Chapter 9
Academic Problems

Many practical optimization problems involve nonsmooth functions. In this chapter, we give an extensive collection of problems for nonsmooth minimization which can be used to test nonsmooth optimization solvers. The general formula for these problems is written by

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in S, \end{cases} \tag{9.1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous on the feasible region $S \subseteq \mathbb{R}^n$. Note that no differentiability or convexity assumptions are made. All the problems given here are found in the literature and have been used in the past to develop, test, or compare NSO software.

We shall use a classification of test problems modified from that of [111]. That is we use a sequence of letters

O[O]-C-R-S,

where Table 9.1 give all possible abbreviations that could replace the letters O, C, R, and S (the brackets mean an optional abbreviation).

We first give a summary of all the problems in Table 9.2, where n denotes the number of variables and $f(\boldsymbol{x}^*)$ is the minimum value of the objective function. In addition, the classification and the references to the origin of the problems in each case are given in Table 9.2. Then, in Sects. 9.1 (small unconstrained problems), 9.2 (bound constraints), 9.3 (linearly constrained problems), 9.4 (large-scale unconstrained problems), and 9.5 (inequality constraints), we present the formulation of the objective function f, possible constraint functions g_j $(j = 1, \ldots, p)$, and the starting point $\boldsymbol{x}^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})^T$ for each problem. We also give the minimum point $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)^T$ for the problems with the precision of at least four decimals if possible (in larger cases this is not always practicable).

In what follows, we denote by div (i, j) the integer division for positive integers i and j, that is, the maximum integer not greater than i/j, and by mod (i, j) the remainder after integer division, that is, mod $(i, j) = j(i/j - \text{div}\,(i, j))$.

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Table 9.1 Classification of test problems

O	**Information about the objective**
L	Piecewise linear objective function
Q	Piecewise quadratic objective function
P	Generalized polynomial objective function
G	General objective function
D	Difference of two convex functions (DC)
M	Min-max- type objective function

C	**Information about constraint functions**
U	Unconstrained problem ($S = \mathbb{R}^n$ in (9.1))
B	Upper and lower bounds only
	($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^l \leq x_i \leq x_i^u$ for all $i = 1, \ldots, n\}$ in (9.1)
L	Linear constraint functions
Q	Quadratic constraint functions
G	General constraint functions
	($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \leq 0$ for all $j = 1, \ldots, p\}$ in (9.1)

R	**Regularity of the problem**
X	Convex problem
Z	Nonconvex problem

S	**Information about the solution**
E	Exact solution known
N	Only approximate numerical solution known

9.1 Small Unconstrained Problems

In this section we describe 40 small-scale nonsmooth unconstrained test problems. The number of variables varies from 2 to 50.

1 CB2 (Charalambous/Bandler) [59]

Classification: GM-U-X-N,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{x_1^2 + x_2^4,\ (2 - x_1)^2 + (2 - x_2)^2,\ 2e^{x_2 - x_1}\right\}$,
Starting point 1: $\boldsymbol{x}^{(1)} = (2, 2)^T$,
Starting point 2: $\boldsymbol{x}^{(1)} = (1, -0.1)^T$,
Optimum point: $\boldsymbol{x}^* = (1.139286, 0.899365)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 1.9522245$.

Table 9.2 Condensed list of all test problems

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
1 CB2	GM-U-X-N	2	1.9522245	[59]
2 CB3	GM-U-X-E	2	2	[59]
3 DEM	QM-U-X-E	2	-3	[168]
4 QL	QM-U-X-E	2	7.2	[168]
5 LQ	QM-U-X-E	2	$-\sqrt{2}$	[168]
6 Mifflin 1	G-U-X-E	2	-1	[168]
7 Wolfe	G-U-X-E	2	-8	[159]
8 Rosen-Suzuki	QM-U-X-E	4	-44	[205]
9 Davidon 2	QM-U-X-N	4	115.70644	[159]
10 Shor	QM-U-X-N	5	22.600162	[168]
11 Maxquad	QM-U-X-N	10	-0.8414083	[146, 168]
12 Polak 2	GM-U-X-E	10	54.598150	[198]
13 Polak 3	GM-U-X-N	11	3.7034924	[198]
14 Wong 1	PM-U-X-N	7	680.63006	[7, 159]
15 Wong 2	QM-U-X-N	10	24.306209	[7, 159]
16 Wong 3	PM-U-X-N	20	133.72828	[7, 159]
17 Maxq	QM-U-X-E	20	0	[208]
18 Maxl	LM-U-X-E	20	0	[168]
19 TR48	LD-U-X-N	48	-638565.0	[146]
20 Goffin	LD-U-X-E	50	0	[168]
21 Crescent	QM-U-Z-E	2	0	[131]
22 Mifflin 2	G-U-Z-E	2	-1	[168]
23 WF	GM-U-Z-E	2	0	[159]
24 SPIRAL	GM-U-Z-E	2	0	[159]
25 EVD52	PM-U-Z-N	3	3.5997193	[159]
26 PBC3	GM-U-Z-N	3	$0.42021427 \cdot 10^{-2}$	[159]
27 Bard	GM-U-Z-N	3	$0.50816327 \cdot 10^{-1}$	[159]
28 Kowalik-Osborne	GM-U-Z-N	4	$0.80843684 \cdot 10^{-2}$	[159]
29 Polak 6	PM-U-Z-E	4	-44	[198]
30 OET5	QM-U-Z-N	4	$0.26359735 \cdot 10^{-2}$	[159]
31 OET6	GM-U-Z-N	4	$0.20160753 \cdot 10^{-2}$	[159]
32 EXP	GM-U-Z-N	5	$0.12237125 \cdot 10^{-3}$	[159]
33 PBC1	GM-U-Z-N	5	$0.22340496 \cdot 10^{-1}$	[159]
34 HS78	G-U-Z-N	5	-2.9197004	[111, 159]
35 El-Attar	G-U-Z-N	6	0.5598131	[159]
36 EVD61	GM-U-Z-N	6	$0.34904926 \cdot 10^{-1}$	[159]
37 Gill	PM-U-Z-N	10	9.7857721	[159]
38 Problem 1 in [21]	GD-U-Z-E	2	2	[21]
39 L1 Rosenbrock	LD-U-Z-E	2	0	[21]

(Continued)

Table 9.2 (Continued)

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
40 L1 Wood	LD-U-Z-E	4	0	[21]
41 Wong 2C	QM-L-X-N	10	24.306209	[159]
42 Wong 3C	PM-L-X-N	20	133.72828	[159]
43 MAD1	GM-L-Z-N	2	−0.38965952	[159]
44 MAD2	GM-L-Z-N	2	−0.33035714	[159]
45 MAD4	GM-L-Z-N	2	−0.44891079	[159]
46 MAD5	GM-L-Z-N	2	−0.42928061	[159]
47 PENTAGON	GM-L-Z-N	6	−1.85961870	[159]
48 MAD6	GM-L-Z-N	7	0.10183089	[159]
49 Dembo 3	GM-L-Z-N	7	1227.2260	[159]
50 Dembo 5	GM-L-Z-N	8	7049.2480	[159]
51 EQUIL	GM-L-Z-N	8	0	[146, 159]
52 HS114	GM-L-Z-N	10	−1768.8070	[159]
53 Dembo 7	GM-L-Z-N	16	174.78699	[159]
54 MAD8	QM-B-Z-N	20	0.50694799	[159]
55 Gen. of MAXL	LM-U-X-E	any	0	[155]
56 Gen. of L1HILB	L-U-X-E	any	0	[155]
57 Gen. of MAXQ	QM-U-X-E	any	0	[98]
58 Gen. of MXHILB	LM-U-X-E	any	0	[98]
59 Chained LQ	G-U-X-E	any	$-(n-1)2^{1/2}$	[98]
60 Chained CB3 I	G-U-X-E	any	$2(n-1)$	[98]
61 Chained CB3 II	GM-U-X-E	any	$2(n-1)$	[98]
62 Number of Active Faces	GM-U-Z-E	any	0	[95]
63 Gen. of Brown Function 2	G-U-Z-E	any	0	[98]
64 Chained Mifflin 2	G-U-Z-N	any	varies*	[98]
65 Chained Crescent I	QM-U-Z-E	any	0	[98]
66 Chained Crescent II	G-U-Z-E	any	0	[98]
67 Problem 6 in `test29`	QM-U-Z-N	any	0	[155]
68 Problem 17 in `test29`	GM-U-Z-E	any	0	[155]
69 Problem 19 in `test29`	GM-U-Z-N	any	0	[155]
70 Problem 20 in `test29`	GM-U-Z-N	any	0	[155]
71 Problem 22 in `test29`	GM-U-Z-N	any	0	[155]
72 Problem 24 in `test29`	GM-U-Z-N	any	0	[155]
73 DC Maxl	LD-U-Z-E	any	0	[21]
74 DC Maxq	QD-U-Z-E	any	0	[26]
75 Problem 6 in [26]	LD-U-Z-E	any	0	[26]
76 Problem 7 in [26]	LD-U-Z-E	any	0	[26]
57 + 77	QM-Q-Z-N	any	0.500065**	[122, 126]
58 + 77	LM-Q-Z-N	any	0.000163**	[122, 126]

(Continued)

Table 9.2 (Continued)

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
59 + 77	G-Q-Z-N	any	−1408.63**	[122, 126]
60 + 77	G-Q-Z-N	any	2003.24**	[122, 126]
61 + 77	GM-Q-Z-N	any	1998.36**	[122, 126]
62 + 77	GM-Q-Z-N	any	0.534851**	[122, 126]
63 + 77	G-Q-Z-N	any	5.00248**	[122, 126]
64 + 77	G-Q-Z-N	any	−680.628**	[122, 126]
65 + 77	QM-Q-Z-N	any	1.56604**	[122, 126]
66 + 77	G-Q-Z-N	any	5.99059**	[122, 126]
57 + 78	QM-Q-R-N	any	0.880569**	[122, 126]
58 + 78	LM-Q-Z-N	any	0.008487**	[122, 126]
59 + 78	G-Q-Z-N	any	−735.874**	[122, 126]
60 + 78	G-Q-Z-N	any	2808.45**	[122, 126]
61 + 78	GM-Q-Z-N	any	2796.35**	[122, 126]
62 + 78	GM-Q-Z-N	any	2.77674**	[122, 126]
63 + 78	G-Q-Z-N	any	not avail.	[122, 126]
64 + 78	G-Q-Z-N	any	4466.99**	[122, 126]
65 + 78	QM-Q-Z-N	any	483.441**	[122, 126]
66 + 78	G-Q-Z-N	any	not avail.	[122, 126]
57 + 79	QM-G-R-N	any	not avail.	[122, 126]
58 + 79	LM-G-Z-N	any	0.007981**	[122, 126]
59 + 79	G-G-Z-N	any	−1412.14**	[122, 126]
60 + 79	G-G-Z-N	any	2001.63**	[122, 126]
61 + 79	GM-G-Z-N	any	not avail.	[122, 126]
62 + 79	GM-G-Z-N	any	0.405473**	[122, 126]
63 + 79	G-G-Z-N	any	not avail.	[122, 126]
64 + 79	G-G-Z-N	any	−705.910**	[122, 126]
65 + 79	QM-G-Z-N	any	0.250063**	[122, 126]
66 + 79	G-G-Z-N	any	1.85396**	[122, 126]
57 + 80	QM-G-R-N	any	0.388891**	[122, 126]
58 + 80	LM-G-Z-N	any	0.007981**	[122, 126]
59 + 80	G-G-Z-N	any	−1412.13**	[122, 126]
60 + 80	G-G-Z-N	any	2001.72**	[122, 126]
61 + 80	GM-G-Z-N	any	not avail.	[122, 126]
62 + 80	GM-G-Z-N	any	0.405549**	[122, 126]
63 + 80	G-G-Z-N	any	not avail.	[122, 126]
64 + 80	G-G-Z-N	any	−705.926**	[122, 126]
65 + 80	QM-G-Z-N	any	0.250222**	[122, 126]
66 + 80	G-G-Z-N	any	1.39342**	[122, 126]
57 + 81	QM-Q-R-N	any	0.138009**	[122, 126]

