t \int_\Omega [h(u(\tau)) + h(z(\tau))] \,\mathrm{d}x\mathrm{d}\tau.$$

Further, for any fixed functions $w_1, w_2, \dots w_m \in V$ (if (A_3) is satisfied) and $w_1, w_2, \dots w_m \in V \cap L^{\lambda+1}(\Omega)$ (if (A_3') holds), respectively, for every $K > 0$, there exists $\psi_K \in L^1(0, T)$ such that if

$$\|c_k\|_{L^\infty(0,T)} \leq K, \quad \|d_k\|_{L^\infty(0,T)} \leq K$$

then

$$\left[\int_{\Omega}\left|H\left(t,x;\sum_{k=1}^{m}c_k w_k,\sum_{k=1}^{m}d_k w_k\right)\right|^2\,\mathrm{d}x\right]^{1/2}\le \psi_K(t),\quad t\in[0,T].$$

Finally, $(u_k)\to u$, $(z_k)\to z$ in $L^2(Q_T)$ imply for a subsequence

$$H(t,x;u_k,z_k)\to H(t,x;u,z) \text{ for a.a. } (t,x)\in Q_T.$$

(B_5) $G: Q_T\times L^2(Q_T)\times L^2(Q_T)\times L^\infty(0,T;H)\to\mathbb{R}$ is a function satisfying $(t,x)\mapsto G(t,x;u,z,w)$ which is measurable for all fixed $u,z\in L^2(Q_T)$ and $w\in L^\infty(0,T;H)$, and G has the Volterra property: for all $t\in[0,T]$, $G(t,x;u,z,w)$ depends only on the restriction of u,z,w to $(0,t)$ and

$$|G(t,x;u,z,w)|\le c_5|w(t)|+c_6$$

with some constants c_5,c_6. Further, if

$$(u_k)\to u,\quad (z_k)\to z \text{ in } L^2(Q_T),\quad (w_k)\to w \text{ weakly in } L^\infty(0,T;H)$$

in the sense that for all fixed $g_1\in L^1(0,T;H)$

$$\int_0^T\langle g_1(t),w_k(t)\rangle\mathrm{d}t\to\int_0^T\langle g_1(t),w(t)\rangle\mathrm{d}t,$$

then for a subsequence and a.a. $(t,x)\in Q_T$

$$G(t,x;u_k,z_k,w_k)\to G(t,x;u,z,w).$$

(G) $\gamma_j: L^2(Q_T)\to C_a[0,T]$ are continuous (nonlinear) operators such that

$$0\le[\gamma_j(u)](t)\le t,\quad [\gamma_j(u)]'(t)\ge c_0\quad (j=1,2)$$

with some constant $c_0>0$. ($C_a[0,T]$ denotes the set of absolutely continuous functions in $[0,T]$ with the supremum norm.)

Condition (G) is fulfilled, e.g., by the operators of the form

$$[\gamma_j](u)(t)=t\beta\left(\int_{Q_t}\Gamma(t,\tau,\xi)u^2(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi\right),$$

where Γ, $\frac{\partial\Gamma}{\partial t}$ are continuous and nonnegative, and $\beta\in C^1(\mathbb{R})$ satisfies $\delta_1\le\beta\le 1$ with some constant $\delta_1>0$ and $\beta'\ge 0$.

Theorem 11.1. *Assume (A_1), (A_2), (A_3), (B_4), (B_5), (G). Then for all $F \in L^2(0,T;H)$, $u_0 \in V$, $u_1 \in H$, there exists $u \in L^\infty(0,T;V)$ such that*

$$u' \in L^\infty(0,T;H), \quad u'' \in L^2(0,T;V^\star),$$

u satisfies (11.1) in the sense: for a.a. $t \in [0,T]$, all $v \in V$

$$\begin{aligned}&\langle u''(t), v\rangle + \langle \tilde{Q}(u(t)), v\rangle + \int_\Omega \varphi(x)h'(u(t))v\,\mathrm{d}x + \int_\Omega H(t,x;u,u(\gamma_1(u)))v\,\mathrm{d}x \\ &+ \int_\Omega G(t,x;u,u(\gamma_2(u)),u')v\,\mathrm{d}x = (F(t),v) \qquad (11.3)\end{aligned}$$

and the initial condition (11.2) is fulfilled.

If (A_1), (A_2), (A_3'), (B_4), (B_5), (G) are satisfied, then for all $F \in L^2(0,T;H)$, $u_0 \in V \cap L^{\lambda+1}(\Omega)$, $u_1 \in H$ there exists $u \in L^\infty(0,T;V \cap L^{\lambda+1}(\Omega))$ such that

$$u' \in L^\infty(0,T;H),$$

$$u'' \in L^2(0,T;V^\star) + L^\infty(0,T;L^{\frac{\lambda+1}{\lambda}}(\Omega)) \subset L^2\left(0,T;[V\cap L^{\lambda+1}(\Omega)]^\star\right)$$

and u satisfies (11.1) in the sense: for a.a. $t \in [0,T]$, all $v \in V \cap L^{\lambda+1}(\Omega)$ (11.3) holds; further, the initial condition (11.2) is fulfilled.

Remark 11.1. $u'' \in L^2(0,T;V^\star) + L^\infty(0,T;L^{\frac{\lambda+1}{\lambda}}(\Omega))$ means that for the distributional derivative $u'' = D_t^2 u$ we have

$$u'' = u_1 + u_2 \text{ where } u_1 \in L^2(0,T;V^\star) \text{ and } u_2 \in L^\infty(0,T;L^{\frac{\lambda+1}{\lambda}}(\Omega)).$$

Since in this case

$$(u')' = u'' \in L^2\left(0,T;[V\cap L^{\lambda+1}(\Omega)]^\star\right) \text{ and}$$

$$u' \in L^\infty(0,T;L^2(\Omega)) \subset L^2\left(0,T;[V\cap L^{\lambda+1}(\Omega)]^\star\right),$$

by Lemma 1.2 in Chap. 1 of [7]

$$u' \in C([0,T];[V\cap L^{\lambda+1}(\Omega)]^\star),$$

thus the initial condition $u'(0) = u_1 \in H$ makes sense since $H \subset [V \cap L^{\lambda+1}(\Omega)]^\star$.

Similarly, if (A_3) is satisfied by

$$u'' \in L^2(0,T;V^\star), \quad u' \in L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;V^\star),$$

we have $u' \in C([0,T];V^\star)$, so the initial condition $u'(0) = u_1 \in H$ makes sense.

In the proof of Theorem 11.1 we shall use the following lemma (for the proof see [10]).

Lemma 11.1. *Assume that $\gamma : L^2(Q_T) \to C_a[0,T]$ satisfies (G). If $(u_k) \to u$ in $L^2(Q_T)$ and $(z_k) \to z$ in $L^2(Q_T)$, then*

$$z_k([\gamma(u_k)](t), x) \to z([\gamma(u)](t), x) \text{ in } L^2(Q_T).$$

Further, $z([\gamma(u)](t)$ is bounded in $L^2(Q_T)$ if u, z are bounded in $L^2(Q_T)$.

