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Introduction to Nonsmooth Optimization

Theory, Practice and Software

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Adil Bagirov
School of Information Technology and Mathematical Sciences, Centre for Informatics and Applied Optimization
University of Ballarat
Ballarat, VC
Australia

Napsu Karmitsa
Marko M. Mäkelä
Department of Mathematics and Statistics
University of Turku
Turku
Finland

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Preface

Nonsmooth optimization refers to the general problem of minimizing (or maximizing) functions that are typically not differentiable at their minimizers (maximizers). These kinds of functions can be found in many applied fields, for example in image denoising, optimal control, neural network training, data mining, economics, and computational chemistry and physics. Since classical theory of optimization presumes certain differentiability and strong regularity assumptions for the functions to be optimized, it cannot be directly utilized. The aim of this book is to provide an easy-to-read introduction to the theory of nonsmooth optimization and also to present the current state of numerical nonsmooth optimization. In addition, the most common cases where nonsmoothness is involved in practical computations are introduced. In preparing this book, all efforts have been made to ensure that it is self-contained.

The book is organized into three parts: Part I deals with nonsmooth optimization theory. We first provide an easy-to-read introduction to convex and nonconvex analysis with many numerical examples and illustrative figures. Then we discuss nonsmooth optimality conditions from both analytical and geometrical viewpoints. We also generalize the concept of convexity for nonsmooth functions. At the end of the part, we give brief surveys of different generalizations of subdifferentials and approximations to subdifferentials.

In Part II, we consider nonsmooth optimization problems. First, we introduce some real-life nonsmooth optimization problems, for instance, the molecular distance geometry problem, protein structural alignment, data mining, hemivariational inequalities, the power unit-commitment problem, image restoration, and the nonlinear income tax problem. Then we discuss some formulations which lead to nonsmooth optimization problems even though the original problem is smooth (continuously differentiable). Examples here include exact penalty formulations. We also represent the maximum eigenvalue problem, which is an important component of many engineering design problems and graph theoretical applications. We refer to these problems as semi-academic problems. Finally, a comprehensive list of test problems—that is, academic problems—used in nonsmooth optimization is given.

Part III is a guide to nonsmooth optimization software. First, we give short descriptions and the pseudo-codes of the most commonly used methods for nonsmooth optimization. These include different subgradient methods, cutting plane methods, bundle methods, and the gradient sampling method, as well as some hybrid methods and discrete gradient methods. In addition, we introduce some common ways of dealing with constrained nonsmooth optimization problems. We also compare implementations of different nonsmooth optimization methods for solving unconstrained problems. At the end of the part, we provide a table enabling the quick selection of suitable software for different types of nonsmooth optimization problems.

The book is ideal for anyone teaching or attending courses in nonsmooth optimization. As a comprehensible introduction to the field, it is also well-suited for self-access learning for practitioners who know the basics of optimization. Furthermore, it can serve as a reference text for anyone—including experts—dealing with nonsmooth optimization.

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Ballarat, April 2014 — Adil Bagirov
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Marko M. Mäkelä

Contents

Acronyms and Symbols

$\mathbb{R}^n$	n-Dimensional Euclidean space
$\mathbb{N}$	Set of natural numbers
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	(column) Vectors
$\boldsymbol{x}^T$	Transposed vector
$\boldsymbol{x}^T\boldsymbol{y}$	Inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$
$\lVert \boldsymbol{x} \rVert$	Norm of $\boldsymbol{x}$ in $\mathbb{R}^n$, $\lVert \boldsymbol{x} \rVert = (\boldsymbol{x}^T\boldsymbol{x})^{\frac{1}{2}}$
x_i	ith Component of vector $\boldsymbol{x}$
$(\boldsymbol{x}_k)$	Sequence of vectors
$\mathbf{0}$	Zero vector
$a, b, c, \alpha, \varepsilon, \lambda$	Scalars
$t \downarrow 0$	$t \rightarrow 0_+$
$A,\ B$	Matrices
$(A)_{ij}$	Element of matrix A in row i of column j
A^T	Transposed matrix
A^{-1}	Inverse of matrix A
$\mathrm{tr}\,A$	Trace of matrix A
$\lVert A \rVert_{m\times n}$	Matrix norm $\lVert A \rVert_{m\times n} = \left(\sum_{i=1}^{m} \lVert A_i \rVert^2\right)^{\frac{1}{2}}$
I	Identity matrix
$\boldsymbol{e}_i$	ith Column of the identity matrix
$\mathrm{diag}[\theta_1, \ldots, \theta_n]$	Diagonal matrix with diagonal elements $\theta_1, \ldots, \theta_n$
$B(\boldsymbol{x};\ \mathrm{r})$	Open ball with radius r and central point $\boldsymbol{x}$
$\bar{B}(\boldsymbol{x}; r)$	Closed ball with radius r and central point $\boldsymbol{x}$
S_1	Sphere of the unit ball
$(a,\ b)$	Open interval
$[a,\ b]$	Closed interval
$[a,\ b)$, $(a,\ b]$	Half-open intervals
$H(\boldsymbol{p},\ \alpha)$	Hyperplane
$H^+(\boldsymbol{p},\ \alpha)$, $H^-(\boldsymbol{p},\ \alpha)$	Halfspaces
S, U	Sets

cl S	Closure of set S
int S	Interior of set S
bd S	Boundary of set S
$\mathcal{P}(S)$	Power set
$\cap_{i=1}^{m} S_i$	Intersection of sets S_i, $i = 1, \ldots, m$
$S \div U$	Demyanov difference
conv S	Convex hull of set S
cone S	Conic hull of set S
ray S	Ray of the set S
S°	Polar cone of the set S
$K_S(\boldsymbol{x})$	Contingent cone of set S at $\boldsymbol{x}$
$T_S(\boldsymbol{x})$	Tangent cone of set S at $\boldsymbol{x}$
$N_S(\boldsymbol{x})$	Normal cone of set S at $\boldsymbol{x}$
$G_S(\boldsymbol{x})$	Cone of globally feasible directions of set S at $\boldsymbol{x}$
$F_S(\boldsymbol{x})$	Cone of locally feasible directions of set S at $\boldsymbol{x}$
$D_S(\boldsymbol{x})$	Cone of descent directions at $\boldsymbol{x} \in S$
$D_S^\circ(\boldsymbol{x})$	Cone of polar subgradient directions at $\boldsymbol{x} \in S$
$F_S^\circ(\boldsymbol{x})$	Cone of polar constraint subgradient directions at $\boldsymbol{x} \in S$
$\text{lev}_\alpha f$	Level set of f with parameter α
epi f	Epigraph of f
$\mathcal{I}, \mathcal{J}, \mathcal{K}$	Sets of indices
$\lvert\mathcal{I}\rvert$	Number of elements in set $\mathcal{I}$
$f(\boldsymbol{x})$	Objective function value at $\boldsymbol{x}$
arg min $f(\boldsymbol{x})$	Point where function f attains its minimum value
$\nabla f(\boldsymbol{x})$	Gradient of function f at $\boldsymbol{x}$
$\frac{\partial}{\partial x_i} f(\boldsymbol{x})$	Partial derivative of function f with respect to x_i
$\nabla^2 f(\boldsymbol{x})$	Hessian matrix of function f at $\boldsymbol{x}$
$\frac{\partial^2}{\partial x_i \partial x_j} f(\boldsymbol{x})$	Second partial derivative of function f with respect to x_i and x_j
$C^m(\mathbb{R}^n)$	The space of functions $f : \mathbb{R}^n \to \mathbb{R}$ with continuous partial derivatives up to order m
$\mathcal{L}(\mathbb{R}^n, \mathbb{R})$	The space of linear mappings from $\mathbb{R}^n \to \mathbb{R}$
D_k	(generalized) Variable metric approximation of the inverse of the Hessian matrix
$f'(\boldsymbol{x}; \boldsymbol{d})$	Directional derivative of function f at $\boldsymbol{x}$ in the direction $\boldsymbol{d}$
$f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d})$	ε-Directional derivative of function f at $\boldsymbol{x}$ in the direction $\boldsymbol{d}$
$f^\circ(\boldsymbol{x}; \boldsymbol{d})$	Generalized directional derivative of function f at $\boldsymbol{x}$ in the direction $\boldsymbol{d}$
$d_H(A, B)$	Hausdorff distance (distance between sets A and B)
$d_S(\boldsymbol{x})$	Distance function (distance of $\boldsymbol{x}$ to the set S)
$d(\boldsymbol{x}, \boldsymbol{y})$	Distance function (distance between $\boldsymbol{x}$ and $\boldsymbol{y}$)
$\partial_c f(\boldsymbol{x})$	Subdifferential of convex function f at $\boldsymbol{x}$
$\partial f(\boldsymbol{x})$	Subdifferential of function f at $\boldsymbol{x}$

$\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$	Subgradient of function f at $\boldsymbol{x}$
$\partial_\varepsilon f(\boldsymbol{x})$	ε-Subdifferential of convex function f at $\boldsymbol{x}$
$\partial_\varepsilon^G f(\boldsymbol{x})$	Goldstein ε-subdifferential of function f at $\boldsymbol{x}$
$\underline{\partial} f(\boldsymbol{x})$	Subdifferential of quasidifferentiable function f at $\boldsymbol{x}$
$\bar{\partial} f(\boldsymbol{x})$	Superdifferential of quasidifferentiable function f at $\boldsymbol{x}$
$\mathcal{D}f(\boldsymbol{x})$	$\mathcal{D}f(\boldsymbol{x}) = [\underline{\partial} f(\boldsymbol{x}), \bar{\partial} f(\boldsymbol{x})]$ Quasidifferential of function f at $\boldsymbol{x}$
$\underline{d} f(\boldsymbol{x})$	Hypodifferential of codifferentiable function f at $\boldsymbol{x}$
$\bar{d} f(\boldsymbol{x})$	Hyperdifferential of codifferentiable function f at $\boldsymbol{x}$
$Df(\boldsymbol{x})$	$Df(\boldsymbol{x}) = [\underline{d} f(\boldsymbol{x}), \bar{d} f(\boldsymbol{x})]$ Codifferential of function f at $\boldsymbol{x}$
$\partial_b f(\boldsymbol{x})$	Basic (limiting) subdifferential of f at $\boldsymbol{x}$
$\partial^\infty f(\boldsymbol{x})$	Singular subdifferential of f at $\boldsymbol{x}$
$\boldsymbol{v} = \Gamma(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, z, \zeta, \alpha)$	Discrete gradient of function f at $\boldsymbol{x}$ in direction $\boldsymbol{g}$
$D_0(\boldsymbol{x}, \lambda)$	Set of discrete gradients
$\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{g}, h)$	Quasi-secant of function f at $\boldsymbol{x}$
$QSec(\boldsymbol{x}, h)$	Set of quasi-secants
$QSL(\boldsymbol{x})$	Set of limit points of quasi-secants as $h \downarrow 0$
P	Set of univariate positive infinitesimal functions
G	Set of all vertices of the unit hypercube in $\mathbb{R}^n$
Ω_f	A set in $\mathbb{R}^n$ where function f is not differentiable
$\hat{f}_k(\boldsymbol{x})$	Piecewise linear cutting plane model of function f at $\boldsymbol{x}$
$\tilde{f}_k(\boldsymbol{x})$	Piecewise quadratic model of function f at $\boldsymbol{x}$
$\nabla \boldsymbol{h}(\boldsymbol{x})$	Jacobian matrix of function $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ at $\boldsymbol{x}$
$\partial \boldsymbol{h}(\boldsymbol{x})$	Generalized Jacobian matrix of function $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ at $\boldsymbol{x}$
$A(\boldsymbol{x})$	Real symmetric matrix-valued affine function of $\boldsymbol{x}$
$\lambda_i(A(\boldsymbol{x}))$	i:th Eigenvalue of $A(\boldsymbol{x})$
$\lambda_{\max}(A(\boldsymbol{x}))$	Eigenvalue of $A(\boldsymbol{x})$ with the largest absolute value
max	Maximum
min	Minimum
sup	Supremum
inf	Infimum
div(i, j)	Integer division for positive integers i and j
mod(i, j)	Remainder after integer division, $\text{mod}(i,j) = j(i/j - \text{div}(i,j))$
ln	Natural logarithm
DC	Difference of convex functions
FJ	Fritz John optimality conditions
KKT	Karush–Kuhn–Tucker optimality conditions
LOVO	Low order value optimization
MDGP	Molecular distance geometry problem
MINLP	Mixed integer nonlinear programming
NC	Nonconstancy
NSO	Nonsmooth optimization
PLP	Piecewise linear potential
LC, LNC	Large-scale convex and nonconvex problems, $n = 1000$
MC, MNC	Medium-scale convex and nonconvex problems, $n = 200$

SC, SNC	Small-scale convex and nonconvex problems, $n = 50$
XLC, XLNC	Extra–large convex and nonconvex problems, $n = 4000$
XSC, XSNC	Extra-small convex and nonconvex problems, $n \leq 20$
BNEW	Bundle–Newton method
BT	Bundle trust method
CP	(standard) Cutting plane method
CPPC	Cutting plane method with proximity control
DGM	Discrete gradient method
GS	Gradient sampling method
LMBM	Limited memory bundle method
LDGB	Limited memory discrete gradient bundle method
NERML	Non-Euclidean restricted memory level method
PBM	Proximal bundle method
QSM	Quasi-secant method
VMBM	Variable metric bundle method

Introduction

Nonsmooth optimization is among the most difficult tasks in optimization. It deals with optimization problems where objective and/or constraint functions have discontinuous gradients. Nonsmooth optimization dates back to the early 1960s, when the concept of the subdifferential was introduced by R.T. Rockafellar and W. Fenchel and the first nonsmooth optimization method—the subgradient method was developed by N. Shor, Y. Ermolyev, and their colleagues in Kyev, Ukraine (in the former Soviet Union at that time). In the 1960s and in early 1970s, nonsmooth optimization was mainly applied to solve minimax and large linear problems using decomposition. Such problems can also be solved using other optimization techniques.

The most important developments in nonsmooth optimization started with the introduction of the bundle methods in the mid-1970s by C. Lemarechal (and also by P. Wolfe and R. Mifflin). In its original form, the bundle method was introduced to solve nonsmooth convex problems. The 1970s and early 1980s were an important period for new developments in nonsmooth analysis. Various generalizations of subdifferentials were introduced, including the Clarke subdifferential and Demyanov–Rubinov quasidifferential. The use of the Clarke subdifferential allowed the extension of bundle methods to solve nonconvex nonsmooth optimization problems.

Since the early 1990s, nonsmooth optimization has been widely applied to solve many practical problems. Such applications, for example, include mechanics, economics, computational chemistry, engineering, machine learning, and data mining. In most of these applications, nonsmooth optimization approaches allow the significant reduction of the number of decision variables in comparison with any other approaches, and thus facilitate the design of efficient algorithms for their solution. Therefore, in these applications, optimization problems cannot be solved by other optimization techniques as efficiently as they can be solved using nonsmooth optimization techniques. Undoubtedly, nonsmooth optimization has now become an indispensable tool for solving problems in diverse fields.

Nonsmoothness appears in the modeling of many practical problems in a very natural way. The source of nonsmoothness can be divided into four classes:

inherent, technological, methodological, and numerical nonsmoothness. In inherent nonsmoothness, the original phenomenon under consideration itself contains various discontinuities and irregularities. Typical examples of inherent nonsmoothness are the phase changes of materials in the continuous casting of steel, piecewise linear tax models in economics, cluster analysis, supervised data classification, and clusterwise linear regression in data mining and machine learning. Technological nonsmoothness in a model is usually caused by extra technological constraints. These constraints may cause a nonsmooth dependence between variables and functions, even though the functions were originally continuously differentiable. Examples of this include so-called obstacle problems in optimal shape design and discrete feasible sets in product planning. On the other hand, some solution algorithms for constrained optimization may also lead to a nonsmooth problem. Examples of methodological nonsmoothness are the exact penalty function method and the Lagrange decomposition method. Finally, problems may be analytically smooth but numerically nonsmooth. That is the case with, for instance, noisy input data or so-called "stiff problems," which are numerically unstable and behave like nonsmooth problems.

Despite huge developments in nonsmooth optimization in recent decades and wide application of its techniques, only a very few books have been specifically written about it. Some of these books are out of date and do not contain the most recent developments in the area. Moreover, all of these books were written in a way that requires from the audience a high level of knowledge of the subject. Our aim in writing this book is to give an overview of the current state of numerical nonsmooth optimization to a much wider audience, including practitioners.

The book is divided into three major parts dealing, respectively, with theory of nonsmooth optimization (convex and nonsmooth analysis, optimality conditions), practical nonsmooth optimization problems (including applications to real world problems and descriptions of academic test problems) and methods of nonsmooth optimization (description of methods and their pseudo-codes, as well as comparison of different implementations). In preparing this book, all efforts have been made to ensure that it is self-contained.

Within each chapter of the first part, exercises, numerical examples and graphical illustrations have been provided to help the reader to understand the concepts, practical problems, and methods discussed. At the end of each part, notes and references are presented to aid the reader in their further study. In addition, the book contains an extensive bibliography.

Part I
Nonsmooth Analysis and Optimization

Introduction

Convexity plays a crucial role in mathematical optimization. Especially, convexity is the most important concept in constructing optimality conditions. In smooth (continuously differentiable) optimization theory, differentiation entails locally linearizing the functions by the gradients, leading to a lower approximation of a convex function. These ideas can be generalized for nonsmooth convex functions resulting in the concepts of *subgradients* and *subdifferentials*. A subgradient preserves the property of the gradient, providing a lower approximation of the function, but in the nonsmooth case it is not unique anymore. Thus, instead of one gradient vector we end up with a set of subgradients called subdifferentials.

Unfortunately, convexity is often too demanding an assumption in practical applications, and we have to be able to deal with nonconvex functions as well. From a practical point of view, locally Lipschitz continuous functions are proved to be a suitable and sufficiently general class of nonconvex functions. In a convex case, differentiation is based on the linearization of a function. For nonconvex functions, the differentiation can be seen as convexification. All of the concepts of convex analysis can be generalized for a nonconvex case in a very natural way. As in the convex analysis, the links between nonconvex analysis and geometry are strong: subgradients and generalized directional derivatives have a one-to-one interpretation to the tangent and normal cones of epigraphs and level sets.

Our aim here is to present the theory of nonsmooth analysis for optimization in a compact and "easy-to-understand" form. The reader is assumed to have some basic knowledge of linear algebra, elementary real analysis, and smooth nonlinear optimization. To give as self-contained a description as possible, we define every concept used and prove almost all theorems and lemmas.

This part is organized as follows: after recalling some basic results and notions from smooth analysis, we concentrate on "Convex Analysis" in Chap. 2. We start our consideration with geometry by defining convex sets and cones. The main result—and foundation for the forthcoming theory—is the separation theorem stating that two convex sets can always be separated by a hyperplane. Next we turn to analytical concepts, by defining subgradients and subdifferentials. Finally, we show that those geometrical and analytical notions can be connected utilizing epigraphs, level sets, and the distance function.

Chapter 3 is devoted to "Nonconvex Analysis." We generalize all of the convex concepts to locally Lipschitz continuous functions. We also show that nonsmooth analysis is a natural enlargement of classical differential theory by generalizing all of the familiar derivation rules such as the mean-value theorem and the chain rule. The price we have to pay when relaxing the assumptions of differentiability and convexity is that instead of equalities, we only obtain inclusions in most of the results. However, by posing some mild regularity assumptions, we can turn the inclusions back into equalities. After the analytical consideration, we generalize, in a similar way to the convex case, all of the geometrical concepts, and give links between analysis and geometry. At the end of the chapter, we recall the definitions of quasidifferentials, codifferentials, and limiting subdifferentials all of which can be used to generalize the subdifferential of convex functions to the nonconvex case.

Chapter 4 concentrates on the theory of nonsmooth optimization. Optimality conditions are an essential part of mathematical optimization theory, heavily affecting, for example, the development of optimization methods. We formulate the necessary conditions for locally Lipschitz continuous functions to attain local minima both in unconstrained and constrained cases. These conditions are proved to be sufficient for convex problems, and the found minima are global. We formulate both geometrical and analytical conditions based on cones and subdifferentials, respectively. We consider both general geometrical constraint sets and analytical constraints determined by functions and inequalities. In the latter case, we end up to generalizing the classical Fritz John (FJ) and Karush-Kuhn-Tucker (KKT) conditions.

The question, "Can we still attain sufficient optimality conditions when relaxing the convexity assumptions?" has been the initiating force behind the study of generalized convexities. In Chap. 5, we define and analyze the properties of the generalized pseudo- and quasiconvexities for locally Lipschitz continuous functions. Finally, we formulate relaxed versions of the sufficient optimality conditions, where the convexity is replaced by generalized pseudo- and quasiconvexities.

The last chapter of this part, Chap. 6, deals with "Approximations of Subdifferentials." We introduce the concept of continuous approximations to subdifferential, and a discrete gradient that can be used as an approximation to the subgradient at a given point.

Chapter 1
Theoretical Background

In this chapter, we first collect some notations and basic results of smooth analysis. We also recall fundamentals from matrix calculus.

1.1 Notations and Definitions

All the vectors $\boldsymbol{x}$ are considered as column vectors and, correspondingly, all the transposed vectors $\boldsymbol{x}^T$ are considered as row vectors (bolded symbols are used for vectors). We denote by $\boldsymbol{x}^T\boldsymbol{y}$ the usual inner product and by $\|\boldsymbol{x}\|$ the norm in the n-dimensional real Euclidean space $\mathbb{R}^n$. In other words,

$$\boldsymbol{x}^T\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad \|\boldsymbol{x}\| = \left(\boldsymbol{x}^T\boldsymbol{x}\right)^{\frac{1}{2}},$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $\mathbb{R}^n$ and $x_i, y_i \in \mathbb{R}$ are the ith components of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

An *open (closed) ball* with center $\boldsymbol{x} \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(\boldsymbol{x}; r)$ ($\bar{B}(\boldsymbol{x}; r)$). That is,

$$B(\boldsymbol{x}; r) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\| < r\} \quad \text{and} \quad \bar{B}(\boldsymbol{x}; r) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\| \leq r\}.$$

We also denote by S_1 the sphere of the unit ball. That is,

$$S_1 = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y}\| = 1\}.$$

We denote by $[\boldsymbol{x}, \boldsymbol{y}]$ the *closed line-segment* joining $\boldsymbol{x}$ and $\boldsymbol{y}$, that is,

$$[\boldsymbol{x}, \boldsymbol{y}] = \left\{\boldsymbol{z} \in \mathbb{R}^n \mid \boldsymbol{z} = \lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \;\text{ for }\; 0 \leq \lambda \leq 1\right\},$$

and by $(\boldsymbol{x}, \boldsymbol{y})$ the corresponding *open line-segment*.

Every nonzero vector $\boldsymbol{p} \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ define a unique *hyperplane*

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$$H(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T \boldsymbol{x} = \alpha\}$$

or equivalently

$$H(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T (\boldsymbol{x} - \boldsymbol{x}_0) = 0\},$$

where $\boldsymbol{p}^T \boldsymbol{x}_0 = \alpha$ and $\boldsymbol{p}$ is called *normal vector* of the hyperplane. A hyperplane divides tho whole space $\mathbb{R}^n$ into two closed (or open) halfspaces

$$H^+(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T (\boldsymbol{x} - \boldsymbol{x}_0) \geq 0\} \quad \text{and}$$
$$H^-(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T (\boldsymbol{x} - \boldsymbol{x}_0) \leq 0\}.$$

Example 1.1 (*Hyperplane*). If $n = 1$ the hyperplane $H(p, \alpha) = \{x \in \mathbb{R} \mid px = \alpha\}$ is the singleton $\{\alpha/p\}$ and the halfspaces are $H^+(p, \alpha) = [\alpha/p, \infty)$ and $H^-(p, \alpha) = (-\infty, \alpha/p]$.

If $n = 2$ the hyperplane is a line and in the case $n = 3$ it is a plane.

The *closure*, *interior* and *boundary* of a given set $S \subseteq \mathbb{R}^n$ are denoted by cl S, int S and bd S, respectively. Notice that we have

$$\text{bd } S = \text{cl } S \setminus \text{int } S.$$

The *power set* of $S \subseteq \mathbb{R}^n$ is denoted by $\mathcal{P}(S)$ and it is the set of all subsets of S including the empty set and S itself.

Example 1.2 (*Power set*). Let S be the set $\{x, y, z\}$. Then the subsets of S are $\{\}$ (also denoted by $\emptyset$, the empty set), $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, and $\{x, y, z\}$, and hence the power set of S is

$$\mathcal{P}(S) = \{\{\}, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}.$$

1.2 Matrix Calculus

A matrix, for which horizontal and vertical dimensions are the same (that is, an $n \times n$ matrix) is called a *square matrix of order* n.

A square matrix $A \in \mathbb{R}^{n\times n}$ is called *symmetric* if $A = A^T$, that is, $(A)_{ij} = (A)_{ji}$ for all $i, j \in \{1, \ldots, n\}$ and $(A)_{ij}$ is the element of matrix A in row i of column j. The matrix $A^T \in \mathbb{R}^{n\times n}$ is called the *transpose* of A.

A square matrix $A \in \mathbb{R}^{n\times n}$ is called *positive definite* if

$$\boldsymbol{x}^T A\boldsymbol{x} > 0 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{x} \neq \mathbf{0}$$

and *negative definite* if

$$\boldsymbol{x}^T A\boldsymbol{x} < 0 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{x} \neq \mathbf{0}.$$

Correspondingly, a square matrix $A \in \mathbb{R}^{n\times n}$ is called *positive semidefinite* if

$$\boldsymbol{x}^T A\boldsymbol{x} \geq 0 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n$$

and *negative semidefinite* if

$$\boldsymbol{x}^T A\boldsymbol{x} \leq 0 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n.$$

A matrix which is neither positive or negative semidefinite is called *indefinite*.

If the matrix $A \in \mathbb{R}^{n\times n}$ is positive definite, then all the submatrices of the matrix A obtained by deleting the corresponding rows and columns of the matrix are also positive definite and all the elements on the leading diagonal of the matrix are positive (that is, $(A)_{ii} > 0$ for all $i \in \{1, \ldots, n\}$). If the square matrices A and B are positive definite, then so is $A + B$.

An *inverse* of matrix $A \in \mathbb{R}^{n\times n}$ is a matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that

$$AA^{-1} = A^{-1}A = I,$$

where $I \in \mathbb{R}^{n\times n}$ is the *identity matrix*. If a square matrix has an inverse it is called *invertible* or *nonsingular*. Otherwise, it is called *singular*. A positive definite matrix is always nonsingular and its inverse is positive definite.

A scalar λ is called an *eigenvalue* of the matrix $A \in \mathbb{R}^{n\times n}$ if

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

for some nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$. The vector $\boldsymbol{x}$ is called an *eigenvector* associated to the eigenvalue λ. The eigenvalues of a symmetric matrix are real and a symmetric matrix is positive definite if and only if all its eigenvalues are positive. A matrix is said to be *bounded* if its eigenvalues lie in the compact interval that does not contain zero.

The *trace* of matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\operatorname{tr} A$ and it is the sum of the diagonal elements of the matrix, that is,

$$\text{tr } A = \sum_{i=1}^{n} (A)_{ii}.$$

The trace of a matrix equals to the sum of its eigenvalues. For square matrices A and B, we have $\text{tr}(A + B) = \text{tr } A + \text{tr } B$.

1.3 Hausdorff Metrics

Let $A, B \subset \mathbb{R}^n$ be given sets. The *Hausdorff distance* $d_H(A, B)$ between the sets A and B is defined as follows:

$$d_H(A, B) = \max \left\{ \sup_{\boldsymbol{a} \in A} \inf_{\boldsymbol{b} \in B} \|\boldsymbol{a} - \boldsymbol{b}\|, \sup_{\boldsymbol{b} \in B} \inf_{\boldsymbol{a} \in A} \|\boldsymbol{b} - \boldsymbol{a}\| \right\}.$$

Consider a set-valued mapping $G: \mathbb{R}^n \to 2^{\mathbb{R}^m}$. This mapping is called *Hausdorff continuous* at a point $\boldsymbol{x} \in \mathbb{R}^n$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_H(G(\boldsymbol{y}), G(\boldsymbol{x})) < \varepsilon \quad \text{for all} \quad \boldsymbol{y} \in B(\boldsymbol{x}; \delta).$$

1.4 Functions and Derivatives

In what follows the considered functions are assumed to be locally Lipschitz continuous. A function $f: \mathbb{R}^n \to \mathbb{R}$ is *locally Lipschitz continuous at a point* $\boldsymbol{x} \in \mathbb{R}^n$ if there exist scalars $K > 0$ and $\varepsilon > 0$ such that

$$|f(\boldsymbol{y}) - f(\boldsymbol{z})| \leq K\|\boldsymbol{y} - \boldsymbol{z}\| \quad \text{for all} \quad \boldsymbol{y}, \boldsymbol{z} \in B(\boldsymbol{x}; \varepsilon).$$

Function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *locally Lipschitz continuous on a set* $U \subseteq \mathbb{R}^n$ if it is locally Lipschitz continuous at every point belonging to the set U. Furthermore, if $U = \mathbb{R}^n$ the function is called *locally Lipschitz continuous*.

Function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *Lipschitz continuous on a set* $U \subseteq \mathbb{R}^n$ if there exists a scalar K such that

$$|f(\boldsymbol{y}) - f(\boldsymbol{z})| \leq K\|\boldsymbol{y} - \boldsymbol{z}\| \quad \text{for all} \quad \boldsymbol{y}, \boldsymbol{z} \in U.$$

If $U = \mathbb{R}^n$ then f is said to be *Lipschitz continuous*.

A function $f: \mathbb{R}^n \to \mathbb{R}$ is *positively homogeneous* if

$$f(\lambda \boldsymbol{x}) = \lambda f(\boldsymbol{x})$$

for all $\lambda \geq 0$ and *subadditive* if

$$f(\boldsymbol{x}+\boldsymbol{y}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$$

for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$. A function is said to be *sublinear* if it is both positively homogeneous and subadditive.

A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *upper semicontinuous* at $\boldsymbol{x} \in \mathbb{R}^n$ if for every sequence $(\boldsymbol{x}_k)$ converging to $\boldsymbol{x}$ the following holds

$$\limsup_{k\to\infty} f(\boldsymbol{x}_k) \leq f(\boldsymbol{x})$$

and *lower semicontinuous* if

$$f(\boldsymbol{x}) \leq \liminf_{k\to\infty} f(\boldsymbol{x}_k).$$

A both upper and lower semicontinuous function is *continuous*. Notice that a locally Lipschitz continuous function is always continuous.

A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *differentiable* at $\boldsymbol{x} \in \mathbb{R}^n$ if there exists a vector $\nabla f(\boldsymbol{x}) \in \mathbb{R}^n$ and a function $\varepsilon: \mathbb{R}^n \to \mathbb{R}$ such that for all $\boldsymbol{d} \in \mathbb{R}^n$

$$f(\boldsymbol{x}+\boldsymbol{d}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T \boldsymbol{d} + \|\boldsymbol{d}\|\varepsilon(\boldsymbol{d})$$

and $\varepsilon(\boldsymbol{d}) \to 0$ whenever $\|\boldsymbol{d}\| \to 0$. The vector $\nabla f(\boldsymbol{x})$ is called the *gradient vector* of the function f at $\boldsymbol{x}$ and it has the following formula

$$\nabla f(\boldsymbol{x}) = \left(\frac{\partial}{\partial x_1} f(\boldsymbol{x}), \ldots, \frac{\partial}{\partial x_n} f(\boldsymbol{x})\right)^T,$$

where the components $\frac{\partial}{\partial x_i} f(\boldsymbol{x})$ for $i = 1, \ldots, n$, are called *partial derivatives* of the function f. If the function is differentiable and all the partial derivatives are continuous, then the function is said to be *continuously differentiable* or *smooth* ($f \in C^1(\mathbb{R}^n)$). Notice that a smooth function is always locally Lipschitz continuous.

Lemma 1.1 *If a function $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\boldsymbol{x}$, then f is locally Lipschitz continuous at $\boldsymbol{x}$.*

Proof Continuous differentiability means that the linear valued derivative mapping $\nabla f: \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is continuous on a neighborhood of $\boldsymbol{x}$. It follows that there exist constants $\varepsilon > 0$ and $M > 0$ such that

$$\|\nabla f(\boldsymbol{w})\| \leq M \quad \text{for all} \quad \boldsymbol{w} \in B(\boldsymbol{x}; \varepsilon).$$

Suppose now that $\boldsymbol{y}, \boldsymbol{y}' \in B(\boldsymbol{x}; \varepsilon)$. Then, by the classical Mean-Value Theorem, there is $\boldsymbol{z} \in (\boldsymbol{y}, \boldsymbol{y}') \subset B(\boldsymbol{x}; \varepsilon)$ such that

$$f(\boldsymbol{y}) - f(\boldsymbol{y}') = \nabla f(\boldsymbol{z})^T (\boldsymbol{y} - \boldsymbol{y}').$$

We now have

$$|f(\boldsymbol{y}) - f(\boldsymbol{y}')| \le \|\nabla f(\boldsymbol{z})\| \, \|\boldsymbol{y} - \boldsymbol{y}'\| \le M \, \|\boldsymbol{y} - \boldsymbol{y}'\|,$$

which is the Lipschitz condition at $\boldsymbol{x}$. □

The limit

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \lim_{t \downarrow 0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t}$$

(if it exists) is called the *directional derivative* of f at $\boldsymbol{x} \in \mathbb{R}^n$ in the direction $\boldsymbol{d} \in \mathbb{R}^n$. If a function f is differentiable at $\boldsymbol{x}$, then the directional derivative exists in every direction $\boldsymbol{d} \in \mathbb{R}^n$ and

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \nabla f(\boldsymbol{x})^T \boldsymbol{d}.$$

Lemma 1.2 *Let $\boldsymbol{x} \in \mathbb{R}^n$ be a point, where $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and differentiable. Let K be the Lipschitz constant of the function f at the point $\boldsymbol{x}$. Then the function $\boldsymbol{d} \mapsto f'(\boldsymbol{x}; \boldsymbol{d})$ is positively homogeneous and Lipschitz continuous with the constant K.*

Proof Since f is differentiable at the point $\boldsymbol{x}$ the directional derivatives $f'(\boldsymbol{x}; \boldsymbol{d})$ exist for all $\boldsymbol{d} \in \mathbb{R}^n$. Let $\lambda > 0$, then

$$\begin{aligned} f'(\boldsymbol{x}; \lambda\boldsymbol{d}) &= \lim_{t \downarrow 0} \frac{f(\boldsymbol{x} + t\lambda\boldsymbol{d}) - f(\boldsymbol{x})}{t} = \lim_{t \downarrow 0} \lambda \frac{f(\boldsymbol{x} + t\lambda\boldsymbol{d}) - f(\boldsymbol{x})}{\lambda t} \\ &= \lambda \lim_{t \downarrow 0} \frac{f(\boldsymbol{x} + t\lambda\boldsymbol{d}) - f(\boldsymbol{x})}{\lambda t} = \lambda f'(\boldsymbol{x}; \boldsymbol{d}), \end{aligned}$$

which proves the positive homogeneity.

Let $\boldsymbol{u}, \boldsymbol{w} \in \mathbb{R}^n$ be arbitrary. Since f is locally Lipschitz continuous there exists $\varepsilon > 0$ such that the Lipschitz condition holds in $B(\boldsymbol{x}; \varepsilon)$. Furthermore, there exists $t^0 > 0$ such that $\boldsymbol{x} + t\boldsymbol{w}, \boldsymbol{x} + t\boldsymbol{u} \in B(\boldsymbol{x}; \varepsilon)$ when $0 < t < t^0$. Then

$$f(\boldsymbol{x} + t\boldsymbol{u}) - f(\boldsymbol{x} + t\boldsymbol{w}) \le Kt \, \|\boldsymbol{u} - \boldsymbol{w}\| \,,$$

and, thus,

$$\lim_{t \downarrow 0} \frac{f(\boldsymbol{x} + \boldsymbol{u}t) - f(\boldsymbol{x})}{t} \le \lim_{t \downarrow 0} \frac{f(\boldsymbol{x} + \boldsymbol{w}t) - f(\boldsymbol{x})}{t} + K \, \|\boldsymbol{u} - \boldsymbol{w}\|$$

whence

$$f'(\boldsymbol{x}; \boldsymbol{u}) - f'(\boldsymbol{x}; \boldsymbol{w}) \le K \, \|\boldsymbol{u} - \boldsymbol{w}\|.$$

Reversing the roles of $\boldsymbol{u}$ and $\boldsymbol{w}$ we obtain

$$f'(\boldsymbol{x};\boldsymbol{w}) - f'(\boldsymbol{x};\boldsymbol{u}) \le K\,\|\boldsymbol{u}-\boldsymbol{w}\|.$$

Thus

$$\left|f'(\boldsymbol{x};\boldsymbol{w}) - f'(\boldsymbol{x};\boldsymbol{u})\right| \le K\,\|\boldsymbol{u}-\boldsymbol{w}\|$$

completing the proof of the Lipschitz continuity. □

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *twice differentiable* at $\boldsymbol{x} \in \mathbb{R}^n$ if there exists a vector $\nabla f(\boldsymbol{x}) \in \mathbb{R}^n$, a symmetric matrix $\nabla^2 f(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$, and a function $\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$ such that for all $\boldsymbol{d} \in \mathbb{R}^n$

$$f(\boldsymbol{x}+\boldsymbol{d}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T\boldsymbol{d} + \frac{1}{2}\boldsymbol{d}^T\nabla^2 f(\boldsymbol{x})\boldsymbol{d} + \|\boldsymbol{d}\|^2\varepsilon(\boldsymbol{d}),$$

where $\varepsilon(\boldsymbol{d}) \to 0$ whenever $\|\boldsymbol{d}\| \to 0$. The matrix $\nabla^2 f(\boldsymbol{x})$ is called the *Hessian matrix* of the function f at $\boldsymbol{x}$ and it is defined to consist of *second order partial derivatives* of f, that is,

$$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(\boldsymbol{x}) & \dots & \frac{\partial^2}{\partial x_1 \partial x_n} f(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1} f(\boldsymbol{x}) & \dots & \frac{\partial^2}{\partial x_n^2} f(\boldsymbol{x}) \end{bmatrix}.$$

If the function is twice differentiable and all the second order partial derivatives are continuous, then the function is said to be *twice continuously differentiable* ($f \in C^2(\mathbb{R}^n)$).

To the end of this chapter we give the famous Weierstrass' Theorem, which guarantees the existence of the solution of the general optimization problem.

Theorem 1.1 (Weierstrass) *If $S \subset \mathbb{R}^n$ is a nonempty compact set and $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous, then f attains its minimum and maximum over S.*

Chapter 2
Convex Analysis

The theory of nonsmooth analysis is based on convex analysis. Thus, we start this chapter by giving basic concepts and results of convexity (for further readings see also [202, 204]). We take a geometrical viewpoint by examining the tangent and normal cones of convex sets. Then we generalize the concepts of differential calculus for convex, not necessarily differentiable functions [204]. We define subgradients and subdifferentials and present some basic results. At the end of this chapter, we link these analytical and geometrical concepts together.

2.1 Convex Sets

We start this section by recalling the definition of a convex set.

Definition 2.1 Let S be a subset of $\mathbb{R}^n$. The set S is said to be *convex* if

$$\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in S,$$

for all $\boldsymbol{x}, \boldsymbol{y} \in S$ and $\lambda \in [0, 1]$.

Geometrically this means that the set is convex if the closed line-segment $[\boldsymbol{x}, \boldsymbol{y}]$ is entirely contained in S whenever its endpoints $\boldsymbol{x}$ and $\boldsymbol{y}$ are in S (see Fig. 2.1).

Example 2.1 (*Convex sets*). Evidently the empty set $\emptyset$, a singleton $\{\boldsymbol{x}\}$, the whole space $\mathbb{R}^n$, linear subspaces, open and closed balls and halfspaces are convex sets. Furthermore, if S is a convex set also cl S and int S are convex.

Theorem 2.1 *Let $S_i \subseteq \mathbb{R}^n$ be convex sets for $i = 1, \ldots, m$. Then their intersection*

$$\bigcap_{i=1}^{m} S_i \tag{2.1}$$

A. Bagirov et al., *Introduction to Nonsmooth Optimization*,
DOI: 10.1007/978-3-319-08114-4_2

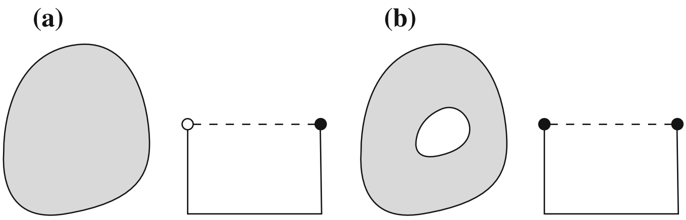


Fig. 2.1 Illustration of convex and nonconvex sets. (**a**) Convex. (**b**) Not convex

is also convex.

Proof Let $\boldsymbol{x}, \boldsymbol{y} \in \bigcap_{i=1}^{m} S_i$ and $\lambda \in [0, 1]$ be arbitrary. Because $\boldsymbol{x}, \boldsymbol{y} \in S_i$ and S_i is convex for all $i = 1, \ldots, m$, we have $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in S_i$ for all $i = 1, \ldots, m$. This implies that

$$\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in \bigcap_{i=1}^{m} S_i$$

and the proof is complete. □

Example 2.2 (*Intersection of convex sets*). The hyperplane

$$H(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) = 0\},$$

where $\boldsymbol{x}_0, \boldsymbol{p} \in \mathbb{R}^n$ and $\boldsymbol{p} \neq \boldsymbol{0}$ is convex, since it can be represent as an intersection of two convex closed halfspaces as

$$\begin{aligned} H(\boldsymbol{p}, \alpha) &= H^+(\boldsymbol{p}, \alpha) \cap H^-(\boldsymbol{p}, \alpha) \\ &= \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \geq 0\} \cap \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0\}. \end{aligned}$$

The next theorem shows that the space of convex sets has some linear properties. This is due to fact that the space of convex sets is a subspace of the power set $\mathcal{P}(\mathbb{R}^n)$ consisting of all subsets of $\mathbb{R}^n$.

Theorem 2.2 *Let $S_1, S_2 \subseteq \mathbb{R}^n$ be nonempty convex sets and $\mu_1, \mu_2 \in \mathbb{R}$. Then the set $\mu_1 S_1 + \mu_2 S_2$ is also convex.*

Proof Let the points $\boldsymbol{x}, \boldsymbol{y} \in \mu_1 S_1 + \mu_2 S_2$ and $\lambda \in [0, 1]$. Then $\boldsymbol{x}$ and $\boldsymbol{y}$ can be written

$$\begin{cases} \boldsymbol{x} &= \mu_1\boldsymbol{x}_1 + \mu_2\boldsymbol{x}_2, \quad \text{where } \boldsymbol{x}_1 \in S_1 \text{ and } \boldsymbol{x}_2 \in S_2 \\ \boldsymbol{y} &= \mu_1\boldsymbol{y}_1 + \mu_2\boldsymbol{y}_2, \quad \text{where } \boldsymbol{y}_1 \in S_1 \text{ and } \boldsymbol{y}_2 \in S_2 \end{cases}$$

and

$$\begin{aligned}\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} &= \lambda(\mu_1 \boldsymbol{x}_1 + \mu_2 \boldsymbol{x}_2) + (1-\lambda)(\mu_1 \boldsymbol{y}_1 + \mu_2 \boldsymbol{y}_2)\\ &= \mu_1(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{y}_1) + \mu_2(\lambda \boldsymbol{x}_2 + (1-\lambda)\boldsymbol{y}_2)\\ &\in \mu_1 S_1 + \mu_2 S_2.\end{aligned}$$

Thus the set $\mu_1 S_1 + \mu_2 S_2$ is convex. □

2.1.1 Convex Hulls

A linear combination $\sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ is called a *convex combination* of elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$ if each $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. The convex hull generated by a set is defined as a set of convex combinations as follows.

Definition 2.2 The *convex hull* of a set $S \subseteq \mathbb{R}^n$ is

$$\text{conv } S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i,\ \sum_{i=1}^{k} \lambda_i = 1,\ \boldsymbol{x}_i \in S,\ \lambda_i \geq 0, k > 0\}.$$

The proof of the next lemma is left as an exercise.

Lemma 2.1 *If* $S \subseteq \mathbb{R}^n$*, then* conv S *is a convex set and* S *is convex if and only if*

$$S = \text{conv } S.$$

Proof Exercise. □

The next theorem shows that the convex hull is actually the intersection of all the convex sets containing the set, in other words, it is the smallest convex set containing the set itself (see Fig. 2.2).

Theorem 2.3 *If* $S \subseteq \mathbb{R}^n$*, then*

$$\text{conv } S = \bigcap_{\substack{S \subseteq \hat{S} \\ \hat{S} \text{ convex}}} \hat{S}.$$

Proof Let $\hat{S}$ be convex such that $S \subseteq \hat{S}$. Then due to Lemma 2.1 we have conv $S \subseteq$ conv $\hat{S} = \hat{S}$ and thus we have

$$\text{conv } S \subseteq \bigcap_{\substack{S \subseteq \hat{S} \\ \hat{S} \text{ convex}}} \hat{S}.$$

On the other hand, it is evident that $S \subseteq \text{conv } S$ and due to Lemma 2.1 conv S is a convex set. Then conv S is one of the sets $\hat{S}$ forming the intersection and thus

$$\bigcap_{\substack{S\subseteq\hat{S} \\ \hat{S}\text{ convex}}} \hat{S} = \bigcap_{\substack{S\subseteq\hat{S} \\ \hat{S}\text{ convex}}} \hat{S} \cap \text{conv } S \subseteq \text{conv } S$$

and the proof is complete. □

2.1.2 Separating and Supporting Hyperplanes

Next we consider some nice properties of hyperplanes. Before those we need the concept of distance function.

Definition 2.3 Let $S \subseteq \mathbb{R}^n$ be a nonempty set. The *distance function* $d_S : \mathbb{R}^n \to \mathbb{R}$ to the set S is defined by

$$d_S(\boldsymbol{x}) := \inf \{\|\boldsymbol{x} - \boldsymbol{y}\| \mid \boldsymbol{y} \in S\} \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{2.2}$$

The following lemma shows that a closed convex set always has a unique closest point.

Lemma 2.2 *Let $S \subset \mathbb{R}^n$ be a nonempty, closed convex set and $\boldsymbol{x}^* \notin S$. Then there exists a unique $\boldsymbol{y}^* \in \text{bd } S$ minimizing the distance to $\boldsymbol{x}^*$. In other words*

$$d_S(\boldsymbol{x}^*) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|.$$

Moreover, a necessary and sufficient condition for a such $\boldsymbol{y}^*$ is that

$$(\boldsymbol{x}^* - \boldsymbol{y}^*)^T(\boldsymbol{x} - \boldsymbol{y}^*) \leq 0 \quad \text{for all } \boldsymbol{x} \in S. \tag{2.3}$$

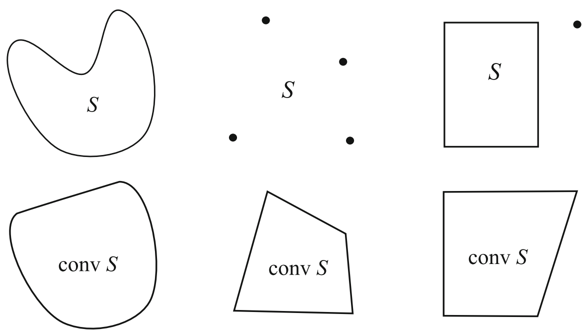


Fig. 2.2 Examples of convex hulls

Proof First we prove the existence of a closest point. Since $S \neq \emptyset$, there exists $\hat{\boldsymbol{x}} \in S$ and we can define $\hat{S} := S \cap \operatorname{cl} B(\boldsymbol{x}^*; r)$, where $r := \|\boldsymbol{x}^* - \hat{\boldsymbol{x}}\| > 0$. Then $\hat{S} \neq \emptyset$ since $\hat{\boldsymbol{x}} \in \hat{S}$. Moreover, $\hat{S}$ is closed, since both S and $\operatorname{cl} B(\boldsymbol{x}^*; r)$ are closed, and bounded, since $\hat{S} \subseteq \operatorname{cl} B(\boldsymbol{x}^*; r)$, thus $\hat{S}$ is a nonempty compact set. Then, due to Weierstrass' Theorem 3.1 the continuous function

$$g(\boldsymbol{y}) := \|\boldsymbol{x}^* - \boldsymbol{y}\|$$

attains its minimum over $\hat{S}$ at some $\boldsymbol{y}^* \in \hat{S}$ and we have

$$d_{\hat{S}}(\boldsymbol{x}^*) = g(\boldsymbol{y}^*) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|.$$

If $\boldsymbol{y} \in S \setminus \hat{S}$, it means that $\boldsymbol{y} \notin \operatorname{cl} B(\boldsymbol{x}^*; r)$, in other words

$$g(\boldsymbol{y}) > r \geq g(\boldsymbol{y}^*)$$

and thus

$$d_S(\boldsymbol{x}^*) = g(\boldsymbol{y}^*) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|.$$

In order to show the uniqueness, suppose that there exists another $\boldsymbol{z}^* \in S$ such that $\boldsymbol{z}^* \neq \boldsymbol{y}^*$ and $g(\boldsymbol{z}^*) = g(\boldsymbol{y}^*)$. Then due to convexity we have $\frac{1}{2}(\boldsymbol{y}^* + \boldsymbol{z}^*) \in S$ and by triangle inequality

$$\begin{aligned} g\left(\tfrac{1}{2}(\boldsymbol{y}^* + \boldsymbol{z}^*)\right) &= \|\boldsymbol{x}^* - \tfrac{1}{2}(\boldsymbol{y}^* + \boldsymbol{z}^*)\| \leq \tfrac{1}{2}\|\boldsymbol{x}^* - \boldsymbol{y}^*\| + \tfrac{1}{2}\|\boldsymbol{x}^* - \boldsymbol{z}^*\| \\ &= \tfrac{1}{2}g(\boldsymbol{y}^*) + \tfrac{1}{2}g(\boldsymbol{z}^*) = g(\boldsymbol{y}^*). \end{aligned}$$

The strict inequality cannot hold since g attains its minimum over S at $\boldsymbol{y}^*$. Thus we have

$$\|(\boldsymbol{x}^* - \boldsymbol{y}^*) + (\boldsymbol{x}^* - \boldsymbol{z}^*)\| = \|\boldsymbol{x}^* - \boldsymbol{y}^*\| + \|\boldsymbol{x}^* - \boldsymbol{z}^*\|,$$

which is possible only if the vectors $\boldsymbol{x}^* - \boldsymbol{y}^*$ and $\boldsymbol{x}^* - \boldsymbol{z}^*$ are collinear. In other words $\boldsymbol{x}^* - \boldsymbol{y}^* = \lambda(\boldsymbol{x}^* - \boldsymbol{z}^*)$ for some $\lambda \in \mathbb{R}$. Since

$$\|\boldsymbol{x}^* - \boldsymbol{y}^*\| = \|\boldsymbol{x}^* - \boldsymbol{z}^*\|$$

we have $\lambda = \pm 1$. If $\lambda = -1$ we have

$$\boldsymbol{x}^* = \tfrac{1}{2}(\boldsymbol{y}^* + \boldsymbol{z}^*) \in S,$$

which contradicts the assumption $\boldsymbol{x}^* \notin S$, and if $\lambda = 1$, we have $\boldsymbol{z}^* = \boldsymbol{y}^*$, thus $\boldsymbol{y}^*$ is a unique closest point.

Next we show that $\boldsymbol{y}^* \in \text{bd } S$. Suppose, by contradiction, that $\boldsymbol{y}^* \in \text{int } S$. Then there exists $\varepsilon > 0$ such that $B(\boldsymbol{y}^*; \varepsilon) \subset S$. Because $g(\boldsymbol{y}^*) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\| > 0$ we can define

$$\boldsymbol{w}^* := \boldsymbol{y}^* + \frac{\varepsilon}{2g(\boldsymbol{y}^*)}(\boldsymbol{x}^* - \boldsymbol{y}^*)$$

and we have $\boldsymbol{w}^* \in B(\boldsymbol{y}^*; \varepsilon)$ since

$$\begin{aligned}\|\boldsymbol{w}^* - \boldsymbol{y}^*\| &= \left\|\boldsymbol{y}^* + \frac{\varepsilon}{2g(\boldsymbol{y}^*)}(\boldsymbol{x}^* - \boldsymbol{y}^*) - \boldsymbol{y}^*\right\| \\ &= \frac{\varepsilon}{2g(\boldsymbol{y}^*)}\|\boldsymbol{x}^* - \boldsymbol{y}^*\| = \frac{\varepsilon}{2}.\end{aligned}$$

Thus $\boldsymbol{w}^* \in S$ and, moreover

$$\begin{aligned}g(\boldsymbol{w}^*) &= \left\|\boldsymbol{x}^* - \boldsymbol{y}^* - \frac{\varepsilon}{2g(\boldsymbol{y}^*)}(\boldsymbol{x}^* - \boldsymbol{y}^*)\right\| \\ &= (1 - \frac{\varepsilon}{2g(\boldsymbol{y}^*)})g(\boldsymbol{y}^*) = g(\boldsymbol{y}^*) - \frac{\varepsilon}{2} < g(\boldsymbol{y}^*),\end{aligned}$$

which is impossible, since g attains its minimum over S at $\boldsymbol{y}^*$. Thus we have $\boldsymbol{y}^* \in \text{bd } S$.

In order to prove that (2.3) is a sufficient condition, let $\boldsymbol{x} \in S$. Then (2.3) implies

$$\begin{aligned}g(\boldsymbol{x})^2 &= \|\boldsymbol{x}^* - \boldsymbol{y}^* + \boldsymbol{y}^* - \boldsymbol{x}\|^2 \\ &= \|\boldsymbol{x}^* - \boldsymbol{y}^*\|^2 + \|\boldsymbol{y}^* - \boldsymbol{x}\|^2 + 2(\boldsymbol{x}^* - \boldsymbol{y}^*)^T(\boldsymbol{y}^* - \boldsymbol{x}) \\ &\geq \|\boldsymbol{x}^* - \boldsymbol{y}^*\|^2 \\ &= g(\boldsymbol{y}^*)^2,\end{aligned}$$

which means that $\boldsymbol{y}^*$ is the closest point.

On the other hand, if $\boldsymbol{y}^*$ is the closest point, we have

$$g(\boldsymbol{x}) \geq g(\boldsymbol{y}^*) \quad \text{for all } \boldsymbol{x} \in S.$$

Let $\boldsymbol{x} \in S$ be arbitrary. The convexity of S implies that

$$\boldsymbol{y}^* + \lambda(\boldsymbol{x} - \boldsymbol{y}^*) = \lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}^* \in S \quad \text{for all } \lambda \in [0, 1]$$

and thus

$$g\left(\boldsymbol{y}^* + \lambda(\boldsymbol{x} - \boldsymbol{y}^*)\right) \geq g(\boldsymbol{y}^*). \tag{2.4}$$

Furthermore, we have

$$g\left(\boldsymbol{y}^* + \lambda(\boldsymbol{x} - \boldsymbol{y}^*)\right)^2 = \|\boldsymbol{x}^* - \boldsymbol{y}^* - \lambda(\boldsymbol{x} - \boldsymbol{y}^*)\|^2$$
$$= g(\boldsymbol{y}^*)^2 + \lambda^2\|\boldsymbol{x} - \boldsymbol{y}^*\|^2 - 2\lambda(\boldsymbol{x}^* - \boldsymbol{y}^*)^T(\boldsymbol{x} - \boldsymbol{y}^*)$$

and combining this with (2.4) we get

$$2\lambda(\boldsymbol{x}^* - \boldsymbol{y}^*)^T(\boldsymbol{x} - \boldsymbol{y}^*) \leq \lambda^2\|\boldsymbol{x} - \boldsymbol{y}^*\|^2 \quad \text{for all } \lambda \in [0, 1]. \tag{2.5}$$

Dividing (2.5) by $\lambda > 0$ and letting $\lambda \downarrow 0$ we get (2.3). □

Next we define separating and supporting hyperplanes.

Definition 2.4 Let $S_1, S_2 \subset \mathbb{R}^n$ be nonempty sets. A hyperplane

$$H(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) = 0\},$$

where $\boldsymbol{p} \neq \boldsymbol{0}$ and $\boldsymbol{p}^T\boldsymbol{x}_0 = \alpha$, *separates* S_1 and S_2 if $S_1 \subseteq H^+(\boldsymbol{p}, \alpha)$ and $S_2 \subseteq H^-(\boldsymbol{p}, \alpha)$, in other words

$$\boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \geq 0 \quad \text{for all } \boldsymbol{x} \in S_1 \quad \text{and}$$
$$\boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0 \quad \text{for all } \boldsymbol{x} \in S_2.$$

Moreover, the separation is *strict* if $S_1 \cap H(\boldsymbol{p}, \alpha) = \emptyset$ and $S_2 \cap H(\boldsymbol{p}, \alpha) = \emptyset$.

Example 2.3 (*Separation of convex sets*). Let $S_1 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid \frac{1}{4}x_1^2 + x_2^2 \leq 1\}$ and $S_2 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - 4)^2 + (x_2 - 2)^2 \leq 1\}$. Then the hyperplane $H((1, 1)^T, 3\frac{1}{2})$, in other words the line $x_2 = -x_1 + 3\frac{1}{2}$ separates S_1 and S_2 (see Fig. 2.3). Notice that $H((1, 1)^T, 3\frac{1}{2})$ is not unique but there exist infinitely many hyperplanes separating S_1 and S_2.

Definition 2.5 Let $S \subset \mathbb{R}^n$ be a nonempty set and $\boldsymbol{x}_0 \in \text{bd } S$. A hyperplane

$$H(\boldsymbol{p}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) = 0\},$$

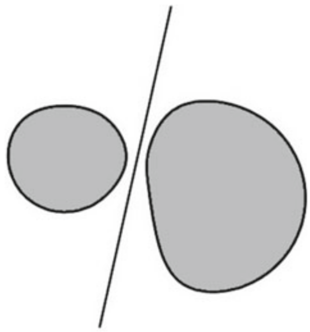
where $\boldsymbol{p} \neq \boldsymbol{0}$ and $\boldsymbol{p}^T\boldsymbol{x}_0 = \alpha$, *supports* S at $\boldsymbol{x}_0$ if either $S \subseteq H^+(\boldsymbol{p}, \alpha)$, in other words

$$\boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \geq 0 \quad \text{for all } \boldsymbol{x} \in S$$

or $S \subseteq H^-(\boldsymbol{p}, \alpha)$, in other words

$$\boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0 \quad \text{for all } \boldsymbol{x} \in S.$$

Fig. 2.3 Separation of convex sets



Example 2.4 (*Supporting hyperplanes*). Let $S := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$. Then the hyperplane $H((0, 1)^T, 1)$, in other words the line $x_2 = 1$ supports S at $\boldsymbol{x}_0 = (0, 1)^T$. Notice that $H((0, 1)^T, 1)$ is the unique supporting hyperplane of S at $\boldsymbol{x}_0 = (0, 1)^T$.

Theorem 2.4 *Let $S \subset \mathbb{R}^n$ be a nonempty, closed convex set and $\boldsymbol{x}^* \notin S$. Then there exists a hyperplane $H(\boldsymbol{p}, \alpha)$ supporting S at some $\boldsymbol{y}^* \in \mathrm{bd}\ S$ and separating S and $\{\boldsymbol{x}^*\}$.*

Proof According to Lemma 2.2 there exists a unique $\boldsymbol{y}^* \in \mathrm{bd}\ S$ minimizing the distance to $\boldsymbol{x}^*$. Let $\boldsymbol{p} := \boldsymbol{x}^* - \boldsymbol{y}^* \neq \mathbf{0}$ and $\alpha := \boldsymbol{p}^T \boldsymbol{y}^*$. Then due to (2.3) we have

$$\boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{y}^*) = (\boldsymbol{x}^* - \boldsymbol{y}^*)^T(\boldsymbol{x} - \boldsymbol{y}^*) \leq 0 \quad \text{for all } \boldsymbol{x} \in S, \tag{2.6}$$

in other words $S \subseteq H^-(\boldsymbol{p}, \alpha)$. This means that $H(\boldsymbol{p}, \alpha)$ supports S at $\boldsymbol{y}^*$. Moreover, we have

$$\boldsymbol{p}^T \boldsymbol{x}^* = \boldsymbol{p}^T(\boldsymbol{x}^* - \boldsymbol{y}^*) + \boldsymbol{p}^T \boldsymbol{y}^* = \|\boldsymbol{p}\|^2 + \alpha > \alpha \tag{2.7}$$

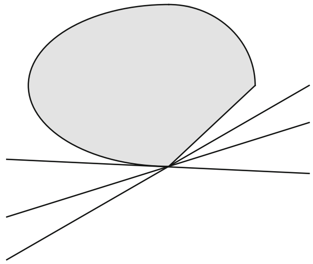
in other words $\{\boldsymbol{x}^*\} \subset H^+(\boldsymbol{p}, \alpha)$ and thus $H(\boldsymbol{p}, \alpha)$ separates S and $\{\boldsymbol{x}^*\}$. □

Next we prove a little bit stronger result, namely that there always exists a hyperplane strictly separating a point and a closed convex set.

Theorem 2.5 *Let $S \subset \mathbb{R}^n$ be a nonempty, closed convex set and $\boldsymbol{x}^* \notin S$. Then there exists a hyperplane $H(\boldsymbol{p}, \beta)$ strictly separating S and $\{\boldsymbol{x}^*\}$.*

Proof Using Lemma 2.2 we get a unique $\boldsymbol{y}^* \in \mathrm{bd}\ S$ minimizing the distance to $\boldsymbol{x}^*$. As in the previous proof let $\boldsymbol{p} := \boldsymbol{x}^* - \boldsymbol{y}^* \neq \mathbf{0}$ but choose now $\beta := \boldsymbol{p}^T \boldsymbol{w}^*$, where $\boldsymbol{w}^* = \frac{1}{2}(\boldsymbol{x}^* + \boldsymbol{y}^*)$. Then due to (2.3) we have

Fig. 2.4 Supporting hyperplanes



$$\begin{aligned}\boldsymbol{p}^T(\boldsymbol{x}-\boldsymbol{w}^*) &= \boldsymbol{p}^T(\boldsymbol{x}-\boldsymbol{y}^*-\tfrac{1}{2}\boldsymbol{p})\\ &= (\boldsymbol{x}^*-\boldsymbol{y}^*)^T(\boldsymbol{x}-\boldsymbol{y}^*)-\tfrac{1}{2}\boldsymbol{p}^T\boldsymbol{p}\\ &\le -\tfrac{1}{2}\|\boldsymbol{p}\|^2 < 0 \quad \text{for all } \boldsymbol{x}\in S,\end{aligned}$$

in other words $S \subset H^-(\boldsymbol{p}, \beta)$ and $S \cap H(\boldsymbol{p}, \beta) = \emptyset$. Moreover, we have

$$\begin{aligned}\boldsymbol{p}^T(\boldsymbol{x}^*-\boldsymbol{w}^*) &= \boldsymbol{p}^T(\boldsymbol{x}^*-\tfrac{1}{2}\boldsymbol{x}^*-\tfrac{1}{2}\boldsymbol{y}^*)\\ &= \tfrac{1}{2}\boldsymbol{p}^T(\boldsymbol{x}^*-\boldsymbol{y}^*)\\ &= \tfrac{1}{2}\|\boldsymbol{p}\|^2 > 0,\end{aligned}$$

which means that $\{\boldsymbol{x}^*\} \subset H^+(\boldsymbol{p}, \beta)$ and $\{\boldsymbol{x}^*\} \cap H(\boldsymbol{p}, \beta) = \emptyset$. Thus $H(\boldsymbol{p}, \beta)$ strictly separates S and $\{\boldsymbol{x}^*\}$. □

Replacing S by cl conv S in Theorem 2.5 we obtain the following result.

Corollary 2.1 *Let $S \subset \mathbb{R}^n$ be a nonempty set and $\boldsymbol{x}^* \notin \text{cl conv } S$. Then there exists a hyperplane $H(\boldsymbol{p}, \beta)$ strictly separating S and $\{\boldsymbol{x}^*\}$.*

The next theorem is very similar to Theorem 2.3 showing that the closure of convex hull is actually the intersection of all the closed halfspaces containing the set.

Theorem 2.6 *If $S \subset \mathbb{R}^n$, then*

$$\text{cl conv } S = \bigcap_{\substack{S\subseteq H^-(\boldsymbol{p},\alpha)\\ \boldsymbol{p}\neq\boldsymbol{0},\ \alpha\in\mathbb{R}}} H^-(\boldsymbol{p}, \alpha).$$

Proof Due to Theorem 2.3 we have

$$\text{conv } S = \bigcap_{\substack{S\subseteq\hat{S}\\ \hat{S}\text{ convex}}} \hat{S} \subseteq \bigcap_{\substack{S\subseteq H^-(\boldsymbol{p},\alpha)\\ \boldsymbol{p}\neq\boldsymbol{0},\ \alpha\in\mathbb{R}}} H^-(\boldsymbol{p}, \alpha) =: T.$$

Since T is closed as an intersection of closed sets, we have

$$\text{cl conv } S \subseteq \text{cl } T = T.$$

Next we show that also $T \subseteq \text{cl conv } S$. To the contrary suppose that there exists $\boldsymbol{x}^* \in T$ but $\boldsymbol{x}^* \notin \text{cl conv } S$. Then due to Corollary 2.1 there exists a closed halfspace $H^-(\boldsymbol{p}, \beta)$ such that $S \subseteq H^-(\boldsymbol{p}, \beta)$ and $\boldsymbol{x}^* \notin H^-(\boldsymbol{p}, \beta)$, thus $\boldsymbol{x}^* \notin T \subseteq H^-(\boldsymbol{p}, \beta)$, which is a contradiction and the proof is complete. □

We can also strengthen the supporting property of Theorem 2.4, namely there exists actually a supporting hyperplane at every boundary point.

Theorem 2.7 *Let $S \subset \mathbb{R}^n$ be a nonempty convex set and $\boldsymbol{x}_0 \in \text{bd } S$. Then there exists a hyperplane $H(\boldsymbol{p}, \alpha)$ supporting* cl S *at $\boldsymbol{x}_0$.*

Proof Since $\boldsymbol{x}_0 \in \text{bd } S$ there exists a sequence $(\boldsymbol{x}_k)$ such that $\boldsymbol{x}_k \notin \text{cl } S$ and $\boldsymbol{x}_k \to \boldsymbol{x}_0$. Then due to Theorem 2.4 for each $\boldsymbol{x}_k$ there exists $\boldsymbol{y}_k \in \text{bd } S$ such that the hyperplane $H(\boldsymbol{q}_k, \beta_k)$, where $\boldsymbol{q}_k := \boldsymbol{x}_k - \boldsymbol{y}_k$ and $\beta_k := \boldsymbol{q}_k^T \boldsymbol{y}_k$ supports cl S at $\boldsymbol{y}_k$. Then inequality (2.6) implies that

$$0 \geq \boldsymbol{q}_k^T(\boldsymbol{x} - \boldsymbol{y}_k) = \boldsymbol{q}_k^T \boldsymbol{x} - \beta_k \quad \text{for all } \boldsymbol{x} \in \text{cl } S,$$

and thus

$$\boldsymbol{q}_k^T \boldsymbol{x} \leq \beta_k \quad \text{for all } \boldsymbol{x} \in \text{cl } S.$$

On the other hand, according to (2.7) we get $\boldsymbol{q}_k^T \boldsymbol{x}_k > \beta_k$, thus we have

$$\boldsymbol{q}_k^T \boldsymbol{x} < \boldsymbol{q}_k^T \boldsymbol{x}_k \quad \text{for all } \boldsymbol{x} \in \text{cl } S. \tag{2.8}$$

Next we normalize vectors $\boldsymbol{q}_k$ by defining $\boldsymbol{p}_k := \boldsymbol{q}_k / \|\boldsymbol{q}_k\|$. Then $\|\boldsymbol{p}_k\| = 1$, which means that the sequence $(\boldsymbol{p}_k)$ is bounded having a convergent subsequence $(\boldsymbol{p}_{k_j})$, in other words there exists a limit $\boldsymbol{p} \in \mathbb{R}^n$ such that $\boldsymbol{p}_{k_j} \to \boldsymbol{p}$ and $\|\boldsymbol{p}\| = 1$. It is easy to verify, that (2.8) holds also for $\boldsymbol{p}_{k_j}$, in other words

$$\boldsymbol{p}_{k_j}^T \boldsymbol{x} < \boldsymbol{p}_{k_j}^T \boldsymbol{x}_{k_j} \quad \text{for all } \boldsymbol{x} \in \text{cl } S. \tag{2.9}$$

Fixing now $\boldsymbol{x} \in \text{cl } S$ in (2.9) and letting $j \to \infty$ we get $\boldsymbol{p}^T \boldsymbol{x} \leq \boldsymbol{p}^T \boldsymbol{x}_0$. In other words

$$\boldsymbol{p}^T(\boldsymbol{x} - \boldsymbol{x}_0) \leq 0,$$

which means that cl $S \subseteq H^-(\boldsymbol{p}, \alpha)$, where $\alpha := \boldsymbol{p}^T \boldsymbol{x}_0$ and thus $H(\boldsymbol{p}, \alpha)$ supports cl S at $\boldsymbol{x}_0$. □

Finally we consider a nice property of convex sets, namely two disjoint convex sets can always be separated by a hyperplane. For strict separation it is not enough to suppose the closedness of the sets, but at least one of the sets should be bounded as well.

Theorem 2.8 *Let $S_1, S_2 \subset \mathbb{R}^n$ be nonempty convex sets. If $S_1 \cap S_2 = \emptyset$, then there exists a hyperplane $H(\boldsymbol{p}, \alpha)$ separating S_1 and S_2. If, in addition, S_1 and S_2 are closed and S_1 is bounded, then the separation is strict.*

Proof It follows from Theorem 2.2, that the set

$$S := S_1 - S_2 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2,\ \boldsymbol{x}_1 \in S_1, \boldsymbol{x}_2 \in S_2\}$$

is convex. Furthermore, $\mathbf{0} \notin S$, since otherwise there would exist $\boldsymbol{x}_1 \in S_1$ and $\boldsymbol{x}_2 \in S_2$ such that $\mathbf{0} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, in other words $\boldsymbol{x}_1 = \boldsymbol{x}_2 \in S_1 \cap S_2 = \emptyset$, which is impossible.

If $\mathbf{0} \notin \operatorname{cl} S$, then due to Corollary 2.1 there exists a hyperplane $H(\boldsymbol{p}, \alpha)$ strictly separating S and $\{\mathbf{0}\}$, in other words

$$\boldsymbol{p}^T\boldsymbol{x} < \alpha < \boldsymbol{p}^T\mathbf{0} = 0 \quad \text{for all } \boldsymbol{x} \in S.$$

Since $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, where $\boldsymbol{x}_1 \in S_1$ and $\boldsymbol{x}_2 \in S_2$, we get

$$\boldsymbol{p}^T\boldsymbol{x}_1 < \alpha < \boldsymbol{p}^T\boldsymbol{x}_2 \quad \text{for all } \boldsymbol{x}_1 \in S_1, \boldsymbol{x}_2 \in S_2,$$

and thus $H(\boldsymbol{p}, \alpha)$ strictly separates S_1 and S_2.

On the other hand, if $\mathbf{0} \in \operatorname{cl} S$ it must hold that $\mathbf{0} \in \operatorname{bd} S$ (since $\mathbf{0} \notin \operatorname{int} S$). Then due to Theorem 2.7 there exists a hyperplane $H(\boldsymbol{p}, \beta)$ supporting $\operatorname{cl} S$ at $\mathbf{0}$, in other words

$$\boldsymbol{p}^T(\boldsymbol{x} - \mathbf{0}) \le 0 \quad \text{for all } \boldsymbol{x} \in \operatorname{cl} S.$$

Denoting again $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, where $\boldsymbol{x}_1 \in S_1$ and $\boldsymbol{x}_2 \in S_2$, we get

$$\boldsymbol{p}^T\boldsymbol{x}_1 \le \boldsymbol{p}^T\boldsymbol{x}_2 \quad \text{for all } \boldsymbol{x}_1 \in S_1, \boldsymbol{x}_2 \in S_2.$$

Since the set of real numbers $\{\boldsymbol{p}^T\boldsymbol{x}_1 \mid \boldsymbol{x}_1 \in S_1\}$ is bounded above by some number $\boldsymbol{p}^T\boldsymbol{x}_2$, where $\boldsymbol{x}_2 \in S_2 \neq \emptyset$ it has a finite supremum. Defining $\alpha := \sup\{\boldsymbol{p}^T\boldsymbol{x}_1 \mid \boldsymbol{x}_1 \in S_1\}$ we get

$$\boldsymbol{p}^T\boldsymbol{x}_1 \le \alpha \le \boldsymbol{p}^T\boldsymbol{x}_2 \quad \text{for all } \boldsymbol{x}_1 \in S_1, \boldsymbol{x}_2 \in S_2,$$

and thus $H(\boldsymbol{p}, \alpha)$ separates S_1 and S_2.

Suppose next, that S_1 and S_2 are closed and S_1 is bounded. In order to show that S is closed suppose, that there exists a sequence $(\boldsymbol{x}_k) \subset S$ and a limit $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x}_k \to \boldsymbol{x}$. Then due to the definition of S we have $\boldsymbol{x}_k = \boldsymbol{x}_{1_k} - \boldsymbol{x}_{2_k}$, where $\boldsymbol{x}_{1_k} \in S_1$ and $\boldsymbol{x}_{2_k} \in S_2$. Since S_1 is compact, there exists a convergent subsequence $(\boldsymbol{x}_{1_{k_j}})$ and a limit $\boldsymbol{x}_1 \in S_1$ such that $\boldsymbol{x}_{1_{k_j}} \to \boldsymbol{x}_1$. Then we have

$\boldsymbol{x}_{2_{kj}} = \boldsymbol{x}_{1_{kj}} - \boldsymbol{x}_{k_j} \to \boldsymbol{x}_1 - \boldsymbol{x} := \boldsymbol{x}_2$. Since S_2 is closed $\boldsymbol{x}_2 \in S_2$. Thus $\boldsymbol{x} \in S$ and S is closed. Now the case $\mathbf{0} \notin \text{cl } S = S$ given above is the only possibility and thus we can find $H(\boldsymbol{p}, \alpha)$ strictly separating S_1 and S_2. □

The next two examples show that both closedness and compactness assumptions actually are essential for strict separation.

Example 2.5 (*Strict separation, counter example 1*). Let $S_1 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 > 0 \text{ and } x_2 \geq 1/x_1\}$ and $S_2 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 = 0\}$. Then both S_1 and S_2 are closed but neither of them is bounded. It follows that $S_1 - S_2 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 > 0\}$ is not closed and there does not exist any strictly separating hyperplane.

Example 2.6 (*Strict separation, counter example 2*). Let $S_1 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ and $S_2 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - 2)^2 + x_2^2 < 1\}$. Then both S_1 and S_2 are bounded but S_2 is not closed and it follows again that $S_1 - S_2 = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 + 2)^2 + x_2^2 < 4\}$ is not closed and there does not exist any strictly separating hyperplane.

2.1.3 Convex Cones

Next we define the notion of a cone, which is a set containing all the rays passing through its points emanating from the origin.

Definition 2.6 A set $C \subseteq \mathbb{R}^n$ is a *cone* if $\lambda \boldsymbol{x} \in C$ for all $\boldsymbol{x} \in C$ and $\lambda \geq 0$. Moreover, if C is convex, then it is called a *convex cone*.

Example 2.7 (*Convex cones*). It is easy to show that a singleton $\{\mathbf{0}\}$, the whole space $\mathbb{R}^n$, closed halfspaces $H^+(\boldsymbol{p}, 0)$ and $H^-(\boldsymbol{p}, 0)$, the nonnegative orthant $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq 0,\ i = 1 \ldots, n\}$ and halflines starting from the origin are examples of closed convex cones.

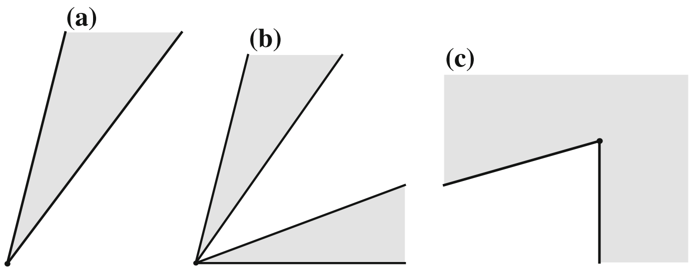


Fig. 2.5 Illustration of convex and nonconvex cones. (**a**) Convex. (**b**) Not convex. (**c**) Not convex

Theorem 2.9 *A set $C \subseteq \mathbb{R}^n$ is a convex cone if and only if*

$$\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in C \quad \textit{for all}\ \boldsymbol{x}, \boldsymbol{y} \in C\ \textit{and}\ \lambda, \mu \geq 0. \tag{2.10}$$

Proof Evidently (2.10) implies that C is a convex cone.

Next, let C be a convex cone and suppose that $\boldsymbol{x}, \boldsymbol{y} \in C$ and $\lambda, \mu \geq 0$. Since C is a cone we have $\lambda \boldsymbol{x} \in C$ and $\mu \boldsymbol{y} \in C$. Furthermore, since C is convex we have

$$\tfrac{1}{2}\lambda \boldsymbol{x} + (1 - \tfrac{1}{2})\mu \boldsymbol{y} \in C \tag{2.11}$$

and again using the cone property we get

$$\lambda \boldsymbol{x} + \mu \boldsymbol{y} = 2\left(\tfrac{1}{2}\lambda \boldsymbol{x} + (1 - \tfrac{1}{2})\mu \boldsymbol{y}\right) \in C \tag{2.12}$$

and the proof is complete. □

Via the next definition we get a connection between sets and cones. Namely a set generates a cone, when every point of the set is replaced by a ray emanating from the origin.

Definition 2.7 The *ray* of a set $S \subseteq \mathbb{R}^n$ is

$$\text{ray } S = \bigcup_{\lambda \geq 0} \lambda S = \{\lambda \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \in S,\ \lambda \geq 0\}.$$

The proof of the next lemma is left as an exercise.

Lemma 2.3 *If $S \subseteq \mathbb{R}^n$, then* ray S *is a cone and $C \subseteq \mathbb{R}^n$ is cone if and only if*

$$C = \text{ray } C.$$

Proof Exercise. □

The next theorem shows that the ray of a set is actually the intersection of all the cones containing S, in other words, it is the smallest cone containing S (see Fig. 2.6).

Theorem 2.10 *If* $S \subset \mathbb{R}^n$, *then*

$$\text{ray } S = \bigcap_{\substack{S \subseteq C \\ C \text{ cone}}} C.$$

Proof Let C be a cone such that $S \subseteq C$. Then due to Lemma 2.3 we have ray $S \subseteq$ ray $C = C$ and thus we have

$$\text{ray } S \subseteq \bigcap_{\substack{S \subseteq C \\ C \text{ cone}}} C.$$

On the other hand, it is evident that $S \subseteq$ conv S and due to Lemma 2.3 ray S is a cone. Then ray S is one of the cones C forming the intersection and thus

$$\bigcap_{\substack{S \subseteq C \\ C \text{ cone}}} C = \bigcap_{\substack{S \subseteq C \\ C \text{ cone}}} C \cap \text{ray } S \subseteq \text{ray } S$$

and the proof is complete. □

It can be seen from Fig. 2.6 that a ray is not necessarily convex. However, if the set is convex, then also its ray is convex.