(Continued)

Table 9.2 (Continued)

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
58 + 81	LM-Q-Z-N	any	0.600611**	[122, 126]
59 + 81	G-Q-Z-N	any	−1153.55**	[122, 126]
60 + 81	G-Q-Z-N	any	4043.82**	[122, 126]
61 + 81	GM-Q-Z-N	any	4043.82**	[122, 126]
62 + 81	GM-Q-Z-N	any	5.81129**	[122, 126]
63 + 81	G-Q-Z-N	any	589.469**	[122, 126]
64 + 81	G-Q-Z-N	any	−660.307**	[122, 126]
65 + 81	QM-Q-Z-N	any	490.173**	[122, 126]
66 + 81	G-Q-Z-N	any	not avail.	[122, 126]

* $f(\boldsymbol{x}^*) \approx -34.795$ for $n = 50$, $f(\boldsymbol{x}^*) \approx -140.86$ for $n = 200$, and $f(\boldsymbol{x}^*) \approx -706.55$ for $n = 1000$.
** $f(\boldsymbol{x}^*)$ for $n = 1000$.

Table 9.3 Values of vectors $\boldsymbol{s}$ and $\boldsymbol{d}$ for problem 19

i	s_i	d_i	i	s_i	d_i	i	s_i	d_i
1	22	61	17	95	32	33	30	52
2	53	67	18	34	21	34	88	66
3	64	24	19	59	61	35	74	89
4	15	84	20	36	21	36	59	65
5	66	13	21	22	51	37	93	63
6	37	86	22	94	14	38	54	47
7	16	89	23	28	89	39	89	7
8	23	46	24	34	79	40	30	61
9	67	48	25	36	38	41	79	87
10	18	50	26	38	20	42	46	19
11	52	74	27	55	97	43	35	36
12	69	75	28	77	19	44	41	43
13	17	88	29	45	10	45	99	9
14	29	40	30	34	73	46	52	12
15	50	29	31	32	59	47	76	8
16	13	45	32	58	92	48	93	67

2 **CB3** (Charalambous/Bandler) [59]

Classification: GM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{x_1^4 + x_2^2,\ (2-x_1)^2 + (2-x_2)^2,\ 2e^{x_2 - x_1}\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (2, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 2$.

Table 9.4 Data for symmetric cost matrix A for problem 19

273	1272	744	1138	1972	1580	1878	1539	1457	429	1129	1251	1421	588	334	837
1364	229	961	754	1169	1488	720	1280	816	664	1178	939	1698	983	1119	1029
1815	721	1753	330	1499	1107	1576	942	484	617	896	1184	1030	1718	604	999
809	866	1722	1338	1640	1266	1185	440	894	992	1173	334	358	626	1124	358
847	533	915	1219	481	1009	543	937	915	667	1441	812	848	776	1560	526
1494	598	1244	1304	1306	685	668	444	1157	1359	1176	1475	335	1519	140	937
697	951	267	227	1229	587	369	554	721	1212	739	596	1291	1114	701	426
285	676	155	456	1936	319	337	604	907	214	424	748	817	666	1592	521
2172	356	467	1583	882	2139	2182	1961	781	678	1425	1861	1473	1713	1761	1617
370	1073	1304	1369	1092	453	798	1283	973	565	1315	1204	1796	846	1447	1143
959	1275	1213	2085	742	1309	1479	1760	703	1727	872	1479	686	1698	1057	387
1252	904	668	443	1600	930	1052	776	1049	402	361	1119	578	406	618	581
1095	670	641	1152	1060	567	433	374	579	235	325	1802	331	217	665	862
182	312	864	732	783	1456	608	2066	491	400	1466	744	2013	2082	1865	875
552	400	182	820	721	1735	851	740	551	1551	1769	1159	613	2072	1300	1605
807	1017	1251	818	1259	2596	826	1137	1255	1123	943	1359	188	1282	271	2300
483	2540	609	1038	2099	1766	2699	2493	2266	264	1398	304	699	538	1335	454
393	173	1198	1370	760	216	1692	919	1286	435	879	861	548	913	2198	483
803	1181	731	627	1086	292	883	279	1906	178	2156	490	662	1699	1430	2300
2117	1888	138	1023	884	755	1612	749	690	476	1501	1654	1049	516	1995	1149
1580	739	1079	1161	815	1214	2485	780	1100	1347	985	916	1361	260	1171	328
2202	445	2385	665	966	1969	1729	2568	2333	2108	177	1327	177	1486	757	506
609	981	1474	967	681	1552	1317	936	594	197	928	316	723	2203	500	604
482	1104	455	630	641	1058	562	1857	528	2425	220	704	1845	1122	2405	2428

(Continued)

Table 9.4 (Continued)

2204	738	945	1362	587	335	435	930	1358	819	504	1496	1153	927	428	341
803	180	649	2119	343	521	652	939	340	649	533	918	451	1783	362	2290
130	568	1727	1105	2301	2285	2059	595	853	891	1082	1199	726	96	583	1125
653	563	947	986	1493	560	1183	813	882	1033	902	1763	642	1032	1131	1604
463	1556	663	1298	947	1461	795	371	882	967	973	768	1472	588	252	308
803	920	309	238	1252	569	940	165	863	414	454	552	1745	269	482	1188
355	397	833	713	432	666	1453	410	1758	642	262	1260	1051	1858	1737	1508
592	598	222	814	1094	510	235	1335	820	892	100	626	541	219	524	1897
90	410	952	605	238	706	570	622	503	1581	257	1985	396	309	1453	1039
2043	1972	1744	514	661	1025	1227	617	90	1525	835	1114	263	770	700	400
740	2049	311	630	1087	630	459	924	405	739	360	1749	115	2055	428	492
1568	1256	2166	2026	1796	303	853	663	632	999	572	972	225	763	908	451
767	293	1240	726	420	1111	862	617	443	1374	586	1299	887	1070	1633	1057
547	999	252	1483	1681	1489	1326	236	610	1156	557	642	879	1000	1467	558
1178	780	831	1038	879	1726	700	1023	1082	1631	488	1579	586	1320	982	1463
796	371	802	949	1021	826	1508	550	546	983	397	821	411	1023	180	651
478	1438	476	485	1333	235	525	827	1022	123	973	1155	715	1475	902	273
953	882	1550	1467	1240	898	396	1479	745	1105	240	831	645	442	723	1983
316	623	1152	543	470	939	482	669	443	1690	205	1969	510	455	1492	1238
2091	1938	1709	354	813	1163	676	1264	1473	839	1326	847	801	1254	976	1643
1157	1169	983	1905	878	1836	346	1590	1286	1621	1034	689	503	995	1376	1239
1828	674	1183	725	1399	549	1004	869	1427	818	882	1716	214	902	1222	1210
390	1184	1225	949	1239	1210	660	863	1207	1446	1197	969	1042	741	865	821
644	790	388	1374	803	484	968	1056	665	318	1420	794	1341	1017	1137	1836

(Continued)

Table 9.4 (Continued)

1056	679	1200	189	1645	1891	1704	1403	442	699	453	290	483	1809	107	384
1024	511	251	712	646	525	585	1499	330	1885	495	231	1356	999	1949	1872
1644	567	591	950	410	690	2147	594	590	326	1191	499	504	838	1098	758
1794	703	2439	414	751	1837	1011	2374	2455	2237	928	921	624	325	1356	480
369	1241	413	473	680	1097	166	1038	1049	781	1497	905	238	925	702	1506
1506	1287	998	216	479	1941	188	350	736	792	161	547	632	745	552	1607
375	2115	296	392	1547	959	2121	2114	1890	641	676	1480	435	129	949	708
325	355	1081	492	1007	1137	779	1759	774	291	1148	516	1688	1785	1573	1038
231	1829	1603	2339	1524	1780	1673	2421	1315	2394	357	2136	825	2237	1589	579
1204	347	959	940	2336	1266	320	919	605	154	623	652	580	582	1508	344
1950	429	242	1402	949	1986	1943	1717	603	582	872	699	197	358	957	529
881	1263	660	1849	645	240	1250	631	1802	1867	1650	923	341	1511	815	669
1092	1397	1019	1982	1010	2708	695	1061	2089	1148	2594	2734	2520	1212	1176	697
1051	1018	290	985	1280	743	1427	996	466	987	1110	1584	1395	1166	861	626
469	761	607	685	1446	472	1969	457	254	1393	823	1963	1975	1752	739	515
1171	847	1089	1316	919	2063	776	598	1434	507	1926	2101	1898	1187	548	1144
83	2145	317	2445	426	875	1972	1584	2571	2408	2179	194	1231	1094	1036	836
1371	1008	354	833	828	1429	1369	1146	1021	352	2083	259	2412	345	811	1925
1507	2523	2380	2151	220	1163	1628	1005	1903	1272	504	849	653	1114	1019	2044
932	2165	330	559	1668	1291	2264	2138	1908	268	917	2377	1723	636	1720	534
145	290	2281	1531	667	1829	1235	2410	2367	2139	519	972	1162	792	1744	1724
1500	796	361	1087	600	701	550	1835	917	1490	1787	1614	1553	486	678	727
2435	1461	229	2238	1560	2010	1353	1157								

Matrix A is symmetric, that is $a(i, j) = a(j, i)$. Data is given for lower triangular

3 DEM [168]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{5x_1 + x_2,\ -5x_1 + x_2,\ x_1^2 + x_2^2 + 4x_2\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, -3)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -3$.