Proof (Theorem 11.1). We show that the assumptions (A_4), (A_5) of Theorem 2.1 in [11] are fulfilled for

$$\tilde{H}(t,x;u) = H(t,x;u,u(\gamma_1(u))), \quad \tilde{G}(t,x;u,w) = G(t,x;u,u(\gamma_2(u)),w)$$

(instead of H and G, respectively), i.e.,

(A_4) $\tilde{H} : Q_T \times L^2(Q_T) \to \mathbb{R}$ is a function for which $(t,x) \mapsto \tilde{H}(t,x;u)$ is measurable for all fixed $u \in L^2(Q_T)$, $\tilde{H}$ has the Volterra property, i.e. for all $t \in [0,T]$, $\tilde{H}(t,x;u)$ depends only on the restriction of u to $(0,t)$; the following inequality holds for all $t \in [0,T]$ and $u \in L^2(Q_T)$:

$$\int_0^t \int_\Omega |\tilde{H}(\tau,x;u)|^2 \, \mathrm{d}x\mathrm{d}\tau \le \text{const} \int_0^t \int_\Omega h(u(\tau)) \, \mathrm{d}x\mathrm{d}\tau.$$

Further, for any fixed functions $w_1, w_2, \dots w_m \in V$ (if (A_3) is satisfied) and $w_1, w_2, \dots w_m \in V \cap L^{\lambda+1}(\Omega)$ (if (A_3') holds), respectively, for every $K > 0$, there exists $\psi_K \in L^1(0,T)$ such that if $\|c_k\|_{L^\infty(0,T)} < K$ then

$$\left[\int_\Omega \left| \tilde{H}\left(t,x;\sum_{k=1}^m c_k w_k\right) \right|^2 \mathrm{d}x \right]^{1/2} \le \psi_K(t), \quad t \in [0,T].$$

Finally, if $(u_k) \to u$ in $L^2(Q_T)$, then for a subsequence

$$\tilde{H}(t,x;u_k) \to \tilde{H}(t,x;u) \text{ for a.a. } (t,x) \in Q_T.$$

(A_5) $\tilde{G} : Q_T \times L^2(Q_T) \times L^\infty(0,T;H) \to \mathbb{R}$ is a function satisfying that $(t,x) \mapsto \tilde{G}(t,x;u,w)$ is measurable for all fixed $u \in L^2(Q_T)$, $w \in L^\infty(0,T;H)$, $\tilde{G}$ has the Volterra property: for all $t \in [0,T]$, $\tilde{G}(t,x;u,w)$ depends only on the restriction of u, w to $(0,t)$, and

$$|\tilde{G}(t,x;u,w)| \le c_5|w(t)| + c_6$$

with some constants c_5, c_6.
Further, if

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ and } (w_k) \to w \text{ weakly in } L^\infty(0,T;H)$$

in the sense that for all fixed $g_1 \in L^1(0,T;H)$

$$\int_0^T \langle g_1(t), w_k(t)\rangle \, \mathrm{d}t \to \int_0^T \langle g_1(t), w(t)\rangle \, \mathrm{d}t,$$

then for a subsequence, a.a. $(t,x) \in Q_T$

$$\tilde{G}(t,x;u_k,w_k) \to \tilde{G}(t,x;u,w).$$

Indeed, by (B_4)

$$\int_0^t \int_\Omega |\tilde{H}(t,x;u)|^2 \, \mathrm{d}x\mathrm{d}\tau = \int_0^t \int_\Omega |H(t,x;u,u((\gamma_1(u))|^2 \, \mathrm{d}x\mathrm{d}\tau \le$$
$$\text{const} \int_0^t \int_\Omega [h(u(\tau)) + h(u([\gamma_1(u)](\tau),x)] \, \mathrm{d}x\mathrm{d}\tau.$$

By using the notation $\psi_1(\tau) = [\gamma_1(u)](\tau)$, we obtain by using (G) and the substitution $\psi_1(\tau) = s$

$$\int_0^t h(u(\psi_1(\tau)),x) \, \mathrm{d}\tau \le \frac{1}{c_0} \int_0^t h(u(\psi_1(\tau)),x) \frac{\partial \psi_1}{\partial \tau} \, \mathrm{d}\tau \le \frac{1}{c_0} \int_0^t h(u(s,x) \, \mathrm{d}s,$$

thus

$$\int_0^t \int_\Omega |\tilde{H}(t,x;u)|^2 \, \mathrm{d}x\mathrm{d}\tau \le \text{const} \int_0^t \int_\Omega h(u(\tau)) \, \mathrm{d}x\mathrm{d}\tau. \tag{11.4}$$

Further, consider arbitrary fixed functions $w_1, w_2, \dots w_m \in V$ (in the case when (A_3) is satisfied) and $w_1, w_2, \dots w_m \in V \cap L^{\lambda+1}(\Omega)$ [in the case (A_3')]. Then, by using the notation $u = \sum_{k=1}^m c_k w_k$, (B_4) implies that there exists $\psi_K \in L^1(0,T)$ such that

$$\left[\left| \tilde{H}\left((t,x; \sum_{k=1}^m c_k w_k \right) \right|^2 dx \right]^{1/2}$$
$$= \left[\left| H\left((t,x; \sum_{k=1}^m c_k w_k, \sum_{k=1}^m c_k(\gamma_1(u)) w_k \right) \right|^2 dx \right]^{1/2} \le \psi_K(t) \tag{11.5}$$

if $t \in [0,T]$, $\|c_k\|_{L^\infty(0,T)} < K$ since for $d_k = c_k(\gamma_1(u))$ we have $\|d_k\|_{L^\infty(0,T)} < K$.

Finally, by Lemma 11.1 $(u_k) \to u$ in $L^2(\Omega)$ implies $u_k(\gamma_1(u_k)) \to u(\gamma_1(u))$ in $L^2(\Omega)$; thus by (B_4) for a subsequence we have

$$\tilde{H}(t,x;u_k) = H(t,x;u_k,u_k(\gamma_1(u_k))) \to H(t,x;u,u(\gamma_1(u))) = \tilde{H}(t,x;u) \tag{11.6}$$

for a.a. $(t,x) \in Q_T$.

According to (11.4)–(11.6) the function $\tilde{H}$ satisfies the assumption (A_4) of [11].

Similarly can be shown that by (B_5) $\tilde{G}$ satisfies the assumption (A_5) of [11]. Since all the assumptions of Theorem 2.1 in [11] are fulfilled, from Theorem 2.1 in [11], we obtain Theorem 11.1. □

Now we describe the main steps of the proof of Theorem 2.1 of [11]. The proof is based on Galerkin's method. Let $w_1, w_2, \dots$ be a linearly independent system in V if (A_3) is satisfied and in $V \cap L^{\lambda+1}(\Omega)$ if (A_3') is satisfied such that the linear combinations are dense in V and $V \cap L^{\lambda+1}(\Omega)$, respectively. Define

$$u_m(t) = \sum_{l=1}^{m} g_{lm}(t) w_l, \tag{11.7}$$

where $g_{lm} \in W^{2,2}(0,T)$ if (A_3) is satisfied and $g_{lm} \in W^{2,2}(0,T) \cap L^\infty(0,T)$ if (A_3') is fulfilled such that for all $j = 1 \dots, m$

$$\begin{aligned} &\langle u_m''(t), w_j \rangle + \langle \tilde{Q}(u_m(t)), w_j \rangle + \int_\Omega \varphi(x) h'(u_m(t)) w_j \,\mathrm{d}x \\ &+ \int_\Omega \tilde{H}(t,x;u_m) w_j \,\mathrm{d}x + \int_\Omega \tilde{G}(t,x;u_m,u_m') w_j \,\mathrm{d}x = \langle F(t), w_j \rangle, \end{aligned} \tag{11.8}$$

$$u_m(0) = u_{m0}, \quad u_m'(0) = u_{m1}, \tag{11.9}$$

where the linear combinations u_{m0}, u_{m1} $(m = 1, 2, \dots)$ of $w_1, w_2 \dots, w_m$ satisfy

$$(u_{m0}) \to u_0 \text{ in } V \text{ and } V \cap L^{\lambda+1}(\Omega), \text{ respectively, as } m \to \infty \text{ and} \tag{11.10}$$

$$(u_{m1}) \to u_1 \text{ in } H \text{ as } m \to \infty. \tag{11.11}$$

By using the existence theorem for a system of functional differential equations with Carathéodory conditions (see [6]), we obtain that there exists a solution of (11.8) and (11.9) in a neighborhood of 0. Further, the maximal solution of (11.8), (11.9) is defined in $[0,T]$. Indeed, multiplying (11.8) by $g_{lm}'(t)$ and taking the sum with respect to j, we obtain

$$\begin{aligned} &\langle u_m''(t), u_m'(t) \rangle + \langle \tilde{Q}(u_m(t)), u_m'(t) \rangle + \int_\Omega \varphi(x) h'(u_m(t)) u_m'(t) \,\mathrm{d}x \\ &+ \int_\Omega \tilde{H}(t,x;u_m) u_m'(t) \,\mathrm{d}x + \int_\Omega \tilde{G}(t,x;u_m,u_m') u_m'(t) \,\mathrm{d}x \\ &= (F(t), u_m'(t)). \end{aligned} \tag{11.12}$$