Theorem 2.11 *If* $S \subseteq \mathbb{R}^n$ *is convex, then* ray S *is a convex cone.*

Proof Due to Lemma 2.3 ray S is a cone. For convexity let $\boldsymbol{x}, \boldsymbol{y} \in$ ray S and $\lambda, \mu \geq 0$. Then $\boldsymbol{x} = \alpha \boldsymbol{u}$ and $\boldsymbol{y} = \beta \boldsymbol{v}$, where $\boldsymbol{u}, \boldsymbol{v} \in S$ and $\alpha, \beta \geq 0$. Since S is convex we have

$$\boldsymbol{z} := \frac{\lambda\alpha}{\lambda\alpha + \mu\beta}\boldsymbol{u} + \left(1 - \frac{\lambda\alpha}{\lambda\alpha + \mu\beta}\right)\boldsymbol{v} \in S.$$

The fact that ray S is cone implies that $(\lambda\alpha + \mu\beta)\boldsymbol{z} \in$ ray S, in other words

$$(\lambda\alpha + \mu\beta)\boldsymbol{z} = \lambda\alpha\boldsymbol{u} + \mu\beta\boldsymbol{v} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} \in \text{ray } S.$$

According to Theorem 2.9 this means, that ray S is convex. □

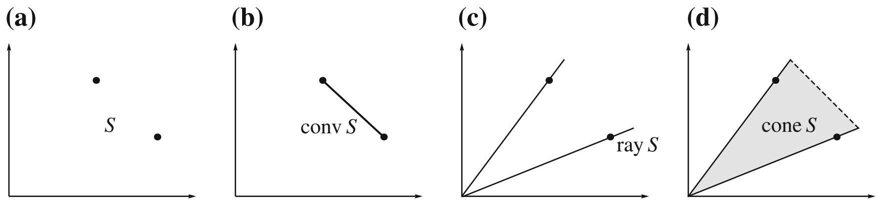


Fig. 2.6 Convex hull, ray and conic hull of a set. (**a**) Set. (**b**) Convex hull. (**c**) Ray. (**d**) Conic hull

It is also easy to show that a ray is not necessarily closed. However, if the set is compact not including the origin its ray is closed.

Theorem 2.12 *If* $S \subset \mathbb{R}^n$ *is compact such that* $\mathbf{0} \notin S$, *then* ray S *is closed.*

Proof Let $(\boldsymbol{x}_j) \subset \text{ray } S$ be a sequence such that $\boldsymbol{x}_j \to \boldsymbol{x}$. Next we show that $\boldsymbol{x} \in \text{ray } S$. The fact that $\boldsymbol{x}_j \in \text{ray } S$ means that $\boldsymbol{x}_j = \lambda_j \boldsymbol{y}_j$ where $\lambda_j \geq 0$ and $\boldsymbol{y}_j \in S$ for all $j \in \mathbb{N}$. Since S is compact the sequence $\boldsymbol{y}_j$ is bounded, thus there exists a subsequence $(\boldsymbol{y}_{j_i}) \subset S$ such that $\boldsymbol{y}_{j_i} \to \boldsymbol{y}$. Because S is closed, it follows that $\boldsymbol{y} \in S$. Furthermore, since $\mathbf{0} \notin S$ one has $\boldsymbol{y} \neq \mathbf{0}$, thus the sequence λ_{j_i} is also converging to some $\lambda \geq 0$. Then $\lambda_{j_i} \boldsymbol{y}_{j_i} \to \lambda \boldsymbol{y} = \boldsymbol{x}$, which means that $\boldsymbol{x} \in \text{ray } S$, in other words S is closed. □

Similarly to the convex combination we say that the linear combination $\sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ is a *conic combination* of elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$ if each $\lambda_i \geq 0$ and the conic hull generated by a set is defined as a set of conic combinations as follows.

Definition 2.8 The *conic hull* of a set $S \subseteq \mathbb{R}^n$ is

$$\text{cone } S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i, \ \boldsymbol{x}_i \in S, \ \lambda_i \geq 0, k > 0\}.$$

The proof of the next lemma is again left as an exercise.

Lemma 2.4 *If* $S \subseteq \mathbb{R}^n$, *then* cone S *is a convex cone and* $C \subseteq \mathbb{R}^n$ *is convex cone if and only if*

$$C = \text{cone } C.$$

Proof Exercise. □

The next theorem shows that the conic hull cone S is actually the intersection of all the convex cones containing S, in other words, it is the smallest convex cone containing S (see Fig. 2.6).

Theorem 2.13 *If* $S \subset \mathbb{R}^n$, *then*

$$\text{cone } S = \bigcap_{\substack{S \subseteq C \\ C \text{ convex cone}}} C.$$

Proof Let C be a convex cone such that $S \subseteq C$. Then due to Lemma 2.4 we have cone $S \subseteq$ cone $C = C$ and thus we have

$$\text{cone } S \subseteq \bigcap_{\substack{S \subseteq C \\ C \text{ convex cone}}} C.$$

On the other hand, it is evident that $S \subseteq \text{cone } S$ and due to Lemma 2.4 cone S is a convex cone. Then cone S is one of the convex cones forming the intersection and thus

$$\bigcap_{\substack{S \subseteq C \\ C \text{ convex cone}}} C = \bigcap_{\substack{S \subseteq C \\ C \text{ convex cone}}} C \cap \text{cone } S \subseteq \text{cone } S$$

and the proof is complete. □

Note, that according to Lemma 2.1, Theorems 2.10 and 2.13, and Definitions 2.7 and 2.8 we get the following result.

Corollary 2.2 *If $S \subseteq \mathbb{R}^n$, then*

$$\text{cone } S = \text{conv ray } S.$$

Finally we get another connection between sets and cones. Namely, every set generates also so called polar cone.

Definition 2.9 The *polar cone* of a nonempty set $S \subseteq \mathbb{R}^n$ is

$$S^\circ = \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y}^T\boldsymbol{x} \le 0 \text{ for all } \boldsymbol{x} \in S\}.$$

The polar cone $\emptyset^0$ of the empty set $\emptyset$ is the whole space $\mathbb{R}^n$.

The next lemma gives some basic properties of polar cones (see Fig. 2.7). The proof is left as an exercise.

Lemma 2.5 *If $S \subseteq \mathbb{R}^n$, then S° is a closed convex cone and $S \subseteq S^{\circ\circ}$.*

Proof Exercise. □

Theorem 2.14 *The set $C \subseteq \mathbb{R}^n$ is a closed convex cone if and only if*

$$C = C^{\circ\circ}.$$

Proof Suppose first that $C = C^{\circ\circ} = (C^\circ)^\circ$. Then due to Lemma 2.5 C is a closed convex cone.

Suppose next, that C is a closed convex cone. Lemma 2.5 implies that $C \subseteq C^{\circ\circ}$. We shall prove next that $C^{\circ\circ} \subseteq C$. Clearly $\emptyset^{\circ\circ} = (\mathbb{R}^n)^\circ = \emptyset$ and thus we can assume that C is nonempty. Suppose, by contradiction, that there exists $\boldsymbol{x} \in C^{\circ\circ}$ such that $\boldsymbol{x} \notin C$. Then due to Theorem 2.4 there exists a hyperplane $H(\boldsymbol{p}, \alpha)$ separating C and $\{\boldsymbol{x}\}$, in other words there exist $\boldsymbol{p} \neq \boldsymbol{0}$ and $\alpha \in \mathbb{R}$ such that

$$\boldsymbol{p}^T\boldsymbol{y} \le \alpha \text{ for all } \boldsymbol{y} \in C \quad \text{and} \quad \boldsymbol{p}^T\boldsymbol{x} > \alpha.$$

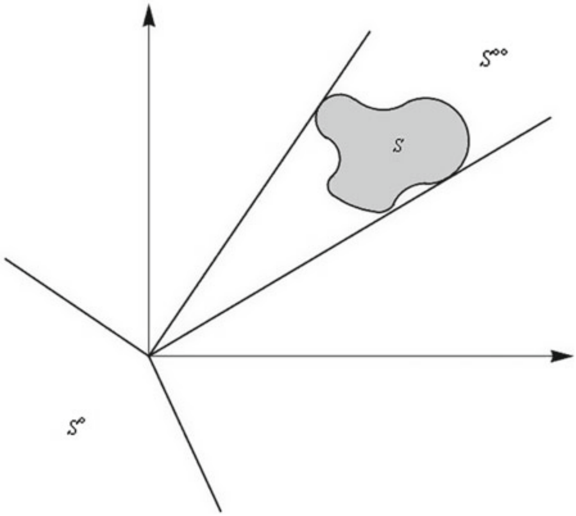


Fig. 2.7 Polar cones of the set

Since $\mathbf{0} \in C$ we have $\alpha \geq \boldsymbol{p}^T\mathbf{0} = 0$ and thus

$$\boldsymbol{p}^T\boldsymbol{x} > 0. \tag{2.13}$$

If $\boldsymbol{p} \notin C^{\circ}$ then due to the definition of the polar cone there exists $\boldsymbol{z} \in C$ such that $\boldsymbol{p}^T\boldsymbol{z} > 0$. Since C is cone we have $\lambda\boldsymbol{z} \in C$ for all $\lambda \geq 0$. Then $\boldsymbol{p}^T(\lambda\boldsymbol{z}) > 0$ can grow arbitrary large when $\lambda \to \infty$, which contradicts the fact that $\boldsymbol{p}^T\boldsymbol{y} \leq \alpha$ for all $\boldsymbol{y} \in C$. Therefore we have $\boldsymbol{p} \in C^{\circ}$. On the other hand

$$\boldsymbol{x} \in C^{\circ\circ} = \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y}^T\boldsymbol{v} \leq 0 \text{ for all } \boldsymbol{v} \in C^{\circ}\}$$

and thus $\boldsymbol{p}^T\boldsymbol{x} \leq 0$, which contradicts (2.13). We conclude that $\boldsymbol{x} \in C$ and the proof is complete. □

2.1.4 Contingent and Normal Cones

In this subsection we consider tangents and normals of convex sets. First we define a classical notion of contingent cone consisting of the tangent vectors (see Fig. 2.8).

Definition 2.10 The *contingent cone* of the nonempty set S at $\boldsymbol{x} \in S$ is given by the formula

$$K_S(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid \text{there exist } t_i \downarrow 0 \text{ and } \boldsymbol{d}_i \to \boldsymbol{d} \text{ such that } \boldsymbol{x} + t_i\boldsymbol{d}_i \in S\}. \tag{2.14}$$

The elements of $K_S(\boldsymbol{x})$ are called *tangent vectors*.

Several elementary facts about the contingent cone will now be listed.

Theorem 2.15 *The contingent cone $K_S(\boldsymbol{x})$ of the nonempty convex set S at $\boldsymbol{x} \in S$ is a closed convex cone.*

Proof We begin by proving that $K_S(\boldsymbol{x})$ is closed. To see this, let $(\boldsymbol{d}_i)$ be a sequence in $K_S(\boldsymbol{x})$ converging to $\boldsymbol{d} \in \mathbb{R}^n$. Next we show that $\boldsymbol{d} \in K_S(\boldsymbol{x})$. The fact that $\boldsymbol{d}_i \to \boldsymbol{d}$ implies that for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that

$$\|\boldsymbol{d} - \boldsymbol{d}_i\| < \varepsilon/2 \qquad \text{for all} \quad i \geq i_0.$$

On the other hand, $\boldsymbol{d}_i \in K_S(\boldsymbol{x})$, thus for each $i \in \mathbb{N}$ there exist sequences $(\boldsymbol{d}_{i_j}) \subset \mathbb{R}^n$ and $(t_{i_j}) \subset \mathbb{R}$ such that $\boldsymbol{d}_{i_j} \to \boldsymbol{d}_i$, $t_{i_j} \downarrow 0$ and $\boldsymbol{x} + t_{i_j}\boldsymbol{d}_{i_j} \in S$ for all $j \in \mathbb{N}$. Then there exist $j_i^{\boldsymbol{y}} \in \mathbb{N}$ and $j_i^t \in \mathbb{N}$ such that for all $i \in \mathbb{N}$

$$\|\boldsymbol{d}_i - \boldsymbol{d}_{i_j}\| < \varepsilon/2 \qquad \text{for all} \quad j \geq j_i^{\boldsymbol{y}}$$

and

$$|t_{i_j}| < 1/i \qquad \text{for all} \quad j \geq j_i^t.$$

Let us choose $j_i := \max\{j_i^{\boldsymbol{y}}, j_i^t\}$. Then $t_{i_{j_i}} \downarrow 0$ and for all $i \geq i_0$

$$\|\boldsymbol{d} - \boldsymbol{d}_{i_{j_i}}\| \leq \|\boldsymbol{d} - \boldsymbol{d}_i\| + \|\boldsymbol{d}_i - \boldsymbol{d}_{i_{j_i}}\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which implies that $\boldsymbol{d}_{i_{j_i}} \to \boldsymbol{d}$ and, moreover, $\boldsymbol{x} + t_{i_{j_i}}\boldsymbol{d}_{i_{j_i}} \in S$. By the definition of the contingent cone, this means that $\boldsymbol{d} \in K_S(\boldsymbol{x})$ and thus $K_S(\boldsymbol{x})$ is closed.

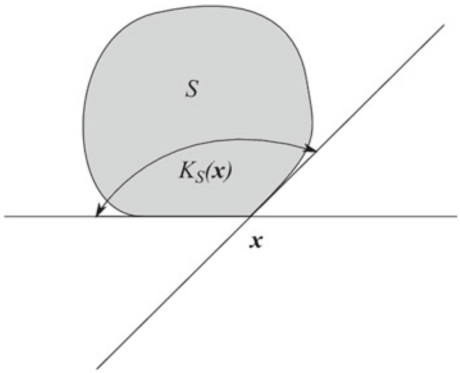


Fig. 2.8 Contingent cone $K_S(\boldsymbol{x})$ of a convex set

We continue by proving that $K_S(\boldsymbol{x})$ is a cone. If $\boldsymbol{d} \in K_S(\boldsymbol{x})$ is arbitrary then there exist sequences $(\boldsymbol{d}_j) \subset \mathbb{R}^n$ and $(t_j) \subset \mathbb{R}$ such that $\boldsymbol{d}_j \to \boldsymbol{d}$, $t_j \downarrow 0$ and $\boldsymbol{x}+t_j\boldsymbol{d}_j \in S$ for all $j \in \mathbb{N}$. Let $\lambda > 0$ be fixed and define $\boldsymbol{d}'_j := \lambda \boldsymbol{d}_j$ and $t'_j := t_j/\lambda$. Since $t'_j \downarrow 0$,

$$\|\boldsymbol{d}'_j - \lambda\boldsymbol{d}\| = \lambda\|\boldsymbol{d}_j - \boldsymbol{y}\| \longrightarrow 0 \quad \text{whenever} \quad j \to \infty$$

and

$$\boldsymbol{x} + t'_j\boldsymbol{d}'_j = \boldsymbol{x} + \frac{t_j}{\lambda} \cdot \lambda\boldsymbol{d}_j \in S$$

it follows that $\lambda\boldsymbol{d} \in K_S(\boldsymbol{x})$. Thus $K_S(\boldsymbol{x})$ is a cone.

For convexity let $\lambda \in [0, 1]$ and $\boldsymbol{d}^1, \boldsymbol{d}^2 \in K_S(\boldsymbol{x})$. We need to show that $\boldsymbol{d} := (1-\lambda)\boldsymbol{d}^1 + \lambda\boldsymbol{d}^2$ belongs to $K_S(\boldsymbol{x})$. By the definition of $K_S(\boldsymbol{x})$ there exist sequences $(\boldsymbol{d}^1_j), (\boldsymbol{d}^2_j) \subset \mathbb{R}^n$ and $(t^1_j), (t^2_j) \subset \mathbb{R}$ such that $\boldsymbol{d}^i_j \to \boldsymbol{d}^i$, $t^i_j \downarrow 0$ and $\boldsymbol{x} + t^i_j\boldsymbol{d}^i_j \in S$ for all $j \in \mathbb{N}$ and $i = 1, 2$. Define

$$\boldsymbol{d}_j := (1-\lambda)\boldsymbol{d}^1_j + \lambda\boldsymbol{d}^2_j \qquad \text{and} \qquad t_j := \min\{t^1_j, t^2_j\}.$$

Then we have

$$\boldsymbol{x} + t_j\boldsymbol{d}_j = (1-\lambda)(\boldsymbol{x} + t_j\boldsymbol{d}^1_j) + \lambda(\boldsymbol{x} + t_j\boldsymbol{d}^2_j) \in S$$

because S is convex and

$$\boldsymbol{x} + t_j\boldsymbol{d}^i_j = (1 - \frac{t_j}{t^i_j})\boldsymbol{x} + \frac{t_j}{t^i_j}(\boldsymbol{x} + t^i_j\boldsymbol{d}^i_j) \in S$$

because $\frac{t_j}{t^i_j} \in [0, 1]$ and S is convex. Moreover, we have

$$\begin{aligned}\|\boldsymbol{d}_j - \boldsymbol{d}\| &= \|(1-\lambda)\boldsymbol{d}^1_j + \lambda\boldsymbol{d}^2_j - (1-\lambda)\boldsymbol{d}^1 - \lambda\boldsymbol{d}^2\| \\ &\le (1-\lambda)\|\boldsymbol{d}^1_j - \boldsymbol{d}^1\| + \lambda\|\boldsymbol{d}^2_j - \boldsymbol{d}^2\| \longrightarrow 0,\end{aligned}$$

when $j \to \infty$, in other words $\boldsymbol{d}_j \to \boldsymbol{d}$. Since $t_j \downarrow 0$ we have $\boldsymbol{d} \in K_S(\boldsymbol{x})$ and thus $K_S(\boldsymbol{x})$ is convex. □

The following cone of feasible directions is very useful in optimization when seeking for feasible search directions.

Definition 2.11 The *cone of globally feasible directions* of the nonempty set S at $\boldsymbol{x} \in S$ is given by the formula

$$G_S(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid \text{there exists } t > 0 \text{ such that } \boldsymbol{x} + t\boldsymbol{d} \in S\}.$$

The cone of globally feasible directions has the same properties as the contingent cone but it is not necessarily closed. The proof of the next theorem is very similar to that of Theorem 2.15 and it is left as an exercise.

Theorem 2.16 *The cone of globally feasible directions $G_S(\boldsymbol{x})$ of the nonempty convex set S at $\boldsymbol{x} \in S$ is a convex cone.*

Proof Exercise. □

We have the following connection between the contingent cone and the cone of feasible directions.

Theorem 2.17 *If S is a nonempty set and $\boldsymbol{x} \in S$, then*

$$K_S(\boldsymbol{x}) \subseteq \text{cl } G_S(\boldsymbol{x}).$$

If, in addition, S is convex then

$$K_S(\boldsymbol{x}) = \text{cl } G_S(\boldsymbol{x}).$$

Proof If $\boldsymbol{d} \in K_S(\boldsymbol{x})$ is arbitrary, then there exist sequences $\boldsymbol{d}_j \to \boldsymbol{d}$ and $t_j \downarrow 0$ such that $\boldsymbol{x} + t_j \boldsymbol{d}_j \in S$ for all $j \in \mathbb{N}$, thus $\boldsymbol{d} \in \text{cl } G_S(\boldsymbol{x})$.

To see the equality, let S be convex and $\boldsymbol{d} \in \text{cl } G_S(\boldsymbol{x})$. Then there exist sequences $\boldsymbol{d}_j \to \boldsymbol{d}$ and $t_j > 0$ such that $\boldsymbol{x} + t_j \boldsymbol{d}_j \in S$ for all $j \in \mathbb{N}$. It suffices now to find a sequence t'_j such that $t'_j \downarrow 0$ and $\boldsymbol{x} + t'_j \boldsymbol{d}_j \in S$. Choose $t'_j := \min \{\frac{1}{j}, t_j\}$, which implies that

$$|t'_j| \leq \frac{1}{j} \longrightarrow 0$$

and by the convexity of S it follows that

$$\boldsymbol{x} + t'_j \boldsymbol{d}_j = (1 - \frac{t'_j}{t_j})\boldsymbol{x} + \frac{t'_j}{t_j}(\boldsymbol{x} + t_j \boldsymbol{d}_j) \in S,$$

which proves the assertion. □

Next we shall define the concept of normal cone (see Fig. 2.9). As we already have the contingent cone, it is natural to use polarity to define the normal vectors.

Definition 2.12 The *normal cone* of the nonempty set S at $\boldsymbol{x} \in S$ is the set

$$N_S(\boldsymbol{x}) := K_S(\boldsymbol{x})^\circ = \{\boldsymbol{z} \in \mathbb{R}^n \mid \boldsymbol{z}^T \boldsymbol{d} \leq 0 \text{ for all } \boldsymbol{d} \in K_S(\boldsymbol{x})\}. \tag{2.15}$$

The elements of $N_S(\boldsymbol{x})$ are called *normal vectors*.

The natural corollary of the polarity is that the normal cone has the same properties as the contingent cone.

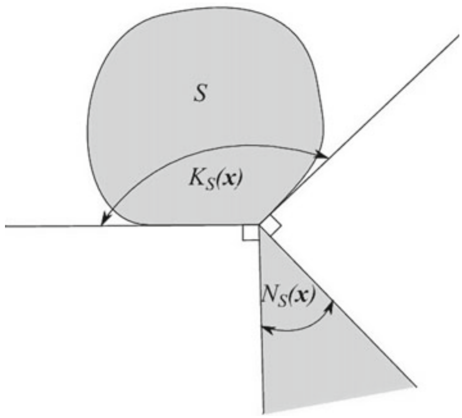


Fig. 2.9 Contingent and normal cones of a convex set

Theorem 2.18 *The normal cone $N_S(\boldsymbol{x})$ of the nonempty convex set S at $\boldsymbol{x} \in S$ is a closed convex cone.*

Proof Follows directly from Lemma 2.5. □

Notice that if $\boldsymbol{x} \in \text{int } S$, then clearly $K_S(\boldsymbol{x}) = \mathbb{R}^n$ and $N_S(\boldsymbol{x}) = \emptyset$. Thus the only interesting cases are those when $\boldsymbol{x} \in \text{bd } S$.

Next we present the following alternative characterization to the normal cone.

Theorem 2.19 *The normal cone of the nonempty convex set S at $\boldsymbol{x} \in S$ can also be written as follows*

$$N_S(\boldsymbol{x}) = \{\boldsymbol{z} \in \mathbb{R}^n \mid \boldsymbol{z}^T(\boldsymbol{y} - \boldsymbol{x}) \leq 0 \ \ for \ all \, \boldsymbol{y} \in S\}. \tag{2.16}$$

Proof Let us denote

$$Z := \{\boldsymbol{z} \in \mathbb{R}^n \mid \boldsymbol{z}^T(\boldsymbol{y} - \boldsymbol{x})^T \leq 0 \text{ for all } \boldsymbol{y} \in S\}.$$

If $\boldsymbol{z} \in N_S(\boldsymbol{x})$ is an arbitrary point, then by the definition of the normal cone we have

$$\boldsymbol{z}^T\boldsymbol{d} \leq 0 \quad \text{for all} \quad \boldsymbol{d} \in K_S(\boldsymbol{x}).$$

Now let $\boldsymbol{y} \in S$, set $\boldsymbol{d} := \boldsymbol{y} - \boldsymbol{x}$ and choose $t := 1$. Then

$$\boldsymbol{x} + t\boldsymbol{d} = \boldsymbol{x} + t\boldsymbol{y} - t\boldsymbol{x} = \boldsymbol{y} \in S,$$

thus $\boldsymbol{d} \in G_S(\boldsymbol{x}) \subseteq \text{cl } G_S(\boldsymbol{x}) = K_S(\boldsymbol{x})$ by Theorem 2.17. Since $\boldsymbol{z} \in N_S(\boldsymbol{x})$ one has

$$\boldsymbol{z}^T(\boldsymbol{y} - \boldsymbol{x})^T = \boldsymbol{z}^T\boldsymbol{d} \leq 0,$$

thus $\boldsymbol{z} \in Z$ and we have $N_S(\boldsymbol{x}) \subseteq Z$.

On the other hand, if $\boldsymbol{z} \in Z$ and $\boldsymbol{d} \in K_S(\boldsymbol{x})$ then there exist sequences $(\boldsymbol{d}_j) \subset \mathbb{R}^n$ and $(t_j) \subset \mathbb{R}$ such that $\boldsymbol{d}_j \to \boldsymbol{d}$, $t_j > 0$ and $\boldsymbol{x} + t_j \boldsymbol{d}_j \in S$ for all $j \in \mathbb{N}$. Let us set $\boldsymbol{y}_j := \boldsymbol{x} + t_j \boldsymbol{d}_j \in S$. Since $\boldsymbol{z} \in Z$ we have

$$t_j \boldsymbol{z}^T \boldsymbol{d}_j = \boldsymbol{z}^T (\boldsymbol{y}_j - \boldsymbol{x}) \leq 0.$$

Because t_j is positive, it implies that $\boldsymbol{z}^T \boldsymbol{d}_j \leq 0$ for all $j \in \mathbb{N}$. Then

$$\begin{aligned} \boldsymbol{z}^T \boldsymbol{d} &= \boldsymbol{z}^T \boldsymbol{d}_j + \boldsymbol{z}^T (\boldsymbol{d} - \boldsymbol{d}_j) \\ &\leq \|\boldsymbol{z}\| \, \|\boldsymbol{d} - \boldsymbol{d}_j\|, \end{aligned}$$

where $\|\boldsymbol{d} - \boldsymbol{d}_j\| \longrightarrow 0$ as $j \to \infty$. This means that

$$\boldsymbol{z}^T \boldsymbol{d} \leq 0 \quad \text{for all} \quad \boldsymbol{d} \in K_S(\boldsymbol{x}).$$

In other words, we have $\boldsymbol{z} \in N_S(\boldsymbol{x})$ and thus $Z \subseteq N_S(\boldsymbol{x})$, which completes the proof. □

The main difference between the groups of cones ray S, cone S, S° and $K_S(\boldsymbol{x})$, $G_S(\boldsymbol{x})$, $N_S(\boldsymbol{x})$ is, that the origin is the vertex point of the cone in the first group and the point $\boldsymbol{x} \in S$ in the second group. If we shift $\boldsymbol{x}$ to the origin, we get the following connections between these two groups.

Theorem 2.20 *If S is a nonempty convex set such that $\mathbf{0} \in S$, then*

(i) $G_S(\mathbf{0}) = \text{ray } S$,
(ii) $K_S(\mathbf{0}) = \text{cl ray } S$,
(iii) $N_S(\mathbf{0}) = S^\circ$.

Proof Exercise. □

2.2 Convex Functions

A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1 - \lambda) f(\boldsymbol{y}) \tag{2.17}$$

whenever $\boldsymbol{x}$ and $\boldsymbol{y}$ are in $\mathbb{R}^n$ and $\lambda \in [0, 1]$. If a strict inequality holds in (2.17) for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{x} \neq \boldsymbol{y}$ and $\lambda \in (0, 1)$, the function f is said to be *strictly convex*. A function $f: \mathbb{R}^n \to \mathbb{R}$ is *(strictly) concave* if $-f$ is (strictly) convex (see Fig. 2.10).

Next we give an equivalent definition of a convex function.

Theorem 2.21 *(Jensen's inequality) A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if and only if*

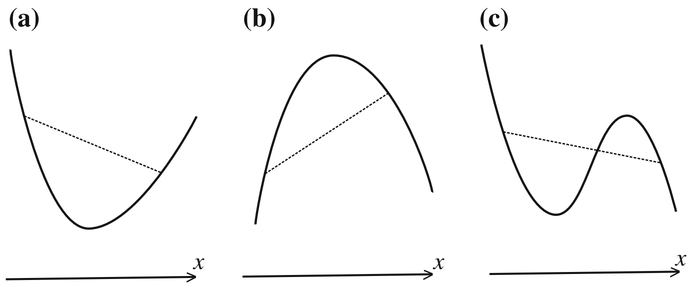


Fig. 2.10 Examples of different functions. (**a**) Convex. (**b**) Concave. (**c**) Neither convex or concave

$$f\left(\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i\right) \leq \sum_{i=1}^{m} \lambda_i f(\boldsymbol{x}_i), \tag{2.18}$$

whenever $\boldsymbol{x}_i \in \mathbb{R}^n$, $\lambda_i \in [0, 1]$ *for all* $i = 1, \ldots, m$ *and* $\sum_{i=1}^{m} \lambda_i = 1$.

Proof Follows by induction from the definition of convex function. □

Next we show that a convex function is always locally Lipschitz continuous.

Theorem 2.22 *Let* $f \colon \mathbb{R}^n \to \mathbb{R}$ *be a convex function. Then for any* $\boldsymbol{x}$ *in* $\mathbb{R}^n$, f *is locally Lipschitz continuous at* $\boldsymbol{x}$.

Proof Let $\boldsymbol{u} \in \mathbb{R}^n$ be arbitrary. We begin by proving that f is bounded on a neighborhood of $\boldsymbol{u}$. Let $\varepsilon > 0$ and define the hypercube

$$S_\varepsilon := \{\boldsymbol{y} \in \mathbb{R}^n \mid |y_i - u_i| \leq \varepsilon \text{ for all } i = 1, \ldots, n\}.$$

Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ denote the $m = 2^n$ vertices of S_ε and let

$$M := \max\{f(\boldsymbol{u}_i) \mid i = 1, \ldots, m\}.$$

Since each $\boldsymbol{y} \in S_\varepsilon$ can be expressed as $\boldsymbol{y} = \sum_{i=1}^{m} \lambda_i \boldsymbol{u}_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$, by Theorem 2.21. we obtain

$$f(\boldsymbol{y}) = f\left(\sum_{i=1}^{m} \lambda_i \boldsymbol{u}_i\right) \leq \sum_{i=1}^{m} \lambda_i f(\boldsymbol{u}_i) \leq M \sum_{i=1}^{m} \lambda_i = M.$$

Since $B(\boldsymbol{u}; \varepsilon) \subset S_\varepsilon$, we have an upper bound M of f on an ε-neighborhood of $\boldsymbol{u}$, that is

$$f(\boldsymbol{x}') \leq M \quad \text{for all} \quad \boldsymbol{x}' \in B(\boldsymbol{u}; \varepsilon).$$

Now let $\boldsymbol{x} \in \mathbb{R}^n$, choose $\rho > 1$ and $\boldsymbol{y} \in \mathbb{R}^n$ so that $\boldsymbol{y} = \rho \boldsymbol{x}$. Define

$$\lambda := 1/\rho \quad \text{and}$$
$$V := \{\boldsymbol{v} \mid \boldsymbol{v} = (1-\lambda)(\boldsymbol{x}' - \boldsymbol{u}) + \boldsymbol{x}, \text{ where } \boldsymbol{x}' \in B(\boldsymbol{u}; \varepsilon)\}.$$

The set V is a neighborhood of $\boldsymbol{x} = \lambda \boldsymbol{y}$ with radius $(1-\lambda)\varepsilon$. By convexity one has for all $\boldsymbol{v} \in V$

$$\begin{aligned} f(\boldsymbol{v}) &= f((1-\lambda)(\boldsymbol{x}' - \boldsymbol{u}) + \lambda \boldsymbol{y}) \\ &= f((1-\lambda)\boldsymbol{x}' + \lambda(\boldsymbol{y} + \boldsymbol{u} - \tfrac{1}{\lambda}\boldsymbol{u})) \\ &\leq (1-\lambda) f(\boldsymbol{x}') + \lambda f(\boldsymbol{y} + \boldsymbol{u} - \tfrac{1}{\lambda}\boldsymbol{u}). \end{aligned}$$

Now $f(\boldsymbol{x}') \leq M$ and $f(\boldsymbol{y} + \boldsymbol{u} - \frac{1}{\lambda}\boldsymbol{u}) = \text{constant} =: K$ and thus

$$f(\boldsymbol{v}) \leq M + \lambda K.$$

In other words, f is bounded above on a neighborhood of $\boldsymbol{x}$.

Let us next show that f is also bounded below. Let $\boldsymbol{z} \in B(\boldsymbol{x}; (1-\lambda)\varepsilon)$ and define $\boldsymbol{z}' := 2\boldsymbol{x} - \boldsymbol{z}$. Then

$$\|\boldsymbol{z}' - \boldsymbol{x}\| = \|\boldsymbol{x} - \boldsymbol{z}\| \leq (1-\lambda)\varepsilon.$$

Thus $\boldsymbol{z}' \in B(\boldsymbol{x}; (1-\lambda)\varepsilon)$ and $\boldsymbol{x} = (\boldsymbol{z} + \boldsymbol{z}')/2$. The convexity of f implies that

$$f(\boldsymbol{x}) = f((\boldsymbol{z} + \boldsymbol{z}')/2) \leq \tfrac{1}{2} f(\boldsymbol{z}) + \tfrac{1}{2} f(\boldsymbol{z}'),$$

and

$$f(\boldsymbol{z}) \geq 2f(\boldsymbol{x}) - f(\boldsymbol{z}') \geq 2f(\boldsymbol{x}) - M - \lambda K$$

so that f is also bounded below on a neighborhood of $\boldsymbol{x}$. Thus we have proved that f is bounded on a neighborhood of $\boldsymbol{x}$.

Let $N > 0$ be a bound of $|f|$ so that

$$|f(\boldsymbol{x}')| \leq N \quad \text{for all} \quad \boldsymbol{x}' \in B(\boldsymbol{x}; 2\delta),$$

where $\delta > 0$, and let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\boldsymbol{x}; \delta)$ with $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. Define

$$\boldsymbol{x}_3 := \boldsymbol{x}_2 + (\delta/\alpha)(\boldsymbol{x}_2 - \boldsymbol{x}_1),$$

where $\alpha := \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$. Then

$$\begin{aligned}\|\boldsymbol{x}_3 - \boldsymbol{x}\| &= \|\boldsymbol{x}_2 + (\delta/\alpha)(\boldsymbol{x}_2 - \boldsymbol{x}_1) - \boldsymbol{x}\| \\ &\le \|\boldsymbol{x}_2 - \boldsymbol{x}\| + (\delta/\alpha)\|\boldsymbol{x}_2 - \boldsymbol{x}_1\| \\ &< \delta + \frac{\delta}{\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\| \\ &= 2\delta,\end{aligned}$$

thus $\boldsymbol{x}_3 \in B(\boldsymbol{x}; 2\delta)$. Solving for $\boldsymbol{x}_2$ gives

$$\boldsymbol{x}_2 = \frac{\delta}{\alpha+\delta}\boldsymbol{x}_1 + \frac{\alpha}{\alpha+\delta}\boldsymbol{x}_3,$$

and by the convexity we get

$$f(\boldsymbol{x}_2) \le \frac{\delta}{\alpha+\delta} f(\boldsymbol{x}_1) + \frac{\alpha}{\alpha+\delta} f(\boldsymbol{x}_3).$$

Then

$$\begin{aligned}f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1) &\le \frac{\alpha}{\alpha+\delta}[f(\boldsymbol{x}_3) - f(\boldsymbol{x}_1)] \\ &\le \frac{\alpha}{\delta}|f(\boldsymbol{x}_3) - f(\boldsymbol{x}_1)| \\ &\le \frac{\alpha}{\delta}(|f(\boldsymbol{x}_3)| + |f(\boldsymbol{x}_1)|).\end{aligned}$$

Since $\boldsymbol{x}_1, \boldsymbol{x}_3 \in B(\boldsymbol{x}; 2\delta)$ we have $|f(\boldsymbol{x}_3)| < N$ and $|f(\boldsymbol{x}_1)| < N$, thus

$$f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1) \le \frac{2N}{\delta}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|.$$

By changing the roles of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we have

$$|f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1)| \le \frac{2N}{\delta}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|,$$

showing that the function f is locally Lipschitz continuous at $\boldsymbol{x}$. □

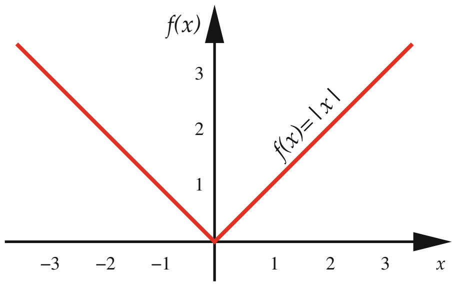


Fig. 2.11 Absolute-value function $f(x) = |x|$

The simplest example of nonsmooth function is the absolute-value function on reals (see Fig. 2.11).

Example 2.8 (*Absolute-value function*). Let us consider the absolute-value function

$$f(x) = |x|$$

on reals.

The gradient of function f is

$$\nabla f(x) = \begin{cases} 1, & \text{when } x > 0, \\ -1, & \text{when } x < 0. \end{cases}$$

Function f is not differentiable at $x = 0$.

We now show that function f is both convex and (locally) Lipschitz continuous. Let $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}$. By triangle inequality we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= |\lambda x + (1-\lambda)y| \\ &\le |\lambda x| + |(1-\lambda)y| \\ &= |\lambda||x| + |1-\lambda||y| \\ &= \lambda|x| + (1-\lambda)|y| \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

Thus, function f is convex. Furthermore, by triangle inequality we also have

$$|f(x) - f(y)| = ||x| - |y|| \le |x - y|$$

for all $x, y \in \mathbb{R}$. In space $\mathbb{R}$, the right-hand side equals to the norm $\|\boldsymbol{x} - \boldsymbol{y}\|$. Thus, we have the Lipschitz constant $K = 1 > 0$ and function f is Lipschitz continuous.

2.2.1 Level Sets and Epigraphs

Next we consider two sets, namely level sets and epigraphs, closely related to convex functions.

Definition 2.13 The *level set* of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with a parameter $\alpha \in \mathbb{R}$ is defined as

$$\operatorname{lev}_\alpha f := \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \leq \alpha\}.$$

We have the following connection between the convexity of functions and level sets.

Theorem 2.23 *If* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is a convex function, then the level set* $\operatorname{lev}_\alpha f$ *is a convex set for all* $\alpha \in \mathbb{R}$.

Proof If $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{lev}_\alpha f$ and $\lambda \in [0, 1]$ we have $f(\boldsymbol{x}) \leq \alpha$ and $f(\boldsymbol{y}) \leq \alpha$. Let $\boldsymbol{z} := \lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$ with some $\lambda \in [0, 1]$. Then the convexity of f implies that

$$f(\boldsymbol{z}) \leq \lambda f(\boldsymbol{x}) + (1 - \lambda) f(\boldsymbol{y}) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha,$$

in other words $\boldsymbol{z} \in \operatorname{lev}_\alpha f$ and thus $\operatorname{lev}_\alpha f$ is convex. □

The previous result can not be inverted since there exist nonconvex functions with convex level sets (see Fig. 2.12). The equivalence can be achieved by replacing the level set with the so called epigraph being a subset of $\mathbb{R}^n \times \mathbb{R}$ (see Fig. 2.13).

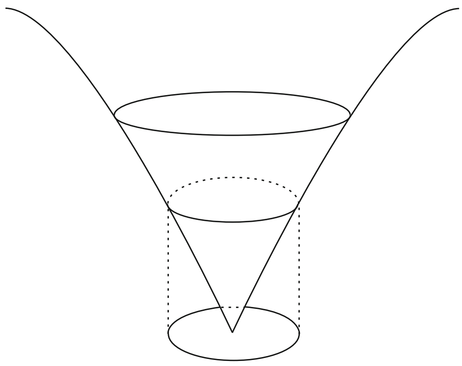


Fig. 2.12 Nonconvex function with convex level sets

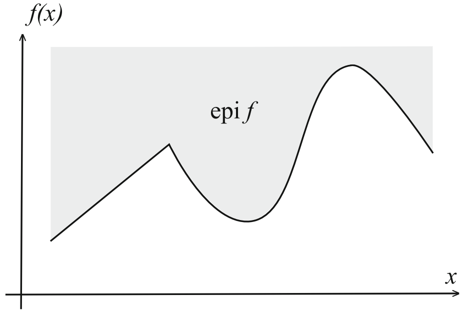


Fig. 2.13 Epigraph of the function

Definition 2.14 The *epigraph* of a function $f: \mathbb{R}^n \to \mathbb{R}$ is the following subset of $\mathbb{R}^n \times \mathbb{R}$:

$$\text{epi } f := \{(\boldsymbol{x}, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(\boldsymbol{x}) \leq r\}. \tag{2.19}$$

Theorem 2.24 *The function* $f: \mathbb{R}^n \to \mathbb{R}$ *is convex if and only if the epigraph* epi f *is a convex set.*

Proof Exercise. □

Notice, that we have the following connection between the epigraph and level sets of a function $f: \mathbb{R}^n \to \mathbb{R}$ at $\boldsymbol{x} \in \mathbb{R}^n$

$$\text{lev}_{f(\boldsymbol{x})} f = \{\boldsymbol{y} \in \mathbb{R}^n \mid (\boldsymbol{y}, f(\boldsymbol{x})) \in \text{epi } f\}.$$

2.2.2 Subgradients and Directional Derivatives

In this section we shall generalize the classical notion of gradient for convex but not necessarily differentiable functions. Before that we consider some properties related to the directional derivative of convex functions.

Theorem 2.25 *If* $f: \mathbb{R}^n \to \mathbb{R}$ *is a convex function, then the directional derivative* $f'(\boldsymbol{x}; \boldsymbol{d})$ *exists in every direction* $\boldsymbol{d} \in \mathbb{R}^n$ *and it satisfies*

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \inf_{t>0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t}. \tag{2.20}$$

Proof Let $\boldsymbol{d} \in \mathbb{R}^n$ be an arbitrary direction. Define $\varphi: \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) := \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t}.$$

We begin by proving that φ is bounded below at t when $t \downarrow 0$. Let $\varepsilon > 0$ and let constants t_1 and t_2 be such that $0 < t_1 < t_2 < \varepsilon$. By the convexity of f we have

$$\begin{aligned}\varphi(t_2) - \varphi(t_1) &= \frac{1}{t_1 t_2}[t_1 f(\boldsymbol{x} + t_2\boldsymbol{d}) - t_2 f(\boldsymbol{x} + t_1\boldsymbol{d}) + (t_2 - t_1)f(\boldsymbol{x})] \\ &= \frac{1}{t_1}\Big\{\big(\frac{t_1}{t_2} f(\boldsymbol{x} + t_2\boldsymbol{d}) + (1 - \frac{t_1}{t_2})f(\boldsymbol{x})\big) \\ &\quad - f\big(\frac{t_1}{t_2}(\boldsymbol{x} + t_2\boldsymbol{d}) + (1 - \frac{t_1}{t_2})\boldsymbol{x}\big)\Big\} \\ &\geq 0,\end{aligned}$$

thus the function $\varphi(t)$ decreases as $t \downarrow 0$. Then for all $0 < t < \varepsilon$ one has

$$\begin{aligned}\varphi(t) - \varphi(-\varepsilon/2) &= \frac{\frac{1}{2}f(\boldsymbol{x} + t\boldsymbol{d}) + \frac{1}{2}f(\boldsymbol{x}) + \frac{t}{\varepsilon}f(\boldsymbol{x} - \frac{\varepsilon}{2}\boldsymbol{d}) + (1 - \frac{t}{\varepsilon})f(\boldsymbol{x}) - 2f(\boldsymbol{x})}{t/2} \\ &\geq \frac{\frac{1}{2}f(\boldsymbol{x} + \frac{t}{2}\boldsymbol{d}) + \frac{1}{2}f(\boldsymbol{x} - \frac{t}{2}\boldsymbol{d}) - f(\boldsymbol{x})}{t/4} \\ &\geq \frac{f(\boldsymbol{x}) - f(\boldsymbol{x})}{t/4} = 0,\end{aligned}$$

which means that the function φ is bounded below for $0 < t < \varepsilon$. This implies that there exists the limit

$$\lim_{t\downarrow 0} \varphi(t) = f'(\boldsymbol{x}; \boldsymbol{d}) \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n$$

and since the function $\varphi(t)$ decreases as $t \downarrow 0$ we deduce that

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \inf_{t>0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t}.$$

□

Theorem 2.26 *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a convex function with a Lipschitz constant* K *at* $\boldsymbol{x} \in \mathbb{R}^n$. *Then the function* $\boldsymbol{d} \mapsto f'(\boldsymbol{x}; \boldsymbol{d})$ *is positively homogeneous and subadditive on* $\mathbb{R}^n$ *with*

$$|f'(\boldsymbol{x}; \boldsymbol{d})| \leq K\|\boldsymbol{d}\|.$$

Proof We start by proving the inequality. From the Lipschitz condition we obtain

$$\begin{aligned}|f'(\boldsymbol{x}; \boldsymbol{d})| &\leq \lim_{t\downarrow 0} \frac{|f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})|}{t} \\ &\leq \lim_{t\downarrow 0} \frac{K\|\boldsymbol{x} + t\boldsymbol{d} - \boldsymbol{x}\|}{t} \\ &\leq K\|\boldsymbol{d}\|.\end{aligned}$$

Next we show that $f'(\boldsymbol{x};\cdot)$ is positively homogeneous. To see this, let $\lambda > 0$. Then

$$\begin{aligned} f'(\boldsymbol{x};\lambda\boldsymbol{d}) &= \lim_{t\downarrow 0}\frac{f(\boldsymbol{x}+t\lambda\boldsymbol{d})-f(\boldsymbol{x})}{t} \\ &= \lim_{t\downarrow 0}\lambda\cdot\left\{\frac{f(\boldsymbol{x}+t\lambda\boldsymbol{d})-f(\boldsymbol{x})}{t\lambda}\right\} \\ &= \lambda\cdot\lim_{t\downarrow 0}\frac{f(\boldsymbol{x}+t\lambda\boldsymbol{d})-f(\boldsymbol{x})}{t\lambda} \\ &= \lambda\cdot f'(\boldsymbol{x};\boldsymbol{d}). \end{aligned}$$

We turn now to the subadditivity. Let $\boldsymbol{d},\boldsymbol{p}\in\mathbb{R}^n$ be arbitrary directions, then by convexity

$$\begin{aligned} f'(\boldsymbol{x};\boldsymbol{d}+\boldsymbol{p}) &= \lim_{t\downarrow 0}\frac{f(\boldsymbol{x}+t(\boldsymbol{d}+\boldsymbol{p}))-f(\boldsymbol{x})}{t} \\ &= \lim_{t\downarrow 0}\frac{f(\frac{1}{2}(\boldsymbol{x}+2t\boldsymbol{d})+\frac{1}{2}(\boldsymbol{x}+2t\boldsymbol{p}))-f(\boldsymbol{x})}{t} \\ &\le \lim_{t\downarrow 0}\frac{f(\boldsymbol{x}+2t\boldsymbol{d})-f(\boldsymbol{x})}{2t}+\lim_{t\downarrow 0}\frac{f(\boldsymbol{x}+2t\boldsymbol{p})-f(\boldsymbol{x})}{2t} \\ &= f'(\boldsymbol{x};\boldsymbol{d})+f'(\boldsymbol{x};\boldsymbol{p}). \end{aligned}$$

Thus $\boldsymbol{d}\mapsto f'(\boldsymbol{x};\boldsymbol{d})$ is subadditive. □

From the previous theorem we derive the following consequence.

Corollary 2.3 *If* $f:\mathbb{R}^n\to\mathbb{R}$ *is a convex function, then the function* $\boldsymbol{d}\mapsto f'(\boldsymbol{x};\boldsymbol{d})$ *is convex, its epigraph* epi $f'(\boldsymbol{x};\cdot)$ *is a convex cone and we have*

$$f'(\boldsymbol{x};-\boldsymbol{d})\ge -f'(\boldsymbol{x};\boldsymbol{d}) \quad \textit{for all } \boldsymbol{x}\in\mathbb{R}^n.$$

Proof Exercise. □

Next we define the subgradient and the subdifferential of a convex function. Note the analogy to the smooth differential theory, namely if a function $f:\mathbb{R}^n\to\mathbb{R}$ is both convex and differentiable, then for all $\boldsymbol{y}\in\mathbb{R}^n$ we have

$$f(\boldsymbol{y})\ge f(\boldsymbol{x})+\nabla f(\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}).$$

Figure 2.14 illustrates the meaning of the definition of the subdifferential.

Definition 2.15 The *subdifferential of a convex function* $f:\mathbb{R}^n\to\mathbb{R}$ at $\boldsymbol{x}\in\mathbb{R}^n$ is the set $\partial_c f(\boldsymbol{x})$ of vectors $\boldsymbol{\xi}\in\mathbb{R}^n$ such that

$$\partial_c f(\boldsymbol{x})=\left\{\boldsymbol{\xi}\in\mathbb{R}^n \mid f(\boldsymbol{y})\ge f(\boldsymbol{x})+\boldsymbol{\xi}^T(\boldsymbol{y}-\boldsymbol{x}) \text{ for all } \boldsymbol{y}\in\mathbb{R}^n\right\}.$$

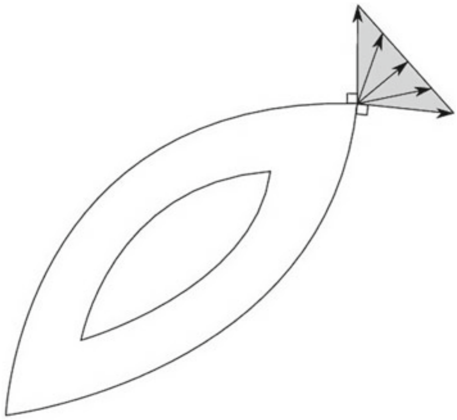


Fig. 2.14 Subdifferential

Each vector $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$ is called a *subgradient* of f at $\boldsymbol{x}$.

Example 2.9 (*Absolute-value function*). As noted in Example 2.8 function $f(x) = |x|$ is convex and differentiable when $x \neq 0$. By the definition of subdifferential we have

$$\begin{aligned}\xi \in \partial_c f(0) &\iff |y| \geq |0| + \xi \cdot (y - 0) \quad \text{for all } y \in \mathbb{R}\\ &\iff |y| \geq \xi \cdot y \quad \text{for all } y \in \mathbb{R}\\ &\iff \xi \leq 1 \quad \text{and} \quad \xi \geq -1.\end{aligned}$$

Thus, $\partial_c f(0) = [-1, 1]$.

Theorem 2.27 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function with a Lipschitz constant K at $\boldsymbol{x}^* \in \mathbb{R}^n$. Then the subdifferential $\partial_c f(\boldsymbol{x}^*)$ is a nonempty, convex, and compact set such that*

$$\partial_c f(\boldsymbol{x}^*) \subseteq B(\mathbf{0}; K).$$

Proof We show first that there exists a subgradient $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)$, in other words $\partial_c f(\boldsymbol{x}^*)$ is nonempty. By Theorem 2.24 epi f is a convex set and by Theorem 2.22 and Exercise 2.29 it is closed. Since $(\boldsymbol{x}^*, f(\boldsymbol{x}^*)) \in$ epi f it is also nonempty, furthermore we have $(\boldsymbol{x}^*, f(\boldsymbol{x}^*)) \in$ bd epi f. Then due to Theorem 2.7 there exists a hyperplane supporting epi f at $(\boldsymbol{x}^*, f(\boldsymbol{x}^*))$. In other words there exists $(\boldsymbol{\xi}^*, \mu) \neq (\mathbf{0}, 0)$ where $\boldsymbol{\xi}^* \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ such that for all $(\boldsymbol{x}, r) \in$ epi f we have

$$(\boldsymbol{\xi}^*, \mu)^T \big((\boldsymbol{x}, r) - (\boldsymbol{x}^*, f(\boldsymbol{x}^*))\big) = (\boldsymbol{\xi}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) + \mu(r - f(\boldsymbol{x}^*)) \leq 0. \quad (2.21)$$

In the above inequality r can be chosen as large as possible, thus μ must be nonpositive. If $\mu = 0$ then (2.21) reduces to

$$(\boldsymbol{\xi}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \leq 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

If we choose $\boldsymbol{x} := \boldsymbol{x}^* + \boldsymbol{\xi}^*$ we get $(\boldsymbol{\xi}^*)^T\boldsymbol{\xi}^* = \|\boldsymbol{\xi}^*\|^2 \leq 0$. This means that $\boldsymbol{\xi}^* = \mathbf{0}$, which is impossible because $(\boldsymbol{\xi}^*, \mu) \neq (\mathbf{0}, 0)$, thus we have $\mu < 0$. Dividing the inequality (2.21) by $|\mu|$ and noting $\boldsymbol{\xi} := \boldsymbol{\xi}^*/|\mu|$ we get

$$\boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{x}^*) - r + f(\boldsymbol{x}^*) \leq 0 \quad \text{for all } (\boldsymbol{x}, r) \in \text{epi } f.$$

If we choose now $r := f(\boldsymbol{x})$ we get

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{x}^*) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n,$$

which means that $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)$.

To see the convexity let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \partial_c f(\boldsymbol{x}^*)$ and $\lambda \in [0, 1]$. Then we have

$$\begin{aligned} f(\boldsymbol{y}) &\geq f(\boldsymbol{x}^*) + \boldsymbol{\xi}_1^T(\boldsymbol{y} - \boldsymbol{x}^*) \quad \text{for all } \boldsymbol{y} \in \mathbb{R}^n \quad \text{and} \\ f(\boldsymbol{y}) &\geq f(\boldsymbol{x}^*) + \boldsymbol{\xi}_2^T(\boldsymbol{y} - \boldsymbol{x}^*) \quad \text{for all } \boldsymbol{y} \in \mathbb{R}^n. \end{aligned}$$

Multiplying the above two inequalities by λ and $(1 - \lambda)$, respectively, and adding them together, we obtain

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}^*) + \big(\lambda\boldsymbol{\xi}_1 + (1 - \lambda)\boldsymbol{\xi}_2\big)^T(\boldsymbol{y} - \boldsymbol{x}^*) \quad \text{for all } \boldsymbol{y} \in \mathbb{R}^n,$$

in other words

$$\lambda\boldsymbol{\xi}_1 + (1 - \lambda)\boldsymbol{\xi}_2 \in \partial_c f(\boldsymbol{x}^*)$$

and thus $\partial_c f(\boldsymbol{x}^*)$ is convex.

If $\boldsymbol{d} \in \mathbb{R}^n$ we get from the definition of the subdifferential

$$\varphi(t) := \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t} \geq \frac{\boldsymbol{\xi}^T(t\boldsymbol{d})}{t} = \boldsymbol{\xi}^T\boldsymbol{d} \quad \text{for all } \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*).$$

Since $\varphi(t) \to f'(\boldsymbol{x}^*; \boldsymbol{d})$ when $t \downarrow 0$ we obtain

$$f'(\boldsymbol{x}^*; \boldsymbol{d}) \geq \boldsymbol{\xi}^T\boldsymbol{d} \quad \text{for all } \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*). \tag{2.22}$$

Thus for an arbitrary $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)$ we get

$$\|\boldsymbol{\xi}\|^2 = |\boldsymbol{\xi}^T\boldsymbol{\xi}| \leq |f'(\boldsymbol{x}^*; \boldsymbol{\xi})| \leq K\|\boldsymbol{\xi}\|$$

by Theorem 2.26. This means that $\partial_c f(\boldsymbol{x}^*)$ is bounded and we have

$$\partial_c f(\boldsymbol{x}^*) \subseteq B(\mathbf{0}; K).$$

Thus, for compactness it suffices to show that $\partial_c f(\boldsymbol{x}^*)$ is closed. To see this let $(\boldsymbol{\xi}_i) \subset \partial_c f(\boldsymbol{x}^*)$ such that $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$. Then for all $\boldsymbol{y} \in \mathbb{R}^n$ we have

$$f(\boldsymbol{y}) - f(\boldsymbol{x}^*) \geq \boldsymbol{\xi}_i^T(\boldsymbol{y} - \boldsymbol{x}) \to \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}),$$

whenever $i \to \infty$, thus $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)$ and $\partial_c f(\boldsymbol{x}^*)$ is closed. □

The next theorem shows the relationship between the subdifferential and the directional derivative. It turns out that knowing $f'(\boldsymbol{x}; \boldsymbol{d})$ is equivalent to knowing $\partial_c f(\boldsymbol{x})$.

Theorem 2.28 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for all $\boldsymbol{x} \in \mathbb{R}^n$*

(i) $\partial_c f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f'(\boldsymbol{x}, \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d}$ *for all* $\boldsymbol{d} \in \mathbb{R}^n\}$, *and*
(ii) $f'(\boldsymbol{x}; \boldsymbol{d}) = \max\,\{\boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})\}$ *for any* $\boldsymbol{d} \in \mathbb{R}^n$.

Proof (i) Set

$$S := \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f'(\boldsymbol{x}; \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d} \text{ for all } \boldsymbol{d} \in \mathbb{R}^n\}$$

and let $\boldsymbol{\xi} \in S$ be arbitrary. Then it follows from convexity that, for all $\boldsymbol{d} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\boldsymbol{\xi}^T \boldsymbol{d} &\leq f'(\boldsymbol{x}; \boldsymbol{d}) \\
&= \lim_{t \downarrow 0} \frac{f\big((1-t)\boldsymbol{x} + t(\boldsymbol{x} + \boldsymbol{d})\big) - f(\boldsymbol{x})}{t} \\
&\leq \lim_{t \downarrow 0} \frac{(1-t)f(\boldsymbol{x}) + tf(\boldsymbol{x} + \boldsymbol{d}) - f(\boldsymbol{x})}{t} \\
&= f(\boldsymbol{x} + \boldsymbol{d}) - f(\boldsymbol{x}),
\end{aligned}$$

whenever $t \leq 1$. By choosing $\boldsymbol{d} := \boldsymbol{y} - \boldsymbol{x}$ we derive $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$. On the other hand, if $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$ then due to (2.22) we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d} \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

Thus $\boldsymbol{\xi} \in S$, which establishes (i).

(ii) First we state that since the subdifferential is compact and nonempty set (Theorem 2.27) the maximum of the linear function $\boldsymbol{d} \mapsto \boldsymbol{\xi}^T \boldsymbol{d}$ is well-defined due to the Weierstrass' Theorem 1.1. Again from (2.22) we deduce that for each $\boldsymbol{d} \in \mathbb{R}^n$ we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) \geq \max\,\{\boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})\}.$$

Suppose next that there were $\boldsymbol{d}^* \in \mathbb{R}^n$ for which

$$f'(\boldsymbol{x}; \boldsymbol{d}^*) > \max \{\boldsymbol{\xi}^T \boldsymbol{d}^* \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})\}. \tag{2.23}$$

By Corollary 2.3 function $\boldsymbol{d} \mapsto f'(\boldsymbol{x}; \boldsymbol{d})$ is convex and thus by Theorem 2.24 epi $f'(\boldsymbol{x}; \cdot)$ is a convex set and by Theorem 2.22 and Exercise 2.29 it is closed. Since $(\boldsymbol{d}^*, f'(\boldsymbol{x}; \boldsymbol{d}^*)) \in$ epi $f'(\boldsymbol{x}; \cdot)$ it is also nonempty, furthermore we have $(\boldsymbol{d}^*, f'(\boldsymbol{x}; \boldsymbol{d}^*)) \in$ bd epi $f'(\boldsymbol{x}; \cdot)$. Then due to Theorem 2.7 there exists a hyperplane supporting epi $f'(\boldsymbol{x}; \cdot)$ at $(\boldsymbol{d}^*, f'(\boldsymbol{x}; \boldsymbol{d}^*))$, in other words there exists $(\boldsymbol{\xi}^*, \mu) \neq (\boldsymbol{0}, 0)$ where $\boldsymbol{\xi}^* \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ such that for all $(\boldsymbol{d}, r) \in$ epi $f'(\boldsymbol{x}; \cdot)$ we have

$$\begin{aligned}(\boldsymbol{\xi}^*, \mu)^T \big((\boldsymbol{d}, r) - (\boldsymbol{d}^*, f'(\boldsymbol{x}; \boldsymbol{d}^*))\big) &= (\boldsymbol{\xi}^*)^T (\boldsymbol{d} - \boldsymbol{d}^*) + \mu(r - f'(\boldsymbol{x}; \boldsymbol{d}^*)) \\ &\leq 0.\end{aligned} \tag{2.24}$$

Just like in the proof of Theorem 2.27 we can deduce that $\mu < 0$. Again dividing the inequality (2.24) by $|\mu|$ and noting $\boldsymbol{\xi} := \boldsymbol{\xi}^*/|\mu|$ we get

$$\boldsymbol{\xi}^T (\boldsymbol{d} - \boldsymbol{d}^*) - r + f'(\boldsymbol{x}; \boldsymbol{d}^*) \leq 0 \quad \text{for all } (\boldsymbol{d}, r) \in \text{epi } f'(\boldsymbol{x}; \cdot).$$

If we choose now $r := f'(\boldsymbol{x}; \boldsymbol{d})$ we get

$$f'(\boldsymbol{x}; \boldsymbol{d}) - f'(\boldsymbol{x}; \boldsymbol{d}^*) \geq \boldsymbol{\xi}^T (\boldsymbol{d} - \boldsymbol{d}^*) \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n. \tag{2.25}$$

Then from the subadditivity of the directional derivative (Theorem 2.26) we obtain

$$f'(\boldsymbol{x}; \boldsymbol{d} - \boldsymbol{d}^*) \geq \boldsymbol{\xi}^T (\boldsymbol{d} - \boldsymbol{d}^*) \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n,$$

which by assertion (i) means that $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$. On the other hand from Eqs. (2.25) and (2.23) we get

$$f'(\boldsymbol{x}; \boldsymbol{d}) - \boldsymbol{\xi}^T \boldsymbol{d} \geq f'(\boldsymbol{x}; \boldsymbol{d}^*) - \boldsymbol{\xi}^T \boldsymbol{d}^* > 0 \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n,$$

in other words we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) > \boldsymbol{\xi}^T \boldsymbol{d} \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

Now by choosing $\boldsymbol{d} := \boldsymbol{0}$ we get '$0 > 0$', which is impossible, thus by the contradiction (2.23) is wrong and we have the equality

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \max \{\boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})\} \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n$$

and the proof is complete. □

Example 2.10 (*Absolute-value function*). By Theorem 2.28 (i) we have

$$\xi \in \partial_c f(0) \iff f'(0, d) \geq \xi \cdot d \quad \text{for all } d \in \mathbb{R}.$$

Now

$$f'(0, d) = \lim_{t \downarrow 0} \frac{|0 + td| - |0|}{t} = \lim_{t \downarrow 0} \frac{t|d|}{t} = |d|$$

and, thus,

$$\begin{aligned} \xi \in \partial_c f(0) &\iff |d| \geq \xi \cdot d \quad \text{for all } d \in \mathbb{R} \\ &\iff \xi \in [-1, 1]. \end{aligned}$$

The next theorem shows that the subgradients really are generalizations of the classical gradient.

Theorem 2.29 *If* $f: \mathbb{R}^n \to \mathbb{R}$ *is convex and differentiable at* $\boldsymbol{x} \in \mathbb{R}^n$*, then*

$$\partial_c f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}. \tag{2.26}$$

Proof According to Theorem 2.25 the directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ of a convex function exists in every direction $\boldsymbol{d} \in \mathbb{R}^n$. From the definition of differentiability we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \nabla f(\boldsymbol{x})^T \boldsymbol{d} \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n,$$

which implies, by Theorem 2.28 (i) that $\nabla f(\boldsymbol{x}) \in \partial_c f(\boldsymbol{x})$. Suppose next that there exists another $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$ such that $\boldsymbol{\xi} \neq \nabla f(\boldsymbol{x})$. Then by Theorem 2.28 (i) we have

$$\boldsymbol{\xi}^T \boldsymbol{d} \leq f'(\boldsymbol{x}; \boldsymbol{d}) = \nabla f(\boldsymbol{x})^T \boldsymbol{d} \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n,$$

in other words

$$(\boldsymbol{\xi} - \nabla f(\boldsymbol{x}))^T \boldsymbol{d} \leq 0 \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

By choosing $\boldsymbol{d} := \boldsymbol{\xi} - \nabla f(\boldsymbol{x})$ we get

$$\|\boldsymbol{\xi} - \nabla f(\boldsymbol{x}))\|^2 \leq 0,$$

implying that $\boldsymbol{\xi} = \nabla f(\boldsymbol{x})$, which contradicts the assumption. Thus

$$\partial_c f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}. \qquad \square$$

Example 2.11 (*Absolute-value function*). Let us define the whole subdifferential $\partial f(x)$ of the function $f(x) = |x|$. Function f is differentiable in everywhere except in $x = 0$, and

$$\nabla f(x) = \begin{cases} 1, & \text{when } x > 0 \\ -1, & \text{when } x < 0. \end{cases}$$

In Examples 2.9 and 2.10, we have computed the subdifferential at $x = 0$, that is, $\partial_c f(0) = [-1, 1]$. Thus, the subdifferential of f is

$$\partial f(x) = \begin{cases} \{-1\}, & \text{when } x < 0 \\ [-1, 1], & \text{when } x = 0 \\ \{1\}, & \text{when } x > 0 \end{cases}$$

(see also Fig. 2.15).

We are now ready to present a very useful result in developing optimization methods. It gives a representation to a convex function by using subgradients.

Theorem 2.30 *If* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is convex then for all* $\boldsymbol{y} \in \mathbb{R}^n$

$$f(\boldsymbol{y}) = \max\{f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})\}. \tag{2.27}$$

Proof Suppose that $\boldsymbol{y} \in \mathbb{R}^n$ is an arbitrary point and $\boldsymbol{\zeta} \in \partial f(\boldsymbol{y})$. Let

$$S := \{f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^n\}.$$

By the definition of subdifferential of a convex function we have

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n \text{and } \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$$

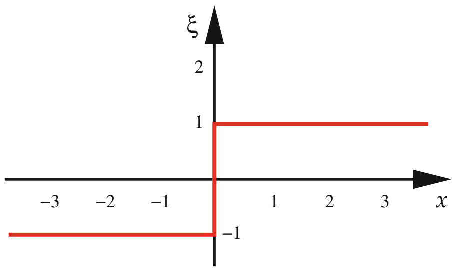


Fig. 2.15 Subdifferential $\partial_c f(x)$ of $f(x) = |x|$

implying that the set S is bounded from above and

$$\sup S \leq f(\boldsymbol{y}).$$

On the other hand, we have

$$f(\boldsymbol{y}) = f(\boldsymbol{y}) + \boldsymbol{\zeta}^T(\boldsymbol{y} - \boldsymbol{y}) \in S,$$

which means that $f(\boldsymbol{y}) \leq \sup\ S$. Thus

$$f(\boldsymbol{y}) = \max\,\{f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^n\}. \qquad \square$$

2.2.3 ε-Subdifferentials

In nonsmooth optimization, so called bundle methods are based on the concept of ε-subdifferential, which is an extension of the ordinary subdifferential. Therefore we now give the definition of ε-subdifferential and present some of its basic properties.

We start by generalizing the ordinary directional derivative. Note the analogy with the property (2.20).

Definition 2.16 Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. The *ε-directional derivative* of f at $\boldsymbol{x}$ in the direction $\boldsymbol{d} \in \mathbb{R}^n$ is defined by

$$f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d}) = \inf_{t>0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x}) + \varepsilon}{t}. \tag{2.28}$$

Now we can reach the same results as in Theorem 2.26 and Corollary 2.3 also for the ε-directional derivative.

Theorem 2.31 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function with a Lipschitz constant K at $\boldsymbol{x} \in \mathbb{R}^n$. Then the function $\boldsymbol{d} \mapsto f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d})$ is*

(i) *positively homogeneous and subadditive on $\mathbb{R}^n$ with*

$$|f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d})| \leq K\|\boldsymbol{d}\|,$$

(ii) *convex, its epigraph* epi $f'_\varepsilon(\boldsymbol{x};\ \cdot)$ *is a convex cone and*

$$f'_\varepsilon(\boldsymbol{x}; -\boldsymbol{d}) \geq -f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d}) \quad \textit{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

Proof These results follow immediately from Theorem 2.26, Corollary 2.3 and the fact that for all $\varepsilon > 0$ we have $\inf_{t>0} \varepsilon/t = 0$. $\square$

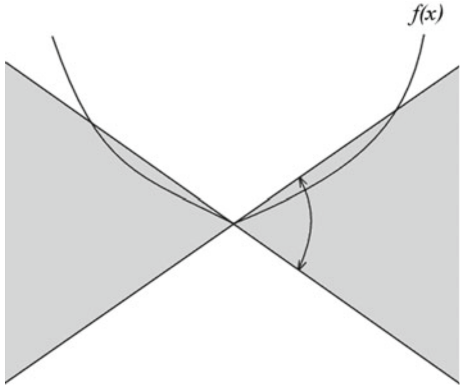


Fig. 2.16 Illustration of ε-subdifferential

As before we now generalize the subgradient and the subdifferential of a convex function. We illustrate the meaning of the definition in Fig. 2.16.

Definition 2.17 Let $\varepsilon \geq 0$, then the ε- *subdifferential of the convex function* $f: \mathbb{R}^n \to \mathbb{R}$ at $\boldsymbol{x} \in \mathbb{R}^n$ is the set

$$\partial_\varepsilon f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f(\boldsymbol{x}') \geq f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{x}' - \boldsymbol{x}) - \varepsilon \quad \text{for all} \quad \boldsymbol{x}' \in \mathbb{R}^n\}. \tag{2.29}$$

Each element $\boldsymbol{\xi} \in \partial_\varepsilon f(\boldsymbol{x})$ is called an ε-*subgradient of* f *at* $\boldsymbol{x}$.

The following summarizes some basic properties of the ε-subdifferential.

Theorem 2.32 *Let* $f: \mathbb{R}^n \to \mathbb{R}$ *be convex function with a Lipschitz constant* K *at* $\boldsymbol{x} \in \mathbb{R}^n$. *Then*

(i) $\partial_0 f(\boldsymbol{x}) = \partial_c f(\boldsymbol{x})$.
(ii) *If* $\varepsilon_1 \leq \varepsilon_2$, *then* $\partial_{\varepsilon_1} f(\boldsymbol{x}) \subseteq \partial_{\varepsilon_2} f(\boldsymbol{x})$.
(iii) $\partial_\varepsilon f(\boldsymbol{x})$ *is a nonempty, convex, and compact set such that* $\partial_\varepsilon f(\boldsymbol{x}) \subseteq B(\mathbf{0}; K)$.
(iv) $\partial_\varepsilon f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d}$ *for all* $\boldsymbol{d} \in \mathbb{R}^n\}$.
(v) $f'_\varepsilon(\boldsymbol{x}; \boldsymbol{d}) = \max\{\boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial_\varepsilon f(\boldsymbol{x})\}$ *for all* $\boldsymbol{d} \in \mathbb{R}^n$.

Proof The definition of the ε-subdifferential implies directly the assertions (i) and (ii) and the proofs of assertions (iv) and (v) are the same as for $\varepsilon = 0$ in Theorem 2.28 (i) and (ii), respectively. By Theorem 2.27 $\partial_c f(\boldsymbol{x})$ is nonempty which implies by assertion (i) that $\partial_\varepsilon f(\boldsymbol{x})$ is also nonempty. The proofs of the convexity and compactness are also same as in Theorem 2.27. □

The following shows that the ε-subdifferential contains in a compressed form the subgradient information from the whole neighborhood.

Theorem 2.33 *Let* $f: \mathbb{R}^n \to \mathbb{R}$ *be convex with Lipschitz constant* K *at* $\boldsymbol{x}$. *Then for all* $\varepsilon \geq 0$ *we have*

$$\partial_c f(\boldsymbol{y}) \subseteq \partial_\varepsilon f(\boldsymbol{x}) \qquad \text{for all} \quad \boldsymbol{y} \in B\left(\boldsymbol{x}; \tfrac{\varepsilon}{2K}\right). \tag{2.30}$$

Proof Let $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{y})$ and $\boldsymbol{y} \in B\left(\boldsymbol{x}; \frac{\varepsilon}{2K}\right)$. Then for all $\boldsymbol{z} \in \mathbb{R}^n$ it holds

$$\begin{aligned} f(\boldsymbol{z}) &\geq f(\boldsymbol{y}) + \boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{y}) \\ &= f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{x}) - (f(\boldsymbol{x}) - f(\boldsymbol{y}) + \boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{x}) - \boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{y})) \end{aligned}$$

and, using the Lipschitz condition and Theorem 2.27 we calculate

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y}) + &\boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{x}) - \boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{y})| \\ &\leq |f(\boldsymbol{x}) - f(\boldsymbol{y})| + |\boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{x}) - \boldsymbol{\xi}^T(\boldsymbol{z} - \boldsymbol{y})| \\ &\leq K\,\|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{\xi}\|\,\|\boldsymbol{x} - \boldsymbol{y}\| \\ &\leq 2K\,\|\boldsymbol{x} - \boldsymbol{y}\| \\ &\leq 2K \cdot \frac{\varepsilon}{2K} = \varepsilon, \end{aligned}$$

which gives $\boldsymbol{\xi} \in \partial_\varepsilon f(\boldsymbol{x})$. □

2.3 Links Between Geometry and Analysis

In this section we are going to show that the analytical and geometrical concepts defined in the previous sections are actually equivalent. We have already showed that the level sets of a convex function are convex, the epigraph of the directional derivative is a convex cone and a function is convex if and only if its epigraph is convex. In what follows we give some more connections, on the one hand, between directional derivatives and contingent cones, and on the other hand, between subdifferentials and normal cones in terms of epigraph, level sets and the distance function.

2.3.1 Epigraphs

The next two theorems describe how one could equivalently define tangents and normals by using the epigraph of a convex function (see Figs. 2.17 and 2.18). First result show that the contingent cone of the epigraph is the epigraph of the directional derivative.

Theorem 2.34 *If the function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is convex, then*

$$K_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x})) = \text{epi } f'(\boldsymbol{x}; \cdot). \tag{2.31}$$

Proof Suppose first that $(\boldsymbol{d}, r) \in K_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x}))$. By the definition of the contingent cone there exist sequences $(\boldsymbol{d}_j, r_j) \to (\boldsymbol{d}, r)$ and $t_j \downarrow 0$ such that

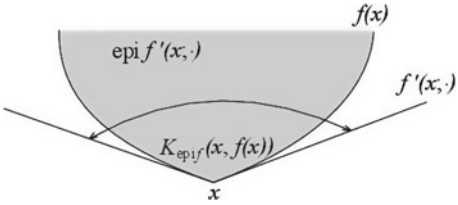


Fig. 2.17 Contingent cone of the epigraph

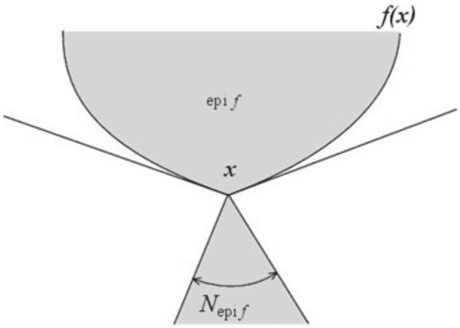


Fig. 2.18 Normal cone of the epigraph

$$(\boldsymbol{x}, f(\boldsymbol{x})) + t_j(\boldsymbol{d}_j, r_j) \in \text{epi } f \quad \text{for all} \quad j \in \mathbb{N},$$

in other words

$$f(\boldsymbol{x} + t_j\boldsymbol{d}_j) \le f(\boldsymbol{x}) + t_j r_j.$$

Now by using (2.20) we can calculate

$$\begin{aligned} f'(\boldsymbol{x}; \boldsymbol{d}) &= \inf_{t>0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t} \\ &= \lim_{j\to\infty} \frac{f(\boldsymbol{x} + t_j\boldsymbol{d}_j) - f(\boldsymbol{x})}{t_j} \\ &\le \lim_{j\to\infty} r_j = r, \end{aligned}$$

which implies that $(\boldsymbol{d}, r) \in \text{epi } f'(\boldsymbol{x}; \cdot)$.

Suppose, next, that $(\boldsymbol{d}, r) \in \text{epi } f'(\boldsymbol{x}; \cdot)$, which means that

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \lim_{t\downarrow 0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t} \le r.$$

Then there exists a sequence $t_j \downarrow 0$ such that

$$\frac{f(\boldsymbol{x} + t_j\boldsymbol{d}) - f(\boldsymbol{x})}{t_j} \le r + \frac{1}{j},$$

which yields

$$f(\boldsymbol{x}+t_j\boldsymbol{d}) \le f(\boldsymbol{x}) + t_j(r+\frac{1}{j})$$

and thus $(\boldsymbol{x}, f(\boldsymbol{x}))+t_j(\boldsymbol{d}, r+\frac{1}{j}) \in \text{epi } f$. This and the fact that $(\boldsymbol{d}, r+\frac{1}{j}) \to (\boldsymbol{d}, r)$ shows that $(\boldsymbol{d}, r) \in K_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x}))$ and we obtain the desired conclusion. □

Next we show that the subgradient is essentially a normal vector of the epigraph.

Theorem 2.35 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is convex, then*

$$\partial_c f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid (\boldsymbol{\xi}, -1) \in N_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x}))\}. \tag{2.32}$$

Proof By Theorem 2.28 (i) we know that $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$ if and only if for any $\boldsymbol{d} \in \mathbb{R}^n$ we have $f'(\boldsymbol{x}; \boldsymbol{d}) \ge \boldsymbol{\xi}^T\boldsymbol{d}$. This is equivalent to the condition that for any $\boldsymbol{d} \in \mathbb{R}^n$ and $r \ge f'(\boldsymbol{x}; \boldsymbol{d})$ we have $r \ge \boldsymbol{\xi}^T\boldsymbol{d}$, that is, for any $\boldsymbol{d} \in \mathbb{R}^n$ and $r \ge f'(\boldsymbol{x}; \boldsymbol{d})$ we have

$$(\boldsymbol{\xi}, -1)^T(\boldsymbol{d}, r) \le 0.$$

By the definition of the epigraph and Theorem 2.34 we have $(\boldsymbol{d}, r) \in \text{ epi } f'(\boldsymbol{x};\ \cdot) = K_{\text{epi } f}(\boldsymbol{x}; f(\boldsymbol{x}))$. This and the last inequality means, by the definition of the normal cone, that $(\boldsymbol{\xi}, -1)$ lies in $N_{\text{epi } f}(\boldsymbol{x}; f(\boldsymbol{x}))$. □

2.3.2 Level Sets

In the following theorem we give the relationship between the directional derivative and the contingent cone via the level sets.

Theorem 2.36 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is convex, then*

$$K_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) \subseteq \text{lev}_0\, f'(\boldsymbol{x};\ \cdot). \tag{2.33}$$

If, in addition, $\mathbf{0} \notin \partial_c f(\boldsymbol{x})$*, then*

$$K_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) = \text{lev}_0\, f'(\boldsymbol{x};\ \cdot). \tag{2.34}$$

Proof Suppose first that $\boldsymbol{d} \in K_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$. By the definition of the contingent cone there exist sequences $\boldsymbol{d}_j \to \boldsymbol{d}$ and $t_j \downarrow 0$ such that

$$\boldsymbol{x} + t_j\boldsymbol{d}_j \in \text{lev}_{f(\boldsymbol{x})} f \quad \text{for all} \quad j \in \mathbb{N},$$

in other words

$$f(\boldsymbol{x}+t_j\boldsymbol{d}_j) \le f(\boldsymbol{x}).$$

Now by using (2.20) we can calculate

$$
\begin{aligned}
f'(\boldsymbol{x};\boldsymbol{d}) &= \inf_{t>0} \frac{f(\boldsymbol{x}+t\boldsymbol{d})-f(\boldsymbol{x})}{t} \\
&= \lim_{j\to\infty} \frac{f(\boldsymbol{x}+t_j\boldsymbol{d}_j)-f(\boldsymbol{x})}{t_j} \\
&\leq \lim_{j\to\infty} r_j = r,
\end{aligned}
$$

which implies that $\boldsymbol{d} \in \operatorname{lev}_0 f'(\boldsymbol{x};\,\cdot)$.