4 QL [168]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 3} f_i(\boldsymbol{x})$,
where $f_1(\boldsymbol{x}) = x_1^2 + x_2^2$,
$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(-4x_1 - x_2 + 4)$,
$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(-x_1 - 2x_2 + 6)$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, 5)^T$,
Optimum point: $\boldsymbol{x}^* = (1.2, 2.4)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 7.2$.

5 LQ [168]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{-x_1 - x_2,\ -x_1 - x_2 + x_1^2 + x_2^2 - 1\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (-0.5, -0.5)^T$,
Optimum point: $\boldsymbol{x}^* = (1/\sqrt{2}, 1/\sqrt{2})^T$,
Optimum value: $f(\boldsymbol{x}^*) = -\sqrt{2}$.

6 Mifflin 1 [168]

Classification: G-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = -x_1 + 20 \max\left\{x_1^2 + x_2^2 - 1,\ 0\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (0.8, 0.6)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -1$.

7 Wolfe [159]

Classification: G-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = 5\sqrt{9x_1^2 + 16x_2^2}$, when $x_1 \geq |x_2|$,
$f(\boldsymbol{x}) = 9x_1 + 16|x_2|$, when $0 < x_1 \leq |x_2|$,
$f(\boldsymbol{x}) = 9x_1 + 16|x_2| - x_1^9$, when $x_1 \leq 0$,
Starting point: $\boldsymbol{x}^{(1)} = (3, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (-1, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -8$.

8 Rosen-Suzuki [205]

Classification: QM-U-X-E,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 4} f_i(\boldsymbol{x})$,
where
$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$,
$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3,$
$-x_4 - 8)$,
$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10)$,
$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5)$,
Starting point: $\boldsymbol{x}^{(1)} = (0, 0, 0, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 1, 2, -1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -44$.

9 Davidon 2 [159]

Classification: QM-U-X-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 20} |f_i(\boldsymbol{x})|$,
where
$f_i(\boldsymbol{x}) = \left(x_1 + x_2 t_i - e^{t_i}\right)^2 + (x_3 + x_4 \sin(t_i) - \cos(t_i))^2$ and
$t_i = 0.2i$, for $i = 1, \ldots, 20$,
Starting point: $\boldsymbol{x}^{(1)} = (25, 5, -5, -1)^T$,
Optimum point: $\boldsymbol{x}^* = (-12.2437, 14.0218, -0.4515, -0.0105)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 115.70644$.

10 Shor [168]

Classification: QM-U-X-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 10} \left\{ b_i \sum_{j=1}^{5} (x_j - a_{ij})^2 \right\}$,

where

$$A = \begin{pmatrix} 0&0&0&0&0 \\ 2&1&1&1&3 \\ 1&2&1&1&2 \\ 1&4&1&2&2 \\ 3&2&1&0&1 \\ 0&2&1&0&1 \\ 1&1&1&1&1 \\ 1&0&1&2&1 \\ 0&0&2&1&0 \\ 1&1&2&0&0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 5 \\ 10 \\ 2 \\ 4 \\ 3 \\ 1.7 \\ 2.5 \\ 6 \\ 3.5 \end{pmatrix},$$

Starting point: $\boldsymbol{x}^{(1)} = (0, 0, 0, 0, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (1.1244, 0.9795, 1.4777, 0.9202, 1.1243)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 22.600162$.

11 Maxquad [146,168]

Classification: QM-U-X-N,
Dimension: 10,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 5} \boldsymbol{x}^T A_i \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{b}_i$,
where $A_{ijk} = A_{ikj} = e^{j/k} \cos(jk) \sin(i)$, $j < k$,
$A_{ijj} = \frac{j}{10} |\sin(i)| + \sum_{k \neq j} |A_{ijk}|$, and
$b_{ij} = e^{i/j} \sin(ij)$,
Starting point: $\boldsymbol{x}^{(1)} = (0, 0, \dots, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (-0.1263, -0.0344, -0.0069, 0.0264, 0.0673, -0.2784, 0.0742, 0.1385, 0.0840, 0.0386)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -0.8414083$.

12 Polak 2 [198]

Classification: GM-U-X-E,
Dimension: 10,
Objective function: $f(\boldsymbol{x}) = \max \{ g(\boldsymbol{x} + 2\boldsymbol{e}_2),\ g(\boldsymbol{x} - 2\boldsymbol{e}_2) \}$,
where $g(\boldsymbol{x}) = e^{10^{-8} x_1^2 + x_2^2 + x_3^2 + 4x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2}$,
$\boldsymbol{e}_2$ = second column of the identity matrix,
Starting point: $x_1^{(1)} = 100.0$ and
$x_i^{(1)} = 0.1$ for $i = 2, \dots, 10$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \dots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 54.598150 = e^4$.

13 Polak 3 [198]

Classification: GM-U-X-N,
Dimension: 11,

Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 10} f_i(\boldsymbol{x})$,

where $f_i(\boldsymbol{x}) = \sum_{j=0}^{10} \frac{1}{i+j} e^{\left(x_{j+1} - \sin(i-1+2j)\right)^2}$,

Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,

Optimum point: $\boldsymbol{x}^* = (0.0124, 0.2904, -0.3347, -0.1265, 0.2331, -0.2766, -0.1666, 0.2291, -0.1858, -0.1704, 0.2402)^T$,

Optimum value: $f(\boldsymbol{x}^*) = 3.7034924$.

Note: A different optimal value $f(\boldsymbol{x}^*) = 261.08258$ has been given in [159]. This is due to erroneous code used in calculations.

14 Wong 1 [7,159]

Classification: PM-U-X-N,

Dimension: 7,

Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 5} f_i(\boldsymbol{x})$,

where

$$f_1(\boldsymbol{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7,$$

$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127),$$

$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282),$$

$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196),$$

$$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7),$$

Starting point: $\boldsymbol{x}^{(1)} = (1, 2, 0, 4, 0, 1, 1)^T$,

Optimum point: $\boldsymbol{x}^* = (2.3305, 1.9514, -0.4775, 4.3657, -0.6245, 1.0381, 1.5942)^T$,

Optimum value: $f(\boldsymbol{x}^*) = 680.63006$.

15 Wong 2 [7,159]

Classification: QM-U-X-N,

Dimension: 10,

Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 9} f_i(\boldsymbol{x})$,

where

$$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45,$$

$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120\right),$$

$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40\right),$$

$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6, -30\right),$$

$$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6\right),$$

$$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\,(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105),$$

$$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\,(10x_1 - 8x_2 - 17x_7 + 2x_8),$$

$$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}\right),$$

$$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\,(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12),$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1720, 2.3637, 8.7739, 5.0960, 0.9907, 1.4306, 1.3217, 9.8287, 8.2801, 8.3759)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 24.306209$.

16 Wong 3 [7,159]

Classification: PM-U-X-N,
Dimension: 20,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 18} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) = {} & x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 \\
& + 4(x_4-5)^2 + (x_5-3)^2 + 2(x_6-1)^2 \\
& + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 \\
& + (x_{11}-9)^2 + 10(x_{12}-1)^2 + 5(x_{13}-7)^2 \\
& + 4(x_{14}-14)^2 + 27(x_{15}-1)^2 + x_{16}^4 + (x_{17}-2)^2 \\
& + 13(x_{18}-2)^2 + (x_{19}-3)^2 + x_{20}^2 + 95,
\end{aligned}$$

$$\begin{aligned}
f_2(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120\right),\\
f_3(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40\right),\\
f_4(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 - 30\right),\\
f_5(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6\right),\\
f_6(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105),\\
f_7(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(10x_1 - 8x_2 - 17x_7 + 2x_8),\\
f_8(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(-3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10}\right),\\
f_9(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12),\\
f_{10}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(x_1 + x_2 + 4x_{11} - 21x_{12}),\\
f_{11}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 + 15x_{11} - 8x_{12} - 28\right),\\
f_{12}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(4x_1 + 9x_2 + 5x_{13}^2 - 9x_{14} - 87\right),\\
f_{13}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(3x_1 + 4x_2 + 3(x_{13}-6)^2 - 14x_{14} - 10\right),\\
f_{14}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(14x_1^2 + 35x_{15} - 79x_{16} - 92\right),\\
f_{15}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(15x_2^2 + 11x_{15} - 61x_{16} - 54\right),\\
f_{16}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(5x_1^2 + 2x_2 + 9x_{17}^4 - x_{18} - 68\right),\\
f_{17}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 - x_2 + 19x_{19} - 20x_{20} + 19\right),\\
f_{18}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 + 5x_2^2 + x_{19}^2 - 30x_{20}\right),
\end{aligned}$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10, 2, 2, 6, 15, 1, 2, 1, 2, 1, 3)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1752, 2.3529, 8.7665, 5.0669, 0.9887, 1.4310, 1.3295, 9.8359, 8.2873, 8.3702, 2.2758, 1.3586, 6.0772, 14.1708, 0.9962, 0.6557, 1.4666, 2.0004, 1.0466, 2.0632)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 133.72828$.

17 MAXQ [208]

Classification: QM-U-X-E,

Dimension:	20,	
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 20}\ x_i^2$,	
Starting point:	$x_i^{(1)} = i$	for $i = 1, \dots, 10$ and
	$x_i^{(1)} = -i$	for $i = 11, \dots, 20$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \dots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

18 MAXL [168]

Classification:	LM-U-X-E,	
Dimension:	20,	
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 20}\ \lvert x_i \rvert$,	
Starting point:	$x_i^{(1)} = i$	for $i = 1, \dots, 10$ and
	$x_i^{(1)} = -i$	for $i = 11, \dots, 20$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \dots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

19 TR48 [146]

Classification:	LD-U-X-N,
Dimension:	48,
Objective function:	$f(\boldsymbol{x}) = \sum_{j=1}^{48} d_j \max_{1 \le i \le 48} (x_i - a_{ij}) - \sum_{i=1}^{48} s_i x_i$,
Starting point:	See Table 9.5,
Optimum point:	See Table 9.5,
Optimum value:	$f(\boldsymbol{x}^*) = -638565.0$.