Integrating the above equality over $(0,t)$ we find by Young's inequality and by using the formulas

$$\int_0^t \langle \tilde{Q}(u_m(\tau)), u_m'(\tau)\rangle \, d\tau = \frac{1}{2}\langle \tilde{Q}(u_m(t)), u_m(t), u_m(t)\rangle - \frac{1}{2}\langle \tilde{Q}(u_m(t)), u_m(0), u_m(0)\rangle,$$

$$\int_0^t \langle u_m''(\tau), u_m'(\tau)\rangle \, d\tau = \frac{1}{2}\|u_m'(t)\|_H^2 - \frac{1}{2}\|u_m'(0)\|_H^2$$

(see [10]):

$$\begin{aligned}
\frac{1}{2}\|u_m'(t)\|_H^2 &+ \frac{1}{2}\langle \tilde{Q}(u_m(t)), u_m(t)\rangle + \int_\Omega \varphi(x)h(u_m(t))\, dx \\
&+ \int_0^t \left[\int_\Omega \tilde{H}(\tau, x; u_m)u_m'(\tau)\, dx\right] d\tau \\
&+ \int_0^t \left[\int_\Omega \tilde{G}(\tau, x; u_m, u_m')u_m'(\tau)\, dx\right] d\tau \\
&= \int_0^t (F(\tau), u_m'(\tau))\, d\tau + \frac{1}{2}\|u_m'(0)\|_H^2 + \frac{1}{2}\langle (Qu_m)(0), u_m(0)\rangle \\
&+ \int_\Omega \varphi(x)h(u_m(0))\, dx. \qquad (11.13)
\end{aligned}$$

By using (A_2), (A_4), (A_5) and Cauchy–Schwarz inequality, Young's inequality, Sobolev's imbedding theorem and Gronwall's lemma we obtain from (11.13)

$$\|u_m'(t)\|_H^2 + \int_\Omega h(u_m(t))\, dx \le \text{const} \qquad (11.14)$$

where the constant is not depending on m and t; consequently, the maximal solution exists in $[0, T]$ and

$$\|u_m(t)\|_V \le \text{const}, \quad \|u_m(t)\|_{V\cap L^{\lambda+1}(\Omega)} \le \text{const}, \qquad (11.15)$$

respectively.

From (11.14), (11.15) it follows that there exists a subsequence of (u_m), again denoted by (u_m) and $u \in L^\infty(0, T; V)$, $u \in L^\infty(0, T; V \cap L^{\lambda+1}(\Omega))$ such that

$$(u_m) \to u \text{ weakly in } L^\infty(0, T; V) \text{ and in } L^\infty(0, T; V \cap L^{\lambda+1}(\Omega)), \qquad (11.16)$$

respectively and

$$(u_m') \to u' \text{ weakly in } L^\infty(0, T; H) \qquad (11.17)$$

in the following sense: for any fixed $g \in L^1(0,T;V^\star)$, $g \in L^1(0,T;(V \cap L^{\lambda+1}(\Omega))^\star)$, respectively, and $g_1 \in L^1(0,T;H)$

$$\int_0^T \langle g(t), u_m(t)\rangle \, \mathrm{d}t \to \int_0^T \langle g(t), u(t)\rangle \, \mathrm{d}t,$$
$$\int_0^T (g_1(t), u_m'(t)) \, \mathrm{d}t \to \int_0^T (g_1(t), u'(t)) \, \mathrm{d}t.$$

Since the imbedding $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact, by (11.16), (11.17), we have for a subsequence

$$(u_m) \to u \text{ in } L^2(0,T;H) = L^2(Q_T) \text{ and a.e. in } Q_T. \tag{11.18}$$

Finally, by using (11.8), (11.9), (11.16)–(11.18) one obtains as $m \to \infty$ that u is a solution of (11.1), (11.2).

Remark 11.2. The uniqueness of the solution of (11.1), (11.2) is proved in [11] in the particular case when (11.1) does not contain $u(\gamma_1(u))$, $u(\gamma_2(u))$.

11.3 Examples

Let the operator $\tilde{Q}$ be defined by

$$\begin{aligned}\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = &\int_\Omega \left[\sum_{j,l=1}^n a_{jl}(x)(D_l\tilde{u})(D_j\tilde{v}) + d(x)\tilde{u}\tilde{v}\right] \mathrm{d}x\\ &+\sum_{j=1}^n \int_\Omega \left[D_j\tilde{v}(x)\int_\Omega K_j(x,y)D_j\tilde{u}(y)\,\mathrm{d}y\right]\mathrm{d}x\\ &+\int_\Omega \left[\tilde{v}(x)\int_\Omega K_0(x,y)\tilde{u}(y)\,\mathrm{d}y\right]\mathrm{d}x,\end{aligned}$$

where $a_{jl}, d \in L^\infty(\Omega)$, $a_{jl} = a_{lj}$, $\sum_{j,l=1}^n a_{jl}(x)\xi_j\xi_l \geq c_0|\xi|^2$, $d \geq c_0$ with some positive constant c_0 and the functions $K_j \in L^2(\Omega \times \Omega)$ satisfy

$$K_j(x,y) = K_j(y,x) \text{ for a.a. } x, y \in \Omega \text{ and } \int_{\Omega\times\Omega} K_j(x,y)w(x)w(y)\,\mathrm{d}x\mathrm{d}y \geq 0$$

for all $w \in L^2(\Omega)$. (The last assumption means that the integral operators defined by the kernels K_j are self-adjoint and positive.) Then, clearly, assumption (A_1) is satisfied.

If h is a C^1 function such that $h(\eta) = |\eta|^{\lambda+1}$ if $|\eta| > 1$, then (A_3), (A_3'), respectively, are satisfied.

Further, let $\tilde{h}_j : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\text{const}\, |\eta|^{\frac{\lambda+1}{2}} \le |\tilde{h}_j(\eta)| \le \text{const}\, |\eta|^{\frac{\lambda+1}{2}} \text{ for } |\eta| > 1$$

with some positive constants. It is not difficult to show that operators H are defined by one of the formulas

$$H(t,x;u,z) = \chi_1(t,x)\tilde{h}_1\left(\int_{Q_t} u(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi\right) + \chi_2(t,x)\tilde{h}_2\left(\int_{Q_t} z(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi\right),$$

$$H(t,x;u,z) = \chi_1(t,x)\tilde{h}_1\left(\int_0^t u(\tau,x)d\tau\right) + \chi_2(t,x)\tilde{h}_2\left(\int_0^t z(\tau,x)\,\mathrm{d}\tau\right),$$

$$H(t,x;u,z) = \chi_1(t,x)\tilde{h}_1\left(\int_\Omega u(t,\xi)\,\mathrm{d}\xi\right) + \chi_2(t,x)\tilde{h}_2\left(\int_\Omega z(t,\xi)\,\mathrm{d}\xi\right),$$

$$H(t,x;u,z) = \chi_1(t,x)\tilde{h}_1(u(\tau_1(t),x)) + \chi_2(t,x)\tilde{h}_2(u(\tau_2(t),x))$$

where $\tau_j \in C^1,\ 0 \le \tau_j(t) \le t,\ \tau_j'(t) \ge c_1 > 0$ satisfy (B_4) if $\chi_j \in L^\infty(Q_T)$.