Suppose, next, that $\boldsymbol{0} \notin \partial_c f(\boldsymbol{x})$ and $\boldsymbol{d} \in \operatorname{lev}_0 f'(\boldsymbol{x};\,\cdot)$, which means that $f'(\boldsymbol{x};\boldsymbol{d}) \leq 0$. Since $\boldsymbol{0} \notin \partial_c f(\boldsymbol{x})$ by Theorem 2.28 (i) we have

$$f'(\boldsymbol{x};\boldsymbol{d}) = \lim_{t\downarrow 0} \frac{f(\boldsymbol{x}+t\boldsymbol{d})-f(\boldsymbol{x})}{t} < 0.$$

Then there exists a sequence $t_j \downarrow 0$ such that

$$\frac{f(\boldsymbol{x}+t_j\boldsymbol{d})-f(\boldsymbol{x})}{t_j} \leq 0,$$

which yields

$$f(\boldsymbol{x}+t_j\boldsymbol{d}) \leq f(\boldsymbol{x})$$

and thus $\boldsymbol{x}+t_j\boldsymbol{d} \in \operatorname{lev}_{f(\boldsymbol{x})} f$. This means that $\boldsymbol{d} \in K_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$ and the proof is complete. □

Next theorem shows the connection between subgradients and normal vectors of the level sets.

Theorem 2.37 *If the function* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is convex, then*

$$N_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) \supseteq \operatorname{ray} \partial_c f(\boldsymbol{x}).$$

If, in addition, $\boldsymbol{0} \notin \partial_c f(\boldsymbol{x})$*, then*

$$N_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) = \operatorname{ray} \partial_c f(\boldsymbol{x}).$$

Proof If $\boldsymbol{z} \in \operatorname{ray} \partial_c f(\boldsymbol{x})$ then $\boldsymbol{z} = \lambda\boldsymbol{\xi}$, where $\lambda \geq 0$ and $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})$. Let now $\boldsymbol{d} \in K_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$, which means due to Theorem 2.36 that $\boldsymbol{d} \in \operatorname{lev}_0 f'(\boldsymbol{x};\,\cdot)$. Then using Theorem 2.28 (i) we get

$$\boldsymbol{z}^T\boldsymbol{d} = \lambda\boldsymbol{\xi}^T\boldsymbol{d} \leq \lambda f'(\boldsymbol{x};\boldsymbol{d}) \leq 0,$$

in other words $\boldsymbol{z} \in N_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$.

Suppose next that $\boldsymbol{0} \notin \partial_c f(\boldsymbol{x})$ and there exists $\boldsymbol{z} \in N_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$ such that $\boldsymbol{z} \notin \operatorname{ray} \partial_c f(\boldsymbol{x})$. According to Theorem 2.27 $\partial_c f(\boldsymbol{x})$ is a convex and compact set. Since $\boldsymbol{0} \notin \partial_c f(\boldsymbol{x})$ Theorems 2.11 and 2.12 implies that $\operatorname{ray} \partial_c f(\boldsymbol{x})$ is closed and convex, respectively. As a cone it is nonempty since $\mathbf{0} \in \text{ray } \partial_c f(\boldsymbol{x})$. Then by Theorem 2.4 there exists a hyperplane separating $\{\boldsymbol{z}\}$ and ray $\partial_c f(\boldsymbol{x})$, in other words there exist $\boldsymbol{p} \neq \mathbf{0}$ and $\alpha \in \mathbb{R}$ such that

$$\boldsymbol{y}^T \boldsymbol{p} \leq \alpha \quad \text{for all} \quad \boldsymbol{y} \in \text{ray } \partial_c f(\boldsymbol{x}) \tag{2.35}$$

and

$$\boldsymbol{z}^T \boldsymbol{p} > \alpha. \tag{2.36}$$

Since ray $\partial_c f(\boldsymbol{x})$ is cone the components of $\boldsymbol{y}$ can be chosen as large as possible in (2.35), thus $\alpha \leq 0$. On the other hand $\mathbf{0} \in \text{ray } \partial_c f(\boldsymbol{x})$ implying $\alpha \geq \boldsymbol{p}^T \mathbf{0} = 0$, thus $\alpha = 0$. Since $\partial_c f(\boldsymbol{x}) \subseteq \text{ray } \partial_c f(\boldsymbol{x})$ Theorem 2.28 (ii) and (2.35) imply

$$f'(\boldsymbol{x}; \boldsymbol{p}) = \max_{\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x})} \boldsymbol{\xi}^T \boldsymbol{p} \leq \max_{\boldsymbol{y} \in \text{ray } \partial_c f(\boldsymbol{x})} \boldsymbol{y}^T \boldsymbol{p} \leq 0.$$

This means that $\boldsymbol{p} \in \text{lev}_0 f'(\boldsymbol{x}; \cdot)$ and thus due to Theorem 2.36 we have $\boldsymbol{p} \in K_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$. Since $\boldsymbol{z} \in N_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$ it follows from the definition of the normal cone that

$$\boldsymbol{z}^T \boldsymbol{p} \leq 0$$

contradicting with inequality (2.36). Thus, $\boldsymbol{z} \in \text{ray } \partial_c f(\boldsymbol{x})$ and the theorem is proved. □

2.3.3 Distance Function

Finally we study the third link between analysis and geometry, namely the distance function defined by (2.2). First we give some important properties of the distance function.

Theorem 2.38 *If $S \subseteq \mathbb{R}^n$ is a nonempty set, then the distance function d_S is Lipschitz continuous with constant $K = 1$, in other words*

$$|d_S(\boldsymbol{x}) - d_S(\boldsymbol{y})| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{for all} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n. \tag{2.37}$$

If in addition the set S is convex then the function d_S is also convex.

Proof Let any $\varepsilon > 0$ and $\boldsymbol{y} \in \mathbb{R}^n$ be given. By definition, there exists a point $\boldsymbol{z} \in S$ such that

$$d_S(\boldsymbol{y}) \geq \|\boldsymbol{y} - \boldsymbol{z}\| - \varepsilon.$$

Now we have

$$\begin{aligned} d_S(\boldsymbol{x}) &\le \|\boldsymbol{x}-\boldsymbol{z}\| \le \|\boldsymbol{x}-\boldsymbol{y}\| + \|\boldsymbol{y}-\boldsymbol{z}\| \\ &\le \|\boldsymbol{x}-\boldsymbol{y}\| + d_S(\boldsymbol{y}) + \varepsilon \end{aligned}$$

which establishes the Lipschitz condition as $\varepsilon > 0$ is arbitrary.

Suppose now that S is a convex set and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\lambda \in [0, 1]$ and $\varepsilon > 0$ be given. Choose points $\boldsymbol{z}_{\boldsymbol{x}}, \boldsymbol{z}_{\boldsymbol{y}} \in S$ such that

$$\|\boldsymbol{z}_{\boldsymbol{x}} - \boldsymbol{x}\| \le d_S(\boldsymbol{x}) + \varepsilon \quad \text{and} \quad \|\boldsymbol{z}_{\boldsymbol{y}} - \boldsymbol{x}\| \le d_S(\boldsymbol{y}) + \varepsilon$$

and define $\boldsymbol{z} := (1-\lambda)\boldsymbol{z}_{\boldsymbol{x}} + \lambda \boldsymbol{z}_{\boldsymbol{y}} \in S$. Then

$$\begin{aligned} d_S\big((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}\big) &\le \|\boldsymbol{c} - (1-\lambda)\boldsymbol{x} - \lambda\boldsymbol{y}\| \\ &\le (1-\lambda)\|\boldsymbol{z}_{\boldsymbol{x}} - \boldsymbol{x}\| + \lambda\|\boldsymbol{z}_{\boldsymbol{y}} - \boldsymbol{y}\| \\ &\le (1-\lambda)d_S(\boldsymbol{x}) + \lambda d_S(\boldsymbol{y}) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, d_S is convex. □

Lemma 2.6 *If $S \subseteq \mathbb{R}^n$ is closed, then*

$$\boldsymbol{x} \in S \iff d_S(\boldsymbol{x}) = 0. \tag{2.38}$$

Proof Let $\boldsymbol{x} \in Z$ be arbitrary. Then

$$0 \le d_S(\boldsymbol{x}) \le \|\boldsymbol{x} - \boldsymbol{x}\| = 0$$

and thus $d_S(\boldsymbol{x}) = 0$.

On the other hand if $d_S(\boldsymbol{x}) = 0$, then there exists a sequence $(\boldsymbol{y}_j) \subset S$ such that

$$\|\boldsymbol{x} - \boldsymbol{y}_j\| < 1/j \longrightarrow 0, \qquad \text{when } j \to \infty.$$

Thus, the sequence $(\boldsymbol{y}_j)$ converges to $\boldsymbol{x}$ and $\boldsymbol{x} \in \operatorname{cl} S = S$. □

The next two theorems show how one could equivalently define tangents and normals by using the distance function.

Theorem 2.39 *The contingent cone of the convex set S at $\boldsymbol{x} \in S$ can also be written as*

$$K_S(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n \mid d'_S(\boldsymbol{x}; \boldsymbol{y}) = 0\}. \tag{2.39}$$

Proof Let $Z := \{\boldsymbol{y} \in \mathbb{R}^n \mid d'_S(\boldsymbol{x}; \boldsymbol{y}) = 0\}$ and let $\boldsymbol{y} \in K_S(\boldsymbol{x})$ be arbitrary. Then there exist sequences $(\boldsymbol{y}_j) \subset \mathbb{R}^n$ and $(t_j) \subset \mathbb{R}$ such that $\boldsymbol{y}_j \to \boldsymbol{y}$, $t_j \downarrow 0$ and $\boldsymbol{x} + t_j\boldsymbol{y}_j \in S$ for all $j \in \mathbb{N}$. It is evident that $d'_S(\boldsymbol{x}; \boldsymbol{y})$ is always nonnegative thus it suffices to show that $d'_S(\boldsymbol{x}; \boldsymbol{y}) \le 0$. Since $\boldsymbol{x} \in S$ we have

$$\begin{aligned} d'_S(\boldsymbol{x}; \boldsymbol{y}) &= \lim_{t\downarrow 0} \frac{d_S(\boldsymbol{x} + t\boldsymbol{y}) - d_S(\boldsymbol{x})}{t} \\ &= \lim_{t\downarrow 0} \frac{\inf_{\boldsymbol{z}\in S}\{\|\boldsymbol{x} + t\boldsymbol{y} - \boldsymbol{z}\|\}}{t} \\ &\le \lim_{t\downarrow 0} \frac{\inf_{\boldsymbol{z}\in S}\{\|\boldsymbol{x} + t\boldsymbol{y}_j - \boldsymbol{z}\|\} + \|t\boldsymbol{y} - t\boldsymbol{y}_j\|}{t} \end{aligned}$$

and

$$\inf_{\boldsymbol{z}\in S}\{\|\boldsymbol{x} + t\boldsymbol{y}_j - \boldsymbol{z}\|\} = \inf_{\boldsymbol{z}\in S}\{\|(1 - \frac{t}{t_j})\boldsymbol{x} + \frac{t}{t_j}(\boldsymbol{x} + t_j\boldsymbol{y}_j) - \boldsymbol{c}\|\}.$$

Since $\boldsymbol{x} \in S$, $\boldsymbol{x} + t_j\boldsymbol{y}_j \in S$ and $t/t_j \in [0, 1]$ whenever $0 \le t \le t_j$, the convexity of S implies that

$$(1 - \frac{t}{t_j})\boldsymbol{x} + \frac{t}{t_j}(\boldsymbol{x} + t_j\boldsymbol{y}_j) \in S,$$

and thus

$$\inf_{\boldsymbol{z}\in S} \|(1 - \frac{t}{t_j})\boldsymbol{x} + \frac{t}{t_j}(\boldsymbol{x} + t_j\boldsymbol{y}_j) - \boldsymbol{z}\| = 0.$$

Therefore

$$d'_S(\boldsymbol{x}; \boldsymbol{y}) \le t\|\boldsymbol{y} - \boldsymbol{y}_j\| \longrightarrow 0,$$

when $j \to \infty$. Thus $d'_S(\boldsymbol{x}; \boldsymbol{y}) = 0$ and $K_S(\boldsymbol{x}) \subseteq Z$.

For the converse let $\boldsymbol{y} \in Z$ and $(t_j) \subset \mathbb{R}$ be such that $t_j \downarrow 0$. By the definition of Z we get

$$d'_S(\boldsymbol{x}; \boldsymbol{y}) = \lim_{t_j\downarrow 0} \frac{d_S(\boldsymbol{x} + t_j\boldsymbol{y})}{t_j} = 0.$$

By the definition of d_S we can choose points $\boldsymbol{z}_j \in S$ such that

$$\|\boldsymbol{x} + t_j\boldsymbol{y} - \boldsymbol{z}_j\| \le d_S(\boldsymbol{x} + t_j\boldsymbol{y}) + \frac{t_j}{j}.$$

By setting

$$\boldsymbol{y}_j := \frac{\boldsymbol{z}_j - \boldsymbol{x}}{t_j},$$

we have

$$\boldsymbol{x} + t_j\boldsymbol{y}_j = \boldsymbol{x} + t_j\frac{\boldsymbol{z}_j - \boldsymbol{x}}{t_j} = \boldsymbol{z}_j \in S$$

and

$$\begin{aligned}\|\boldsymbol{y}-\boldsymbol{y}_j\| &= \|\boldsymbol{y}-\frac{\boldsymbol{z}_j-\boldsymbol{x}}{t_j}\| \\ &= \frac{\|\boldsymbol{x}+t_j\boldsymbol{y}-\boldsymbol{z}_j\|}{t_j} \\ &\le \frac{d_S(\boldsymbol{x}+t_j\boldsymbol{y})}{t_j}+\frac{1}{j} \longrightarrow 0,\end{aligned}$$

as $j \to \infty$. Thus $\boldsymbol{y} \in K_S(\boldsymbol{x})$ and $Z = K_S(\boldsymbol{x})$. □

Theorem 2.40 *The normal cone of the convex set S at $\boldsymbol{x} \in S$ can also be written as*

$$N_S(\boldsymbol{x}) = \operatorname{cl}\operatorname{ray}\partial_c d_S(\boldsymbol{x}). \tag{2.40}$$

Proof First, let $\boldsymbol{z} \in \partial_c d_S(\boldsymbol{x})$. Then by Theorem 2.28 (i)

$$\boldsymbol{z}^T\boldsymbol{y} \le d'_S(\boldsymbol{x};\boldsymbol{y}) \quad \text{for all} \quad \boldsymbol{y} \in \mathbb{R}^n.$$

If one has $\boldsymbol{y} \in K_S(\boldsymbol{x})$ then by Theorem 2.39 $d'_S(\boldsymbol{x};\boldsymbol{y}) = 0$. Thus $\boldsymbol{z}^T\boldsymbol{y} \le 0$ for all $\boldsymbol{y} \in K_S(\boldsymbol{x})$ which implies that $\boldsymbol{z} \in N_S(\boldsymbol{x})$. By Theorem 2.27 $\partial_c d_S(\boldsymbol{x})$ is a convex set and then by Theorem 2.11 ray $\partial_c d_S(\boldsymbol{x})$ is a convex cone. Furthermore, by Theorem 2.10 ray $\partial_c d_S(\boldsymbol{x})$ is the smallest cone containing $\partial_c d_S(\boldsymbol{x})$. Then, because $N_S(\boldsymbol{x})$ is also a convex cone (Theorem 2.18), we have

$$\operatorname{ray}\partial_c d_S(\boldsymbol{x}) \subseteq N_S(\boldsymbol{x}).$$

On the other hand, if $N_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$ we have clearly $N_S(\boldsymbol{x}) \subseteq \operatorname{ray}\partial_c d_S(\boldsymbol{x})$. Suppose next that $N_S(\boldsymbol{x}) \neq \{\boldsymbol{0}\}$ and let $\boldsymbol{z} \in N_S(\boldsymbol{x})\setminus\{\boldsymbol{0}\}$ be arbitrary. Since S is convex due to Theorem 2.19 we have

$$\boldsymbol{z}^T(\boldsymbol{y}-\boldsymbol{x}) \le 0 \quad \text{for all} \quad \boldsymbol{y} \in S$$

and hence $S \subseteq H^-(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x})$. Since $d_S(\boldsymbol{y}) \ge 0$ for all $\boldsymbol{y} \in \mathbb{R}^n$ we have

$$\lambda\boldsymbol{z}^T(\boldsymbol{y}-\boldsymbol{x}) \le 0 \le d_S(\boldsymbol{y}) \quad \text{for all} \quad \boldsymbol{y} \in H^-(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x}) \text{ and } \lambda \ge 0.$$

Suppose next that $\boldsymbol{y} \in H^+(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x})$. Since $S \subseteq H^-(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x})$ we have clearly $d_{H^-(\boldsymbol{z},\boldsymbol{z}^T\boldsymbol{x})}(\boldsymbol{y}) \le d_S(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathbb{R}^n$. On the other hand (see Exercise 2.3)

$$d_{H^-(\boldsymbol{z},\boldsymbol{z}^T\boldsymbol{x})}(\boldsymbol{y}) = \frac{1}{\|\boldsymbol{z}\|}\boldsymbol{z}^T(\boldsymbol{y}-\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{y} \in H^+(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x}).$$

Thus, for any $\boldsymbol{y} \in \mathbb{R}^n = H^-(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x}) \cup H^+(\boldsymbol{z}, \boldsymbol{z}^T\boldsymbol{x})$ we have

$$\frac{1}{\|\boldsymbol{z}\|}\boldsymbol{z}^T(\boldsymbol{y}-\boldsymbol{x}) \leq d_S(\boldsymbol{y}) = d_S(\boldsymbol{y}) - d_S(\boldsymbol{x}).$$

Then the definition of subdifferential of convex function and the convexity of d_S imply that

$$\frac{1}{\|\boldsymbol{z}\|}\boldsymbol{z} \in \partial_c d_S(\boldsymbol{x}),$$

thus $N_S(\boldsymbol{x}) \subseteq \text{ray } \partial_c d_S(\boldsymbol{x})$ and the proof is complete. □

Note that since $N_S(\boldsymbol{x})$ is always closed, we deduce that also ray $\partial_c d_S(\boldsymbol{x})$ is closed if S is convex.

2.4 Summary

This chapter contains the basic results from convex analysis. First we have concentrated on geometrical concepts and started by considering convex sets and cones. The main results are the existence of separating and supporting hyperplanes (Theorems 2.4, 2.7 and 2.8). We have defined tangents and normals in the form of contingent and normal cones. Next we moved to analytical concepts and defined subgradients and subdifferentials of convex functions. Finally we showed that everything is one by connecting these geometrical and analytical concepts via epigraphs, level sets and the distance functions. We have proved, for example, that the contingent cone of the epigrapf is the epigrapf of the directional derivative (Theorem 2.34), the contingent cone of the zero level set is zero level set of the directional derivative (Theorem 2.36), and the contigent cone of a convex set consist of the points where the directional derivative of the distance function vanish (Theorem 2.39).

Exercises

2.1 Show that open and closed balls and halfspaces are convex sets.

2.2 (Lemma 2.1) Prove that if $S \subseteq \mathbb{R}^n$, then conv S is a convex set and S is convex if and only if $S = \text{conv } S$.

2.3 Let $\boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{p} \neq \boldsymbol{0}$ and $\alpha \in \mathbb{R}$. Prove that

$$d_{H^-(\boldsymbol{p},\alpha)}(\boldsymbol{y}) = \frac{1}{\|\boldsymbol{p}\|}(\boldsymbol{p}^T\boldsymbol{y} - \alpha) \quad \text{for all} \quad \boldsymbol{y} \in H^+(\boldsymbol{p},\alpha).$$

2.4 (Farkas' Lemma) Let $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{c} \in \mathbb{R}^n$. Prove that either

$$A\boldsymbol{x} \leq \boldsymbol{0} \quad \text{and} \quad \boldsymbol{c}^T\boldsymbol{x} > 0 \quad \text{for some } \boldsymbol{x} \in \mathbb{R}^n$$

or

$$A^T \boldsymbol{y} = \boldsymbol{c} \quad \text{and} \quad \boldsymbol{y} \geq 0 \quad \text{for some } \boldsymbol{y} \in \mathbb{R}^n.$$

2.5 (Gordan's Lemma) Let $A \in \mathbb{R}^{n \times n}$. Prove that either

$$A\boldsymbol{x} < \boldsymbol{0} \quad \text{and} \quad \boldsymbol{c}^T \boldsymbol{x} > 0 \quad \text{for some } \boldsymbol{x} \in \mathbb{R}^n$$

or

$$A^T \boldsymbol{y} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{0} \neq \boldsymbol{y} \geq 0 \quad \text{for some } \boldsymbol{y} \in \mathbb{R}^n.$$

2.6 Show that closed halfspaces $H^+(\boldsymbol{p}, 0)$ and $H^-(\boldsymbol{p}, 0)$, the nonnegative orthant $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq 0,\ i = 1 \ldots, n\}$ and halflines starting from the origin are closed convex cones.

2.7 (Lemma 2.3) Prove that if $S \subseteq \mathbb{R}^n$, then ray S is a cone and $C \subseteq \mathbb{R}^n$ is cone if and only if $C = \text{ray } C$.

2.8 (Lemma 2.4) Prove that if $S \subseteq \mathbb{R}^n$, then cone S is a convex cone and $C \subseteq \mathbb{R}^n$ is convex cone if and only if $C = \text{cone } C$.

2.9 (Corollary 2.2) Prove that if $S \subseteq \mathbb{R}^n$, then cone $S = \text{conv ray } S$.

2.10 Show that $S_1 \subseteq S_2$ implies $S_2^\circ \subseteq S_1^\circ$.

2.11 (Lemma 2.5) Prove that if $S \subseteq \mathbb{R}^n$, then S° is a closed convex cone and $S \subseteq S^{\circ\circ}$.

2.12 Specify the sets conv S, ray S, cone S and S° when

(a) $S = \{(1, 1)\}$
(b) $S = \{(1, 1), (1, 2), (2, 1)\}$
(c) $S = \text{int } \mathbb{R}^2_+ \cup \{(0, 0)\}$.

2.13 Let $C \subseteq \mathbb{R}^n$ be a closed convex cone. Show that $K_C(\boldsymbol{0}) = C$.

2.14 (Theorem 2.16) Prove that the cone of global feasible directions $G_S(\boldsymbol{x})$ of the nonempty convex set S at $\boldsymbol{x} \in S$ is a convex cone.

2.15 Let $S \subseteq \mathbb{R}^n$ be convex. Show that $K_S(\boldsymbol{x}) = N_S(\boldsymbol{x})^\circ$.

2.16 Specify the sets $K_{\mathbb{R}^2_+}(\boldsymbol{0})$ and $N_{\mathbb{R}^2_+}(\boldsymbol{0})$.

2.17 Let $S \subseteq \mathbb{R}^n$ be convex and $\boldsymbol{x} \in \text{int } S$. Show that $K_S(\boldsymbol{x}) = \mathbb{R}^n$ and $N_S(\boldsymbol{x}) = \emptyset$.

2.18 Let $S_1, S_2 \subseteq \mathbb{R}^n$ be convex and $\boldsymbol{x} \in S_1 \cap S_2$. Show that

(a) $K_{S_1 \cap S_2}(\boldsymbol{x}) \subseteq K_{S_1}(\boldsymbol{x}) \cap K_{S_2}(\boldsymbol{x})$,

(b) $N_{S_1 \cap S_2}(\boldsymbol{x}) \supseteq N_{S_1}(\boldsymbol{x}) + N_{S_2}(\boldsymbol{x})$.

2.19 (Theorem 2.20) Prove that if S is a nonempty convex set such that $\mathbf{0} \in S$, then

(a) $G_S(\mathbf{0}) = \text{ray } S$,

(b) $K_S(\mathbf{0}) = \text{cl ray } S$,

(c) $N_S(\mathbf{0}) = S^\circ$.

2.20 Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := e^{x^2}$$

is convex.

2.21 By exploiting Exercise 2.20 show that for all $x, y > 0$ we have

$$\frac{x}{4} + \frac{3y}{4} \leq \sqrt{\ln\left(\frac{e^{x^2}}{4} + \frac{3e^{y^2}}{4}\right)}.$$

2.22 (Theorem 2.24) Prove that the function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph epi f is a convex set.

2.23 How should the concept of a ‘concave set’ to be defined?

2.24 (Corollary 2.3) Prove that if $f: \mathbb{R}^n \to \mathbb{R}$ is a convex function, then the function $\boldsymbol{d} \mapsto f'(\boldsymbol{x}; \boldsymbol{d})$ is convex, its epigraph epi $f'(\boldsymbol{x}; \cdot)$ is a convex cone and we have

$$f'(\boldsymbol{x}; -\boldsymbol{d}) \geq -f'(\boldsymbol{x}; \boldsymbol{d}) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

2.25 Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \max\{|x|, x^2\}$$

is convex. Calculate $f'(1; \pm 1)$ and $\partial_c f(1)$.

2.26 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be such that

$$f(x, y) := \max\{-\max\{-x, y\}, y - x\}.$$

Calculate $\partial_c f(0, 0)$.

2.27 Let $f: \mathbb{R}^n \to \mathbb{R}$ be such that $f(\boldsymbol{x}) := \|\boldsymbol{x}\|$ and $g: \mathbb{R} \to \mathbb{R}$ such that $g(x) := x^2$. Calculate $\partial_c f(0)$ and $\partial_c g(f(0))$.

2.28 Let $f: \mathbb{R}^n \to \mathbb{R}$ be convex. Show that the mapping $\boldsymbol{x} \mapsto \partial_c f(\boldsymbol{x})$ is monotonic, in other words for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we have

$$(\boldsymbol{\xi}_{\boldsymbol{x}} - \boldsymbol{\xi}_{\boldsymbol{y}})^T(\boldsymbol{x} - \boldsymbol{y}) \geq 0 \quad \text{for all} \quad \boldsymbol{\xi}_{\boldsymbol{x}} \in \partial_c f(\boldsymbol{x}),\ \boldsymbol{\xi}_{\boldsymbol{y}} \in \partial_c f(\boldsymbol{y}).$$

2.29 Prove that if $f: \mathbb{R}^n \to \mathbb{R}$ is continuous, then epi f and $\text{lev}_\alpha f$ are closed for all $\alpha \in \mathbb{R}$.

2.30 Let the functions $f_i: \mathbb{R}^n \to \mathbb{R}$ be convex for all $i = 1, \ldots, m$ and define $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(\boldsymbol{x}) := \max\{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}.$$

Show that

(a) lev $f = \bigcap_{i=1}^m$ lev f_i,

(b) epi $f = \bigcap_{i=1}^m$ epi f_i.

2.31 Show that the equality does not hold in Theorem 2.36 without the extra assumption $\mathbf{0} \notin \partial_c f(\boldsymbol{x})$. In other words, if the function $f: \mathbb{R}^n \to \mathbb{R}$ is convex, then

$$K_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) \not\supseteq \text{lev}_0 f'(\boldsymbol{x};\ \cdot).$$

(Hint: Consider the function $f(\boldsymbol{x}) := \|\boldsymbol{x}\|^2$).

2.32 Let $S \subseteq \mathbb{R}^n$ convex and $x \in S$. Show that if $\mathbf{0} \notin \partial_c d_S(\boldsymbol{x})$, then

(a) $K_S(\boldsymbol{x}) = K_{\text{lev}_{d_S(\boldsymbol{x})} d_S}(\boldsymbol{x}) \cap K_{\text{lev}_{-d_S(\boldsymbol{x})} -d_S}(\boldsymbol{x})$,

(a) $N_S(\boldsymbol{x}) = N_{\text{lev}_{d_S(\boldsymbol{x})} d_S}(\boldsymbol{x})$.

2.33 Let

$$S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 \leq x_2 \text{ and } |x_1| \leq x_2\}.$$

Calculate $K_S((1, 1))$ and $N_S((1, 1))$.

2.34 Let

$$S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \leq 2,\ x_1 \geq -2x_2 \text{ and } x_1 \geq 2x_2\}.$$

Calculate $K_S((0, 0))$ and $N_S((0, 0))$.

Chapter 3
Nonconvex Analysis

In this chapter, we generalize the convex concepts defined in the previous chapter for nonconvex locally Lipschitz continuous functions. Since the classical directional derivative does not necessarily exist for locally Lipschitz continuous functions, we first define a generalized directional derivative. Then we generalize the subdifferential analogously. We use the approach of Clarke in a finite dimensional case. This is done in Sect. 3.1 In Sect. 3.2 we give several derivation rules to help the calculation of subgradients in practice. The main result of this chapter is the Theorem 3.9, which tells how one can compute the subdifferential by using the limits of ordinary gradients. In addition, in Sect. 2.2.3 we define the so-called ε-subdifferential approximating the ordinary subdifferentials. In addition to the Clarke subdifferential, many different generalizations of the subdifferential for nonconvex nonsmooth functions exist. In Sect. 3.4, we briefly recall some of them. More specifically, we give definitions of the quasidifferential, the codifferential, the basic (limiting) and the singular subdifferentials.

3.1 Generalization of Derivatives

In this section, we give the generalized directional derivative, subdifferentials, ε-subdifferentials and Jacobian matrices.

3.1.1 Generalized Directional Derivative

We start by generalizing the ordinary directional derivative. Note that this generalized derivative always exists for locally Lipschitz continuous functions.

Definition 3.1 (*Clarke*) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. The *generalized directional derivative* of f at $\boldsymbol{x}$ in the direction of $\boldsymbol{d} \in \mathbb{R}^n$ is defined by

A. Bagirov et al., *Introduction to Nonsmooth Optimization*,
DOI: 10.1007/978-3-319-08114-4_3

$$f^\circ(\boldsymbol{x};\boldsymbol{d}) = \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\boldsymbol{d})-f(\boldsymbol{y})}{t}.$$

The following summarizes some basic properties of the generalized directional derivative. Note, that the results are identical to those in convex case (see Theorem 2.26).

Theorem 3.1 *Let f be locally Lipschitz continuous at $\boldsymbol{x}$ with constant K. Then the function $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x};\boldsymbol{d})$ is positively homogeneous and subadditive on $\mathbb{R}^n$ with*

$$|f^\circ(\boldsymbol{x};\boldsymbol{d})| \le K\|\boldsymbol{d}\|.$$

Proof We start by proving the inequality. From the Lipschitz condition we obtain

$$\begin{aligned}
\big|f^\circ(\boldsymbol{x};\boldsymbol{d})\big| &= \left|\limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\boldsymbol{d})-f(\boldsymbol{y})}{t}\right| \\
&\le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{|f(\boldsymbol{y}+t\boldsymbol{d})-f(\boldsymbol{y})|}{t} \\
&\le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{K\|\boldsymbol{y}+t\boldsymbol{d}-\boldsymbol{y}\|}{t},
\end{aligned}$$

whenever $\boldsymbol{y}, \boldsymbol{y}+t\boldsymbol{d} \in B(\boldsymbol{x};\varepsilon)$ with some $\varepsilon > 0$. Thus

$$|f^\circ(\boldsymbol{x};\boldsymbol{d})| \le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{Kt\|\boldsymbol{d}\|}{t} = K\|\boldsymbol{d}\|.$$

Next we show that the derivative is positively homogeneous. To see this, let $\lambda > 0$. Then

$$\begin{aligned}
f^\circ(\boldsymbol{x};\lambda\boldsymbol{d}) &= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\lambda\boldsymbol{d})-f(\boldsymbol{y})}{t} \\
&= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \lambda\cdot\left\{\frac{f(\boldsymbol{y}+t\lambda\boldsymbol{d})-f(\boldsymbol{y})}{\lambda t}\right\} \\
&= \lambda\cdot\limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\lambda\boldsymbol{d})-f(\boldsymbol{y})}{\lambda t} = \lambda\cdot f^\circ(\boldsymbol{x};\boldsymbol{d}).
\end{aligned}$$

We turn now to the subadditivity. Let $\boldsymbol{d}, \boldsymbol{p} \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned} f^\circ(\boldsymbol{x}; \boldsymbol{d}+\boldsymbol{p}) &= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t(\boldsymbol{d}+\boldsymbol{p}))-f(\boldsymbol{y})}{t} \\ &= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\boldsymbol{d}+t\boldsymbol{p})-f(\boldsymbol{y}+t\boldsymbol{p})+f(\boldsymbol{y}+t\boldsymbol{p})-f(\boldsymbol{y})}{t} \\ &\le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f((\boldsymbol{y}+t\boldsymbol{p})+t\boldsymbol{d})-f(\boldsymbol{y}+t\boldsymbol{p})}{t} \\ &\quad + \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\boldsymbol{p})-f(\boldsymbol{y})}{t} \\ &= f^\circ(\boldsymbol{x}; \boldsymbol{d}) + f^\circ(\boldsymbol{x}; \boldsymbol{p}). \end{aligned}$$

Thus $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ is subadditive. □

Based on the previous theorem it is easy to prove the following consequence.

Corollary 3.1 *If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}$, then the function $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ is convex, its epigraph* epi $f^\circ(\boldsymbol{x}; \cdot)$ *is a convex cone and we have*

$$f^\circ(\boldsymbol{x}; -\boldsymbol{d}) = (-f)^\circ(\boldsymbol{x}; \boldsymbol{d}).$$

Proof Exercise. □

We obtain also the following useful continuity property.

Theorem 3.2 *Let f be locally Lipschitz continuous at $\boldsymbol{x}$ with constant K. Then the function $(\boldsymbol{x}, \boldsymbol{d}) \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ is upper semicontinuous.*

Proof Let $(\boldsymbol{x}_i), (\boldsymbol{d}_i) \subset \mathbb{R}^n$ be sequences such that $\boldsymbol{x}_i \to \boldsymbol{x}$ and $\boldsymbol{d}_i \to \boldsymbol{d}$. By definition of upper limit, there exist sequences $(\boldsymbol{y}_i) \subset \mathbb{R}^n$ and $(t_i) \subset \mathbb{R}$ such that $t_i > 0$,

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}_i) \le [f(\boldsymbol{y}_i + t\boldsymbol{d}_i) - f(\boldsymbol{y}_i)]/t_i + 1/i$$

and

$$\|\boldsymbol{y}_i - \boldsymbol{x}_i\| + t_i < 1/i \quad \text{for all} \quad i \in \mathbb{N}.$$

Now we have

$$\begin{aligned} f^\circ(\boldsymbol{x}_i; \boldsymbol{d}_i) - \frac{1}{i} &= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}_i\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t\boldsymbol{d}_i)-f(\boldsymbol{y})}{t} - \frac{1}{i} \\ &\le \frac{f(\boldsymbol{y}_i+t_i\boldsymbol{d}_i)-f(\boldsymbol{y}_i)}{t_i} \end{aligned}$$

$$= \frac{f(\boldsymbol{y}_i + t_i\boldsymbol{d}) - f(\boldsymbol{y}_i)}{t_i} + \frac{f(\boldsymbol{y}_i + t_i\boldsymbol{d}_i) - f(\boldsymbol{y}_i + t_i\boldsymbol{d})}{t_i}$$

and, by the Lipschitz condition

$$\frac{|f(\boldsymbol{y}_i + t_i\boldsymbol{d}_i) - f(\boldsymbol{y}_i + t_i\boldsymbol{d})|}{t_i} \leq \frac{K\|t_i\boldsymbol{d}_i - t_i\boldsymbol{d}\|}{t_i} = K\|\boldsymbol{d}_i - \boldsymbol{d}\| \longrightarrow 0,$$

as $i \to \infty$ provided $\boldsymbol{y}_i + t_i\boldsymbol{d}_i,\ \boldsymbol{y}_i + t_i\boldsymbol{d} \in B(\boldsymbol{x}; \varepsilon)$ with some $\varepsilon > 0$. On taking upper limits (as $i \to \infty$), we obtain

$$\limsup_{i\to\infty} f^\circ(\boldsymbol{x}_i; \boldsymbol{d}_i) \leq \limsup_{i\to\infty} \frac{f(\boldsymbol{y}_i + t_i\boldsymbol{d}) - f(\boldsymbol{y}_i)}{t_i} \leq f^\circ(\boldsymbol{x}; \boldsymbol{d}),$$

which establishes the upper semicontinuity. □

3.1.2 Generalized Subgradients

We are now ready to generalize the subdifferential to nonconvex locally Lipschitz continuous functions. Note that the definition is analogous to the property in Theorem 2.28 (i) for convex functions, with the directional derivative replaced by the generalized directional derivative. In what follows we sometimes refer to this subdifferential as *Clarke subdifferential*.

Definition 3.2 (*Clarke*) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at a point $\boldsymbol{x} \in \mathbb{R}^n$. Then the *subdifferential* of f at $\boldsymbol{x}$ is the set $\partial f(\boldsymbol{x})$ of vectors $\boldsymbol{\xi} \in \mathbb{R}^n$ such that

$$\partial f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid f^\circ(\boldsymbol{x}; \boldsymbol{d}) \geq \boldsymbol{\xi}^T\boldsymbol{d} \text{ for all } \boldsymbol{d} \in \mathbb{R}^n\}.$$

Each vector $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$ is again called a *subgradient* of f at $\boldsymbol{x}$.

The subdifferential has the same basic properties than in convex case.

Theorem 3.3 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$ with a Lipschitz constant K. Then the subdifferential $\partial f(\boldsymbol{x})$ is a nonempty, convex, and compact set such that*

$$\partial f(\boldsymbol{x}) \subseteq B(\boldsymbol{0}; K).$$

Proof By Corollary 3.1 function $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ is convex and thus by Theorem 2.24 epi $f^\circ(\boldsymbol{x}; \cdot)$ is a convex set and by Theorem 2.22 and Exercise 4.29 it is closed. If $\boldsymbol{d}^* \in \mathbb{R}^n$ is fixed, then $(\boldsymbol{d}^*, f^\circ(\boldsymbol{x}; \boldsymbol{d}^*)) \in$ epi $f^\circ(\boldsymbol{x}; \cdot)$ is nonempty, furthermore we have $(\boldsymbol{d}^*, f^\circ(\boldsymbol{x}; \boldsymbol{d}^*)) \in$ bd epi $f^\circ(\boldsymbol{x}; \cdot)$. Then due to Theorem 2.7 there exists a hyperplane supporting epi $f^\circ(\boldsymbol{x}; \cdot)$ at $(\boldsymbol{d}^*, f^\circ(\boldsymbol{x}; \boldsymbol{d}^*))$, in other words there exists

$(\boldsymbol{\xi}^*, \mu) \neq (\mathbf{0}, 0)$ where $\boldsymbol{\xi}^* \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ such that for all $(\boldsymbol{d}, r) \in \text{epi } f^\circ(\boldsymbol{x};\ \cdot)$ we have

$$\begin{aligned}(\boldsymbol{\xi}^*, \mu)^T \big((\boldsymbol{d}, r) - (\boldsymbol{d}^*, f^\circ(\boldsymbol{x}; \boldsymbol{d}^*))\big)& \\ &= (\boldsymbol{\xi}^*)^T (\boldsymbol{d} - \boldsymbol{d}^*) + \mu(r - f^\circ(\boldsymbol{x}; \boldsymbol{d}^*)) \\ &\leq 0.\end{aligned} \tag{3.1}$$

Just like in the proof of Theorem 2.27 we can deduce that $\mu < 0$. Dividing the inequality (3.1) by $|\mu|$ and noting $\boldsymbol{\xi} := \boldsymbol{\xi}^*/|\mu|$ we get

$$\boldsymbol{\xi}^T (\boldsymbol{d} - \boldsymbol{d}^*) - r + f^\circ(\boldsymbol{x}; \boldsymbol{d}^*) \leq 0 \quad \text{for all } (\boldsymbol{d}, r) \in \text{epi } f^\circ(\boldsymbol{x};\ \cdot).$$

Now choosing $r := f^\circ(\boldsymbol{x}; \boldsymbol{d})$ we get

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) - f^\circ(\boldsymbol{x}; \boldsymbol{d}^*) \geq \boldsymbol{\xi}^T (\boldsymbol{d} - \boldsymbol{d}^*) \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

Then from the subadditivity of the generalized directional derivative (Theorem 3.1) we obtain

$$f^\circ(\boldsymbol{x}; \boldsymbol{d} - \boldsymbol{d}^*) \geq \boldsymbol{\xi}^T (\boldsymbol{d} - \boldsymbol{d}^*) \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n,$$

which by the Definition 3.2 means that $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$.

For convexity, let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \partial f(\boldsymbol{x})$ and $\lambda \in [0, 1]$. Then for all $\boldsymbol{d} \in \mathbb{R}^n$ we have

$$\begin{aligned}(\lambda\boldsymbol{\xi}_1 + (1 - \lambda)\boldsymbol{\xi}_2)^T \boldsymbol{d} &= \lambda\boldsymbol{\xi}_1^T \boldsymbol{d} + (1 - \lambda)\boldsymbol{\xi}_2^T \boldsymbol{d} \\ &\leq \lambda f^\circ(\boldsymbol{x}; \boldsymbol{d}) + (1 - \lambda) f^\circ(\boldsymbol{x}; \boldsymbol{d}) \\ &= f^\circ(\boldsymbol{x}; \boldsymbol{d}),\end{aligned}$$

whence $\lambda\boldsymbol{\xi}_1 + (1 - \lambda)\boldsymbol{\xi}_2 \in \partial f(\boldsymbol{x})$; thus $\partial f(\boldsymbol{x})$ is convex.

For an arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$ we get by Theorem 3.1 (i)

$$\|\boldsymbol{\xi}\|^2 = |\boldsymbol{\xi}^T \boldsymbol{\xi}| \leq |f^\circ(\boldsymbol{x}; \boldsymbol{\xi})| \leq K\|\boldsymbol{\xi}\|.$$

In other words, $\partial f(\boldsymbol{x})$ is bounded such that

$$\partial f(\boldsymbol{x}) \subseteq B(\mathbf{0}; K).$$

For the compactness it now suffices to show that $\partial f(\boldsymbol{x})$ is closed. To see this, let $(\boldsymbol{\xi}_i) \subset \partial f(\boldsymbol{x})$ be a sequence such that $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$. Then

$$\boldsymbol{\xi}^T \boldsymbol{d} = \lim_{i \to \infty} \boldsymbol{\xi}_i^T \boldsymbol{d} \leq \lim_{i \to \infty} f^\circ(\boldsymbol{x}; \boldsymbol{d}) = f^\circ(\boldsymbol{x}; \boldsymbol{d}),$$

which means that $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$ and thus $\partial f(\boldsymbol{x})$ is closed. □

The generalized directional derivative can be calculated from the subdifferential similarly to the directional derivative in Theorem 2.28 (ii).

Theorem 3.4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Then*

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \max \{\boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{x})\} \quad \textit{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

Proof The proof is identical with the proof of Theorem 2.28 (ii) replacing $f'(\boldsymbol{x}; \boldsymbol{d})$ by $f^\circ(\boldsymbol{x}; \boldsymbol{d})$ and $\partial_c f(\boldsymbol{x})$ by $\partial f(\boldsymbol{x})$, respectively. □

The upper semicontinuity of the generalized directional derivative implies the same property of the subdifferential mapping.

Theorem 3.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Then the mapping $\partial f : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is upper semicontinuous.*

Proof Let $(\boldsymbol{y}_i) \subset \mathbb{R}^n$ and $(\boldsymbol{\xi}_i) \subset \partial f(\boldsymbol{y}_i)$ be sequences such that $\boldsymbol{y}_i \to \boldsymbol{x}$ and $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$. Then for all $\boldsymbol{d} \in \mathbb{R}^n$ we have

$$\boldsymbol{\xi}^T \boldsymbol{d} = \lim_{i\to\infty} \boldsymbol{\xi}_i^T \boldsymbol{d} \le \limsup_{i\to\infty} f^\circ(\boldsymbol{y}_i; \boldsymbol{d}).$$

By Theorem 3.2 the function $f^\circ(\boldsymbol{x}; \cdot)$ is upper semicontinuous, hence

$$\boldsymbol{\xi}^T \boldsymbol{d} \le f^\circ(\boldsymbol{x}; \boldsymbol{d}).$$

Thus the mapping $\partial f(\cdot)$ is also upper semicontinuous and the proof is complete. □

The next two theorems show that the subdifferential really is a generalization of the classical derivative.

Theorem 3.6 *Let f be locally Lipschitz continuous and differentiable at $\boldsymbol{x}$. Then*

$$\nabla f(\boldsymbol{x}) \in \partial f(\boldsymbol{x}).$$

Proof By the definition of differentiability the directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ exists for all $\boldsymbol{d} \in \mathbb{R}^n$ and $f'(\boldsymbol{x}; \boldsymbol{d}) = \nabla f(\boldsymbol{x})^T \boldsymbol{v}$. Since $f' \le f^\circ$, it follows that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) \ge \nabla f(\boldsymbol{x})^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n$$

and thus $\nabla f(\boldsymbol{x}) \in \partial f(\boldsymbol{x})$. □

Theorem 3.7 *If f is continuously differentiable at $\boldsymbol{x}$, then*

$$\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}.$$

Proof In view of the Lemma 1.1 the function f is locally Lipschitz continuous at $\boldsymbol{x}$. Then due to Theorem 3.6 we have $\nabla f(\boldsymbol{x}) \in \partial f(\boldsymbol{x})$. The continuous differentiability means that if $\boldsymbol{x}_i \to \boldsymbol{x}$, then the gradient sequence $\nabla f(\boldsymbol{x}_i)$ converges to $\nabla f(\boldsymbol{x})$ and for all $\boldsymbol{d} \in \mathbb{R}^n$ we have

$$\begin{aligned}\lim_{\boldsymbol{x}_i \to \boldsymbol{x}} f'(\boldsymbol{x}_i; \boldsymbol{d}) &= \lim_{\boldsymbol{x}_i \to \boldsymbol{x}} \lim_{t \downarrow 0} \frac{f(\boldsymbol{x}_i + t\boldsymbol{d}) - f(\boldsymbol{x}_i)}{t} \\ &= \lim_{\boldsymbol{x}_i \to \boldsymbol{x}} \nabla f(\boldsymbol{x}_i)^T \boldsymbol{d} \\ &= \nabla f(\boldsymbol{x})^T \boldsymbol{d} = f'(\boldsymbol{x}; \boldsymbol{d}).\end{aligned}$$

Thus for all $\boldsymbol{d} \in \mathbb{R}^n$ we get

$$\begin{aligned}f'(\boldsymbol{x}; \boldsymbol{v}) &= \lim_{\boldsymbol{x}_i \to \boldsymbol{x}} f'(\boldsymbol{x}_i; \boldsymbol{v}) \\ &= \lim_{\substack{\boldsymbol{x}_i \to \boldsymbol{x} \\ t \downarrow 0}} \frac{f(\boldsymbol{x}_i + t\boldsymbol{v}) - f(\boldsymbol{x}_i)}{t} \\ &= \limsup_{\substack{\boldsymbol{x}_i \to \boldsymbol{x} \\ t \downarrow 0}} \frac{f(\boldsymbol{x}_i + t\boldsymbol{v}) - f(\boldsymbol{x}_i)}{t} \\ &= f^\circ(\boldsymbol{x}; \boldsymbol{v}),\end{aligned}$$

in other words $f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \nabla f(\boldsymbol{x})^T \boldsymbol{d}$ for all $\boldsymbol{d} \in \mathbb{R}^n$. Suppose now, that there exists another subgradient $\xi \in \partial f(\boldsymbol{x})$ such that $\xi \neq \nabla f(\boldsymbol{x})$. Then

$$\xi^T \boldsymbol{d} \leq f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \nabla f(\boldsymbol{x})^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n$$

and thus

$$(\xi - \nabla f(\boldsymbol{x}))^T \boldsymbol{d} \leq 0 \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

By choosing $\boldsymbol{d} := \xi - \nabla f(\boldsymbol{x})$ we get

$$\|\xi - \nabla f(\boldsymbol{x})\|^2 \leq 0,$$

implying that $\xi = \nabla f(\boldsymbol{x})$, which contradicts the assumption. Thus $\nabla f(\boldsymbol{x})$ is the unique subgradient at $\boldsymbol{x}$. □

The continuous differentiability is critical in Theorem 3.7 as the next example shows.

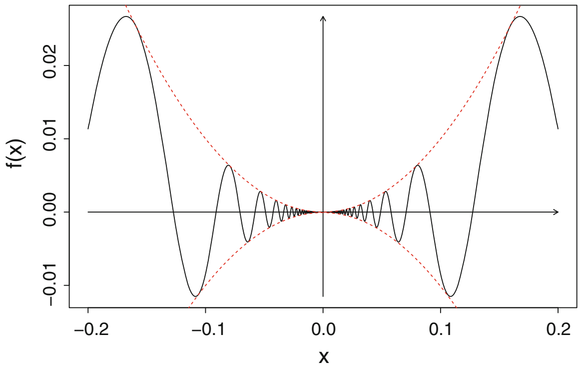


Fig. 3.1 A plot of function f when $x \in [-0.2, 0.2]$

Example 3.1 (*Differentiable but nonsmooth function*) We now proof that the function (see Fig. 3.1)

$$f(x) = \begin{cases} 0, & x = 0 \\ x^2 \cos(\frac{1}{x}), & x \neq 0 \end{cases}$$

is locally Lipschitz continuous, differentiable everywhere but nonsmooth (not continuously differentiable) and there exists a point $y \in \mathbb{R}$ such that $\partial f(y) \neq \{\nabla f(y)\}$.

We first show the differentiability. Function $g(x) := x^2 \cos\left(\frac{1}{x}\right)$ is differentiable everywhere but at 0 and it's derivative is

$$g'(x) = \sin\left(\frac{1}{x}\right) + 2x \cos\left(\frac{1}{x}\right). \tag{3.2}$$

The derivative is also continuous when $x \neq 0$. Thus, f is continuously differentiable when $x \neq 0$. Since

$$f(0 + x) - f(0) = x^2 \cos\left(\frac{1}{x}\right)$$

and $\lim_{x \to 0} |x| \cos\left(\frac{1}{x}\right) = 0$ the function f is differentiable at the point 0 and $f'(0) = 0$.

However, from (3.2) we see that the limit $\lim_{x \to 0} f'(x)$ does not exist implying that f is not continuously differentiable.

Next we prove that f is locally Lipschitz continuous. As stated before (Lemma 1.1), continuously differentiable function is locally Lipschitz continuous. Hence, f is locally Lipschitz continuous, when $x \neq 0$. We prove that f is locally Lipschitz continuous also at the point $x = 0$ by considering Lipschitz condition with different values of $y, z \in (-1, 1), y \neq z$.

Let $-1 < y < z < 0$. The function f is continuously differentiable on the interval (y, z). Then,

$$\begin{aligned} |f(z) - f(y)| &= \left| \int_y^z f'(x)dx \right| \leq \int_y^z \max_{x \in [y,z]} \left\{ \left| f'(x) \right| \right\} dx \\ &= \max_{x \in [y,z]} \left\{ \left| \sin(\frac{1}{x}) + 2x \cos(\frac{1}{x}) \right| \right\} (z - y) \\ &\leq (1 + 2 \cdot 1 \cdot 1) |z - y| = 3 |z - y|, \end{aligned}$$

hence the Lipschitz condition holds. Due to the symmetry of the function f the Lipschitz condition holds also when $0 < y < z < 1$.

Now, let $-1 < y < 0$ and $0 < z < 1$. Then $|y + z| < |y - z|$ and the symmetry implies $f(-z) = f(z)$. Thus,

$$|f(y) - f(z)| = |f(y) - f(-z)| \leq 3 |y + z| \leq 3 |y - z|,$$

where the first inequality follows from the consideration of the case $-1 < y < z < 0$. Thus, the Lipschitz condition holds when $-1 < y < 0$ and $0 < z < 1$. Finally, let $y = 0$ and $z \in (-1, 1) \setminus \{0\}$. Then

$$|f(0) - f(z)| = \left| z^2 \cos\left(\frac{1}{z}\right) \right| \leq |z| \, 1 \cdot 1 = |0 - z|,$$

and the Lipschitz condition holds for this case too. Thus, the function f is locally Lipschitz continuous. Consider the subdifferential of the function f at the point 0. By choosing the sequence $x^i = \left(\frac{1}{2i\pi + \frac{\pi}{2}}\right)$, $i \in \mathbb{N}$ we see that $\lim_{i \to \infty} f'(x^i) = 1$. Correspondingly, by choosing the sequence $x^i = \left(\frac{1}{2i\pi - \frac{\pi}{2}}\right)$, $i \in \mathbb{N}$ we see that $\lim_{i \to \infty} f'(x) = -1$. By Theorem 3.9 this means that

$$[-1, 1] \subseteq \partial f(0).$$

Particularly, we have $\partial f(0) \neq f'(0)$.

The following theorem shows that the subdifferential for Lipschitz continuous functions is a generalization of the subdifferential for convex functions.

Theorem 3.8 *If the function* $f : \mathbb{R}^n \to \mathbb{R}$ *is convex, then*

(i) $f'(\boldsymbol{x}; \boldsymbol{d}) = f^\circ(\boldsymbol{x}; \boldsymbol{d})$ *for all* $\boldsymbol{d} \in \mathbb{R}^n$ *and*
(ii) $\partial_c f(\boldsymbol{x}) = \partial f(\boldsymbol{x})$.

Proof Note first, that due to Theorem 2.22 f is locally Lipschitz continuous at any $\boldsymbol{x} \in \mathbb{R}^n$. Then, if (i) is true, (ii) follows from the definition of subdifferential and Theorem 2.28 (i), thus it suffices to prove (i). By the definition of the generalized directional derivative, one has $f^\circ(\boldsymbol{x}; \boldsymbol{d}) \geq f'(\boldsymbol{x}; \boldsymbol{d})$ for all $\boldsymbol{d} \in \mathbb{R}^n$. On the other hand, if $\delta > 0$ is fixed, then

$$\begin{aligned} f^\circ(\boldsymbol{x}; \boldsymbol{d}) &= \limsup_{\substack{\boldsymbol{x}' \to \boldsymbol{x} \\ t \downarrow 0}} \frac{f(\boldsymbol{x}' + t\boldsymbol{d}) - f(\boldsymbol{x}')}{t} \\ &= \lim_{\varepsilon \downarrow 0} \sup_{\|\boldsymbol{x}' - \boldsymbol{x}\| < \varepsilon\delta} \sup_{0 < t < \varepsilon} \frac{f(\boldsymbol{x}' + t\boldsymbol{d}) - f(\boldsymbol{x}')}{t}. \end{aligned}$$

From the proof of Theorem 2.25 we get that the function $\varphi(t) = (1/t)(f(\boldsymbol{x}' + t\boldsymbol{d}) - f(\boldsymbol{x}'))$ is nondecreasing and hence we can write

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \lim_{\varepsilon \downarrow 0} \sup_{\|\boldsymbol{x}' - \boldsymbol{x}\| < \varepsilon\delta} \frac{f(\boldsymbol{x}' + \varepsilon\boldsymbol{d}) - f(\boldsymbol{x}')}{\varepsilon}.$$

Now by the Lipschitz condition, for any $\boldsymbol{x}' \in B(\boldsymbol{x}; \varepsilon\delta)$ with some $\varepsilon > 0$, one has

$$\begin{aligned} \left| \frac{f(\boldsymbol{x}' + \varepsilon\boldsymbol{d}) - f(\boldsymbol{x}')}{\varepsilon} - \frac{f(\boldsymbol{x} + \varepsilon\boldsymbol{d}) - f(\boldsymbol{x})}{\varepsilon} \right| &\leq \left| \frac{f(\boldsymbol{x}' + \varepsilon\boldsymbol{d}) - f(\boldsymbol{x} + \varepsilon\boldsymbol{d})}{\varepsilon} \right| \\ &\quad + \left| \frac{f(\boldsymbol{x}) - f(\boldsymbol{x}')}{\varepsilon} \right| \\ &\leq \frac{K}{\varepsilon}\|\boldsymbol{x}' - \boldsymbol{x}\| + \frac{K}{\varepsilon}\|\boldsymbol{x}' - \boldsymbol{x}\| \\ &\leq \frac{2K}{\varepsilon}\varepsilon\delta = 2K\delta \end{aligned}$$

so that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) \leq \lim_{\varepsilon \downarrow 0} \frac{f(\boldsymbol{x} + \varepsilon\boldsymbol{d}) - f(\boldsymbol{x})}{\varepsilon} + 2\delta K = f'(\boldsymbol{x}; \boldsymbol{d}) + 2\delta K.$$

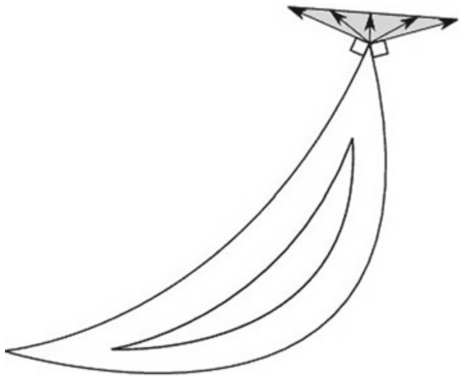


Fig. 3.2 Subdifferential of nonconvex function

Since $\delta > 0$ is arbitrary, we deduce

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) \le f'(\boldsymbol{x}; \boldsymbol{d})$$

and the proof is complete. □

The following result is essential when calculating subgradients in practice. Namely the subdifferential can be constructed as a convex hull of all possible limits of gradients at point $\boldsymbol{x}_i$ converging to $\boldsymbol{x}$. Figure 3.2 illustrates the subdifferential of nonconvex function. Let

$$\Omega_f = \{\boldsymbol{x} \in \mathbb{R}^n \mid f \text{is not differentiable at the point } \boldsymbol{x}\}$$

be the set of points where f is not differentiable. By Rademacher's Theorem [82] a function which is Lipschitz continuous on a set $U \subseteq \mathbb{R}^n$ is differentiable almost everywhere on U, in other words, meas$(\Omega_f) = 0$ in U.

Theorem 3.9 *Let* $f: \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz continuous at* $\boldsymbol{x} \in \mathbb{R}^n$. *Then*

$$\partial f(\boldsymbol{x}) = \text{conv}\,\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \textit{there exists } (\boldsymbol{x}_i) \subset \mathbb{R}^n \setminus \Omega_f \textit{ such that } \boldsymbol{x}_i \to \boldsymbol{x} \textit{ and } \nabla f(\boldsymbol{x}_i) \to \boldsymbol{\xi}\}.$$

Proof Let

$$S := \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \exists\, (\boldsymbol{x}_i) \subset \mathbb{R}^n \setminus \Omega_f \text{ s.}t.\ \boldsymbol{x}_i \to \boldsymbol{x} \text{ and } \nabla f(\boldsymbol{x}_i) \to \boldsymbol{\xi}\}.$$

We shall first show that S is nonempty. It follows from Rademacher's Theorem that the measure of the set Ω_f is zero. Then there exists a sequence $(\boldsymbol{x}_i) \subset \mathbb{R}^n$ such that $\boldsymbol{x}_i \notin \Omega_f$ and $\boldsymbol{x}_i \to \boldsymbol{x}$. Since f is locally Lipschitz continuous at $\boldsymbol{x}$, Theorem 3.3 implies that there exists $\varepsilon > 0$ such that for any $\boldsymbol{x}_i \in B(\boldsymbol{x}; \varepsilon)$

$$\|\boldsymbol{\xi}_i\| \le K \quad \text{for all} \quad \boldsymbol{\xi}_i \in \partial f(\boldsymbol{x}_i). \tag{3.3}$$

This means that the point-to-set mapping ∂f is locally bounded on $B(\boldsymbol{x}; \varepsilon)$. By Theorem 3.6 we have

$$\nabla f(\boldsymbol{x}_i) \in \partial f(\boldsymbol{x}_i)$$

and thus by (3.3) the sequence $\{\nabla f(\boldsymbol{x}_i)\}$ is bounded. Then the sequence $\{\nabla f(\boldsymbol{x}_i)\}$ admits a convergent subsequence $\{\nabla f(\boldsymbol{x}_{i_k})\}$ and there exists $\boldsymbol{\xi} \in \mathbb{R}^n$ such that $\nabla f(\boldsymbol{x}_{i_k}) \to \boldsymbol{\xi}$. From Theorem 3.5 we know that ∂f is upper semicontinuous which means that $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$.

Now we have proved that S is nonempty, bounded set and $S \subseteq \partial f(\boldsymbol{x})$. By Theorem 3.3 the set $\partial f(\boldsymbol{x})$ is convex and thus we have

$$\text{conv } S \subseteq \text{conv } \partial f(\boldsymbol{x}) = \partial f(\boldsymbol{x}).$$

For the reverse inclusion we show that S is also closed, hence compact. To see this, let $(\boldsymbol{\xi}_j) \in S$ be a sequence such that $\boldsymbol{\xi}_j \to \boldsymbol{\xi}$. Then

$$\boldsymbol{\xi}_j = \lim_{i\to\infty} \nabla f\big(\boldsymbol{x}_i^j\big), \quad \text{where} \quad \boldsymbol{x}_i^j \to \boldsymbol{x} \quad \text{as } i \to \infty \quad \text{and} \quad \boldsymbol{x}_i^j \notin \Omega_f.$$

Extracting subsequences if necessary, there are points $\boldsymbol{x}_i \in \mathbb{R}^n$ such that

$$\boldsymbol{x}_i := \lim_{j\to\infty} \boldsymbol{x}_i^j \quad \text{for each} \quad i \in \mathbb{N}.$$

Then it holds that $\boldsymbol{x}_i \to \boldsymbol{x}$, $\boldsymbol{x}_i \notin \Omega_f$ and

$$\boldsymbol{\xi} = \lim_{j\to\infty} \lim_{i\to\infty} \nabla f\big(\boldsymbol{x}_i^j\big) = \lim_{i\to\infty} \lim_{j\to\infty} \nabla f(\boldsymbol{x}_i^j) = \lim_{i\to\infty} \nabla f(\boldsymbol{x}_i).$$

Thus $\boldsymbol{\xi} \in S$ and the set S is closed and compact. Then its convex hull conv S is also compact and we only need to show that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \max_{\boldsymbol{\xi}\in\partial f(\boldsymbol{x})} \boldsymbol{\xi}^T\boldsymbol{d} \le \max_{\boldsymbol{\eta}\in\text{conv } S} \boldsymbol{\eta}^T\boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

This follows from the next lemma. □

Lemma 3.1 *Let a function* $f : \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz continuous,* $\varepsilon > 0$ *and* $\boldsymbol{d} \in \mathbb{R}^n$ *be such that* $\boldsymbol{d} \neq \mathbf{0}$. *Then*

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) - \varepsilon \le \limsup \{\nabla f(\boldsymbol{y})^T\boldsymbol{d} \mid \boldsymbol{y} \to \boldsymbol{x},\ \boldsymbol{y} \notin \Omega_f\}. \tag{3.4}$$

Proof Let $\alpha := \limsup \{\nabla f(\boldsymbol{y})^T\boldsymbol{d} \mid \boldsymbol{y} \to \boldsymbol{x},\ \boldsymbol{y} \notin \Omega_f\}$. Then, by definition, there exists $\delta > 0$ such that the conditions

$$\boldsymbol{y} \in B(\boldsymbol{x}; \delta) \quad \text{and} \quad \boldsymbol{y} \notin \Omega_f$$

imply $\nabla f(\boldsymbol{y})^T \boldsymbol{d} \le \alpha + \varepsilon$. We also choose δ small enough so that Ω_f has measure zero in $B(\boldsymbol{x}; \delta)$. Now consider the line segments $L_{\boldsymbol{y}} = \{\boldsymbol{y} + t\boldsymbol{d} \mid 0 < t < \delta/(2|\boldsymbol{d}|)\}$. Since Ω_f has measure zero in $B(\boldsymbol{x}; \delta)$, it follows from Fubini's Theorem (see e.g. [202] p. 78) that for almost every $\boldsymbol{y}$ in $B(\boldsymbol{x}; \delta/2)$, the line segment $L_{\boldsymbol{y}}$ meets Ω_f in a set of zero one-dimensional measure. Let $\boldsymbol{y}$ be any point in $B(\boldsymbol{x}; \delta/2)$ having this property and let t lie in $\big(0, \delta/(2|\boldsymbol{d}|)\big)$. Then

$$f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y}) = \int_0^t \nabla f(\boldsymbol{y} + s\boldsymbol{d})^T \boldsymbol{d}\, ds,$$

since ∇f exists almost everywhere on $L_{\boldsymbol{y}}$. Since one has $|\boldsymbol{y} + s\boldsymbol{d} - \boldsymbol{x}| < \delta$ for $0 < s < t$, it follows that $\nabla f(\boldsymbol{y} + s\boldsymbol{d})^T \boldsymbol{d} \le \alpha + \varepsilon$, whence

$$f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y}) \le t(\alpha + \varepsilon).$$

Since this is true for almost all $\boldsymbol{y}$ within $\delta/2$ of $\boldsymbol{x}$ and for all t in $\big(0, \delta/(2|\boldsymbol{d}|)\big)$, and since f is continuous, it is in fact true for all such $\boldsymbol{y}$ and t. We deduce that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) \le \alpha + \varepsilon,$$

which completes the proof. □

Example 3.2 (*Absolute-value function*) Let us again consider the absolute-value function on reals

$$f(x) = |x|.$$

The subdifferential of this function at $x = 0$ is given by

$$\partial f(0) = \text{conv}\,\{-1, 1\} = [-1, 1].$$

3.1.3 ε-Subdifferentials

It would be possible to generalize the ε-subdifferential for convex functions analogously also for Lipschitz continuous functions by using the generalized directional derivative. However, the theory of nonsmooth optimization has shown that it will be more useful to use the Goldstein ε-subdifferential for nonconvex functions. For this reason we shall now define and examine that instead of the natural generalization.

Definition 3.3 Let a function $f: \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $\boldsymbol{x} \in \mathbb{R}^n$ and let $\varepsilon \geq 0$. Then the *Goldstein ε-subdifferential* of f is the set

$$\partial_\varepsilon^G f(\boldsymbol{x}) = \text{cl conv}\,\{\partial f(\boldsymbol{y}) \mid \boldsymbol{y} \in B(\boldsymbol{x}; \varepsilon)\}.$$

Each element $\boldsymbol{\xi} \in \partial_\varepsilon^G f(\boldsymbol{x})$ is called an *ε-subgradient* of the function f at $\boldsymbol{x}$.

The following theorem summarizes some basic properties of the Goldstein ε-subdifferential.

Theorem 3.10 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$ with a Lipschitz constant K. Then*

(i) $\partial_0^G f(\boldsymbol{x}) = \partial f(\boldsymbol{x})$.
(ii) *if* $\varepsilon_1 \leq \varepsilon_2$, *then* $\partial_{\varepsilon_1}^G f(\boldsymbol{x}) \subseteq \partial_{\varepsilon_2}^G f(\boldsymbol{x})$.
(iii) $\partial_\varepsilon^G f(\boldsymbol{x})$ *is a nonempty, convex, and compact set such that* $\partial_\varepsilon^G f(\boldsymbol{x}) \subseteq B(\mathbf{0}; K)$.
(iv) *the mapping* $\partial_\varepsilon^G f: \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ *is upper semicontinuous.*

Proof Assertions (i) and (ii) follow directly from the definition of the Goldstein ε-subdifferential. We turn now to (iii). Assertion (i) implies that for all $\varepsilon \geq 0$

$$\partial f(\boldsymbol{x}) = \partial_0^G f(\boldsymbol{x}) \subseteq \partial_\varepsilon^G f(\boldsymbol{x}).$$

From Theorem 3.3 we know that $\partial f(\boldsymbol{x})$ is nonempty and so is $\partial_\varepsilon^G f(\boldsymbol{x})$. Because $\partial_\varepsilon^G f(\boldsymbol{x})$ is the convex hull of a set, it is evidently convex and the compactness follows from the same property of $\partial f(\boldsymbol{x})$.

Let $\boldsymbol{\xi} \in \partial_\varepsilon^G f(\boldsymbol{x})$ be arbitrary. Then $\boldsymbol{\xi} = \sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i$, where $\boldsymbol{\xi}_i \in \partial f(\boldsymbol{y}_i)$, $\boldsymbol{y}_i \in B(\boldsymbol{x}; \varepsilon)$, $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Now by Theorem 3.3 we have

$$\|\boldsymbol{\xi}\| = \|\sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i\| \leq \sum_{i=1}^m \lambda_i \|\boldsymbol{\xi}_i\| \leq \sum_{i=1}^m \lambda_i \cdot K = K,$$

in other words, $\partial_\varepsilon^G f(\boldsymbol{x}) \subseteq B(\mathbf{0}; K)$. Assertion (iv) follows directly from the same property of $\partial f(\boldsymbol{x})$ (Theorem 3.3) and thus the proof is complete. □

As a corollary to Theorem 3.9 we obtain the following result.

Corollary 3.2 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Then*

$$\partial_\varepsilon^G f(\boldsymbol{x}) = \text{cl conv}\,\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \textit{there exists } (\boldsymbol{y}_i) \subset \mathbb{R}^n \setminus \Omega_f \textit{ such that } \boldsymbol{y}_i \to \boldsymbol{y},\ \nabla f(\boldsymbol{y}_i) \to \boldsymbol{\xi},\ \textit{and } \boldsymbol{y} \in B(\boldsymbol{x}; \varepsilon)\}.$$

As in the convex case, the Goldstein ε-subdifferential contains in a compressed form the subgradient information from the whole neighborhood of $\boldsymbol{x}$.

Theorem 3.11 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $\boldsymbol{x} \in \mathbb{R}^n$. Then for any $\varepsilon \geq 0$ we have*

$$\partial f(\boldsymbol{y}) \subseteq \partial_\varepsilon^G f(\boldsymbol{x}) \quad \textit{for all} \quad \boldsymbol{y} \in B(\boldsymbol{x}; \varepsilon).$$

Proof This follows directly from definition of the Goldstein ε-subdifferential. □

We conclude this section by considering the relationship between the ε-subdifferential for convex functions and the Goldstein ε-subdifferential.

Theorem 3.12 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex with Lipschitz constant K at $\boldsymbol{x}$. Then for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon^G f(\boldsymbol{x}) \subseteq \partial_{2K\varepsilon} f(\boldsymbol{x}). \tag{3.5}$$

Proof Suppose, that $\boldsymbol{\xi} \in \partial_\varepsilon^G f(\boldsymbol{x})$. Then $\boldsymbol{\xi} = \sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i$, where $\boldsymbol{\xi}_i \in \partial f(\boldsymbol{y}_i)$, $\boldsymbol{y}_i \in B(\boldsymbol{x}; \varepsilon)$, $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Since $\boldsymbol{\xi}_i \in \partial f(\boldsymbol{y}_i)$, for all $i = 1, \ldots, m$ one has

$$f(\boldsymbol{z}) \geq f(\boldsymbol{y}_i) + \boldsymbol{\xi}_i^T (\boldsymbol{z} - \boldsymbol{y}_i) \quad \text{for all} \quad \boldsymbol{z} \in \mathbb{R}^n.$$

We multiply both sides by λ_i and sum over i to get

$$\begin{aligned} f(\boldsymbol{z}) &= \sum_{i=1}^m \lambda_i f(\boldsymbol{z}) \\ &\geq \sum_{i=1}^m \lambda_i f(\boldsymbol{y}_i) + \sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i^T (\boldsymbol{z} - \boldsymbol{y}_i) \\ &= f(\boldsymbol{x}) + \boldsymbol{\xi}^T (\boldsymbol{z} - \boldsymbol{x}) \\ &\quad - \big(f(\boldsymbol{x}) - \sum_{i=1}^m \lambda_i f(\boldsymbol{y}_i) + \boldsymbol{\xi}^T (\boldsymbol{z} - \boldsymbol{x}) - \sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i^T (\boldsymbol{z} - \boldsymbol{y}_i) \big) \end{aligned}$$

and by using the Lipschitz condition and Theorem 3.3 we obtain

$$\begin{aligned} |f(\boldsymbol{x}) - &\sum_{i=1}^m \lambda_i f(\boldsymbol{y}_i) + \boldsymbol{\xi}^T (\boldsymbol{z} - \boldsymbol{x}) - \sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i^T (\boldsymbol{z} - \boldsymbol{y}_i)| \\ &\leq |f(\boldsymbol{x}) - \sum_{i=1}^m \lambda_i f(\boldsymbol{y}_i)| + |\sum_{i=1}^m \lambda_i \boldsymbol{\xi}^T (\boldsymbol{z} - \boldsymbol{x}) - \sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i^T (\boldsymbol{z} - \boldsymbol{y}_i)| \\ &\leq \sum_{i=1}^m \lambda_i |f(\boldsymbol{x}) - f(\boldsymbol{y}_i)| + \sum_{i=1}^m \lambda_i |\boldsymbol{\xi}_i^T (\boldsymbol{x} - \boldsymbol{y}_i)| \end{aligned}$$

$$\leq \sum_{i=1}^{m} \lambda_i \big(K \,\|\boldsymbol{x} - \boldsymbol{y}_i\| + \|\boldsymbol{\xi}\| \,\|\boldsymbol{x} - \boldsymbol{y}_i\| \big) = 2K\varepsilon,$$

which means that $\boldsymbol{\xi} \in \partial_{2K\varepsilon} f(\boldsymbol{x})$. □

3.1.4 Generalized Jacobians

In what follows we shall need derivatives of a nonsmooth vector-valued function $\boldsymbol{h}\colon \mathbb{R}^n \to \mathbb{R}^m$, written in terms of component functions as $\boldsymbol{h}(\boldsymbol{x}) = \big(h_1(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x})\big)^T$. Each component function h_i for $i = 1, \ldots, m$ (and hence $\boldsymbol{h}$) is supposed to be locally Lipschitz continuous. Then, due to the Rademacher's Theorem [82] we conclude that $\boldsymbol{h}$ is differentiable almost everywhere. We denote again by $\Omega_{\boldsymbol{h}}$ the set in $\mathbb{R}^n$ where $\boldsymbol{h}$ fails to be differentiable and by $\nabla \boldsymbol{h}(\boldsymbol{x})$ for $\boldsymbol{x} \notin \Omega_{\boldsymbol{h}}$ the usual $m \times n$ Jacobian matrix. Based on Theorem 3.9 we generalize now the derivative of $\boldsymbol{h}$.

Definition 3.4 Let $\boldsymbol{h}\colon \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous at a point $\boldsymbol{x} \in \mathbb{R}^n$. Then the *generalized Jacobian* of $\boldsymbol{h}$ at $\boldsymbol{x}$ is the set

$$\partial \boldsymbol{h}(\boldsymbol{x}) := \operatorname{conv} \big\{ A \in \mathbb{R}^{m \times n} \mid \text{there exists } (\boldsymbol{x}_i) \subset \mathbb{R}^n \setminus \Omega_{\boldsymbol{h}} \text{ such that } \boldsymbol{x}_i \to \boldsymbol{x} \text{ and } \nabla \boldsymbol{h}(\boldsymbol{x}_i) \to A \big\}. \tag{3.6}$$

The space of $m \times n$ matrices is endowed with the norm

$$\|A\|_{m \times n} := \left(\sum_{i=1}^{m} \|A_i\|^2 \right)^{\frac{1}{2}}, \tag{3.7}$$

where A_i is the ith row of A. Some basic properties of $\partial \boldsymbol{h}(\boldsymbol{x})$ will now be listed.

Corollary 3.3 *Let h_i for $i = 1, \ldots, m$ be locally Lipschitz continuous at $\boldsymbol{x}$ with constant K_i. Then*

(i) $\boldsymbol{h}(\boldsymbol{x}) = \big(h_1(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x})\big)^T$ *is locally Lipschitz continuous at $\boldsymbol{x}$ with constant $K = \|\big(K_1, \ldots, K_m\big)^T\|$,*
(ii) $\partial \boldsymbol{h}(\boldsymbol{x})$ *is a nonempty, convex, and compact subset of $\mathbb{R}^{m \times n}$,*
(iii) *the mapping $\partial \boldsymbol{h}(\cdot)\colon \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is upper semicontinuous.*

Proof Follows directly from Theorems 3.3, 3.5 and 3.9. □

3.2 Subdifferential Calculus

In this section we shall derive an assortment of formulas that facilitate greatly the calculation of subdifferentials in practice. Note that due to Theorems 3.7 and 3.8 all the classical smooth and convex derivation rules can be obtained from these results as special cases. However, for locally Lipschitz continuous functions we have to be content with inclusions instead of equalities.

3.2.1 Subdifferential Regularity

In order to maintain equalities instead of inclusions in subderivation rules we need the following regularity property.

Definition 3.5 The function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be *subdifferentially regular* at $\boldsymbol{x} \in \mathbb{R}^n$ if it is locally Lipschitz continuous at $\boldsymbol{x}$ and for all $\boldsymbol{d} \in \mathbb{R}^n$ the classical directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ exists and we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) = f^\circ(\boldsymbol{x}; \boldsymbol{d}). \tag{3.8}$$

Note, that the equality (3.8) is not necessarily valid in general even if $f'(\boldsymbol{x}; \boldsymbol{d})$ exists. This is the case, for instance, with concave nonsmooth functions. For example, the function $f(x) = -|x|$ has the directional derivative $f'(0; 1) = -1$, but the generalized directional derivative is $f^\circ(0; 1) = 1$.

We now note some sufficient conditions for f to be subdifferentially regular.

Theorem 3.13 *The function* $f: \mathbb{R}^n \to \mathbb{R}$ *is subdifferentially regular at* $\boldsymbol{x}$ *if*

(i) f *is continuously differentiable at* $\boldsymbol{x}$*,*
(ii) f *is convex, or*
(iii) $f = \sum_{i=1}^m \lambda_i f_i$*, where* $\lambda_i > 0$ *and* f_i *is subdifferentially regular at* $\boldsymbol{x}$ *for each* $i = 1, \ldots, m$*.*

Proof (i) If f is continuously differentiable, then due to Lemma 3.1 f is locally Lipschitz continuous at $\boldsymbol{x}$. Furthermore, the directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ exists for all $\boldsymbol{d} \in \mathbb{R}^n$ and by the proof of Theorem 3.7 $f^\circ(\boldsymbol{x}; \boldsymbol{d}) = f'(\boldsymbol{x}; \boldsymbol{d})$ for all $\boldsymbol{d} \in \mathbb{R}^n$.
(ii) This follows from Theorems 2.25 and 3.8 (i).
(iii) It suffices to prove the formula for $m = 2$; the general case follows by induction. Clearly $f_1 + f_2$ is locally Lipschitz continuous at $\boldsymbol{x}$. If f is subdifferentially regular at $\boldsymbol{x}$ and $\lambda > 0$, then

$$(\lambda f)^\circ(\boldsymbol{x}; \boldsymbol{d}) = \lambda \cdot f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \lambda \cdot f'(\boldsymbol{x}; \boldsymbol{d}) = (\lambda f)'(\boldsymbol{x}; \boldsymbol{d}) \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

It is evident that $(f_1 + f_2)'$ always exists and $(f_1 + f_2)' = f_1' + f_2'$. By the definition of the generalized directional derivative $(f_1 + f_2)^\circ \geq (f_1 + f_2)'$. On the other hand

$$
\begin{aligned}
(f_1+f_2)^\circ(\boldsymbol{x};\boldsymbol{d}) &= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{(f_1+f_2)(\boldsymbol{y}+t\boldsymbol{d})-(f_1+f_2)(\boldsymbol{y})}{t}\\
&= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f_1(\boldsymbol{y}+t\boldsymbol{d})+f_2(\boldsymbol{y}+t\boldsymbol{d})-f_1(\boldsymbol{y})-f_2(\boldsymbol{y})}{t}\\
&\le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f_1(\boldsymbol{y}+t\boldsymbol{d})-f_1(\boldsymbol{y})}{t} + \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f_2(\boldsymbol{y}+t\boldsymbol{d})-f_2(\boldsymbol{y})}{t}\\
&= f_1^\circ(\boldsymbol{x};\boldsymbol{d})+f_2^\circ(\boldsymbol{x};\boldsymbol{d}).
\end{aligned}
$$

Then we have

$$(f_1+f_2)' = f_1'+f_2' = f_1^\circ + f_2^\circ \ge (f_1+f_2)^\circ,$$

thus

$$(f_1+f_2)' = (f_1+f_2)^\circ$$

and the proof is complete. □

Hence, convexity, as well as smoothness, implies subdifferential regularity. Furthermore, we are now able to formulate the following necessary and sufficient condition for convexity.