20 Goffin [168]

Classification:	LD-U-X-E,	
Dimension:	50,	
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 50}\ x_i + \sum_{i=1}^{50} x_i$,	
Starting point:	$x_i^{(1)} = i - 25.5$	for $i = 1, \dots, 50$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \dots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

21 Crescent [131]

Classification:	QM-U-Z-E,
Dimension:	2,
Objective function:	$f(\boldsymbol{x}) = \max\left\{x_1^2 + (x_2 - 1)^2 + x_2 - 1, -x_1^2 - (x_2 - 1)^2 + x_2 + 1\right\}$,
Starting point:	$\boldsymbol{x}^{(1)} = (-1.5, 2)$,

Table 9.5 Initialization and optimal point for 19

i	Starting point 1	Starting point 2	Optimum point x^*
1	0.0	11.19	144.0
2	0.0	127.20	257.0
3	0.0	−129.70	0.0
4	0.0	344.50	483.0
5	0.0	−40.72	89.0
6	0.0	−295.30	−165.0
7	0.0	−202.30	−72.0
8	0.0	−382.30	−252.0
9	0.0	−217.70	−88.0
10	0.0	−307.70	−178.0
11	0.0	178.10	−311.0
12	0.0	−4.36	126.0
13	0.0	−123.30	7.0
14	0.0	−265.30	−135.0
15	0.0	28.28	158.0
16	0.0	70.57	209.0
17	0.0	−31.81	101.0
18	0.0	−222.30	−92.0
19	0.0	96.19	229.0
20	0.0	−52.79	80.0
21	0.0	−34.71	95.0
22	0.0	−59.16	71.0
23	0.0	−373.70	−244.0
24	0.0	−28.35	102.0
25	0.0	−141.70	−12.0
26	0.0	2.28	132.0
27	0.0	198.50	337.0
28	0.0	−69.16	61.0
29	0.0	−26.35	104.0
30	0.0	−88.72	41.0
31	0.0	130.80	261.0
32	0.0	−12.35	118.0
33	0.0	−30.70	99.0
34	0.0	−376.30	−246.0
35	0.0	23.18	156.0
36	0.0	−400.30	−270.0
37	0.0	197.10	330.0
38	0.0	−260.30	−130.0
39	0.0	813.50	952.0

(Continued)

Table 9.5 (Continued)

i	Starting point 1	Starting point 2	Optimum point $\boldsymbol{x}^*$
40	0.0	−191.70	−62.0
41	0.0	31.29	161.0
42	0.0	345.50	484.0
43	0.0	−7.72	122.0
44	0.0	335.50	474.0
45	0.0	947.50	1086.0
46	0.0	722.50	861.0
47	0.0	−300.30	−170.0
48	0.0	73.20	206.0

Optimum point: $\boldsymbol{x}^* = (0, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

22 Mifflin 2 [168]

Classification: G-U-Z-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = -x_1 + 2(x_1^2 + x_2^2 - 1) + 1.75|x_1^2 + x_2^2 - 1|$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -1$.

23 WF [159]

Classification: GM-U-Z-E,
Dimension: 2,
Objective function:
$$f(\boldsymbol{x}) = \max \left\{ \tfrac{1}{2}\left(x_1 + \tfrac{10x_1}{x_1+0.1} + 2x_2^2\right), \tfrac{1}{2}\left(-x_1 + \tfrac{10x_1}{x_1+0.1} + 2x_2^2\right), \tfrac{1}{2}\left(x_1 - \tfrac{10x_1}{x_1+0.1} + 2x_2^2\right)\right\},$$
Starting point: $\boldsymbol{x}^{(1)} = (3, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

24 SPIRAL [159]

Classification: GM-U-Z-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max \{ f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \}$,
where
$$f_1(\boldsymbol{x}) = \left(x_1 - \sqrt{x_1^2 + x_2^2}\cos\sqrt{x_1^2 + x_2^2}\right)^2 + 0.005(x_1^2 + x_2^2),$$
$$f_2(\boldsymbol{x}) = \left(x_2 - \sqrt{x_1^2 + x_2^2}\sin\sqrt{x_1^2 + x_2^2}\right)^2 + 0.005(x_1^2 + x_2^2),$$

Starting point: $\boldsymbol{x}^{(1)} = (1.411831, -4.79462)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

25 EVD52 [159]

Classification: PM-U-Z-N,
Dimension: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 6} f_i(\boldsymbol{x})$,
where
$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 - 1$,
$f_2(\boldsymbol{x}) = x_1^2 + x_2^2 + (x_3 - 2)^2$,
$f_3(\boldsymbol{x}) = x_1 + x_2 + x_3 - 1$,
$f_4(\boldsymbol{x}) = x_1 + x_2 - x_3 - 1$,
$f_5(\boldsymbol{x}) = 2x_1^3 + 6x_2^2 + 2(5x_3 - x_1 + 1)^2$,
$f_6(\boldsymbol{x}) = x_1^2 - 9x_3$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.3283, 0.0000, 0.1313)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 3.5997193$.

26 PBC3 [159]

Classification: GM-U-Z-N,
Dimension: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} f_i(\boldsymbol{x})$,
where
$f_i(\boldsymbol{x}) = \frac{x_3}{x_2} e^{-t_i x_1} \sin(t_i x_2) - y_i$,
$y_i = \frac{3}{20} e^{-t_i} + \frac{1}{52} e^{-5t_i} - \frac{1}{65} e^{-2t_i} (3\sin(2t_i) + 11\cos(2t_i))$,
and $t_i = 10(i-1)/20$, for $i = 1, \ldots, 21$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.9516, 0.8761, 0.1623)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.42021427 \cdot 10^{-2}$.

27 Bard [159]

Classification: PM-U-Z-N,
Dimension: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 15} |f_i(\boldsymbol{x})|$,
where
$f_i(\boldsymbol{x}) = x_1 + \frac{i}{(16-i)x_2 + u_i x_3}$, for $i = 1, \ldots, 15$,
$\boldsymbol{u} = (1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1)^T$, and
$\boldsymbol{y} = (0.14, 0.18, 0.22, 0.25, 0.29, 0.32, 0.35, 0.39, 0.37, 0.58, 0.73, 0.96, 1.34, 2.10, 4.39)^T$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.0535, 1.5106, 1.9894)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.50816327 \cdot 10^{-1}$.

28 Kowalik-Osborne [159]

Classification: GM-U-Z-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 11} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = \frac{x_1(u_i^2 + x_2 u_i)}{u_i^2 + x_3 u_i + x_4} - y_i$,
$\boldsymbol{u} = (4.0000, 2.0000, 1.0000, 0.5000, 0.2500, 0.1670, 0.1250, 0.1000, 0.0833, 0.0714, 0.0625)^T$,
$\boldsymbol{y} = (0.1957, 0.1947, 0.1735, 0.1600, 0.0844, 0.0627, 0.0456, 0.0342, 0.0323, 0.0235, 0.0246)^T$,
Starting point: $\boldsymbol{x}^{(1)} = (0.250, 0.390, 0.415, 0.390)^T$,
Optimum point: $\boldsymbol{x}^* = (0.1846, 0.1052, 0.0196, 0.1118)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.80843684 \cdot 10^{-2}$.

29 Polak 6 [198]

Classification: PM-U-Z-E,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 4} f_i(\boldsymbol{x})$,
where
$$\begin{aligned} f_1(\boldsymbol{x}) = {} & (x_1 - (x_4+1)^4)^2 + \left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 + 2x_3^2 \\ & + x_4^2 - 5(x_1 - (x_4+1)^4) \\ & - 5\left(x_2 - (x_1 - (x_4+1)^4)^4\right) - 21x_3 + 7x_4, \\ f_2(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10\left((x_1 - (x_4+1)^4)^2\right. \\ & + \left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 \\ & + x_3^2 + x_4^2 + (x_1 - (x_4+1)^4) \\ & - \left(x_2 - (x_1 - (x_4+1)^4)^4\right) \\ & \left. + x_3 - x_4 - 8\right), \\ f_3(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10\left((x_1 - (x_4+1)^4)^2\right. \\ & + 2\left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 \\ & \left. + x_3^2 + 2x_4^2 - (x_1 - (x_4+1)^4) - x_4 - 10\right), \\ f_4(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10\left((x_1 - (x_4+1)^4)^2\right. \\ & + \left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 \\ & + x_3^2 + 2(x_1 - (x_4+1)^4) - \left(x_2 - (x_1 - (x_4+1)^4)^4\right) \\ & \left. - x_4 - 5\right), \end{aligned}$$
Starting point: $\boldsymbol{x}^{(1)} = (0, 0, 0, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 1, 2, -1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -44$.

30 OET5 [159]

Classification: QM-U-Z-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = x_4 - (x_1 t_i^2 + x_2 t_i + x_3)^2 - \sqrt{t_i}$ and
$t_i = 0.25 + 0.75(i-1)/20 \quad \text{for } i = 1, \ldots, 21$,

Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.0876, -0.497, 1.1155, 1.4963,)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.26359735 \cdot 10^{-2}$.

31 OET6 [159]

Classification: GM-U-Z-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = x_1 e^{x_3 t_i} + x_2 e^{x_4 t_i} - \frac{1}{1+t_i}$ and
$t_i = -0.5 + (i-1)/20, \quad \text{for } i = 1, \dots, 21$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, -3, -1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.0987, 0.9009, -4.0619, -0.6477)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.20160753 \cdot 10^{-2}$.

32 EXP [159]

Classification: GM-U-Z-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} f_i(\boldsymbol{x})$,
where $f_i(\boldsymbol{x}) = \frac{x_1 + x_2 t_i}{1 + x_3 t_i + x_4 t_i^2 + x_5 t_i^3} - e^{t_i}$ and
$t_i = -1 + (i-1)/10, \quad i = 1, \dots, 21$,
Starting point: $\boldsymbol{x}^{(1)} = (0.5, 0, 0, 0, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (0.9999, 0.2536, -0.7466, 0.2452, -0.0375)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.12237125 \cdot 10^{-3}$.

33 PBC1 [159]

Classification: GM-U-Z-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 30} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = \frac{x_1 + x_2 t_i + x_3 t_i^2}{1 + x_4 t_i + x_5 t_i^2} - \frac{\sqrt{(8t_i - 1)^2 + 1} \arctan(8t_i)}{8t_i}$ and
$t_i = -1 + 2(i-1)/29, \quad \text{for } i = 1, \dots, 30$,
Starting point: $\boldsymbol{x}^{(1)} = (0, -1, 10, 1, 10)^T$,
Optimum point: $\boldsymbol{x}^* = (1.4136, -10.5797, 40.7117, -4.0213, 27.6150)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.22340496 \cdot 10^{-1}$.

34 HS78 [111,159]

Classification: G-U-Z-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = x_1 x_2 x_3 x_4 x_5 + 10 \sum_{i=1}^{3} |f_i(\boldsymbol{x})|$,
where $f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10$,

$f_2(\boldsymbol{x}) = x_2x_3 - 5x_4x_5,$

$f_3(\boldsymbol{x}) = x_1^3 + x_2^3 + 1,$

Starting point: $\boldsymbol{x}^{(1)} = (-2.0, 1.5, 2.0, -1.0, -1.0)^T,$

Optimum point: $\boldsymbol{x}^* = (-1.7171, 1.5957, 1.8272, -0.7636, -0.7636)^T,$

Optimum value: $f(\boldsymbol{x}^*) = -2.9197004.$

Note: Function HS78 is unbounded from below. The reported minimum is a local one.