The operator G may have the form

$$G(t,x;u,z,w) = \psi_1(t,x;u,z)w(t) + \psi_2(t,x;u,z)$$

where the values of the operators $\psi_1, \psi_2 : Q_T \times L^2(Q_T) \times L^2(Q_T) \to \mathbb{R}$ (of Volterra type) are bounded, and they may have the form similar to the above forms of H with bounded continuous functions $\tilde{h}_j$. Then (B_5) is fulfilled.

Remark 11.3. Instead of $\int_{Q_t} u(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi$ one may consider

$$\int_{Q_t} K(t,x;\tau,\xi)u(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi$$

with a "sufficiently good" kernel K. Similar generalizations of $\int_0^t u(\tau,x)\,\mathrm{d}\tau$ and $\int_\Omega u(t,\xi)\,\mathrm{d}\xi$ can be considered.

11.4 Solutions in $(0,\infty)$

Now we formulate and prove existence of solutions for $t \in (0,\infty)$. Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $u : (0,\infty) \to V$ such that for each fixed finite $T > 0$, their restrictions to $(0,T)$ satisfy $u|_{(0,T)} \in L^p(0,T;V)$ and let $Q_\infty = (0,\infty) \times \Omega$, $L^\alpha_{loc}(Q_\infty)$ be the set of functions $u : Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T.

Now we formulate assumptions on H and G.

(C_4) The function $H : Q_\infty \times L^2_{loc}(Q_\infty) \times L^2_{loc}(Q_\infty) \to \mathbb{R}$ is such that for all fixed $u, z \in L^2_{loc}(Q_\infty)$ the function $(t,x) \mapsto H(t,x;u,z)$ is measurable, H has the Volterra property (see (B_4)), and for each fixed finite $T > 0$, the restriction H_T of H to $Q_T \times L^2(Q_T) \times L^2(Q_T)$ satisfies (B_4).

Remark 11.4. Since H has the Volterra property, this restriction H_T is well defined by the formula

$$H_T(t,x;\tilde{u},\tilde{z}) = H(t,x;u,z), \quad (t,x) \in Q_T \quad \tilde{u},\tilde{z} \in L^2(Q_T),$$

where $u, z \in L^2_{loc}(Q_\infty)$ may be any functions satisfying $u(t,x) = \tilde{u}(t,x)$, $z(t,x) = \tilde{z}(t,x)$ for $(t,x) \in Q_T$.

(C_5) The operator

$$G : Q_\infty \times L^2_{loc}(Q_\infty) \times L^2_{loc}(Q_\infty) \times L^\infty_{loc}(0,\infty;H) \to \mathbb{R}$$

is such that for all fixed $u, z \in L^2_{loc}(Q_\infty)$, $w \in L^\infty_{loc}(0,\infty;H)$, the function $(t,x) \mapsto G(t,x;u,z,w)$ is measurable, G has the Volterra property and for each fixed finite $T > 0$, the restriction G_T of G to $Q_T \times L^2(Q_T) \times L^2(Q_T) \times L^\infty(0,T;H)$ satisfies (B_5).

Theorem 11.2. *Assume (A_1)–(A_3), (C_4), (C_5). Then for all $u_0 \in V$, $u_1 \in H$, $F \in L^2_{loc}(0,\infty;H)$ there exists*

$$u \in L^\infty_{loc}(0,\infty;V) \text{ such that } u' \in L^\infty_{loc}(0,\infty;H), \quad u'' \in L^2_{loc}(0,\infty;V^\star),$$

u satisfies (11.1) for a.a. $t \in (0,\infty)$ (in the sense, formulated in Theorem 11.1) and the initial condition (11.2).

If (A_1), (A_2), (A'_3), (C_4), (C_5) are fulfilled then for all $F \in L^2_{loc}(0,\infty;H)$, $u_0 \in V \cap L^{\lambda+1}(\Omega)$, $u_1 \in H$ there exists

$$u \in L^\infty_{loc}(0,\infty;V \cap L^{\lambda+1}(\Omega)) \text{ such that } u' \in L^\infty_{loc}(0,\infty;H),$$

$$u'' \in L^2_{loc}(0,\infty;V^\star) + L^\infty_{loc}(0,\infty;L^{\frac{\lambda+1}{\lambda}}(\Omega)) \subset L^2_{loc}(0,\infty;[V \cap L^{\lambda+1}(\Omega)]^\star),$$

u satisfies (11.1) for a.a. $t \in (0,\infty)$ (in the sense, formulated in Theorem 11.1) and the initial condition (11.2).

If there exists a finite $T_0 > 0$ such that

$$\text{for a.a. } t > T_0, \quad F(t) = 0, \quad H(t,x;u,z) = 0, \tag{11.19}$$

$$\text{for a.a. } t > T_0, \quad G(t,x;u,z,u')u'(t) \geq -\chi_2(t), \text{ where } \chi_2 \in L^1(0,\infty), \tag{11.20}$$

then for the above solution u we have

$$u \in L^\infty(0,\infty;V),\ u \in L^\infty(0,\infty;V \cap L^{\lambda+1}(\Omega)), \textit{ resp., and } u' \in L^\infty(0,\infty;H). \tag{11.21}$$

Further, if instead of (11.19) the condition

$$|H(t,x;u,z)| \le \chi_1(t) \textit{ is satisfied for } t > T_0 \textit{ with some } \chi_1 \in L^2(T_0,\infty), \tag{11.22}$$

$$|G(t,x;u,z,u')| \le c_5|u'(t)| + \chi_1(t), \quad G(t,x;u,z,u')u'(t) \ge \tilde{c}u'(t)^2 - \chi_2(t) \tag{11.23}$$

with some constant $\tilde{c} > 0$, *finally, there exist* $F_\infty \in H$ *and* $u_\infty \in V$ *such that*

$$F - F_\infty \in L^2(0,\infty;H) \textit{ and } \tilde{Q}u_\infty = F_\infty, \tag{11.24}$$

then

$$\|u'(t)\|_H \le \textit{const}\, e^{-\tilde{c}t}, \quad t \in (0,\infty), \tag{11.25}$$

and there exists $w_0 \in V$ *such that*

$$u(t) \to w_0 \textit{ in } H \textit{ as } t \to \infty, \quad \|u(t) - w_0\|_H \le \textit{const}\, e^{-\tilde{c}t}, \tag{11.26}$$

and w_0 *satisfies*

$$\tilde{Q}(w_0) + \varphi h'(w_0) = F_\infty. \tag{11.27}$$

Proof. Similarly to the proof of Theorem 11.1, we apply Galerkin's method and we want to find the mth approximation of solution u for $t \in (0,\infty)$ in the form [see (11.7)].

$$u_m(t) = \sum_{l=1}^{m} g_{lm}(t)w_l$$

where $g_{lm} \in W^{2,2}_{loc}(0,\infty)$ if (A_3) is satisfied and $g_{lm} \in W^{2,2}_{loc}(0,\infty) \cap L^\infty_{loc}(0,\infty)$ if (A_3') is satisfied. Here $W^{2,2}_{loc}(0,\infty)$ and $L^\infty_{loc}(0,\infty)$ denote the set of functions $g : (0,\infty) \to \mathbb{R}$ such that the restriction of g to $(0,T)$ belongs to $W^{2,2}(0,T)$, $L^\infty(0,T)$, respectively. According to the arguments in the proof of Theorem 11.1, the maximal solutions of (11.8), (11.9) are defined in $[0,T]$ and the estimates (11.14), (11.15) hold for any finite $T > 0$ (if $t \in [0,T]$) where on the right-hand side are finite constants (depending on T).