Theorem 3.14 *Let $f:\mathbb{R}^n \to \mathbb{R}$ be subdifferentially regular at all $\boldsymbol{x}\in\mathbb{R}^n$. Then f is convex if and only if for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ we have*

$$f(\boldsymbol{y})-f(\boldsymbol{x}) \ge f'(\boldsymbol{x};\boldsymbol{y}-\boldsymbol{x}). \tag{3.9}$$

Proof Suppose first, that f is convex. Then due to the definition of the subdifferential for any $\boldsymbol{x}\in\mathbb{R}^n$ we have

$$f(\boldsymbol{y}) \ge f(\boldsymbol{x}) + \boldsymbol{\xi}^T(\boldsymbol{y}-\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{\xi}\in\partial_c f(\boldsymbol{x}) \text{ and } \boldsymbol{y}\in\mathbb{R}^n.$$

Then Theorem 2.28 (ii) and subdifferential regularity imply that

$$f(\boldsymbol{y})-f(\boldsymbol{x}) \ge f'(\boldsymbol{x};\boldsymbol{y}-\boldsymbol{x}) = f^\circ(\boldsymbol{x};\boldsymbol{y}-\boldsymbol{x}).$$

Suppose next that inequality (3.9) is valid and let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ and $\lambda\in[0,1]$. Then due to the subdifferential regularity $f'(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y};\boldsymbol{d})$ exists for all $\boldsymbol{d}\in\mathbb{R}^n$ and by (3.9) and positive homogeneity of the directional derivative (Theorem 2.26) we have

$$
\begin{aligned}
f(\boldsymbol{x})-f(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}) &\ge f'(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y};(1-\lambda)\boldsymbol{x}-\boldsymbol{y})\\
&= (1-\lambda)f'(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y};\boldsymbol{x}-\boldsymbol{y}).
\end{aligned} \tag{3.10}
$$

Moreover, by Theorem 2.3 we have also

$$\begin{aligned} f(\boldsymbol{y}) - f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &\geq f'(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}; -\lambda(\boldsymbol{x}-\boldsymbol{y}) \\ &\geq -\lambda f'(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}; \boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{3.11}$$

Multiplying (3.10) by λ and (3.11) by $1-\lambda$ and summing up them we obtain

$$\lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) \geq f(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}),$$

in other words, f is convex. □

Subdifferential regularity guarantees that the gradient is the only subgradient of a differentiable function.

Corollary 3.4 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is differentiable and subdifferentially regular at* $\boldsymbol{x}$*, then*

$$\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}. \tag{3.12}$$

Proof This follows immediately from subdifferential regularity and the proof of Theorem 3.7. □

3.2.2 Subderivation Rules

Next we go through classical derivation rules for locally Lipschitz continuous functions.

Theorem 3.15 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous at* $\boldsymbol{x}$*, then for all* $\lambda \in \mathbb{R}$

$$\partial(\lambda f)(\boldsymbol{x}) = \lambda \partial f(\boldsymbol{x}). \tag{3.13}$$

Proof It is evident that the function λf is also locally Lipschitz continuous at $\boldsymbol{x}$. If $\lambda \geq 0$ then clearly $(\lambda f)^\circ = \lambda \cdot f^\circ$, so $\partial(\lambda f)(\boldsymbol{x}) = \lambda \partial f(\boldsymbol{x})$ for all $\lambda \geq 0$. It suffices now to prove the formula for $\lambda = -1$. We calculate

$$\begin{aligned} \boldsymbol{\xi} \in \partial(-f)(\boldsymbol{x}) &\iff (-f)^\circ(\boldsymbol{x}; \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n \\ &\iff f^\circ(\boldsymbol{x}; -\boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n \\ &\iff f^\circ(\boldsymbol{x}; -\boldsymbol{d}) \geq (-\boldsymbol{\xi})^T(-\boldsymbol{d}) \quad \text{for all} \quad -\boldsymbol{d} \in \mathbb{R}^n \\ &\iff -\boldsymbol{\xi} \in \partial f(\boldsymbol{x}) \\ &\iff \boldsymbol{\xi} \in -\partial f(\boldsymbol{x}) \end{aligned}$$

and the assertion follows. □

Second rule considers the derivation of a linear combination.

Theorem 3.16 *Let $f_i: \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $\boldsymbol{x}$ and $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, m$. Then the function*

$$f(\boldsymbol{x}) := \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x})$$

is locally Lipschitz continuous at $\boldsymbol{x}$ and

$$\partial f(\boldsymbol{x}) \subseteq \sum_{i=1}^{m} \lambda_i \partial f_i(\boldsymbol{x}). \tag{3.14}$$

In addition, if f_i is subdifferentially regular at $\boldsymbol{x}$ and $\lambda_i \geq 0$ for all $i = 1, \ldots, m$, then f is also subdifferentially regular at $\boldsymbol{x}$ and equality holds in (3.14).

Proof Again, it is evident that the function f is also locally Lipschitz continuous at $\boldsymbol{x}$. It suffices now to prove the formula for $m = 2$; the general case follows by induction. In the proof of Theorem 3.13 we observed that

$$(f_1 + f_2)^\circ(\boldsymbol{x}; \boldsymbol{d}) \leq f_1^\circ(\boldsymbol{x}; \boldsymbol{d}) + f_2^\circ(\boldsymbol{x}; \boldsymbol{d}),$$

whence, by the definition of subdifferential

$$\partial(f_1 + f_2)(\boldsymbol{x}) \subseteq \partial f_1(\boldsymbol{x}) + \partial f_2(\boldsymbol{x}).$$

In view of Theorem 3.15 we have

$$\partial(\lambda_1 f_1 + \lambda_2 f_2)(\boldsymbol{x}) \subseteq \partial(\lambda_1 f_1)(\boldsymbol{x}) + \partial(\lambda_2 f_2)(\boldsymbol{x}) = \lambda_1 \partial f_1(\boldsymbol{x}) + \lambda_2 \partial f_2(\boldsymbol{x}).$$

Suppose next that f_i is subdifferentially regular at $\boldsymbol{x}$ and $\lambda_i > 0$ for $i = 1, 2$. By Theorem 3.13 the function $\lambda_1 f_1 + \lambda_2 f_2$ is subdifferentially regular; in other words

$$(\lambda_1 f_1 + \lambda_2 f_2)^\circ = (\lambda_1 f_1 + \lambda_2 f_2)' = \lambda_1 f_1' + \lambda_2 f_2' = \lambda_1 f_1^\circ + \lambda_2 f_2^\circ,$$

and it follows that

$$\partial(\lambda_1 f_1 + \lambda_2 f_2)(\boldsymbol{x}) = \lambda_1 \partial f_1(\boldsymbol{x}) + \lambda_2 \partial f_2(\boldsymbol{x}).$$

Thus the proof is complete. □

The following results is one the most important results in optimization theory. However, here we need it only in the proof of forthcoming derivation rules.

Theorem 3.17 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous and attains its extremum at* $\boldsymbol{x}$*, then*

$$\mathbf{0} \in \partial f(\boldsymbol{x}). \tag{3.15}$$

Proof Suppose first that f attains a local minimum at $\boldsymbol{x}$. Then there exists $\varepsilon > 0$ such that $f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x}) \geq 0$ for all $0 < t < \varepsilon$ and $\boldsymbol{d} \in \mathbb{R}^n$. Now we have

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \limsup_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ t \downarrow 0}} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t} \geq \limsup_{t \downarrow 0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t} \geq 0$$

and thus

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) \geq 0 = \mathbf{0}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n,$$

which means by the definition of subdifferential that $\mathbf{0} \in \partial f(\boldsymbol{x})$.

Suppose next that f attains a local maximum at $\boldsymbol{x}$. Then $-f$ attains a local minimum at $\boldsymbol{x}$ and, as above $\mathbf{0} \in \partial(-f)(\boldsymbol{x})$. The statement follows then from Theorem 3.15.

Evidently global minima and maxima are also local minima and maxima, respectively. □

Next we present one of the key results of differential calculus, namely the *mean-value theorem*.

Theorem 3.18 (Mean-Value Theorem) *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *be such that* $\boldsymbol{x} \neq \boldsymbol{y}$ *and let the function* $f: \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz continuous on an open set* $U \subseteq \mathbb{R}^n$ *such that the line segment* $[\boldsymbol{x}, \boldsymbol{y}] \subset U$*. Then there exists a point* $\boldsymbol{z} \in (\boldsymbol{x}, \boldsymbol{y})$ *such that*

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) \in \partial f(\boldsymbol{z})^T (\boldsymbol{y} - \boldsymbol{x}).$$

In the proof of mean-value theorem we need the following lemma.

Lemma 3.2 *The function* $g: [0, 1] \to \mathbb{R}$ *defined by* $g(t) := f(\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}))$*, is Lipschitz continuous on* $(0, 1)$ *and*

$$\partial g(t) \subseteq \partial f(\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}))^T (\boldsymbol{y} - \boldsymbol{x}). \tag{3.16}$$

Proof We denote $\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x})$ by $\boldsymbol{x}_t$. The function g is Lipschitz continuous on $(0, 1)$ because

$$\begin{aligned}
|g(t) - g(t')| &= |f(\boldsymbol{x}_t) - f(\boldsymbol{x}_{t'})| \\
&\leq K \, \|\boldsymbol{x}_t - \boldsymbol{x}_{t'}\| \\
&= K \, \|(t - t')(\boldsymbol{y} - \boldsymbol{x})\| \\
&= K \, \|\boldsymbol{y} - \boldsymbol{x}\| \, |t - t'|
\end{aligned}$$

$$= \tilde{K}\,|t - t'| \quad \text{for all} \quad t, t' \in (0, 1),$$

where $\tilde{K} := K\,\|\boldsymbol{y} - \boldsymbol{x}\|$.

From Theorem 3.3 we get that the sets $\partial g(t)$ and $\partial f(\boldsymbol{x}_t)^T(\boldsymbol{y} - \boldsymbol{x})$ are compact and convex. Since they belong to $\mathbb{R}$, they must be closed intervals in $\mathbb{R}$ and thus it suffices to prove that for $\mu = \pm 1$, we have

$$\max\{\partial g(t)\mu\} \le \max\{\partial f(\boldsymbol{x}_t)^T(\boldsymbol{y} - \boldsymbol{x})\mu\}.$$

By Theorem 3.3 we have $\max\{\partial g(t)\mu\} = g^\circ(t; \mu)$ and thus

$$\begin{aligned}
\max\{\partial g(t)\mu\} &= \limsup_{\substack{s \to t \\ \lambda \downarrow 0}} \frac{g(s + \lambda\mu) - g(s)}{\lambda} \\
&= \limsup_{\substack{s \to t \\ \lambda \downarrow 0}} \frac{f(\boldsymbol{x} + [s + \lambda\mu](\boldsymbol{y} - \boldsymbol{x})) - f(\boldsymbol{x} + s(\boldsymbol{y} - \boldsymbol{x}))}{\lambda} \\
&\le \limsup_{\substack{\boldsymbol{y}' \to \boldsymbol{x}_t \\ \lambda \downarrow 0}} \frac{f(\boldsymbol{y}' + \lambda\mu(\boldsymbol{y} - \boldsymbol{x})) - f(\boldsymbol{y}')}{\lambda} \\
&= f^\circ(\boldsymbol{x}_t; \mu(\boldsymbol{y} - \boldsymbol{x})).
\end{aligned}$$

Furthermore it follows again from Theorem 3.3 that

$$f^\circ(\boldsymbol{x}_t; \mu(\boldsymbol{y} - \boldsymbol{x})) = \max\{\partial f(\boldsymbol{x}_t)^T(\mu(\boldsymbol{y} - \boldsymbol{x}))\},$$

and thus

$$\max\{\partial g(t)\mu\} \le \max\{\partial f(\boldsymbol{x}_t)^T(\boldsymbol{y} - \boldsymbol{x})\mu\}.$$

□

Proof (Mean-Value Theorem) Let us define the function $\Theta\colon [0, 1] \to \mathbb{R}$ such that $\Theta(t) := f(\boldsymbol{x}_t) + t[f(\boldsymbol{x}) - f(\boldsymbol{y})]$. Then it is evident that Θ is continuous and

$$\begin{aligned}
\Theta(0) &= f(\boldsymbol{x}_0) = f(\boldsymbol{x}) \\
\Theta(1) &= f(\boldsymbol{x}_1) + f(\boldsymbol{x}) - f(\boldsymbol{y}) = f(\boldsymbol{x}).
\end{aligned}$$

Then it follows that there exists $t_0 \in (0, 1)$ such that Θ attains a local extremum at t_0 and by Theorem 3.17 we have $0 \in \partial\Theta(t_0)$. Now by using Theorem 3.16 we get

$$\partial\Theta(t) = \partial[f(\boldsymbol{x}_t) + t(f(\boldsymbol{x}) - f(\boldsymbol{y}))] \subset \partial f(\boldsymbol{x}_t) + [f(\boldsymbol{x}) - f(\boldsymbol{y})]\partial(t)$$

and furthermore by Lemma 3.2 we have

$$0 \in \partial f(\boldsymbol{x}_t)^T(\boldsymbol{y} - \boldsymbol{x}) + [f(\boldsymbol{x}) - f(\boldsymbol{y})] \cdot \partial(t).$$

Then from the fact that $\partial(t) = 1$, it follows that

$$f(\boldsymbol{y}) - f(\boldsymbol{x}) \in \partial f(\boldsymbol{z})^T(\boldsymbol{y} - \boldsymbol{x}),$$

where $\boldsymbol{z} := \boldsymbol{x}_t \in (\boldsymbol{x}, \boldsymbol{y})$, which is the assertion of the theorem. □

Now is the turn of another main result of differential calculus, namely the *chain rule*.

Theorem 3.19 (Chain Rule) *Let $f: \mathbb{R}^n \to \mathbb{R}$ be such that $f = g \circ \boldsymbol{h}$, where $\boldsymbol{h}: \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz continuous at $\boldsymbol{x} \in \mathbb{R}^n$ and $g: \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{h}(\boldsymbol{x}) \in \mathbb{R}^m$. Then f is locally Lipschitz continuous at $\boldsymbol{x}$ and*

$$\partial f(\boldsymbol{x}) \subseteq \operatorname{conv} \left\{ \partial \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big) \right\} \tag{3.17}$$

Proof It is evident that f is locally Lipschitz continuous at $\boldsymbol{x}$. Denote

$$S := \left\{ \partial \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big) \right\}.$$

The fact that $\partial \boldsymbol{h}(\boldsymbol{x})$ and $\partial g(\boldsymbol{h}(\boldsymbol{x}))$ are compact sets implies that S is also compact, and hence its convex hull is a convex compact set (see e.g. [202] p. 78). Then it suffices to prove that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) \le \max_{\boldsymbol{\eta} \in \operatorname{conv} S} \boldsymbol{\eta}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n. \tag{3.18}$$

To see this let $\boldsymbol{\eta} \in \operatorname{conv} S$. Then we have $\boldsymbol{\eta} = \sum_{j=1}^k \lambda_j \boldsymbol{s}_j$ with $\boldsymbol{s}_j \in S, \sum_{j=1}^k \lambda_j = 1$ and $\lambda_j \ge 0$ and for all $\boldsymbol{d} \in \mathbb{R}^n$ we obtain

$$\boldsymbol{\eta}^T \boldsymbol{d} = \sum_{j=1}^k \lambda_j \boldsymbol{s}_j^T \boldsymbol{d} \le \sum_{j=1}^k \lambda_j \max_{\boldsymbol{s} \in S} \boldsymbol{s}^T \boldsymbol{d} = \max_{\boldsymbol{s} \in S} \boldsymbol{s}^T \boldsymbol{d}.$$

Thus

$$\max_{\boldsymbol{\eta} \in \operatorname{conv} S} \boldsymbol{\eta}^T \boldsymbol{d} = \max_{\boldsymbol{s} \in S} \boldsymbol{s}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

Define

$$q_\varepsilon(\boldsymbol{d}) := \max \left\{ \sum_{i=1}^m \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}_i),\ \boldsymbol{\alpha} \in \partial g(\boldsymbol{u}),\ \boldsymbol{x}_i \in B(\boldsymbol{x}; \varepsilon), \right.$$
$$\left. \boldsymbol{u} \in B(\boldsymbol{h}(\boldsymbol{x}); \varepsilon) \right\}.$$

Then we have

$$\begin{aligned} q_0(\boldsymbol{d}) &= \max \left\{ \sum_{i=1}^{m} \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}),\ \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x})) \right\} \\ &= \max \{\partial \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big)\boldsymbol{d}\} \\ &= \max_{\boldsymbol{s} \in S} \boldsymbol{s}^T \boldsymbol{d}. \end{aligned}$$

This will imply (3.18), if we show that for all $\varepsilon > 0$

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) - \varepsilon \le q_\varepsilon(\boldsymbol{d}) \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n, \tag{3.19}$$

and that $q_\varepsilon(\boldsymbol{d}) \to q_0(\boldsymbol{d})$ as $\varepsilon \downarrow 0$, for all $\boldsymbol{d} \in \mathbb{R}^n$. The last claim is proved in the following lemma.

Lemma 3.3

$$\lim_{\varepsilon \downarrow 0} q_\varepsilon = q_0.$$

Proof To see this let $\delta > 0$ and $\boldsymbol{d} \in \mathbb{R}^n$ be given. Because each h_i is locally Lipschitz continuous at $\boldsymbol{x}$, g is locally Lipschitz continuous at $\boldsymbol{h}(\boldsymbol{x})$ and the function $h_i^\circ(\cdot\,;\cdot)$ is upper semicontinuous by Theorem 3.2, we can choose $\varepsilon > 0$ such that each h_i is Lipschitz continuous on $B(\boldsymbol{x}; \varepsilon)$ and g is Lipschitz continuous on $B(\boldsymbol{h}(\boldsymbol{x}); \varepsilon)$ with the same constant K, and such that for all $i = 1, \ldots, m$ one has

$$h_i^\circ(\boldsymbol{x}_i; \pm\boldsymbol{d}) \le h_i^\circ(\boldsymbol{x}; \pm\boldsymbol{d}) + \delta/K \quad \text{for all} \quad \boldsymbol{x}_i \in B(\boldsymbol{x}; \varepsilon).$$

If $\boldsymbol{\alpha} \in \partial g\,(B(\boldsymbol{h}(\boldsymbol{x}); \varepsilon))$ then by Theorem 3.3 we have $|\alpha_i| \le K$ for all $i = 1, \ldots, m$. By Theorem 3.1 we know that $h_i^\circ(\boldsymbol{y}; \cdot)$ is positively homogeneous. Then multiplying across by $|\alpha_i|$ gives

$$h_i^\circ(\boldsymbol{x}_i; \alpha_i\boldsymbol{d}) \le h_i^\circ(\boldsymbol{x}; \alpha_i\boldsymbol{d}) + |\alpha_i|\,\delta/K \le h_i^\circ(\boldsymbol{x}; \alpha_i\boldsymbol{d}) + \delta.$$

On the other hand, we know by Theorem 3.5 that the mapping $\partial g(\cdot)$ is upper semicontinuous, from which it follows that we can also choose ε small enough to guarantee that $\partial g\,(B(\boldsymbol{h}(\boldsymbol{x}); \varepsilon)) \subset B\,(\partial g(\boldsymbol{h}(\boldsymbol{x})); \delta)$. We may now calculate

$$\begin{aligned} q_0 &\le q_\varepsilon(\boldsymbol{d}) \\ &= \max \Big\{ \sum_{i=1}^{m} \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}_i),\ \boldsymbol{\alpha} \in \partial g(\boldsymbol{u}), \\ &\qquad\qquad \boldsymbol{x}_i \in B(\boldsymbol{x}; \varepsilon),\ \boldsymbol{u} \in B(\boldsymbol{h}(\boldsymbol{x}); \varepsilon) \Big\} \end{aligned}$$

$$\le \max\Big\{\sum_{i=1}^{m} \max\{\alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}_i),\ \boldsymbol{x}_i \in B(\boldsymbol{x};\varepsilon)\} \mid \boldsymbol{\alpha} \in B(\partial g(\boldsymbol{h}(\boldsymbol{x}));\delta)\Big\}$$

$$\le \max\Big\{\sum_{i=1}^{m} (h_i^\circ(\boldsymbol{x};\alpha_i\boldsymbol{d}) + \delta) \mid \boldsymbol{\alpha} \in B(\partial g(\boldsymbol{h}(\boldsymbol{x}));\delta)\Big\}$$

$$\le \max\Big\{\sum_{i=1}^{m} \max\{\alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x})\} \mid \boldsymbol{\alpha} \in B(\partial g(\boldsymbol{h}(\boldsymbol{x}));\delta)\Big\} + m\delta$$

$$\le q_0 + m\delta K|\boldsymbol{d}| + m\delta \longrightarrow q_0, \quad \text{whenever} \quad \delta \to 0,$$

which completes the proof of the lemma. □

Now we turn back to the proof of the chain rule. We are going to show that inequality (3.19) holds. To see this let $\varepsilon > 0$. Then by the definition of the generalized directional derivative there exist $\boldsymbol{y} \in \mathbb{R}^n$ and $t > 0$ such that

$$f^\circ(\boldsymbol{x};\boldsymbol{d}) \le \frac{f(\boldsymbol{y}+t\boldsymbol{d}) - f(\boldsymbol{y})}{t} + \varepsilon \tag{3.20}$$

and

$$\begin{cases} \boldsymbol{y},\ \boldsymbol{y}+t\boldsymbol{d} \in B(\boldsymbol{x};\varepsilon) \\ \boldsymbol{h}(\boldsymbol{y}), \boldsymbol{h}(\boldsymbol{y}+t\boldsymbol{d}) \in B(\boldsymbol{h}(\boldsymbol{x});\varepsilon). \end{cases}$$

By the mean-value Theorem 3.8 there exists $\boldsymbol{\alpha} \in \partial g(\boldsymbol{u})$ such that $\boldsymbol{u} \in [\boldsymbol{h}(\boldsymbol{y}+t\boldsymbol{d}), \boldsymbol{h}(\boldsymbol{y})] \subset B(\boldsymbol{h}(\boldsymbol{x});\varepsilon)$ and

$$\begin{aligned} f(\boldsymbol{y}+t\boldsymbol{d}) - f(\boldsymbol{y}) &= g(\boldsymbol{h}(\boldsymbol{y}+t\boldsymbol{d})) - g(\boldsymbol{h}(\boldsymbol{y})) \\ &= \boldsymbol{\alpha}^T(\boldsymbol{h}(\boldsymbol{y}+t\boldsymbol{d}) - \boldsymbol{h}(\boldsymbol{y})) \\ &= \sum_{i=1}^{m} \alpha_i[h_i(\boldsymbol{y}+t\boldsymbol{d}) - h_i(\boldsymbol{y})]. \end{aligned}$$

We apply the mean-value theorem again to the functions $h_i,\ i = 1,\dots,m$. Then there exist subgradients $\boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}_i)$ such that $\boldsymbol{x}_i \in [\boldsymbol{y}+t\boldsymbol{d}, \boldsymbol{y}] \subset B(\boldsymbol{x};\varepsilon)$ and

$$\begin{aligned} f(\boldsymbol{y}+t\boldsymbol{d}) - f(\boldsymbol{y}) &= \sum_{i=1}^{m} \alpha_i[h_i(\boldsymbol{y}+t\boldsymbol{d}) - h_i(\boldsymbol{y})] \\ &= \sum_{i=1}^{m} \alpha_i \boldsymbol{\xi}_i^T(\boldsymbol{y}+t\boldsymbol{d} - \boldsymbol{y}) \end{aligned}$$

$$= \sum_{i=1}^{m} \alpha_i \boldsymbol{\xi}_i^T (t\boldsymbol{d}).$$

Now it follows from (3.20) that

$$\begin{aligned} f^\circ(\boldsymbol{x}; \boldsymbol{d}) &\le \frac{f(\boldsymbol{y}+t\boldsymbol{d}) - f(\boldsymbol{y})}{t} + \varepsilon \\ &= \frac{\sum_{i=1}^m \alpha_i \boldsymbol{\xi}_i^T (t\boldsymbol{d})}{t} + \varepsilon \\ &= \frac{t\sum_{i=1}^m \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d}}{t} + \varepsilon \\ &= \sum_{i=1}^{m} \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} + \varepsilon \\ &\le q_\varepsilon(\boldsymbol{d}) + \varepsilon \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n, \end{aligned} \tag{3.21}$$

which establishes (3.19) and thus the proof is complete. □

We have several possibilities to achieve equality in (3.17) as the following theorem advises.

Theorem 3.20 *Suppose, that the assumptions of Theorem 3.19 are valid. If*

(i) *the function g is subdifferentially regular at $\boldsymbol{h}(\boldsymbol{x})$, each h_i is subdifferentially regular at $\boldsymbol{x}$ and for any $\boldsymbol{\alpha} \in \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big)$ we have $\alpha_i \ge 0$ for all $i = 1, \ldots, m$. Then also f is subdifferentially regular at $\boldsymbol{x}$ and we have*

$$\partial f(\boldsymbol{x}) = \operatorname{conv} \left\{ \partial \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big) \right\}.$$

(ii) *the function g is subdifferentially regular at $\boldsymbol{h}(\boldsymbol{x})$ and h_i is continuously differentiable at $\boldsymbol{x}$ for all $i = 1, \ldots, m$. Then*

$$\partial f(\boldsymbol{x}) = \nabla \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big).$$

(iii) *$m = 1$ and g is continuously differentiable at $h(\boldsymbol{x})$. Then*

$$\partial f(\boldsymbol{x}) = g'\big(h(\boldsymbol{x})\big) \partial h(\boldsymbol{x}).$$

Proof (i) Suppose first that g is subdifferentially regular at $\boldsymbol{h}(\boldsymbol{x})$, each h_i is subdifferentially regular at $\boldsymbol{x}$ and for any $\boldsymbol{\alpha} \in \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big)$ we have $\alpha_i > 0$ for all $i = 1, \ldots, m$. To prove the equality in (3.17) it suffices to show that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = q_0(\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

From above we found that $f^\circ(\boldsymbol{x}; \boldsymbol{d}) \le q_0(\boldsymbol{d})$ for all $\boldsymbol{d} \in \mathbb{R}^n$. On the other hand, the fact that $\alpha_i \ge 0$ for all $i = 1, \ldots, m$, h_i is subdifferentially regular at $\boldsymbol{x}$ and g is subdifferentially regular at $\boldsymbol{h}(\boldsymbol{x})$ imply

$$
\begin{aligned}
q_0(\boldsymbol{d}) &= \max\Big\{\sum_{i=1}^m \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}),\ \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\Big\} \\
&\le \max\Big\{\sum_{i=1}^m \alpha_i \max_{\boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x})} \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\Big\} \\
&= \max\Big\{\sum_{i=1}^m \alpha_i h_i^\circ(\boldsymbol{x}; \boldsymbol{d}) \mid \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\Big\} \\
&= \max\Big\{\sum_{i=1}^m \alpha_i h_i'(\boldsymbol{x}; \boldsymbol{d}) \mid \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\Big\} \\
&= g^\circ(\boldsymbol{h}(\boldsymbol{x}); \boldsymbol{h}'(\boldsymbol{x}; \boldsymbol{d})) = g'(\boldsymbol{h}(\boldsymbol{x}); \boldsymbol{v}),
\end{aligned}
$$

where $v_i := h_i'(\boldsymbol{x}; \boldsymbol{d})$. Then by definition the directional derivative

$$
\begin{aligned}
g'(\boldsymbol{h}(\boldsymbol{x}); \boldsymbol{v}) &= \lim_{t \downarrow 0} \frac{g(\boldsymbol{h}(\boldsymbol{x}) + t\boldsymbol{v}) - g(\boldsymbol{h}(\boldsymbol{x}))}{t} \\
&= \lim_{t \downarrow 0} \left\{ \frac{g(\boldsymbol{h}(\boldsymbol{x} + t\boldsymbol{d})) - g(\boldsymbol{h}(\boldsymbol{x}))}{t} + T \right\},
\end{aligned}
$$

where $T := g(\boldsymbol{h}(\boldsymbol{x}) + t\boldsymbol{v}) - g(\boldsymbol{h}(\boldsymbol{x} + t\boldsymbol{d}))/t$. We obtain an upper estimate of T and show that it goes to zero, when $t \to 0$. Due to the Lipschitz property of the function g one has

$$
\begin{aligned}
T &\le \frac{|g(\boldsymbol{h}(\boldsymbol{x}) + t\boldsymbol{v}) - g(\boldsymbol{h}(\boldsymbol{x} + t\boldsymbol{d}))|}{t} \le \frac{K\,\|\boldsymbol{h}(\boldsymbol{x}) + t\boldsymbol{v} - \boldsymbol{h}(\boldsymbol{x} + t\boldsymbol{d})\|}{t} \\
&= K \left\| \boldsymbol{v} - \frac{\boldsymbol{h}(\boldsymbol{x} + t\boldsymbol{d}) - \boldsymbol{h}(\boldsymbol{x})}{t} \right\| \longrightarrow K\,\|\boldsymbol{h}'(\boldsymbol{x}, \boldsymbol{d}) - \boldsymbol{h}'(\boldsymbol{x}, \boldsymbol{d})\| = 0,
\end{aligned}
$$

as $t \to 0$. Thus,

$$
q_0(\boldsymbol{d}) \le \lim_{t \downarrow 0} \frac{g(\boldsymbol{h}(\boldsymbol{x} + t\boldsymbol{d})) - g(\boldsymbol{h}(\boldsymbol{x}))}{t} = f'(\boldsymbol{x}; \boldsymbol{d}) \le f^\circ(\boldsymbol{x}; \boldsymbol{d})
$$

and by (3.21) we have $q_0(\boldsymbol{d}) = f'(\boldsymbol{x}; \boldsymbol{d}) = f^\circ(\boldsymbol{x}; \boldsymbol{d})$ for all $\boldsymbol{d} \in \mathbb{R}^n$. In other words, f is subdifferentially regular at $\boldsymbol{x}$ and equality holds in (3.17), which establishes (i).

(ii) Suppose next that the function g is subdifferentially regular at $\boldsymbol{h}(\boldsymbol{x})$ and each h_i is continuously differentiable at $\boldsymbol{x}$ for all $i = 1, \dots, m$. Then by Theorem 3.7 we have

$$
\begin{aligned}
q_0(\boldsymbol{d}) &= \max\left\{\sum_{i=1}^{m} \alpha_i \boldsymbol{\xi}_i^T \boldsymbol{d} \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}) = \{\nabla h_i(\boldsymbol{x})\},\ \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\right\} \\
&= \max\left\{\sum_{i=1}^{m} \alpha_i \nabla h_i(\boldsymbol{x})^T \boldsymbol{d} \mid \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\right\} \\
&= \max\left\{\sum_{i=1}^{m} \alpha_i h_i'(\boldsymbol{x}, \boldsymbol{d}) \mid \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\right\}.
\end{aligned}
$$

Now we can continue in the same way as in (i) to get

$$
q_0(\boldsymbol{v}) \le f^\circ(\boldsymbol{x}; \boldsymbol{d}),
$$

which means that

$$
\partial f(\boldsymbol{x}) = \operatorname{conv}\left\{\nabla \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big)\right\}.
$$

Then by Theorems 3.4 and 2.2 the set $\nabla \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big)$ is convex and thus

$$
\partial f(\boldsymbol{x}) = \nabla \boldsymbol{h}(\boldsymbol{x})^T \partial g\big(\boldsymbol{h}(\boldsymbol{x})\big).
$$

which proves assertion (ii).

(iii) Finally, if $m = 1$ and the function g is continuously differentiable at $h(\boldsymbol{x})$, then

$$
\alpha = g'(h(\boldsymbol{x})) = \lim_{\boldsymbol{y} \to \boldsymbol{x}} \frac{g(h(\boldsymbol{x})) - g(h(\boldsymbol{y}))}{h(\boldsymbol{x}) - h(\boldsymbol{y})}
$$

and $\lim_{\boldsymbol{z} \to \boldsymbol{x}} g'(h(\boldsymbol{z})) = \alpha$. We may assume that $\alpha \ge 0$. Then we calculate

$$
\begin{aligned}
q_0(\boldsymbol{d}) &= \max\{\alpha \boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial h(\boldsymbol{x})\} = \alpha \cdot h^\circ(\boldsymbol{x}; \boldsymbol{d}) \\
&= \limsup_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ t \downarrow 0}} \frac{\alpha[h(\boldsymbol{y} + t\boldsymbol{d}) - h(\boldsymbol{y})]}{t} = \limsup_{\substack{\boldsymbol{x}' \to \boldsymbol{x} \\ t \downarrow 0}} \frac{g'(h(\boldsymbol{y}))[h(\boldsymbol{y} + t\boldsymbol{d}) - h(\boldsymbol{y})]}{t} \\
&= \limsup_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ t \downarrow 0}} \frac{g(h(\boldsymbol{y} + t\boldsymbol{d})) - g(h(\boldsymbol{y}))}{t} = f^\circ(\boldsymbol{x}; \boldsymbol{d}) \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n
\end{aligned}
$$

and the theorem is proved. □

Next we give an example of the chain rule.

Example 3.3 (Chain Rule) Let $f: \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\boldsymbol{x}) := \ln(\|\boldsymbol{x}\| + 2).$$

The subdifferential $\partial f(\mathbf{0})$ can be calculated as follows:

Let us define $h(\boldsymbol{x}) := \|\boldsymbol{x}\| + 2$ and $g(x) := \ln x$. Then for all $\boldsymbol{x}, \boldsymbol{y}, \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$\begin{aligned} h(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= \|\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}\| + 2 \\ &\leq \lambda(\|\boldsymbol{x}\| + 2) + (1-\lambda)(\|\boldsymbol{y}\| + 2) \\ &= \lambda h(\boldsymbol{x}) + (1-\lambda)h(\boldsymbol{y}), \end{aligned}$$

in other words h is convex and by Theorem 2.22 locally Lipschitz continuous at $\mathbf{0}$. Let $\boldsymbol{\xi} \in \text{cl } B(\mathbf{0}; 1)$ meaning that $\|\boldsymbol{\xi}\| \leq 1$. Then for all $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$\boldsymbol{\xi}^T \boldsymbol{x} \leq \|\boldsymbol{\xi}\|\|\boldsymbol{x}\| \leq \|\boldsymbol{x}\| \tag{3.22}$$

and thus

$$h(\boldsymbol{x}) = \|\boldsymbol{x}\| + 2 \geq \boldsymbol{\xi}^T \boldsymbol{x} + 2 = h(\mathbf{0}) + \boldsymbol{\xi}^T(\boldsymbol{x} - \mathbf{0}).$$

Then by the definition of the subdifferential of the convex function (Definition 2.15) we have $\boldsymbol{\xi} \in \partial_c h(\mathbf{0})$, in other words $\text{cl } B(\mathbf{0}; 1) \subseteq \partial_c h(\mathbf{0})$. On the other hand, if $\boldsymbol{\xi} \notin \text{cl } B(\mathbf{0}; 1)$ then $\|\boldsymbol{\xi}\| > 1$. By choosing $\boldsymbol{x} := \boldsymbol{\xi}$ we have

$$\boldsymbol{\xi}^T \boldsymbol{x} = \boldsymbol{\xi}^T \boldsymbol{\xi} = \|\boldsymbol{\xi}\|^2 > \|\boldsymbol{\xi}\| = \|\boldsymbol{x}\|$$

and thus (3.22) is not valid and $\boldsymbol{\xi}$ can not be a subgradient. This means that $\partial h(\mathbf{0}) = \partial_c h(\mathbf{0}) = \text{cl } B(\mathbf{0}; 1)$. Furthermore, g is clearly continuously differentiable and

$$g'(x) = \frac{1}{x}.$$

Then due to Theorem 3.20 (iii) we have

$$\partial f(\mathbf{0}) = g'\big(h(\mathbf{0})\big)\partial h(\mathbf{0}) = \tfrac{1}{2} \cdot \text{cl } B(\mathbf{0}; 1) = \text{cl } B\left(\mathbf{0}; \tfrac{1}{2}\right).$$

Based on the chain rule we can prove the generalization of the classical derivation rules of products and quotients.

Theorem 3.21 (Products) *Let f_1 and f_2 be locally Lipschitz continuous at $\boldsymbol{x} \in \mathbb{R}^n$. Then the function $f_1 f_2$ is locally Lipschitz continuous at $\boldsymbol{x}$ and*

$$\partial(f_1 f_2)(\boldsymbol{x}) \subseteq f_2(\boldsymbol{x})\partial f_1(\boldsymbol{x}) + f_1(\boldsymbol{x})\partial f_2(\boldsymbol{x}). \tag{3.23}$$

If in addition $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \geq 0$ *and* f_1, f_2 *are both subdifferentially regular at* $\boldsymbol{x}$*, then the function* $f_1 f_2$ *is subdifferentially regular at* $\boldsymbol{x}$ *and equality holds in* (3.23)*.*

Proof Define the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ by $g(\boldsymbol{u}_1, \boldsymbol{u}_2) := \boldsymbol{u}_1^T \boldsymbol{u}_2$. Then g is continuously differentiable and by Theorem 3.7 it is locally Lipschitz continuous at $(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$ with

$$\partial g(f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) = \{\nabla g(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))\} = \{(f_2(\boldsymbol{x}), f_1(\boldsymbol{x}))\}.$$

Next define the function $\boldsymbol{h}\colon \mathbb{R}^n \to \mathbb{R}^2$ by $\boldsymbol{h}(\boldsymbol{x}) := (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$. Now we have $f_1 \cdot f_2 = g \circ \boldsymbol{h}$. By the chain rule (Theorem 3.19) the function $f_1 \cdot f_2$ is locally Lipschitz continuous at $\boldsymbol{x}$ and

$$\begin{aligned}\partial(f_1 f_2)(\boldsymbol{x}) &\subseteq \operatorname{conv}\Big\{\sum_{i=1}^{2} \alpha_i \boldsymbol{\xi}_i \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}),\ \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\Big\}\\ &= \operatorname{conv}\{f_2(\boldsymbol{x})\partial f_1(\boldsymbol{x}) + f_1(\boldsymbol{x})\partial f_2(\boldsymbol{x})\}.\end{aligned}$$

Then by Theorems 3.3 and 2.2 the set $f_2(\boldsymbol{x})\partial f_1(\boldsymbol{x}) + f_1(\boldsymbol{x})\partial f_2(\boldsymbol{x})$ is convex and thus

$$\partial(f_1 f_2)(\boldsymbol{x}) \subseteq f_2(\boldsymbol{x})\partial f_1(\boldsymbol{x}) + f_1(\boldsymbol{x})\partial f_2(\boldsymbol{x}).$$

Suppose next that $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \geq 0$ and that f_1, f_2 are subdifferentially regular at $\boldsymbol{x}$. The function g is subdifferentially regular by Theorem 3.13 (i). Then by Theorem 3.20 (i) the function $f_1 f_2$ is subdifferentially regular at $\boldsymbol{x}$ and equality holds in (3.23). □

The proof for quotients is nearly the same as for products.

Theorem 3.22 (Quotients) *Let* f_1 *and* f_2 *be locally Lipschitz continuous at* $\boldsymbol{x} \in \mathbb{R}^n$ *and* $f_2(\boldsymbol{x}) \neq 0$*. Then the function* f_1/f_2 *is locally Lipschitz continuous at* $\boldsymbol{x}$ *and*

$$\partial\Big(\frac{f_1}{f_2}\Big)(\boldsymbol{x}) \subseteq \frac{f_2(\boldsymbol{x})\partial f_1(\boldsymbol{x}) - f_1(\boldsymbol{x})\partial f_2(\boldsymbol{x})}{f_2^2(\boldsymbol{x})}. \tag{3.24}$$

If in addition $f_1(\boldsymbol{x}) \geq 0$*,* $f_2(\boldsymbol{x}) > 0$ *and* f_1, f_2 *are both subdifferentially regular at* $\boldsymbol{x}$*, then the function* f_1/f_2 *is subdifferentially regular at* $\boldsymbol{x}$ *and equality holds in* (3.24)*.*

Proof Exercise. □

The following theorem deals with a class of functions which are frequently encountered in nonsmooth optimization, namely max-functions. The problem of minimizing such a function is usually called the *min–max problem*.

Theorem 3.23 (max-function) *Let $f_i: \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $\boldsymbol{x}$ for all $i = 1, \dots, m$. Then the function*

$$f(\boldsymbol{x}) := \max \{f_i(\boldsymbol{x}) \mid i = 1, \dots, m\}$$

is locally Lipschitz continuous at $\boldsymbol{x}$ and

$$\partial f(\boldsymbol{x}) \subseteq \operatorname{conv} \{\partial f_i(\boldsymbol{x}) \mid i \in \mathcal{I}(\boldsymbol{x})\}, \tag{3.25}$$

where

$$\mathcal{I}(\boldsymbol{x}) := \{i \in \{1, \dots, m\} \mid f_i(\boldsymbol{x}) = f(\boldsymbol{x})\}.$$

In addition, if f_i is subdifferentially regular at $\boldsymbol{x}$ for all $i = 1, \dots, m$, then f is also subdifferentially regular at $\boldsymbol{x}$ and equality holds in (3.25).

Proof The function f is evidently locally Lipschitz continuous at $\boldsymbol{x}$ (see Exercise 3.5). Define $g: \mathbb{R}^m \to \mathbb{R}$ and $\boldsymbol{h}: \mathbb{R}^n \to \mathbb{R}^m$ by

$$\begin{aligned} g(\boldsymbol{u}) &:= \max_{i=1,\dots,m} \{u_i\} \\ h(\boldsymbol{x}) &:= \big(f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\big). \end{aligned}$$

Now we have $f = g \circ \boldsymbol{h}$. For all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$ and $\lambda \in [0, 1]$ it holds

$$\begin{aligned} g(\lambda \boldsymbol{u} + (1-\lambda)\boldsymbol{v}) &= \max_{i=1,\dots,m} \{\lambda u_i + (1-\lambda) v_i\} \\ &\leq \lambda \max_{i=1,\dots,m} \{u_i\} + (1-\lambda) \max_{i=1,\dots,m} \{v_i\} \\ &= \lambda g(\boldsymbol{u}) + (1-\lambda) g(\boldsymbol{v}), \end{aligned}$$

which means that g is convex and by Theorem 2.22 locally Lipschitz continuous at $\boldsymbol{h}(\boldsymbol{x})$. Let $\mathcal{J}(\boldsymbol{u}) = \{i \in \{1, \dots, m\} \mid u_i = g(\boldsymbol{u})\}$. Then the directional derivative is

$$\begin{aligned} g'(\boldsymbol{u}; \boldsymbol{d}) = \lim_{t \downarrow 0} \frac{g(\boldsymbol{u} + t\boldsymbol{d}) - g(\boldsymbol{u})}{t} &= \lim_{t \downarrow 0} \max_{i=1,\dots,m} \frac{\{u_i + td_i\} - g(\boldsymbol{u})}{t} \\ = \lim_{t \downarrow 0} \max_{i \in \mathcal{J}(\boldsymbol{u})} \frac{\{u_i + td_i\} - g(\boldsymbol{u})}{t} &= \lim_{t \downarrow 0} \max_{i \in \mathcal{J}(\boldsymbol{u})} \frac{\{u_i + td_i - u_i\}}{t}. \end{aligned}$$

Thus

$$g'(\boldsymbol{u}; \boldsymbol{d}) = \max_{i \in \mathcal{J}(\boldsymbol{u})} d_i$$

and by Theorem 3.8 (i) we have $g^\circ = g'$, which gives

$$\partial g(\boldsymbol{u}) = \{\boldsymbol{\alpha} \in \mathbb{R}^m \mid \max_{i \in \mathcal{J}(\boldsymbol{u})} d_i \geq \boldsymbol{\alpha}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^m\}.$$

Now it is easy to see that

$$\boldsymbol{\alpha} \in \partial g(\boldsymbol{u}) \iff \begin{cases} \alpha_i \geq 0, & i = 1, \ldots, m, \\ \sum_{i=1}^m \alpha_i = 1, \\ \alpha_i = 0, & \text{when } i \notin \mathcal{J}(\boldsymbol{u}) \end{cases}$$

and so we can calculate the subdifferential of g at $\boldsymbol{h}(\boldsymbol{x}) \in \mathbb{R}^m$ by

$$\partial g(\boldsymbol{h}(\boldsymbol{x})) = \Big\{\boldsymbol{\alpha} \in \mathbb{R}^m \mid \alpha_i \geq 0, \ \sum_{i=1}^m \alpha_i = 1 \quad \text{and} \quad \alpha_i = 0 \quad \text{if } i \notin \mathcal{I}(\boldsymbol{x})\Big\}.$$

By applying Theorem 3.19 to f we get

$$\begin{aligned}
\partial f(\boldsymbol{x}) &\subseteq \operatorname{conv}\Big\{\sum_{i=1}^m \alpha_i \boldsymbol{\xi}_i \mid \boldsymbol{\xi}_i \in \partial h_i(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))\Big\} \\
&= \operatorname{conv}\Big\{\sum_{i \in \mathcal{I}(\boldsymbol{x})} \alpha_i \partial f_i(\boldsymbol{x}) \mid \alpha_i \geq 0 \quad \text{and} \quad \sum_{i \in \mathcal{I}(\boldsymbol{x})} \alpha_i = 1\Big\} \\
&= \operatorname{conv}\{\partial f_i(\boldsymbol{x}) \mid i \in \mathcal{I}(\boldsymbol{x})\}.
\end{aligned}$$

Suppose next that f_i is, in addition, subdifferentially regular at $\boldsymbol{x}$ for all $i \in \mathcal{I}(\boldsymbol{x})$. Because g is convex, it is, by Theorem 3.13 (ii), also subdifferentially regular at $\boldsymbol{h}(\boldsymbol{x})$. Then the fact that $\alpha_i \geq 0$ for all $\boldsymbol{\alpha} \in \partial g(\boldsymbol{h}(\boldsymbol{x}))$ and Theorem 3.20 (i) imply that f is subdifferentially regular at $\boldsymbol{x}$ and equality holds in (3.25). □

Corollary 3.5 *Suppose that the functions* $f_i\colon \mathbb{R}^n \to \mathbb{R}$ *are continuously differentiable at* $\boldsymbol{x}$ *and* $g_i\colon \mathbb{R}^n \to \mathbb{R}$ *are convex for each* $i = 1, \ldots, m$. *Define the functions* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and* $g\colon \mathbb{R}^n \to \mathbb{R}$ *by*

$$\begin{aligned}
f(\boldsymbol{x}) &= \max\{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\} \quad \textit{and} \\
g(\boldsymbol{x}) &= \max\{g_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\partial f(\boldsymbol{x}) &= \operatorname{conv}\{\nabla f_i(\boldsymbol{x}) \mid i \in \mathcal{I}(\boldsymbol{x})\} \quad \textit{and} \\
\partial_c g(\boldsymbol{x}) &= \operatorname{conv}\{\partial_c g_i(\boldsymbol{x}) \mid i \in \mathcal{J}(\boldsymbol{x})\},
\end{aligned} \tag{3.26}$$

where $\mathcal{I}(\boldsymbol{x}) = \{i \in \{1, \ldots, m\} \mid f_i(\boldsymbol{x}) = f(\boldsymbol{x})\}$ *and* $\mathcal{J}(\boldsymbol{x}) = \{i \in \{1, \ldots, m\} \mid g_i(\boldsymbol{x}) = g(\boldsymbol{x})\}$.

Proof Exercise. □

The next example shows how we can utilize the subdifferential calculus and subderivation rules in practice.

Example 3.4 (*Subderivation rules*) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \frac{\max\{|x|, x^2\}}{\sqrt{|x|}}.$$

We calculate next $\partial f(1)$.

Let us define $g_1(x) := |x|$ and $g_2(x) := x^2$. Then due to Example 2.8 g_1 is convex and locally Lipschitz continuous and thus, by Theorem 3.13 (ii) it is subdifferentially regular at $x = 1$. Furthermore, g_2 is clearly continuously differentiable and thus, due to Theorem 1.1 it is locally Lipschitz continuous and by Theorem 3.13 (i) it is subdifferentially regular at $x = 1$. Then by Theorem 3.7 we have

$$\begin{aligned} \partial g_1(1) &= \{\nabla g_1(1)\} = \{1\} \quad \text{and} \\ \partial g_2(1) &= \{\nabla g_2(1)\} = \{2\}. \end{aligned}$$

Define next $f_1(x) := \max\{|x|, x^2\}$. By Theorem 3.23 f_1 is locally Lipschitz continuous, subdifferentially regular at $x = 1$ and we have

$$\begin{aligned} \partial f_1(1) &= \operatorname{conv}\{\partial g_i(1) \mid i \in \mathcal{I}(1)\} \\ &= \operatorname{conv}\{\partial g_1(1), \partial g_2(1)\} \\ &= \operatorname{conv}\{1, 2\} \\ &= [1, 2]. \end{aligned}$$

Finally, function $f_2(x) := \sqrt{|x|}$ is clearly continuously differentiable at $x = 1$ and thus, due to Theorem 1.1 it is locally Lipschitz continuous, by Theorem 3.13 (i) it is subdifferentially regular at $x = 1$ and by Theorem 3.7 we have

$$\partial f_2(1) = \{\nabla f_2(1)\} = \{\tfrac{1}{2}\}.$$

Since $f_1(1) = 1 \geq 0$ and $f_2(1) = 1 > 0$ Theorem 3.22 implies that $f = f_1/f_2$ is subdifferentially regular at $x = 1$ and

$$\partial f(1) = \frac{f_2(1)\partial f_1(1) - f_1(1)\partial f_2(1)}{f_2^2(1)} = \frac{[1, 2] - \frac{1}{2}}{1} = \left[\tfrac{1}{2}, 1\tfrac{1}{2}\right].$$

3.3 Nonconvex Geometry

This chapter is devoted to geometrical concepts in nonconvex case. We show how the geometrical concepts can analogously be generalized in nonconvex analysis.

3.3.1 Tangent and Normal Cones

In Definition 2.10 we defined the notation of a contingent cone $K_S(\boldsymbol{x})$ of an arbitrary nonempty set S at a point $\boldsymbol{x} \in S$ and its elements were called tangent vectors. In Theorem 2.15 we proved that if S is convex, then $K_S(\boldsymbol{x})$ is a convex cone. Due to the convexity of $K_S(\boldsymbol{x})$ it was possible to define the normal cone $N_S(\boldsymbol{x})$ as a polar cone to the contingent cone.

For nonconvex sets we cannot guarantee the convexity of $K_S(\boldsymbol{x})$ and thus the nonemptyness of $N_S(\boldsymbol{x})$, but we need a new concept for tangents. Next we define the tangent cone by using the distance function. Note that this definition is based on Theorem 2.39.

Definition 3.6 The (Clarke) *tangent cone* of the nonempty set S at $\boldsymbol{x} \in S$ is given by the formula

$$T_S(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid d_S^\circ(\boldsymbol{x}; \boldsymbol{d}) = 0\}.$$

The elements of $T_S(\boldsymbol{x})$ are called again *tangent vectors*.

The tangent cone has the same elementary properties as in the convex case.

Theorem 3.24 *The tangent cone $T_S(\boldsymbol{x})$ of the nonempty set S at $\boldsymbol{x} \in S$ is a closed convex cone.*

Proof Exercise. (Hint: Use the convexity of the generalized directional derivative.) □

As in the convex case we can define the normal cone utilizing polarity.

Definition 3.7 The *normal cone* of the nonempty set S at $\boldsymbol{x} \in S$ is the set

$$N_S(\boldsymbol{x}) := T_S(\boldsymbol{x})^\circ = \{\boldsymbol{z} \in \mathbb{R}^n \mid \boldsymbol{z}^T\boldsymbol{d} \leq 0 \text{ for all } \boldsymbol{d} \in T_S(\boldsymbol{x})\}. \tag{3.27}$$

The elements of $N_S(\boldsymbol{x})$ are called again *normal vectors*.

Also the normal cone now has the same properties as before.

Theorem 3.25 *The normal cone $N_S(\boldsymbol{x})$ of the nonempty set S at $\boldsymbol{x} \in S$ is a closed convex cone.*

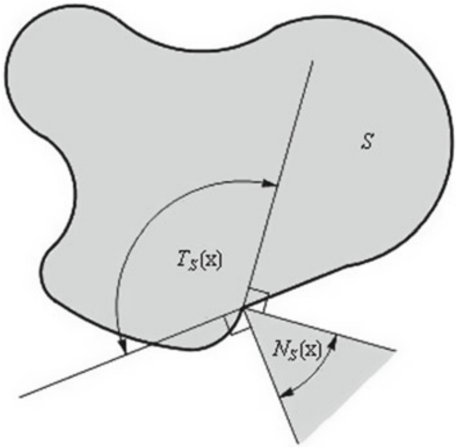


Fig. 3.3 Tangent and normal cones of a nonconvex set

Proof Follows directly from Lemma 4.5. □

Next we present alternative characterizations to the tangent and normal cones. The following reformulation of the tangent cone is similar to the definition of the contingent cone (Definition 2.10).

Theorem 3.26 *The tangent cone $T_S(\boldsymbol{x})$ of the nonempty set S at $\boldsymbol{x} \in S$ can also be written as*

$$T_S(\boldsymbol{x}) = \{\boldsymbol{d} \in \mathbb{R}^n \mid \textit{for all } t_i \downarrow 0 \textit{ and } \boldsymbol{x}_i \to \boldsymbol{x} \textit{ with } \boldsymbol{x}_i \in S, \\ \textit{there exists } \boldsymbol{d}_i \to \boldsymbol{d} \textit{ with } \boldsymbol{x}_i + t_i\boldsymbol{d}_i \in S\}.$$

Proof Let

$$Z := \{\boldsymbol{d} \in \mathbb{R}^n \mid \text{for all } t_i \downarrow 0 \text{ and } \boldsymbol{x}_i \to \boldsymbol{x} \text{ with } \boldsymbol{x}_i \in S, \\ \text{there exists } \boldsymbol{d}_i \to \boldsymbol{d} \text{ with } \boldsymbol{x}_i + t_i\boldsymbol{d}_i \in S\}.$$

Suppose first that $\boldsymbol{d} \in T_S(\boldsymbol{x})$, and that sequences $\boldsymbol{x}_i \to x$ with $x_i \in S$ and $t_i \downarrow 0$ are given. Then $d_S^\circ(\boldsymbol{x}; \boldsymbol{d}) = 0$ due to Definition 3.6 and since $\boldsymbol{x}_i \in S$ we have

$$\begin{aligned} 0 \le \lim_{i\to\infty} \frac{d_S(\boldsymbol{x}_i + t_i\boldsymbol{d})}{t_i} &= \lim_{i\to\infty} \frac{d_S(\boldsymbol{x}_i + t_i\boldsymbol{d}) - d_S(\boldsymbol{x}_i)}{t_i} \\ &\le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x} \\ t\downarrow 0}} \frac{d_S(\boldsymbol{y} + t\boldsymbol{d}) - d_S(\boldsymbol{y})}{t} = d_S^\circ(\boldsymbol{x}; \boldsymbol{d}) = 0. \end{aligned}$$

It follows that the limit exists and is zero. Then for all $i \in \mathbb{N}$ there exists $\boldsymbol{z}_i \in S$ such that

$$\|\boldsymbol{x}_i + t_i\boldsymbol{d} - \boldsymbol{z}_i\| \le d_S(\boldsymbol{x}_i + t_i\boldsymbol{d}) + \frac{t_i}{i}.$$

If we now define

$$\boldsymbol{d}_i := \frac{\boldsymbol{z}_i - \boldsymbol{x}_i}{t_i},$$

we have

$$\begin{aligned}\|\boldsymbol{d} - \boldsymbol{d}_i\| &= \left\| \boldsymbol{d} - \frac{\boldsymbol{z}_i - \boldsymbol{x}_i}{t_i} \right\| \\ &= \frac{\|\boldsymbol{x}_i + t_i - \boldsymbol{z}_i\|}{t_i} \le \frac{d_S(\boldsymbol{x}_i + t_i\boldsymbol{d})}{t_i} + \frac{1}{i} \longrightarrow 0\end{aligned}$$

as $i \to \infty$ and

$$\boldsymbol{x}_i + t_i\boldsymbol{d}_i = \boldsymbol{x}_i + t_i\left(\frac{\boldsymbol{z}_i - \boldsymbol{x}_i}{t_i}\right) = \boldsymbol{z}_i \in S,$$

thus $\boldsymbol{d} \in Z$.

Now for the converse. Suppose that $\boldsymbol{d} \in Z$ and choose sequences $\boldsymbol{x}_i \to x$ and $t_i \downarrow 0$ such that

$$\lim_{i\to\infty} \frac{d_S(\boldsymbol{x}_i + t_i\boldsymbol{d}) - d_S(\boldsymbol{x}_i)}{t_i} = d_S^\circ(\boldsymbol{x}; \boldsymbol{d}). \tag{3.28}$$

In order prove that $d_S^\circ(\boldsymbol{x}; \boldsymbol{d}) = 0$, it suffices to show that the quantity in (3.28) is nonpositive. To see this, choose $\boldsymbol{z}_i \in S$ such that

$$\|\boldsymbol{z}_i - \boldsymbol{x}_i\| \le d_S(\boldsymbol{x}_i) + \frac{t_i}{i}.$$

Then we have

$$\|\boldsymbol{x} - \boldsymbol{z}_i\| \le \|\boldsymbol{x} - \boldsymbol{x}_i\| + \|\boldsymbol{x}_i - \boldsymbol{z}_i\| \le \|\boldsymbol{x} - \boldsymbol{x}_i\| + d_S(\boldsymbol{x}_i) + \frac{t_i}{i} \longrightarrow 0$$

as $i \to \infty$. Then by the assumption there exists a sequence $\boldsymbol{d}_i$ converging to $\boldsymbol{d}$ such that $\boldsymbol{d}_i + t_i\boldsymbol{d}_i \in S$. By Theorem 2.38 the distance function d_S is Lipschitz continuous with constant $K = 1$ we get

$$\begin{aligned}d_S(\boldsymbol{x}_i + t_i\boldsymbol{d}) &\le d_S(\boldsymbol{z}_i + t_i\boldsymbol{d}_i) + \|\boldsymbol{x}_i - \boldsymbol{z}_i\| + t_i\|\boldsymbol{d} - \boldsymbol{d}_i\| \\ &\le d_S(\boldsymbol{x}_i) + t_i\left(\|\boldsymbol{d} - \boldsymbol{d}_i\| + \frac{1}{i}\right).\end{aligned}$$

This implies that the quantity in (3.28) is nonpositive and thus we have $d_S^\circ(\boldsymbol{x}; \boldsymbol{d}) = 0$, in other words $\boldsymbol{d} \in T_S(\boldsymbol{x})$. □

Now we are ready to show the connection between contingent and tangent cones, and that in convex case those concepts are equivalent.

Theorem 3.27 *If S is a nonempty set and $\boldsymbol{x} \in S$, then*

$$T_S(\boldsymbol{x}) \subseteq K_S(\boldsymbol{x}).$$

If, in addition, S is convex then

$$T_S(\boldsymbol{x}) = K_S(\boldsymbol{x}).$$

Proof Suppose that $\boldsymbol{d} \in T_S(\boldsymbol{x})$, and we have a sequence $t_i \downarrow 0$. Define $\boldsymbol{x}_i := \boldsymbol{x}$ for all $i \in \mathbb{N}$. Then by Theorem 3.26 there exists $\boldsymbol{d}_i \to \boldsymbol{d}$ such that $\boldsymbol{x}_i + t_i \boldsymbol{d}_i \in S$, which means by the definition of the contingent cone (Definition 2.10) that $\boldsymbol{d} \in K_S(\boldsymbol{x})$.

Suppose next that S is convex and $\boldsymbol{x} \in S$. Then by Theorem 2.38 the function d_S is convex and by Theorem 3.8 (i) we have $d'_S(\boldsymbol{x}; \boldsymbol{d}) = d^\circ_S(\boldsymbol{x}; \boldsymbol{d})$ for all $\boldsymbol{d} \in \mathbb{R}^n$. Then the equality of the sets follows from Theorem 2.39 and Definition 3.6. □

The next reformulation of the normal cone is similar to that of in convex case (Theorem 2.40).

Theorem 3.28 *The normal cone of the set S at $\boldsymbol{x} \in S$ can also be written as*

$$N_S(\boldsymbol{x}) = \text{cl ray } \partial d_S(\boldsymbol{x}). \tag{3.29}$$

Proof Let $z \in \partial d_S(\boldsymbol{x})$ be given. Then, by the definition of the subdifferential,

$$\boldsymbol{z}^T \boldsymbol{d} \leq d^\circ_S(\boldsymbol{x}; \boldsymbol{d}) \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n.$$

If one has $\boldsymbol{d} \in T_S(\boldsymbol{x})$ then by the definition of the tangent cone $d^\circ_S(\boldsymbol{x}; \boldsymbol{d}) = 0$. Thus $\boldsymbol{z}^T \boldsymbol{d} \leq 0$ for all $\boldsymbol{d} \in T_S(\boldsymbol{x})$ which implies that $\boldsymbol{z} \in N_S(\boldsymbol{x})$. By Theorem 3.3 $\partial d_S(\boldsymbol{x})$ is a convex set and then by Theorem 2.11 ray $\partial_c d_S(\boldsymbol{x})$ is a convex cone. Furthermore, by Theorem 2.10 ray $\partial d_S(\boldsymbol{x})$ is the smallest cone containing $\partial d_S(\boldsymbol{x})$. Then, because $N_S(\boldsymbol{x})$ is also a convex cone (Theorem 3.25), we have

$$\text{ray } \partial d_S(\boldsymbol{x}) \subseteq N_S(\boldsymbol{x}).$$

For the converse, denote $Z := \text{cl ray } \partial d_S(\boldsymbol{x})$. Suppose that $\boldsymbol{z} \in N_S(\boldsymbol{x})$, but $\boldsymbol{z} \notin Z$. Clearly Z is closed and due to Theorems 3.3 and 2.11 convex cone. Because $\boldsymbol{0} \in Z$ it is also nonempty and thus, by Theorem 2.4 there exists a hyperplane separating $\{\boldsymbol{z}\}$ and Z. In other words there exists $\boldsymbol{d} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\boldsymbol{d}^T \boldsymbol{z} > \alpha \tag{3.30}$$

and

$$\boldsymbol{d}^T \boldsymbol{y} \leq \alpha \quad \text{for all} \quad \boldsymbol{y} \in Z. \tag{3.31}$$

Since Z is a cone the components of $\boldsymbol{d}$ can be chosen as large as possible in (3.31), thus $\alpha \leq 0$. On the other hand $\boldsymbol{0} \in Z$ implying $\alpha \geq \boldsymbol{p}^T \boldsymbol{0} = 0$, thus $\alpha = 0$.

Furthermore, we have $\partial d_S(\boldsymbol{x}) \subset Z$ and the inequality (3.31) implies by Theorem 3.4 that $d^\circ(\boldsymbol{x};\boldsymbol{d}) \leq 0$. Since $\boldsymbol{x} \in S$ we have $d^\circ(\boldsymbol{x};\boldsymbol{d}) \geq 0$. Hence, $d^\circ(\boldsymbol{x};\boldsymbol{d}) = 0$ implying $\boldsymbol{d} \in T_S(x)$. This contradicts with the inequality (3.30). Thus, $\boldsymbol{z} \in Z$ and theorem is proven. □

In what follows we shall need the next two properties of tangents and normals.

Theorem 3.29 *If* $\boldsymbol{x} \in \operatorname{int} S$, *then*

$$T_S(\boldsymbol{x}) = \mathbb{R}^n \quad \textit{and} \quad N_S(\boldsymbol{x}) = \{\mathbf{0}\}.$$

Proof Let $\boldsymbol{d} \in \mathbb{R}^n$ be arbitrary. If $\boldsymbol{x} \in \operatorname{int} S$, then there exists $\delta > 0$ such that $B(\boldsymbol{x};\delta) \subset S$. Choose sequences $\boldsymbol{x}_i \to \boldsymbol{x}$ and $t_i \downarrow 0$ such that

$$d_S^\circ(\boldsymbol{x};\boldsymbol{d}) = \limsup_{i\to\infty} \frac{d_S(\boldsymbol{x}_i + t_i\boldsymbol{d}) - d_S(\boldsymbol{x}_i)}{t_i}.$$

Then there exists $i_0 \in \mathbb{N}$ such that $\boldsymbol{x}_i,\ \boldsymbol{x}_i + t_i\boldsymbol{d} \in B(\boldsymbol{x};\delta) \subset S$ for all $i \geq i_0$. Thus we have $d_S(\boldsymbol{x}_i) = d(\boldsymbol{x}_i + t_i\boldsymbol{d}) = 0$ for all $i \geq i_0$, so $d^\circ(\boldsymbol{x};\boldsymbol{d}) = 0$ and thus $\boldsymbol{d} \in T_S(\boldsymbol{x})$.

Next suppose that $\boldsymbol{z} \in N_S(\boldsymbol{x})$, then by definition we have

$$\boldsymbol{z}^T\boldsymbol{d} \leq 0 \quad \text{for all} \quad \boldsymbol{d} \in T_S(\boldsymbol{x}) = \mathbb{R}^n.$$

Applying this property for any $\boldsymbol{d} \in \mathbb{R}^n$ and $-\boldsymbol{d} \in \mathbb{R}^n$, we get $\boldsymbol{z} = \mathbf{0}$ and the proof is complete. □

Theorem 3.30 *If* $S_1, S_2 \subseteq \mathbb{R}^n$ *are such that* $\boldsymbol{x} \in S_1 \cap S_2$ *and* $\boldsymbol{x} \in \operatorname{int} S_2$, *then*

$$T_{S_1}(\boldsymbol{x}) = T_{S_1\cap S_2}(\boldsymbol{x}) \quad \textit{and} \quad N_{S_1}(\boldsymbol{x}) = N_{S_1\cap S_2}(\boldsymbol{x}).$$

Proof Exercise. □

3.3.2 Epigraphs and Level Sets

In this subsection we shall present corresponding results to those of Chap. 2 concerning the concepts of epigraphs and level sets.

The next relationship between the the tangent cone of the epigraph and the epigraph of the generalized directional derivative is similar to that in convex case (Theorem 2.34).

Theorem 3.31 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous at* $\boldsymbol{x}$, *then*

$$T_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x})) = \text{epi } f^\circ(\boldsymbol{x};\ \cdot).$$

Proof Suppose first that $(\boldsymbol{d}, r) \in T_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x}))$. Choose sequences $\boldsymbol{x}_i \to \boldsymbol{x}$ and $t_i \downarrow 0$ such that

$$\lim_{i\to\infty} \frac{f(\boldsymbol{x}_i + t_i\boldsymbol{d}) - f(\boldsymbol{x}_i)}{t_i} = \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t} = f^\circ(\boldsymbol{x}; \boldsymbol{d}).$$

By the Lipschitz condition we have

$$\begin{aligned} \| (\boldsymbol{x}_i, f(\boldsymbol{x}_i)) - (\boldsymbol{x}, f(\boldsymbol{x})) \|^2 &= \|\boldsymbol{x}_i - \boldsymbol{x}\|^2 + |f(\boldsymbol{x}_i) - f(\boldsymbol{x})|^2 \\ &\le (1 + K^2)\|\boldsymbol{x}_i - \boldsymbol{x}\|^2 \longrightarrow 0, \end{aligned}$$

as $i \to \infty$, which means that the sequence $(\boldsymbol{x}_i, f(\boldsymbol{x}_i)) \in \text{epi } f$ is converging to $(\boldsymbol{x}, f(\boldsymbol{x}))$. By Theorem 3.26 there exists a sequence $(\boldsymbol{d}_i, r_i)$ converging to $(\boldsymbol{d}, r)$ such that

$$(\boldsymbol{x}_i, f(\boldsymbol{x}_i)) + t_i(\boldsymbol{d}_i, r_i) \in \text{epi } f \quad \text{for all} \quad i \in \mathbb{N},$$

thus we have

$$f(\boldsymbol{x}_i + t_i\boldsymbol{d}_i) \le f(\boldsymbol{x}_i) + t_i r_i.$$

Now we can calculate

$$\begin{aligned} f^\circ(\boldsymbol{x}; \boldsymbol{d}) &= \lim_{i\to\infty} \frac{f(\boldsymbol{x}_i + t_i\boldsymbol{d}_i) - f(\boldsymbol{x}_i)}{t_i} \\ &\le \lim_{i\to\infty} r_i = r, \end{aligned}$$

which implies that $(\boldsymbol{d}, r) \in \text{epi } f^\circ(\boldsymbol{x};\ \cdot)$.

Suppose next that $(\boldsymbol{d}, r) \in \text{epi } f^\circ(\boldsymbol{x};\ \cdot)$, which means that $f^\circ(\boldsymbol{x}; \boldsymbol{d}) \le r$. Define $\delta \ge 0$ such that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) + \delta = r$$

and let $t_i \downarrow 0$ and $(\boldsymbol{x}_i, s_i) \in \text{epi } f$ be arbitrary sequences such that $(\boldsymbol{x}_i, s_i) \to (\boldsymbol{x}, f(\boldsymbol{x}))$. Define sequences $\boldsymbol{d}_i := \boldsymbol{d}$ and

$$r_i := \max\{f^\circ(\boldsymbol{x}; \boldsymbol{d}) + \delta, \frac{f(\boldsymbol{x}_i + t_i\boldsymbol{d}) - f(\boldsymbol{x}_i)}{t_i}\}.$$

Then the fact that

$$\limsup_{i\to\infty} \frac{f(\boldsymbol{x}_i + t_i\boldsymbol{d}) - f(\boldsymbol{x}_i)}{t_i} \le f^\circ(\boldsymbol{x}; \boldsymbol{d})$$

shows that $r_i \to f^\circ(\boldsymbol{x}; \boldsymbol{d}) + \delta = r$ and, since $(\boldsymbol{x}_i, s_i) \in \text{epi } f$, we have

$$s_i + t_i r_i \ge s_i + [f(\boldsymbol{x}_i + t_i\boldsymbol{d}) - f(\boldsymbol{x}_i)] \ge f(\boldsymbol{x}_i) - f(\boldsymbol{x}_i) + f(\boldsymbol{x}_i + t_i\boldsymbol{d}),$$

which means that $(\boldsymbol{x}_i, s_i) + t_i(\boldsymbol{d}_i, r_i) \in \text{epi}\ f$. Now Theorem 3.26 implies that $(\boldsymbol{d}, r) \in T_{\text{epi}\ f}(\boldsymbol{x}, f(\boldsymbol{x}))$ and we obtain the desired conclusion. □

The next connection between the subdifferential and normal vectors of the epigraph is the same as in convex case (Theorem 2.35).

Theorem 3.32 *If the function $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}$, then*

$$\partial f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid (\boldsymbol{\xi}, -1) \in N_{\text{epi}\ f}(\boldsymbol{x}, f(\boldsymbol{x}))\}. \tag{3.32}$$

Proof By the definition of the subdifferential we know that $\boldsymbol{\xi}$ belongs to $\partial f(x)$ if and only if, for any $\boldsymbol{d} \in \mathbb{R}^n$ we have $f^\circ(\boldsymbol{x}; \boldsymbol{d}) \geq \boldsymbol{\xi}^T \boldsymbol{d}$. This is equivalent to the condition that for any $\boldsymbol{d} \in \mathbb{R}^n$ and $r \geq f^\circ(\boldsymbol{x}; \boldsymbol{d})$ we have $r \geq \boldsymbol{\xi}^T \boldsymbol{d}$, that is, for any $\boldsymbol{d} \in \mathbb{R}^n$ and $r \geq f^\circ(\boldsymbol{x}; \boldsymbol{d})$ we have

$$(\boldsymbol{\xi}, -1)^T (\boldsymbol{d}, r) \leq 0.$$

By the definition of the epigraph and Theorem 3.31 we have $(\boldsymbol{d}, r) \in \text{epi}\ f^\circ(\boldsymbol{x}; \cdot) = T_{\text{epi}\ f}(\boldsymbol{x}, f(\boldsymbol{x}))$. This and the last inequality mean, by the definition of the normal cone, that $(\boldsymbol{\xi}, -1)$ lies in $N_{\text{epi}\ f}(\boldsymbol{x}, f(\boldsymbol{x}))$. □

In the following theorem we give the relationship between generalized directional derivative and tangent vectors of the level sets. Note that the direction of the inclusion is opposite to that in convex case (Theorem 2.36).

Theorem 3.33 *If the function $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}$ and $\boldsymbol{0} \notin \partial f(\boldsymbol{x})$, then*

$$T_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) \supseteq \text{lev}_0\ f^\circ(\boldsymbol{x}; \cdot).$$

If, in addition, f is subdifferentially regular at $\boldsymbol{x}$ then

$$T_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) = \text{lev}_0\ f^\circ(\boldsymbol{x}; \cdot).$$

Proof Let $\boldsymbol{d} \in \text{lev}_0\ f^\circ(\boldsymbol{x}; \cdot)$, which means that $f^\circ(\boldsymbol{x}; \boldsymbol{d}) \leq 0$. Suppose first that

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \limsup_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ t \downarrow 0}} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t} < 0.$$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t} < -\delta \quad \text{for all} \quad \boldsymbol{y} \in B(\boldsymbol{x}; \varepsilon) \quad \text{and} \quad t \in (0, \varepsilon). \tag{3.33}$$

Let $\boldsymbol{x}_i \to \boldsymbol{x}$ and $t_i \downarrow 0$ be arbitrary sequences such that $\boldsymbol{x}_i \in \text{lev}_{f(\boldsymbol{x})} f$. Then there exists $i_0 \in \mathbb{N}$ such that $\boldsymbol{x}_i \in B(\boldsymbol{x}; \varepsilon)$ and $t_i \in (0, \varepsilon)$ for all $i \geq i_0$ and, by the definition of the set $\operatorname{lev}_{f(\boldsymbol{x})} f$, one has $f(\boldsymbol{x}_i) \leq f(\boldsymbol{x})$. By (3.33) we have for all $i \geq i_0$

$$f(\boldsymbol{x}_i + t_i\boldsymbol{d}) \leq f(\boldsymbol{x}_i) - \delta t_i \leq f(\boldsymbol{x}) - \delta t_i \leq f(\boldsymbol{x}),$$

thus $\boldsymbol{x}_i + t_i\boldsymbol{d} \in \operatorname{lev}_{f(\boldsymbol{x})} f$ for all $i \geq i_0$. Then, setting $\boldsymbol{d}_i := \boldsymbol{d}$, we deduce from Theorem 3.26 that $\boldsymbol{d} \in T_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$.

Suppose next that $f^\circ(\boldsymbol{x}; \boldsymbol{d}) = 0$. If there were $f^\circ(\boldsymbol{x}; \boldsymbol{p}) \geq 0$ for all $\boldsymbol{p} \in \mathbb{R}^n$, then by the definition of the subdifferential one would have $\mathbf{0} \in \partial f(\boldsymbol{x})$, contradicting the assertion. Thus there always exists $\hat{\boldsymbol{p}} \in \mathbb{R}^n$ such that $f^\circ(\boldsymbol{x}; \hat{\boldsymbol{p}}) < 0$. Now define the sequence $\boldsymbol{d}_i := \boldsymbol{d} + \frac{1}{i}\hat{\boldsymbol{p}}$. Then $\boldsymbol{d}_i \to \boldsymbol{d}$ and due to subadditivity and positive homogeneity of the generalized directional derivative (Theorem 3.1) we have

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}_i) = f^\circ\left(\boldsymbol{x}; \boldsymbol{d} + \frac{1}{i}\hat{\boldsymbol{p}}\right) \leq f^\circ(\boldsymbol{x}; \boldsymbol{d}) + \frac{1}{i} f^\circ(\boldsymbol{x}; \hat{\boldsymbol{p}}) < 0,$$

thus, as at the beginning of the proof, we get $\boldsymbol{d}_i \in T_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$. By Theorem 3.24 the tangent cone $T_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$ is closed, which implies that also $\boldsymbol{d} \in T_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$.

Finally, suppose that f is subdifferentially regular at $\boldsymbol{x}$ and $\boldsymbol{d} \in T_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$. Then the directional derivative at $\boldsymbol{x}$ exists for $\boldsymbol{d}$ and $f'(\boldsymbol{x}; \boldsymbol{d}) = f^\circ(\boldsymbol{x}; \boldsymbol{d})$. Let $t_i \downarrow 0$, then by Theorem 3.26 there exists a sequence $\boldsymbol{d}_i \to \boldsymbol{d}$ such that $\boldsymbol{x} + t_i\boldsymbol{d}_i \in \operatorname{lev}_{f(\boldsymbol{x})} f$, thus $f(\boldsymbol{x} + t_i\boldsymbol{d}_i) \leq f(\boldsymbol{x})$ for all $i \in \mathbb{N}$. By the Lipschitz condition we get

$$\begin{aligned}
\frac{f(\boldsymbol{x} + t_i\boldsymbol{d}) - f(\boldsymbol{x})}{t_i} &= \frac{f(\boldsymbol{x} + t_i\boldsymbol{d}) - f(\boldsymbol{x} + t_i\boldsymbol{d}_i) + f(\boldsymbol{x} + t_i\boldsymbol{d}_i) - f(\boldsymbol{x})}{t_i} \\
&\leq \frac{f(\boldsymbol{x} + t_i\boldsymbol{d}_i) - f(\boldsymbol{x})}{t_i} + \frac{K\|\boldsymbol{x} + t_i\boldsymbol{d} - \boldsymbol{x} - t_i\boldsymbol{d}_i\|}{t_i} \\
&\leq 0 + K\|\boldsymbol{d} - \boldsymbol{d}_i\|,
\end{aligned}$$

whence taking the limit as $i \to \infty$, one has $f^\circ(\boldsymbol{x}; \boldsymbol{d}) = f'(\boldsymbol{x}; \boldsymbol{d}) \leq 0$. In other words $\boldsymbol{d} \in \operatorname{lev}_0 f^\circ(\boldsymbol{x}; \cdot)$ and the proof is complete. □

To the end of this subsection we give the relationship between subgradients and normal vectors of the level sets. Note that again the direction of the inclusion is opposite to that in convex case (Theorem 2.37).

Theorem 3.34 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous at* $\boldsymbol{x}$ *and* $\mathbf{0} \notin \partial f(\boldsymbol{x})$*, then*

$$N_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) \subseteq \operatorname{ray} \partial f(\boldsymbol{x}).$$

If, in addition, f *is subdifferentially regular at* $\boldsymbol{x}$ *then*

$$N_{\operatorname{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) = \operatorname{ray} \partial f(\boldsymbol{x}).$$

Proof Exercise. (Hint: In the proof of Theorem 2.37 replace $\partial_c f(\boldsymbol{x})$, $f'(\boldsymbol{x};\ \cdot)$ and $K_{\text{lev}_{f(\boldsymbol{x})}\, f}(\boldsymbol{x})$ by $\partial f(\boldsymbol{x})$, $f^\circ(\boldsymbol{x};\ \cdot)$ and $T_{\text{lev}_{f(\boldsymbol{x})}\, f}(\boldsymbol{x})$, respectively.) □

3.3.3 Cones of Feasible Directions

To the end of this chapter we will discuss more about the cones of feasible directions. In Definition 2.11 defined the cone of globally feasible directions $G_S(\boldsymbol{x})$ and due to Theorems 2.17 and 3.27 we know that for a nonempty set S at $\boldsymbol{x} \in S$ it holds

$$T_S(\boldsymbol{x}) \subseteq K_S(\boldsymbol{x}) \subseteq \text{cl } G_S(\boldsymbol{x})$$

and if S is convex, then

$$T_S(\boldsymbol{x}) = K_S(\boldsymbol{x}) = \text{cl } G_S(\boldsymbol{x}).$$

Next we define another cone of feasible directions.