35 El-Attar [159]

Classification: G-U-Z-N,

Dimension: 6,

Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{50} \left| x_1 e^{-x_2 t_i} \cos(x_3 3 t_i + x_4) + x_5 e^{-x_6 t_i} - y_i \right|,$

where $y_i = 0.5e^{-t_i} - e^{-2t_i} + 0.5e^{-3t_i} + 1.5e^{-1.5t_i} \sin(7t_i)$
$+ e^{-2.5t_i} \sin(5t_i),$

$t_i = (i-1)/10, \quad \text{for } i = 1, \ldots, 51,$

Starting point: $\boldsymbol{x}^{(1)} = (2, 2, 7, 0, -2, 1)^T,$

Optimum point: $\boldsymbol{x}^* = (2.2407, 1.8577, 6.7701, -1.6449, 0.1659, 0.7423)^T,$

Optimum value: $f(\boldsymbol{x}^*) = 0.5598131.$

36 EVD61 [159]

Classification: GM-U-Z-N,

Dimension: 6,

Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 51} |f_i(\boldsymbol{x})|,$

where $f_i(\boldsymbol{x}) = x_1 e^{-x_2 t_i} \cos(x_3 t_i + x_4) + x_5 e^{-x_6 t_i} - y_i,$

$y_i = 0.5e^{-t_i} - e^{-2t_i} + 0.5e^{-3t_i} + 1.5e^{-1.5t_i} \sin(7t_i)$
$+ e^{-2.5t_i} \sin(5t_i),$

$t_i = (i-1)/10, \quad \text{for } i = 1, \ldots, 51,$

Starting point: $\boldsymbol{x}^{(1)} = (2, 2, 7, 0, -2, 1)^T,$

Optimum point: $\boldsymbol{x}^* = (2.2759, 1.8993, 6.8482, -1.6503, 0.1457, 0.5170)^T,$

Optimum value: $f(\boldsymbol{x}^*) = 0.34904926 \cdot 10^{-1}.$

37 Gill [159]

Classification: PM-U-Z-N,

Dimension: 10,

Objective function: $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x})\},$

where $f_1(\boldsymbol{x}) = \sum_{i=1}^{10} (x_i - 1)^2 + 10^{-3} \sum_{i=1}^{10} \left(x_i^2 - \frac{1}{4}\right)^2,$

$$f_2(\boldsymbol{x}) = \sum_{i=2}^{30} \left[\sum_{j=2}^{10} x_j (j-1) \left(\frac{i-1}{29}\right)^{j-2} - \left(\sum_{j=1}^{10} x_j \left(\frac{i-1}{29}\right)^{j-1} \right)^2 - 1 \right]^2 + x_1^2 + (x_2 - x_1^2 - 1)^2,$$

	$f_3(\boldsymbol{x}) = \sum_{i=2}^{10} \left[100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right]$,
Starting point:	$\boldsymbol{x}^{(1)} = (-0.1, -0.1, \ldots, -0.1)^T$,
Optimum point:	$\boldsymbol{x}^* = (-0.6022, 0.4907, 0.3096, 0.1416, 0.0542,$ $0.0287, 0.0197, 0.0137, 0.0087, 0.0045)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 9.7857721$.

38 Problem 1 in [21]

Classification:	GD-U-Z-E,
Dimension:	2,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 3} f_i(\boldsymbol{x}) + \min_{4 \le i \le 6} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = x_1^4 + x_2^2$, and
	$f_2(\boldsymbol{x}) = (2 - x_1)^2 + (2 - x_2)^2$,
	$f_3(\boldsymbol{x}) = 2e^{-x_1 + x_2}$,
	$f_4(\boldsymbol{x}) = x_1^2 - 2x_1 + x_2^2 - 4x_2 + 4$,
	$f_5(\boldsymbol{x}) = 2x_1^2 - 5x_1 + x_2^2 - 2x_2 + 4$,
	$f_6(\boldsymbol{x}) = x_1^2 + 2x_2^2 - 4x_2 + 1$,
Starting point:	$\boldsymbol{x}^{(1)} = (2, 2)^T$,
Optimum point:	$\boldsymbol{x}^* = (1, 1)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 2$.

39 L1 version of Rosenbrock function [21]

Classification:	LD-U-Z-E,
Dimension:	2,
Objective function:	$f(\boldsymbol{x}) = \lvert x_1 - 1\rvert + 100\lvert x_2 - \lvert x_1\rvert\rvert$,
Starting point:	$\boldsymbol{x}^{(1)} = (-1.2, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (1, 1)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 0$.
Note:	DC representation can be found in [21].

40 L1 version of Wood function [21]

Classification:	LD-U-Z-E,
Dimension:	4,
Objective function:	$f(\boldsymbol{x}) = \lvert x_1 - 1\rvert + 100\lvert x_2 - \lvert x_1\rvert\rvert + 90\lvert x_4 - \lvert x - 3\rvert\rvert$ $+ \lvert x_3 - 1\rvert + 10.1(\lvert x_2 - 1\rvert + \lvert x_4 - 1\rvert)$ $+ 4.95(\lvert x_2 + x_4 - 2\rvert - \lvert x_2 - x_4\rvert)$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 3, 3, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (1, 1, 1, 1)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 0$.
Note:	DC representation can be found in [21].

9.2 Bound Constrained Problems

Bound constrained problems ($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^l \le x_i \le x_i^u$ for all $i = 1, \dots, n\}$ in (9.1)) are easily constructed from the problems given above, for instance, by inclosing the bounds

$$x_i^* + 0.1 \le x_i \le x_i^* + 1.1 \qquad \text{for all odd } i.$$

If the starting point $\boldsymbol{x}^{(1)}$ is not feasible, it can simply be projected to the feasible region (if a strictly feasible starting point is needed an additional safeguard of 0.0001 may be added). The classification of the bound constrained problems is the same as that of unconstrained problems (see Sect. 9.1 and also Sect. 9.4) but, naturally, the information about constraint functions should be replaced with B (see Table 9.1).

9.3 Linearly Constrained Problems

In this section we present small-scale nonsmooth linearly constrained test problems ($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \le \boldsymbol{b}\}$ with the inequality taken component-wise in (9.1)). The number of variables varies from 2 to 20 and there are up to 15 constraint functions.

41 Wong 2C [159]

Classification: QM-L-X-N,
Dimension: 10,
No. of constraints: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 6} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 \\
&\quad + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 \\
&\quad + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45, \\
f_2(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10(3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120), \\
f_3(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10(5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40), \\
f_4(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10(0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30), \\
f_5(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10(x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6), \\
f_6(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}),
\end{aligned}$$

Constraint function:

$$\begin{aligned}
g_1(\boldsymbol{x}) &= 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 105, \\
g_2(\boldsymbol{x}) &= 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0, \\
g_3(\boldsymbol{x}) &= -8x_1 + 2x_2 + 5x_9 - 2x_{10} \le 12,
\end{aligned}$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1722, 2.3634, 8.7737, 5.0959, 0.9906, 1.4307, 1.3219, 9.8289, 8.2803, 8.3756)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 24.306209$.

42 Wong 3C [159]

Classification: PM-L-X-N,
Dimension: 20,
No. of constraints: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 14} f_i(\boldsymbol{x})$,
where

$$\begin{aligned} f_1(\boldsymbol{x}) = {} & x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 \\ & + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 \\ & + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 \\ & + (x_{11} - 9)^2 + 10(x_{12} - 1)^2 + 5(x_{13} - 7)^2 \\ & + 4(x_{14} - 14)^2 + 27(x_{15} - 1)^2 + x_{16}^4 + (x_{17} - 2)^2 \\ & + 13(x_{18} - 2)^2 + (x_{19} - 3)^2 - x_{20}^2 + 95, \end{aligned}$$

$$\begin{aligned} f_2(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 \\ & - 120), \end{aligned}$$

$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40),$$

$$\begin{aligned} f_4(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 \\ & - 30), \end{aligned}$$

$$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6),$$

$$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}),$$

$$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + 5x_{11} - 8x_{12} - 28),$$

$$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(4x_1 + 9x_2 + 5x_{13}^2 - 9x_{14} - 87),$$

$$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(3x_1 + 4x_2 + 3(x_{13} - 6)^2 - 14x_{14} - 10),$$

$$f_{10}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(14x_1^2 + 35x_{15} - 79x_{16} - 92),$$

$$f_{11}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(15x_2^2 + 11x_{15} - 61x_{16} - 54),$$

$$f_{12}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(5x_1^2 + 2x_2 + 9x_{17}^4 - x_{18} - 68),$$

$$f_{13}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 - x_9 + 19x_{19} - 20x_{20} + 19),$$

$$f_{14}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(7x_1^2 + 5x_2^2 + x_{19}^2 - 30x_{20}),$$

Constraint function:

$$g_1(\boldsymbol{x}) = 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 105,$$

$$g_2(\boldsymbol{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0,$$

$$g_3(\boldsymbol{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} \le 12,$$

$$g_4(\boldsymbol{x}) = x_1 + x_2 + 4x_{11} - 21x_{21} \le 0,$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10, 2, 2, 6, 15, 1, 2, 1, 2, 1, 3)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1749, 2.3537, 8.7666, 5.0669, 0.9888, 1.4309, 1.3288, 9.8354, 8.2867, 8.3709, 2.2759, 1.3586, 6.0771, 14.1708, 0.9962, 0.6566, 1.4666, 2.0004, 1.0471, 2.0636)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 24.306209$.

43 MAD1 [159]

Classification: GM-L-Z-N,
Dimension: 2,
No. of constraints: 1,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 3} f_i(\boldsymbol{x})$,
where

$$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 1,$$

$$f_2(\boldsymbol{x}) = \sin x_1,$$

	$f_3(\boldsymbol{x}) = -\cos x_2$,
Constraint function:	$g_1(\boldsymbol{x}) = x_1 + x_2 - 0.5 \geq 0$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 2)^T$,
Optimum point:	$\boldsymbol{x}^* = (-0.4003, 0.9003)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -0.38965952$.

44 **MAD2** [159]

Classification:	GM-L-Z-N,
Dimension:	2,
No. of constraints:	1,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1 x_2 - 1$,
	$f_2(\boldsymbol{x}) = \sin x_1$,
	$f_3(\boldsymbol{x}) = -\cos x_2$,
Constraint function:	$g_1(\boldsymbol{x}) = -3x_1 - x_2 + 2.5 \geq 0$,
Starting point:	$\boldsymbol{x}^{(1)} = (-2, -1)^T$,
Optimum point:	$\boldsymbol{x}^* = (-0.8929, 0.1786)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -0.33035714$.