Let $(T_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence, converging to $+\infty$. By (11.14), (11.15) there is a subsequence (u_{m1}) of (u_m) for which (11.16), (11.17), and (11.18) hold, respectively, with $T = T_1$. Further, there is a subsequence (u_{m2}) of (u_{m1}) for which (11.16), (11.17) and (11.18) hold, respectively, with $T = T_2$. By a diagonal process we obtain a sequence $(u_{mm})_{m\in\mathbb{N}}$ such that (11.16), (11.17), (11.18) hold for every fixed $T > 0$; further,

$$u \in L^\infty_{loc}(0,\infty;V), \quad u' \in L^\infty_{loc}(0,\infty;H), \quad u'' \in L^2_{loc}(0,\infty;V^\star) \text{ and}$$

$$u \in L^\infty_{loc}(0,\infty;V \cap L^{\lambda+1}(\Omega)), \quad u' \in L^\infty_{loc}(0,\infty;H),$$

$$u'' \in L^2_{loc}(0,\infty;V^\star) + L^\infty_{loc}(0,\infty;L^{\frac{\lambda+1}{\lambda}}(\Omega)),$$

respectively, u is a solution of (11.1), (11.2) for $t \in (0,\infty)$.

Now we consider the case when (11.19) and (11.20) hold. Then by (11.13) and Cauchy–Schwarz inequality we obtain for all $t > 0$

$$\begin{aligned}
&\frac{1}{2}\|u_m'(t)\|_H^2 + \frac{1}{2}\langle (Qu_m)(t), u_m(t)\rangle + c_1 \int_\Omega h(u_m(t))\,\mathrm{d}x \\
&\quad \le \|F\|_{L^2(0,T_0;H)}\|u_m'\|_{L^2(0,T_0;H)} + \frac{1}{2}\|u_m'(0)\|_H^2 \\
&\qquad + \frac{1}{2}\langle (Qu_m)(0), u_m(0)\rangle + c_2 \int_\Omega h(u_m(0))\,\mathrm{d}x \\
&\qquad + \text{const} \int_0^{T_0}\int_\Omega h(u_m(\tau))\,\mathrm{d}x\mathrm{d}\tau + \|\chi_2\|_{L^1(0,\infty)} \\
&\quad \le \text{const},
\end{aligned}$$

which implies (11.21).

Finally, if (11.22), (11.23), (11.24) are satisfied then by (11.13) and Young's inequality we obtain for $w_m = u_m - u_\infty$ (since $w_m' = u_m'$):

$$\begin{aligned}
&\frac{1}{2}\|w_m'(t)\|_H^2 + \frac{1}{2}\langle \tilde{Q}(w_m(t)), w_m(t)\rangle + \int_\Omega \varphi(x)h(u_m(t))\,\mathrm{d}x \\
&\qquad + \tilde{c}\int_0^t \left[\int_\Omega |w_m'(\tau)|^2\,\mathrm{d}x\right]\mathrm{d}\tau \\
&\le \quad \varepsilon \int_0^t \left[\int_\Omega |w_m'(\tau)|^2\,\mathrm{d}x\right]\mathrm{d}\tau + C_1(\varepsilon)\int_0^t \|F(\tau) - F_\infty\|_H^2\,\mathrm{d}\tau \\
&\qquad + \frac{1}{2}\|u_m'(0)\|_H^2 + \frac{1}{2}\langle (Qu_m)(0), u_m(0)\rangle + c_2\int_\Omega h(u_m(0))\,\mathrm{d}x \\
&\qquad + \text{const}\left\{\int_0^{T_0}\left[\int_\Omega h(u_m(\tau))\,\mathrm{d}x\right]d\tau\right\}^{1/2}\|w_m'\|_{L^2(0,T_0;H)} \\
&\qquad + C_2(\varepsilon)\|\chi_1\|^2_{L^2(T_0,\infty)} + \|\chi_2\|_{L^1(0,\infty)}.
\end{aligned} \tag{11.28}$$

Choosing $\varepsilon = \tilde{c}/2$ we obtain

$$\int_0^t \left[\int_\Omega |w_m'(\tau)|^2 dx \right] \mathrm{d}\tau \le \text{const} \tag{11.29}$$

for all $t > 0$, m which implies $u' \in L^2(0, \infty; H)$ because for every finite $T > 0$

$$w_m' = u_m' \to u' \text{ weakly in } L^\infty(0, T; H).$$

Further, from (11.28), (11.29) we obtain

$$\|u_m'(t)\|_H^2 + \tilde{c} \int_0^t \|u_m'(\tau)\|_H^2 d\tau \le c^\star$$

with some positive constant $c^\star$ not depending on m and t. Thus by Gronwall's lemma we find

$$\|u_m'(t)\|_H^2 = \|w_m'(t)\|_H^2 \le c^\star \mathrm{e}^{-\tilde{c}t}, \quad t > 0$$

which implies (11.25).

Further, for arbitrary $T_1 < T_2$

$$\begin{aligned}
\|u(T_2) - u(T_1)\|_H^2 &= (u(T_2), u(T_2) - u(T_1))_H - (u(T_1), u(T_2) - u(T_1))_H \\
&= \int_{T_1}^{T_2} \langle u'(t), u(T_2) - u(T_1) \rangle \,\mathrm{d}t \\
&= \int_{T_1}^{T_2} (u'(t), u(T_2) - u(T_1))_H \,\mathrm{d}t \\
&\le \|u(T_2) - u(T_1)\|_H \int_{T_1}^{T_2} \|u'(t)\|_H \,\mathrm{d}t,
\end{aligned}$$

which implies

$$\|u(T_2) - u(T_1)\|_H \le \int_{T_1}^{T_2} \|u'(t)\|_H \,\mathrm{d}t, \tag{11.30}$$

and by (11.25)

$$\|u(T_2) - u(T_1)\|_H \to 0 \text{ as } T_1, T_2 \to \infty.$$

So there exists $w_0 \in H$ such that

$$u(T) \to w_0 \text{ in } H \text{ as } T \to \infty, \tag{11.31}$$

and by (11.30)

$$\|u(T) - w_0\|_H \le \int_T^\infty \|u'(t)\|_H dt \le \text{const } \mathrm{e}^{-\tilde{c}T}.$$

Since $u \in L^\infty(0, \infty; V)$,

$$u(T_k) \to w_0^\star \text{ weakly in } V, \quad w_0^\star \in V \tag{11.32}$$

for some sequence (T_k), $\lim(T_k) = +\infty$. Clearly, (11.32) implies

$$u(T_k) \to w_0^\star \text{ weakly in } H$$

thus by (11.31) $w_0 = w_0^\star \in V$ and (11.32) holds for arbitrary sequence (T_k).

In order to prove (11.27), consider arbitrary fixed $v \in V$, $v \in V \cap L^{\lambda+1}(\Omega)$, respectively and

$$\chi_T(t) = \chi(t-T) \text{ where } \chi \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\chi \subset [0,1], \quad \int_0^1 \chi(t)\,\mathrm{d}t = 1.$$

Multiply (11.3) by $\chi_T(t)$ and integrate with respect to t on $(0,\infty)$; then we obtain

$$\begin{aligned}
&\int_0^\infty \langle u''(t), v\rangle \chi_T(t)\,\mathrm{d}t + \int_0^\infty \langle \tilde{Q}(u(t)), v\rangle \chi_T(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty \left[\int_\Omega \varphi(x)h'(u(t))v\,\mathrm{d}x\right] \chi_T(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty \left[\int_\Omega H(t,x;u,u(\gamma_1(u)))v\,\mathrm{d}x\right] \chi_T(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty \left[\int_\Omega G(t,x;u,u(\gamma_2(u)),u')v\,\mathrm{d}x\right] \chi_T(t)\,\mathrm{d}t \\
&= \int_0^\infty (F(t),v)\chi_T(t)\,\mathrm{d}t.
\end{aligned} \tag{11.33}$$

Let (T_k) be an arbitrary sequence converging to $+\infty$.