Definition 3.8 The *cone of locally feasible directions* of the nonempty set S at $\boldsymbol{x} \in S$ is given by the formula

$$F_S(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid \text{there exists } \varepsilon > 0 \text{ such that } \boldsymbol{x} + t\boldsymbol{d} \in S \text{ for all } t \in (0, \varepsilon]\}.$$

The cone of locally feasible directions has the same properties as the cone of globally feasible directions (Theorem 2.16).

Theorem 3.35 *The cone of locally feasible directions $F_S(\boldsymbol{x})$ of the nonempty convex set S at $\boldsymbol{x} \in S$ is a convex cone.*

Proof Exercise. □

We have the following connection between the cones of feasible directions and the contingent cone.

Theorem 3.36 *If S is a nonempty set and $\boldsymbol{x} \in S$, then*

$$\text{cl } F_S(\boldsymbol{x}) \subseteq K_S(\boldsymbol{x}). \tag{3.34}$$

If, in addition, S is convex then

$$F_S(\boldsymbol{x}) = G_S(\boldsymbol{x}).$$

Proof If $\boldsymbol{d} \in F_S(\boldsymbol{x})$ is arbitrary, then there exists $\varepsilon > 0$ such that $\boldsymbol{x} + t\boldsymbol{d} \in S$ for all $t \in (0, \varepsilon]$. Define $t_i := \varepsilon/i$ and $\boldsymbol{d}_i := \boldsymbol{d}$ for all $i \in \mathbb{N}$. Then clearly $t_i \downarrow 0$,

$\boldsymbol{d}_i \to \boldsymbol{d}$ and $\boldsymbol{x} + t_i \boldsymbol{d}_i \in S$, and thus by the definition of the contingent cone we have $\boldsymbol{d} \in K_S(\boldsymbol{x})$, in other words

$$F_S(\boldsymbol{x}) \subseteq K_S(\boldsymbol{x}). \tag{3.35}$$

Since, due to Theorem 2.15, $K_S(\boldsymbol{x})$ is closed, we get the assertion (3.34) by taking the closure from both sides of (3.35).

Due to the definitions it is clear that $F_S(\boldsymbol{x}) \subseteq G_S(\boldsymbol{x})$. To see the converse, let S be convex and $\boldsymbol{d} \in G_S(\boldsymbol{x})$. Then there exists $\hat{t} > 0$ such that $\boldsymbol{x} + \hat{t}\boldsymbol{d} \in S$. The convexity of S implies that for all $\lambda \in [0, 1]$ we have

$$\lambda \boldsymbol{x} + (1 - \lambda)(\boldsymbol{x} + \hat{t}\boldsymbol{d}) = \boldsymbol{x} + (1 - \lambda)\hat{t}\boldsymbol{d} \in S.$$

Now we can choose $\varepsilon := \hat{t}$ and $t := (1 - \lambda)\hat{t}$ and we have $\boldsymbol{x} + t\boldsymbol{d} \in S$ for all $t \in (0, \varepsilon]$, in other words, $\boldsymbol{d} \in F_S(\boldsymbol{x})$, which proves the assertion. □

In order to sum up all the results above for a nonempty set S at $\boldsymbol{x} \in S$ we get the inclusions

$$\text{cl } F_S(\boldsymbol{x}) \subseteq K_S(\boldsymbol{x}) \subseteq \text{cl } G_S(\boldsymbol{x}) \quad \text{and} \quad T_S(\boldsymbol{x}) \subseteq K_S(\boldsymbol{x}) \subseteq \text{cl } G_S(\boldsymbol{x}).$$

If in addition S is convex, then

$$\text{cl } F_S(\boldsymbol{x}) = T_S(\boldsymbol{x}) = K_S(\boldsymbol{x}) = \text{cl } G_S(\boldsymbol{x}) \quad \text{and} \quad F_S(\boldsymbol{x}) = G_S(\boldsymbol{x}).$$

Example 3.5 (*Cones*) Let $S_1 \subset \mathbb{R}^2$ defined by

$$S_1 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 \le -x_1^3\} \cap \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 \ge x_1^3\}.$$

Then it is easy to calculate (also see Fig. 3.4a) that

$$F_{S_1}(\boldsymbol{0}) = T_{S_1}(\boldsymbol{0}) = K_{S_1}(\boldsymbol{0}) = \{\boldsymbol{d} \in \mathbb{R}^2 \mid d_1 \le 0, d_2 = 0\}$$
$$G_{S_1}(\boldsymbol{0}) = \{\boldsymbol{d} \in \mathbb{R}^2 \mid d_1 < 0\} \cup \{\boldsymbol{0}\}.$$

On the other hand, if $S_2 \subset \mathbb{R}^2$ is defined by

$$S_2 := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 \le -x_1^3\} \cup \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 \ge x_1^3\},$$

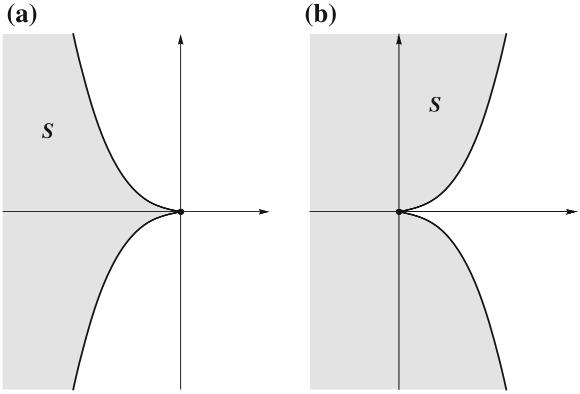


Fig. 3.4 Illustration of sets S_1 and S_2 in Example 3.5. (**a**) Set S_1. (**b**) Set S_2

(see Fig. 3.4b) then we have

$$\begin{aligned} T_{S_2}(\mathbf{0}) &= \{\boldsymbol{d} \in \mathbb{R}^2 \mid d_1 \leq 0, d_2 = 0\} \\ F_{S_2}(\mathbf{0}) &= G_{S_2}(\mathbf{0}) = \mathbb{R}^2 \setminus \{\boldsymbol{d} \in \mathbb{R}^2 \mid d_1 > 0, d_2 = 0\} \\ K_{S_2}(\mathbf{0}) &= \mathbb{R}^2. \end{aligned}$$

Note in the last case we have $K_{S_2}(\mathbf{0}) \supset G_{S_2}(\mathbf{0})$, but $K_{S_2}(\mathbf{0}) = \text{cl } G_{S_2}(\mathbf{0})$.

3.4 Other Generalized Subdifferentials

In addition to the Clarke subdifferential (see Definition 3.2), many different generalizations of the subdifferential for nonconvex nonsmooth functions exist. In this section we briefly recall some of them. More specifically we give definitions of the quasidifferential, the codifferential, the basic (limiting) and the singular subdifferentials. We also give some results on the relationship between the Clarke subdifferential and the quasidifferential. We have omitted the proofs since they can be found in [72].

3.4.1 Quasidifferentials

Let a function f be defined on an open set $X \subset \mathbb{R}^n$ and be directionally differentiable at a point $\boldsymbol{x} \in X$. That is, the directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ exists. We now define the quasidifferentiable function.

Definition 3.9 The function is called *quasidifferentiable* at $\boldsymbol{x}$ if there exists the pair of compact convex sets $(\underline{\partial} f(\boldsymbol{x}), \overline{\partial} f(\boldsymbol{x}))$ such that the directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ of the function f at $\boldsymbol{x}$ in the direction $\boldsymbol{d} \in \mathbb{R}^n$ can be represented in the form:

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \max \left\{\xi^T \boldsymbol{d} \mid \xi \in \underline{\partial} f(\boldsymbol{x})\right\} + \min \left\{\nu^T \boldsymbol{d} \mid \nu \in \overline{\partial} f(\boldsymbol{x})\right\}. \tag{3.36}$$

The set $\underline{\partial} f(\boldsymbol{x})$ is called the *subdifferential* of the function f at $\boldsymbol{x}$ and the $\overline{\partial} f(\boldsymbol{x})$ is called the *superdifferential* of the function f at $\boldsymbol{x}$. The pair $\mathcal{D} f(\boldsymbol{x}) = [\underline{\partial} f(\boldsymbol{x}), \overline{\partial} f(\boldsymbol{x})]$ is called the *quasidifferential* of the function f at $\boldsymbol{x}$.

The quasidifferential mapping is not uniquely defined. Indeed, if $\mathcal{D} f(\boldsymbol{x}) = [\underline{\partial} f(\boldsymbol{x}), \overline{\partial} f(\boldsymbol{x})]$ is a quasidifferential of the function f at a point $\boldsymbol{x}$, then for any compact convex set $A \subset \mathbb{R}^n$ the pair $\mathcal{D} f(\boldsymbol{x}) = [\underline{\partial} f(\boldsymbol{x}) + A, \overline{\partial} f(\boldsymbol{x}) - A]$ is also a quasidifferential of f at $\boldsymbol{x}$.

The limit

$$\lim_{\alpha \downarrow 0, \boldsymbol{d}' \to \boldsymbol{d}} \frac{1}{\alpha} \left[f(\boldsymbol{x} + \alpha \boldsymbol{d}') - f(\boldsymbol{x}) \right] \tag{3.37}$$

is called the *Hadamard derivative* of a function f at a point $\boldsymbol{x}$ in a direction $\boldsymbol{d}$. A function f is called *Hadamard quasidifferentiable* if in (3.36) the Hadamard derivative is used instead of the usual directional derivative.

The quasidifferential mapping enjoys the full calculus in a sense that the equalities can be used instead of inclusions (cf. subdifferentially regular functions with Clarke subdifferential in Sect. 3.2). The following theorem is presented in [72].

Theorem 3.37 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be such that $f = g \circ \boldsymbol{h}$, where $\boldsymbol{h} = (h_1, \dots, h_m)$: $\mathbb{R}^n \to \mathbb{R}^m$, all functions h_i, $i = 1, \dots, m$ are quasidifferentiable at a point $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $g: \mathbb{R}^m \to \mathbb{R}$ is Hadamard quasidifferentiable at the point $\boldsymbol{y}_0 = h(\boldsymbol{x}_0)$. Then the function f is quasidifferentiable at the point $\boldsymbol{x}_0$ and the quasidifferential $\mathcal{D} f(\boldsymbol{x}_0) = \left[\underline{\partial} f(\boldsymbol{x}_0), \overline{\partial} f(\boldsymbol{x}_0)\right]$ is given by the following formulas*

$$\underline{\partial} f(\boldsymbol{x}_0) = \Bigg\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \boldsymbol{\xi} = \sum_{i=1}^{m} ((\boldsymbol{u}_i + \boldsymbol{v}_i) p_i - \boldsymbol{u}_i \underline{p_i} - \mu_i \overline{p^i})$$

$$\boldsymbol{p} = (p_1, \dots, p_m) \in \underline{\partial} g(\boldsymbol{y}_0), \boldsymbol{u}_i \in \underline{\partial} h_i(\boldsymbol{x}_0), \boldsymbol{v}_i \in \overline{\partial} h_i(\boldsymbol{x}_0)\Bigg\},$$

$$\overline{\partial} f(\boldsymbol{x}_0) = \Bigg\{\nu \in \mathbb{R}^n \mid \nu = \sum_{i=1}^{m} ((\boldsymbol{u}_i + \boldsymbol{v}_i) p_i + \boldsymbol{u}_i \underline{p_i} + \mu_i \overline{p^i})$$

$$\boldsymbol{p} = (p_1, \dots, p_m) \in \overline{\partial} g(\boldsymbol{y}_0), \boldsymbol{u}_i \in \underline{\partial} h_i(\boldsymbol{x}_0), \boldsymbol{v}_i \in \overline{\partial} h_i(\boldsymbol{x}_0)\Bigg\}.$$

Here $\underline{\boldsymbol{p}}$ and $\overline{\boldsymbol{p}}$ are arbitrary vectors such that $\underline{\boldsymbol{p}} \le \boldsymbol{p} \le \overline{\boldsymbol{p}}$ for all $\boldsymbol{p} \in \underline{\partial} g(\boldsymbol{y}_0) \bigcup (-\overline{\partial} g(\boldsymbol{y}_0))$.

The following theorems about quasidifferential calculus follow from Theorem 3.37.

Theorem 3.38 *Let functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be quasidifferentiable at a point $\boldsymbol{x}$. Then*

(i) *The function $f_1 + f_2$ is quasidifferentiable at $\boldsymbol{x}$ and*

$$\mathcal{D}(f_1 + f_2)(\boldsymbol{x}) = \mathcal{D}f_1(\boldsymbol{x}) + \mathcal{D}f_2(\boldsymbol{x}).$$

In other words, if $[\underline{\partial} f_1(\boldsymbol{x}), \overline{\partial} f_1(\boldsymbol{x})]$ and $[\underline{\partial} f_2(\boldsymbol{x}), \overline{\partial} f_2(\boldsymbol{x})]$ are quasidifferentials of the functions f_1 and f_2 at $\boldsymbol{x}$, respectively, then

$$\underline{\partial}(f_1 + f_2)(\boldsymbol{x}) = \underline{\partial} f_1(\boldsymbol{x}) + \underline{\partial} f_2(\boldsymbol{x}),$$

$$\overline{\partial}(f_1 + f_2)(\boldsymbol{x}) = \overline{\partial} f_1(\boldsymbol{x}) + \overline{\partial} f_2(\boldsymbol{x}).$$

(ii) *The function $f_1 \cdot f_2$ is quasidifferentiable at $\boldsymbol{x}$, and*

$$\mathcal{D}(f_1 \cdot f_2)(\boldsymbol{x}) = f_1(\boldsymbol{x})\mathcal{D}f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\mathcal{D}f_1(\boldsymbol{x})$$

and

$$\underline{\partial}(f_1 \cdot f_2)(\boldsymbol{x}) = \begin{cases} f_1(\boldsymbol{x})\underline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\underline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \geq 0,\ f_2(\boldsymbol{x}) \geq 0, \\ f_1(\boldsymbol{x})\overline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\underline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \leq 0,\ f_2(\boldsymbol{x}) \geq 0, \\ f_1(\boldsymbol{x})\overline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\overline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \leq 0,\ f_2(\boldsymbol{x}) \leq 0, \\ f_1(\boldsymbol{x})\underline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\overline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \geq 0,\ f_2(\boldsymbol{x}) \leq 0, \end{cases}$$

$$\overline{\partial}(f_1 \cdot f_2)(\boldsymbol{x}) = \begin{cases} f_1(\boldsymbol{x})\overline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\overline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \geq 0,\ f_2(\boldsymbol{x}) \geq 0, \\ f_1(\boldsymbol{x})\underline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\overline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \leq 0,\ f_2(\boldsymbol{x}) \geq 0, \\ f_1(\boldsymbol{x})\underline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\underline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \leq 0,\ f_2(\boldsymbol{x}) \leq 0, \\ f_1(\boldsymbol{x})\overline{\partial} f_2(\boldsymbol{x}) + f_2(\boldsymbol{x})\underline{\partial} f_1(\boldsymbol{x}), & \text{if } f_1(\boldsymbol{x}) \geq 0,\ f_2(\boldsymbol{x}) \leq 0. \end{cases}$$

Proof Exercise. □

Example 3.6 (*Quasidifferentials of functions*) Applying Theorem 3.38 one can get formulae for quasidifferentials of functions:

(i) $f(\boldsymbol{x}) = \lambda f_1(\boldsymbol{x})$ where λ is any real number and f_1 is quasidifferentiable at $\boldsymbol{x}$;
(ii) $f(\boldsymbol{x}) = f_1(\boldsymbol{x}) - f_2(\boldsymbol{x})$ where functions f_1 and f_2 are quasidifferentiable at $\boldsymbol{x}$.

(iii) $f(\boldsymbol{x}) = f_1(\boldsymbol{x})/f_2(\boldsymbol{x})$, where functions f_1, f_2 are quasidifferentiable at $\boldsymbol{x}$ and $f_2(\boldsymbol{x}) \neq 0$.

Theorem 3.39 *Let functions $f_i, i = 1, \dots, m$ be defined on an open set $X \subset \mathbb{R}^n$ and quasidifferentiable at a point $\boldsymbol{x} \in X$. Let*

$$\varphi_1(\boldsymbol{x}) = \max_{i=1,\dots,m} f_i(x), \quad \varphi_2(\boldsymbol{x}) = \min_{i=1,\dots,m} f_i(x).$$

Then the functions φ_1 and φ_2 are quasidifferentiable at $\boldsymbol{x}$ and

$$\mathcal{D}\varphi_1(\boldsymbol{x}) = [\underline{\partial}\varphi_1(\boldsymbol{x}), \overline{\partial}\varphi_1(\boldsymbol{x})], \quad \mathcal{D}\varphi_2(\boldsymbol{x}) = [\underline{\partial}\varphi_2(\boldsymbol{x}), \overline{\partial}\varphi_2(\boldsymbol{x})]$$

where

$$\begin{aligned}
\underline{\partial}\varphi_1(\boldsymbol{x}) &= \operatorname{conv} \bigcup_{k\in R(\boldsymbol{x})} \left(\underline{\partial} f_k(\boldsymbol{x}) - \sum_{i\in R(\boldsymbol{x}), i\neq k} \overline{\partial} f_i(\boldsymbol{x}) \right), \\
\overline{\partial}\varphi_1(\boldsymbol{x}) &= \bigcup_{k\in R(\boldsymbol{x})} \overline{\partial} f_k(\boldsymbol{x}), \\
\underline{\partial}\varphi_2(\boldsymbol{x}) &= \bigcup_{k\in Q(\boldsymbol{x})} \underline{\partial} f_k(\boldsymbol{x}), \\
\overline{\partial}\varphi_2(\boldsymbol{x}) &= \operatorname{conv} \bigcup_{k\in Q(\boldsymbol{x})} \left(\overline{\partial} f_k(\boldsymbol{x}) - \sum_{i\in Q(\boldsymbol{x}), i\neq k} \underline{\partial} f_i(\boldsymbol{x}) \right).
\end{aligned}$$

Here $[\underline{\partial} f_k(\boldsymbol{x}), \overline{\partial} f_k(\boldsymbol{x})]$ is a quasidifferential of the function f_k at the point $\boldsymbol{x}$, and

$$\begin{aligned}
R(\boldsymbol{x}) &= \{i \in \{1, \dots, m\} \mid f_i(\boldsymbol{x}) = \varphi_1(\boldsymbol{x})\}, \ \textit{and} \\
Q(\boldsymbol{x}) &= \{i \in \{1, \dots, m\} \mid f_i(\boldsymbol{x}) = \varphi_2(\boldsymbol{x})\}.
\end{aligned}$$

Proof Exercise. □

Example 3.7 (*Quasidifferential of composite functions*) Consider the function $f: \mathbb{R}^n \to \mathbb{R}$ defined as in Theorem 3.37, where the functions $h_i,\ i = 1, \ldots, m$ are subdifferentially regular functions (see Definition 3.5) and the function g is continuously differentiable. Take a point $\boldsymbol{x} \in \mathbb{R}^n$, put $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \in \mathbb{R}^m$ and consider the following index sets:

$$\mathcal{I}_+(\boldsymbol{x}) = \left\{ i \in \mathcal{I} \mid \frac{\partial g(\boldsymbol{y})}{\partial y_i} > 0 \right\},\ \mathcal{I}_-(\boldsymbol{x}) = \left\{ i \in \mathcal{I} \mid \frac{\partial g(\boldsymbol{y})}{\partial y_i} < 0 \right\}$$

$$\mathcal{I}_0(\boldsymbol{x}) = \left\{ i \in \mathcal{I} \mid \frac{\partial g(\boldsymbol{y})}{\partial y_i} = 0 \right\}.$$

Then the quasidifferential of the function f at the point $\boldsymbol{x}$ is $\mathcal{D} f(\boldsymbol{x}) = [\underline{\partial} f(\boldsymbol{x}), \overline{\partial} f(\boldsymbol{x})]$ where

$$\underline{\partial} f(\boldsymbol{x}) = \left\{ \xi \in \mathbb{R}^n \mid \xi = \sum_{i \in \mathcal{I}_+(\boldsymbol{x})} \frac{\partial g(\boldsymbol{y})}{\partial y_i} \partial f_i(\boldsymbol{x}) \right\},$$

and

$$\overline{\partial} f(\boldsymbol{x}) = \left\{ \nu \in \mathbb{R}^n \mid \nu = \sum_{i \in \mathcal{I}_-(\boldsymbol{x})} \frac{\partial g(\boldsymbol{y})}{\partial y_i} \partial f_i(\boldsymbol{x}) \right\}.$$

Example 3.8 (*Quasidifferential of convex functions*) If a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex then it is quasidifferentiable and its quasidifferential at a point $\boldsymbol{x} \in \mathbb{R}^n$ is

$$\mathcal{D} f(\boldsymbol{x}) = [\partial f(\boldsymbol{x}), \{\mathbf{0}\}].$$

Example 3.9 (*Quasidifferential of DC functions*) If a function $f : \mathbb{R}^n \to \mathbb{R}$ can be represented as a difference of two convex functions

$$f(\boldsymbol{x}) = f_1(\boldsymbol{x}) - f_2(\boldsymbol{x}),$$

then it is quasidifferentiable and its quasidifferential at a point $\boldsymbol{x} \in \mathbb{R}^n$ is

$$\mathcal{D} f(\boldsymbol{x}) = [\partial f_1(\boldsymbol{x}), -\partial f_2(\boldsymbol{x})].$$

3.4.2 Relationship Between Quasidifferential and Clarke Subdifferential

In order to establish relationship between the quasidifferential and Clarke subdifferential we consider the so-called *Demyanov difference* between two compact convex sets. Given two convex compact sets $U, V \subset \mathbb{R}^n$ consider their support functions:

$$p_U(\boldsymbol{d}) = \max_{\boldsymbol{u} \in U} \boldsymbol{u}^T \boldsymbol{d}, \quad p_V(\boldsymbol{d}) = \max_{\boldsymbol{v} \in V} \boldsymbol{v}^T \boldsymbol{d}, \quad \boldsymbol{d} \in \mathbb{R}^n.$$

Since both functions p_U and p_V are locally Lipschitz continuous they are differentiable almost everywhere. Let T be a set of full measure such that at every point $\boldsymbol{d} \in T$ there exist the gradients $\nabla p_U(\boldsymbol{d})$ and $\nabla p_V(\boldsymbol{d})$. The Demyanov difference $U \dot{-} V$ between the sets U and V is defined as follows:

$$U \dot{-} V := \text{cl conv}\left\{\nabla p_U(\boldsymbol{d}) - \nabla p_V(\boldsymbol{d}),\ \boldsymbol{d} \in T\right\}.$$

Now, let us assume that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasidifferentiable at a point $\boldsymbol{x} \in \mathbb{R}^n$ with a quasidifferential $\mathcal{D} f(\boldsymbol{x}) = [\underline{\partial} f(\boldsymbol{x}), \overline{\partial} f(\boldsymbol{x})]$. We denote $U := \underline{\partial} f(\boldsymbol{x})$ and $V := \overline{\partial} f(\boldsymbol{x})$. Let us also assume that $\boldsymbol{d} \in \mathbb{R}^n$ is such that the linear function $\boldsymbol{u}^T \boldsymbol{d}$ attains its maximal value on the set U at a unique point $\bar{\boldsymbol{u}}(\boldsymbol{d})$, and the linear function $\boldsymbol{v}^T \boldsymbol{d}$ attains its minimal value on the set V at a unique point $\bar{\boldsymbol{v}}(\boldsymbol{d})$. This implies that $\nabla p_U(\boldsymbol{d}) = \bar{\boldsymbol{u}}(\boldsymbol{d})$ and $\nabla p_V(\boldsymbol{d}) = \bar{\boldsymbol{v}}(\boldsymbol{d})$.

In addition, we assume that a set $T \subset \mathbb{R}^n$ satisfies the following conditions:

(1) the Lebesgue measure of the set $\mathbb{R}^n \setminus T$ is zero;
(2) for $\boldsymbol{d} \in T$ points $\bar{\boldsymbol{u}}(\boldsymbol{d})$ and $\bar{\boldsymbol{v}}(\boldsymbol{d})$ are unique.

Let $X \subset \mathbb{R}^n$ be an open set and $\boldsymbol{x} \in X$. We denote by $M(\boldsymbol{x})$ the family of functions f defined on X such that

(i) f is quasidifferentiable at $\boldsymbol{x}$;
(ii) f is locally Lipschitz continuous in some neighborhood $B(\boldsymbol{x}; \delta)$ of the point $\boldsymbol{x}$;
(iii) there exists a set $Q \subset \Omega_f$, where Ω_f is the set of points where the gradient of f exists, of full measure (with respect to $B(\boldsymbol{x}; \delta)$); a quasidifferential $[U, V]$ of the function f at the point $\boldsymbol{x}$; and a set T possessing properties (1) and (2) with respect to the pair $[U, V]$ such that the condition

$$\boldsymbol{d}_k \to \boldsymbol{d}, \quad \alpha_k \downarrow 0, \quad \boldsymbol{x}_k = \boldsymbol{x} + \alpha_k \boldsymbol{d}_k \in Q, \quad \boldsymbol{d} \in T$$

imply

$$\nabla f(\boldsymbol{x}_k) \to \bar{\boldsymbol{u}}(\boldsymbol{d}) + \bar{\boldsymbol{v}}(\boldsymbol{d}).$$

Let a function f be locally Lipschitz continuous on $B(\boldsymbol{x}; \delta)$, $Q \subset B(\boldsymbol{x}; \delta)$ and $T \subset \mathbb{R}^n$ such that

$$\mu(B(\boldsymbol{x};\delta)\setminus Q)=0,\quad \mu(\mathbb{R}^n\setminus T)=0,$$

where μ is the Lebesgue measure. Denote $\partial_T f(\boldsymbol{x}) :=$ cl conv D_T where

$$D_T=\{\boldsymbol{v}\in\mathbb{R}^n \mid \exists \boldsymbol{x}_k=\boldsymbol{x}+\alpha_k\boldsymbol{d}_k\in Q, \boldsymbol{d}_k\to\boldsymbol{d}\in T, \alpha_k\downarrow 0, \nabla f(\boldsymbol{x}_k)\to\boldsymbol{v}\}.$$

Theorem 3.40 *Let $f\in M(\boldsymbol{x})$, $Q\subset \Omega_f$ and a set T possessing properties (1) and (2) with respect to a quasidifferential $[U,V]$ be such that condition (iii) holds. Then*

$$\partial_T f(\boldsymbol{x})=\underline{\partial} f(\boldsymbol{x})\dotdiv\overline{\partial} f(\boldsymbol{x}).$$

Let $f\in M(\boldsymbol{x})$ then

$$\underline{\partial} f(\boldsymbol{x})\dotdiv\overline{\partial} f(\boldsymbol{x})\subset\partial f(\boldsymbol{x}).$$

This leads to the following theorem which shows the relationship between the Clarke subdifferential and quasidifferential.

Theorem 3.41 *If $f\in M(\boldsymbol{x})$ and there exists a pair $[\underline{\partial} f(\boldsymbol{x}),\overline{\partial} f(\boldsymbol{x})]$ such that (iii) holds for $T=\mathbb{R}^n$, then*

$$\partial f(\boldsymbol{x})=\underline{\partial} f(\boldsymbol{x})\dotdiv\overline{\partial} f(\boldsymbol{x}).$$

3.4.3 Codifferentials

The quasidifferential mapping need not to be even upper continuous. The notion of *codifferential* can be considered as a modification of the quasidifferential that enjoys Hausdorff continuity (see Sect. 1.3). Here we briefly recall the definition of codifferentiable functions.

Definition 3.10 Let $X\subset\mathbb{R}^n$ be an open set and $\boldsymbol{x}\in X$. Assume that a function f is defined on X and it is finite. We say that this function is *codifferentiable* at a point $\boldsymbol{x}\in X$, if there exist convex compact sets $\underline{d} f(\boldsymbol{x})\subset\mathbb{R}^{n+1}$ and $\overline{d} f(\boldsymbol{x})\subset\mathbb{R}^{n+1}$ such that

$$f(\boldsymbol{x}+\Delta)=f(\boldsymbol{x})+\Phi_{\boldsymbol{x}}(\Delta)+o_{\boldsymbol{x}}(\Delta), \tag{3.38}$$

where

$$\Phi_{\boldsymbol{x}}(\Delta)=\max_{(\eta,\boldsymbol{v})\in\underline{d} f(\boldsymbol{x})}\left[\eta+\boldsymbol{v}^T\Delta\right]+\min_{(\theta,\boldsymbol{w})\in\overline{d} f(\boldsymbol{x})}\left[\theta+\boldsymbol{w}^T\Delta\right],$$

$$\frac{o_{\boldsymbol{x}}(\alpha\Delta)}{\alpha} \to 0, \quad \text{as } \alpha \downarrow 0 \ \forall \Delta \in \mathbb{R}^n. \tag{3.39}$$

Here $\eta, \theta \in \mathbb{R}$, $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. We assume that $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{x} + \Delta\} \subset X$.

The pair $Df(\boldsymbol{x}) = [\underline{d}f(\boldsymbol{x}), \overline{d}f(\boldsymbol{x})]$ is called a *codifferential*, the set $\underline{d}f(\boldsymbol{x})$ a *hypodifferential* and the set $\overline{d}f(\boldsymbol{x})$ a *hyperdifferential* of function f at $\boldsymbol{x}$.

Note that similarly to quasidifferential mappings the codifferential mapping is not unique.

If $\overline{d}f(\boldsymbol{x}) = \{\boldsymbol{0}\}$ then f is called *hypodifferentiable* at $\boldsymbol{x}$ and if $\underline{d}f(\boldsymbol{x}) = \{\boldsymbol{0}\}$ then f is called *hyperdifferentiable* at $\boldsymbol{x}$. A proper convex function is hypodifferentiable, and a proper concave function is hyperdifferentiable. The classes of quasidifferentiable and codifferentiable functions coincide. Moreover, if $\mathcal{D}f(\boldsymbol{x}) = [\underline{\partial}f(\boldsymbol{x}), \overline{\partial}f(\boldsymbol{x})]$ is the quasidifferential of the function f at the point $\boldsymbol{x}$ then we have

$$\underline{\partial}f(\boldsymbol{x}) = \{\boldsymbol{u} \in \mathbb{R}^n \mid (\boldsymbol{0}, \boldsymbol{u}) \in \underline{d}f(\boldsymbol{x})\}$$

and

$$\overline{\partial}f(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n \mid (\boldsymbol{0}, \boldsymbol{v}) \in \overline{d}f(\boldsymbol{x})\}.$$

Likewise quasidifferential mapping, the codifferential mapping enjoys a full calculus. However, it is not always easy to apply this calculus to compute codifferentials as it involves operations over polytopes in $(n + 1)$-dimensional space.

Definition 3.11 Let $X \subset \mathbb{R}^n$ be an open set and $\boldsymbol{x} \in X$. Assume that a function f is defined on X and it is finite. We say that a function f is *directionally uniformly codifferentiable* at $\boldsymbol{x} \in X$ if (3.39) holds uniformly with respect to $\Delta \in S_1$, where S_1 is the sphere of the unit ball. In addition, we say that a function f is *continuously codifferentiable* at a point $\boldsymbol{x} \in X$, if it is codifferentiable in some neighborhood of this point and mappings $\boldsymbol{x} \mapsto \underline{d}f(\boldsymbol{x})$, $\boldsymbol{x} \mapsto \overline{d}f(\boldsymbol{x})$ are Hausdorff continuous at $\boldsymbol{x}$.

The class of functions admitting a Hausdorff continuous codifferentials includes nonsmooth convex and concave functions as well as functions represented as a max–min of a finite number of smooth functions.

For some nonsmooth functions computation of codifferentials is straightforward.

Example 3.10 (*Codifferentials*) Consider the following maximum function:

$$f(\boldsymbol{x}) = \max_{i=1,\dots,m} f_i(\boldsymbol{x}),$$

where functions f_i, $i = 1, \dots, m$ are continuously differentiable. This function is hypodifferentiable and its hypodifferential at $\boldsymbol{x}$ is as follows:

$$\underline{d}f(\boldsymbol{x}) = \text{conv}\left\{(\eta, \boldsymbol{v}) \in \mathbb{R}^{n+1} \mid \eta = f_i(\boldsymbol{x}) - f(\boldsymbol{x}),\ \boldsymbol{v} \in \nabla f_i(\boldsymbol{x}),\ i = 1, \ldots, m\right\}.$$

3.4.4 Basic and Singular Subdifferentials

Another important generalizations of the subdifferential are the so-called *basic* (*limiting*) and *singular subdifferentials* introduced in [182]. The basic and singular subdifferentials of an extended real-valued function are defined through basic normals to its epigraph.

Definition 3.12 Consider a function $f : X \to \overline{\mathbb{R}}$, where $X \subset \mathbb{R}^n$ is an open set and $\overline{\mathbb{R}} = [-\infty, \infty]$ extended real line. Let a point $\boldsymbol{x} \in X$ be such that $|f(\boldsymbol{x})| < \infty$.

(i) The set

$$\partial_b f(\boldsymbol{x}) = \left\{\boldsymbol{\xi} \in \mathbb{R}^n \mid (\boldsymbol{\xi}, -1) \in N((\boldsymbol{x}, f(\boldsymbol{x})), \text{epi } f)\right\}$$

is the *basic* (*limiting*) *subdifferential* of f at $\boldsymbol{x}$, and its elements are *basic subgradients* of f at this point. We put $\partial_b f(\boldsymbol{x}) = \emptyset$ if $|f(\boldsymbol{x})| = \infty$.

(ii) The set

$$\partial^\infty f(\boldsymbol{x}) = \left\{\boldsymbol{\xi} \in \mathbb{R}^n \mid (\boldsymbol{\xi}, 0) \in N((\boldsymbol{x}, f(\boldsymbol{x})), \text{epi } f)\right\}$$

is the *singular subdifferential* of f at $\boldsymbol{x}$, and its elements are *singular subgradients* of f at this point. We put $\partial^\infty f(\boldsymbol{x}) = \emptyset$ if $|f(\boldsymbol{x})| = \infty$.

The basic (limiting) subdifferential agrees with the classical gradient for strictly differentiable functions as well as with the subdifferential of convex analysis (see Definition 2.15) when f is convex. The singular subdifferential is useful for the study of non-Lipschitzian functions. Both basic and singular subdifferentials can be used to study general classes of nonsmooth functions.

3.5 Summary

In this chapter we have generalized all the convex concepts defined in the previous chapter to nonconvex locally Lipschitz continuous functions. We have shown that nonsmooth analysis is a natural enlargement of the classical differential theory by generalizing all the familiar derivation rules like the mean-value theorem (Theorem 3.18) and the chain rule (Theorem 3.19). Instead of the equalities we only get inclusions in most of the results. However, the subdifferential regularity assumption can guarantee equalities. From practical point of view the main result of this chapter is the Theorem 3.9, which tells how one can compute the Clarke subdifferential by using limits of ordinary gradients. After the analytical part we have generalized all the geometrical concepts and given links between analysis and geometry.

In addition, we have presented three generalizations of the subdifferential: the quasidifferential, the codifferential and the basic (limiting) and singular subdifferentials. Unlike the Clarke subdifferential all these three generalized subdifferentials can be considered as nonconvex subdifferentials, because they contain more than one convex compact set.

The quasidifferential mapping enjoys a full calculus (see Theorem 3.37) in a sense that equalities can be used instead of inclusions. We also have presented the result on the relationship between the Clarke subdifferential and the quasidifferential.

Exercises

3.1 (Corollary 3.1) *Prove that if* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is locally Lipschitz continuous at* $\boldsymbol{x}$*, then the function* $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ *is convex, its epigraph* epi $f^\circ(\boldsymbol{x};\,\cdot)$ *is a convex cone and we have*

$$f^\circ(\boldsymbol{x}; -\boldsymbol{d}) = (-f)^\circ(\boldsymbol{x}; \boldsymbol{d}).$$

3.2 Let f_1 *and* f_2 *be locally Lipschitz continuous at* $\boldsymbol{x} \in \mathbb{R}^n$. *Show that*

$$\partial(f_1 + f_2)(\boldsymbol{x}) \neq \partial f_1(\boldsymbol{x}) + \partial f_2(\boldsymbol{x}).$$

(Hint: Consider the functions $f_1(x) := |x|$ and $f_2(x) := -|x|$).

3.3 (Theorem 3.22) Let f_1 and f_2 be locally Lipschitz continuous at $\boldsymbol{x} \in \mathbb{R}^n$ and $f_2(\boldsymbol{x}) \neq 0$. Prove that the function f_1/f_2 is locally Lipschitz continuous at $\boldsymbol{x}$ and

$$\partial\Big(\frac{f_1}{f_2}\Big)(\boldsymbol{x}) \subseteq \frac{f_2(\boldsymbol{x})\partial f_1(\boldsymbol{x}) - f_1(\boldsymbol{x})\partial f_2(\boldsymbol{x})}{f_2^2(\boldsymbol{x})}. \tag{3.40}$$

If in addition $f_1(\boldsymbol{x}) \geq 0$, $f_2(\boldsymbol{x}) > 0$ and f_1, f_2 are both subdifferentially regular at $\boldsymbol{x}$, prove that the function f_1/f_2 is also subdifferentially regular at $\boldsymbol{x}$ and equality holds in (3.40).

3.4 (Corollary 3.5) Suppose that the functions $f_i\colon \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable at $\boldsymbol{x}$ and $g_i\colon \mathbb{R}^n \to \mathbb{R}$ are convex for each $i = 1, \ldots, m$. Define the functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(\boldsymbol{x}) = \max\,\{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\} \quad \text{and}$$
$$g(\boldsymbol{x}) = \max\,\{g_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}.$$

Then we have

$$\partial f(\boldsymbol{x}) = \text{conv}\,\{\nabla f_i(\boldsymbol{x}) \mid i \in I(\boldsymbol{x})\} \quad \text{and}$$
$$\partial_c g(\boldsymbol{x}) = \text{conv}\,\{\partial_c g_i(\boldsymbol{x}) \mid i \in \mathcal{J}(\boldsymbol{x})\},$$

where $I(\boldsymbol{x}) = \{i \in \{1, \dots, m\} \mid f_i(\boldsymbol{x}) = f(\boldsymbol{x})\}$ and $\mathcal{J}(\boldsymbol{x}) = \{i \in \{1, \dots, m\} \mid g_i(\boldsymbol{x}) = g(\boldsymbol{x})\}$.

3.5 (Theorem 3.23) Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous at $\boldsymbol{x}$ for all $i = 1, \dots, m$. Prove that the function

$$f(\boldsymbol{x}) := \max\,\{f_i(\boldsymbol{x}) \mid i = 1, \dots, m\}$$

is locally Lipschitz continuous at $\boldsymbol{x}$.

3.6 Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(\boldsymbol{x}) := \max_{i=1,\dots,n}\,\{x_i\}.$$

Show that f is convex and we have

$$\partial_c f(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \xi_i \geq 0, \sum_{i=1}^{n} \xi_i = 1 \text{ and } \boldsymbol{\xi}_{\mathrm{i}} = 0, \text{if i} \notin \mathrm{I}(\boldsymbol{x})\},$$

where

$$I(\boldsymbol{x}) := \{i \in \{1, \dots, n\} \mid x_i = f(\boldsymbol{x})\}.$$

3.7 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \frac{\max\,\{\sqrt{|x|}, x^3\}}{\ln(|x| + 2)}.$$

Calculate $\partial f(0)$.

3.8 (Theorem 3.24) Prove that the tangent cone $T_S(\boldsymbol{x})$ of the nonempty set S at $\boldsymbol{x} \in S$ is a closed convex cone.

3.9 (Theorem 3.30) If S_1,$S_2 \subseteq \mathbb{R}^n$ are such that $\boldsymbol{x} \in S_1 \cap S_2$ and $\boldsymbol{x} \in \text{int } S_2$, prove that

$$T_{S_1}(\boldsymbol{x}) = T_{S_1 \cap S_2}(\boldsymbol{x}) \quad \text{and} \quad N_{S_1}(\boldsymbol{x}) = N_{S_1 \cap S_2}(\boldsymbol{x}).$$

3.10 Let S_1, $S_2 \subseteq \mathbb{R}^n$ and $\boldsymbol{x} \in S_1 \cap S_2$. Show that
(a) $T_{S_1 \cap S_2}(\boldsymbol{x}) \subseteq T_{S_1}(\boldsymbol{x}) \cap T_{S_2}(\boldsymbol{x})$,
(b) $N_{S_1 \cap S_2}(\boldsymbol{x}) \supseteq N_{S_1}(\boldsymbol{x}) + N_{S_2}(\boldsymbol{x})$.

3.11 If the function $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}$, prove that

$$N_{\text{epi } f}(\boldsymbol{x}, f(\boldsymbol{x})) = \{(\lambda\boldsymbol{\xi}, -\lambda) \in \mathbb{R}^{n+1} \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{x}), \lambda \geq 0\}.$$

3.12 (Theorem 3.34) If the function $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}$ and $\boldsymbol{0} \notin \partial f(\boldsymbol{x})$, prove that

$$N_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) \subseteq \text{ray}\, \partial f(\boldsymbol{x}).$$

If, in addition, f is subdifferentially regular at $\boldsymbol{x}$ prove that

$$N_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x}) = \text{ray}\, \partial f(\boldsymbol{x}).$$

3.13 (Theorem 3.35) Prove that the cone of locally feasible directions $F_S(\boldsymbol{x})$ of the nonempty convex set S at $\boldsymbol{x} \in S$ is a convex cone.

3.14 Let $S \subset \mathbb{R}^2$ defined by

$$S := \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 < 0\} \cup \{\boldsymbol{0}\}.$$

Calculate the cones $F_S(\boldsymbol{0})$, $T_S(\boldsymbol{0})$, $K_S(\boldsymbol{0})$, $G_S(\boldsymbol{0})$ and $N_S(\boldsymbol{0})$.

3.15 Prove Theorem 3.38.

3.16 Prove Theorem 3.39.

3.17 Find DC representation of the following functions:

(a) $f(\boldsymbol{x}) = \max\{\min\{2x_1 + x_2, -3x_1 + 2x_2\}, \min\{x_1 - 4x_2, 3x_1 - x_2\}\}$,
$\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$;
(b) $f(\boldsymbol{x}) = \min\{x_1 - x_2, 3x_1 + x_2\} + \min\{-x_1 + 2x_2, -3x_1 - x_2\}\}$,
$\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$;
(c) $f(\boldsymbol{x}) = |\min\{x_1 - 4x_2 + 2x_3, x_1 - 2x_2 - x_3\}|$,
$\boldsymbol{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$;
(d) $f(\boldsymbol{x}) = \min\{x_1^2 - 2x_2 - 2x_1 - x_2^2, 3x_1^2 + 5x_2 - x_1 - x_2^2\}|$,
$\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$.

3.18 Find quasidifferentials of the following functions:

(a) $f(\boldsymbol{x}) = \max\{\min\{x_1 - x_2, 4x_1 - 3x_2\}, \min\{2x_1 - x_2, x_1 - x_2\}\}$,
$\boldsymbol{x} = (x_1, x_2)^T = (0, 0)^T \in \mathbb{R}^2$;
(b) $f(\boldsymbol{x}) = \min\{x_1 - x_2 - 1, x_1 + x_2 - 3\} + \min\{-x_1 + 3x_2 + 1, -2x_1 - x_2 + 8\}\}$,
$\boldsymbol{x} = (x_1, x_2)^T = (3, 1)^T \in \mathbb{R}^2$;
(c) $f(\boldsymbol{x}) = |\min\{x_1 - 4x_2 + 2x_3 - 1, -2x_1 - x_2 - x_3 + 2\}|$,
$\boldsymbol{x} = (x_1, x_2, x_3)^T = (1, 0, 0)^T \in \mathbb{R}^3$;
(d) $f(\boldsymbol{x}) = \min\{x_1^2 - x_2 - x_1 - x_2^2 + 8, x_1^2 + 2x_2 - x_1^2 - x_2^2 + 2\}|$,
$\boldsymbol{x} = (x_1, x_2)^T = (0, 2)^T \in \mathbb{R}^2$.

3.19 Find codifferentials of the following functions:

(a) $f(\boldsymbol{x}) = \max\{\min\{-3x_1 + 2x_2, 2x_1 - 3x_2\}, \min\{4x_1 - 3x_2 - 2, x_1 - x_2 - 1\}\}$, $\boldsymbol{x} = (x_1, x_2)^T = (1, 1)^T \in \mathbb{R}^2$;

(b) $f(\boldsymbol{x}) = \min\{x_1 - 2x_2 - 2, 2x_1 + x_2\} + \min\{-2x_1 + x_2 + 1, -4x_1 - 2x_2 + 6\}\}$, $\boldsymbol{x} = (x_1, x_2)^T = (1, -1)^T \in \mathbb{R}^2$;

(c) $f(\boldsymbol{x}) = |\min\{x_1 - x_2 + 3x_3 - 3, -x_1 - 2x_2 - 2x_3 + 3\}|$, $\boldsymbol{x} = (x_1, x_2, x_3)^T = (1, 0, 1)^T \in \mathbb{R}^3$;

(d) $f(\boldsymbol{x}) = \min\{2x_1^2 - x_2 - 3x_1 - 2x_2^2 + 3, 3x_1^2 + x_2 - 2x_1^2 - 3x_2^2\}|$, $\boldsymbol{x} = (x_1, x_2)^T = (2, 1)^T \in \mathbb{R}^2$.

Chapter 4
Optimality Conditions

In this chapter, we present some results connecting the theories of nonsmooth analysis and optimization. We first define global and local minima of functions. After that, we generalize the classical first order optimality conditions for unconstrained nonsmooth optimization. Furthermore, we define linearizations for locally Lipschitz continuous functions by using subgradient information, and present their basic properties. These linearizations are suitable for function approximation and they will be used in nonsmooth optimization methods in Part III. At the end of this chapter, we define the notion of a descent direction and show how to find it for a locally Lipschitz continuous function.

We consider a nonsmooth optimization problem of the form

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in S, \end{cases} \tag{4.1}$$

where the *objective function* $f: \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous and the *feasible region* $S \subseteq \mathbb{R}^n$ is nonempty. If f is a convex function and S is a convex set, then the problem (4.1) is called *convex*.

Definition 4.1 A point $\boldsymbol{x}^* \in S$ is a *global optimum* of the problem (4.1) if it satisfies

$$f(\boldsymbol{x}^*) \le f(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in S.$$

Definition 4.2 A point $\boldsymbol{x}^* \in \mathbb{R}^n$ is a *local optimum* of the problem (4.1) if there exists $\delta > 0$ such that

$$f(\boldsymbol{x}^*) \le f(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*; \delta).$$

4.1 Unconstrained Optimization

We consider first the unconstrained version of the problem (4.1), in other words the case $S = \mathbb{R}^n$. Then we are actually looking for local and global minima of a locally Lipschitz continuous function.

A. Bagirov et al., *Introduction to Nonsmooth Optimization*,
DOI: 10.1007/978-3-319-08114-4_4

4.1.1 Analytical Optimality Conditions

The necessary conditions for a locally Lipschitz continuous function to attain its local minimum are given in the next theorem. For convex functions these conditions are also sufficient and the minimum is global.

Theorem 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x}^* \in \mathbb{R}^n$. If f attains its local minimum at $\boldsymbol{x}^*$, then*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) \quad \textit{and} \quad f^\circ(\boldsymbol{x}^*; \boldsymbol{d}) \geq 0 \quad \textit{for all } \boldsymbol{d} \in \mathbb{R}^n. \tag{4.2}$$

Proof Follows directly from the proof of Theorem 3.17. □

In what follows we are seeking for the points satisfying the necessary optimality condition (4.2).

Definition 4.3 A point $\boldsymbol{x} \in \mathbb{R}^n$ satisfying $\mathbf{0} \in \partial f(\boldsymbol{x})$ is called a *stationary point* of f.

Next we formulate the following sufficient optimality condition utilizing convexity.

Theorem 4.2 *If the function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then f attains its global minimum at $\boldsymbol{x}^*$ if and only if*

$$\mathbf{0} \in \partial_c f(\boldsymbol{x}^*) \quad \textit{or} \quad f'(\boldsymbol{x}^*; \boldsymbol{d}) \geq 0 \quad \textit{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

Proof If f attains its global minimum at $\boldsymbol{x}^*$, then it follows from Theorem 4.1 that $\mathbf{0} \in \partial_c f(\boldsymbol{x}^*)$ and by Theorem 2.28 (ii) we have for all $\boldsymbol{d} \in \mathbb{R}^n$

$$f'(\boldsymbol{x}^*; \boldsymbol{d}) = \max\{\boldsymbol{\xi}^T \boldsymbol{d} \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)\} \geq \mathbf{0}^T \boldsymbol{d} = 0. \tag{4.3}$$

Suppose next, that $\boldsymbol{x} \in \mathbb{R}^n$ is arbitrary and define $\boldsymbol{d} := \boldsymbol{x} - \boldsymbol{x}^*$, then due to (4.3) we have

$$0 \leq f'(\boldsymbol{x}^*; \boldsymbol{x} - \boldsymbol{x}^*) = \max\{\boldsymbol{\xi}^T (\boldsymbol{x} - \boldsymbol{x}^*) \mid \boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)\}$$

and thus there exists $\boldsymbol{\xi}_* \in \partial_c f(\boldsymbol{x}^*)$ such that

$$0 \leq f'(\boldsymbol{x}^*; \boldsymbol{x} - \boldsymbol{x}^*) = \boldsymbol{\xi}_*^T (\boldsymbol{x} - \boldsymbol{x}^*).$$

Then due to the definition of the subdifferential of a convex function (Definition 2.15) we get

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \boldsymbol{\xi}_*^T(\boldsymbol{x} - \boldsymbol{x}^*) \geq f(\boldsymbol{x}^*),$$

in other words, f attains its global minimum at $\boldsymbol{x}^*$. Now we have proved that all these three conditions are equivalent. □

Example 4.1 (*Absolute-value function*). Function $f(x) = |x|$ is convex.

$$\begin{aligned} &\text{The point } x = 0 \text{ is a global minimum of } f. \\ \iff\ & 0 \in [-1, 1] = \partial f(0). \end{aligned}$$

A necessary optimality condition can be presented also with the aid of the Goldstein ε-subdifferential.

Theorem 4.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x}^* \in \mathbb{R}^n$. If f attains its local minimum at $\boldsymbol{x}^*$, then for all $\varepsilon \geq 0$ and $\boldsymbol{x} \in B(\boldsymbol{x}^*; \varepsilon)$ we have*

$$\mathbf{0} \in \partial_\varepsilon^G f(\boldsymbol{x}).$$

Proof Let $\varepsilon \geq 0$ and $\boldsymbol{x} \in B(\boldsymbol{x}^*; \varepsilon)$, then clearly $\boldsymbol{x}^* \in B(\boldsymbol{x}; \varepsilon)$ and according to Theorems 4.1 and 3.11 we have

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) \subseteq \partial_\varepsilon^G f(\boldsymbol{x}).$$ □

The Goldstein ε-subdifferential measures the distance in the variable space, while the ε-subdifferential of a convex function in the objective space. Due to these differences we need the following ε-optimality.

Definition 4.4 If $\varepsilon \geq 0$, then a point $\boldsymbol{x}^* \in S$ is a *global ε-optimum* of the problem (4.1) if it satisfies

$$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) + \varepsilon \quad \text{for all } \boldsymbol{x} \in S.$$

Note, that similarly we can define also local ε-optimality.

Theorem 4.4 *If the function $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\varepsilon \geq 0$, then f attains its global ε-minimum at $\boldsymbol{x}^*$ if and only if*

$$\mathbf{0} \in \partial_\varepsilon f(\boldsymbol{x}^*) \quad \textit{or} \quad f'_\varepsilon(\boldsymbol{x}^*; \mathbf{d}) \geq 0 \quad \textit{for all } \mathbf{d} \in \mathbb{R}^n.$$

Proof Exercise. [Hint: In the proof of Theorem 4.2 replace $\partial_c f(\boldsymbol{x}^*)$ and $f'(\boldsymbol{x}^*;\ \cdot)$ by $\partial_\varepsilon f(\boldsymbol{x}^*)$ and $f'_\varepsilon(\boldsymbol{x}^*;\ \cdot)$, respectively.] □

4.1.2 Descent Directions

If we have not yet found the optimal solution of the problem (4.1), an essential part of iterative optimization methods is finding a direction such that the objective function values improve when moving in that direction. Next we define *a descent direction* for an objective function and show how to find it for a locally Lipschitz continuous function.

Definition 4.5 The direction $\boldsymbol{d} \in \mathbb{R}^n$ is called a *descent direction* for $f\colon \mathbb{R}^n \to \mathbb{R}$ at $\boldsymbol{x} \in \mathbb{R}^n$, if there exists $\varepsilon > 0$ such that for all $t \in (0, \varepsilon]$

$$f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x}).$$

Theorem 4.5 *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. The direction $\boldsymbol{d} \in \mathbb{R}^n$ is a descent direction for f at $\boldsymbol{x}$ if*

$$\boldsymbol{\xi}^T\boldsymbol{d} < 0 \quad \textit{for all } \boldsymbol{\xi} \in \partial f(\boldsymbol{x}) \quad \textit{or} \quad f^\circ(\boldsymbol{x}; \boldsymbol{d}) < 0.$$

Proof Suppose first, that $f^\circ(\boldsymbol{x}; \boldsymbol{d}) < 0$. By the definition of the generalized directional derivative we have

$$\limsup_{t\downarrow 0} \frac{f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x})}{t} \le \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t} = f^\circ(\boldsymbol{x}; \boldsymbol{d}) < 0.$$

Then, by the definition of upper limit, there exists $\varepsilon > 0$ such that $f(\boldsymbol{x} + t\boldsymbol{d}) - f(\boldsymbol{x}) < 0$ for all $t \in (0, \varepsilon]$, which means that $\boldsymbol{d}$ is a descent direction for f at $\boldsymbol{x}$.

Suppose next, that $\boldsymbol{\xi}^T\boldsymbol{d} < 0$ for all $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$. Then due to Theorem 3.4 we have

$$f^\circ(\boldsymbol{x}; \boldsymbol{d}) = \max\,\{\boldsymbol{\xi}^T\boldsymbol{d} \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{x})\} < 0$$

and thus, by the first part of the proof, $\boldsymbol{d}$ is a descent direction for f at $\boldsymbol{x}$. □

From the above results we get the consequence, that either we have a stationary point or we can find a descent direction.

Corollary 4.1 *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Then either $\mathbf{0} \in \partial f(\boldsymbol{x})$ or there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for f at $\boldsymbol{x}$.*

Proof Exercise. (Hint: Use the Theorems 4.1 and 4.5.) □

Note, that in convex case Corollary 4.1 means, that either we have a global minimum point or we can find a descent direction.

Corollary 4.2 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then either f attains its global minimum at $\boldsymbol{x} \in \mathbb{R}^n$ or there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for f at $\boldsymbol{x}$.*

Proof Exercise. (Hint: Use Corollary 4.1 and Theorem 4.2.) □

To the end of this subsection we define two cones related to descent directions.

Definition 4.6 Consider the optimization problem (4.1). The *cone of descent directions* at $\boldsymbol{x} \in S$ is

$$D_S(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid \boldsymbol{d} = 0 \text{ or there exists } \varepsilon > 0 \\ \text{such that } f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x}) \text{ for all } t \in (0, \varepsilon]\}$$

and the *cone of polar subgradient directions* at $\boldsymbol{x} \in S$ is

$$D_S^\circ(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid \boldsymbol{d} = 0 \text{ or } \boldsymbol{\xi}^T\boldsymbol{d} < 0 \text{ for all } \boldsymbol{\xi} \in \partial f(\boldsymbol{x})\}.$$

It is easy to show (Exercise 4.5), that $D_S(\boldsymbol{x})$ and $D_S^\circ(\boldsymbol{x})$ are cones and they are both convex, if f is a convex function. We left also as an exercise (Exercise 4.6) to show that

$$D_S(\boldsymbol{x}) \subseteq F_{\text{lev}_{f(\boldsymbol{x})}\, f}(\boldsymbol{x}) \quad \text{and} \quad D_S^\circ(\boldsymbol{x}) \subseteq \partial f(\boldsymbol{x})^\circ.$$

Moreover, by Theorem 4.5 we have $D_S^\circ(\boldsymbol{x}) \subseteq D_S(\boldsymbol{x})$.

Finally, we formulate the following geometrical optimality condition.

Corollary 4.3 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x}^* \in \mathbb{R}^n$. If f attains its local minimum at $\boldsymbol{x}^*$, then*

$$D_S^\circ(\boldsymbol{x}^*) = D_S(\boldsymbol{x}^*) = \{\mathbf{0}\}. \tag{4.4}$$

If, in addition, f is convex, then the condition (4.4) is sufficient for $\boldsymbol{x}^*$ to be a global minimum of f.

Proof Exercise. (Hint: Use the Theorems 4.1 and 4.2, Corollary 4.1 and the fact that $\mathbf{0} \in D_S^\circ(\boldsymbol{x}) \subseteq D_S(\boldsymbol{x})$.) □

4.2 Geometrical Constraints

Next we consider the problem (4.1) when the feasible set is not the whole space $\mathbb{R}^n$, in other words $S \subset \mathbb{R}^n$. In this subsection we do not assume any special structure of S, but consider it as a general set.

4.2.1 Geometrical Optimality Conditions

In unconstrained case we were looking for descent directions, in other words directions from the cone $D_S(\boldsymbol{x})$, but in constrained case those directions should also be feasible. In Sect. 3.3.3 we defined the cone of locally feasible directions $F_S(\boldsymbol{x})$. Thus we are interested in finding directions from the intersection $D_S(\boldsymbol{x}) \cap F_S(\boldsymbol{x})$.

First we generalize the geometrical optimality condition (4.3) to the constrained problem (4.1) (see also Fig. 4.1).

Theorem 4.6 *Let $\boldsymbol{x}^*$ be a local optimum of the problem* (4.1)*, where $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}^* \in S \neq \emptyset$. Then*

$$D_S^\circ(\boldsymbol{x}^*) \cap F_S(x^*) = \{\mathbf{0}\}. \tag{4.5}$$

If, in addition, the problem (4.1) *is convex, then the condition* (4.5) *implies that $\boldsymbol{x}^*$ is a global optimum of the problem* (4.1).

Proof By contradiction, suppose that there exists $\boldsymbol{d} \in D_S^\circ(\boldsymbol{x}^*) \cap F_S(x^*)$ such that $\boldsymbol{d} \neq \mathbf{0}$. By Theorem 4.5 we have $D_S^\circ(\boldsymbol{x}^*) \subseteq D_S(\boldsymbol{x}^*)$, which means by the definition of the cone of descent directions that there exists $\varepsilon_1 > 0$ such that

$$f(\boldsymbol{x}^* + t\boldsymbol{d}) < f(\boldsymbol{x}^*) \quad \text{for all } t \in (0, \varepsilon_1]. \tag{4.6}$$

On the other hand, due to the definition of the cone of locally feasible directions $F_S(\boldsymbol{x}^*)$ there exists $\varepsilon_2 > 0$ such that

$$\boldsymbol{x}^* + t\boldsymbol{d} \in S \quad \text{for all } t \in (0, \varepsilon_2]. \tag{4.7}$$

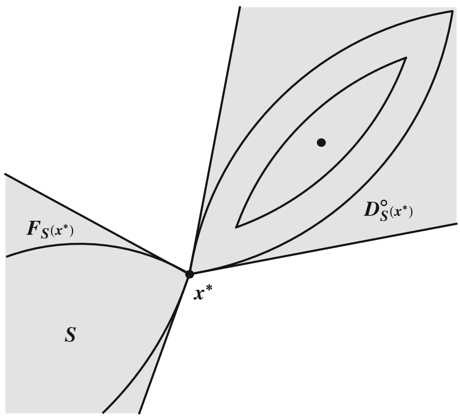


Fig. 4.1 Geometrical optimality condition

By choosing $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ we see that both (4.6) and (4.7) are valid for all $t \in (0, \varepsilon]$ and thus, due to Definition 4.2 $\boldsymbol{x}^*$ can not be a local optimum of the problem (4.1). Thus $D_S^\circ(\boldsymbol{x}^*) \cap F_S(x^*) = \{\boldsymbol{0}\}$.

Suppose next, that the problem (4.1) is convex and the condition (4.5) is valid. If $\boldsymbol{x}^*$ is not a global optimum there exist $\boldsymbol{y} \in S$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x}^*)$. Then $\boldsymbol{d} := \boldsymbol{y} - \boldsymbol{x}^* \neq \boldsymbol{0}$ and since S is convex we have

$$\boldsymbol{x}^* + t\boldsymbol{d} = \boldsymbol{x}^* + t(\boldsymbol{y} - \boldsymbol{x}^*) = t\boldsymbol{y} + (1-t)\boldsymbol{x}^* \in S,$$

whenever $t \in (0, 1]$, thus $\boldsymbol{d} \in F_S(\boldsymbol{x}^*)$. On the other hand, due to the definition of subdifferential of convex function (Definition 2.15) for all $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)$ we have

$$0 > f(\boldsymbol{y}) - f(\boldsymbol{x}^*) = f(\boldsymbol{x}^* + \boldsymbol{d}) - f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}^*) + \boldsymbol{\xi}^T\boldsymbol{d} - f(\boldsymbol{x}^*) = \boldsymbol{\xi}^T\boldsymbol{d}.$$

By Theorem 3.8 (ii) we have $\partial_c f(\boldsymbol{x}^*) = \partial f(\boldsymbol{x}^*)$ and we conclude that $\boldsymbol{d} \in D_S^\circ(\boldsymbol{x}^*)$. Thus we have found $\boldsymbol{d} \neq \boldsymbol{0}$ such that $\boldsymbol{d} \in D_S^\circ(\boldsymbol{x}^*) \cap F_S(\boldsymbol{x}^*)$. This contradicts the condition (4.5), meaning that $\boldsymbol{x}^*$ must be a global optimum of the problem (4.1). □

4.2.2 Mixed Optimality Conditions

Next we formulate a mixed-analytical-geometrical optimality conditions. Before that, we need the following two lemmas.

Lemma 4.1 *Let $S_1 \subset S_2 \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous on S_2 with constant K. If $\boldsymbol{x}^* \in S_1$ is a local optimum of the problem*

$$\begin{cases} \textit{minimize} & f(\boldsymbol{x}) \\ \textit{subject to} & \boldsymbol{x} \in S_1, \end{cases} \tag{4.8}$$

then, it is also a local optimum of the problem

$$\begin{cases} \textit{minimize} & f(\boldsymbol{x}) + K d_{S_1}(\boldsymbol{x}) \\ \textit{subject to} & \boldsymbol{x} \in S_2. \end{cases} \tag{4.9}$$

Proof By contradiction, suppose that there exists $\boldsymbol{y} \in S_2$ and $\varepsilon > 0$ such that

$$f(\boldsymbol{y}) + K d_{S_1}(\boldsymbol{y}) < f(\boldsymbol{x}^*) + K d_{S_1}(\boldsymbol{x}^*) - K\varepsilon = f(\boldsymbol{x}^*) - K\varepsilon.$$

Let $\boldsymbol{z} \in S_1$ be such that

$$\|\boldsymbol{y} - \boldsymbol{z}\| \leq d_{S_1}(\boldsymbol{y}) + \varepsilon.$$

Then due to the Lipschitz continuity we get

$$\begin{aligned} f(\boldsymbol{z}) &\le f(\boldsymbol{y}) + K\|\boldsymbol{y} - \boldsymbol{z}\| \\ &\le f(\boldsymbol{y}) + K(d_{S_1}(\boldsymbol{y}) + \varepsilon) \\ &< f(\boldsymbol{x}^*) + K\varepsilon - K\varepsilon \\ &= f(\boldsymbol{x}^*), \end{aligned}$$

implying that $\boldsymbol{x}^*$ can not be a local optimum of the problem (4.8). □

For an open feasible region S we get the same optimality conditions than in unconstrained case.

Lemma 4.2 *Let $\boldsymbol{x}^*$ be a local optimum of the problem* (4.1)*, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}^* \in S \neq \emptyset$ and $S \subseteq \mathbb{R}^n$ is open. Then*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) \quad \textit{and} \quad f^\circ(\boldsymbol{x}^*; \boldsymbol{d}) \ge 0 \quad \textit{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

Proof Since $\boldsymbol{x}^*$ is a local optimum of the problem (4.1) there exists $\delta_1 > 0$ such that

$$f(\boldsymbol{x}^*) \le f(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*; \delta_1). \tag{4.10}$$

On the other hand, since S is open there exists $\delta_2 > 0$ such that $B(\boldsymbol{x}^*; \delta_2) \subset S$. By choosing $\delta := \min\{\delta_1, \delta_2\}$ the condition (4.10) is valid for all

$$\boldsymbol{x} \in S \cap B(\boldsymbol{x}^*; \delta) = B(\boldsymbol{x}^*; \delta).$$

Then we have $f(\boldsymbol{x}^* + t\boldsymbol{d}) - f(\boldsymbol{x}^*) \ge 0$ for all $t \in (0, \delta]$ and $\boldsymbol{d} \in \mathbb{R}^n$, and thus

$$f^\circ(\boldsymbol{x}^*; \boldsymbol{d}) = \limsup_{\substack{\boldsymbol{y} \to \boldsymbol{x}^* \\ t \downarrow 0}} \frac{f(\boldsymbol{y} + t\boldsymbol{d}) - f(\boldsymbol{y})}{t} \ge \limsup_{t \downarrow 0} \frac{f(\boldsymbol{x}^* + t\boldsymbol{d}) - f(\boldsymbol{x}^*)}{t} \ge 0.$$

Moreover

$$f^\circ(\boldsymbol{x}^*; \boldsymbol{d}) \ge 0 = \mathbf{0}^T \boldsymbol{d} \quad \text{for all} \quad \boldsymbol{d} \in \mathbb{R}^n,$$

which means by the definition of subdifferential that $\mathbf{0} \in \partial f(\boldsymbol{x}^*)$. □

Finally we are ready to proof the main results of this subsection, namely the necessary mixed-analytical-geometrical optimality condition.

Theorem 4.7 *Let $\boldsymbol{x}^*$ be a local optimum of the problem* (4.1)*, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}^* \in S \neq \emptyset$. Then*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$

Proof Since f is locally Lipschitz continuous at $\boldsymbol{x}^*$, we can find an open set $S^* \subset \mathbb{R}^n$ such that $\boldsymbol{x}^* \in S^*$ and f is Lipschitz continuous on the set S^* with constant K. Now we have $S \cap S^* \subseteq S$ and thus $\boldsymbol{x}^* \in S \cap S^*$ is a local optimum of the problem

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in S \cap S^*. \end{cases}$$

Moreover, $S \cap S^* \subseteq S^*$ and due to Lemma 4.1 $\boldsymbol{x}^*$ is also a local optimum of the problem

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) + K d_{S \cap S^*}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in S^*. \end{cases}$$

Because the feasible region S^* is open, Lemma 4.2 implies that

$$\mathbf{0} \in \partial(f(\boldsymbol{x}^*) + K d_{S \cap S^*}(\boldsymbol{x}^*)).$$

Due to Theorem 2.38 the distance function $d_{S \cap S^*}$ is locally Lipschitz continuous everywhere and thus, due to Theorem 3.16

$$\mathbf{0} \in \partial(f(\boldsymbol{x}^*) + K d_{S \cap S^*}(\boldsymbol{x}^*)) \subseteq \partial f(\boldsymbol{x}^*) + \partial K d_{S \cap S^*}(\boldsymbol{x}^*).$$

Moreover, due to Theorem 3.28 we have

$$N_{S \cap S^*}(\boldsymbol{x}^*) = \text{cl ray } \partial d_{S \cap S^*}(\boldsymbol{x}^*),$$

implying that

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + \partial K d_{S \cap S^*}(\boldsymbol{x}^*) \subseteq \partial f(\boldsymbol{x}^*) + N_{S \cap S^*}(\boldsymbol{x}^*).$$

Since S^* is open, $\boldsymbol{x}^* \in \text{int } S^*$ and due to Theorem 3.30 $N_{S \cap S^*}(\boldsymbol{x}^*) = N_S(\boldsymbol{x}^*)$ and thus

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$ □

To the end of this subsection we formulate the following necessary and sufficient mixed-analytical-geometrical optimality condition utilizing convexity.

Theorem 4.8 *If the problem* (4.1) *is convex, then* $\boldsymbol{x}^* \in S$ *is a global optimum of the problem* (4.1) *if and only if*

$$\mathbf{0} \in \partial_c f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$

Proof The necessity part follows directly from Theorem 4.7. For sufficiency suppose, that

$$\mathbf{0} \in \partial_c f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$

This means, that there exist $\boldsymbol{\xi} \in \partial_c f(\boldsymbol{x}^*)$ and $\boldsymbol{z} \in N_S(\boldsymbol{x}^*)$ such that $\boldsymbol{\xi} = -\boldsymbol{z}$. Then due to the definition of the subdifferential of a convex function (Definition 2.15) we get

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{x}^*) = f(\boldsymbol{x}^*) - \boldsymbol{z}^T(\boldsymbol{x} - \boldsymbol{x}^*) \quad \text{for all } \boldsymbol{x} \in S.$$

Since S is convex and $\boldsymbol{z} \in N_S(\boldsymbol{x}^*)$ Theorem 2.19 implies that

$$\boldsymbol{z}^T(\boldsymbol{x} - \boldsymbol{x}^*) \leq 0 \quad \text{for all } \boldsymbol{x} \in S,$$

and thus

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) - \boldsymbol{z}^T(\boldsymbol{x} - \boldsymbol{x}^*) \geq f(\boldsymbol{x}^*) \quad \text{for all } \boldsymbol{x} \in S.$$

In other words, $\boldsymbol{x}^*$ is a global optimum of the problem (4.1). □

4.3 Analytical Constraints

In this subsection we assume a special structure of S determined with inequality constraints. Thus we consider a nonsmooth optimization problem of the form

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leq 0 \quad \text{for } i = 1, \ldots, m \\ & \boldsymbol{x} \in X, \end{cases} \tag{4.11}$$

where the *constraint functions* $g_i\colon \mathbb{R}^n \to \mathbb{R}$ are supposed to be locally Lipschitz continuous for all $i = 1, \ldots, m$ and $X \subseteq \mathbb{R}^n$ is a nonempty open set. Without loosing generality, we can scalarize the multiple constraints by introducing the *total constraint function* $g\colon \mathbb{R}^n \to \mathbb{R}$ in the form

$$g(\boldsymbol{x}) := \max \{g_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}.$$

Then the problem (4.11) is a special case of (4.1) with

$$S = \{\boldsymbol{x} \in X \mid g(\boldsymbol{x}) \leq 0\} = \operatorname{lev}_0 g.$$

4.3.1 Geometrical Optimality Conditions

First we generalize the geometrical optimality condition of Theorem 4.6 to the analytically constrained problem (4.11). In order to do that, we need one more cone.

Definition 4.7 Consider the optimization problem (4.11). The *cone of polar constraint subgradient directions* at $\boldsymbol{x} \in S$ is

$$F_S^\circ(\boldsymbol{x}) := \{\boldsymbol{d} \in \mathbb{R}^n \mid \boldsymbol{d} = 0 \text{ or } \boldsymbol{\xi}_i^T \boldsymbol{d} < 0 \text{ for all } \boldsymbol{\xi}_i \in \partial g_i(\boldsymbol{x}) \text{ and } i \in \mathcal{I}^\circ(\boldsymbol{x})\},$$

where

$$\mathcal{I}^\circ(\boldsymbol{x}) = \{i \in \{1, \dots, m\} \mid g_i(\boldsymbol{x}) = 0\}$$

is the set of *active constraints*.

Theorem 4.9 *Let $\boldsymbol{x}^*$ be a local optimum of the problem (4.11), where $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ are supposed to be locally Lipschitz continuous at $\boldsymbol{x}^* \in S \neq \emptyset$. Then*

$$D_S^\circ(\boldsymbol{x}^*) \cap F_S^\circ(x^*) = \{\mathbf{0}\}. \tag{4.12}$$

Proof We show first, that $F_S^\circ(\boldsymbol{x}^*) \subseteq F_S(\boldsymbol{x}^*)$. Clearly $\mathbf{0} \in F_S^\circ(\boldsymbol{x}^*) \cap F_S(\boldsymbol{x}^*)$ and so we can suppose that $\mathbf{0} \neq \boldsymbol{d} \in F_S^\circ(\boldsymbol{x}^*)$. Since $\boldsymbol{x}^* \in X$ and X is open, there exists $\varepsilon_1 > 0$ such that

$$\boldsymbol{x}^* + t\boldsymbol{d} \in X \quad \text{for all } t \in (0, \varepsilon_1]. \tag{4.13}$$

Suppose next, that $i \notin \mathcal{I}^\circ(\boldsymbol{x}^*)$ implying that $g_i(\boldsymbol{x}^*) < 0$. Since locally Lipschitz continuous function is continuous, there exists $\varepsilon_2 > 0$ such that

$$g_i(\boldsymbol{x}^* + t\boldsymbol{d}) < 0 \quad \text{for all } t \in (0, \varepsilon_2]. \tag{4.14}$$

Finally, if $i \in \mathcal{I}^\circ(\boldsymbol{x}^*)$, then we have

$$\boldsymbol{\xi}_i^T \boldsymbol{d} < 0 \text{ for all } \boldsymbol{\xi}_i \in \partial g_i(\boldsymbol{x}),$$

which means by Theorem 4.5, that $\boldsymbol{d}$ is a descent direction for g_i at $\boldsymbol{x}^*$. Then there exists $\varepsilon_3 > 0$ such that

$$g_i(\boldsymbol{x}^* + t\boldsymbol{d}) < g_i(\boldsymbol{x}^*) = 0 \quad \text{for all } t \in (0, \varepsilon_3]. \tag{4.15}$$

By choosing $\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ the conditions (4.13)–(4.15) imply

$$\boldsymbol{x}^* + t\boldsymbol{d} \in S \quad \text{for all } t \in (0, \varepsilon],$$

in other words, $\boldsymbol{d} \in F_S(\boldsymbol{x}^*)$ and thus $F_S^\circ(\boldsymbol{x}^*) \subseteq F_S(\boldsymbol{x}^*)$. On the other hand, due to Theorem 4.6 we have $D_S^\circ(\boldsymbol{x}^*) \cap F_S(\boldsymbol{x}^*) = \{\mathbf{0}\}$ implying that also $D_S^\circ(\boldsymbol{x}^*) \cap F_S^\circ(\boldsymbol{x}^*) = \{\mathbf{0}\}$. □

The next example shows, that even in the convex case the condition (4.12) is not sufficient for optimality.

Example 4.2 (*Necessary but not sufficient condition*). Let $n = m = 1$, $f(x) = -x$, $g_1(x) = x^3 - 1$ and $X = \mathbb{R}$. Then clearly f is a convex function and $S = (-\infty, 1]$ is a convex set. Furthermore, the global optimum of the problem (4.11) is $x^* = 1$. On the other hand, we have $\partial g_1(0) = \{0\}$ and thus $F_S^\circ(0) = \{0\}$. Then the condition (4.12) is trivially valid at a nonoptimal point $x = 0$.

4.3.2 Fritz John Optimality Conditions

Next we formulate necessary analytical optimality conditions generalizing the classical Fritz John (FJ) optimality conditions for nonsmooth optimization.

Theorem 4.10 (FJ Necessary Conditions) *Let $\boldsymbol{x}^*$ be a local optimum of the problem (4.11), where $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ are supposed to be locally Lipschitz continuous at $\boldsymbol{x}^* \in S \neq \emptyset$. Then there exist multipliers $\lambda_i \geq 0$ for $i = 0, \ldots, m$ such that $\sum_{i=0}^m \lambda_i = 1$, $\lambda_i g_i(\boldsymbol{x}^*) = 0$ for $i = 1, \ldots, m$ and*

$$\mathbf{0} \in \lambda_0 \partial f(\boldsymbol{x}^*) + \sum_{i=1}^m \lambda_i \partial g_i(\boldsymbol{x}^*).$$

Proof According to Theorem 3.23 function $h_{\boldsymbol{x}^*}: \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_{\boldsymbol{x}^*}(\boldsymbol{x}) := \max\{f(\boldsymbol{x}) - f(\boldsymbol{x}^*), g(\boldsymbol{x})\}$$

is locally Lipschitz continuous at $\boldsymbol{x}^*$. Since $\boldsymbol{x}^* \in \mathbb{R}^n$ is a local optimum of the problem (4.1), there exists $\delta > 0$ such that

$$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*; \delta).$$

Then

$$\begin{aligned} h_{\boldsymbol{x}^*}(\boldsymbol{x}) &= \max\{f(\boldsymbol{x}) - f(\boldsymbol{x}^*), g(\boldsymbol{x})\} \\ &\geq f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \\ &\geq 0 \qquad \text{for all } \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*; \delta). \end{aligned}$$

Since $\boldsymbol{x}^* \in S$ we have $g(\boldsymbol{x}^*) \leq 0$ implying

$$h_{\boldsymbol{x}^*}(\boldsymbol{x}^*) = \max\{f(\boldsymbol{x}^*) - f(\boldsymbol{x}^*), g(\boldsymbol{x}^*)\} = 0,$$

in other words, $h_{\boldsymbol{x}^*}$ attains its local minimum at $\boldsymbol{x}^*$. Then due to Theorem 4.1 we have

$$\mathbf{0} \in \partial h_{\boldsymbol{x}^*}(\boldsymbol{x}^*).$$

If $g(\boldsymbol{x}^*) < 0$ we have $g(\boldsymbol{x}^*) < f(\boldsymbol{x}^*) - f(\boldsymbol{x}^*)$ and thus, due to Theorem 3.23 we get

$$\mathbf{0} \in \partial h_{\boldsymbol{x}^*}(\boldsymbol{x}^*) \subseteq \partial f(\boldsymbol{x}^*).$$

Then the assertion of the theorem is proved by choosing $\lambda_0 := 1$ and $\lambda_i := 0$ for $i = 1, \ldots, m$.

On the other hand, if $g(\boldsymbol{x}^*) = 0$ we have $g(\boldsymbol{x}^*) = f(\boldsymbol{x}^*) - f(\boldsymbol{x}^*)$ and thus, due to Theorem 3.23 we get

$$\mathbf{0} \in \partial h_{\boldsymbol{x}^*}(\boldsymbol{x}^*) \subseteq \operatorname{conv}\{\partial f(\boldsymbol{x}^*) \cup \partial g(\boldsymbol{x}^*)\}.$$

Furthermore, we have

$$\partial g(\boldsymbol{x}^*) \subseteq \operatorname{conv}\{\partial g_i(\boldsymbol{x}^*) \mid i \in \mathcal{I}(\boldsymbol{x}^*)\},$$

where

$$\mathcal{I}(\boldsymbol{x}^*) = \{i \in \{1, \ldots, m\} \mid g_i(\boldsymbol{x}^*) = g(\boldsymbol{x}^*)\}.$$

Then, due to the definition of convex hull (Definition 2.2) there exist $\lambda_0 \geq 0$ and $\lambda_i \geq 0$ for $i \in \mathcal{I}(\boldsymbol{x}^*)$ such that $\lambda_0 + \sum_{i \in \mathcal{I}(\boldsymbol{x}^*)} \lambda_i = 1$, $\lambda_i g_i(\boldsymbol{x}^*) = 0$ for $i \in \mathcal{I}(\boldsymbol{x}^*)$ and

$$\mathbf{0} \in \lambda_0 \partial f(\boldsymbol{x}^*) + \sum_{i \in \mathcal{I}(\boldsymbol{x}^*)} \lambda_i \partial g_i(\boldsymbol{x}^*).$$

The assertion of the theorem is now proved by choosing $\lambda_i := 0$ for $i \notin \mathcal{I}(\boldsymbol{x}^*)$. $\square$

4.3.3 Karush-Kuhn-Tucker Optimality Conditions

As in the classical case, the disadvantage of the FJ conditions are, that λ_0 can be zero. Then the conditions tell nothing about the objective function. The conditions are valid, for example, in those (nonoptimal) points $\boldsymbol{x}$, where we have $\mathbf{0} \in \partial g_i(\boldsymbol{x}) = 0$ for some $i \in \mathcal{I}(\boldsymbol{x})$. In order to get information about the objective function, we need some regularization assumptions, called *constraint qualifications*.