45 **MAD4** [159]

Classification:	GM-L-Z-N,
Dimension:	2,
No. of constraints:	1,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = -\exp(x_1 - x_2)$,
	$f_2(\boldsymbol{x}) = \sinh(x_1 - 1) - 1$,
	$f_3(\boldsymbol{x}) = -\log(x_2) - 1$,
Constraint function:	$g_1(\boldsymbol{x}) = 0.05x_1 - x_2 + 0.5 \geq 0$,
Starting point:	$\boldsymbol{x}^{(1)} = (-1, 0.01)^T$,
Optimum point:	$\boldsymbol{x}^* = (1.5264, 0.5763)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -0.44891079$.

46 **MAD5** [159]

Classification:	GM-L-Z-N,
Dimension:	2,
No. of constraints:	1,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = -\exp(x_1 - x_2)$,
	$f_2(\boldsymbol{x}) = \sinh(x_1 - 1) - 1$,
	$f_3(\boldsymbol{x}) = -\log(x_2) - 1$,
Constraint function:	$g_1(\boldsymbol{x}) = -0.9x_1 + x_2 - 1 \geq 0$,

Starting point: $\boldsymbol{x}^{(1)} = (-1, 3)^T$,
Optimum point: $\boldsymbol{x}^* = (1.5436, 2.3892)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -0.42928061$.

47 PENTAGON [159]

Classification: GM-L-Z-N,
Dimension: 6,
No. of constraints: 15,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 3} f_i(\boldsymbol{x})$,
where $f_1(\boldsymbol{x}) = -\sqrt{(x_1 - x_3)^2 + (x_2 - x_4)^2}$,
$f_2(\boldsymbol{x}) = -\sqrt{(x_3 - x_5)^2 + (x_4 - x_6)^2}$,
$f_3(\boldsymbol{x}) = -\sqrt{(x_5 - x_1)^2 + (x_6 - x_2)^2}$,
Constraint function: $g_{ij}(\boldsymbol{x}) = x_i \cos \frac{2\pi j}{5} + x_{i+1} \sin \frac{2\pi j}{5} \le 1$,
where $i = 1, 3, 5$, $j = 0, 1, 2, 3, 4$
Starting point: $\boldsymbol{x}^{(1)} = (-1, 0, 0, -1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (-0.9723, 0.2436, 0.5322, -0.8494, 0.7265, 1.0000)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -1.85961870$.

48 MAD6 [159]

Classification: GM-L-Z-N,
Dimension: 7,
No. of constraints: 9,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 163} f_i(\boldsymbol{x})$,
where $f_i(\boldsymbol{x}) = \frac{1}{15} + \frac{2}{15} \sum_{i=1}^{7} \cos(2\pi x_j \sin \vartheta_i)$,
$\vartheta_i = \frac{\pi}{180}(8.5 + i0.5)$, $1 \le i \le 163$,
Constraint function: $g_1(\boldsymbol{x}) = x_1 \ge 0.4$,
$g_2(\boldsymbol{x}) = -x_1 + x_2 \ge 0.4$,
$g_3(\boldsymbol{x}) = -x_2 + x_3 \ge 0.4$,
$g_4(\boldsymbol{x}) = -x_3 + x_4 \ge 0.4$,
$g_5(\boldsymbol{x}) = -x_4 + x_5 \ge 0.4$,
$g_6(\boldsymbol{x}) = -x_5 + x_6 \ge 0.4$,
$g_7(\boldsymbol{x}) = -x_6 + x_7 \ge 0.4$,
$g_8(\boldsymbol{x}) = -x_4 + x_6 = 1.0$,
$g_9(\boldsymbol{x}) = x_7 = 3.5$,
Starting point: $\boldsymbol{x}^{(1)} = (0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5)^T$,
Optimum point: $\boldsymbol{x}^* = (0.4000, 0.8198, 1.2198, 1.6940, 2.0940, 2.6940, 3.5000)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.10183089$.

49 Dembo 3 [159]

Classification: GM-L-Z-N,
Dimension: 7,
No. of constraints: 2 (+ bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 13} f_i(\boldsymbol{x})$,
where

$$f_1(\boldsymbol{x}) = a_1x_1 + a_2x_1x_6 + a_3x_3 + a_4x_2 + a_5 + a_6x_3x_5,$$
$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_7x_6^2 + a_8x_1^{-1}x_3 + a_9x_6 - 1),$$
$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{10}x_1x_3^{-1} + a_{11}x_1x_3^{-1}x_6 + a_{12}x_1x_3^{-1}x_6^2 - 1),$$
$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{13}x_6^2 + a_{14}x_5 + a_{15}x_4 + a_{16}x_6 - 1),$$
$$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{17}x_5^{-1} + a_{18}x_5^{-1}x_6 + a_{19}x_4x_5^{-1} + a_{20}x_5^{-1}x_6^2 - 1),$$
$$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{21}x_7 + a_{22}x_2x_3^{-1}x_4^{-1} + a_{23}x_2x_3^{-1} - 1),$$
$$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{24}x_7^{-1} + a_{25}x_2x_3^{-1}x_7^{-1} + a_{26}x_2x_3^{-1}x_4^{-1}x_7^{-1} - 1),$$
$$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{27}x_5^{-1} + a_{28}x_5^{-1}x_7 - 1),$$
$$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{33}x_1x_3^{-1} + a_{34}x_3^{-1} - 1),$$
$$f_{10}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{35}x_2x_3^{-1}x_4^{-1} + a_{36}x_2x_3^{-1} - 1),$$
$$f_{11}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{37}x_4 + a_{38}x_3^{-1}x_3x_4 - 1),$$
$$f_{12}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{39}x_1x_6 + a_{40}x_1 + a_{41}x_3 - 1),$$
$$f_{13}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(a_{42}x_1^{-1}x_3 + a_{43}x_1 + a_{44}x_6 - 1),$$

Constraint function:
$$g_1(\boldsymbol{x}) = a_{29}x_5 + a_{30}x_7 \le 1,$$
$$g_2(\boldsymbol{x}) = a_{31}x_3 + a_{32}x_1 \le 1,$$
$1 \le x_1 \le 2000$, $1 \le x_2 \le 120$, $1 \le x_3 \le 5000$,
$85 \le x_4 \le 93$, $90 \le x_5 \le 95$, $3 \le x_6 \le 12$,
$145 \le x_7 \le 162$,

Starting point: $\boldsymbol{x}^{(1)} = (1745, 110, 3048, 89, 92, 8, 145)^T$,
Optimum point: $\boldsymbol{x}^* = (1698.0025, 53.7482, 3031.1493, 90.1212, 95.0000, 10.4870, 153.5354)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 1227.2260$.

50 Dembo 5 [159]

Classification: GM-L-Z-N,
Dimension: 8,
No. of constraints: 3 (+ bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 4} f_i(\boldsymbol{x})$,
where

$$f_1(\boldsymbol{x}) = x_1 + x_2 + x_3,$$
$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(ax_1^{-1}x_4x_6^{-1} + 100x_6^{-1} + bx_1^{-1}x_6^{-1} - 1),$$
$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(x_4x_7^{-1} + 1250(x_5 - x_4)x_2^{-1}x_7^{-1} - 1),$$
$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^5(cx_3^{-1}x_8^{-1} + x_5x_8^{-1} - 2500x_3^{-1}x_5x_8^{-1} - 1),$$
$a = 833.33252$, $b = -83333.333$, and $c = 1250000.0$,

Constraint function:
$$g_1(\boldsymbol{x}) = 0.0025(x_4 + x_5) \le 1,$$

$$
\begin{aligned}
& g_2(\boldsymbol{x}) = 0.0025(x_5 - x_7 - x_4) \le 1, \\
& g_3(\boldsymbol{x}) = 0.01(x_8 - x_5) \le 1, \\
& 100 \le x_1 \le 10000, \quad 1000 \le x_2 \le 10000, \\
& 1000 \le x_3 \le 10000, \quad 10 \le x_4 \le 1000, \\
& 10 \le x_5 \le 1000, \quad 10 \le x_6 \le 1000, \\
& 10 \le x_7 \le 1000, \quad 10 \le x_8 \le 1000,
\end{aligned}
$$

Starting point: $\boldsymbol{x}^{(1)} = (5000, 5000, 5000, 200, 359, 150, 225, 425)^T$,

Optimum point: $\boldsymbol{x}^* = (581.1358, 1358.8591, 5109.2561, 182.1702, 295.6298, 217.8298, 286.5404, 395.6298)^T$,

Optimum value: $f(\boldsymbol{x}^*) = 7049.2480$.

51 EQUIL [146,159]

Classification: GM-L-Z-N,

Dimension: 8,

No. of constraints: 1 (+ bound constraints),

Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 8} f_i(\boldsymbol{x})$,

where $f_i(\boldsymbol{x}) = \sum_{j=1}^{5} \left(\frac{a_{ji} \sum_{k=1}^{8} w_{jk} x_k}{x_i^{b_j} \sum_{k=1}^{8} a_{jk} x_k^{1-b_j}} - w_{ji} \right)$, for $i = 1, \ldots, 8$,

$$
W = [w_{jk}] = \begin{pmatrix}
3 & 1 & 0.1 & 0.1 & 5 & 0.1 & 0.1 & 6 \\
0.1 & 10 & 0.1 & 0.1 & 5 & 0.1 & 0.1 & 0.1 \\
0.1 & 9 & 10 & 0.1 & 4 & 0.1 & 7 & 0.1 \\
0.1 & 0.1 & 0.1 & 10 & 0.1 & 3 & 0.1 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 11
\end{pmatrix},
$$

$$
A = [a_{jk}] = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 0.8 & 1 & 0.5 & 1 & 1 & 1 & 1 \\
1 & 1.2 & 0.8 & 1.2 & 1.6 & 2 & 0.6 & 0.1 \\
2 & 0.1 & 0.6 & 2 & 1 & 1 & 1 & 2 \\
1.2 & 1.2 & 0.8 & 1 & 1.2 & 0.1 & 3 & 4
\end{pmatrix},
$$

$b = [b_j] = (0.5, 1.2, 0.8, 2.0, 1.5)^T$,

Constraint function: $g_1(\boldsymbol{x}) = \sum_{i=1}^{8} x_i = 1$,

$x_i \ge 0$, for $i = 1, \ldots, 8$,

Starting point: $x_i^{(1)} = 0.125$, for $i = 1, \ldots, 8$,

Optimum point: $\boldsymbol{x}^* = (0.2712, 0.0296, 0.0629, 0.0931, 0.0672, 0.3059, 0.1044, 0.0657)^T$,

Optimum value: $f(\boldsymbol{x}^*) = 0$.