For the first term on the left-hand side of (11.33) we have by (11.25)

$$\int_0^\infty \langle u''(t), v\rangle \chi_{T_k}(t)\,\mathrm{d}t = -\int_0^\infty \langle u'(t), v\rangle \chi'_{T_k}(t)\,\mathrm{d}t \to 0 \text{ as } k \to \infty \tag{11.34}$$

further, by (A_1), (11.32) and Lebesgue's dominated convergence theorem

$$\begin{aligned}
\int_0^\infty \langle \tilde{Q}(u(t)), v\rangle \chi_{T_k}(t)\,\mathrm{d}t &= \int_0^\infty \langle \tilde{Q}v), u(t)\rangle \chi_{T_k}(t)\,\mathrm{d}t \\
&= \int_0^1 \langle \tilde{Q}v, u(T_k+\tau)\rangle \chi(\tau)\,\mathrm{d}\tau \\
&\to \int_0^1 \langle \tilde{Q}v, w_0\rangle \chi(\tau)\,\mathrm{d}\tau \\
&= \langle \tilde{Q}v, w_0\rangle = \langle \tilde{Q}w_0, v\rangle
\end{aligned} \tag{11.35}$$

as $k \to \infty$.

For the third term on the left-hand side of (11.33) we have

$$\int_0^\infty \left[\int_\Omega \varphi(x)h'(u(t))v\,\mathrm{d}x\right]\chi_{T_k}(t)\,\mathrm{d}t = \int_0^1 \left[\int_\Omega \varphi(x)h'(u(T_k+\tau))v\,\mathrm{d}x\right]\chi(t)\,\mathrm{d}\tau$$
$$\to \int_0^1 \left[\int_\Omega \varphi(x)h'(w_0))v\,\mathrm{d}x\right]\chi(\tau)\,\mathrm{d}\tau$$
$$= \int_\Omega \varphi(x)h'(w_0))v\,\mathrm{d}x \tag{11.36}$$

as $k \to \infty$, since by (11.26)

$$u(T_k+\tau) \to w_0 \text{ in } L^2((0,1)\times\Omega) \text{ as } k \to \infty$$

and thus for a.e. $(\tau, x) \in (0,1)\times\Omega$ (for a subsequence), consequently,

$$h'(u(T_k+\tau,x)) \to h'(w_0(x)) \text{ for a.e. } (\tau,x) \in (0,1)\times\Omega. \tag{11.37}$$

By using Hölder's inequality, (A_3), (A_3'), respectively, and Vitali's theorem we obtain (11.36) from (11.37).

The fourth and fifth terms on the left-hand side of (11.33) can be estimated by (11.22), (11.25) as follows: for sufficiently large k

$$\left|\int_0^\infty \left[\int_\Omega H(t,x;u,u(\gamma_1(u)))v\,\mathrm{d}x\right]\chi_{T_k}(t)\,\mathrm{d}t\right|$$
$$\le \left|\int_0^1 \left[\int_\Omega H(T_k+\tau,x;u,u(\gamma_1(u)))v\,\mathrm{d}x\right]\chi(\tau)\,\mathrm{d}\tau\right|$$
$$\le \int_0^1 \left[\int_\Omega \chi_1(T_k+\tau)|v|\,\mathrm{d}x\right]|\chi(\tau)|\,\mathrm{d}\tau$$
$$\to 0 \tag{11.38}$$

and

$$\left|\int_0^\infty \left[\int_\Omega G(t,x;u,u(\gamma_2(u)),u')v\,\mathrm{d}x\right]\chi_{T_k}(t)\,\mathrm{d}t\right|$$
$$\le \int_0^1 \left[\int_\Omega \{c_5|u'(T_k+\tau)| + \chi_1(T_k+\tau)\}|v|\,\mathrm{d}x\right]|\chi(\tau)|\,\mathrm{d}\tau$$
$$\to 0 \tag{11.39}$$

as $k \to \infty$. Finally, for the right-hand side of (11.33), we obtain, by using (11.24) and the Cauchy–Schwarz inequality

$$\int_0^\infty \langle F(t), v\rangle \chi_{T_k}\,\mathrm{d}t = \int_0^\infty (F(t), v)\chi_{T_k}\,\mathrm{d}t = \int_0^1 (F(T_k+\tau), v)\chi(\tau)\,\mathrm{d}\tau$$
$$\to \int_0^1 (F_\infty, v)\chi(\tau)\,\mathrm{d}\tau = (F_\infty, v). \qquad (11.40)$$

From (11.33)–(11.36) and (11.38)–(11.40) one obtains (11.27). □

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Chapter 12
A Fast Parallel Algorithm for Delay Partial Differential Equations Modeling the Cell Cycle in Cell Lines Derived from Human Tumors

Barbara Zubik-Kowal

Abstract We present a fast numerical algorithm for solving delay partial differential equations that model the growth of human tumor cells. The undetermined model parameters need to be estimated according to experimental data and it is desired to shorten the computational time needed in estimating them. To speed up the computations, we present an algorithm invoking parallelization designed for arbitrary numbers of available processors. The presented numerical results demonstrate the efficiency of the algorithm.

Keywords Delay partial differential equations • Cell division cycle • Parallel algorithm • Parallel computations • Computational efficiency

12.1 Introduction

Malignant tumors are composed of cancer cells that divide quickly and grow uncontrollably, invade and destroy nearby normal tissues, and are likely to spread to other parts of the body establishing new metastatic tumors. A crucial part of the study of the dynamics of cancer tumor growth is an understanding and analysis of the cell division cycle. The cell division cycle is dominated by four phases: G_1, S, G_2, and M, respectively. The G_1-phase is characterized by an increased production of proteins, during which the cell is getting ready to divide. The S-phase is characterized by an increased production of DNA and consequently the new cells that will be made as a result of the cell production in the M-phase will have the same DNA. During the G_2-phase, the cell continues to grow in a state shortly before it

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splits into two cells. Though the transition from the G_2-phase to the M-phase cannot be easily measured, a characteristic of the M-phase is the initiation of splitting of the cell into two identical cells.

The influence of the cell cycle on cell response to anticancer drugs or radiation has been investigated by scientists from many disciplines. The cell cycle dynamics have been also studied mathematically, for example, Basse et al. [1] have developed a mathematical model with four compartments representing the G_1, S, G_2, and M subpopulations of cells and have determined its parameters by utilizing cell lines that have been established from malignant tumors taken from humans and exposed to the anticancer drug paclitaxel. The correspondence obtained in [1] between experimental and predicted data has shown the potential of the model to describe the evolution of human tumor cells and their responses to therapy. The goal of such mathematical models is to gain an intuitive understanding of why some patients fail to respond to anticancer therapy and predict its effect.

Anticancer therapy is cautiously evaluated for each individual patient and, depending on the stage of the disease, may involve surgical treatment, radiotherapy, chemotherapy, hyperthermia therapy, immunotherapy, and combined treatment. Combining radiotherapy with chemotherapeutic drugs increases patient survival by, for example, decreasing distant metastases. Moreover, radiotherapy and chemotherapeutic drugs are more powerful in damaging proliferating than non-proliferating cells; thus, their effectiveness depends on the cell cycle phase. Specifically, cells in the G_2- and M-phases are approximately three times more sensitive than cells in the S-phase, [19]. Some drugs cause significant accumulation of cells in the M-phase several hours after the treatment and radiation delivered at this time period increases its effectiveness. Any combination of anticancer drugs with radiotherapy is subjected to preclinical studies and evaluated with in vitro cell culture systems followed by in vivo testing for the drug's interaction with radiation.