Definition 4.8 The problem (4.11) satisfies the Slater constraint qualification if there exists $\boldsymbol{x} \in S$ such that $g(\boldsymbol{x}) < 0$.

Definition 4.9 The problem (4.11) satisfies the Cottle constraint qualification at $\boldsymbol{x} \in S$ if either $g(\boldsymbol{x}) < 0$ or $\mathbf{0} \notin \partial g(\boldsymbol{x})$.

Note, that Slater condition is global while Cottle is defined pointwisely. Next we give the relationship between those two qualifications.

Lemma 4.3 *If the problem* (4.11) *satisfies the Cottle constraint qualification at some* $\boldsymbol{x} \in S$*, then it satisfies also the Slater constraint qualification. If the functions* g_i *are convex for all* $i = 1, \ldots, m$ *and the problem* (4.11) *satisfies the Slater constraint qualification, then it satisfies also the Cottle constraint qualification at every* $\boldsymbol{x} \in S$.

Proof Suppose first, that the problem (4.11) satisfies the Cottle constraint qualification at $\boldsymbol{x} \in S$. If $g(\boldsymbol{x}) < 0$, then the Slater condition is fulfilled directly. On the other hand, if $\mathbf{0} \notin \partial g(\boldsymbol{x})$, then due to Corollary 4.1 there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for g at $\boldsymbol{x}$, in other words, there exists $\varepsilon > 0$ such that for all $t \in (0, \varepsilon]$

$$g(\boldsymbol{x} + t\boldsymbol{d}) < g(\boldsymbol{x}) = 0$$

and thus, the Slater condition is valid.

Suppose next, that the functions g_i are convex for all $i = 1, \ldots, m$, the problem (4.11) satisfies the Slater constraint qualification and $\boldsymbol{x} \in S$ is arbitrary. If $g(\boldsymbol{x}) = 0$, then due to Slater condition there exists $\boldsymbol{y} \in S$ such that $g(\boldsymbol{y}) < 0 = g(\boldsymbol{x})$, thus $\boldsymbol{x}$ is not a global minimum of function g. Since g is convex, due to Theorem 4.2 we have $\mathbf{0} \notin \partial g(\boldsymbol{x})$. □

Now we are ready to generalize Karush-Kuhn-Tucker (KKT) optimality conditions for nonsmooth optimization.

Theorem 4.11 (KKT Necessary Conditions) *Let the problem* (4.11) *satisfy the Cottle constraint qualification at a local optimum* $\boldsymbol{x}^*$*, where* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $g_i : \mathbb{R}^n \to \mathbb{R}$ *for* $i = 1, \ldots, m$ *are supposed to be locally Lipschitz continuous at* $\boldsymbol{x}^* \in S \neq \emptyset$. *Then there exist multipliers* $\lambda_i \geq 0$ *for* $i = 1, \ldots, m$ *such that* $\lambda_i g_i(\boldsymbol{x}^*) = 0$ *and*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(\boldsymbol{x}^*).$$

Proof According to Theorem 4.7 we have

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$

If $g(\boldsymbol{x}^*) < 0$ we have $\boldsymbol{x}^* \in \text{int } S$ and thus, due to Theorem 3.29 $N_S(\boldsymbol{x}^*) = \{\mathbf{0}\}$, in other words $\mathbf{0} \in \partial f(\boldsymbol{x}^*)$. Then the assertion of the theorem is proved by choosing $\lambda_i := 0$ for all $i = 1, \ldots, m$.

On the other hand, if $g(\boldsymbol{x}^*) = 0$ we have $S = \text{lev}_{g(\boldsymbol{x}^*)}\, g$ and due to the Cottle constraint qualification $\mathbf{0} \notin \partial g(\boldsymbol{x}^*)$. Then by Theorem 3.34

$$N_S(\boldsymbol{x}^*) = N_{\text{lev}_{g(\boldsymbol{x}^*)}\, g}(\boldsymbol{x}^*) \subseteq \text{ray } \partial g(\boldsymbol{x}^*).$$

Furthermore, due to Theorem 3.23 we get

$$\partial g(\boldsymbol{x}^*) = \text{conv } \{\partial g_i(\boldsymbol{x}^*) \mid i \in \mathcal{I}(\boldsymbol{x}^*)\},$$

where

$$\mathcal{I}(\boldsymbol{x}^*) = \{i \in \{1, \ldots, m\} \mid g_i(\boldsymbol{x}^*) = g(\boldsymbol{x}^*)\}.$$

Then, due to the definition of ray (Definition 2.7) there exist $\lambda_i \geq 0$ for all $i \in \mathcal{I}(\boldsymbol{x}^*)$ such that $\lambda_i g_i(\boldsymbol{x}^*) = 0$ and

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + \sum_{i \in \mathcal{I}(\boldsymbol{x}^*)} \lambda_i \partial g_i(\boldsymbol{x}^*).$$

The assertion of the theorem is now proved by choosing $\lambda_i := 0$ for $i \notin \mathcal{I}(\boldsymbol{x}^*)$. □

To this end we formulate sufficient KKT optimality conditions utilizing convexity.

Theorem 4.12 (KKT Sufficient Conditions) *Let* $f: \mathbb{R}^n \to \mathbb{R}$ *and* $g_i: \mathbb{R}^n \to \mathbb{R}$ *for* $i = 1, \ldots, m$ *be convex functions. If at* $\boldsymbol{x}^* \in S \neq \emptyset$ *there exist multipliers* $\lambda_i \geq 0$ *for all* $i = 1, \ldots, m$ *such that* $\lambda_i g_i(\boldsymbol{x}^*) = 0$ *and*

$$\mathbf{0} \in \partial_c f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \partial_c g_i(\boldsymbol{x}^*), \tag{4.16}$$

then $\boldsymbol{x}^*$ *is a global optimum of the problem* (4.11).

Proof As maximum of the convex constraint functions $g_i: \mathbb{R}^n \to \mathbb{R}$ the total constraint function g is convex and as a level set of a convex function the feasible set $S = \text{lev}_0\, g$ is convex by Theorem 2.23. If $g(\boldsymbol{x}^*) < 0$ we have $g_i(\boldsymbol{x}^*) < 0$ for all $i = 1, \ldots, m$ and since $\lambda_i g_i(\boldsymbol{x}^*) = 0$ we deduce that $\lambda_i = 0$ for all $i = 1, \ldots, m$. Then (4.16) implies that $\mathbf{0} \in \partial_c f(\boldsymbol{x}^*)$. Furthermore, $\boldsymbol{x}^* \in \text{int } S$ and thus, due to Theorem 3.29 we have $N_S(\boldsymbol{x}^*) = \{\mathbf{0}\}$, in other words

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*)$$

and then $\boldsymbol{x}^*$ is a global optimum of the problem (4.11) due to Theorem 4.8.

On the other hand, if $g(\boldsymbol{x}^*) = 0$ we have $S = \operatorname{lev}_{g(\boldsymbol{x}^*)} g$ and due to Theorem 3.23 we get

$$\partial g(\boldsymbol{x}^*) = \operatorname{conv} \{\partial g_i(\boldsymbol{x}^*) \mid i \in \mathcal{I}(\boldsymbol{x}^*)\},$$

where

$$\mathcal{I}(\boldsymbol{x}^*) = \{i \in \{1, \ldots, m\} \mid g_i(\boldsymbol{x}^*) = g(\boldsymbol{x}^*)\}.$$

Then Theorem 2.37 implies that

$$\sum_{i=1}^{m} \lambda_i \partial_c g_i(\boldsymbol{x}^*) \subseteq \operatorname{ray} \, \partial g(\boldsymbol{x}^*) = N_{\operatorname{lev}_{g(\boldsymbol{x}^*)} g}(\boldsymbol{x}^*) = N_S(\boldsymbol{x}^*)$$

and thus

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*)$$

which implies that $\boldsymbol{x}^*$ is a global optimum of the problem (4.11) due to Theorem 4.8. □

Finally, we can combine the necessary and sufficient conditions. Note that here, unlike in sufficient conditions, we cannot avoid the constraint qualification.

Corollary 4.4 (KKT Necessary and Sufficient Conditions) *Suppose, that the problem (4.11) satisfies the Slater constraint qualification and let $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ be convex functions. Then $\boldsymbol{x}^*$ is a global optimum of the problem (4.11) if and only if there exist multipliers $\lambda_i \geq 0$ for all $i = 1, \ldots, m$ such that $\lambda_i g_i(\boldsymbol{x}^*) = 0$ and*

$$\mathbf{0} \in \partial_c f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \partial_c g_i(\boldsymbol{x}^*),$$

Proof The result follows from Theorems 4.11 and 4.12, and Lemma 4.3. □

Example 4.3 (*Optimality conditions*). Consider the problem

$$\begin{cases} \text{minimize} & f(x_1, x_2) = \max\{x_1, x_2\} \\ \text{subject to} & (x_1 - 3)^2 + (x_2 - 3)^2 - 5 \le 0 \\ & x_1 - x_2 + 1 \le 0 \\ & x_1^2 + 2x_2 - 14 \le 0. \end{cases} \tag{4.17}$$

Clearly f is convex and according to Exercise 3.6 we have

$$\boldsymbol{\xi} \in \partial_c f(x_1, x_2) \iff \begin{cases} \boldsymbol{\xi} = (0, 1)^T, & \text{if } x_1 < x_2 \\ \boldsymbol{\xi} = (\mu, 1 - \mu)^T, & \text{if } x_1 = x_2 \\ \boldsymbol{\xi} = (1, 0)^T, & \text{if } x_1 > x_2, \end{cases}$$

where $\mu \in [0, 1]$. Let us denote

$$\begin{aligned} g_1(x_1, x_2) &:= (x_1 - 3)^2 + (x_2 - 3)^2 - 5, \\ g_2(x_1, x_2) &:= x_1 - x_2 + 1, \\ g_3(x_1, x_2) &:= x_1^2 + 2x_2 - 14. \end{aligned}$$

Since functions g_i are convex and differentiable for all $i = 1, 2, 3$ we deduce from Theorem 2.29 that

$$\partial_c g_i(x_1, x_2) = \{\nabla g_i(x_1, x_2)\},$$

where

$$\begin{aligned} \nabla g_1(x_1, x_2) &= (2(x_1 - 3), 2(x_2 - 3))^T \\ \nabla g_2(x_1, x_2) &= (1, -1)^T \\ \nabla g_3(x_1, x_2) &= (2x_1, 2)^T. \end{aligned}$$

Then from Theorem 4.12 we get the following KKT conditions

$$\begin{cases} \xi_1 + 2\lambda_1(x_1 - 3) + \lambda_2 + 2\lambda_3 x_1 & = 0 \\ \xi_2 + 2\lambda_1(x_2 - 3) - \lambda_2 + 2\lambda_3 & = 0 \\ \lambda_1((x_1 - 3)^2 + (x_2 - 3)^2 - 5) & = 0 \\ \lambda_2(x_1 - x_2 + 1) & = 0 \\ \lambda_3(x_1^2 + 2x_2 - 14) & = 0 \\ \lambda_1, \lambda_2, \lambda_3 & \geq 0 \end{cases}$$

If we calculate the conditions at the point $\boldsymbol{x}^* = (1, 2)^T$, we have $x_1 < x_2$ and thus $\boldsymbol{\xi} = (0, 1)^T$. Then we get

$$\begin{cases} \lambda_1 = \frac{1}{6} \geq 0 \\ \lambda_2 = \frac{2}{3} \geq 0 \\ \lambda_3 = 0 \geq 0. \end{cases}$$

In other words we have found $\lambda_i \geq 0$ for all $i = 1, 2, 3$ such that $\lambda_i g_i(\boldsymbol{x}^*) = 0$ and

$$\mathbf{0} \in \partial_c f(\boldsymbol{x}^*) + \sum_{i=1}^{3} \lambda_i \partial_c g_i(\boldsymbol{x}^*).$$

Thus due to Theorem 4.12 $\boldsymbol{x}^* = (1, 2)^T$ is a global optimum of the problem (4.17).

4.4 Optimality Conditions Using Quasidifferentials

The quasidifferential (see Definition 3.9) can be used to formulate optimality conditions for a minimum both in unconstrained and constrained problems. First, we formulate such conditions for unconstrained optimization problems. That is, we consider the problem (4.1) with $S = \mathbb{R}^n$. In addition, we assume that the objective function f is quasidifferentiable on $\mathbb{R}^n$.

Theorem 4.13 *For the function f to attain its smallest value on $\mathbb{R}^n$ at a point $\boldsymbol{x}^*$ it is necessary that*

$$-\overline{\partial} f(\boldsymbol{x}^*) \subset \underline{\partial} f(\boldsymbol{x}^*).$$

If, in addition, f is locally Lipschitz continuous and

$$-\overline{\partial} f(\boldsymbol{x}^*) \subset \text{int } \underline{\partial} f(\boldsymbol{x}^*),$$

then the point $\boldsymbol{x}^*$ is a strict local minimizer of f on $\mathbb{R}^n$.

Proof See [72]. □

In order to formulate optimality conditions for constrained minimization problems, we introduce the notion of a quasidifferentiable set.

Definition 4.10 A set $\Omega \subset \mathbb{R}^n$ is called a *quasidifferentiable set* if it can be represented in the form

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leq 0\},$$

where h is a quasidifferentiable function.

Let us now consider the problem (4.1) with $S = \Omega$. Let us take $\boldsymbol{x} \in \Omega$ and consider the cones

$$\gamma_1(\boldsymbol{x}) = \{\boldsymbol{d} \in \mathbb{R}^n \mid h'(\boldsymbol{x}; \boldsymbol{d}) < 0\}, \qquad \gamma_2(\boldsymbol{x}) = \{\boldsymbol{d} \in \mathbb{R}^n \mid h'(\boldsymbol{x}; \boldsymbol{d}) \leq 0\}.$$

Let $h(\boldsymbol{x}) = 0$. We say that the *regularity condition* is satisfied at the point $\boldsymbol{x}$ if $\text{cl } \gamma_1(\boldsymbol{x}) = \gamma_2(\boldsymbol{x})$.

Theorem 4.14 *Let functions f and h be locally Lipschitz continuous and quasidifferentiable in some neighborhood of a point $\boldsymbol{x}^* \in \Omega$ and $h(\boldsymbol{x}^*) = 0$. Assume also that the regularity condition holds at $\boldsymbol{x}^*$. For the function f to attain its smallest value on Ω at the point $\boldsymbol{x}^*$ it is necessary that*

$$\left(\underline{\partial} f(\boldsymbol{x}^*) + \boldsymbol{w}\right) \bigcap \left[- \text{cl}\,(\text{cone}(\underline{\partial} h(\boldsymbol{x}^*) + \boldsymbol{w}'))\right] \neq \emptyset$$

for all $\boldsymbol{w} \in \overline{\partial} f(\boldsymbol{x}^*)$, $\boldsymbol{w}' \in \overline{\partial} h(\boldsymbol{x}^*)$.

Proof See [72]. □

4.5 Summary

In this chapter we have concentrated on the theory of nonsmooth optimization. We have formulated necessary and sufficient optimality conditions for a Lipschitz functions to attain local and global minima both in the unconstrained and constrained cases. We have formulated both geometrical and analytical conditions based on cones and subdifferentials, respectively. We have also considered both geometrical and analytical constraints and generalized the classical Fritz John (FJ) (Theorem 4.10) and Karush-Kuhn-Tucker (KKT) optimality conditions (Theorems 4.11 and 4.12). In addition, we have formulated the optimality conditions using quasidifferentials.

Exercises

4.1 (Theorem 4.4) Prove that if the function $f: \mathbb{R}^n \to \mathbb{R}$ is convex and $\varepsilon \geq 0$, then f attains its global ε-minimum at $\boldsymbol{x}^*$ if and only if

$$\mathbf{0} \in \partial_\varepsilon f(\boldsymbol{x}^*) \quad \text{or} \quad f'_\varepsilon(\boldsymbol{x}^*; \boldsymbol{d}) \geq 0 \quad \text{for all } \boldsymbol{d} \in \mathbb{R}^n.$$

4.2 (Corollary 4.1) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Prove that either $\mathbf{0} \in \partial f(\boldsymbol{x})$ or there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for f at $\boldsymbol{x}$.

4.3 (Corollary 4.2) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that either f attains its global minimum at $\boldsymbol{x} \in \mathbb{R}^n$ or there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for f at $\boldsymbol{x}$.

4.4 Find the point where the function

$$f(x_1, x_2) = \max\,\{x_1^2 + (x_2 - 1)^2, 3 - 2x_1\}$$

attains its global minimum.

4.5 Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Prove that $D_S(\boldsymbol{x})$ and $D_S^\circ(\boldsymbol{x})$ are cones and they are both convex, if f is a convex function.

4.6 Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$. Prove that

(a) $D_S(\boldsymbol{x}) \subseteq F_{\text{lev}_{f(\boldsymbol{x})} f}(\boldsymbol{x})$

(b) $D_S^\circ(\boldsymbol{x}) \subseteq \partial f(\boldsymbol{x})^\circ$

(c) $D_S^\circ(\boldsymbol{x}) \subseteq D_S(\boldsymbol{x})$.

4.7 (Corollary 4.3) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x}^* \in \mathbb{R}^n$. Prove that if f attains its local minimum at $\boldsymbol{x}^*$, then

$$D_S^\circ(\boldsymbol{x}^*) = D_S(\boldsymbol{x}^*) = \{\mathbf{0}\}. \tag{4.18}$$

If, in addition, f is convex, prove that the condition (4.18) is sufficient for $\boldsymbol{x}^*$ to be a global minimum of f.

4.8 Let $g_i: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x} \in \mathbb{R}^n$ for all $i = 1, \ldots, m$. Prove that $F_S^\circ(\boldsymbol{x})$ is a cone. Is it convex if $g_i: \mathbb{R}^n \to \mathbb{R}$ are convex for all $i \in \mathcal{I}^\circ$?

4.9 Consider the problem

$$\begin{cases} \text{minimize} & f(x_1, x_2) = \max\,\{|x_1 - 3|, |x_2 - 2|\} \\ \text{subject to} & x_1^2 + x_2^2 - 5 \leq 0 \\ & x_1 + x_2 - 3 \leq 0 \\ & x_1, x_2 \geq 0. \end{cases}$$

Determine the cones $D_S(\boldsymbol{x})$, $D_S^\circ(\boldsymbol{x})$, $F_S(\boldsymbol{x})$ and $F_S^\circ(\boldsymbol{x})$ at $\boldsymbol{x} = (2, 1)^T$. What can you say about the optimality of $\boldsymbol{x}$?

4.10 Show that $\boldsymbol{x}^* = (1, 0)^T$ is a global optimum of the problem

$$\begin{cases} \text{minimize} & f(x_1, x_2) = |x_1| + |x_2| \\ \text{subject to} & x_1^2 + x_2^2 - 1 \leq 0 \\ & (x_1 - 2)^2 + x_2^2 - 1 \leq 0. \end{cases}$$

However, show that the necessary KKT optimality conditions are not valid. Why?

4.11 Consider the problem

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) = \max\{\|\boldsymbol{x}\|, \|\boldsymbol{x}\|^2\} \\ \text{subject to} & \|\boldsymbol{x}\| \geq 1. \end{cases}$$

Find the points satisfying the necessary KKT optimality condition. What can you say about the optimality of them?

4.12 Solve the problem

$$\begin{cases} \text{minimize} & f(x_1, x_2, x_3) = |x_1| + x_2^2 - 2x_3 \\ \text{subject to} & |x_1 + x_3| - 2 \leq 0 \\ & \max\{x_2, x_3\} - 5 \leq 0 \\ & x_1 \geq 1, x_2 \geq 2. \end{cases}$$

Chapter 5
Generalized Convexities

Convexity plays a crucial role in mathematical optimization theory. Especially, in duality theory and in constructing optimality conditions, convexity has been the most important concept since the basic reference by Rockafellar [204] was published. Different types of generalized convexities has proved to be the main tool when constructing optimality conditions, particularly sufficient conditions for optimality. In this chapter we analyze the properties of the generalized pseudo- and quasiconvexities for nondifferentiable locally Lipschitz continuous functions. The treatment is based on the Clarke subdifferentials and generalized directional derivatives.

5.1 Generalized Pseudoconvexity

We start this section by recalling the most famous definition of pseudoconvexity for smooth functions

Definition 5.1 A continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *pseudoconvex*, if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f(\boldsymbol{y}) < f(\boldsymbol{x}) \quad \text{implies} \quad \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) < 0.$$

The main result for a smooth pseudoconvex function f is that the convexity assumption of Theorem 4.2 can be relaxed, in other words, a smooth pseudoconvex function f attains its global minimum at $\boldsymbol{x}^*$, if and only if $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$ (see Fig. 5.1).

Now we extend the concept of pseudoconvexity for nonsmooth functions.

Definition 5.2 A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is f°*-pseudoconvex*, if it is locally Lipschitz continuous and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f(\boldsymbol{y}) < f(\boldsymbol{x}) \quad \text{implies} \quad f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) < 0.$$

Note that due to (3.9) a convex function is always f°-pseudoconvex. The next result shows that f°-pseudoconvexity is a natural extension of pseudoconvexity.

A. Bagirov et al., *Introduction to Nonsmooth Optimization*,
DOI: 10.1007/978-3-319-08114-4_5

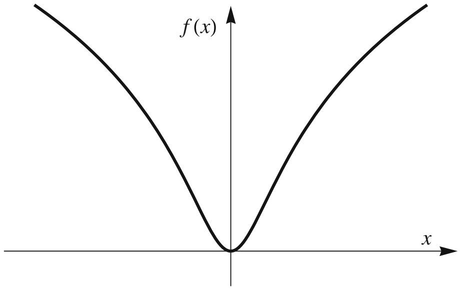


Fig. 5.1 Pseudoconvex function

Theorem 5.1 *If $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, then f is f°-pseudoconvex, if and only if f is pseudoconvex.*

Proof Follows immediately from Theorem 3.4, since for a smooth f we have $f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) = f'(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) = \nabla f(\boldsymbol{x})^T (\boldsymbol{y} - \boldsymbol{x})$. □

Sometimes the reasoning chain in the definition of f°-pseudoconvexity needs to be converted.

Lemma 5.1 *A locally Lipschitz continuous function $f: \mathbb{R}^n \to \mathbb{R}$ is f°-pseudoconvex, if and only if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$*

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \geq 0 \quad \textit{implies} \quad f(\boldsymbol{y}) \geq f(\boldsymbol{x}).$$

Proof Follows directly from the definition of f°-pseudoconvexity. □

The important sufficient extremum property of convexity (Theorem 4.2) and pseudoconvexity remains also for f°-pseudoconvexity.

Theorem 5.2 *If the function $f: \mathbb{R}^n \to \mathbb{R}$ is f°-pseudoconvex, then f attains its global minimum at $\boldsymbol{x}^*$, if and only if*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*).$$

Proof If f attains its global minimum at $\boldsymbol{x}^*$, then by Theorem 4.1 we have $\mathbf{0} \in \partial f(\boldsymbol{x}^*)$. On the other hand, if $\mathbf{0} \in \partial f(\boldsymbol{x}^*)$ and $\boldsymbol{y} \in \mathbb{R}^n$, then by Definition 3.2

$$f^\circ(\boldsymbol{x}^*; \boldsymbol{y} - \boldsymbol{x}^*) \geq \mathbf{0}^T (\boldsymbol{y} - \boldsymbol{x}^*) = 0$$

and, thus by Lemma 5.1 we have

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}^*).$$

□

The following example shows, that f°-pseudoconvexity is a more general property than pseudoconvexity.

Example 5.1 (f°*-pseudoconvexity*). Define $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) := \min\{|x|, x^2\}$ (see Fig. 5.2). Then f is clearly locally Lipschitz continuous but not convex nor pseudoconvex. However, for all $y > x$ we have

$$f^\circ(x; y - x) = \begin{cases} -1, & x \in (-\infty, -1] \\ 2x, & x \in (-1, 1] \\ 1, & x \in (1, \infty), \end{cases}$$

and thus, due to the symmetricity of the function f and Lemma 5.1, f is f°-pseudoconvex. Furthermore, for the unique global minimum $x^* = 0$ we have $\partial f(x^*) = \{0\}$.

The notion of monotonicity is closely related to convexity.

Definition 5.3 The generalized directional derivative f° is called *pseudomonotone*, if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \geq 0 \quad \text{implies} \quad f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y}) \leq 0$$

or, equivalently

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) > 0 \quad \text{implies} \quad f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y}) < 0.$$

Furthermore, f° is *strictly pseudomonotone*, if

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \geq 0 \quad \text{implies} \quad f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y}) < 0.$$

Theorem 5.3 *If $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous such that f° is pseudomonotone, then f is f°-pseudoconvex.*

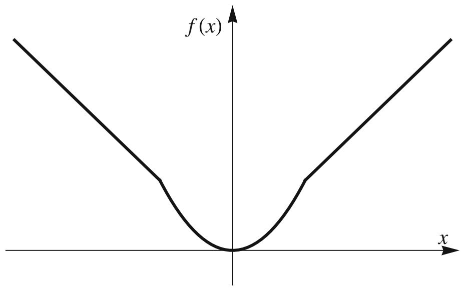


Fig. 5.2 f°-pseudoconvex but not pseudoconvex function

Proof Let us, on the contrary, assume that f is not f°-pseudoconvex. Then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x})$ and

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \geq 0. \tag{5.1}$$

Then by the Mean-Value Theorem 3.18 there exists $\lambda \in (0, 1)$ such that $\boldsymbol{z} = \boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x})$ and

$$f(\boldsymbol{x}) - f(\boldsymbol{y}) \in \partial f(\boldsymbol{z})^T(\boldsymbol{x} - \boldsymbol{y}).$$

This means that due to the definition of the Clarke subdifferential there exists $\boldsymbol{\xi} \in \partial f(\boldsymbol{z})$ such that

$$0 < f(\boldsymbol{x}) - f(\boldsymbol{y}) = \boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{y}) \leq f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{y}). \tag{5.2}$$

On the other hand, from (5.1) and the positive homogeneity of $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ (see Theorem 3.1) we deduce that

$$f^\circ(\boldsymbol{x}; \boldsymbol{z} - \boldsymbol{x}) = f^\circ(\boldsymbol{x}; \lambda(\boldsymbol{y} - \boldsymbol{x})) = \lambda f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \geq 0.$$

Then the pseudomonotonicity, the positive homogeneity of $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ and (5.2) imply that

$$0 \geq f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{z}) = \lambda f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{y}) > 0,$$

which is impossible. Thus f is f°-pseudoconvex. □

The converse result is also true, but before we can prove it we need few lemmas.

Lemma 5.2 *If the function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* f°*-pseudoconvex,* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\boldsymbol{z} = \lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$*, where* $\lambda \in (0, 1)$*, then*

$$f(\boldsymbol{z}) \leq \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}.$$

Proof On the contrary assume that $f(\boldsymbol{z}) > \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$. Since f is f°-pseudoconvex and $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ is positively homogeneous by Theorem 3.1, we have

$$0 > f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{z}) = f^\circ(\boldsymbol{z}; (1 - \lambda)(\boldsymbol{x} - \boldsymbol{y})) = (1 - \lambda) f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{y})$$

and thus

$$f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{y}) < 0.$$

Correspondingly, we obtain

$$0 > f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{z}) = f^\circ(\boldsymbol{z}; \lambda(\boldsymbol{y} - \boldsymbol{x})) = \lambda f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{x})$$

and thus

$$f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{x}) < 0.$$

Since $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ is subadditive by Theorem 3.1, we have

$$0 > f^\circ(\boldsymbol{z}; \boldsymbol{x} - \boldsymbol{y}) + f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{x}) \geq f^\circ(\boldsymbol{z}; (\boldsymbol{x} - \boldsymbol{y}) + (\boldsymbol{y} - \boldsymbol{x})) = f^\circ(\boldsymbol{z}; \mathbf{0}) = 0,$$

which is impossible. In other words, $f(\boldsymbol{z}) \leq \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$. □

Lemma 5.3 *If the function* $f: \mathbb{R}^n \to \mathbb{R}$ *is* f°*-pseudoconvex, then for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) \quad \textit{implies} \quad f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \leq 0.$$

Proof On the contrary, assume that there exist points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\alpha > 0$ such that $f(\boldsymbol{y}) = f(\boldsymbol{x})$ and $f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) = \alpha > 0$. Since f is locally Lipschitz continuous there exist $\beta, K > 0$ such that K is the Lipschitz constant in the ball $B(\boldsymbol{x}; \beta)$. Since $f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) = \alpha$ Lemma 3.1 implies that there exists a sequence $(\boldsymbol{x}_i)$ of points where f is differentiable and $I \in \mathbb{N}$ such that $\boldsymbol{x}_i \to \boldsymbol{x}$ and

$$f'(\boldsymbol{x}_i; \boldsymbol{y} - \boldsymbol{x}) = \nabla f(\boldsymbol{x}_i)^T(\boldsymbol{y} - \boldsymbol{x}) > \frac{\alpha}{2} \tag{5.3}$$

holds when $i \geq I$. Let

$$\delta = \min\left\{\beta, \frac{\alpha}{2K}\right\}$$

and $\boldsymbol{z} \in B(\boldsymbol{x}; \delta) \cap \{(\boldsymbol{x}_i) \mid i \geq I\}$. According to Lemma 1.2 $f'(\boldsymbol{z}; \cdot)$ is Lipschitz continuous with the constant K. Hence,

$$\begin{aligned} \left|f'(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{x}) - f'(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{z})\right| &\leq K\,\|\boldsymbol{y} - \boldsymbol{x} - (\boldsymbol{y} - \boldsymbol{z})\| \\ &= K\,\|\boldsymbol{z} - \boldsymbol{x}\| < K\frac{\alpha}{2K} = \frac{\alpha}{2}. \end{aligned} \tag{5.4}$$

Thus, $f'(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{z}) > 0$ according to (5.3) and (5.4). Since $f'(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{z}) > 0$ there exists $\mu \in (0, 1)$ such that

$$f(\mu\boldsymbol{z} + (1 - \mu)\boldsymbol{y}) > f(\boldsymbol{z}). \tag{5.5}$$

Due to Corollary 3.1 $f^\circ(\boldsymbol{x}; \cdot)$ is convex, and thus by Theorem 2.22 it is continuous. Then the fact that $f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) = \alpha$ implies that there exists $\varepsilon > 0$ such that $f^\circ(\boldsymbol{x}; \boldsymbol{d}) > 0$ when $\boldsymbol{d} \in B(\boldsymbol{y} - \boldsymbol{x}; \varepsilon)$. Let now $\boldsymbol{v} \in B(\boldsymbol{y}; \varepsilon)$ be arbitrary. Since

$$\|\boldsymbol{v} - \boldsymbol{x} - (\boldsymbol{y} - \boldsymbol{x})\| = \|\boldsymbol{v} - \boldsymbol{y}\| < \varepsilon,$$

it follows that $\boldsymbol{v} - \boldsymbol{x} \in B(\boldsymbol{y} - \boldsymbol{x}; \varepsilon)$. Thus, $f^\circ(\boldsymbol{x}; \boldsymbol{v} - \boldsymbol{x}) > 0$ and the f°-pseudoconvexity of the function f implies that $f(\boldsymbol{v}) \geq f(\boldsymbol{x}) = f(\boldsymbol{y})$. Thus, $\boldsymbol{y}$ is a local minimum for the function f and Theorem 4.1 implies that $\mathbf{0} \in \partial f(\boldsymbol{y})$. Due to Theorem 5.2 $\boldsymbol{y}$ is also a global minimum. Thus, we have $f(\boldsymbol{y}) \leq f(\boldsymbol{z})$ and the inequality (5.5) implies that

$$f(\mu\boldsymbol{z} + (1-\mu)\boldsymbol{y}) > \max\{f(\boldsymbol{z}), f(\boldsymbol{y})\},$$

which is impossible by Lemma 5.2. □

Now we are ready to prove the converse result of Theorem 5.3.

Theorem 5.4 *The generalized directional derivative of a f°-pseudoconvex function $f\colon \mathbb{R}^n \to \mathbb{R}$ is pseudomonotone.*

Proof Let f be f°-pseudoconvex and, on the contrary, assume that there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $f^\circ(\boldsymbol{x}; \boldsymbol{y}-\boldsymbol{x}) \geq 0$ and $f^\circ(\boldsymbol{y}; \boldsymbol{x}-\boldsymbol{y}) > 0$. Then, by f°-pseudoconvexity $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ and $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$, hence $f(\boldsymbol{x}) = f(\boldsymbol{y})$. Thus, we have $f^\circ(\boldsymbol{y}; \boldsymbol{x}-\boldsymbol{y}) > 0$ and $f(\boldsymbol{x}) = f(\boldsymbol{y})$, which contradicts Lemma 5.3. □

Note, that in Lemma 5.3 the differentiability at the points $\boldsymbol{x}_i$ is crucial in order to attain the existence of the directional derivative. Unlike the convexity, the f°-pseudoconvexity does not guarantee that directional derivatives exist at every point as shown in the next example.

Example 5.2 (f°*-pseudoconvex function with no directional derivative*). Consider the following piecewise linear function

$$f(x) = \begin{cases} x & ,\text{if } x \leq 0 \\ 2^{(-1)^\alpha} \frac{1}{10^\alpha} + \left(\frac{5}{4} + (-1)^\alpha \frac{11}{12}\right)\left(x - \frac{1}{10^\alpha}\right) & ,\text{if } 0 < x < \frac{1}{10} \\ x - \frac{1}{20} & ,\text{if } x \geq \frac{1}{10}, \end{cases}$$

where

$$\alpha = \alpha(x) = \lfloor -\log_{10}(x) \rfloor.$$

The function f is drawn in Fig. 5.3. The dashed lines represents lines $y = 2x$ and $y = \frac{1}{2}x$. Function f always lies between these two lines.

We now show that function f is f°-pseudoconvex, but its directional derivative $f'(\boldsymbol{x}; \boldsymbol{d})$ is not defined at every point.

The function f is not differentiable at points $\left(\frac{1}{10^i}\right)$, $i = 1, 2, \ldots$ and 0. From the definition of the function f we see that everywhere but at 0 the classical directional derivative $f'(x; 1)$ has an upper bound

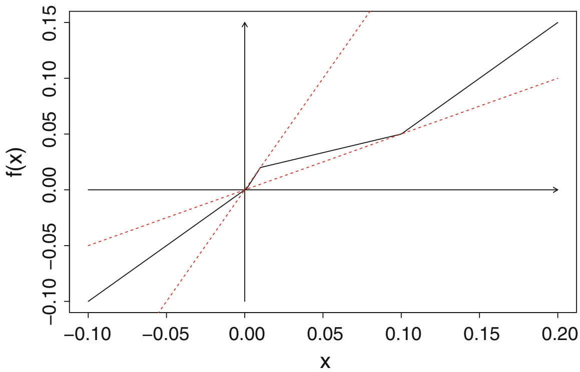


Fig. 5.3 f°-pseudoconvex function with no directional derivative

$$\max\left\{1, \frac{5}{4} + \frac{11}{12}\right\} = \frac{5}{4} + \frac{11}{12} = \frac{13}{6}$$

and a lower bound

$$\min\left\{1, \frac{5}{4} - \frac{11}{12}\right\} = \frac{1}{3}.$$

The directional derivative $f'(x; -1)$ has lower and upper bounds $-\frac{13}{6}$ and $-\frac{1}{3}$, respectively. When $x = 0$ we see from the Fig. 5.3 that $|f(y)| \le 2\,|y|$ for all $y \in \mathbb{R}$. Thus, we see that when $x, y \in \mathbb{R}$ the inequality

$$|f(x) - f(y)| \le \frac{13}{6}\,|x - y|$$

holds and f is Lipschitz continuous. Actually, at an arbitrary point x^0 the function lies between the lines

$$y = \frac{13}{6}(x - x^0) + f(x^0) \quad \text{and} \quad y = \frac{1}{3}(x - x^0) + f(x^0).$$

Next, we prove the f°-pseudoconvexity of the function f. As stated previously, for all $x \in \mathbb{R}$, $t > 0$ the inequalities

$$-\frac{13}{6} \le \frac{f(x-t) - f(x)}{t} \le -\frac{1}{3},$$

holds implying $f^\circ(x,-1) \le -\frac{1}{3}$. Now, the f°-pseudoconvexity follows from the fact that $f(y) < f(x)$ if and only if $y < x$.

Finally, we prove that directional derivative $f'(0;1)$ does not exist. Consider the limit

$$\lim_{t\downarrow 0} \varphi(t) = \lim_{t\downarrow 0} \frac{f(0+t)-f(0)}{t} \tag{5.6}$$

with different sequences (t^i). Let the sequence be $t^i = \frac{1}{10^{2i}}$, $i \in \mathbb{N}$. Then

$$\begin{aligned}
\alpha(t^i) &= 2i \\
f(t^i) &= 2\frac{1}{10^{2i}} + \frac{13}{6}\Big(\frac{1}{10^{2i}} - \frac{1}{10^{2i}}\Big) = 2\frac{1}{10^{2i}} \\
\varphi(t^i) &= \frac{2\frac{1}{10^{2i}}}{\frac{1}{10^{2i}}} = 2,
\end{aligned}$$

and the limit (5.6) is 2. Now, let the sequence be $s^i = \frac{1}{10^{2i+1}}$, $i \in \mathbb{N}$. Then

$$\begin{aligned}
\alpha(s^i) &= 2i+1 \\
f(s^i) &= \frac{1}{2}\frac{1}{10^{2i+1}} + \frac{1}{3}\Big(\frac{1}{10^{2i+1}} - \frac{1}{10^{2i+1}}\Big) = \frac{1}{2}\frac{1}{10^{2i+1}} \\
\varphi(s^i) &= \frac{1}{2}\frac{\frac{1}{10^{2i+1}}}{\frac{1}{10^{2i+1}}} = \frac{1}{2},
\end{aligned}$$

and the limit (5.6) is $\frac{1}{2}$. The sequences (t^i) and (s^i) generates different limits and thus, the function f does not have the directional derivative $f'(0;1)$.

In what follows we consider how to verify the f°-pseudoconvexity in practice. Before that, however, we need the following result.

Lemma 5.4 *A locally Lipschitz continuous function $g\colon \mathbb{R} \to \mathbb{R}$ is f°-pseudoconvex and strictly increasing, if and only if $\alpha > 0$ for all $\alpha \in \partial g(x)$ and $x \in \mathbb{R}$.*

Proof Suppose first that g is both f°-pseudoconvex and strictly increasing and let $d < 0$. Then for every $x \in \mathbb{R}$ we have $g(x+d) < g(x)$ and due to f°-pseudoconvexity $g^\circ(x;d) < 0$. By the definition of the subdifferential for all $\alpha \in \partial g(x)$ we have

$$\alpha d \le g^\circ(x;d) < 0,$$

which implies $\alpha > 0$.

On the other hand, let all the subgradients of g be positive. We first prove that g is strictly increasing. Suppose, on the contrary, that there exist $y, x \in \mathbb{R}$ such that $y < x$ and $g(y) \geq g(x)$. By the Mean-Value Theorem 3.18 there exists $z \in (y, x)$ such that

$$g(x) - g(y) \in \partial g(z)(x - y).$$

This means that there exists $\alpha \in \partial g(z)$ such that $\alpha > 0$ and

$$0 \geq g(x) - g(y) = \alpha(x - y) > 0,$$

which is impossible. Thus, g is strictly increasing.

Since g is strictly increasing we have $g(y) < g(x)$ if and only if $y < x$, where $x, y \in \mathbb{R}$. Thus, to prove f°-pseudoconvexity we need to show that $y < x$ implies $f^\circ(x; y - x) < 0$. Let $x, y \in \mathbb{R}$ be arbitrary such that $y < x$. By Theorem 3.4

$$f^\circ(x; y - x) = \max\{\alpha(y - x) \mid \alpha \in \partial f(x)\} < 0$$

which proves the f°-pseudoconvexity. □

Theorem 5.5 *Let $h: \mathbb{R}^n \to \mathbb{R}$ be f°-pseudoconvex and $g: \mathbb{R} \to \mathbb{R}$ be f°-pseudoconvex and strictly increasing. Then the composite function $f := g \circ h: \mathbb{R}^n \to \mathbb{R}$ is also f°-pseudoconvex.*

Proof According to Theorem 3.19 function f is locally Lipschitz continuous. Suppose now that $f(\boldsymbol{y}) < f(\boldsymbol{x})$. Then $g(h(\boldsymbol{y})) = f(\boldsymbol{y}) < f(\boldsymbol{x}) = g(h(\boldsymbol{x}))$ and since g is strictly increasing we have

$$h(\boldsymbol{y}) < h(\boldsymbol{x}). \tag{5.7}$$

From Theorems 3.4 and 3.19 we deduce that

$$\begin{aligned} f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) &= \max\{\boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{x})\} \\ &\leq \max\left\{\boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \operatorname{conv}\{\partial g(h(\boldsymbol{x}))\partial h(\boldsymbol{x})\}\right\}. \end{aligned} \tag{5.8}$$

Due to the definition of a convex hull the right hand side of (5.8) is equivalent to

$$\begin{aligned} &\max\Bigg\{\Big(\sum_{i=1}^m \lambda_i \alpha_i \boldsymbol{\xi}_i\Big)^T(\boldsymbol{y} - \boldsymbol{x}) \mid \\ &\qquad\qquad \alpha_i \in \partial g(h(\boldsymbol{x})),\ \boldsymbol{\xi}_i \in \partial h(\boldsymbol{x}),\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1\Bigg\} \\ &\leq \max\Bigg\{\Big(\sum_{i=1}^m \lambda_i \alpha_i\Big) \cdot \max_{\boldsymbol{\xi}_i \in \partial h(\boldsymbol{x})} \boldsymbol{\xi}_i^T(\boldsymbol{y} - \boldsymbol{x}) \mid \end{aligned}$$

$$\alpha_i \in \partial g(h(\boldsymbol{x})),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1\Big\}$$

$$= \max \Big\{ \Big(\sum_{i=1}^{m} \lambda_i \alpha_i\Big) h^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \mid \alpha_i \in \partial g(h(\boldsymbol{x})),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \Big\}.$$

Moreover, by Lemma 5.4 we have $\alpha_i > 0$ for all $i = 1, \ldots, m$ and thus

$$\sum_{i=1}^{m} \lambda_i \alpha_i > 0.$$

On the other hand, since h is f°-pseudoconvex, (5.7) implies that $h^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) < 0$. Then

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \leq \max \Big\{ \Big(\sum_{i=1}^{m} \lambda_i \alpha_i\Big) h^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \mid \alpha_i \in \partial g(h(\boldsymbol{x})),\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \Big\} < 0$$

and, thus, f is f°-pseudoconvex. □

Theorem 5.6 *Let $f_i \colon \mathbb{R}^n \to \mathbb{R}$ be f°-pseudoconvex for all $i = 1, \ldots, m$. Then the function*

$$f(\boldsymbol{x}) := \max \{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}$$

is also f°-pseudoconvex.

Proof According to Theorem 3.23 f is locally Lipschitz continuous. Suppose that $f(\boldsymbol{y}) < f(\boldsymbol{x})$ and define the index set

$$I(\boldsymbol{x}) := \{i \in \{1, \ldots, m\} \mid f_i(\boldsymbol{x}) = f(\boldsymbol{x})\}.$$

Then for all $i \in I(\boldsymbol{x})$ we have

$$f_i(\boldsymbol{y}) \leq f(\boldsymbol{y}) < f(\boldsymbol{x}) = f_i(\boldsymbol{x}). \tag{5.9}$$

From Theorems 3.4 and 3.23, the definition of a convex hull, f°-pseudoconvexity of f_i, and (5.9) we deduce that

$$\begin{aligned} f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) &= \max \{\boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{x})\} \\ &\leq \max \big\{\boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \operatorname{conv} \{\partial f_i(\boldsymbol{x})) \mid i \in I(\boldsymbol{x})\}\big\} \\ &= \max \Big\{ \Big(\sum_{i \in I(\boldsymbol{x})} \lambda_i \boldsymbol{\xi}_i\Big)^T (\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi}_i \in \partial f(\boldsymbol{x}), \lambda_i \geq 0,\ \sum_{i \in I(\boldsymbol{x})} \lambda_i = 1 \Big\} \end{aligned}$$

$$\le \max \Big\{ \sum_{i \in I(\boldsymbol{x})} \lambda_i \cdot \max_{\boldsymbol{\xi}_i \in \partial f_i(\boldsymbol{x})} \boldsymbol{\xi}_i^T (\boldsymbol{y} - \boldsymbol{x}) \mid \lambda_i \ge 0, \sum_{i \in I(\boldsymbol{x})} \lambda_i = 1 \Big\}$$

$$= \max \Big\{ \sum_{i \in I(\boldsymbol{x})} \lambda_i f_i^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \mid \lambda_i \ge 0, \sum_{i \in I(\boldsymbol{x})} \lambda_i = 1 \Big\} < 0.$$

Thus, f is f°-pseudoconvex. □

Due to the fact that the sum of f°-pseudoconvex functions is not necessarily f°-pseudoconvex we need the following new property.

Definition 5.4 The functions $f_i \colon \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ are said to be *additively strictly monotone*, if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda_i \ge 0$, $i = 1, \dots, m$

$$\sum_{i=1}^m \lambda_i f_i(\boldsymbol{y}) < \sum_{i=1}^m \lambda_i f_i(\boldsymbol{x}) \quad \text{implies} \quad f_i(\boldsymbol{y}) < f_i(\boldsymbol{x}).$$

Theorem 5.7 *Let $f_i \colon \mathbb{R}^n \to \mathbb{R}$ be f°-pseudoconvex and additively strictly monotone, and let $\lambda_i \ge 0$ for all $i = 1, \dots, m$. Then the function*

$$f(\boldsymbol{x}) := \sum_{i=1}^m \lambda_i f_i(\boldsymbol{x})$$

is f°-pseudoconvex.

Proof According to Theorem 3.16 f is locally Lipschitz continuous. Suppose that $f(\boldsymbol{y}) < f(\boldsymbol{x})$. Then the additive strict monotonicity implies that for all $i = 1, \dots, m$ we have

$$f_i(\boldsymbol{y}) < f_i(\boldsymbol{x}). \tag{5.10}$$

From Theorems 3.4 and 3.16, nonnegativity of λ_i, f°-pseudoconvexity of f_i, and (5.10) we deduce that

$$\begin{aligned}
f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) &= \max \{\boldsymbol{\xi}^T (\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{x})\} \\
&\le \max \{\boldsymbol{\xi}^T (\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi} \in \sum_{i=1}^m \lambda_i \partial f_i(\boldsymbol{x})\} \\
&= \max \Big\{ \Big(\sum_{i=1}^m \lambda_i \boldsymbol{\xi}_i \Big)^T (\boldsymbol{y} - \boldsymbol{x}) \mid \boldsymbol{\xi}_i \in \partial f_i(\boldsymbol{x}) \Big\} \\
&\le \sum_{i=1}^m \lambda_i \cdot \max_{\boldsymbol{\xi}_i \in \partial f_i(\boldsymbol{x})} \boldsymbol{\xi}_i^T (\boldsymbol{y} - \boldsymbol{x})
\end{aligned}$$

$$= \sum_{i=1}^{m} \lambda_i f_i^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) < 0.$$

Thus, f is f°-pseudoconvex. □

5.2 Generalized Quasiconvexity

The notion of quasiconvexity is the most widely used generalization of convexity, and, thus, there exist various equivalent definitions and characterizations. Next we recall the most commonly used definition of quasiconvexity.

Definition 5.5 A function $f: \mathbb{R}^n \to \mathbb{R}$ is *quasiconvex*, if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$

$$f(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \le \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}.$$

Remark 5.1 Lemma 5.2 implies that f°-pseudoconvex function is also quasiconvex.

Note, that unlike pseudoconvexity, the previous definition of quasiconvexity does not require differentiability or even continuity.

Next we will give a well-known important geometrical characterization to quasiconvexity, namely that the convexity of the level sets is equivalent to the quasiconvexity of the function. In Theorem 2.23 we proved, that the level sets of a convex function are convex, but the converse was not true as illustrated in Fig. 2.12.

Theorem 5.8 *A function* $f: \mathbb{R}^n \to \mathbb{R}$ *is quasiconvex, if and only if the level set* $\text{lev}_\alpha f$ *is a convex set for all* $\alpha \in \mathbb{R}$.

Proof Let f be quasiconvex, $\boldsymbol{x}, \boldsymbol{y} \in \text{lev}_\alpha f$, $\lambda \in [0, 1]$ and $\alpha \in \mathbb{R}$. Then

$$f(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \le \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\} \le \max\{\alpha, \alpha\} = \alpha,$$

thus $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in \text{lev}_\alpha f$.

On the other hand, let $\text{lev}_\alpha f$ be a convex set for all $\alpha \in \mathbb{R}$. By choosing $\beta := \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$ we have $\boldsymbol{x}, \boldsymbol{y} \in \text{lev}_\beta f$. The convexity of $\text{lev}_\beta f$ implies, that $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in \text{lev}_\beta f$ for all $\lambda \in [0, 1]$, in other words

$$f(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \le \beta = \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}.$$

□

We give also a useful result concerning a finite maximum of quasiconvex functions.

Theorem 5.9 *Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be quasiconvex at $\boldsymbol{x}$ for all $i = 1, \ldots, m$. Then the function*

$$f(\boldsymbol{x}) := \max\{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}$$

is also quasiconvex.

Proof Follows directly from the definition of quasiconvexity. □

Analogously to the Definition 5.2 we can define the corresponding generalized quasiconvexity concept.

Definition 5.6 A function $f : \mathbb{R}^n \to \mathbb{R}$ is f°*-quasiconvex*, if it is locally Lipschitz continuous and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f(\boldsymbol{y}) \leq f(\boldsymbol{x}) \qquad \text{implies} \qquad f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \leq 0.$$

Note that (3.9) implies that a convex function is always f°-quasiconvex.

Similarly to f°-pseudoconvexity, the reasoning chain may be converted.

Lemma 5.5 *A locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is f°-quasiconvex, if and only if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$*

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) > 0 \qquad \textit{implies} \qquad f(\boldsymbol{y}) > f(\boldsymbol{x}).$$

Proof Follows directly from the definition of f°-quasiconvexity. □

There is a way, similar to Definition 5.6, to express locally Lipschitz continuous and quasiconvex function.

Definition 5.7 A function $f : \mathbb{R}^n \to \mathbb{R}$ is *l-quasiconvex*, if it is locally Lipschitz continuous and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f(\boldsymbol{y}) < f(\boldsymbol{x}) \qquad \text{implies} \qquad f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) \leq 0.$$

Remark 5.2 Definitions 5.6 and 5.7 imply that an f°-quasiconvex function is always l-quasiconvex.

Next, we prove that l-quasiconvexity coincides with quasiconvexity in locally Lipschitz continuous case.

Theorem 5.10 *If a locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex then it is l-quasiconvex.*

Proof Let f be locally Lipschitz continuous and quasiconvex. Let $\boldsymbol{x}, \boldsymbol{z} \in \mathbb{R}^n$ be such that $f(\boldsymbol{z}) < f(\boldsymbol{x})$. Since local Lipschitz continuity implies continuity there exists $\varepsilon > 0$ such that

$$f(\boldsymbol{z}+\boldsymbol{d}) < f(\boldsymbol{x}+\boldsymbol{d}) \quad \text{for all } \boldsymbol{d} \in B(\boldsymbol{0};\varepsilon). \tag{5.11}$$

For generalized directional derivative $f^\circ(\boldsymbol{x};\boldsymbol{z}-\boldsymbol{x})$ we have

$$\begin{aligned}
f^\circ(\boldsymbol{x};\boldsymbol{z}-\boldsymbol{x}) &= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t(\boldsymbol{z}-\boldsymbol{x}))-f(\boldsymbol{y})}{t}\\
&= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f(\boldsymbol{y}+t(\boldsymbol{z}-\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{y}))-f(\boldsymbol{y})}{t}\\
&= \limsup_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ t\downarrow 0}} \frac{f((1-t)\boldsymbol{y}+t(\boldsymbol{z}+\boldsymbol{y}-\boldsymbol{x}))-f(\boldsymbol{y})}{t}.
\end{aligned}$$

When $t \in (0,1)$ and $\boldsymbol{y}-\boldsymbol{x} \in B(\boldsymbol{0};\varepsilon)$ the quasiconvexity of f and (5.11) implies

$$\begin{aligned}
&\frac{f((1-t)\boldsymbol{y}+t(\boldsymbol{z}+\boldsymbol{y}-\boldsymbol{x}))-f(\boldsymbol{y})}{t}\\
\le\ &\frac{\max\{f(\boldsymbol{y}), f(\boldsymbol{z}+\boldsymbol{y}-\boldsymbol{x})\}-f(\boldsymbol{y}-\boldsymbol{x}+\boldsymbol{x})}{t}\\
=\ &\frac{\max\{0, f(\boldsymbol{z}+\boldsymbol{y}-\boldsymbol{x})-f(\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{x})\}}{t} = 0
\end{aligned}$$

Passing to the limit $t \to 0$ and $\boldsymbol{y} \to \boldsymbol{x}$ we get $f^\circ(\boldsymbol{x};\boldsymbol{z}-\boldsymbol{x}) \le 0$. Thus, f is l-quasiconvex. □

Before we can proof the converse, we need the following lemma.

Lemma 5.6 *If the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous on $[\boldsymbol{x},\boldsymbol{y}]$, where $f(\boldsymbol{x}) < f(\boldsymbol{y})$. Then, there exists a point $\boldsymbol{z} \in (\boldsymbol{x},\boldsymbol{y})$ such that*

$$f(\boldsymbol{z}) > f(\boldsymbol{x}) \quad \textit{and} \quad f^\circ(\boldsymbol{z};\boldsymbol{y}-\boldsymbol{x}) > 0.$$

Proof Consider the set $S := \operatorname{lev}_{f(\boldsymbol{x})} f \cap [\boldsymbol{x},\boldsymbol{y}]$. Since level sets of a continuous function are closed sets (Exercise 2.29) and $[\boldsymbol{x},\boldsymbol{y}]$ is compact, S is a compact set. It is also nonempty, because $\boldsymbol{x} \in S$. Since function $g(\boldsymbol{w}) := \|\boldsymbol{w}-\boldsymbol{y}\|$ is continuous, it attains its minimum over the set S according to the Weierstrass Theorem 1.1. Let this minimum point be $\boldsymbol{v}$. Then $\boldsymbol{v}$ is the nearest point to $\boldsymbol{y}$ on the set S and the continuity of function f implies $f(\boldsymbol{v}) = f(\boldsymbol{x})$. Also, $\boldsymbol{v} \neq \boldsymbol{y}$ since $f(\boldsymbol{x}) < f(\boldsymbol{y})$. The Mean-Value Theorem 3.18 implies that there exist $\boldsymbol{z} \in (\boldsymbol{v},\boldsymbol{y})$ and $\boldsymbol{\xi} \in \partial f(\boldsymbol{z})$ such that

$$f(\boldsymbol{y}) - f(\boldsymbol{v}) = \boldsymbol{\xi}^T(\boldsymbol{y}-\boldsymbol{v}).$$

Since $\|\boldsymbol{y}-\boldsymbol{v}\| \le \|\boldsymbol{y}-\boldsymbol{x}\|$ and $\boldsymbol{v} \in [\boldsymbol{x},\boldsymbol{y})$, there exists $\lambda \in (0,1]$ such that $\boldsymbol{y}-\boldsymbol{v} = \lambda(\boldsymbol{y}-\boldsymbol{x})$. Then by the fact $f(\boldsymbol{v}) < f(\boldsymbol{y})$ and Theorem 3.1 we have

$$0 < f(\boldsymbol{y}) - f(\boldsymbol{v}) = \boldsymbol{\xi}^T(\boldsymbol{y} - \boldsymbol{v}) \leq f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{v})$$
$$= f^\circ(\boldsymbol{z}; \lambda(\boldsymbol{y} - \boldsymbol{x})) = \lambda \cdot f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{x}) \leq f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{x}).$$

Now $\boldsymbol{z} \notin \text{lev}_{f(\boldsymbol{x})} f$, because otherwise we would have $\boldsymbol{z} \in S$ and

$$g(\boldsymbol{z}) = \|\boldsymbol{z} - \boldsymbol{y}\| < \|\boldsymbol{v} - \boldsymbol{y}\| = g(\boldsymbol{v}),$$

which is impossible, since $\boldsymbol{v}$ was the minimum of g over S. Thus we have $f(\boldsymbol{z}) > f(\boldsymbol{x})$ and the lemma has been proven. □

Now we can prove, that l-quasiconvexity implies also quasiconvexity.

Theorem 5.11 *If function $f : \mathbb{R}^n \to \mathbb{R}$ is l-quasiconvex then it is quasiconvex.*

Proof On the contrary assume that an l-quasiconvex function f is not quasiconvex. Then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\mu \in (0, 1)$ such that $f(\boldsymbol{v}) > \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$, where $\boldsymbol{v} = \mu\boldsymbol{x} + (1 - \mu)\boldsymbol{y}$. Without a loss of generality we may assume that $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$. Lemma 5.6 implies that there exists $\boldsymbol{z} \in (\boldsymbol{x}, \boldsymbol{v})$, for which

$$f(\boldsymbol{z}) > f(\boldsymbol{x}) \quad \text{and} \quad f^\circ(\boldsymbol{z}; \boldsymbol{v} - \boldsymbol{x}) > 0.$$

Denote $\boldsymbol{z} = \lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$, where $\lambda \in (\mu, 1)$. From the definitions of points $\boldsymbol{v}$ and $\boldsymbol{z}$ we see that

$$\boldsymbol{v} - \boldsymbol{x} = (1 - \mu)(\boldsymbol{y} - \boldsymbol{x}) \quad \text{and} \quad \boldsymbol{y} - \boldsymbol{z} = \lambda(\boldsymbol{y} - \boldsymbol{x}).$$

Thus,

$$\boldsymbol{v} - \boldsymbol{x} = \frac{1 - \mu}{\lambda}(\boldsymbol{y} - \boldsymbol{z})$$

and

$$0 < f^\circ(\boldsymbol{z}; \boldsymbol{v} - \boldsymbol{x}) = \frac{1 - \mu}{\lambda} f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{z}).$$

Thus, $0 < f^\circ(\boldsymbol{z}; \boldsymbol{y} - \boldsymbol{z})$ and $f(\boldsymbol{z}) > f(\boldsymbol{x}) \geq f(\boldsymbol{y})$ which contradicts the l-quasiconvexity of function f. Hence, f is quasiconvex. □

Corollary 5.1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and quasiconvex if and only if it is l-quasiconvex.*

Proof The result follows directly from Theorems 5.10 and 5.11. □

Corollary 5.2 *If $f : \mathbb{R}^n \to \mathbb{R}$ is f°-quasiconvex, then f is quasiconvex.*

Proof The result follows from Remark 5.2 and Theorem 5.11. □

The converse of this corollary is not necessarily true as the next example shows.

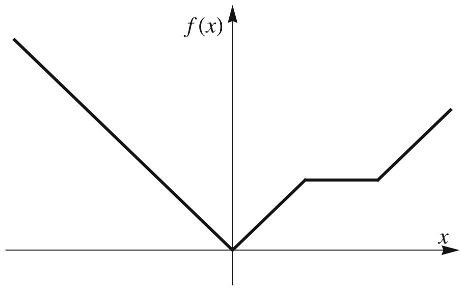


Fig. 5.4 Quasiconvex but not f°-quasiconvex function

Example 5.3 (*Quasiconvex but not f°-quasiconvex function*). Define $f\colon \mathbb{R} \to \mathbb{R}$ such that (see Fig. 5.4)

$$f(x) := \begin{cases} |x|, & x \in (-\infty, 1) \\ 1, & x \in [1, 2] \\ x - 1, & x \in (2, \infty). \end{cases}$$

Then f is clearly locally Lipschitz continuous and quasiconvex. However, by taking $x := 1$ and $y := 2$ we have $f^\circ(x; y - x) = f^\circ(1; 1) = 1 > 0$, but $f(y) = f(2) = 1 \not> 1 = f(1) = f(x)$ and thus, due to Lemma 5.5, f is not f°-quasiconvex.

Furthermore, f is not f°-pseudoconvex, since $0 \in \partial f(1) = [0, 1]$ although $x = 1$ is not a global minimum (cf. Theorem 5.2).

Likewise the pseudomonotonicity there also exists a concept of quasimonotonicity (see [138]).

Definition 5.8 The generalized directional derivative f° is called *quasimonotone*, if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) > 0 \qquad \text{implies} \qquad f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y}) \leq 0$$

or, equivalently

$$\min\{f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}), f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y})\} \leq 0.$$

Note that we could define the strict quasimonotonicity analogously to the pseudomonotonicity (see Definition 5.3), but it would be equivalent to the pseudomonotonicity.

It turns out that the generalized directional derivative f° of the function $f: \mathbb{R}^n \to \mathbb{R}$ is quasimonotone if and only if the function is locally Lipschitz continuous and quasiconvex.

Theorem 5.12 *If $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous such that f° is quasimonotone, then f is quasiconvex.*

Proof Let us, on the contrary assume, that f is not quasiconvex. Then there exist $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and $\lambda \in (0, 1)$ such that

$$f(\boldsymbol{z}) > f(\boldsymbol{x}) \geq f(\boldsymbol{y}),$$

where $\boldsymbol{z} = \boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x})$. Then by the Mean-Value Theorem 3.18 there exist $\boldsymbol{v} \in (\boldsymbol{z}, \boldsymbol{y})$ and $\boldsymbol{w} \in (\boldsymbol{x}, \boldsymbol{z})$ such that

$$f(\boldsymbol{z}) - f(\boldsymbol{y}) \in \partial f(\boldsymbol{v})^T(\boldsymbol{z} - \boldsymbol{y})$$

and

$$f(\boldsymbol{z}) - f(\boldsymbol{x}) \in \partial f(\boldsymbol{w})^T(\boldsymbol{z} - \boldsymbol{x}),$$

where

$$\boldsymbol{v} = \boldsymbol{x} + \mu(\boldsymbol{y} - \boldsymbol{x}), \quad \boldsymbol{w} = \boldsymbol{x} + \nu(\boldsymbol{y} - \boldsymbol{x}), \quad 0 < \nu < \lambda < \mu < 1.$$

This means that, due to the definition of the Clarke subdifferential, there exist $\boldsymbol{\xi}_v \in \partial f(\boldsymbol{v})$ and $\boldsymbol{\xi}_w \in \partial f(\boldsymbol{w})$ such that

$$0 < f(\boldsymbol{z}) - f(\boldsymbol{y}) = \boldsymbol{\xi}_v^T(\boldsymbol{z} - \boldsymbol{y}) \leq f^\circ(\boldsymbol{v}; \boldsymbol{z} - \boldsymbol{y}) = (1 - \lambda) f^\circ(\boldsymbol{v}; \boldsymbol{x} - \boldsymbol{y})$$

and

$$0 < f(\boldsymbol{z}) - f(\boldsymbol{x}) = \boldsymbol{\xi}_w^T(\boldsymbol{z} - \boldsymbol{x}) \leq f^\circ(\boldsymbol{w}; \boldsymbol{z} - \boldsymbol{x}) = \lambda f^\circ(\boldsymbol{w}; \boldsymbol{y} - \boldsymbol{x})$$

by the positive homogeneity of $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ (see Theorem 3.1). Then we deduce that

$$f^\circ(\boldsymbol{v}; \boldsymbol{w} - \boldsymbol{v}) = (\mu - \nu) f^\circ(\boldsymbol{v}; \boldsymbol{x} - \boldsymbol{y}) > 0$$

and

$$f^\circ(\boldsymbol{w}; \boldsymbol{v} - \boldsymbol{w}) = (\mu - \nu) f^\circ(\boldsymbol{w}; \boldsymbol{y} - \boldsymbol{x}) > 0,$$

which contradicts the quasimonotonicity. Thus, f is quasiconvex. □

Theorem 5.13 *If function $f: \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and quasiconvex then the generalized directional derivative f° is quasimonotone.*

Proof On the contrary, assume that f° is not quasimonotone. Then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) > 0$ and $f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y}) > 0$. Let

$$\delta := \min \left\{ f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}),\ f^\circ(\boldsymbol{y}; \boldsymbol{x} - \boldsymbol{y}) \right\} > 0.$$

Let $\varepsilon_1 > 0$ be such that the local Lipschitz condition holds in the ball $B(\boldsymbol{x}; \varepsilon_1)$ with Lipschitz constant $K_1 > 0$. Correspondingly, let $\varepsilon_2 > 0$ be such that the local Lipschitz condition holds in the ball $B(\boldsymbol{y}; \varepsilon_2)$ with Lipschitz constant $K_2 > 0$. Let $K := \max\{K_1, K_2\}$ and $\varepsilon := \min\left\{\frac{\delta}{4K}, \varepsilon_1, \varepsilon_2\right\}$. According to Lemma 1.1 there exists a sequence $(\boldsymbol{z}_1^i)$, such that f is differentiable, $\lim_{i\to\infty} \boldsymbol{z}_1^i = \boldsymbol{x}$ and an index $I \in \mathbb{N}$ such that

$$f'(\boldsymbol{z}_1^i; \boldsymbol{y} - \boldsymbol{x}) = \nabla f(\boldsymbol{z}_1^i)^T(\boldsymbol{y} - \boldsymbol{x}) \geq \frac{\delta}{2}$$

when $i \geq I$. Similarly, there exists a sequence $(\boldsymbol{z}_2^j)$, such that f is differentiable, $\lim_{j\to\infty} \boldsymbol{z}_2^j = \boldsymbol{y}$ and an index $J \in \mathbb{N}$ such that

$$f'(\boldsymbol{z}_2^j; \boldsymbol{x} - \boldsymbol{y}) = \nabla f(\boldsymbol{z}_2^j)^T(\boldsymbol{x} - \boldsymbol{y}) \geq \frac{\delta}{2}$$

when $j \geq J$. Let

$$\boldsymbol{z}_1 \in B(\boldsymbol{x}; \varepsilon) \cap \{(\boldsymbol{z}_1^i) \mid i \geq I\} \quad \text{and}$$
$$\boldsymbol{z}_2 \in B(\boldsymbol{y}; \varepsilon) \cap \{(\boldsymbol{z}_2^j) \mid j \geq J\}.$$

Due to symmetry we may assume that $f(\boldsymbol{z}_1) \geq f(\boldsymbol{z}_2)$ without a loss of generality. According to Lemma 3.2

$$\begin{aligned}\left|f'(\boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) - f'(\boldsymbol{z}_1; \boldsymbol{y} - \boldsymbol{x})\right| &\leq K\,\|\boldsymbol{z}_2 - \boldsymbol{z}_1 - (\boldsymbol{y} - \boldsymbol{x})\| \\ &\leq K\,\|\boldsymbol{x} - \boldsymbol{z}_1\| + K\,\|\boldsymbol{z}_2 - \boldsymbol{y}\| < 2K\frac{\delta}{4K} = \frac{\delta}{2}.\end{aligned}$$

Since $f'(\boldsymbol{z}_1; \boldsymbol{y} - \boldsymbol{x}) > \frac{\delta}{2}$ also $f'(\boldsymbol{z}_1; \boldsymbol{z}_2 - \boldsymbol{z}_1) > 0$. Thus, there exists $\lambda \in (0, 1)$ such that

$$f(\boldsymbol{z}_1 + \lambda(\boldsymbol{z}_2 - \boldsymbol{z}_1)) > f(\boldsymbol{z}_1) \geq f(\boldsymbol{z}_2),$$

which contradicts the quasiconvexity. □

It follows from the previous two theorems, that l-quasiconvexity of the function is equivalent to quasimonotonicity of the generalized directional derivative.

Corollary 5.3 *A function $f: \mathbb{R}^n \to \mathbb{R}$ is l-quasiconvex if and only if f° is quasimonotone.*

Proof The result follows from Corollary 5.1 and Theorems 5.12 and 5.13. □

Corollary 5.4 *If the function $f: \mathbb{R}^n \to \mathbb{R}$ is f°-quasiconvex, then f° is quasimonotone.*

Proof The results follows from Remark 5.2 and Corollary 5.3. □

By Corollary 5.2 f°-quasiconvex function is quasiconvex. The next result shows, that for a subdifferentially regular function quasiconvexity and f°-quasiconvexity coincides.

Theorem 5.14 *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is both quasiconvex and subdifferentially regular, then f is f°-quasiconvex.*

Proof Due to the subdifferential regularity f is locally Lipschitz continuous. Suppose, that $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$. Then the subdifferential regularity and quasiconvexity implies, that

$$
\begin{aligned}
f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) = f'(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) &= \lim_{t \downarrow 0} \frac{f(\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x})) - f(\boldsymbol{x})}{t} \\
&= \lim_{t \downarrow 0} \frac{f(t\boldsymbol{y} + (1-t)\boldsymbol{x}) - f(\boldsymbol{x})}{t} \leq \lim_{t \downarrow 0} \frac{f(\boldsymbol{x}) - f(\boldsymbol{x})}{t} = 0
\end{aligned}
$$

in other words, f is f°-quasiconvex. □

Corollary 5.5 *A subdifferentially regular l-quasiconvex function $f\colon \mathbb{R}^n \to \mathbb{R}$ is f°-quasiconvex.*

Proof The result follows from Corollary 5.1 and Theorem 5.14. □

Corollary 5.6 *A subdifferentially regular function $f\colon \mathbb{R}^n \to \mathbb{R}$ with quasimonotone f° is f°-quasiconvex.*

Proof The result follows from Corollaries 5.3 and 5.5. □

In Theorem 5.14 the subdifferential regularity cannot be omitted, as the next example shows.

Example 5.4 (*Not subdifferentially regular quasiconvex function*). Consider again the function in Example 5.3 (see also Fig. 5.4). As noted in Example 5.3 f is locally Lipschitz continuous and quasiconvex but not f°-quasiconvex. Note that f is not subdifferentially regular since $f'(1;1) = 0 \neq 1 = f^\circ(1;1)$.

As stated in Corollary 5.5 the subdifferential regularity ensures that the l-quasiconvexity implies f°-quasiconvexity. In sequel we show that also the following nonconstancy property has the similar consequence.

Definition 5.9 The function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the *nonconstancy property* (in short, NC-property), if there exists no line segment $[\boldsymbol{x}, \boldsymbol{y}]$ along which f is constant.

Note that the subdifferential regularity (see Definition 3.5) and the NC-property are two separate concepts.

Example 5.5 (*Subdifferential regularity and NC-property*). An example of function which is subdifferentially regular but does not satisfy the NC-property is (see Fig. 5.5)

$$f_1(x) = \begin{cases} (x+1)^2 & , \text{ if } x \le -1 \\ 0 & , \text{ if } -1 \le x \le 1 \\ (x-1)^2 & , \text{ if } x \ge 1. \end{cases}$$

On the other hand, the function (see Fig. 5.6)

$$f_2(x) = \begin{cases} 2x & , \text{ if } x \le 0 \\ \frac{1}{2}x & , \text{ if } x \ge 0 \end{cases}$$

poses the NC-property but it is not subdifferentially regular since $f_2^\circ(0; 1) = 2 \neq \frac{1}{2} = f_2'(0; 1)$.

For the function with NC-property also the quasimonotonicity and the f°-quasiconvexity coincides.

Theorem 5.15 *If the function $f \colon \mathbb{R}^n \to \mathbb{R}$ poses the NC-property and f° is quasimonotone, then f is f°-quasiconvex.*

Proof Let us, on the contrary, assume that f is not f°-quasiconvex. Then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $f(\boldsymbol{y}) \le f(\boldsymbol{x})$ and

$$f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) > 0. \tag{5.12}$$

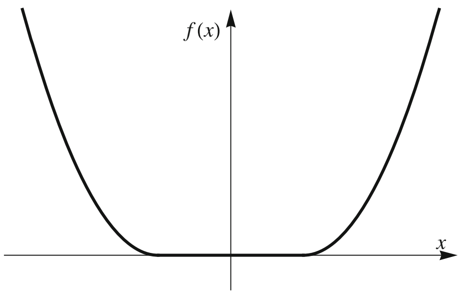


Fig. 5.5 Subdifferentially regular function not posing NC-property

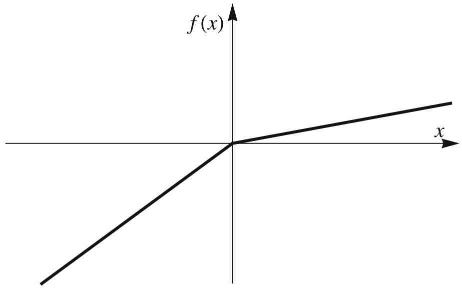


Fig. 5.6 Not subdifferentially regular function posing NC-property

According to Theorem 5.12 f is quasiconvex, which means that $f(\boldsymbol{z}) \leq f(\boldsymbol{x})$ for all $\boldsymbol{z} \in [\boldsymbol{y}, \boldsymbol{x}]$. Thus, due to the NC-property, there exists $\mu \in (0, 1]$ such that $\boldsymbol{v} = \boldsymbol{x} + \mu(\boldsymbol{y} - \boldsymbol{x})$ and $f(\boldsymbol{v}) < f(\boldsymbol{x})$. Furthermore, by the Mean-Value Theorem 3.18 there exists $\boldsymbol{w} \in (\boldsymbol{v}, \boldsymbol{x})$ such that

$$f(\boldsymbol{x}) - f(\boldsymbol{v}) \in \partial f(\boldsymbol{w})^T (\boldsymbol{x} - \boldsymbol{v}),$$

where $\boldsymbol{w} = \boldsymbol{x} + \nu(\boldsymbol{v} - \boldsymbol{x})$ and $\nu \in (0, 1)$. This means that there exists $\boldsymbol{\xi} \in \partial f(\boldsymbol{w})$ such that

$$0 < f(\boldsymbol{x}) - f(\boldsymbol{v}) = \boldsymbol{\xi}^T (\boldsymbol{x} - \boldsymbol{v}) \leq f^\circ(\boldsymbol{w}; \boldsymbol{x} - \boldsymbol{v}). \tag{5.13}$$

On the other hand, from the positive homogeneity of $\boldsymbol{d} \mapsto f^\circ(\boldsymbol{x}; \boldsymbol{d})$ (see Theorem 3.1) and (5.12) we deduce that

$$f^\circ(\boldsymbol{x}; \boldsymbol{w} - \boldsymbol{x}) = \nu f^\circ(\boldsymbol{x}; \boldsymbol{v} - \boldsymbol{x}) = \nu\mu f^\circ(\boldsymbol{x}; \boldsymbol{y} - \boldsymbol{x}) > 0.$$

Then the quasimonotonicity, the positive homogeneity imply that

$$0 \geq f^\circ(\boldsymbol{w}; \boldsymbol{x} - \boldsymbol{w}) = \nu f^\circ(\boldsymbol{w}; \boldsymbol{x} - \boldsymbol{v}) > 0,$$

which is impossible. Thus f is f°-quasiconvex. □

Corollary 5.7 *A l-quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$ with NC-property is f°-quasiconvex.*

Proof The result follows from Corollary 5.3 and Theorem 5.15. □

Corollary 5.8 *A locally Lipschitz continuous and quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$ with NC-property is f°-quasiconvex.*

Proof The result follows from Theorems 5.13 and 5.15. □

Example 5.6 (*NC-property and f°-quasiconvexity*). Consider the function in Example 5.3. Its generalized directional derivative is quasimonotone since the function is quasiconvex and locally Lipschitz continuous. However, the function does not satisfy the NC-property and, thus, it is not guaranteed to be f°-quasiconvex. As shown in Example 5.3 the function is not f°-quasiconvex.

The next results concerning the verification of the f°-quasiconvexity are analogous to those of f°-pseudoconvexity.

Lemma 5.7 *A locally Lipschitz continuous function $g\colon \mathbb{R} \to \mathbb{R}$ is increasing, if and only if $\varsigma \geq 0$ for all $\varsigma \in \partial g(x)$ and $x \in \mathbb{R}$.*

Proof Exercise. (Hint: The proof is similar to that of Lemma 5.4). □

Theorem 5.16 *Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be f°-quasiconvex and $g\colon \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous and increasing. Then the composite function $f := g \circ h\colon \mathbb{R}^n \to \mathbb{R}$ is also f°-quasiconvex.*

Proof Exercise. (Hint: The proof is similar to that of Theorem 5.5). □

Theorem 5.17 *Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be f°-quasiconvex for all $i = 1, \ldots, m$. Then the function*

$$f(\boldsymbol{x}) := \max \{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}$$

is also f°-quasiconvex.

Proof Exercise. (Hint: The proof is similar to that of Theorem 5.6). □

As in the case of f°-pseudoconvexity, the following property guarantees that the sum of f°-quasiconvex functions is also f°-quasiconvex.

Definition 5.10 The functions $f_i\colon \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ are said to be *additively monotone*, if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda_i \geq 0$, $i = 1, \ldots, m$

$$\sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{y}) \leq \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) \qquad \text{implies} \qquad f_i(\boldsymbol{y}) \leq f_i(\boldsymbol{x}).$$

Theorem 5.18 *Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be f°-quasiconvex and additively monotone, and $\lambda_i \geq 0$ for all $i = 1, \ldots, m$. Then the function*

$$f(\boldsymbol{x}) := \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x})$$

is f°-quasiconvex.

Proof Exercise. (Hint: The proof is similar to that of Theorem 5.7). □

Finally we study the relations between pseudo- and quasiconvexity. For differentiable functions pseudoconvexity implies quasiconvexity. Next we show that also f°-pseudoconvexity implies f°-quasiconvexity.

Theorem 5.19 *An f°-pseudoconvex function $f: \mathbb{R}^n \to \mathbb{R}$ is f°-quasiconvex.*

Proof On the contrary, assume that an f°-pseudoconvex function f is not f°-quasiconvex. Then, there exist points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $f^\circ(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}) > 0$ and $f(\boldsymbol{x}) = f(\boldsymbol{y})$. According to Lemma 5.3 this is impossible for f°-pseudoconvex function. Thus, f is f°-quasiconvex. □

Corollary 5.9 *If f° is pseudomonotone then it is also quasimonotone.*

Proof The result follows from Corollary 5.4 and Theorems 5.3 and 5.19. □

The next example shows that the result in Theorem 5.19 cannot be converted.

Example 5.7 (*f°-quasiconvex but not f°-pseudoconvex function*). Define $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) := x^3$ (see Fig. 5.7). Clearly f is quasiconvex and as a smooth function also subdifferentially regular. Thus, by Theorem 5.14 it is f°-quasiconvex. However, by taking $x := 0$ and $y := -1$ we have $f^\circ(x; y-x) = f^\circ(0; -1) = 0$, but $f(y) = f(-1) = -1 \ngeq 0 = f(0) = f(x)$ and thus, due to Lemma 5.1, f is not f°-pseudoconvex.

5.3 Relaxed Optimality Conditions

In this section we consider the sufficient optimality conditions of Chap. 4 in order to relax the convexity assumptions with generalized convexities.