52 HS114 [159]

Classification: GM-L-Z-N,

Dimension: 10,

No. of constraints: 5 (+ bound constraints),

Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 9} f_i(\boldsymbol{x})$,

where $f_1(\boldsymbol{x}) = 5.04x_1 + 0.035x_2 + 10x_3 + 3.36x_5 - 0.063x_4x_7$,

$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(1.12x_1 + 013167x_1x_8 - 0.00667x_1x_8^2 - \tfrac{1}{a}x_4),$$
$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(1.12x_1 + 013167x_1x_8 - 0.00667x_1x_8^2 - ax_4),$$
$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(1.098x_8 - 0.038x_8^2 + 0.352x_6 - \tfrac{1}{a}x_7 + 57.425),$$
$$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(1.098x_8 - 0.038x_8^2 + 0.352x_6 - ax_7 + 57.425),$$
$$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(\tfrac{98000x_3}{x_4x_9+1000x_3} - x_6),$$
$$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(\tfrac{98000x_3}{x_4x_9+1000x_3} - x_6),$$
$$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(\tfrac{x_2+x_5}{x_1} - x_8),$$
$$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(\tfrac{x_2+x_5}{x_1} - x_8),$$

$a = 0.99$, $b = 0.90$,

Constraint function: $g_1(\boldsymbol{x}) = 0.222x_{10} + bx_9 \le 35.82$,
$g_2(\boldsymbol{x}) = 0.222x_{10}\frac{1}{b}x_9 \ge 35.82$,
$g_3(\boldsymbol{x}) = 3x_7 - ax_{10} \ge 133$,
$g_4(\boldsymbol{x}) = 3x_7 - \frac{1}{a}x_{10} \le 133$,
$g_5(\boldsymbol{x}) = 1.22x_4 - x_1 - x_5 = 0$,
$10^{-5} \le x_1 \le 2000$, $10^{-5} \le x_2 \le 16000$,
$10^{-5} \le x_3 \le 120$, $10^{-5} \le x_4 \le 5000$,
$10^{-5} \le x_5 \le 2000$, $85 \le x_6 \le 93$,
$90 \le x_7 \le 95$, $3 \le x_8 \le 12$,
$1.2 \le x_9 \le 4$, $145 \le x_{10} \le 162$,

Starting point: $\boldsymbol{x}^{(1)} = (1745, 12000, 110, 3048, 1974, 89.2, 92.8, 8.0, 3.6, 145)^T$,

Optimum point: $\boldsymbol{x}^* = (1697.8253, 15782.8314, 54.2333, 3031.0044, 2000.0000, 90.1334, 95.0000, 10.4739, 1.5616, 153.5354)^T$,

Optimum value: $f(\boldsymbol{x}^*) = -1768.8070$.

53 Dembo 7 [159]

Classification: GM-L-Z-N,
Dimension: 16,
No. of constraints: 1 (+ bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 19} f_i(\boldsymbol{x})$,
where

$$f_1(\boldsymbol{x}) = a(x_{12} + x_{13} + x_{14} + x_{15} + x_{16}) + b(x_1x_{12} + x_2x_{13} + x_3x_{14} + x_4x_{14} + x_5x_{16})$$
$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_1x_6^{-1} + 100dx_1 - dx_1^2x_6^{-1} - 1),$$
$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_2x_7^{-1} + 100dx_2 - dx_2^2x_7^{-1} - 1)),$$
$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_3x_8^{-1} + 100dx_3 - dx_3^2x_8^{-1} - 1),$$

$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_4x_9^{-1} + 100dx_4 - dx_4^2x_9^{-1} - 1),$
$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_5x_{10}^{-1} + 100dx_5 - dx_5^2x_{10}^{-1} - 1),$
$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_6x_7^{-1} + x_1x_7^{-1}x_{11}^{-1}x_{12}$
$\quad - x_6x_7^{-1}x_{11}^{-1}x_{12} - 1),$
$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_7x_8^{-1} + 0.002(x_7 - x_1)x_8^{-1}x_{12}$
$\quad + 0.002(x_2x_8^{-1} - 1)x_{13} - 1),$
$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_8 + 0.002(x_8 - x_2)x_{13}$
$\quad + 0.002(x_3 - x_9)x_{14} + x_9 - 1),$
$f_{10}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_3^{-1}x_9 + (x_4 - x_8)x_3^{-1}x_{14}^{-1}x_{15}$
$\quad + 500(x_10 - x_9)x_3^{-1}x_{14}^{-1} - 1),$
$f_{11}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3((x_4^{-1}x_5 - 1)x_{15}^{-1}x_{16} + x_4^{-1}x_{10}$
$+ 500(1 - x_4^{-1}x_{10})x_{15}^{-1} - 1),$
$f_{12}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(0.9x_4^{-1} + 0.002(1 - x_4^{-1}x_5)x_{16} - 1),$
$f_{13}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_{11}^{-1}x_{12} - 1),$
$f_{14}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_4x_5^{-1} - 1),$
$f_{15}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_3x_4^{-1} - 1),$
$f_{16}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_2x_3^{-1} - 1),$
$f_{17}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_1x_2^{-1} - 1),$
$f_{18}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_9x_{10}^{-1} - 1),$
$f_{19}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_8x_9^{-1} - 1),$
$a = 1.262626,$ $b = -1.231060,$
$c = 0.034750,$ $d = 0.009750,$

Constraint function: $g_1(\boldsymbol{x}) = 0.002(x_{11} - x_{12}) \le 1,$
$0.1 \le x_1 \le 0.9,$ $0.1 \le x_2 \le 0.9,$
$0.1 \le x_3 \le 0.9,$ $0.1 \le x_4 \le 0.9,$
$0.9 \le x_5 \le 1.0,$ $10^{-4} \le x_6 \le 0.1,$
$0.1 \le x_7 \le 0.9,$ $0.1 \le x_8 \le 0.9,$
$0.1 \le x_9 \le 0.9,$ $0.1 \le x_{10} \le 0.9,$
$1 \le x_{11} \le 1000,$ $10^{-6} \le x_{12} \le 500,$
$1 \le x_{13} \le 500,$ $500 \le x_{14} \le 1000,$
$500 \le x_{15} \le 1000,$ $10^{-6} \le x_{16} \le 500,$

Starting point: $\boldsymbol{x}^{(1)} = (0.80, 0.83, 0.85, 0.87, 0.90, 0.10, 0.12, 0.19, 0.25, 0.29, 512, 13.1, 71.8, 640, 650, 5.7)^T$

Optimum point: $\boldsymbol{x}^* = (0.8038, 0.8161, 0.9000, 0.9000, 0.9000, 0.1000, 0.1070, 0.1908, 0.1908, 0.1908, 505.0526, 5.0526, 72.6358, 500.0000, 500.0000, 0.0000)^T,$

Optimum value: $f(\boldsymbol{x}^*) = 174.78699.$

54 MAD8 [159]

Classification: QM-B-Z-N,
Dimension: 20,
No. of constraints: 0 (only bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 38} |f_i(\boldsymbol{x})|,$
where $f_1(\boldsymbol{x}) = -1 + x_1^2 + \sum_{j=2}^{20} x_j,$

Table 9.6 Values of $\boldsymbol{a}$ for problem 49

i	a_i	i	a_i	i	a_i
1	1.715	16	$-0.19120592 \cdot 10^{-1}$	31	0.00061000
2	0.035	17	$0.56850750 \cdot 10^{2}$	32	−0.0005
3	4.0565	18	1.08702000	33	0.81967200
4	10.0	19	0.32175000	34	0.81967200
5	3000.0	20	−0.03762000	35	24500.0
6	−0.063	21	0.00619800	36	−250.0
7	$0.59553571 \cdot 10^{-2}$	22	$0.24623121 \cdot 10^{4}$	37	$0.10204082 \cdot 10^{-1}$
8	0.88392857	23	$-0.25125634 \cdot 10^{2}$	38	$0.12244898 \cdot 10^{-4}$
9	−0.11756250	24	$0.16118996 \cdot 10^{3}$	39	0.00006250
10	1.10880000	25	5000.0	40	0.00006250
11	0.13035330	26	$-0.48951000 \cdot 10^{6}$	41	-0.00007625
12	−0.00660330	27	$0.44333333 \cdot 10^{2}$	42	1.22
13	$0.66173269 \cdot 10^{-3}$	28	0.33000000	43	1.0
14	$0.17239878 \cdot 10^{-1}$	29	0.02255600	44	−1.0
15	$-0.56595559 \cdot 10^{-2}$	30	−0.00759500		

$f_i(\boldsymbol{x}) = -1 + c_i x_k^2 + \sum_{j=1, j\neq k}^{20} x_j \quad$ for $1 < i < 38$,
$f_{38} = -1 + x_{20}^2 + \sum_{j=1}^{19} x_j$,
$k = (i+2)/2, \quad c_i = 1, \quad$ for $i = 2, 4, ..., 36$,
$k = (i+1)/2, \quad c_i = 2, \quad$ for $i = 3, 5, ..., 37$,

Constraint function: $x_j \geq 0.5 \quad$ for $1 \leq j \leq 10$,
Starting point: $x_j^{(1)} = 100 \quad$ for $j = 1, \ldots, 20$,
Optimum point: $\boldsymbol{x}^* = (0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.5069)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.50694799$.

9.4 Large Problems

In this section we present 21 large-scale nonsmooth unconstrained test problems. The problems can be formulated with any number of variables.

55 Generalization of MAXL [155]

Classification: LM-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1\leq i\leq n} |x_i|$,

Starting point: $x_i^{(1)} = i/n$ for $i = 1, \ldots, n/2$ and
$x_i^{(1)} = -i/n$ for $i = n/2+1, \ldots, n$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

56 Generalization of L1HILB [155]

Classification: L-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \frac{x_j}{i+j-1} \right|$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

57 Generalization of MAXQ [98]

Classification: QM-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} x_i^2$,
Starting point: $x_i^{(1)} = i$ for $i = 1, \ldots, n/2$ and
$x_i^{(1)} = -i$ for $i = n/2+1, \ldots, n$.
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

58 Generalization of MXHILB [98]

Classification: LM-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \frac{x_j}{i+j-1} \right|$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

59 Chained LQ [98]

Classification: G-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \{ -x_i - x_{i+1},$
$-x_i - x_{i+1} + (x_i^2 + x_{i+1}^2 - 1) \}$,
Starting point: $\boldsymbol{x}^{(1)} = (-0.5, -0.5, \ldots, -0.5)^T$,
Optimum point: $\boldsymbol{x}^* = (1/\sqrt{2}, 1/\sqrt{2}, \ldots, 1/\sqrt{2})^T$,
Optimum value: $f(\boldsymbol{x}^*) = -(n-1)\sqrt{2}$.