With the aid of computational tools and numerical simulations, mathematical models became valuable mechanisms in the understanding and qualitative and quantitative analyses of the biology of cancer. As a stepping stone in the discipline, Bellomo and Forni [8] initiated the application of mathematical kinetic theory to modeling tumor progression dynamics and cellular interactions within an active host immune system, using statistical variables related to distributions over physical states modeling interactions at the cellular level. This approach to tumor growth modeling receives growing attention and has been developed in many papers, e.g., [4, 6, 9, 16, 17], and monographs, e.g., [2, 7, 11]. This active area of current research has also been developed in [3, 5, 10] and references provided therein, where macroscopic models of biological tissues have been derived from an underlying statistical mechanics viewpoint with discrete microscopic states using the application of the kinetic theory of active particles.

Various approaches in the mathematical modeling of tumor cell population growth were also validated by comparison to experimental and clinical data, e.g., in [1] utilizing in vitro experimental data and in [13, 15] with in vivo laboratory data, while hospital clinical data of the evolution of human cancer tumors are utilized in [12, 18] to determine the parameters of the model equations. The determination of the parameters involved in the modeling of tumor growth is based on a series of numerical simulations and as such questions regarding the computational time required to conduct the search procedures and how to limit it are of utmost importance.

The challenge is the minimization of search time so that the procedures will be appropriate for wide varieties of patients and be able to satisfy individual physical observations. The goals of the paper are to present a fast numerical algorithm for solving the model developed in [1] that we construct by making use of the time domain decomposition results derived in [22] and to study and demonstrate its efficiency as well as answer the question of how much the computational time can decrease as we increase the number of processors used. The model is a system of delay partial differential equations and its solutions represent four subpopulations of cells, G_1, S, G_2, and M, corresponding to the cell densities in the G_1, S, G_2, and M-phases, respectively. The algorithm is designed in such a way that it may make use of arbitrary numbers of processors working in parallel, thus highly speeding up the computational time.

The paper is organized as follows. The model equations for the four subpopulations of cells G_1, S, G_2, and M and the delay term are presented in Sect. 12.2. The numerical algorithm designed for the model and results of numerical experiments are presented in Sect. 12.3. The paper is completed with concluding remarks sketched in Sect. 12.4.

12.2 Model Equations with Time Delay Terms

The movement of cells between the four phases, namely the G_1, S, G_2, and M-phases, may be modeled as a cell migration process between four compartments, correspondingly named G_1, S, G_2, and M. An illustration of the movement of cells between the four compartments is provided in [1, Fig. 1]. The solutions $G_1(x,t)$, $S(x,t)$, $G_2(x,t)$, $M(x,t)$ of the model equations developed in [1] represent the densities of cells in each compartment, respectively. Here, the independent variable t represents time and x, rather than being a spacial variable, corresponds to the dimensionless relative DNA content. In the mathematical model, DNA content is used as a measure of cell size as the phase changes correspond to changes in DNA content.

The equation for the cell density $G_1(x,t)$ in the G_1-phase is written in the form

$$\frac{\partial G_1(x,t)}{\partial t} = 4b\, M(2x,t) - (k_1 + \mu_{G_1})\, G_1(x,t), \tag{12.1}$$

where b represents the rate at which a cell in the M-phase divides into two daughter cells. Since all cells corresponding to DNA content specified by the range $x \in [2\tilde{x}, 2\tilde{x} + 2\Delta x]$ are doubled at the moment of division and mapped to $[\tilde{x}, \tilde{x} + \Delta x]$, the rate b is multiplied by 2^2 in (12.1) (see [1]). Correspondingly, the term $4b\,M(2x,t)$ describes the influx of cells from the M-phase to the G_1-phase. The term $\mu_{G_1} G_1(x,t)$ describes the death of cells in the G_1-phase and the term $k_1 G_1(x,t)$ describes the outflow of cells from the G_1-phase to the S-phase.

The evolution of the cell density in the S-phase may be summarized by the delay partial differential equation written in the form

$$\begin{aligned}\frac{\partial S(x,t)}{\partial t} = \varepsilon\,\frac{\partial^2 S(x,t)}{\partial x^2} - \mu_S\, S(x,t) - g\frac{\partial S(x,t)}{\partial x}\\ +\; k_1\, G_1(x,t) - I(x,t;T_S).\end{aligned} \tag{12.2}$$

Note the coupling between the cell densities in the S and G_1-phases. Here, ε is the dispersion coefficient appearing in the diffusive term, μ_S is the death rate of cells in the S-phase in a natural decay model of cell death giving rise to exponential decay, g is the average growth rate of DNA, and k_1 is the transition rate from the G_1-phase to the S-phase. The delay term $I(x,t;T_S)$ on the right-hand side of (12.2) is defined by

$$I(x,t;T_S) = \begin{cases} \displaystyle\int_0^\infty k_1\, G_1(y,t-T_S)\,\gamma(T_S,x,y)\,\mathrm{d}y, & \text{for } t \ge T_S,\\ 0, & \text{for } t < T_S,\end{cases}$$

with

$$\begin{aligned}\gamma(\tau,x,y) = \frac{\exp(-\mu_S\tau)}{2\sqrt{\pi\varepsilon\tau}}\Bigg(\exp\Bigg(-\frac{\big((x-g\tau)-y\big)^2}{4\varepsilon\tau}\Bigg)\\ -\big(1+\nu(\tau,x,y)\big)\exp\Bigg(-\frac{\big(x+g\tau)+y\big)^2}{4\varepsilon\tau}\Bigg)\Bigg)\end{aligned}$$

and

$$\nu(\tau,x,y) = \frac{x+y}{g\tau}\Big(1+O(\tau^{-1})\Big)$$

and stands for the subpopulation of cells transferred from the G_1-phase to the S-phase with a delay of T_S units of time and is ready to exit to the G_2-phase.

This delayed exit to the G_2-phase is also represented by the delay term, appearing as the first term on the right-hand side of the following equation describing the dynamics of cell density evolution in the G_2-phase

$$\frac{\partial G_2(x,t)}{\partial t} = I(x,t;T_S) - (k_2+\mu_{G_2})\,G_2(x,t). \tag{12.3}$$

The last terms in (12.3) represent the loss of cells due to their death in the G_2-phase (with the death rate μ_{G_2}) and the loss due to the transition (with the death rate k_2) from G_2 to the M-phase.

The evolution of the cell density in the last phase of the cell division cycle is described by the equation

$$\frac{\partial M(x,t)}{\partial t} = k_2\, G_2(x,t) - b\, M(x,t) - \mu_M\, M(x,t), \tag{12.4}$$

where $k_2\, G_2(x,t)$ becomes the corresponding source term due to the transition from the G_2-phase into the M-phase. The second term on the right-hand side of (12.4) represents the loss term due to the division of cells in the M-phase (with the division rate b) and the third term represents the loss of cells due to their death (with the death rate μ_M).

The goal of the paper is to introduce a parallel algorithm for (12.1)–(12.4) in order to compute fast numerical solutions, which are utilized in the process of determination of the parameters of the model according to experimental data. There have been previous studies focusing on the computational speedup by parallelization for systems with a delay character like that of (12.1)–(12.4). The basis of these ideas in the assignment of equations to processors was discretization in the x-domain followed by the separation of the resulting delay differential equations using waveform relaxation [20,21]. In regard to computational efficiency, the paper [14] demonstrates that pseudospectral semi-discretization is more efficient for delay systems of the form (12.1)–(12.4) than semi-discretization based on finite differences. The above described approach of parallelization is, however, restricted in the sense that the choice of the number of grid points used in the discretization of the x-domain limits the possible number of assignable processors and in the use of iterative processes with slow convergence properties. Motivated by the recent advancements in nowadays' cost-effective technology solutions and the increasing availability of processors, we focus this paper towards the construction of numerical algorithms with the capability of using arbitrary numbers of processors, so that the choice in the number of processors depends on user preferences and availability instead of numerical algorithm structure.