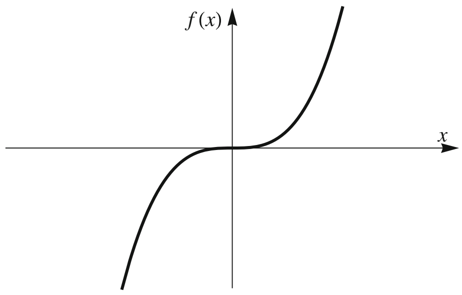


Fig. 5.7 f°-quasiconvex but not f°-pseudoconvex function

5.3.1 Unconstrained Optimization

Let us consider first the unconstrained version of the Problem (4.1). In Theorem 5.2 we already generalized the unconstrained sufficient condition 4.2 and proved that f°-pseudoconvex f attains its global minimum at $\boldsymbol{x}^*$, if and only if

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*).$$

Example 5.8 (Optimality conditions). Consider again Example 3.3, where we defined $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(\boldsymbol{x}) = \ln(\|\boldsymbol{x}\| + 2).$$

The plot of f can be seen in Fig. 5.8 in the case $n = 1$. We denoted $h(\boldsymbol{x}) = \|\boldsymbol{x}\| + 2$ and $g(x) = \ln x$ and proved that h is convex. Then by (3.9) h is f°-pseudoconvex. On the other hand, g is clearly continuously differentiable when $x \geq 2$ and due to Theorem 3.7 we have

$$\partial g(x) = \{g'(x)\} = \{\tfrac{1}{x}\}.$$

Since $\frac{1}{x} > 0$ for all $x \geq 2$ Lemma 5.4 implies that g is f°-pseudoconvex and strictly increasing. Then by Theorem 5.5 $f = g \circ h$ is also f°-pseudoconvex. Moreover, in Example 3.3 we calculated that

$$\partial f(\mathbf{0}) = \text{cl } B(\mathbf{0}; \tfrac{1}{2}).$$

Then we have $\mathbf{0} \in \partial f(\mathbf{0})$ and thus by Theorem 5.2 f attains its global minimum at $\boldsymbol{x}^* = \mathbf{0}$.

The convexity can be relaxed also in Corollary 4.1, in other words a f°-pseudoconvex function either attains its global minimum or we can find a descent direction for it.

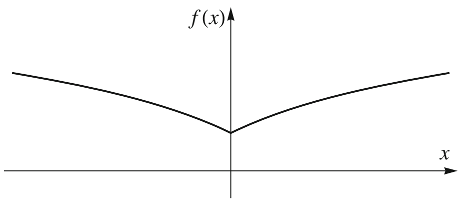


Fig. 5.8 A plot of function $f(x) = \ln(|x| + 2)$

Corollary 5.10 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a f°-pseudoconvex function. Then either f attains its global minimum at $\boldsymbol{x} \in \mathbb{R}^n$ or there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for f at $\boldsymbol{x}$.*

Proof Exercise. (Hint: Use Corollary 4.1 and Theorem 5.2.) □

Finally, we get the following relaxed geometrical optimality condition of Theorem 4.3.

Corollary 5.11 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x}^* \in \mathbb{R}^n$. If f attains its local minimum at $\boldsymbol{x}^*$, then*

$$D_S^\circ(\boldsymbol{x}^*) = D_S(\boldsymbol{x}^*) = \{\boldsymbol{0}\}. \tag{5.14}$$

If, in addition, f is f°-pseudoconvex, then the condition (5.14) is sufficient for $\boldsymbol{x}^*$ to be a global minimum of f.

Proof Exercise. (Hint: Use the Theorems 4.3 and 5.2, and Corollary 5.10 and the fact that $\boldsymbol{0} \in D_S^\circ(\boldsymbol{x}) \subseteq D_S(\boldsymbol{x})$.) □

5.3.2 Geometrical Constraints

Consider next the Problem (4.1) with a general feasible set $S \subset \mathbb{R}^n$. First we formulate a relaxed version of the geometrical optimality condition of Theorem 4.6.

Theorem 5.20 *Let $\boldsymbol{x}^*$ be a local optimum of the* Problem (4.1)*, where $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous at $\boldsymbol{x}^* \in S \neq \emptyset$. Then*

$$D_S^\circ(\boldsymbol{x}^*) \cap F_S(x^*) = \{\boldsymbol{0}\}. \tag{5.15}$$

If, in addition, f is f°-pseudoconvex and S is convex, then the condition (5.15) implies that $\boldsymbol{x}^*$ is a global optimum of the Problem (4.1).

Proof The necessity follows directly from Theorem 4.6. For sufficiency suppose that f is f°-pseudoconvex, S is convex and the condition (5.15) is valid. If $\boldsymbol{x}^*$ is not a global optimum there exist $\boldsymbol{y} \in S$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x}^*)$. Then $\boldsymbol{d} := \boldsymbol{y} - \boldsymbol{x}^* \neq \boldsymbol{0}$ and since S is convex we have

$$\boldsymbol{x}^* + t\boldsymbol{d} = \boldsymbol{x}^* + t(\boldsymbol{y} - \boldsymbol{x}^*) = t\boldsymbol{y} + (1-t)\boldsymbol{x}^* \in S,$$

whenever $t \in (0, 1]$, thus $\boldsymbol{d} \in F_S(\boldsymbol{x}^*)$. On the other hand, due to the definition of f°-pseudoconvexity and the subdifferential we have

$$0 > f^\circ(\boldsymbol{x}^*; \boldsymbol{d}) \geq \boldsymbol{\xi}^T\boldsymbol{d} \quad \text{for all } \boldsymbol{\xi} \in \partial f(\boldsymbol{x}^*),$$

in other words $\boldsymbol{d} \in D_S^\circ(\boldsymbol{x}^*)$. Thus we have found $\boldsymbol{d} \neq \mathbf{0}$ such that $\boldsymbol{d} \in D_S^\circ(\boldsymbol{x}^*) \cap F_S(\boldsymbol{x}^*)$. This contradicts the condition (5.15), meaning that $\boldsymbol{x}^*$ must be a global optimum of the Problem (4.1). □

To the end of this subsection we relax the necessary and sufficient mixed-analytical-geometrical optimality condition of Theorem 4.8.

Theorem 5.21 *If* $f: \mathbb{R}^n \to \mathbb{R}$ *is* f°*-pseudoconvex and* S *is convex, then* $\boldsymbol{x}^* \in S$ *is a global optimum of the* Problem (4.1) *if and only if*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$

Proof The necessity part follows again from Theorem 4.7. For sufficiency suppose, that

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + N_S(\boldsymbol{x}^*).$$

This means, that there exist $\boldsymbol{\xi} \in \partial f(\boldsymbol{x}^*)$ and $\boldsymbol{z} \in N_S(\boldsymbol{x}^*)$ such that $\boldsymbol{\xi} = -\boldsymbol{z}$. Since S is convex and $\boldsymbol{z} \in N_S(\boldsymbol{x}^*)$ Theorem 2.19 implies that

$$\boldsymbol{z}^T(\boldsymbol{x} - \boldsymbol{x}^*) \leq 0 \quad \text{for all } \boldsymbol{x} \in S,$$

and due to the definition of the subdifferential for all $\boldsymbol{x} \in S$ we have

$$0 \leq -\boldsymbol{z}^T(\boldsymbol{x} - \boldsymbol{x}^*) = \boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{x}^*) \leq f^\circ(\boldsymbol{x}^*; \boldsymbol{x} - \boldsymbol{x}^*).$$

Then by Lemma 5.1 the f°-pseudoconvexity of f implies that

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*).$$

In other words, $\boldsymbol{x}^*$ is a global optimum of the Problem (4.1). □

5.3.3 *Analytical Constraints*

Finally we consider the Problem (4.11) including inequality constraints and generalize the necessary and sufficient KKT optimality conditions of Corollary 4.4 utilizing generalized convexities. Recall, that the total constraint function $g: \mathbb{R}^n \to \mathbb{R}$ is defined by

$$g(\boldsymbol{x}) := \max\{g_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}.$$

Then the Problem (4.11) is a special case of (4.1) with

$$S = \{\boldsymbol{x} \in X \mid g(\boldsymbol{x}) \leq 0\} = \operatorname{lev}_0 g.$$

Theorem 5.22 *(KKT Relaxed Necessary and Sufficient Conditions) Suppose, that the Problem (4.11) satisfies the Cottle constraint qualification at $\boldsymbol{x}^* \in S$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be f°-pseudoconvex and $g_i: \mathbb{R}^n \to \mathbb{R}$ be quasiconvex and subdifferentially regular at $\boldsymbol{x}^*$ for all $i = 1, \ldots, m$. Then $\boldsymbol{x}^*$ is a global optimum of the Problem (4.11) if and only if there exist multipliers $\lambda_i \geq 0$ for all $i = 1, \ldots, m$ such that $\lambda_i g_i(\boldsymbol{x}^*) = 0$ and*

$$\mathbf{0} \in \partial f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(\boldsymbol{x}^*). \tag{5.16}$$

Proof Since the functions g_i for $i = 1, \ldots, m$ are subdifferentially regular, they are locally Lipschitz continuous at $\boldsymbol{x}^*$. Then the necessity follows directly from Theorem 4.11. For the sufficiency suppose that (5.16) is valid and denote

$$\lambda := \sum_{i=1}^{m} \lambda_i \geq 0.$$

Assume first, that $\lambda > 0$. Then due to Theorem 3.23 g is subdifferentially regular at $\boldsymbol{x}^*$ and (5.16) implies, that there exists $\boldsymbol{\xi} \in \partial f(\boldsymbol{x}^*)$ such that

$$-\frac{1}{\lambda}\boldsymbol{\xi} \in \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \partial g_i(\boldsymbol{x}^*) \subseteq \text{conv}\, \{\partial g_i(\boldsymbol{x}^*) \mid i \in I(\boldsymbol{x}^*)\} = \partial g(\boldsymbol{x}^*),$$

where

$$I(\boldsymbol{x}^*) := \{i \in \{1, \ldots, m\} \mid g_i(\boldsymbol{x}^*) = g(\boldsymbol{x}^*)\}.$$

Since $\lambda > 0$, we have $\lambda_i > 0$ at least for one $i \in \{1, \ldots, m\}$. Since $\lambda_i g_i(\boldsymbol{x}^*) = 0$ we deduce that $g_i(\boldsymbol{x}^*) = 0$ and thus we have

$$0 = g_i(\boldsymbol{x}^*) \leq g(\boldsymbol{x}^*) \leq 0,$$

in other words $g(\boldsymbol{x}^*) = 0$. This implies due to the Cottle constraint qualification that $\mathbf{0} \notin \partial g(\boldsymbol{x}^*)$. Then by Theorem 3.34

$$\partial g(\boldsymbol{x}^*) \subseteq \text{ray}\ \partial g(\boldsymbol{x}^*) = N_{\text{lev}_{g(\boldsymbol{x}^*)}\, g}(\boldsymbol{x}^*) = N_S(\boldsymbol{x}^*),$$

and thus

$$-\frac{1}{\lambda}\boldsymbol{\xi} \in N_S(\boldsymbol{x}^*).$$

Because $N_S(\boldsymbol{x}^*)$ is a cone and $\lambda > 0$ we deduce that

$$-\,\boldsymbol{\xi} \in N_S(\boldsymbol{x}^*). \tag{5.17}$$

Furthermore, by Theorem 5.9 g is quasiconvex and thus, by Theorem 5.8 $S = \mathrm{lev}_{g(\boldsymbol{x}^*)}\, g$ is a convex set. Then (5.17), Theorem 2.19 and the definition of the subdifferential imply that for all $\boldsymbol{x} \in S$ we have

$$0 \leq \boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{x}^*) = f^\circ(\boldsymbol{x}^*; \boldsymbol{x} - \boldsymbol{x}^*).$$

Then by Lemma 5.1 the f°-pseudoconvexity of f implies that

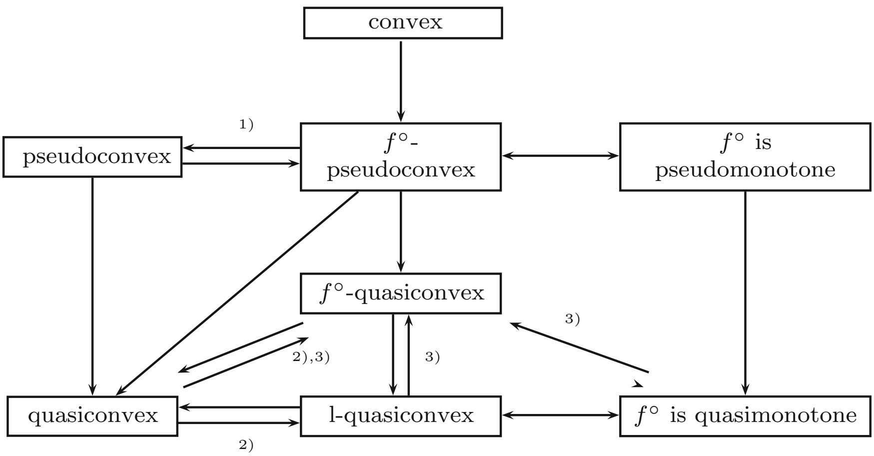
$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*),$$

in other words, $\boldsymbol{x}^*$ is a global optimum of the Problem (4.1).

On the other hand, if $\lambda = 0$ we have $\boldsymbol{\xi} \in \partial f(\boldsymbol{x}^*)$ and due to Theorem 5.2 f°-pseudoconvex f attains its global minimum at $\boldsymbol{x}^*$. Since $\boldsymbol{x}^* \in S$ it is also a global optimum of the Problem (4.1). □

5.4 Summary

To the end of this chapter we summarize all the relationships presented above:



(1) demands continuous differentiability,

(2) demands local Lipschitz continuity,

(3) demands NC-property or subdifferential regularity.

Furthermore, we have formulated the relaxed versions of the sufficient optimality conditions replacing the convexities by generalized convexities in unconstrained case (Theorem 5.2), with general constraints (Theorems 5.20 and 5.21) and with inequality constraints (Theorem 5.22).

Exercises

5.1 Show that the sum of f°-pseudoconvex functions is not necessarily f°-pseudoconvex.

5.2 (Lemma 5.7) Prove that a locally Lipschitz continuous function $g\colon \mathbb{R} \to \mathbb{R}$ is increasing, if and only if $\varsigma \geq 0$ for all $\varsigma \in \partial g(x)$ and $x \in \mathbb{R}$.

5.3 (Theorem 5.16) Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be f°-quasiconvex and $g\colon \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous and increasing. Prove that the composite function $f := g \circ h\colon \mathbb{R}^n \to \mathbb{R}$ is also f°-quasiconvex.

5.4 (Theorem 5.17) Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be f°-quasiconvex for all $i = 1, \ldots, m$. Prove that the function

$$f(\boldsymbol{x}) := \max\{f_i(\boldsymbol{x}) \mid i = 1, \ldots, m\}$$

is also f°-quasiconvex.

5.5 Show that the sum of f°-quasiconvex functions is not necessarily f°-quasiconvex.

5.6 (Theorem 5.18) Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be f°-quasiconvex and additively monotone, and $\lambda_i \geq 0$ for all $i = 1, \ldots, m$. Prove that the function

$$f(\boldsymbol{x}) := \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x})$$

is f°-quasiconvex.

5.7 Classify the following functions according to their degree of convexity.

(a) $f(x) = x + 2014$
(b) $f(x) = 2x^2 + 3x - 1$
(c) $f(x) = e^x$
(d) $f(x) = \frac{1}{3}x^3$
(e) $f(x) = \frac{1}{3}x^3 + x$
(f) $f(x) = \ln x,\ x > 0$
(g) $f(x) = |x| + e^{x^2} + x^2$
(h) $f(x) = \sin x$
(i) $f(x) = \sin^2 x + \cos^2 x$

(j)

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{3}x^3, & x > 0. \end{cases}$$

(k)

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0. \end{cases}$$

5.8 (Corollary 5.10) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a f°-pseudoconvex function. Prove that either f attains its global minimum at $\boldsymbol{x} \in \mathbb{R}^n$ or there exists a descent direction $\boldsymbol{d} \in \mathbb{R}^n$ for f at $\boldsymbol{x}$.

5.9 (Corollary 5.11) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function at $\boldsymbol{x}^* \in \mathbb{R}^n$. Prove that if f attains its local minimum at $\boldsymbol{x}^*$, then

$$D_S^\circ(\boldsymbol{x}^*) = D_S(\boldsymbol{x}^*) = \{\mathbf{0}\}. \tag{5.18}$$

If, in addition, f is f°-pseudoconvex, prove that the condition (5.18) is sufficient for $\boldsymbol{x}^*$ to be a global minimum of f.

5.10 Solve the problem

$$\begin{cases} \text{minimize} & f(x_1, x_2) = |x_1| + x_2^3 \\ \text{subject to} & |x_1 + x_2| \geq 1 \\ & \max\{x_1, x_2\} \geq 0. \end{cases}$$

5.11 Solve the problem

$$\begin{cases} \text{minimize} & f(x_1, x_2) = \ln\left(\sqrt{x_1^2 + x_2^2} + 1\right) \\ \text{subject to} & \max\{-x_1 - x_2 + 1, -x_2 + \frac{1}{2}\} \leq 0. \end{cases}$$

Chapter 6
Approximations of Subdifferentials

In practice, the computation of subdifferential is not an easy task. In this chapter, we first consider some families of set-valued mappings that can be used to approximate subdifferentials. Then we define the concept of a discrete gradient that can be used as an approximation of the subgradient at a given point. We demonstrate how discrete gradients can be used to compute subsets of continuous approximations. From a practical point of view, discrete gradients are useful, since only values of a function are used to compute discrete gradients and no subderivative information is needed. In Chap. 15 of Part III, two NSO algorithms using discrete gradients are introduced. At the end of this chapter, we introduce the notion of piecewise partially separable functions and study their properties. In particular, we describe how to calculate the discrete gradient for a piecewise partially separable function.

6.1 Continuous Approximations of Subdifferential

In this section, we consider some families of set-valued mappings that can be used to approximate subdifferentials. Namely, the *continuous approximations of subdifferentials*, the *uniform continuous approximations of subdifferentials*, and the *strong continuous approximations of subdifferentials*. We will study their basic properties and present the connections between these different approximations.

Let X be a compact subset of the space $\mathbb{R}^n$. We consider a family $C(\boldsymbol{x}, \varepsilon) = C_\varepsilon(\boldsymbol{x})$ of set-valued mappings depending on a parameter $\varepsilon > 0$, that is,

$$C(\cdot, \varepsilon) : X \to 2^{\mathbb{R}^n} \text{ for each } \varepsilon > 0.$$

We suppose that $C(\boldsymbol{x}, \varepsilon)$ is a compact convex subset for all $\boldsymbol{x} \in X$ and $\varepsilon > 0$. It is assumed that there exists a number $K > 0$ such that:

$$\sup \{\|\boldsymbol{v}\| \mid \boldsymbol{v} \in C(\boldsymbol{x}, \varepsilon), \boldsymbol{x} \in X, \varepsilon > 0\} \le K. \tag{6.1}$$

Definition 6.1 The *limit of the family* $C(\boldsymbol{x}, \varepsilon)$ at a point $\boldsymbol{x} \in \mathbb{R}^n$ is defined by the set:

A. Bagirov et al., *Introduction to Nonsmooth Optimization*,
DOI: 10.1007/978-3-319-08114-4_6

$$C_L(\boldsymbol{x}) = \Big\{ \boldsymbol{v} \in \mathbb{R}^n \mid \exists\, \boldsymbol{x}_k \to \boldsymbol{x}, \varepsilon_k \downarrow 0, k \to \infty, \boldsymbol{v}_k \in C(\boldsymbol{x}_k, \varepsilon_k) \text{ such that } \boldsymbol{v} = \lim_{k\to\infty} \boldsymbol{v}_k \Big\}.$$

It is possible that the limit $C_L(\boldsymbol{x})$ is not convex even if all the sets $C(\boldsymbol{x}, \varepsilon)$ are convex. Thus, we consider conv $C_L(\boldsymbol{x})$, the convex hull of the limit $C_L(\boldsymbol{x})$. It follows from the Definition 6.1 and the inequality (6.1) that the mapping conv C_L has compact convex images.

Let f be a locally Lipschitz continuous function defined on an open set $X^0 \subseteq \mathbb{R}^n$ which contains a compact set X. This function is Clarke subdifferentiable on X^0 (see Definition 5.2). We now define the continuous approximation of the subdifferential.

Definition 6.2 A family $Cf(\boldsymbol{x}, \varepsilon)$ is called a *continuous approximation of the subdifferential* ∂f on X, if the following holds:

(i) $Cf(\boldsymbol{x}, \varepsilon)$ is a Hausdorff continuous mapping with respect to $\boldsymbol{x}$ on X for all $\varepsilon > 0$;
(ii) The subdifferential $\partial f(\boldsymbol{x})$ is the convex hull of the limit of the family $Cf(\boldsymbol{x}, \varepsilon)$ on X. That is, for all $\boldsymbol{x} \in X$ we have

$$\partial f(\boldsymbol{x}) = \text{conv } C_L f(\boldsymbol{x}).$$

For a family $Cf(\cdot, \varepsilon)$ at a point $\boldsymbol{x} \in X$, we define the following mapping

$$C_0 f(\boldsymbol{x}) = \text{conv } \Big\{ \boldsymbol{v} \in \mathbb{R}^n \mid \exists\, \varepsilon_k \downarrow 0,\ \boldsymbol{v}_k \in Cf(\boldsymbol{x}, \varepsilon_k) \text{ such that } \boldsymbol{v} = \lim_{k\to\infty} \boldsymbol{v}_k \Big\}.$$

We denote by σ_d the support function of the family $Cf(\cdot, \varepsilon)$ with $\varepsilon > 0$. That is,

$$\sigma_d(\boldsymbol{x}, \varepsilon) = \max_{\boldsymbol{v} \in Cf(\boldsymbol{x}, \varepsilon)} \boldsymbol{v}^T \boldsymbol{d}.$$

We also set

$$Cf(\boldsymbol{x}, 0) = C_0 f(\boldsymbol{x}), \qquad \text{and}$$
$$\sigma_d(\boldsymbol{x}, 0) = \max_{\boldsymbol{v} \in C_0 f(\boldsymbol{x})} \boldsymbol{v}^T \boldsymbol{d}.$$

Theorem 6.1 *Let the family $Cf(\boldsymbol{x}, \varepsilon)$ be a continuous approximation of the subdifferential $\partial f(\boldsymbol{x})$ and the function σ_d be upper semicontinuous at the point $(\boldsymbol{x}, 0)$ for all $\boldsymbol{d} \in \mathbb{R}^n$. Then*

$$\partial f(\boldsymbol{x}) = C_0 f(\boldsymbol{x}).$$

Theorem 6.1 follows from the following assertion.

Theorem 6.2 *The function* σ_d *is upper semicontinuous at the point* $(\boldsymbol{x}, 0)$ *for all* $\boldsymbol{d} \in \mathbb{R}^n$ *if and only if*

$$\text{conv } C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x}).$$

Proof We start by proving *necessity*. Suppose that the function σ_d is upper semicontinuous at the point $(\boldsymbol{x}, 0)$. It is clear that $C_0 f(\boldsymbol{x}) \subseteq \text{conv } C_L f(\boldsymbol{x})$. From the upper semicontinuity of the function σ_d, we obtain that

$$\limsup_{k\to\infty} \sigma_d(\boldsymbol{x}_k, \varepsilon_k) \le \sigma_d(\boldsymbol{x}, 0)$$

for any $\boldsymbol{x}_k \to \boldsymbol{x},\ \varepsilon_k \downarrow 0$ as $k \to \infty$. Thus, for all $\boldsymbol{w}_k \in C f(\boldsymbol{x}_k, \varepsilon_k)$, we have

$$\limsup_{k\to\infty}\ (\boldsymbol{w}_k)^T \boldsymbol{d} \le \sigma_d(\boldsymbol{x}, 0).$$

It follows from this inequality that for all $\boldsymbol{w} \in \text{conv } C_L f(\boldsymbol{x})$:

$$\boldsymbol{w}^T \boldsymbol{d} \le \sigma_d(\boldsymbol{x}, 0).$$

Thus, for all $\boldsymbol{d} \in \mathbb{R}^n$, we have

$$\max_{\boldsymbol{w}\in\text{conv } C_L f(\boldsymbol{x})} \boldsymbol{w}^T \boldsymbol{d} \le \max_{\boldsymbol{v}\in C_0 f(\boldsymbol{x})} \boldsymbol{v}^T \boldsymbol{d}.$$

Thus, $\text{conv } C_L f(\boldsymbol{x}) \subseteq C_0 f(\boldsymbol{x})$ and the necessity is proved.

Next we prove the *sufficiency*. Let $\text{conv } C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ and $(\boldsymbol{x}_k),\ (\varepsilon_k)$ be arbitrary sequences such that $\boldsymbol{x}_k \to \boldsymbol{x},\ \varepsilon_k \downarrow 0$ as $k \to \infty$. Then there exists $\boldsymbol{w}_k \in C f(\boldsymbol{x}_k, \varepsilon_k)$ such that

$$\sigma_d(\boldsymbol{x}_k, \varepsilon_k) = (\boldsymbol{w}_k)^T \boldsymbol{d}.$$

Assume without loss of generality that there exists $\lim_{k\to\infty} \boldsymbol{w}_k = \boldsymbol{w}$. It is clear that $\boldsymbol{w} \in \text{conv } C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ and therefore

$$\boldsymbol{w}^T \boldsymbol{d} \le \sigma_d(\boldsymbol{x}, 0).$$

Thus

$$\limsup_{\boldsymbol{y}\to\boldsymbol{x},\varepsilon\downarrow 0} \sigma_d(\boldsymbol{y}, \varepsilon) \le \sigma_d(\boldsymbol{x}, 0),$$

so $\sigma_d(\boldsymbol{x}, \varepsilon)$ is upper semicontinuous at the point $(\boldsymbol{x}, 0)$. □

We say that the mapping $Cf(\boldsymbol{x},\varepsilon)$ is *monotonically decreasing* as $\varepsilon \downarrow 0$ if $0 < \varepsilon_1 < \varepsilon_2$ implies that

$$Cf(\boldsymbol{x},\varepsilon_1) \subset Cf(\boldsymbol{x},\varepsilon_2) \text{ for all } \boldsymbol{x} \in X.$$

Corollary 6.1 *Let the mapping $Cf(\boldsymbol{x},\varepsilon)$ be continuous with respect to $\boldsymbol{x}$ and monotonically decreasing as $\varepsilon \downarrow 0$. Then the support function σ_d is upper semicontinuous at the point $(\boldsymbol{x}, 0)$.*

Proof The intersection

$$\bigcap_{\varepsilon>0}\{Cf(\boldsymbol{x},\varepsilon)\}$$

is nonempty and coincides with $\lim_{\varepsilon\downarrow 0} Cf(\boldsymbol{x},\varepsilon)$ in the Hausdorff metric (see Sect. 1.3). It is easy to check that

$$\lim_{\varepsilon\downarrow 0} Cf(\boldsymbol{x},\varepsilon) = C_0 f(\boldsymbol{x}).$$

Thus, for each $\tau > 0$ there exists $\varepsilon_0 > 0$ such that the inequality $0 < \varepsilon \leq \varepsilon_0$ implies

$$Cf(\boldsymbol{x},\varepsilon) \subset C_0 f(\boldsymbol{x}) + B(\mathbf{0};\tau).$$

Since $Cf(\boldsymbol{x},\varepsilon)$ is a continuous mapping with respect to $\boldsymbol{x}$, it follows that for any $\tau > 0$ there exists $\delta = \delta(\tau) > 0$ such that

$$Cf(\boldsymbol{y},\varepsilon) \subset Cf(\boldsymbol{x},\varepsilon) + B(\mathbf{0};\tau)$$

for all $\boldsymbol{y} \in B(\boldsymbol{x};\delta)$. Thus, for $\boldsymbol{y} \in B(\boldsymbol{x};\delta)$ we have

$$Cf(\boldsymbol{y},\varepsilon_0) \subset C_0 f(\boldsymbol{x}) + B(\mathbf{0};2\tau).$$

If $0 < \varepsilon < \varepsilon_0$, then $Cf(\boldsymbol{y},\varepsilon) \subset Cf(\boldsymbol{y},\varepsilon_0)$. So for all $0 < \varepsilon \leq \varepsilon_0$ and $\boldsymbol{y} \in B(\boldsymbol{x};\delta)$ we have

$$Cf(\boldsymbol{y},\varepsilon) \subset C_0 f(\boldsymbol{x}) + B(\mathbf{0};2\tau). \tag{6.2}$$

It follows from (6.2) that conv $C_L f(\boldsymbol{x}) \subset C_0 f(\boldsymbol{x})$. The reverse inclusion is obvious. Thus, the desired result follows. □

Let X be a compact subset of the space $\mathbb{R}^n$. For a family $Cf(\boldsymbol{x},\varepsilon),\ \boldsymbol{x} \in X, \varepsilon > 0$ we define

$$Q_\varepsilon(\boldsymbol{x}) := \bigcup\{Cf(\boldsymbol{x},t) \mid\ 0 \leq t \leq \varepsilon\},$$

where $Cf(\boldsymbol{x},0) = C_0 f(\boldsymbol{x})$.

Lemma 6.1 *Let the family* $Cf(\boldsymbol{x}, \varepsilon)$ *be Hausdorff continuous with respect to* $(\boldsymbol{x}, \varepsilon)$, $\boldsymbol{x} \in X$, $\varepsilon > 0$ *and* conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$. *Then for each* $\varepsilon > 0$ *the mapping* $Q_\varepsilon(\boldsymbol{x})$ *is Hausdorff continuous.*

Proof First we will prove that Q_ε is lower semicontinuous. Let $\boldsymbol{x}_k \to \boldsymbol{x}$ and $\boldsymbol{v} \in Q_\varepsilon(\boldsymbol{x})$. Then there exists $t \in [0, \varepsilon]$ such that $\boldsymbol{v} \in Cf(\boldsymbol{x}, t)$. If $t > 0$ then we can exploit the continuity of the mapping $Cf(\boldsymbol{x}, t)$ and find a sequence $\boldsymbol{v}_k \in Cf(\boldsymbol{x}_k, t)$ such that $\boldsymbol{v}_k \to \boldsymbol{v}$. If $t = 0$ then the equality conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ shows that there exist sequences $t_k \in (0, \varepsilon]$ and $\boldsymbol{v}_k \in Cf(\boldsymbol{x}_k, t_k)$ such that $\boldsymbol{v}_k \to \boldsymbol{v}$. Thus, lower semicontinuity of Q_ε has been proved.

Now we will prove that Q_ε is a closed mapping. Let $\boldsymbol{x}_k \to \boldsymbol{x}$, $\boldsymbol{v}_k \in Q_\varepsilon(\boldsymbol{x}_k)$ and $\boldsymbol{v}_k \to \boldsymbol{v}$. It follows from the definition of the mapping Q_ε that there exists a sequence $t_k \in [0, \varepsilon]$ such that $\boldsymbol{v}_k \in Cf(\boldsymbol{x}_k, t_k)$. Assume without loss of generality that $t_k \to t$. First assume $t > 0$. Since $Cf(\cdot, \cdot)$ is continuous, it follows that $\boldsymbol{v} \in Cf(\boldsymbol{x}, t)$ and so $\boldsymbol{v} \in Q_\varepsilon(\boldsymbol{x})$. Now let $t = 0$. If there exists a sequence (t_{k_i}) such that $t_{k_i} > 0$ for all $i = 1, 2, \ldots$, then $\boldsymbol{v} \in$ conv $C_L f(\boldsymbol{x})$ and the equality conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ shows that $v \in Q_\varepsilon(\boldsymbol{x})$. If $t_k = 0$ for all k then $v_k \in C_0 f(\boldsymbol{x}_k) =$ conv $C_L f(\boldsymbol{x}_k)$. Since conv $C_L f$ is upper semicontinuous, it follows that $\boldsymbol{v} \in Q_\varepsilon(\boldsymbol{x})$. We have proved that Q_ε is closed. Since the set $Q_\varepsilon(X)$ is compact, it follows that Q_ε is upper semicontinuous. Thus, the mapping Q_ε is both lower and upper semicontinuous. Therefore this mapping is Hausdorff continuous. □

Let us denote

$$Q(\boldsymbol{x}, \varepsilon) := \text{conv } Q_\varepsilon(\boldsymbol{x}). \tag{6.3}$$

Since the mapping Q_ε is closed, it follows that Q is also closed.

Theorem 6.3 *Let the family* $Cf(\boldsymbol{x}, \varepsilon)$ *be Hausdorff continuous with respect to* $(\boldsymbol{x}, \varepsilon)$, $\boldsymbol{x} \in X$, $\varepsilon > 0$ *and* conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$. *Then for each* $\varepsilon > 0$ *the mapping* $Q(\boldsymbol{x}, \varepsilon)$ *is Hausdorff continuous with respect to* $\boldsymbol{x}$ *and monotonically decreasing as* $\varepsilon \downarrow 0$.

Proof Monotonicity of $Q(\boldsymbol{x}, \varepsilon)$ with respect to ε follows directly from the definition. Lemma 6.1 shows that this mapping is continuous with respect to $\boldsymbol{x} \in X$. □

The set-valued mapping $Cf(\boldsymbol{x}, \varepsilon)$ need not to be monotonically decreasing as $\varepsilon \downarrow 0$. However, uniform and strongly continuous approximations of the subdifferential to be defined next have such a property.

Let f be a locally Lipschitz continuous function defined on an open set which contains a compact set X. We consider a family of set-valued mappings $\mathcal{A}_\varepsilon f : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $\varepsilon > 0$. Assume that the sets $\mathcal{A}_\varepsilon f(\boldsymbol{x})$ are nonempty and compact for all $\varepsilon > 0$ and $\boldsymbol{x} \in X$. We denote

$$\partial f(\boldsymbol{x} + \bar{B}(\boldsymbol{x}; \delta)) := \bigcup \{\partial f(\boldsymbol{y}) \mid \boldsymbol{y} \in \bar{B}(\boldsymbol{x}; \delta)\}.$$

Definition 6.3 We say that the family $\{\mathcal{A}_\varepsilon f(\cdot)\}_{\varepsilon > 0}$ is a *uniform continuous approximation of the subdifferential* ∂f on X, if the following conditions are satisfied:

(i) for each given $\varepsilon > 0$, $\mu > 0$ there exists $\tau > 0$, such that for all $\boldsymbol{x} \in X$ we have

$$\partial f(\boldsymbol{x} + \bar{B}(\mathbf{0}; \tau)) \subset \mathcal{A}_\varepsilon f(\boldsymbol{x}) + \bar{B}(\mathbf{0}; \mu);$$

(ii) for each $\boldsymbol{x} \in X$ and for all $0 < \varepsilon_1 < \varepsilon_2$ we have

$$\mathcal{A}_{\varepsilon_1} f(\boldsymbol{x}) \subset \mathcal{A}_{\varepsilon_2} f(\boldsymbol{x});$$

(iii) $\mathcal{A}_\varepsilon f(\boldsymbol{x})$ is Hausdorff continuous with respect to $\boldsymbol{x}$ on X;

(iv) for each $\boldsymbol{x} \in X$ we have

$$\bigcap_{\varepsilon > 0} \{\mathcal{A}_\varepsilon f(\boldsymbol{x})\} = \partial f(\boldsymbol{x}).$$

Definition 6.4 We say that the family $\{\mathcal{A}_\varepsilon f(\cdot)\}_{\varepsilon>0}$ is a *strong continuous approximation of the subdifferential* ∂f on X, if $\{\mathcal{A}_\varepsilon f(\cdot)\}_{\varepsilon>0}$ satisfies properties (i)–(iii) above and instead of (iv) the following is valid:

(iv') for every $\gamma, \mu > 0$ there exists $\varepsilon > 0$ such that for all $\boldsymbol{x} \in X$:

$$\partial f(\boldsymbol{x}) \subset \mathcal{A}_\varepsilon f(\boldsymbol{x}) \subset \partial f(\boldsymbol{x} + \bar{B}(\mathbf{0}; \gamma)) + \bar{B}(\mathbf{0}; \mu).$$

As already said the set-valued mapping $Cf(\boldsymbol{x}, \varepsilon)$ need not to be monotonically decreasing as $\varepsilon \downarrow 0$. However, we can use this mapping to construct uniform and strongly continuous approximations of the subdifferential. We will now establish connections between these continuous approximations of the subdifferential.

Theorem 6.4 *Let the mapping $\mathcal{A}_\varepsilon f$ be a uniform continuous approximation of the subdifferential ∂f on compact set X. Then $Cf(\boldsymbol{x}, \varepsilon) = \mathcal{A}_\varepsilon f(\boldsymbol{x})$ is a continuous approximation of the subdifferential ∂f in the sense of Definition 6.2.*

Proof It follows from Definition 6.3 (iii) that $Cf(\boldsymbol{x}, \varepsilon)$ is Hausdorff continuous with respect to $\boldsymbol{x}$. Since $Cf(\boldsymbol{x}, \varepsilon)$ is monotonically decreasing as $\varepsilon \downarrow 0$ we have, by applying Corollary 6.1 and Theorem 6.2, the equality conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$. On the other hand, Definition 6.3 (iv) shows that $C_0 f(\boldsymbol{x}) = \partial f(\boldsymbol{x})$. Thus, $\partial f(\boldsymbol{x}) =$ conv $C_L f(\boldsymbol{x})$. □

Corollary 6.2 *A strong continuous approximation is a uniform continuous approximation. Therefore, a strong continuous approximation is a continuous approximation in the sense of Definition 6.2.*

Theorem 6.5 *Let $Cf(\boldsymbol{x}, \varepsilon)$ be a continuous approximation of the subdifferential ∂f on a compact set X and let the mapping $Cf(\boldsymbol{x}, \varepsilon)$ be continuous with respect to $(\boldsymbol{x}, \varepsilon)$. Assume that* conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ *for all $\boldsymbol{x} \in X$. Then the mapping*

$$Q(\boldsymbol{x}, \varepsilon) = \text{conv} \bigcup \{Cf(\boldsymbol{x}, t) : 0 \leq t \leq \varepsilon\}$$

is a uniform continuous approximation of $\partial f(\boldsymbol{x})$ on X.

Proof Theorem 6.3 shows that the mapping $Q(\boldsymbol{x}, \varepsilon)$ is Hausdorff continuous with respect to $\boldsymbol{x}$ and monotonically decreasing as $\varepsilon \downarrow 0$. Since $Cf(\boldsymbol{x}, \varepsilon)$ is a continuous approximation of the subdifferential $\partial f(\boldsymbol{x})$ it follows that items (ii), (iii), and (iv) of Definition 6.3 hold. Let us check the validity of (iv). Assume for the sake of contradiction that there exist $\varepsilon > 0$ and $\mu > 0$ such that for each sequence $\tau_k \downarrow 0$ we can find $\boldsymbol{x}_k \in X$ with the following property:

$$\partial f(\boldsymbol{x}_k + \bar{B}(\mathbf{0}; \tau_k)) \not\subset Q(\boldsymbol{x}_k, \varepsilon) + \bar{B}(\mathbf{0}; \mu).$$

Since X is a compact set there exists a sequence $k_i \to \infty$ and an element $\boldsymbol{x} \in X$ such that $\boldsymbol{x}_{k_i} \to \boldsymbol{x}$. Upper semicontinuity of the subdifferential $\partial f(\boldsymbol{x})$ and Hausdorff continuity of the mapping $Q(\boldsymbol{x}, \varepsilon)$ implies

$$\partial f(\boldsymbol{x}) \not\subset Q(\boldsymbol{x}, \varepsilon) + \bar{B}(\mathbf{0}; \mu).$$

On the other hand, it follows from the definition of a continuous approximation and the equality conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ that $\partial f(\boldsymbol{x}) \subset Q(\boldsymbol{x}, \varepsilon)$. Thus, we have a contradiction which shows that the desired result holds. □

Corollary 6.3 *Let the family $Cf(\boldsymbol{x}, \varepsilon)$ be a continuous approximation of the subdifferential ∂f on a compact set X and the mapping $Cf(\boldsymbol{x}, \varepsilon)$ be a continuous with respect to $(\boldsymbol{x}, \varepsilon)$, $\boldsymbol{x} \in X, \varepsilon > 0$. Assume* conv $C_L f(\boldsymbol{x}) = C_0 f(\boldsymbol{x})$ *for all $\boldsymbol{x} \in X$. Then the mapping Q is upper semicontinuous with respect to $(\boldsymbol{x}, \varepsilon)$ at the point $(\boldsymbol{x}, 0)$.*

6.2 Discrete Gradient and Approximation of Subgradients

In this section we will introduce the notion of *discrete gradient* for a locally Lipschitz continuous function. Before doing so, we will give some grounding results that are used in order to prove that the set of discrete gradients approximates the subdifferential of a quasidifferentiable function.

Let us consider a function $f : \mathbb{R}^n \to \mathbb{R}$ and assume that it is quasidifferentiable (see Definition 5.36). We also assume that both sets $\underline{\partial} f(\boldsymbol{x})$ and $\overline{\partial} f(\boldsymbol{x})$ are polytopes at any $\boldsymbol{x} \in \mathbb{R}^n$. That is, at a point $\boldsymbol{x} \in \mathbb{R}^n$ there exist non-empty sets

$$A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\} \subset \mathbb{R}^n,$$
$$B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_p\} \subset \mathbb{R}^n$$

such that $\underline{\partial} f(\boldsymbol{x}) = \text{conv } A$, and $\overline{\partial} f(\boldsymbol{x}) = \text{conv } B$. We denote by $\mathcal{F}$ the class of all semismooth, quasidifferentiable functions whose subdifferential and superdifferential are polytopes at any $\boldsymbol{x} \in \mathbb{R}^n$. This class contains, for example, functions represented as a maximum, minimum, or max-min of a finite number of smooth functions.

Let

$$G = \{\boldsymbol{e} \in \mathbb{R}^n \mid \boldsymbol{e} = (e_1, \dots, e_n), |e_j| = 1,\ j = 1, \dots, n\}$$

be a set of all vertices of the unit hypercube in $\mathbb{R}^n$. For $\boldsymbol{e} \in G$ we define the sequence of n vectors $\boldsymbol{e}_j = \boldsymbol{e}_j(\alpha),\ j = 1, \dots, n$ with $\alpha \in (0, 1]$, where $\boldsymbol{e}_j = (\alpha e_1, \alpha^2 e_2, \dots, \alpha^j e_j, 0, \dots, 0)$.

We introduce the following sets

$$\underline{R}_0(\boldsymbol{e}) = \underline{R}_0 = A,$$
$$\underline{R}_j(\boldsymbol{e}) = \left\{\boldsymbol{v} \in \underline{R}_{j-1}(\boldsymbol{e}) \mid v_j e_j = \max\{w_j e_j \mid \boldsymbol{w} \in \underline{R}_{j-1}(\boldsymbol{e})\}\right\},$$

and

$$\overline{R}_0(\boldsymbol{e}) = \overline{R}_0 = B,$$
$$\overline{R}_j(\boldsymbol{e}) = \left\{\boldsymbol{v} \in \overline{R}_{j-1}(\boldsymbol{e}) \mid v_j e_j = \min\{w_j e_j \mid \boldsymbol{w} \in \overline{R}_{j-1}(\boldsymbol{e})\}\right\}.$$

It is clear that

$$\underline{R}_j(\boldsymbol{e}) \neq \emptyset, \quad \text{for all } j \in \{0, \dots, n\},$$
$$\underline{R}_j(\boldsymbol{e}) \subseteq \underline{R}_{j-1}(\boldsymbol{e}), \quad \text{for all } j \in \{1, \dots, n\},$$

and

$$\overline{R}_j(\boldsymbol{e}) \neq \emptyset, \quad \text{for all } j \in \{0, \dots, n\},$$
$$\overline{R}_j(\boldsymbol{e}) \subseteq \overline{R}_{j-1}(\boldsymbol{e}), \quad \text{for all } j \in \{1, \dots, n\}.$$

Moreover,

$$v_r = u_r \text{ for all } \boldsymbol{v}, \boldsymbol{u} \in \underline{R}_j(\boldsymbol{e}),\ \text{and}$$
$$w_r = z_r \text{ for all } \boldsymbol{w}, \boldsymbol{z} \in \overline{R}_j(\boldsymbol{e}),\ r = 1, \dots, j. \tag{6.4}$$

Lemma 6.2 *Assume that $f \in \mathcal{F}$. Then $\underline{R}_n(\boldsymbol{e})$ and $\overline{R}_n(\boldsymbol{e})$ are singleton sets.*

Proof The proof immediately follows from (6.4). □

Now, let us consider the two sets

$$\underline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha)) = \left\{\boldsymbol{v} \in A \mid \boldsymbol{v}^T\boldsymbol{e}_j = \max\ \{\boldsymbol{u}^T\boldsymbol{e}_j \mid \boldsymbol{u} \in A\}\right\},$$
$$\overline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha)) = \left\{\boldsymbol{w} \in B \mid \boldsymbol{w}^T\boldsymbol{e}_j = \min\ \{\boldsymbol{u}^T\boldsymbol{e}_j \mid \boldsymbol{u} \in B\}\right\}.$$

We take any $\boldsymbol{a} \in A$. If $\boldsymbol{a} \notin \underline{R}_n(\boldsymbol{e})$ then there exists $r \in \{1, \dots, n\}$ such that $a \in \underline{R}_t(\boldsymbol{e}),\ t = 0, \dots, r-1$, and $\boldsymbol{a} \notin \underline{R}_r(\boldsymbol{e})$. It follows from $\boldsymbol{a} \notin \underline{R}_r(\boldsymbol{e})$ that $v_r e_r > a_r e_r$ for all $\boldsymbol{v} \in \underline{R}_r(\boldsymbol{e})$. For $\boldsymbol{a} \in A,\ \boldsymbol{a} \notin \underline{R}_n(\boldsymbol{e})$, we define $d(\boldsymbol{a}) = v_r e_r - a_r e_r > 0$, and then introduce the following number: $d_1 = \min\{d(\boldsymbol{a}) \mid \boldsymbol{a} \in A \setminus \underline{R}_n(\boldsymbol{e})\}$. Since the set A is finite and $d(\boldsymbol{a}) > 0$ for all $\boldsymbol{a} \in A \setminus \underline{R}_n(\boldsymbol{e})$, it follows that $d_1 > 0$.

We also take any $\boldsymbol{b} \in B$. If $\boldsymbol{b} \notin \overline{R}_n(\boldsymbol{e})$ then there exists $r \in \{1, \dots, n\}$ such that $\boldsymbol{b} \in \overline{R}_t(\boldsymbol{e}),\ t = 0, \dots, r-1$, and $\boldsymbol{b} \notin \overline{R}_r(\boldsymbol{e})$. Then we get $v_r e_r < b_r e_r$ for all $\boldsymbol{v} \in \overline{R}_r(\boldsymbol{e})$. For $\boldsymbol{b} \in B,\ \boldsymbol{b} \notin \overline{R}_n(\boldsymbol{e})$ we define $d(\boldsymbol{b}) = b_r e_r - v_r e_r > 0$ and introduce the number $d_2 = \min\{d(\boldsymbol{b}) \mid \boldsymbol{b} \in B \setminus \overline{R}_n(\boldsymbol{e})\}$. Now, $d_2 > 0$ due to the fact that the set B is finite and $d(\boldsymbol{b}) > 0$ for all $\boldsymbol{b} \in B \setminus \overline{R}_n(\boldsymbol{e})$. Let $\bar{d} = \min\{d_1,\ d_2\}$. Since the subdifferential $\underline{\partial} f(\boldsymbol{x})$ and the superdifferential $\overline{\partial} f(\boldsymbol{x})$ are bounded on any bounded subset $X \subset \mathbb{R}^n$, there exists $D > 0$ such that $\|\boldsymbol{v}\| \le D$ and $\|\boldsymbol{w}\| \le D$ for all $\boldsymbol{v} \in \underline{\partial} f(\boldsymbol{y}),\ \boldsymbol{w} \in \overline{\partial} f(\boldsymbol{y})$ and $\boldsymbol{y} \in X$. Let us take any $r, j \in \{1, \dots, n\}$ such that $r < j$. Then, for all $\boldsymbol{v}, \boldsymbol{w} \in \underline{\partial} f(\boldsymbol{x}),\ \boldsymbol{x} \in X$ and $\alpha \in (0, 1]$, we have

$$\left| \sum_{t=r+1}^{j} (v_t - w_t)\alpha^{t-r} e_t \right| < 2D\alpha n.$$

Let $\alpha_0 = \min\{1, \bar{d}/(4Dn)\}$. Then, for any $\alpha \in (0, \alpha_0]$, we have

$$\left| \sum_{t=r+1}^{j} (v_t - w_t)\alpha^{t-r} e_t \right| < \frac{\bar{d}}{2}. \tag{6.5}$$

In a similar way we can show that for all $\boldsymbol{v}, \boldsymbol{w} \in \overline{\partial} f(\boldsymbol{x}),\ \boldsymbol{x} \in X$ and $\alpha \in (0, \alpha_0]$ we have

$$\left| \sum_{t=r+1}^{j} (v_t - w_t)\alpha^{t-r} e_t \right| < \frac{\bar{d}}{2}. \tag{6.6}$$

Lemma 6.3 *Assume that $f \in \mathcal{F}$. Then there exists $\alpha_0 > 0$ such that $\underline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha)) \subset \underline{R}_j(\boldsymbol{e})$ and $\overline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha)) \subset \overline{R}_j(\boldsymbol{e}),\ j = 1, \dots, n$ for all $\alpha \in (0, \alpha_0]$.*

Proof We will prove the first inclusion. The second inclusion can be proved in a similar way. Assume the contrary. Then there exists $\boldsymbol{y} \in \underline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha))$ such that $\boldsymbol{y} \notin \underline{R}_j(\boldsymbol{e})$. Consequently there exists $r \in \{1, \ldots, n\},\ r \le j$ such that $\boldsymbol{y} \notin \underline{R}_r(\boldsymbol{e})$ and $\boldsymbol{y} \in \underline{R}_t(\boldsymbol{e})$ for any $t = 0, \ldots, r-1$. Take any $\boldsymbol{v} \in \underline{R}_j(\boldsymbol{e})$. From (6.4) we have $v_t e_t = y_t e_t,\ t = 1, \ldots, r-1,\ v_r e_r \ge y_r e_r + \bar{d}$. It follows from (6.5) that

$$\begin{aligned}\boldsymbol{v}^T \boldsymbol{e}_j - \boldsymbol{y}^T \boldsymbol{e}_j &= \sum_{t=1}^{j} (v_t - y_t)\alpha^t e_t \\ &= \alpha^r \left[v_r e_r - y_r e_r + \sum_{t=r+1}^{j} (v_t - y_t)\alpha^{t-r} e_t \right] > \alpha^r \bar{d}/2 > 0.\end{aligned}$$

Since $\boldsymbol{y}^T \boldsymbol{e}_j = \max\{\boldsymbol{u}^T \boldsymbol{e}_j \mid \boldsymbol{u} \in \underline{\partial} f(\boldsymbol{x})\}$ and $\boldsymbol{v} \in \underline{\partial} f(\boldsymbol{x})$, we obtain

$$\boldsymbol{y}^T \boldsymbol{e}_j \ge \boldsymbol{v}^T \boldsymbol{e}_j > \boldsymbol{y}^T \boldsymbol{e}_j + \alpha^r \bar{d}/2,$$

which is the contradiction. □

Corollary 6.4 *Assume that $f \in \mathcal{F}$. Then there exits $\alpha_0 > 0$ such that*

$$f'(\boldsymbol{x}; \boldsymbol{e}_j(\alpha)) = f'(\boldsymbol{x}; \boldsymbol{e}_{j-1}(\alpha)) + v_j \alpha^j e_j + w_j \alpha^j e_j,$$

for all $\boldsymbol{v} \in \underline{R}_j(\boldsymbol{e})$ and $\boldsymbol{w} \in \overline{R}_j(\boldsymbol{e}),\ j = 1, \ldots, n$, and for all $\alpha \in (0, \alpha_0]$.

Proof Lemma 6.3 implies that $\underline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha)) \subset \underline{R}_j(\boldsymbol{e})$ and $\overline{R}(\boldsymbol{x}, \boldsymbol{e}_j(\alpha)) \subset \overline{R}_j(\boldsymbol{e})$, $j = 1, \ldots, n$. Then there exist $\boldsymbol{v} \in \underline{R}_j(\boldsymbol{e})$, $\boldsymbol{w} \in \overline{R}_j(\boldsymbol{e})$, $\boldsymbol{v}^0 \in \underline{R}_{j-1}(\boldsymbol{e})$, $\boldsymbol{w}^0 \in \overline{R}_{j-1}(\boldsymbol{e})$ such that $f'(\boldsymbol{x}; \boldsymbol{e}_j(\alpha)) - f'(\boldsymbol{x}; \boldsymbol{e}_{j-1}(\alpha)) = (\boldsymbol{v}+\boldsymbol{w})^T \boldsymbol{e}_j - (\boldsymbol{v}^0 + \boldsymbol{w}^0)^T \boldsymbol{e}_{j-1}$ and the proof follows from (6.4). □

Let $\boldsymbol{e} \in G$ and $\lambda > 0,\ \alpha > 0$ be given numbers. We consider the following points

$$\boldsymbol{x}_0 = \boldsymbol{x}, \qquad \boldsymbol{x}_j = \boldsymbol{x}_0 + \lambda \boldsymbol{e}_j(\alpha), \quad j = 1, \ldots, n.$$

It is clear that $\boldsymbol{x}_j = \boldsymbol{x}_{j-1} + (0, \ldots, 0, \lambda\alpha^j e_j, 0, \ldots, 0)$, $j = 1, \ldots, n$. Let $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{e}, \alpha, \lambda) \in \mathbb{R}^n$ be a vector with the following coordinates:

$$v_j = (\lambda\alpha^j e_j)^{-1} \left[f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1}) \right], \quad j = 1, \ldots, n. \tag{6.7}$$

For any fixed $\boldsymbol{e} \in G$ and $\alpha > 0$, we introduce the set

$$V(\boldsymbol{e}, \alpha) = \left\{ \boldsymbol{w} \in \mathbb{R}^n \mid \exists \lambda_k \downarrow 0,\ k \to \infty,\ \text{such that } \boldsymbol{w} = \lim_{k \to \infty} \boldsymbol{v}(\boldsymbol{e}, \alpha, \lambda_k) \right\}.$$

Theorem 6.6 *Assume that $f \in \mathcal{F}$. Then there exists $\alpha_0 > 0$ such that*

$$V(\boldsymbol{e}, \alpha) \subset \partial f(\boldsymbol{x}), \text{ for all } \alpha \in (0, \alpha_0].$$

Proof It follows from the definition of vectors $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{e}, \alpha, \lambda)$ that

$$\begin{aligned} v_j &= (\lambda\alpha^j e_j)^{-1}\left[f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1})\right] \\ &= (\lambda\alpha^j e_j)^{-1}\left[f(\boldsymbol{x}_j) - f(\boldsymbol{x}) - (f(\boldsymbol{x}_{j-1}) - f(\boldsymbol{x}))\right] \\ &= (\lambda\alpha^j e_j)^{-1}\left[\lambda f'(\boldsymbol{x}; \boldsymbol{e}_j) - \lambda f'(\boldsymbol{x}; \boldsymbol{e}_{j-1}) + o(\lambda, \boldsymbol{e}_j) - o(\lambda, \boldsymbol{e}^{j-1})\right] \end{aligned}$$

where $\lambda^{-1}o(\lambda, \boldsymbol{e}_i) \to 0, \lambda \downarrow 0, i = j-1, j$. We take $\boldsymbol{w} \in \underline{R}_n(\boldsymbol{e})$ and $\boldsymbol{y} \in \overline{R}_n(\boldsymbol{e})$. By Lemma 6.2 $\boldsymbol{w}$ and $\boldsymbol{y}$ are unique. Since $\underline{R}_n(\boldsymbol{e}) = \underline{R}(\boldsymbol{x}, \boldsymbol{e}^n)$ and $\overline{R}_n(\boldsymbol{e}) = \overline{R}(\boldsymbol{x}, \boldsymbol{e}^n)$ it follows from Theorem 5.41 in that $\boldsymbol{w} + \boldsymbol{y} \in \partial f(\boldsymbol{x})$. The inclusions $\boldsymbol{w} \in \underline{R}_n(\boldsymbol{e})$ and $\boldsymbol{y} \in \overline{R}_n(\boldsymbol{e})$ imply that $\boldsymbol{w} \in \underline{R}_j(\boldsymbol{e})$ and $\boldsymbol{y} \in \overline{R}_j(\boldsymbol{e})$ for all $j \in \{1, \dots, n\}$. It follows from Corollary 6.4 that there exists $\alpha_0 > 0$ such that

$$\begin{aligned} v_j(\boldsymbol{e}, \alpha, \lambda) &= (\lambda\alpha^j e_j)^{-1}\left[\lambda\alpha^j e_j(w_j + y_j) + o(\lambda, \boldsymbol{e}_j) - o(\lambda, \boldsymbol{e}_{j-1})\right] \\ &= w_j + y_j + (\lambda\alpha^j e_j)^{-1}\left[o(\lambda, \boldsymbol{e}_j) - o(\lambda, \boldsymbol{e}_{j-1})\right] \end{aligned}$$

for all $\alpha \in (0, \alpha_0]$. Then for any fixed $\alpha \in (0, \alpha_0]$ we have

$$\lim_{\lambda \downarrow 0} |v_j(\boldsymbol{e}, \alpha, \lambda) - (w_j + y_j)| = 0.$$

Consequently, $\lim_{\lambda\downarrow 0} v(\boldsymbol{e}, \alpha, \lambda) = \boldsymbol{w} + \boldsymbol{y} \in \partial f(\boldsymbol{x})$. □

Next we give the definition of the discrete gradient. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function and let us denote by S_1 the sphere of the unit ball and by

$$P = \{z \mid z : \mathbb{R}_+ \to \mathbb{R}_+,\ \lambda > 0,\ \lambda^{-1}z(\lambda) \to 0,\ \lambda \to 0\}$$

the set of univariate positive infinitesimal functions. We take any $\boldsymbol{g} \in S_1, \boldsymbol{e} \in G, z \in P$, a positive number $\alpha \in (0, 1]$, and we compute $i = \operatorname{argmax}\{|g_k|,\ k = 1, \dots, n\}$. For $\boldsymbol{e} \in G$ we define the sequence of n vectors $\boldsymbol{e}_j(\alpha) = (\alpha e_1, \alpha^2 e_2, \dots, \alpha^j e_j, 0, \dots, 0), j = 1, \dots, n$ as before, and for $\boldsymbol{x} \in \mathbb{R}^n$ and $\lambda > 0$, we consider the points

$$\boldsymbol{x}_0 = \boldsymbol{x} + \lambda\boldsymbol{g}, \qquad \boldsymbol{x}_j = \boldsymbol{x}_0 + z(\lambda)\boldsymbol{e}_j(\alpha), \qquad j = 1, \dots, n.$$

Definition 6.5 Let $\boldsymbol{g} \in S_1$, $\boldsymbol{e} \in G$, $z \in P$, $\alpha \in (0, 1]$, $\lambda > 0$ and take $i = \operatorname{argmax}\{|g_k| \mid k = 1, \dots, n\}$. The *discrete gradient* of the function $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\boldsymbol{x} \in \mathbb{R}^n$ in the direction $\boldsymbol{g}$ is the vector $\Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, z, \lambda, \alpha) =$

$(\Gamma_1^i, \dots, \Gamma_n^i) \in \mathbb{R}^n$ with the following coordinates:

$$\Gamma_j^i = [z(\lambda)\alpha^j e_j)]^{-1}\left[f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1})\right], \qquad j = 1, \dots, n, \ \ j \neq i,$$
$$\Gamma_i^i = (\lambda g_i)^{-1}\left[f(\boldsymbol{x} + \lambda \boldsymbol{g}) - f(\boldsymbol{x}) - \lambda \sum_{j=1, j\neq i}^{n} \Gamma_j^i g_j\right].$$

It follows from Definition 6.5 that

$$f(\boldsymbol{x} + \lambda \boldsymbol{g}) - f(\boldsymbol{x}) = \lambda \Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, z, \lambda, \alpha)^T \boldsymbol{g} \tag{6.8}$$

for all $\boldsymbol{g} \in S_1, \ \boldsymbol{e} \in G, \ z \in P, \ \lambda > 0, \ \alpha > 0.$

Remark 6.1 The discrete gradient is defined with respect to a given direction $\boldsymbol{g} \in S_1$. To compute the discrete gradient we first define a a sequence of points $\boldsymbol{x}_0, \dots, \boldsymbol{x}_n$ and compute the values of the function f at these points. That is, we compute $n+2$ values of f including the point $\boldsymbol{x}$ itself. The $n-1$ coordinates of the discrete gradient are defined similar to those of the vector $\boldsymbol{v}(\boldsymbol{e}, \alpha, \lambda)$ in Eq. (6.7) and the ith coordinate is defined so it satisfies equality (6.8) which can be considered as some version of the mean-value theorem (Theorem 5.18).

Lemma 6.4 *Let f be a locally Lipschitz continuous function defined on $\mathbb{R}^n$ and let $L > 0$ be its Lipschitz constant. Then, for any $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{g} \in S_1$, $\boldsymbol{e} \in G$, $\lambda > 0$, $z \in P$, and $\alpha > 0$ we have*

$$\|\Gamma^i\| \le C(n)L, \ \ C(n) = (n^2 + 2n^{3/2} - 2n^{1/2})^{1/2}.$$

Proof It follows from the definition of the discrete gradients that $|\Gamma_j^i| \le L$ for all $j = 1, \dots, n, \ j \neq i$. For $j = i$ we obtain

$$|\Gamma_i^i| \le L\left(|g_i|^{-1}\|\boldsymbol{g}\| + \sum_{j=1, j\neq i}^{n} |g_i|^{-1}|g_j|\right).$$

Since $|g_i| = \max\{|g_j|, \ j = 1, \dots, n\}$, we have $|g_i|^{-1}|g_j| \le 1, \ j = 1, \dots, n$ and $|g_i|^{-1}\|g\| \le n^{1/2}$. Consequently, $|\Gamma_i^i| \le L(n + n^{1/2} - 1)$. Thus, $\|\Gamma^i\| \le C(n)L$. □

Next we show the connections between discrete gradients and Clarke subdifferential. For a given $\alpha > 0$, we define the following set

$$D(\boldsymbol{x}, \alpha) = \{\boldsymbol{v} \in \mathbb{R}^n \ | \exists(\boldsymbol{g} \in S_1, \ \boldsymbol{e} \in G, \ z_k \in P, \ z_k \downarrow 0, \ \lambda_k \downarrow 0, \ k \to \infty),$$
$$\text{such that } \boldsymbol{v} = \lim_{k\to\infty} \Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, z_k, \lambda_k, \alpha)\}. \tag{6.9}$$

Theorem 6.7 *Assume that* $f \in \mathcal{F}$. *Then, there exists* $\alpha_0 > 0$ *such that*

$$\text{conv } D(\boldsymbol{x}, \alpha) \subset \partial f(\boldsymbol{x}), \textit{ for all } \alpha \in (0, \alpha_0].$$

Proof Since the function f is semismooth it follows that for any $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that $\boldsymbol{v} \in R(\boldsymbol{x}, \boldsymbol{g}) + B(\boldsymbol{0}; \varepsilon)$ for all $\boldsymbol{v} \in \partial f(\boldsymbol{x} + \lambda \boldsymbol{g})$ and $\lambda \in (0, \lambda_0)$. Here $R(\boldsymbol{x}, \boldsymbol{g}) = \{\boldsymbol{v} \in \partial f(\boldsymbol{x}) \mid f'(\boldsymbol{x}; \boldsymbol{g}) = \boldsymbol{v}^T \boldsymbol{g}\}$. We take any $\lambda \in (0, \lambda_0)$. It follows from Theorem 6.6 and the definition of the discrete gradient that there exist $\alpha_0 > 0$ and $z_0(\lambda) \in P$ such that for any $\alpha \in (0, \alpha_0]$, $z \in P$, and $z(\lambda) < z_0(\lambda)$ we can found $\boldsymbol{v} \in \partial f(\boldsymbol{x} + \lambda \boldsymbol{g})$ so that $|\Gamma_j^i - v_j| < \varepsilon$, $j = 1, \ldots, n$, $j \neq i$. Semismoothness of f implies that $\|\boldsymbol{v} - \boldsymbol{w}\| < \varepsilon$ for some $\boldsymbol{w} \in R(x, g)$. Then

$$|\Gamma_j^i - w_j| < 2\varepsilon, \quad j = 1, \ldots, n, \; j \neq i. \tag{6.10}$$

Since $\boldsymbol{w} \in R(\boldsymbol{x}, g)$ and the function f is semismooth, $f'(\boldsymbol{x}; \boldsymbol{g}) = \boldsymbol{w}^T \boldsymbol{g}$ and

$$f(\boldsymbol{x} + \lambda \boldsymbol{g}) - f(\boldsymbol{x}) = \lambda \boldsymbol{w}^T \boldsymbol{g} + o(\lambda, \boldsymbol{g}) \tag{6.11}$$

where $\lambda^{-1} o(\lambda, \boldsymbol{g}) \to 0$ as $\lambda \downarrow 0$. It follows from (6.8) and (6.11) that

$$\Gamma_i^i - w_i = \sum_{j=1, j \neq i}^{n} (w_j - \Gamma_j^i) g_j g_i^{-1} + (\lambda g_i)^{-1} o(\lambda, \boldsymbol{g}).$$

Taking into account (6.10), we obtain

$$|\Gamma_i^i - w_i| \leq 2(n-1)\varepsilon + n^{1/2} \lambda^{-1} |o(\lambda, \boldsymbol{g})|. \tag{6.12}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (6.10) and (6.12) that

$$\lim_{k \to \infty} \Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, z_k, \lambda_k, \alpha) = \boldsymbol{w} \in \partial f(\boldsymbol{x}).$$

□

Remark 6.2 The discrete gradient contains three parameters: $\lambda > 0$, $z \in P$ and $\alpha > 0$. The parameter $z \in P$ is used to exploit semismoothness of the function f. If $f \in \mathcal{F}$ then for any $\delta > 0$ there exists $\alpha_0 > 0$ such that $\alpha \in (0, \alpha_0]$ for all $\boldsymbol{y} \in B(\boldsymbol{x}; \delta)$. In the sequel we assume that $z \in P$ and $\alpha > 0$ are sufficiently small.

Let us consider the closed convex set of discrete gradients at a point $\boldsymbol{x} \in \mathbb{R}^n$. That is, the set

$$D_0(\boldsymbol{x}, \lambda) = \text{cl conv}\{\boldsymbol{v} \in \mathbb{R}^n \mid \exists (\boldsymbol{g} \in S_1, \; \boldsymbol{e} \in G, \; z \in P, \; \alpha > 0) \text{ such that } \boldsymbol{v} = \Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, \lambda, z, \alpha)\}.$$

Lemma 6.4 implies that the set $D_0(\boldsymbol{x}, \lambda)$ is compact and convex for any $\boldsymbol{x} \in \mathbb{R}^n$. The next corollary shows that the set $D_0(\boldsymbol{x}, \lambda)$ is an approximation of the subdifferential $\partial f(\boldsymbol{x})$ for sufficiently small $\lambda > 0$.

Corollary 6.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a semismooth function at $\boldsymbol{x}$. For $\lambda > 0$ and $\boldsymbol{g} \in S_1$ define*

$$o(\lambda, \boldsymbol{g}) = f(\boldsymbol{x} + \lambda \boldsymbol{g}) - f(\boldsymbol{x}) - \lambda f'(\boldsymbol{x}; \boldsymbol{g}).$$

If $\lambda^{-1} o(\lambda, \boldsymbol{g}) \to 0$ uniformly with respect to $\boldsymbol{g}$ as $\lambda \downarrow 0$, then for any $\varepsilon > 0$ there exists $\lambda_0 > 0$ such that

$$D_0(\boldsymbol{x}, \lambda) \subset \partial f(\boldsymbol{x}) + B(\boldsymbol{0}; \varepsilon)$$

for all $\lambda \in (0, \lambda_0)$.

Proof Take $\varepsilon > 0$ and set $\bar{\varepsilon} = \varepsilon / \bar{Q}$, where $\bar{Q} = \left(4n^2 + 4n\sqrt{n} - 6n - 4\sqrt{n} + 3\right)^{1/2}$. It follows from the proof of Theorem 6.7 and upper semicontinuity of the subdifferential $\partial f(\boldsymbol{x})$ that for $\bar{\varepsilon} > 0$ there exists $\lambda_1 > 0$ such that

$$\min \left\{ \sum_{j=1, j \neq i}^{n} \left(\Gamma_j^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, \lambda, z, \alpha) - v_j \right)^2 \mid \boldsymbol{v} \in \partial f(\boldsymbol{x}) \right\} < \bar{\varepsilon}, \tag{6.13}$$

for all $\lambda \in (0, \lambda_1)$. Let

$$A_0 = \operatorname{argmin}_{\boldsymbol{v} \in \partial f(\boldsymbol{x})} \sum_{j=1, j \neq i}^{n} \left(\Gamma_j^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, \lambda, z, \alpha) - v_j \right)^2.$$

It follows from the assumption and Eq. (6.12) that for $\bar{\varepsilon} > 0$ there exists $\lambda_2 > 0$ such that

$$\min \left\{ \left| \Gamma_i^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, \lambda, z, \alpha) - v_i \right| \mid \boldsymbol{v} \in A_0 \right\} \leq \left(2(n-1) + n^{1/2} \right) \bar{\varepsilon} \tag{6.14}$$

for all $\boldsymbol{g} \in S_1$ and $\lambda \in (0, \lambda_2)$. Let $\lambda_0 = \min(\lambda_1, \lambda_2)$. Then (6.13) and (6.14) imply that

$$\min \left\{ \| \Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, \lambda, z, \alpha) - v_i \| \mid \boldsymbol{v} \in \partial f(\boldsymbol{x}) \right\} \leq \varepsilon$$

for all $\boldsymbol{g} \in S_1$ and $\lambda \in (0, \lambda_0)$. □

As said before, Corollary 6.5 shows that the set $D_0(\boldsymbol{x}, \lambda)$ is an approximation of the subdifferential $\partial f(\boldsymbol{x})$ for sufficiently small $\lambda > 0$. However, this is true at a given point $\boldsymbol{x} \in \mathbb{R}^n$. In order to get convergence results for minimization algorithms based on discrete gradients (see Part III, Chap. 15), we need some relationship between the set $D_0(\boldsymbol{x}, \lambda)$ and $\partial f(\boldsymbol{x})$ also in some neighborhood of a given point $\boldsymbol{x}$. We will consider functions satisfying the following assumption.

Assumption 6.1 Let $\boldsymbol{x} \in \mathbb{R}^n$ be a given point. For any $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda_0 > 0$ such that

$$D_0(\boldsymbol{y}, \lambda) \subset \partial f(\boldsymbol{x} + \bar{B}(\boldsymbol{0}; \varepsilon)) + B(\boldsymbol{0}; \varepsilon)$$

for all $\boldsymbol{y} \in B(\boldsymbol{x}; \delta)$ and $\lambda \in (0, \lambda_0)$. Here,

$$\partial f(\boldsymbol{x} + \bar{B}(\boldsymbol{0}; \varepsilon)) = \bigcup_{\boldsymbol{y} \in \bar{B}(\boldsymbol{x}; \varepsilon)} \partial f(\boldsymbol{y}).$$

In what follows, we show the necessary condition to a point $\boldsymbol{x}^*$ to be a minimizer of a function f using the set $D_0(\boldsymbol{x}^*, \lambda)$.

Theorem 6.8 *Let $\boldsymbol{x}^* \in \mathbb{R}^n$ be a local minimizer of the function f. Then there exists $\lambda_0 > 0$ such that $\boldsymbol{0} \in D_0(\boldsymbol{x}^*, \lambda)$ for all $\lambda \in (0, \lambda_0)$.*

Proof The proof follows from the fact that the set $D_0(\boldsymbol{x}^*, \lambda)$ is compact and convex for any $\lambda > 0$. □

The last theorem given in this section shows that the set $D_0(\boldsymbol{x}, \lambda)$ can be used to compute descent directions (see Definition 6.5).

Theorem 6.9 *Let $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda > 0$ and $0 \notin D_0(\boldsymbol{x}, \lambda)$. That is, $\|\boldsymbol{v}^0\| = \min\{\|\boldsymbol{v}\| \mid \boldsymbol{v} \in D_0(\boldsymbol{x}, \lambda)\} > 0$. Then, $\boldsymbol{g}^0 = -\|\boldsymbol{v}^0\|^{-1}\boldsymbol{v}^0$ is a descent direction at $\boldsymbol{x}$.*

6.3 Piecewise Partially Separable Functions and Computation of Discrete Gradients

Some important practical problems can be reduced to NSO problems which contain hundreds or thousands of variables. The cluster analysis problem and the problem of calculation of piecewise linear function separating two sets are among such problems (see Sect. 7.2 in Part II). Most of large-scale optimization problems have a special structure which can be exploited to design efficient algorithms. In this section we will discuss one of such structures: *piecewise partial separability of nonsmooth functions*. In particular, we show how to calculate the discrete gradient for a piecewise partially separable function.

6.3.1 Piecewise Partially Separable Functions

Let f be a scalar function defined on an open set $D_0 \subseteq \mathbb{R}^n$ containing a closed set $D \subseteq \mathbb{R}^n$.

Definition 6.6 The function f is called *partially separable* if there exists a family of $n \times n$ diagonal matrices $U_i, \ i = 1, \ldots, M$ such that the function f can be represented as follows:

$$f(\boldsymbol{x}) = \sum_{i=1}^{M} f_i(U_i \boldsymbol{x}).$$

Without loss of generality we assume that the matrices U_i are binary, that is they contain only 0 and 1. It is also assumed that the number m_i of non-zero elements in the diagonal of the matrix U_i is much smaller than n. In other terms, the function f is called partially separable if it can be represented as the sum of functions of a much smaller number of variables. If $M = n$ and $\text{diag}(U_i) = \boldsymbol{e}_i$ where $\boldsymbol{e}_i$ is the ith orth vector, then the function f is *separable*.

Any function f can be considered as partially separable if we take $M = 1$ and $U_1 = I$, where I is the identity matrix. However, we consider situations where $M > 1$ and $m_i \ll n, \ i = 1, \ldots, M$.

Example 6.1 (*Partially separable function*). Consider the following function

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} \min\{|x_i|, |x_1|\}.$$

This function is partially separable. Indeed, in this case $M = n, \ m_i = 2$, $U_i^{11} = 1, \ U_i^{ii} = 1$, all other elements of U_i are zeros for all $i = 1, \ldots, n$ and $f_i(U_i \boldsymbol{x}) = \min\{|x_i|, |x_1|\}$.

Definition 6.7 The function f is said to be *piecewise partially separable* if there exists a finite family of closed sets $D_1, \ldots, D_m$ such that $\bigcup_{i=1}^{m} D_i = D$ and the function f is partially separable on each set $D_i, \ i = 1, \ldots, m$.

Example 6.2 (*Piecewise partially separable function 1*). All partially separable functions are piecewise partially separable.

Example 6.3 (*Piecewise partially separable function* 2). Consider the following function

$$f(\boldsymbol{x}) = \max_{j=1,\ldots,n} \sum_{i=1}^{n} |x_i - x_j|.$$

The function f is piecewise partially separable. It is clear that the functions

$$\varphi_j(\boldsymbol{x}) = \sum_{i=1}^{n} |x_i - x_j|, \; j = 1, \ldots, n$$

are partially separable with $M = n, m_i = 2$ and $U_i^{ii} = U_i^{jj} = 1$ for all $i = 1, \ldots, n$. In this case the sets $D_i, \; i = 1, \ldots, n$ are defined by

$$D_i = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \varphi_i(\boldsymbol{x}) \geq \varphi_j(\boldsymbol{x}), \; j = 1, \ldots, n, \; j \neq i\right\}.$$

The piecewise partially separability of the function f follows from the fact that the maximum of partially separable functions is piecewise partially separable, which will be proved later on in Theorem 6.14.

6.3.2 Chained and Piecewise Chained Functions

One of the interesting and important classes of partially separable functions is the one of the so-called *chained functions*.

Definition 6.8 The function f is said to be *k-chained*, $k \leq n$, if it can be represented as follows:

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-k+1} f_i(x_i, \ldots, x_{i+k-1}), \; \boldsymbol{x} \in \mathbb{R}^n.$$

For example, if $k = 2$, the function f is

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} f_i(x_i, x_{i+1}).$$

Theorem 6.10 *Any k-chained function is partially separable.*

Proof Indeed for k-chained functions $M = n - k + 1,\ m_i = k$ and the matrices $U_i,\ i = 1, \ldots, M$ are defined by

$$U_i^{jj} = 1,\ j = i, \ldots, i + k - 1$$

and all other elements of U_i are zeros. □

Lemma 6.5 *Any separable function is* 1*-chained.*

Proof Exercise. □

Definition 6.9 The function f is said to be *piecewise k-chained* if there exists a finite family of closed sets $D_1, \ldots, D_m$ such that $\bigcup_{i=1}^m D_i = D$ and the function f is k-chained on each set $D_i,\ i = 1, \ldots, m$.

Theorem 6.11 *Any piecewise k-chained function is piecewise partially separable.*

Proof The proof directly follows from Theorem 6.10. □

The following is an example of piecewise 2-chained function.

Example 6.4 (*Chained Crescent I function*).

$$f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$$

where

$$f_1(\boldsymbol{x}) = \sum_{i=1}^{n-1} \Big(x_i^2 + (x_{i+1} - 1)^2 + x_{i+1} - 1\Big),$$

$$f_2(\boldsymbol{x}) = \sum_{i=1}^{n-1} \Big(-x_i^2 - (x_{i+1} - 1)^2 + x_{i+1} + 1\Big).$$

Both f_1 and f_2 are 2-chained functions. We define two sets as follows:

$$D_1 = \{\boldsymbol{x} \in \mathbb{R}^n \mid\ f_1(\boldsymbol{x}) \geq f_2(\boldsymbol{x})\},$$

$$D_2 = \{\boldsymbol{x} \in \mathbb{R}^n \mid\ f_2(\boldsymbol{x}) \geq f_1(\boldsymbol{x})\}.$$

It is clear that the sets D_1 and D_2 are closed, $f(\boldsymbol{x}) = f_1(\boldsymbol{x})$ for $\boldsymbol{x} \in D_1$ and $f(\boldsymbol{x}) = f_2(\boldsymbol{x})$ for $\boldsymbol{x} \in D_2$. Furthermore $D_1 \bigcup D_2 = D$. Thus, the function f is piecewise 2-chained.

Definition 6.10 The function f is said to be *piecewise separable* if there exists a finite family of closed sets $D_1, \ldots, D_m$ such that $\bigcup_{i=1}^{m} D_i = D$ and the function f is separable on each set $D_i,\ i = 1, \ldots, m$.

Theorem 6.12 *Any piecewise separable function is piecewise* 1*-chained.*

Proof Since any separable function is 1-chained (Lemma 6.5) the proof is straightforward. □

Corollary 6.6 *Any piecewise separable function is piecewise partially separable.*

Lemma 6.6 *All separable functions are piecewise separable. In this case* $m = 1$.

Proof Exercise. □

Example 6.5 (*Piecewise separable function 1*). All piecewise linear functions are piecewise separable. A function $f : D \to \mathbb{R}$ is said to be piecewise linear, if there exists a finite family of closed sets $Q_1, \ldots, Q_p$ such that $\bigcup_{i=1}^{p} Q_i = D$ and the function f is linear on each set $Q_i,\ i = 1, \ldots, p$. Since any linear function is separable, the function f is piecewise separable and in this case $m = p$.

Example 6.6 (*Piecewise separable function 2*). One of the simplest piecewise separable functions is the following maximum function

$$f(\boldsymbol{x}) = \max_{i=1,\ldots,n} x_i^2.$$

Here $m = n$ and

$$D_i = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^2 \geq x_j^2,\ j = 1, \ldots, n,\ j \neq i\}.$$

$f(\boldsymbol{x}) = x_i^2$ for any $\boldsymbol{x} \in D_i$. It is clear that $\bigcup_{i=1}^{m} D_i = \mathbb{R}^n$. It should be noted that function f is neither separable nor piecewise linear.