60 Chained CB3 I [98]

Classification: G-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \left\{x_i^4 + x_{i+1}^2, (2-x_i)^2 + (2-x_{i+1})^2, 2e^{-x_i+x_{i+1}}\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 1, \ldots, 1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 2(n-1)$.

61 Chained CB3 II [98]

Classification: GM-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max \left\{ \sum_{i=1}^{n-1} \left(x_i^4 + x_{i+1}^2\right), \sum_{i=1}^{n-1} \left((2-x_i)^2 + (2-x_{i+1})^2\right), \sum_{i=1}^{n-1} \left(2e^{-x_i+x_{i+1}}\right) \right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 1, \ldots, 1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 2(n-1)$.

62 Number of active faces [95]

Classification: GM-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \left\{ g\left(-\sum_{i=1}^{n} x_i\right), g(x_i) \right\}$,
where $g(y) = \ln(|y| + 1)$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

63 Nonsmooth generalization of Brown function 2 [98]

Classification: G-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(|x_i|^{x_{i+1}^2+1} + |x_{i+1}|^{x_i^2+1}\right)$,
Starting point: $x_i^{(1)} = 1.0$, when $\bmod(i, 2) = 0$ and
$x_i^{(1)} = -1.0$, when $\bmod(i, 2) = 1$, $i = 1, \ldots, n$.
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

64 Chained Mifflin 2 [98]

Classification: G-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \big(-x_i + 2\left(x_i^2 + x_{i+1}^2 - 1\right) + 1.75\left|x_i^2 + x_{i+1}^2 - 1\right|\big)$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \dots, 1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*)$ varies:
$f(\boldsymbol{x}^*) \approx -34.795$, when $n = 50$,
$f(\boldsymbol{x}^*) \approx -140.86$, when $n = 200$, and
$f(\boldsymbol{x}^*) \approx -706.55$, when $n = 1000$.

65 Chained crescent I [98]

Classification: QM-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max\left\{\sum_{i=1}^{n-1}\left(x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1\right), \sum_{i=1}^{n-1}\left(-x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1\right)\right\}$.
Starting point: $x_i^{(1)} = 2.0$, when $\mod(i, 2) = 0$ and
$x_i^{(1)} = -1.5$, when $\mod(i, 2) = 1$, $i = 1, \dots, n$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \dots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

66 Chained crescent II [98]

Classification: G-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max\left\{x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1, -x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1\right\}$,
Starting point: $x_i^{(1)} = 2.0$, when $\mod(i, 2) = 0$ and
$x_i^{(1)} = -1.5$, when $\mod(i, 2) = 1$, $i = 1, \dots, n$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \dots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

67 Problem 6 in `TEST29` [155]

Classification: QM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} |(3 - 2x_i)x_i + 1 - x_{i-1} - x_{i+1}|$,
where $x_0 = x_{n+1} = 0$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1, \dots, -1)^T$,

Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

68 Problem 17 in `TEST29` [155]

Classification: GM-U-Z-E,
Dimension: any, divisible by five
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \Big| 5 - (j+1)(1 - \cos x_i) - \sin x_i - \sum_{k=5j+1}^{5j+5} \cos x_k \Big|$,
where $j = \text{div}\,(i-1, 5)$,
Starting point: $\boldsymbol{x}^{(1)} = (1/n, 1/n, \ldots, 1/n)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

69 Problem 19 in `TEST29` [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} ((3 - 2x_i)x_i - x_{i-1} - 2x_{i+1} + 1)^2$,
where $x_0 = x_{n+1} = 0$.
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1, \ldots, -1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

70 Problem 20 in `TEST29` [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} |(0.5x_i - 3)x_i - 1 + x_{i-1} + 2x_{i+1}|$,
where $x_0 = x_{n+1} = 0$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1, \ldots, -1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

71 Problem 22 in `TEST29` [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \Big| 2x_i + \frac{1}{2(n+1)^2}(x_i + \frac{i}{n+1} + 1)^3 - x_{i-1} - x_{i+1} \Big|$,
where $x_0 = x_{n+1} = 0$,

Starting point: $x_i^{(1)} = \frac{i}{n+1}\left(\frac{i}{n+1} - 1\right)$ for all $i = 1, \ldots, n$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

72 Problem 24 in TEST29 [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le n} \left| 2x_i + \frac{10}{(n+1)^2}\sinh(10x_i) - x_{i-1} - x_{i+1} \right|$,
where $x_0 = 0$ and $x_{n+1} = 1$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

73 DC Maxl [21]

Classification: LD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = n\ \max_{1\le i\le n} |x_i| - \sum_{i=1}^n |x_i|$,
Starting point: $x_i^{(1)} = i$ for $i = 1, \ldots, n/2$,
$x_i^{(1)} = -i$ for $i = n/2 + 1, \ldots, n$,
Optimum point: $\boldsymbol{x}^* = (\alpha, \alpha, \ldots, \alpha)$, $\alpha \in \mathbb{R}$
Optimum value: $f(\boldsymbol{x}^*) = 0$.

74 DC Maxq [26]

Classification: QD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = (n+1)\ \max_{1\le i\le n} x_i^2 - \sum_{i=1}^n x_i^2$,
Starting point: $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ not specified,
Optimum point: $\boldsymbol{x}^* = 0$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

75 Problem 6 in [26]

Classification: LD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = 10\ \max_{1\le j\le 10} \left\{ |\sum_{i=1}^n (x_i - x_i^*) t_j{}^{i-1}| \right\} - \sum_{j=1}^{10} |\sum_{i=1}^n (x_i - x_i^*) t_j{}^{i-1}|$,
$t_j = \max_{1\le j\le 10} \{0.001, 0.1j\}$
Starting point: $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ not specified,

Optimum point: $\boldsymbol{x}^* = (1/n, ..., 1/n)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

76 Problem 7 in [26]

Classification: LD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = 10 \max_{1\le j\le 10} \left\{ |\sum_{i=1}^{n} |x_i - x_i^*| t_j{}^{i-1} \right\}$
$- \sum_{j=1}^{10} | \sum_{i=1}^{n} |x_i - x_i^*| t_j{}^{i-1}$,
$t_j = \max_{1\le j\le 10} \{0.001, 0.1j\}$
Starting point: $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ not specified,
Optimum point: $\boldsymbol{x}^* = (1/n, ..., 1/n)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

Similarly to small-scale problems these problems can be turned to bound constrained ones, for instance, by *inclosing* the additional bounds

$$x_i^* + 0.1 \le x_i \le x_i^* + 1.1 \qquad \text{for all odd } i.$$

9.5 Inequality Constrained Problems

In this section, we describe five nonlinear or nonsmooth inequality constraints (or constraint combinations). The constraints can be combined with the problems 57–66 given in Sect. 9.4 to obtain 50 inequality constrained problems ($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \le 0$ for all $j = 1, \dots, p\}$ in (9.1)).

The constraints are selected such that the original unconstrained minimizers of problems in Sect. 9.4 are not feasible. Note that, due to nonconvexity of the constraints, all the inequality constrained problems formed this way are nonconvex.

The starting points $\boldsymbol{x}^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})^T$ for inequality constrained problems are chosen to be strictly feasible. In what follows, *the starting points for the problems with constraints are the same as those for problems without constraints* (see Sect. 9.4) unless stated otherwise. The optimum values for the problems with different objective functions and $n = 1000$ are given in Table 9.2.

77 Modification of Broyden tridiagonal constraint I [122,126]

Classification: O[O]-Q-Z-N,
No. of constraints: $n - 2$,
Dimension: any,
Objective functions: 57, 58, 62, 63, 65, and 66,
Constraint function: $g_j(\boldsymbol{x}) = (3.0 - 2.0x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 1.0$,
$j = 1, \dots, n-2$,

Objective functions: 59, 60, 61, and 64,
Constraint function: $g_j(\boldsymbol{x}) = (3.0 - 2.0x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 2.5,$
$j = 1, \ldots, n-2,$
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$ for objectives 59 and 64,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$ for objectives 65 and 66,
Starting point: $x_i^{(1)} = -1, i \le n$ and
mod $(i, 2) = 0$ for objective 63.

78 Modification of Broyden tridiagonal constraint II [122,126]

Classification: O[O]-Q-Z-N,
No. of constraints: 1,
Dimension: any,
Objective functions: 57, 58, 62, 63, 65, and 66,
Constraint function: $g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} ((3.0 - 2.0x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 1.0),$
Objective functions: 59, 60, 61, and 64,
Constraint function: $g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} ((3.0 - 2.0x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 2.5),$
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$ for objectives 59 and 64.

79 Modification of MAD1 I [122, 126]

Classification: O[O]-G-Z-N,
No. of constraints: 2,
Dimension: any,
Objective functions: 57–66,
Constraint function: $g_1(\boldsymbol{x}) = \max\left\{x_1^2 + x_2^2 + x_1x_2 - 1.0,\ \sin x_1,\ -\cos x_2\right\},$
$g_2(\boldsymbol{x}) = -x_1 - x_2 + 0.5,$
Starting point: $x_1^{(1)} = -0.5$ and $x_2^{(1)} = 1.1$ for all objectives,
otherwise, the starting points given in Sect. 9.4 are used.

80 Modification of MAD1 II [122, 126]

Classification: O[O]-G-Z-N,
No. of constraints: 4,
Dimension: any,
Objective functions: 57–66,
Constraint function: $g_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 1.0,$
$g_2(\boldsymbol{x}) = \sin x_1,$
$g_3(\boldsymbol{x}) = -\cos x_2,$
$g_4(\boldsymbol{x}) = -x_1 - x_2 + 0.5,$
Starting point: $x_1^{(1)} = -0.5$ and $x_2^{(1)} = 1.1,$ for all objectives,
otherwise the starting points given in Sect. 9.4 are used.

81 Simple modification of MAD1 [122,126]

Classification:	O[O]-Q-Z-N,
No. of constraints:	1,
Dimension:	any,
Objective functions:	57, 58, 62, 63, 65, and 66,
Constraint function:	$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-1} \big(x_i^2 + x_{i+1}^2 + x_i x_{i+1} - 2.0x_i - 2.0x_{i+1} + 1.0\big)$,
Objective functions:	59, 60, 61, and 64,
Constraint function:	$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-1} \big(x_i^2 + x_{i+1}^2 + x_i x_{i+1} - 1.0\big)$,
Starting point:	$\boldsymbol{x}^{(1)} = (0.5, 0.5, \ldots, 0.5)^T$ for objectives 57, 58, 62, 63, 65, and 66,
Starting point:	$\boldsymbol{x}^{(1)} = (0, 0, \ldots, 0)^T$ for objectives 60, 61, and 64.