12.3 Parallel Algorithm

For the new strategy for the model equations for $G_1(x,t)$, $S(x,t)$, $G_2(x,t)$, and $M(x,t)$-phases, we use their natural properties. Since the equations for $G_1(x,t)$, $G_2(x,t)$, and $M(x,t)$ do not include partial derivatives with respect to x, they are isolated for each grid-point x_i and parallelization across the x-domain can be applied with no additional cost.

However, the equation for the S-phase includes first- and second- order derivatives with respect to x and its semi-discretization results in a system of joint equations, which is not natural for parallelization. Therefore, instead of parallelization across the x-domain, we introduce parallelization in t with arbitrary numbers of processors. The delay term is computed from (12.1), (12.3), (12.4) and collected in the vector function $F(t)$. The process of semi-discretization in x of (12.2) leads to

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t}(t) = Av(t) + F(t), \, 0 < t \leq T, \\ v(0) = g, \end{cases} \tag{12.5}$$

where A is an M by M matrix, $v(t)$ is a vector function and its elements are approximations to $S(x_i, t)$, and g is an initial vector corresponding to the initial values $S(x_i, 0)$.

Let n be any number of available processors. Define $N = n - M$ if $M < n$ and in case $M \geq n$, define $N = n$. Next, define $h = T/N$ and the grid-points $t_k = kh$, for $k = 0, 1, \ldots, N$. Let $v_k(t)$, with $k = 1, \ldots, N$, be the solution to the initial-value problem

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}t}(t) = Az(t) + F(t), \, t_{k-1} \leq t \leq t_k, \\ z(t_{k-1}) = \mathbf{0}, \end{cases} \tag{12.6}$$

where $\mathbf{0} \in \mathbb{R}^M$ is the zero vector. The kth solution $v_k(t)$ is computed by the kth processor and each of them works with no communications with other processors.

After all the solutions $v_k(t)$, for $k = 1, \ldots, N$, are computed, the solution to (12.5) is generated from the formula

$$v(t_k) = \exp(hA)v(t_{k-1}) + v_k(t_k). \tag{12.7}$$

The relation (12.7) between v and v_k, $k = 1, \ldots, N$, is proved in [22]. For our numerical solution, we compute the matrix exponential $\exp(hA)$ from the formula

$$\exp(hA) = \left[\, d_1 \; d_2 \, \ldots \, d_M \,\right],$$

with

$$d_k = \tilde{v}_{k+N}(h) - v_1(h)$$

for $k = 1, 2, \ldots, M$, and $\tilde{v}_{k+N}(t)$ is the solution to the initial-value problem

$$\begin{cases} \frac{dz}{dt}(t) = Az(t) + F(t), \, 0 \leq t \leq h, \\ z(0) = e_k, \end{cases} \tag{12.8}$$

where $e_i \in \mathbb{R}^M$, $i = 1, 2, \ldots, M$, are the unit vectors with 1 as the ith component and 0 as the rest of the elements.

In case $n > M$, the columns d_k of the matrix exponential $\exp(hA)$ are computed in parallel by M processors, which work independently from each other

Table 12.1 Results of numerical experiments

N	h	Execution time in seconds
1	150.0	400.51
2	75.0	194.66
3	50.0	126.33
4	37.5	94.10
5	30.0	75.00
6	25.0	62.32
8	18.75	47.76
10	15.0	37.15
12	12.5	30.81
15	10.0	24.56
20	7.5	18.36
25	6.0	14.80
30	5.0	12.28
40	3.75	9.16
50	3.0	7.32
100	1.5	3.67
150	1.0	2.46
300	0.5	1.25

and independently from the N processors working for (12.6). Note that (12.6) and (12.8) form $N + M$ independent tasks, which can be solved by $N + M$ independent processors. The first N processors compute the solutions $v_k(t)$, with $k = 1, \ldots, N$, to (12.6) and the next M processors compute the solutions $\tilde{v}_k(t)$, with $k = N + 1, \ldots, N + M$, to (12.8).

We apply the algorithm and solve the model (12.1)–(12.4). For discretization in $x \in [0, L]$, we use the Chebyshev–Gauss–Lobatto points

$$x_i = \frac{L}{2}\left(1 - \cos\frac{i\pi}{I}\right),$$

with $i = 0, 1, \ldots, I$, and apply pseudospectral differentiation matrices of first and second order to approximate the first- and second-order spacial partial derivatives in (12.1). This leads to the pseudospectral semi-discrete system presented in the form (12.5).

We perform numerical experiments to test the efficiency of the method and apply the algorithm with different numbers N of processors leading to different values of h. The resulting execution time for the different series of cases is presented in Table 12.1, from which we see a noticeable improvement as N increases, illustrating the efficiency of the algorithm. The numerical solutions for the cell count as functions of DNA content obtained by the algorithm are presented in Fig. 12.1 at different values of time.

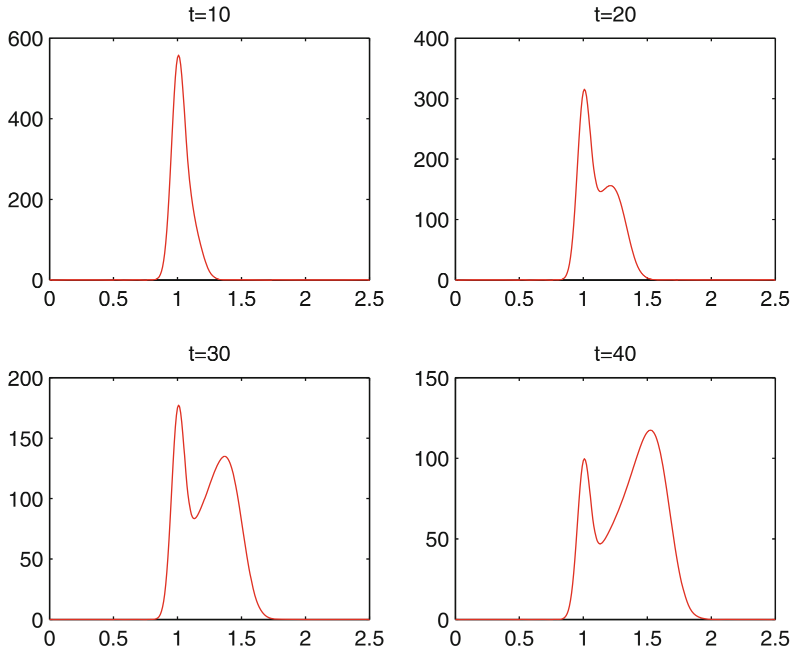

Fig. 12.1 Cell count versus DNA content at time $t = 10$, $t = 20$, $t = 30$, and $t = 40$

12.4 Concluding Remarks

We have presented a fast algorithm for the numerical solution of a system composed of delay partial differential equations that model the evolution and development of human tumor cells. The mathematical model characterizes the cell cycle in cell lines derived from human tumors and depends on a set of biological parameters that are determinable from data obtained experimentally. The problem of estimating the model parameters from experimental and clinical data from different cancer patients relies on efficient numerical schemes that minimize computational time. The algorithm presented in this paper reduces the required computational time by employing parallelization across arbitrary numbers of processors working independently from each other. We confirm the efficiency of the algorithm by a series of numerical experiments. Our future plans include a study of the interaction of human tumor growth with the effects of therapy in the form of radiation, chemotherapy, and anticancer drugs.

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