6.3.3 Properties of Piecewise Partially Separable Functions

We now introduce some interesting properties of piecewise partially separable functions.

Theorem 6.13 *Let f_1 and f_2 be partially separable functions on the closed set $D \subseteq \mathbb{R}^n$. Then the function $f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x})$ is also partially separable on D.*

Proof Since the functions f_1 and f_2 are partially separable there exist families of matrices $U_i^1,\ i = 1, \ldots, M_1$ and $U_j^2,\ j = 1, \ldots, M_2$ such that

$$f_1(\boldsymbol{x}) = \sum_{i=1}^{M_1} f_{1i}(U_i^1 \boldsymbol{x}),$$
$$f_2(\boldsymbol{x}) = \sum_{j=1}^{M_2} f_{2j}(U_j^2 \boldsymbol{x}).$$

Consider the following sets

$$\mathcal{I} = \left\{ i \in \{1, \ldots, M_1\} \mid U_i^1 \neq U_j^2, \text{ for all } j \in \{1, \ldots, M_2\} \right\},$$
$$\mathcal{J} = \left\{ j \in \{1, \ldots, M_2\} \mid U_j^2 \neq U_i^1, \text{ for all } i \in \{1, \ldots, M_1\} \right\},$$
$$\mathcal{H} = \left\{ (i, j),\ i \in \{1, \ldots, M_1\},\ j \in \{1, \ldots, M_2\} \mid U_i^1 = U_j^2 \right\}.$$

It is clear that for any $i \in \mathcal{I}$ there is no $j \in \{1, \ldots, M_2\}$ such that $(i, j) \in \mathcal{H}$ and, similarly, for any $j \in \mathcal{J}$ there is no $i \in \{1, \ldots, M_1\}$ such that $(i, j) \in \mathcal{H}$. Then the function f can be represented as follows

$$f(\boldsymbol{x}) = \sum_{(i,j)\in\mathcal{H}} (f_{1i}(U_i^1 \boldsymbol{x}) + f_{2j}(U_j^2 \boldsymbol{x})) + \sum_{i\in\mathcal{I}} f_{1i}(U_i^1 \boldsymbol{x}) + \sum_{j\in\mathcal{J}} f_{2j}(U_j^2 \boldsymbol{x}).$$

This function is partially separable, that is

$$f(\boldsymbol{x}) = \sum_{k=1}^{M} \bar{f}_k(V_k \boldsymbol{x}),$$

where $M = M_1 + M_2 - |\mathcal{H}|$ and $|\mathcal{H}|$ stands for the cardinality of the set $\mathcal{H}$. The matrices $V_k,\ k = 1, \ldots, M$ can be defined by

$$V_k = \begin{cases} U_i^1 = U_j^2, & k = 1, \ldots, |\mathcal{H}|,\ (i, j) \in \mathcal{H} \\ U_i^1, & k = |\mathcal{H}| + 1, \ldots, M_1,\ i \in \mathcal{I} \\ U_j^2, & k = M_1 + 1, \ldots, M_1 + M_2 - |\mathcal{H}|,\ j \in \mathcal{J}, \end{cases}$$

and

$$\bar{f}_k(V_k \boldsymbol{x}) = \begin{cases} (f_{1i}(U_i^1 \boldsymbol{x}) + f_{2j}(U_j^2 \boldsymbol{x})), & k = 1, \dots, |\mathcal{H}|, \ (i, j) \in \mathcal{H} \\ f_{1i}(U_i^1 \boldsymbol{x}), & k = |\mathcal{H}| + 1, \dots, M_1, \ i \in \mathcal{I} \\ f_{2j}(U_j^2 \boldsymbol{x}), & k = M_1 + 1, \dots, M_1 + M_2 - |\mathcal{H}|, \ j \in \mathcal{J}. \end{cases}$$

□

We say that two partially separable functions f_1 and f_2 have the same structure, if $\mathcal{I} = \mathcal{J} = \emptyset$. These kinds of functions are more interesting from a practical point of view. If f_1 and f_2 have the same structure, then also the function $f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x})$ has the same structure as f_1, f_2. In addition,

$$f(\boldsymbol{x}) = \sum_{(i,j) \in \mathcal{H}} (f_{1i}(U_i^1 \boldsymbol{x}) + f_{2j}(U_j^2 \boldsymbol{x})).$$

For example, if f_1 and f_2 are k-chained then the function f is also k-chained.

Theorem 6.14 *If f and g are piecewise partially separable (piecewise k-chained, piecewise separable) continuous functions on the closed set D, then*

(i) $h(\boldsymbol{x}) = \alpha f(\boldsymbol{x}), \ \alpha \in \mathbb{R}$ *is piecewise partially separable (piecewise k-chained, piecewise separable);*
(ii) $h(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x})$ *is piecewise partially separable (piecewise k-chained, piecewise separable);*
(iii) $h(\boldsymbol{x}) = \max(f(\boldsymbol{x}), g(\boldsymbol{x})), \ h(\boldsymbol{x}) = \min(f(\boldsymbol{x}), g(\boldsymbol{x}))$ *and* $h(\boldsymbol{x}) = |f(\boldsymbol{x})|$ *are piecewise partially separable (piecewise k-chained, piecewise separable).*

Proof (i) The proof is straightforward.

(ii) Since the functions f and g are piecewise partially separable there exist families of closed sets

$$D_i^f, \ i = 1, \dots, m_1, \ \bigcup_{i=1}^{m_1} D_i^f = D$$

and

$$D_j^g, \ j = 1, \dots, m_2, \ \bigcup_{j=1}^{m_2} D_j^g = D$$

such that the function f is partially separable on the sets D_i^f and the function g is partially separable on the sets D_j^g. We define a family of sets Q_{ij}, for $i = 1, \dots, m_1$, $j = 1, \dots, m_2$, where

$$Q_{ij} = D_i^f \bigcap D_j^g.$$

It is clear that

$$\bigcup_{i,j} Q_{ij} = D$$

and the sets Q_{ij} are closed. Since the sum of partially separable functions is partially separable, we get that $f + g$ is partially separable on each set Q_{ij}.

The proofs for piecewise k-chained and piecewise separable functions are similar.

(iii) Consider the following two sets

$$P_1 = \{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \geq g(\boldsymbol{x})\}, \quad \text{and} \quad P_2 = \{\boldsymbol{x} \in D \mid g(\boldsymbol{x}) \geq f(\boldsymbol{x})\}.$$

It is clear that $P_1 \bigcup P_2 = D$. Since the functions f and g are continuous, the sets P_1 and P_2 are closed. We define the following families of sets:

$$Q_i^1 = P_1 \bigcap D_i^f, \ i = 1, \ldots, m_1, \qquad Q_j^2 = P_2 \bigcap D_j^g, \ j = 1, \ldots, m_2.$$

These sets are closed. It can be easily shown that

$$\left(\bigcup_i^{m_1} Q_i^1\right) \bigcup \left(\bigcup_j^{m_2} Q_j^2\right) = D,$$

$h(\boldsymbol{x}) = f(\boldsymbol{x}), \ \boldsymbol{x} \in Q_i^1, \ i = 1, \ldots, m_1$, and f is partially separable on each set Q_i^1. Similarly, $h(\boldsymbol{x}) = g(\boldsymbol{x}), \ \boldsymbol{x} \in Q_j^2, \ j = 1, \ldots, m_2$ and g is partially separable on each set Q_j^2. Thus, the function h is piecewise partially separable.

Since $h(\boldsymbol{x}) = \min(f(\boldsymbol{x}), g(\boldsymbol{x})) = -\max(-f(\boldsymbol{x}), -g(\boldsymbol{x}))$, we get that h is piecewise partially separable. In addition, since $h(\boldsymbol{x}) = |f(\boldsymbol{x})| = \max(f(\boldsymbol{x}), -f(\boldsymbol{x}))$ and both f and $-f$ are piecewise partially separable, it follows that function h is also piecewise partially separable.

Again the proofs for piecewise k-chained and piecewise separable functions are similar. □

Next we study the Lipschitz continuity and the directional differentiability of piecewise partially separable functions. Let us first assume that the function f is partially separable and the functions $f_i, \ i = 1, \ldots, M$ are directionally differentiable. That is, the directional derivative $f_i'(\boldsymbol{x}; \boldsymbol{d})$ exists. Then, the function f is also directionally differentiable and

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \sum_{i=1}^{M} f_i'(U_i\boldsymbol{x}; U_i\boldsymbol{d}). \tag{6.15}$$

It follows from formula (6.15) that if f separable then

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \sum_{i=1}^{n} f_i'(x_i; d_i) \tag{6.16}$$

where

$$f_i'(x_i; d_i) = \begin{cases} f_{i+}'(x_i) & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0, \\ -f_{i-}'(x_i) & \text{if } d_i < 0. \end{cases}$$

and $f_{i+}'(x_i),\ f_{i-}'(x_i)$ are the right and left side derivatives of the function f_i at the point x_i.

Now, let f be a piecewise partially separable function on the closed convex set $D \subset \mathbb{R}^n$. That is, there exists a family of closed sets $D_j,\ j = 1, \ldots, m$ such that $\bigcup_{j=1}^m D_j = D$, $f(\boldsymbol{x}) = f_j(\boldsymbol{x}),\ \boldsymbol{x} \in D_j$ and the functions f_j are partially separable on D_j.

Theorem 6.15 *Let f be continuous and a piecewise partially separable function on the closed convex set $D \subset \mathbb{R}^n$. In addition, let each function f_j be locally Lipschitz continuous on $D_j,\ j = 1, \ldots, m$. Then the function f is locally Lipschitz continuous on D.*

Proof We take any bounded subset $\overline{D} \subset D$. Then there exists a subset of indices $\{j_1, \ldots, j_p\} \subset \{1, \ldots, m\}$ such that $\operatorname{conv} \overline{D} \bigcap D_{j_k} \neq \emptyset,\ k = 1, \ldots, p$. Let $L_{j_k} > 0$ be a Lipschitz constant of the function f_{j_k} on the set $\operatorname{conv} \overline{D} \bigcap D_{j_k},\ k = 1, \ldots, p$. Let

$$L_0 = \max_{k=1,\ldots,p} L_{j_k}.$$

Now we take any two points $\boldsymbol{x}, \boldsymbol{y} \in \overline{D}$. Then there exist indices $j_{k_1},\ j_{k_2} \in \{j_1, \ldots, j_p\}$ such that $\boldsymbol{x} \in D_{j_{k_1}}$ and $\boldsymbol{y} \in D_{j_{k_2}}$. If $k_1 = k_2 = k$ then it is clear that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| = |f_k(\boldsymbol{x}) - f_k(\boldsymbol{y})| \le L_k \|\boldsymbol{x} - \boldsymbol{y}\| \le L_0 \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Otherwise we consider the segment $[\boldsymbol{x}, \boldsymbol{y}] = \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y},\ \alpha \in [0, 1]$ joining these two points and define the following set

$$Z_{[\boldsymbol{x},\boldsymbol{y}]} = \left\{ \boldsymbol{z} \in [\boldsymbol{x}, \boldsymbol{y}] \mid \exists l_1, l_2 \in \{1, \ldots, p\},\ \boldsymbol{z} \in D_{j_{l_1}} \bigcap D_{j_{l_2}} \right\}.$$

It is clear that in this case the set $Z_{[\boldsymbol{x},\boldsymbol{y}]}$ is not empty. Then there exists a sequence of points $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N\} \subset Z_{[\boldsymbol{x},\boldsymbol{y}]},\ N \le p$ such that

(i) $\{\boldsymbol{x}, \boldsymbol{z}_1\} \subset D_{j_{k_1}},\ l_0 = k_1$;
(ii) $\{\boldsymbol{z}_N, \boldsymbol{y}\} \subset D_{j_{k_2}},\ l_N = k_2$;
(iii) for all $i \in \{1, \ldots, N - 1\}$ there exists $l_i \in \{1, \ldots, p\}$ such that $\{\boldsymbol{z}_i, \boldsymbol{z}_{i+1}\} \subset D_{j_{l_i}}$.

Then taking into account the continuity of the function f we have

$$\begin{aligned}|f(\boldsymbol{y})-f(\boldsymbol{x})| &= \left|f(\boldsymbol{y})+\sum_{i=1}^{N}(f(\boldsymbol{z}_i)-f(\boldsymbol{z}_i))-f(\boldsymbol{x})\right| \\ &= \left|f_{j_{k_2}}(\boldsymbol{y})+\sum_{i=1}^{N}(f_{j_{l_{i-1}}}(\boldsymbol{z}_i)-f_{j_{l_i}}(\boldsymbol{z}_i))-f_{j_{k_1}}(\boldsymbol{x})\right| \\ &\le |f_{j_{k_2}}(\boldsymbol{y})-f_{j_{k_2}}(\boldsymbol{z}_N)|+\sum_{i=1}^{N-1}|f_{j_{l_i}}(\boldsymbol{z}_{i+1})-f_{j_{l_i}}(\boldsymbol{z}_i)| \\ &\quad +|f_{j_{k_1}}(\boldsymbol{z}_1)-f_{j_{k_1}}(\boldsymbol{x})| \\ &\le L_{j_1}\|\boldsymbol{y}-\boldsymbol{z}_N\|+\sum_{i=1}^{N-1}L_{j_i}\|\boldsymbol{z}_i-\boldsymbol{z}_{i+1}\|+L_{j_{k_1}}\|\boldsymbol{z}_1-\boldsymbol{x}\| \\ &\le L_0(\|\boldsymbol{y}-\boldsymbol{z}_N\|+\sum_{i=1}^{N-1}\|\boldsymbol{z}_i-\boldsymbol{z}_{i+1}\|+\|\boldsymbol{z}_1-\boldsymbol{x}\|).\end{aligned}$$

Then, as all $\boldsymbol{z}_i$ are aligned on the segment $[\boldsymbol{x}, \boldsymbol{y}]$, we obtain

$$|f(\boldsymbol{y})-f(\boldsymbol{x})| \le L_0\|\boldsymbol{y}-\boldsymbol{x}\|.$$

Since points $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary it follows that the function f is locally Lipschitz continuous. □

Corollary 6.7 *Assume that all conditions of* Theorem 6.15 *are satisfied. Then the function f is Clarke subdifferentiable* (see Definition 5.2).

Theorem 6.16 *Assume that for any two points $\boldsymbol{x}, \boldsymbol{y} \in D$ the set $Z_{[\boldsymbol{x},\boldsymbol{y}]}$ is finite and all functions $f_j,\ j = 1, \dots, m$ are directionally differentiable. Then the function f is also directionally differentiable.*

Proof We take any point $\boldsymbol{x} \in D$ and any direction $\boldsymbol{d} \neq \boldsymbol{0}$ such that $\boldsymbol{x} + \alpha\boldsymbol{d} \in D,\ \alpha \in [0, \bar{\alpha}]$ for some $\bar{\alpha} > 0$. By the definition we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) = \lim_{\alpha\downarrow 0} \frac{f(\boldsymbol{x}+\alpha\boldsymbol{d})-f(\boldsymbol{x})}{\alpha}.$$

Assume that $\boldsymbol{x} \in \bigcap_{k\in\mathcal{K}} D_k$, where $\mathcal{K} \subset \{1, \dots, m\}$. Let $\boldsymbol{y} = \boldsymbol{x} + \bar{\alpha}\boldsymbol{d} \in D$. Since the set $Z_{[\boldsymbol{x},\boldsymbol{y}]}$ is finite there exists a finite sequence of numbers $\alpha_1, \dots, \alpha_l$ such that $\alpha_i \in (0, \bar{\alpha})$ and $\boldsymbol{x} + \alpha_j\boldsymbol{d} \in D_{k_j} \bigcap D_{k_{j+1}},\ j = 1, \dots, l$ and

$$\begin{aligned}&[\boldsymbol{x}, \boldsymbol{x}+\alpha_1\boldsymbol{d}] \subset D_{k_1},\ k_1 \in \mathcal{K}; \\ &[\boldsymbol{x}+\alpha_l\boldsymbol{d}, \boldsymbol{y}] \subset D_{k_{l+1}}; \\ &[\boldsymbol{x}+\alpha_i\boldsymbol{d}, \boldsymbol{x}+\alpha_{i+1}\boldsymbol{d}] \subset D_{k_{i+1}} \text{ for all } i \in \{1, \dots, l-1\}.\end{aligned}$$

This implies that the segment $[\boldsymbol{x}, \boldsymbol{x}+\alpha_1 g] \subset D_{k_1}$. Thus

$$f'(\boldsymbol{x}; \boldsymbol{d}) = f'_{k_1}(\boldsymbol{x}; \boldsymbol{d}).$$

It follows that if the function f is piecewise partially separable then its directional derivative can be calculated using Eq. (6.15) and if this function is piecewise separable then its directional derivative can be calculated using Eq. (6.16). □

In general piecewise partially separable functions are not subdifferentiable regular (see Definition 5.5). The following example demonstrates it.

Example 6.7 (*Nonregularity of piecewise partially separable functions*). Consider the function

$$f(x_1, x_2) = \max\{|x_1| - |x_2|, -|x_1| + |x_2|\}, \ (x_1, x_2) \in \mathbb{R}^2.$$

This function is piecewise separable. However, it is not regular. Indeed, for the direction $\boldsymbol{d} = (1, 1)$ at the point $\boldsymbol{x} = (0, 1)$ we have

$$f'(\boldsymbol{x}; \boldsymbol{d}) = 0 \ \text{ and } \ f^\circ(\boldsymbol{x}; \boldsymbol{d}) = 2,$$

that is $f'(\boldsymbol{x}; \boldsymbol{d}) < f^\circ(\boldsymbol{x}; \boldsymbol{d})$.

This example shows that in general for the subdifferential of piecewise partially separable functions a full calculus does not exist and therefore the computation of their subgradients is a difficult task.

6.3.4 Calculation of the Discrete Gradients

In order to demonstrate how the computation of discrete gradients (see Definition 6.5) can be simplified for piecewise partially separable functions, we consider the following function

$$f(\boldsymbol{x}) = \sum_{i=1}^{M} \max_{j \in \mathcal{J}_i} \ \min_{k \in \mathcal{K}_j} f_{ijk}(\boldsymbol{x}), \tag{6.17}$$

where functions $f_{ijk}, \ i = 1, \ldots, M, \ j \in \mathcal{J}_i, \ k \in \mathcal{K}_j$ are partially separable. That is there exists a family of $n \times n$ matrices $U_{ijkt}, \ t = 1, \ldots, M_{ijk}$ such that

$$f_{ijk}(\boldsymbol{x}) = \sum_{t=1}^{M_{ijk}} f^t_{ijk}(U_{ijkt}\boldsymbol{x}).$$

Now the function f is piecewise partially separable. If all functions f_{ijk} are l-chained (separable), then the function f is piecewise l-chained (piecewise separable).

We take any point $\boldsymbol{x} \in \mathbb{R}^n$ and any direction $\boldsymbol{g} \in S_1$, where as before S_1 is the unit sphere. For the calculation of the discrete gradient of f at $\boldsymbol{x}$ with respect to the direction $\boldsymbol{g}$ first we have to define the sequence

$$\boldsymbol{x}_i^0, \ldots, \boldsymbol{x}_i^{i-1}, \boldsymbol{x}_i^{i+1}, \ldots, \boldsymbol{x}_i^n.$$

Here $i \in \mathcal{I}(\boldsymbol{g}) = \{i \in \{1, \ldots, n\} \mid g_i \neq 0\}$. The point $\boldsymbol{x}_i^p$ differs from $\boldsymbol{x}_i^{p-1}$, $p \in \{1, \ldots, n\}$, $p \neq i$ by one coordinate only.

We will call functions f_{ijk}^t *term functions*. The total number of these functions is

$$N_0 = \sum_{i=1}^{M} \sum_{j \in \mathcal{J}_i} \sum_{k \in \mathcal{K}_j} M_{ijk}.$$

For one evaluation of the function f we have to compute term functions N_0 times. Since for one evaluation of the discrete gradient we compute $n+1$ times the function f, the total number of computation of term functions for one evaluation of the discrete gradient is

$$N_t = (n+1)N_0.$$

For $p \in \{1, \ldots, n\}$ we introduce

$$Q_p^{ijk} = \left\{t \in \{1, \ldots, M_{ijk}\} \mid U_{ijkt}^{pp} = 1\right\},$$

$$\bar{Q}_p^{ijk} = \left\{t \in \{1, \ldots, M_{ijk}\} \mid U_{ijkt}^{pp} = 0\right\}.$$

It is clear that $M_{ijk} = |Q_p^{ijk}| + |\bar{Q}_p^{ijk}|$. One can assume that $|Q_p^{ijk}| \ll |\bar{Q}_p^{ijk}|$.

Example 6.8 (*The number of term functions 1*). If all functions f_{ijk} are l-chained then

$$|Q_p^{ijk}| \leq l \quad \text{and} \quad |\bar{Q}_p^{ijk}| \geq n - l - 1.$$

If these functions are separable then

$$|Q_p^{ijk}| = 1 \quad \text{and} \quad |\bar{Q}_p^{ijk}| = n - 1.$$

The function f_{ijk} can be calculated at the point $\boldsymbol{x}_i^p$ using the following simplified scheme

$$f_{ijk}(\boldsymbol{x}_i^p) = \sum_{t \in Q_p^{ijk}} f_{ijk}^t(U_{ijkt}\boldsymbol{x}_i^p) + \sum_{t \in \bar{Q}_p^{ijk}} f_{ijk}^t(U_{ijkt}\boldsymbol{x}_i^{p-1}). \tag{6.18}$$

That is, we compute only functions $f_{ijk}^t,\ t \in Q_p^{ijk}$ at the point $\boldsymbol{x}_i^p$ and all other functions remain the same as at the point $\boldsymbol{x}_i^{p-1}$. Thus, in order to calculate the function f at the point $\boldsymbol{x}_i^p$, we compute

$$N_s = \sum_{i=1}^{M} \sum_{j \in J_i} \sum_{k \in K_j} |Q_p^{ijk}|$$

times the term functions at this point. Since $|Q_p^{ijk}| \ll M_{ijk}$, one can expect that $N_s \ll N_0$.

Example 6.9 (*The number of term functions 2*). If all functions $f_{ijk},\ i = 1, \ldots, M,\ j \in \mathcal{J}_i,\ k \in \mathcal{K}_j$ are l-chained then

$$N_s \leq l \sum_{i=1}^{M} \sum_{j \in \mathcal{J}_i} |\mathcal{K}_j|.$$

If all these functions are separable then

$$N_s = \sum_{i=1}^{M} \sum_{j \in \mathcal{J}_i} |\mathcal{K}_j|.$$

In order to compute one discrete gradient at the point x with respect to the direction $\boldsymbol{g} \in S_1$ we have to compute the function f at the points $\boldsymbol{x}$ and $x + \lambda \boldsymbol{g}$ using formula (6.17) and at all other points $\boldsymbol{x}_i^p, p = 1, \ldots, n,\ p \neq i$ it can be computed using simplified scheme (6.18). In this case the total number of computation of term functions is

$$N_{ts} = 2N_0 + (n-1)N_s$$

which is significantly less than N_t when n is large.

Example 6.10 (*Special case from cluster analysis*). As a special case of functions (6.17) consider the following function

$$f(\boldsymbol{x}) = \sum_{i=1}^{M} \min_{k \in \bar{\mathcal{K}}} f_{ik}(\boldsymbol{x}_k) \tag{6.19}$$

where $\bar{\mathcal{K}} = \{1, \dots, K\}$, $\boldsymbol{x}_k \in \mathbb{R}^n$, $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_K) \in \mathbb{R}^{K \times n}$ and the functions f_{ik} are separable

$$f_{ik}(\boldsymbol{x}) = \sum_{j=1}^{n} f_{ijk}(x_{kj}).$$

The function (6.10) can be derived from the function (6.17) when

$$\mathcal{J}_i = \{1\}, \quad i = 1, \dots, M, \quad \mathcal{K}_j = \{1, \dots, K\}.$$

For the computation of one discrete gradient without using simplified scheme we have to compute $MK(n+1)$ the term functions f_{ijk}, however the use of the simplified scheme allows one to reduce this number to $2MK+n-1$. Since, for instance, in the cluster analysis the number M is large we can assume that $MK \gg n$ and therefore

$$\frac{MK(n+1)}{2MK+n-1} \approx \frac{n+1}{2}.$$

If n is large then we can significantly reduce computational efforts using the simplified scheme.

6.4 Summary

In this chapter we have introduced the continuous approximations of a subdifferential and the notion of a discrete gradient. We have demonstrated how discrete gradients can be used to compute subsets of continuous approximations. From a practical point of view, discrete gradients may be useful, since only values of a function are used to compute discrete gradients and no subderivative information is needed. In addition, we have introduced a class of piecewise partially separable functions which is an important subclass of general nonsmooth functions. We have demonstrated that for such functions the number of function evaluations for the computation of discrete gradients can be significantly reduced using their specific structures. This is very important when one applies discrete gradients to design algorithms for minimizing large-scale piecewise partially separable functions.

Exercises

6.1 Let the mapping $Cf(\boldsymbol{x}, \varepsilon)$ be continuous with respect to $\boldsymbol{x}$ and monotonically decreasing as $\varepsilon \downarrow 0$. Show that

$$\lim_{\varepsilon \downarrow 0} Cf(\boldsymbol{x}, \varepsilon) = C_0 f(\boldsymbol{x})$$

6.2 Determine piecewise partially separable representation of the following functions by defining matrices U:

(a) $f(\boldsymbol{x}) = \max\{\min\{-4x_1 + x_2, x_1 - 5x_2\}, \min\{x_1 + 3x_2 - 1, x_1 - 2x_2 - 3\}\}$,
$\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$;

(b) $f(\boldsymbol{x}) = \min\{x_1 - 2x_2 - 1, 2x_1 + 4x_2\} + \min\{-2x_1 + 3x_2 + 1, -4x_1 - 2x_2\}\}$,
$\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$;

(c) $f(\boldsymbol{x}) = |\min\{x_1 - x_2 - 2, -x_1 - 2x_2 + 1\}|$,
$\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$;

6.3 (Lemma 6.5) Prove that any separable function is 1-chained.

6.4 (Lemma 6.6) Prove that all separable functions are piecewise separable and in these cases $m = 1$.

Part I
Notes and References

The first part is mainly based on the previous work [168], which in turn was based on the classic books of Rockafellar [204] and Clarke [61]. Compared to [168], we have included more theory about cones in general, separation and supporting hyperplanes, and generalized convexities and optimality conditions. We have also added illustrative figures and examples in order to make the theory more readable. Other related works are, for example, [35, 36, 44, 109, 110].

Quasidifferentials were introduced by Demyanov and Rubinov [69] (see, also [70–73]). The notion of codifferential was first introduced by Demyanov [68] and further studied in [71, 72]. In addition, in [71, 72], it was proved that the classes of quasidifferentiable and codifferentiable functions coincide. The basic and singular subdifferentials were introduced by Mordukhovich [182]. There are many other generalizations of subdifferentials which were not considered in this book: see, for example, [108, 112, 113, 127, 128, 137, 195, 227].

There have been numerous attempts to generalize the concept of convexity. The concept of pseudoconvexity has been extended for nonsmooth cases by many authors: see, for example, [11, 197] and the references therein. One way to do this is by using directional derivatives. The Dini directional derivatives were used, for example, by Diewert [78], Komlósi [136] and Borde and Crouzeix [43]. In [138], this idea was generalized for lower semicontinuous functions via h-pseudoconvexity, where $h(\boldsymbol{x}, \boldsymbol{d})$ is any real-valued bifunction, that is, for example, any directional derivative. In this book, we have used the definition by Hiriart-Urruty [107] for locally Lipschitz continuous functions. For an excellent survey of generalized convexities, we refer to [197]. In addition, see, [37, 207, 233].

The optimality conditions for generalized convex functions have also been studied. There exist a wide amount of papers published for smooth singleobjective cases (see [197] and references therein). For nonsmooth and multiobjective problems, the necessary conditions were derived, for instance, in [190, 191, 214], and both necessary and sufficient conditions were derived, for instance, in [162].

Continuous approximations of the subdifferential are discussed in [12, 13], and the notion of uniform and strong continuous approximations of the subdifferential was introduced in [231]. The notion of discrete gradient was introduced in [12], and it was further developed in [13, 24]. Other related works for approximation of subgradients can be found, for instance, in [186, 215].

Different algorithms have been developed for solving large-scale optimization problems where both objective and constraint functions are twice continuously differentiable (see, for example, [62, 63, 94]). These algorithms strongly rely on the structure of large-scale optimization problems, specifically the sparsity of Hessians of the objective and constraint functions. The problem of computation of Hessians of twice continuously differentiable partially separable functions was discussed by many authors (see, e.g. [1, 62]). Piecewise partially separable functions were studied in [18].

Part II
Nonsmooth Problems

Introduction

We are considering a nonsmooth optimization (NSO) problem of the form

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \end{cases}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is not required to have continuous derivatives. We suppose only that f is a locally Lipschitz continuous function on the feasible set $S \subseteq \mathbb{R}^n$.

NSO problems of this type arise in many applied fields, for example in economics, mechanics, engineering, optimal control, and data mining, as well as in computational chemistry and biology. The source of nonsmoothness can be divided into four classes:

physical,
technological,
methodological, and
numerical nonsmoothness.

In *physical nonsmoothness*, the original phenomenon under consideration itself contains different kinds of discontinuities and irregularities. Typical examples of physical nonsmoothness are the phase changes of materials in continuous casting of steel, and piecewise linear tax models in economics. Although the objective and constraints functions are originally smooth, some external elements may introduce nonsmooth components to the problem. *Technological nonsmoothness* in a model is usually caused by some extra technological constraints, which affect nonsmooth dependence between the variables. Examples of this include so-called obstacle problems in optimal shape design, and discrete feasible sets in product planning. On the other hand, using certain major methodologies for solving difficult smooth problems leads directly to the need to solve NSO problems, which are either smaller in dimension or simpler in structure. Examples of this *methodological nonsmoothness* include decompositions, dual formulations, and exact penalty functions. Finally, the problems may be analytically smooth but *numerically nonsmooth*. That is the case, for instance, with noisy input data and so-called stiff problems, which are numerically unstable and behave like nonsmooth problems.

This part is organized as follows: In Chap. 7, we introduce some real-life NSO problems; that is, some problems from computational chemistry and biology; data mining and regression analysis; engineering and industrial areas; optimal control; image denoising; and economics. Then, in Chap. 8, we give some formulations which lead to NSO problems although the original problem is smooth: exact penalty formulations and Lagrange relaxation. In addition, we represent the maximum eigenvalue problem which is an important component of many engineering design problems and graph theoretical applications. Finally, in Chap. 9, we give a collection of academic test problems that can be and have been used to test NSO solvers.

Chapter 7
Practical Problems

In this chapter, we briefly describe several kinds of application problems which naturally have nonsmooth characteristics or in which a nonsmooth formulation has proven to yield some improvement for the models. These kinds of problems arise in computational chemistry and biology, optimal control, and data analysis, to mention but a few. The interested reader can find more details of each class of problems in the Notes and References at the end of the part.

7.1 Computational Chemistry and Biology

Several problems arising in molecular modeling lead to nonsmooth optimization problems. Here we describe four of them, namely: the polyatomic clustering problem, the molecular distance geometry problem, protein structural alignment, and molecular docking.

7.1.1 Polyatomic Clustering Problem

A cluster is a group of identical molecules or atoms loosely bound by inter-atomic forces. The optimal geometry minimizes the potential energy of the cluster, expressed as a function of Cartesian coordinates

$$E(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} v(r_{ij}), \tag{7.1}$$

where N is the number of atoms (molecules) in the cluster and r_{ij} is the distance between the centers of a pair of atoms (molecules). That is,

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}.$$

A. Bagirov et al., *Introduction to Nonsmooth Optimization*,
DOI: 10.1007/978-3-319-08114-4_7

The simplest (yet extremely difficult to solve) model uses the *Lennard-Jones* pairwise potential energy function

$$v(r_{ij}) = \frac{1}{r_{ij}^{12}} - \frac{2}{r_{ij}^{6}}. \tag{7.2}$$

Variations of this problem include carbon and argon clusters as well as water molecule clusters. In addition, the Lennard-Jones potential represents an important component in many of the potential energy models used for complex molecular conformation and protein folding problems.

The objective function of the Lennard-Jones potential (7.1) and (7.2) is smooth (supposing that $r_{ij} > 0$) and easy to implement. However, it has an extremely complicated landscape with a huge number of local minima. One possibility for a local search method to escape the enormous number of local minima involved in the Lennard-Jones energy landscape is to use a *nonsmooth penalized Lennard-Jones potential*

$$\bar{v}(r) = \frac{1}{r^{2p}} - \frac{2}{r^{p}} + \mu r + \beta \max\{0, r^2 - D^2\}, \tag{7.3}$$

where $p > 0$, $\mu, \beta \geq 0$ are real constants and $D > 0$ is an underestimate of the diameter of the cluster. The local minimum of this modified objective function may be used as a starting point for the local optimization of the Lennard-Jones potential function (7.1) and (7.2). Note that by choosing $p = 6$ and $\mu, \beta = 0$, the penalized Lennard-Jones potential $\bar{v}$ coincides with the Lennard-Jones pairwise potential (7.2).

Parameter p affects the rigidity of the model. By choosing $p < 6$, the atoms (molecules) can be moved more freely, and by decreasing p, the infinite barrier at $r = 0$, which prevents atoms from getting too close to each other, is also decreased. The first penalty term μr gives a penalty to distances between atoms of greater than 1.0. The penalty increases linearly as a function of distance. In turn, the second penalty term adds a penalty to the diameter of the cluster. It has no influence on pairs of atoms close to each other, but it adds a strong penalty to atoms that are far away from each other. In Fig. 7.1 the case $p = 4$, $\mu = 0.2$, $\beta = 1$, $D = 2$ is displayed and compared with the Lennard-Jones pairwise potential (7.2).

7.1.2 Molecular Distance Geometry Problem

Proteins are essential parts of all living organisms and participate in most cellular processes. They are large organic compounds formed by chains of α-amino acid residues bounded by peptide bonds. The chemical structure of a protein is given in Fig. 7.2. The proteins in a living cell contain 20 different residues with side chains having 1–18 atoms. The sequence of residues in a protein is defined by a gene. This sequence is known as the *primary structure* of a protein. Although the chain of

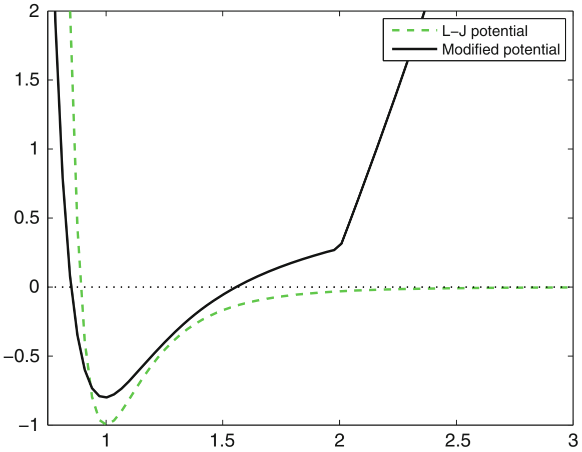


Fig. 7.1 Comparison between Lennard-Jones and modified potentials

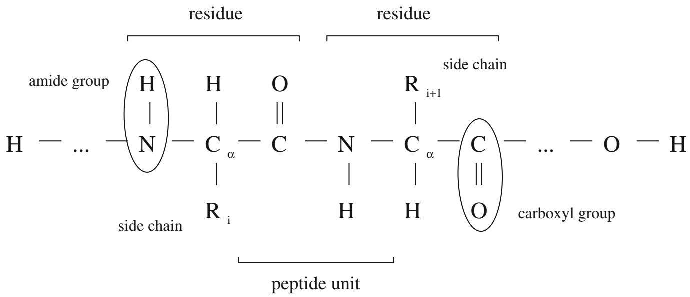


Fig. 7.2 The chemical structure of a protein

residues seems to be linear in Fig. 7.2, which displays the bond structure, in reality, it is bent and twisted by inter-atomic forces in a way that is characteristic for each protein. This three-dimensional configuration is called the *folded state* or the *tertiary structure* of a protein. The alternative structures of the same protein are referred to as its different *conformations*.

The determination of the conformation of a molecule, especially in the protein folding framework, is one of the most important problems in computational chemistry and biology. This is because the conformation is strongly associated with the chemical and biological properties of the molecule. The determination of the molecular conformation can be tackled either by minimizing the potential energy function (if the conformation corresponds to the global minimizer of this function) or by solving a distance geometry problem where some or all distances between the pairs of atoms are known. Both of these methods end in some sort of global optimization problem. The distances required in the distance geometry problem may be obtained e.g. via nuclear magnetic resonance (NMR) spectroscopy.

The *molecular distance geometry problem* (MDGP) can be formulated as follows

Find positions $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ of m atoms in $\mathbb{R}^3$ such that

$$\|\boldsymbol{x}_i - \boldsymbol{x}_j\| = d_{ij} \qquad \text{for all } (i, j) \in S, \tag{7.4}$$

where S is a subset of the atom pairs and d_{ij} with $(i, j) \in S$ is the distance between atoms i and j.

The above formulation corresponds to the *exact MDGP*. If there is an error in the theoretical or experimental data, there may not exist any solution to this problem. This is the case, for instance, if the triangle inequality

$$d_{ij} \leq d_{ik} + d_{kj}$$

is violated for atoms i, j and k. Usually, only a small subset of pairwise distances is known and, in practice, the lower and upper limits for the distances between the atoms are given instead of the exact distances. Hence, a more practical definition of the MDGP is the *general MDGP*, which is formulated as follows

Find positions $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ of m atoms in $\mathbb{R}^3$ such that

$$l_{ij} \leq \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leq u_{ij} \qquad \text{for all } (i, j) \in S, \tag{7.5}$$

where S is a subset of the atom pairs and l_{ij} and u_{ij} are the lower and the upper limits of the distance between atoms i and j, respectively.

This formulation can be reformulated as a NSO problem [211]

$$\begin{aligned} \text{minimize} \qquad & f(\boldsymbol{x}) = \sum_{(i,j)\in S} \max\{l_{ij} - \|\boldsymbol{x}_i - \boldsymbol{x}_j\|, 0\} \\ & \qquad + \max\{\|\boldsymbol{x}_i - \boldsymbol{x}_j\| - u_{ij}, 0\} \end{aligned} \tag{7.6}$$

$$\begin{aligned} \text{where} \qquad & \boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \in \mathbb{R}^{3m}, \quad \text{and} \\ & \boldsymbol{x}_k \in \mathbb{R}^3, \text{ for all } k \in \{1, \ldots, m\} \end{aligned}$$

(some other nonsmooth formulations for both (7.4) and (7.5) are given e.g. in [3, 4]). It is easy to see that $f(\boldsymbol{x}) = 0$ if and only if all of the restrictions $l_{ij} \leq \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leq u_{ij}$ are satisfied.

In particular cases, the MDGP can be solved relatively easily: indeed, when all the distances between all atom pairs of a molecule are known, the exact MDGP (7.4) can be solved using a linear time algorithm. However, when the upper and lower limits are close to each other, the MDGP with relaxed distances belongs to the NP-hard class.

7.1.3 Protein Structural Alignment

The function of different proteins may be the same in spite of different primary structures when they share the same overall three-dimensional structure. Therefore, a fundamental task in structural molecular biology is the *comparison of the structures of proteins* in the hope of finding shared functionalities. This procedure, called *structural alignment*, is carried out on two known structures and is usually based on the Euclidean distance between corresponding residues. Typically, the model is simplified by comparing only one atom per residue, generally but not necessary the C_α-atom (see Fig. 7.2).

Let $A = (a_1, \dots, a_n) \in \mathbb{R}^{3n}$ and $B = (b_1, \dots, b_m) \in \mathbb{R}^{3m}$ be proteins represented by the coordinates of their C_α -atoms. A k-long *subchain* $P = (p_1, \dots, p_k) \in \mathbb{R}^{3k}$ of protein A is a subset of its atoms, arranged by the order of appearance in A, whereupon $1 \le p_1 \le \dots \le p_k \le n$. Let us denote a subchain P of A by $A(P) = (a_{p_1}, \dots, a_{p_k})$ and a subchain Q of B by $B(Q) = (b_{q_1}, \dots, b_{q_k})$. We denote by ϕ a monotone bijection between a subset of $\{1, \dots, n\}$ and a subset $\{1, \dots, m\}$ such that $\phi(p_i) = q_i$. A "*gap*" is two consecutive indices p_i and p_{i+1} (or q_i and q_{i+1}) such that $p_i + 1 < p_{i+1}$ (or $q_i + 1 < q_{i+1}$) and a "*correspondence*" is the two subchains of equal length, $|P| = |Q|$. Some examples of gaps, bijections and correspondences are given in Fig. 7.3. The uppermost correspondence has no gaps, the central correspondence has two gaps, and the lowest correspondence has one gap.

The protein structural alignment problem can be defined as follows [135]:

> Given two proteins A and B, find two subchains P and Q of equal length such that

1. $A(P)$ and $B(Q)$ are similar, and
2. the correspondence length $|P| = |Q|$ is maximal under condition 1.

A protein can be rigidly transformed (that is, rotated and translated) without affecting its inherent structure. Since we are interested in the relative position and orientation of the two proteins, we can keep A fixed and only transform B. Thus, we need to find a transformation D such that some subchain of $D(B) = (D(b_1), \dots, D(b_m))$ fits into some subchain of A.

In practice, the methods for structural alignment are based on the maximization of some *scoring function*. The most commonly used scoring function is the *STRUCTAL-score* [216] associated with the transformation D and a bijection ϕ

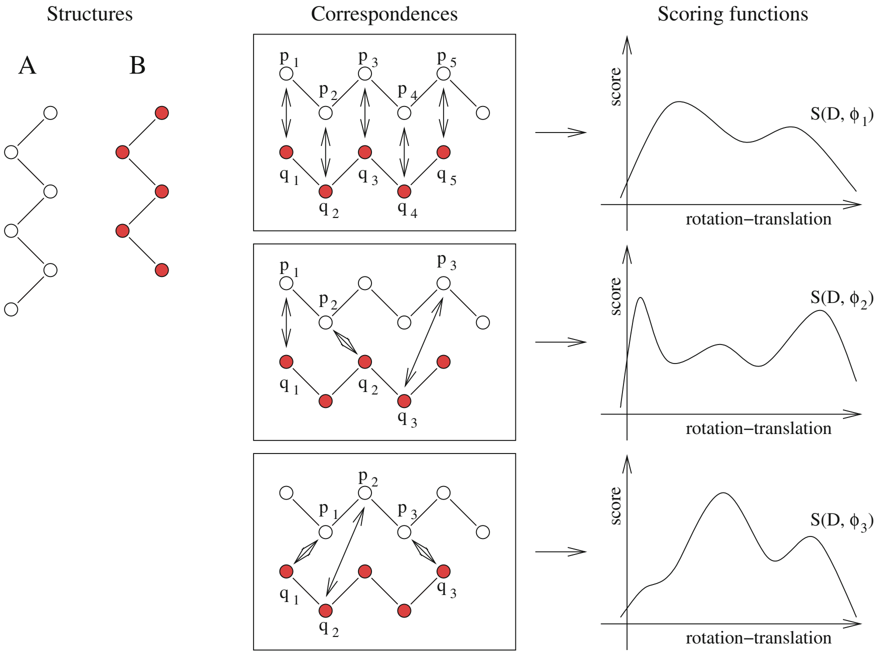


Fig. 7.3 Examples of bijective correspondences. Each correspondence has a smooth score that depends on the rotations and translations of the proteins

$$S(D, \phi) = \sum_{i=1}^{k} \frac{20}{1 + \|a_{p_i} - D(b_{\phi(p_i)})\|^2/5} - 10n_g,$$

where n_g is the number of gaps in a correspondence, that is, in P and Q, and k is the length of the correspondence.

The transformation D may be represented by three translation variables and three angles of rotation. Therefore, the rigid transformation may be represented by a vector $D \in \mathbb{R}^6$. Let $[\phi_1, \ldots, \phi_j]$ be the set of all monotone bijections between subchains of A and B. For each $i = 1, \ldots, j$ and for each rigid transformation D, we define

$$f_i(D) = -S(D, \phi_i)$$

and

$$f_{min}(D) = \min\{f_1(D), \ldots, f_j(D)\}.$$

Now the structural alignment problem can be written as

$$\text{minimize } f_{min}(D) \text{ subject to D} \in \mathbb{R}^6. \tag{7.7}$$

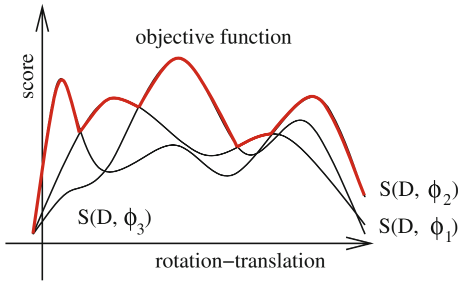


Fig. 7.4 Objective function in the protein structural alignment problem

This is the so-called *low order value optimization (LOVO)* problem, which is a continuous NSO problem. The objective function is the function that assumes the maximum value among all of the possibly scoring functions, as shown in Fig. 7.4.

7.1.4 Molecular Docking

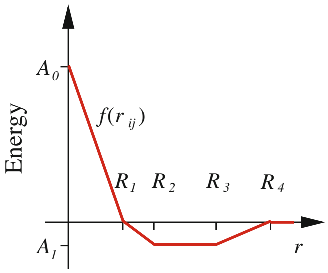
Molecular complexes are composed of two or more molecules that are held together in unique structural relationships by forces other than those of full covalent bonds. The prediction of a small molecule (*ligand*) conformation and orientation relative to the active site of a macromolecular target (*receptor*, usually a protein) is referred to as a *molecular docking problem*. The most important application of the molecular docking problem is in drug discovery, since molecular docking facilitates structure-based ligand design. The idea in drug design is to derive drugs that bind more strongly to a given protein target than the natural substrate.

Molecular docking is basically a conformational sampling procedure in which various docked conformations are explored in order to identify the correct one. This conformational sampling must be guided by a scoring function (or energy function) that identifies the energetically most favorable ligand conformation when bound to a target protein. The general hypothesis is that lower energy scores represent better protein–ligand bindings compared to higher energy ones. Therefore, molecular docking can be formulated as an optimization problem, where the task is to find the ligand-binding mode (i.e. the orientation and the conformation of the ligand relative to the receptor) with the lowest energy.

One frequently used scoring function that allows flexible ligand-protein binding is a *piecewise linear potential* (PLP) as described by Gehlhaar et al. [92], or some modification of it (see e.g. [220, 232]). The scoring function E_{score} is given by

$$E_{score} = E_{tor} + E_{pair}.$$

Here, E_{tor} is the internal *torsional energy* of the ligand that is restricted to $sp^3 - sp^3$ and $sp^2 - sp^3$ bonds:



Interaction type	A_0	A_1	R_1	R_2	R_3	R_4
Steric	20.0	–0.4	3.4	3.6	4.5	5.5
HB	20.0	–2.0	2.3	2.6	3.1	3.4

R_i are in Angstroms. A_j are in arbitrary energy units.

Fig. 7.5 The piecewise linear pairwise potential function used for ligand-protein interactions and the parameters for different interactions

Table 7.1 Pairwise atomic interaction types used in PLP

Atom type	Donor	Acceptor	Both	Nonpolar
Donor	Steric	HB	HB	Steric
Acceptor	HB	Steric	HB	Steric
Both	HB	HB	HB	Steric
Nonpolar	Steric	Steric	Steric	Steric

$$E_{tor} = A \cdot (1 + \cos(n\phi - \phi_0))$$

with $A = 3.0$, $n = 3$, $\phi_0 = \phi$ for $sp^3 - sp^3$ bonds, and $A = 1.5$, $n = 6$, $\phi_0 = 0$ for $sp^2 - sp^3$ bonds. In turn, E_{pair} may be considered as a kind of van der Waals interaction of non-bonded terms. That is,

$$E_{pair} = \sum_{i \neq j} f(r_{ij}),$$

where f is an interval piecewise linear function of the pairwise atom distance r_{ij} of atoms i and j (see Fig. 7.5). The summation runs over all heavy atoms in the ligand and all heavy atoms in the protein. The parameters used in the pairwise potential depend on the atom types involved in the interaction. PLP considers four different atom types: nonpolar, hydrogen-bond-donor, hydrogen-bond-acceptor, and both-acceptor-and-donor. These atom types interact through two types of non-bonded interaction, namely *steric* and *hydrogen bond potentials*. The resulting interactions are given in Table 7.1 and the corresponding parameters are given in Fig. 7.5.

In addition to these terms, the original PLP provides a separate energy term for the internal non-bonded interactions of the ligand by assigning a penalty of 10^4 if two non-bonded ligand atoms come closer than 2.35 Angstrom. However, this kind of energy barrier can be avoided by using the same term for internal ligand-ligand interactions as is used for ligand-protein interactions.

7.2 Data Analysis

Data analysis is a process of gathering, modeling, and transforming data with the goal of highlighting useful information, suggesting conclusions, and supporting decision-making. Related problems of supervised and unsupervised data classification and regression problems arise in many areas, including management science, medicine, chemistry, information retrieval, document extraction, market segmentation, and image segmentation.

In this section, we first consider cluster analysis via nonsmooth optimization, then we describe a supervised data classification problem, and finally, we focus on regression analysis using both piecewise linear and clusterwise linear approximations.

7.2.1 Cluster Analysis via NSO

In this section, we consider data mining as the process of extracting hidden patterns from data. *Clustering* is the *unsupervised classification* of these patterns. Cluster analysis deals with the problems involved in organizing a collection of patterns into clusters based on similarity.

There are different types of clustering problems. We consider unconstrained hard clustering problem which can be formulated as an optimization problem and there are various such formulations: mixed integer nonlinear programming and nonconvex nonsmooth optimization formulations are among them.

In cluster analysis we assume that we have been given a finite set of points A in the n-dimensional space $\mathbb{R}^n$, that is

$$A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}, \text{ where } \boldsymbol{a}_i \in \mathbb{R}^n, \ i = 1, \ldots, m.$$

The subject of cluster analysis is the partition of the set A into a given number k of overlapping or disjoint subsets $A_j, \ j = 1, \ldots, k$ with respect to predefined criteria such that

$$A = \bigcup_{j=1}^{k} A_j.$$

The sets $A_j, \ j = 1, \ldots, k$ are called *clusters*. The clustering problem is said to be *hard clustering problem* if every data point $\boldsymbol{a} \in A$ belongs to one and only one cluster A_j. Unlike hard clustering problem, in the *fuzzy clustering problem* the clusters are allowed to overlap and instances have degrees of appearance in each cluster. Here we will exclusively consider the hard unconstrained clustering problem; that is, we assume that

(i) $A_j \neq \emptyset, \ j = 1, \ldots, k$;
(ii) $A_j \bigcap A_l = \emptyset, \quad \text{for all } j, l = 1, \ldots, k, \ j \neq l$;

(iii) $A = \bigcup_{j=1}^{k} A_j$;

(iv) no constraints are imposed on the clusters $A_j,\ j = 1, \ldots, k$.

Each cluster A_j can be identified by its center (or centroid) $\boldsymbol{x}_j \in \mathbb{R}^n,\ j = 1, \ldots, k$. Data points from the same cluster are similar and data points from different clusters are dissimilar to each other. The similarity between points can be measured using different distance functions. In particular, we can define the *similarity measure* using the squared Euclidean norm, L_1-norm and L_∞-norm:

(i) The distance function using the squared Euclidean norm:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} (\boldsymbol{x}_i - y_i)^2; \tag{7.8}$$

(ii) The distance function using the L_1-norm:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |\boldsymbol{x}_i - y_i|; \tag{7.9}$$

(iii) The distance function using the L_∞-norm:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \max_{i=1,\ldots,n} |\boldsymbol{x}_i - y_i|. \tag{7.10}$$

Note that the distance functions d defined using the L_1 and L_∞ norms are nonsmooth. The use of different distance functions can lead to the finding of different cluster structures in the data set.

The problem of finding k clusters in the set A can be reduced to the following optimization problem:

$$\begin{cases} \text{minimize} & \psi_k(\boldsymbol{x}, \boldsymbol{w}) = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{k} w_{ij} d(\boldsymbol{x}_j, \boldsymbol{a}_i) \\ \text{subject to} & \boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \in \mathbb{R}^{n \times k}, \\ & \sum_{j=1}^{k} w_{ij} = 1,\ i = 1, \ldots, m, \\ & w_{ij} \in \{0, 1\},\ i = 1, \ldots, m,\ j = 1, \ldots, k. \end{cases} \tag{7.11}$$

Here w_{ij} is the association weight of pattern $\boldsymbol{a}_i$ with cluster j (to be found), given by

$$w_{ij} = \begin{cases} 1 & \text{if pattern } \boldsymbol{a}_i \text{ is allocated to cluster } A_j, \\ 0 & \text{otherwise.} \end{cases}$$

and $\boldsymbol{w}$ is an $m \times k$ matrix.

The problem (7.11) is called the *mixed integer nonlinear programming formulation* (MINLP) of the clustering problem. It contains mn integer variables w_{ij}, $i = 1, \dots, m$, $j = 1, \dots, k$ and kn continuous variables $\boldsymbol{x}_j \in \mathbb{R}^n$, $j = 1, \dots, k$.

Cluster centers $\boldsymbol{x}_j,\ j = 1, \dots, k$ can be found by solving the following problem

$$\begin{cases} \text{minimize} & \frac{1}{|A_j|} \sum_{\boldsymbol{a} \in A_j} d(\boldsymbol{y}, \boldsymbol{a}) \\ \text{subject to} & \boldsymbol{y} \in \mathbb{R}^n, \end{cases} \tag{7.12}$$

where $|A_j|$ is a cardinality of the set A_j. If the squared Euclidean distance (7.8) is used for the similarity measure then the center $\boldsymbol{x}_j$ can be found explicitly as follows:

$$\boldsymbol{x}_j = \frac{1}{|A_j|} \sum_{\boldsymbol{a} \in A_j} \boldsymbol{a},\ j = 1, \dots, k. \tag{7.13}$$

In this case the problem (7.11) becomes an *integer programming problem* as cluster centers $\boldsymbol{x}_j$, $j = 1, \dots, k$ are not decision variables.

Problem (7.11) can be reformulated as the following NSO problem

$$\begin{cases} \text{minimize} & f_k(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) \in \mathbb{R}^{n \times k}, \end{cases} \tag{7.14}$$

where

$$f_k(\boldsymbol{x}) = f_k(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) = \frac{1}{m} \sum_{i=1}^{m} \min_{j=1,\dots,k} d(\boldsymbol{x}_j, \boldsymbol{a}_i). \tag{7.15}$$

If $k = 1$ then the objective function f_1 in problem (7.14) is convex. Moreover, if the similarity measure is defined using the squared Euclidean distance (7.8), the function f_k is smooth for any $\boldsymbol{x} \in \mathbb{R}^n$. If $k > 1$, the objective function f_k is nonconvex and nonsmooth due to the minimum operation used. If the similarity measure is defined using the L_1 or L_∞ norms [Eqs. (7.9) or (7.10)] then the function f_k is nonsmooth for all $k \geq 1$. This is due to the minimum operation and the fact that both L_1 and L_∞ norm based distance functions are nonsmooth.

We call objective functions ψ_k and f_k *cluster functions*. Comparing these two functions and also two different formulations of the clustering problem one can note that:

(i) The objective function ψ_k depends on variables $w_{ij},\ i = 1, \dots, m,\ j = 1, \dots, k$ (coefficients, which are integers) and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k,\ \boldsymbol{x}_j \in \mathbb{R}^n,\ j = 1, \dots, k$ (cluster centers, which are continuous variables). However, the function f_k depends only on continuous variables $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$.

(ii) The number of variables in problem (7.11) is $(m + n) \times k$, whereas in problem (7.14) this number is only $n \times k$ and the number of variables does not depend on the number of instances m. It should be noted that in many real-world databases,

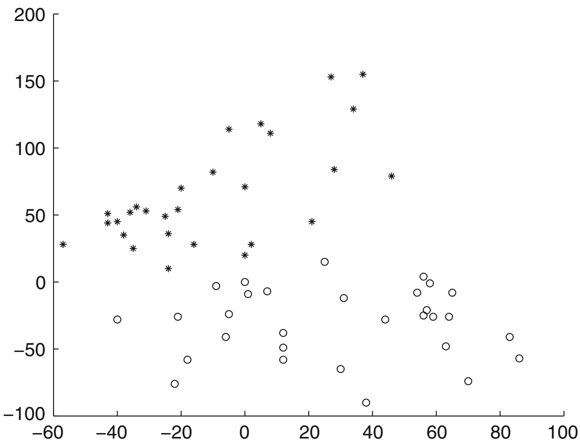


Fig. 7.6 Illustration of clusters

the number of instances m is substantially greater than the number of attributes n.

(iii) Since function f_k is represented as a sum of minima functions it is nonsmooth for $k > 1$. Both functions ψ_k and f_k are nonconvex for $k > 1$.

(iv) Problem (7.11) is MINLP problem and problem (7.14) is nonsmooth global optimization problem. However, they are equivalent in the sense that their global minimizers coincide.

Items (i) and (ii) can be considered as advantages of the nonsmooth optimization formulation (7.14) of the clustering problem. In addition, the objective function f_k in problem (7.14) can be expressed as a difference of two functions as follows:

$$f_k(\boldsymbol{x}) = f_k^1(\boldsymbol{x}) - f_k^2(\boldsymbol{x}), \ \ \boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \in \mathbb{R}^{n \times k},$$

where

$$f_k^1(\boldsymbol{x}) = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{k} d(\boldsymbol{x}_j, \boldsymbol{a}_i), \ \ f_k^2(\boldsymbol{x}) = \frac{1}{m} \sum_{i=1}^{m} \max_{j=1,\ldots,k} \sum_{t=1, t \neq j}^{k} d(\boldsymbol{x}_t, \boldsymbol{a}_i).$$

If the distance function d is convex with respect to $\boldsymbol{x}$, which is the case when one uses the squared Euclidean, L_1 or L_∞ norms, then the functions f_1 and f_2 are convex and therefore, the function f_k can be represented as a difference of convex (DC) functions.

Figure 7.6 illustrates two clusters obtained using the incremental discrete gradient method in the German Town data set which contains 59 points in $\mathbb{R}^2$. The similarity measure was defined using the squared Euclidean norm. Points from different clusters are shown using stars and circles.

7.2.2 *Piecewise Linear Separability in Supervised Data Classification*

The aim of *supervised data classification* is to establish rules for the classification of some observations assuming that the classes of data are known. To find these rules, known training subsets of the given classes are used. We start this section by introducing a concept of piecewise linear separability of sets that can be used to approximate nonlinear decision boundaries between pattern classes. Then, we introduce the so-called classification error function which reduces the problem of piecewise linear separability in supervised data classification to a NSO problem.

7.2.2.1 Piecewise Linear Separability

Piecewise linear functions can be used to approximate nonlinear decision boundaries between pattern classes. One hyperplane provides perfect separation when the convex hull of these pattern classes do not intersect. However, in many real-world applications this is not the case. In many data sets the classes are disjoint, but their convex hulls intersect. In this situation, the decision boundary between the classes is nonlinear. It can be approximated using piecewise linear functions.

Assume that we are given two nonempty disjoint finite point sets $A, B \subset \mathbb{R}^n$, that is $A, B \neq \emptyset$ and $A \bigcap B = \emptyset$. Let $H = \{h_1, \dots, h_l\}$, where $h_j = \{\boldsymbol{x}_j, y_j\},\ j = 1, \dots, l$ with $\boldsymbol{x}_j \in \mathbb{R}^n,\ y_j \in \mathbb{R}$, be a finite set of hyperplanes. Let $\mathcal{J} = \{1, \dots, l\}$. Consider a partition $\mathcal{J}^r = \{\mathcal{J}_1, \dots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ such that

(i) $\mathcal{J}_k \neq \emptyset,\ k = 1, \dots, r;$
(ii) $\mathcal{J}_k \bigcap \mathcal{J}_j = \emptyset, k, j = 1, \dots, r, k \neq j;$
(iii) $\bigcup_{k=1}^r \mathcal{J}_k = \mathcal{J}.$

Let $\mathcal{I} = \{1, \dots, r\}$. A partition $\mathcal{J}^r = \{\mathcal{J}_1, \dots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ defines the following max-min-type function:

$$\varphi(z) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{z} - y_j \right\},\ \ \boldsymbol{z} \in \mathbb{R}^n. \tag{7.16}$$

Definition 7.1 The sets A and B are *max-min linear separable*, if there exist a finite number of hyperplanes $\{\boldsymbol{x}_j, y_j\}$ with $\boldsymbol{x}_j \in \mathbb{R}^n,\ y_j \in \mathbb{R},\ j \in \mathcal{J} = \{1, \dots, l\}$ and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \dots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ such that

(i) for all $i \in \mathcal{I}$ and $\boldsymbol{a} \in A$

$$\min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j \right\} < 0;$$

(ii) for any $\boldsymbol{b} \in B$ there exists at least one $i \in \mathcal{I}$ such that

$$\min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{b} - y_j \right\} > 0.$$

Remark 7.1 It follows from Definition 7.1 that if the sets A and B are max-min linear separable then $\varphi(\boldsymbol{a}) < 0$ for any $\boldsymbol{a} \in A$ and $\varphi(\boldsymbol{b}) > 0$ for any $\boldsymbol{b} \in B$, where the function φ is defined by (7.16). Thus, the sets A and B can be separated by a function represented as a max-min of linear functions.

Next we define the concept of h-polyhedral separability given in [9].

Definition 7.2 The sets A and B are *h-polyhedrally separable* if there exists a set of h hyperplanes $\{\boldsymbol{x}_i, y_i\}$, with $\boldsymbol{x}_i \in \mathbb{R}^n,\ y_i \in \mathbb{R},\ i = 1, \ldots, h$ such that

(i) $(\boldsymbol{x}_i)^T \boldsymbol{a} - y_i < 0$ for all $\boldsymbol{a} \in A$ and $i = 1, \ldots, h$;
(ii) for any $\boldsymbol{b} \in B$ there exists at least one $i \in \{1, \ldots, h\}$ such that

$$(\boldsymbol{x}_i)^T \boldsymbol{b} - y_i > 0.$$

The sets A and B are h-polyhedrally separable, for some $h > 0$, if and only if

$$(\text{conv } A) \bigcap B = \emptyset.$$

Linear and polyhedral separability can be considered as particular cases of the max-min linear separability: if $\mathcal{I} = \{1\}$ and $\mathcal{J}_1 = \{1\}$ then we have the linear separability, and if $\mathcal{I} = \{1, \ldots, h\}$ and $\mathcal{J}_i = \{i\},\ i \in \mathcal{I}$ we obtain the h-polyhedral separability. Since any continuous piecewise linear function can be represented as a max-min of linear functions, the notions of max-min linear and piecewise linear separabilities are equivalent.

Linearly separable sets are illustrated in Fig. 7.7, polyhedrally separable sets are shown in Fig. 7.8, and Fig. 7.9 illustrates max-min linearly separable sets.

Proposition 7.1 *The sets A and B are max-min linear separable if and only if there exists a set of hyperplanes $\{\boldsymbol{x}_j, y_j\}$ with $\boldsymbol{x}_j \in \mathbb{R}^n,\ y_j \in \mathbb{R},\ j \in \mathcal{J}$, and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \ldots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ such that*

(i) $\min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j \right\} \leq -1$ for all $i \in \mathcal{I}$ *and* $\boldsymbol{a} \in A$*;*
(ii) for any $b \in B$ there exists at least one $i \in \mathcal{I}$ such that

$$\min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{b} - y_j \right\} \geq 1.$$

Proof Sufficiency is straightforward.
Necessity. Since A and B are max-min linear separable there exists a set of hyperplanes $\{\bar{\boldsymbol{x}}_j, \bar{y}_j\}$ with $\bar{\boldsymbol{x}}_j \in \mathbb{R}^n,\ \bar{y}_j \in \mathbb{R},\ j \in \mathcal{J}$, a partition $\mathcal{J}^r$ of the set $\mathcal{J}$ and numbers $\delta_1 > 0,\ \delta_2 > 0$ such that

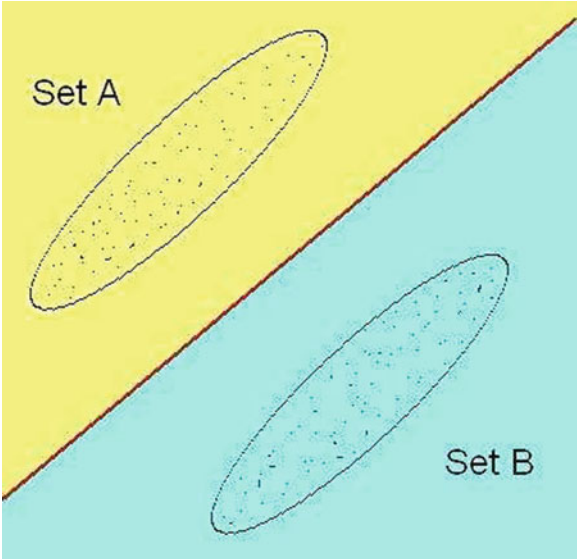


Fig. 7.7 Sets A and B are linearly separable

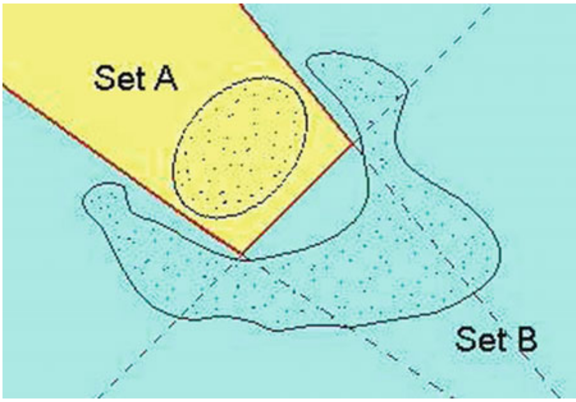


Fig. 7.8 Sets A and B are polyhedral separable

$$\max_{\boldsymbol{a}\in A}\max_{i\in\mathcal{I}}\min_{j\in\mathcal{J}_i}\left\{(\bar{\boldsymbol{x}}_j)^T\boldsymbol{a}-\bar{y}_j\right\}=-\delta_1$$

and

$$\min_{\boldsymbol{b}\in B}\max_{i\in\mathcal{I}}\min_{j\in\mathcal{J}_i}\left\{(\bar{\boldsymbol{x}}_j)^T\boldsymbol{b}-\bar{y}_j\right\}=\delta_2.$$

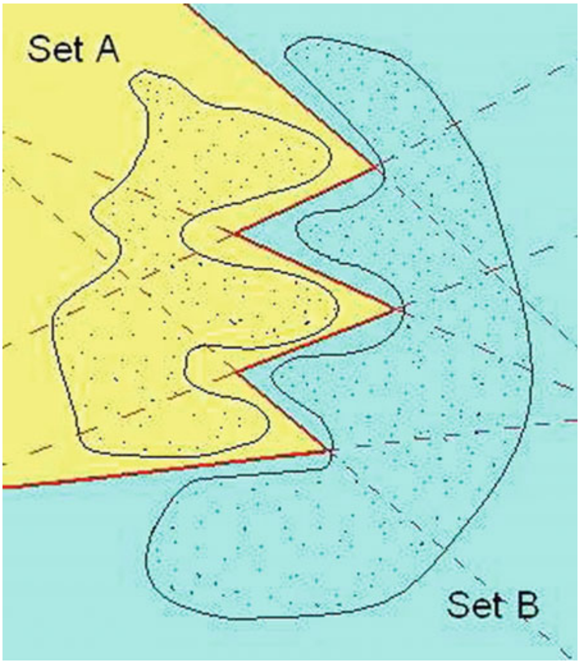


Fig. 7.9 Sets A and B are max-min linear separable

Putting $\delta = \min\{\delta_1, \delta_2\} > 0$ we have

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\bar{\boldsymbol{x}}_j)^T \boldsymbol{a} - \bar{y}_j \right\} \leq -\delta, \quad \text{for all } \boldsymbol{a} \in A, \tag{7.17}$$

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\bar{\boldsymbol{x}}_j)^T \boldsymbol{b} - \bar{y}_j \right\} \geq \delta, \quad \text{for all } \boldsymbol{b} \in B. \tag{7.18}$$

Consider the set of hyperplanes $\{\boldsymbol{x}_j, y_j\}$ with $\boldsymbol{x}_j \in \mathbb{R}^n,\ y_j \in \mathbb{R},\ j \in \mathcal{J}$, defined as: $\boldsymbol{x}_j = \bar{\boldsymbol{x}}_j/\delta, y_j = \bar{y}_j/\delta,\ j \in \mathcal{J}$. Then it follows from (7.17) and (7.18) that

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j \right\} \leq -1, \quad \text{for all } \boldsymbol{a} \in A,$$

and

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{b} - y_j \right\} \geq 1, \quad \text{for all } \boldsymbol{b} \in B,$$

which completes the proof. □

Now we consider special cases when the sets A and B are max-min linearly separable.

Proposition 7.2 *Assume that the set* A *can be represented as a union of sets* $A_i,\ i = 1, \ldots, q$*:*

$$A = \bigcup_{i=1}^{q} A_i$$

and for any $i = 1, \ldots, q$

$$B \bigcap \text{conv } A_i = \emptyset. \tag{7.19}$$

Then the sets A *and* B *are max-min linearly separable.*

Proof It follows from (7.19) that $\boldsymbol{b} \notin \text{conv } A_i$ for all $\boldsymbol{b} \in B$ and $i \in \{1, \ldots, q\}$. Then, for each $\boldsymbol{b} \in B$ and $i \in \{1, \ldots, q\}$ there exists a hyperplane $\{\boldsymbol{x}_i(\boldsymbol{b}), y_i(\boldsymbol{b})\}$ separating $\boldsymbol{b}$ from the set conv A_i, that is

$$(\boldsymbol{x}_i(\boldsymbol{b}))^T \boldsymbol{b} - y_i(\boldsymbol{b}) > 0$$

and

$$(\boldsymbol{x}_i(\boldsymbol{b}))^T \boldsymbol{a} - y_i(\boldsymbol{b}) < 0, \quad \text{for all } \boldsymbol{a} \in \text{conv } A_i,\ i = 1, \ldots, q.$$

Then we have

$$\min_{i=1,\ldots,q} \left\{ (\boldsymbol{x}_i(\boldsymbol{b}))^T \boldsymbol{b} - y_i(\boldsymbol{b}) \right\} > 0$$

and

$$\min_{i=1,\ldots,q} \left\{ (\boldsymbol{x}_i(\boldsymbol{b}))^T \boldsymbol{a} - y_i(\boldsymbol{b}) \right\} < 0, \quad \text{for all } \boldsymbol{a} \in A.$$

Thus for any $\boldsymbol{b} \in B$ there exists a set of q hyperplanes $\{\boldsymbol{x}_i(\boldsymbol{b}), y_i(\boldsymbol{b})\},\ i = 1, \ldots, q$ such that

$$\min_{i=1,\ldots,q} \left\{ (\boldsymbol{x}_i(\boldsymbol{b}))^T \boldsymbol{b} - y_i(\boldsymbol{b}) \right\} > 0 \tag{7.20}$$

and

$$\min_{i=1,\ldots,q} \left\{ (\boldsymbol{x}^i(\boldsymbol{b}))^T \boldsymbol{a} - y_i(\boldsymbol{b}) \right\} < 0, \quad \text{for all } \boldsymbol{a} \in A. \tag{7.21}$$

Consequently we have pq hyperplanes, where p is the number of points in the set B,

$$\{\boldsymbol{x}_i(\boldsymbol{b}), y_i(\boldsymbol{b})\},\ i = 1, \ldots, q,\ \boldsymbol{b} \in B.$$

The set can be rewritten as:

$$H = \{h_1, \dots, h_l\},$$

where

$$h_{i+(j-1)q} = \left\{\boldsymbol{x}_i(\boldsymbol{b}_j), y_i(\boldsymbol{b}_j)\right\}, \quad i = 1, \dots, q, \; j = 1, \dots, p, \; l = pq.$$

Let $\mathcal{J} = \{1, \dots, l\}$, $\mathcal{I} = \{1, \dots, p\}$ and

$$\bar{\boldsymbol{x}}_{i+(j-1)q} = \boldsymbol{x}_i(\boldsymbol{b}_j), \quad \bar{y}_{i+(j-1)q} = y_i(\boldsymbol{b}_j), \quad i = 1, \dots, q, \; j = 1, \dots, p.$$

Consider the following partition of the set $\mathcal{J}$:

$$\mathcal{J}^p = \left\{\mathcal{J}_1, \dots, \mathcal{J}_p\right\}, \quad \mathcal{J}_k = \{(k-1)q + 1, \dots, kq\}, \quad k = 1, \dots, p.$$

It follows from (7.20) and (7.21) that for all $k \in \mathcal{I}$ and $\boldsymbol{a} \in A$

$$\min_{j \in \mathcal{J}_k} \left\{(\bar{\boldsymbol{x}}_j)^T \boldsymbol{a} - \bar{y}_j\right\} < 0$$

and for any $\boldsymbol{b} \in B$ there exists at least one $k \in \mathcal{I}$ such that

$$\min_{j \in \mathcal{J}_k} \left\{(\bar{\boldsymbol{x}}_j)^T \boldsymbol{b} - \bar{y}_j\right\} > 0$$

which means that the sets A and B are max-min linearly separable. □

Corollary 7.1 *The sets A and B are max-min linearly separable if and only if they are disjoint: $A \bigcap B = \emptyset$.*

Proof Necessity is straightforward.
Sufficiency. The set A can be represented as a union of its own points. Since the sets A and B are disjoint the condition (7.19) is satisfied. Then the proof of the corollary follows from Proposition 7.2. □

Proposition 7.3 *Assume that the set A can be represented as a union of sets A_i, $i = 1, \dots, q$ and the set B as a union of sets B_j, $j = 1, \dots, d$ such that*

$$A = \bigcup_{i=1}^{q} A_i, \quad B = \bigcup_{j=1}^{d} B_j$$

and

$$\text{conv } A_i \bigcap \text{conv } B_j = \emptyset \quad \textit{for all } i = 1, \dots, q, \; j = 1, \dots, d. \tag{7.22}$$

Then the sets A and B are max-min linearly separable with no more than qd hyperplanes.

Proof Let $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, d\}$ be any fixed indices. Since conv $A_i \bigcap$ conv $B_j = \emptyset$ there exists a hyperplane $\{\boldsymbol{x}_{ij}, y_{ij}\}$ with $\boldsymbol{x}_{ij} \in \mathbb{R}^n,\ y_{ij} \in \mathbb{R}$ such that

$$(\boldsymbol{x}_{ij})^T \boldsymbol{a} - y_{ij} < 0 \ \text{ for all } \boldsymbol{a} \in \text{conv } A_i$$

and

$$(\boldsymbol{x}_{ij})^T \boldsymbol{b} - y_{ij} > 0 \quad \text{for all } \boldsymbol{b} \in \text{conv } B_j.$$

Consequently for any $j \in \{1, \dots, d\}$ there exists a set of hyperplanes $\{\boldsymbol{x}_{ij}, y_{ij}\},\ i = 1, \dots, q$ such that

$$\min_{i=1,\dots,q} (\boldsymbol{x}_{ij})^T \boldsymbol{b} - y_{ij} > 0, \ \text{ for all } \boldsymbol{b} \in B_j \tag{7.23}$$

and

$$\min_{i=1,\dots,q} (\boldsymbol{x}_{ij})^T \boldsymbol{a} - y_{ij} < 0, \ \text{ for all } \boldsymbol{a} \in A. \tag{7.24}$$

Thus we get a set of $l = dq$ hyperplanes:

$$H = \{h_1, \dots, h_l\}$$

where $h_{i+(j-1)q} = \left\{\boldsymbol{x}_{ij}, y_{ij}\right\},\ i = 1, \dots, q,\ j = 1, \dots, d.$ Let $\mathcal{J} = \{1, \dots, l\}$, $\mathcal{I} = \{1, \dots, d\}$ and

$$\bar{\boldsymbol{x}}_{i+(j-1)q} = \boldsymbol{x}_{ij}, \quad \bar{y}_{i+(j-1)q} = y_{ij}, \quad i = 1, \dots, q,\ j = 1, \dots, d.$$

Consider the following partition of the set $\mathcal{J}$:

$$\mathcal{J}^d = \{\mathcal{J}_1, \dots, \mathcal{J}_d\}, \quad \mathcal{J}_k = \{(k-1)q + 1, \dots, kq\}, \quad k = 1, \dots, d.$$

It follows from (7.23) and (7.24) that for all $k \in \mathcal{I}$ and $\boldsymbol{a} \in A$

$$\min_{j \in \mathcal{J}_k} \left\{(\bar{\boldsymbol{x}}_j)^T \boldsymbol{a} - \bar{y}_j\right\} < 0$$

and for any $\boldsymbol{b} \in B$ there exists at least one $k \in \mathcal{I}$ such that

$$\min_{j \in \mathcal{J}_k} \left\{(\bar{\boldsymbol{x}}_j)^T \boldsymbol{b} - \bar{y}_j\right\} > 0,$$

that is the sets A and B are max-min linearly separable with at most qd hyperplanes. □

7.2.2.2 Classification Error

Given any set of hyperplanes $\{\boldsymbol{x}_j, y_j\},\ j \in \mathcal{J} = \{1, \ldots, l\}$ with $\boldsymbol{x}_j \in \mathbb{R}^n,\ y_j \in \mathbb{R}$ and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \ldots, \mathcal{J}_r\}$ of the set $\mathcal{J}$, we say that a point $\boldsymbol{a} \in A$ is well classified if the following condition satisfied:

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j \right\} + 1 \le 0.$$

Thus, we can define the *classification error for a point* $a \in A$ as follows:

$$\max \left[0, \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \right]. \tag{7.25}$$

To a well-classified point this error is zero. Analogously, a point $\boldsymbol{b} \in B$ is said to be well-classified if the following condition is satisfied:

$$\min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j \right\} + 1 \le 0.$$

Then the *classification error for a point* $\boldsymbol{b} \in B$ can be written as

$$\max \left[0, \min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \right]. \tag{7.26}$$

Thus, an *averaged classification error function* can be defined as

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) = (1/|A|) \sum_{\boldsymbol{a} \in A} \max \left[0, \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} \rangle - y_j + 1 \right\} \right] \\ + (1/|B|) \sum_{\boldsymbol{b} \in B} \max \left[0, \min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} \rangle + y_j + 1 \right\} \right] \end{aligned} \tag{7.27}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l) \in \mathbb{R}^{l \times n}$ and $\boldsymbol{y} = (y_1, \ldots, y_l) \in \mathbb{R}^l$. It is clear that $f(\boldsymbol{x}, \boldsymbol{y}) \ge 0$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Then the problem of max-min linear separability is reduced to the following optimization problem:

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{(n+1) \times l}. \end{cases} \tag{7.28}$$

Proposition 7.4 *The sets A and B are max-min linearly separable if and only if there exist a set of hyperplanes $\{\boldsymbol{x}_j, y_j\}$, $j \in \mathcal{J} = \{1, \ldots, l\}$ and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \ldots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ such that $f(\boldsymbol{x}, \boldsymbol{y}) = 0$.*

Proof Necessity. Assume that the sets A and B are max-min linearly separable. Then it follows from Proposition 7.1 that there exists a set of hyperplanes $\{\boldsymbol{x}_j, y_j\}, j \in \mathcal{J}$ and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \ldots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ such that

$$\min_{j \in \mathcal{J}_i} \{(\boldsymbol{x}_j)^T \boldsymbol{a} - y_j\} \leq -1, \quad \text{for all } \boldsymbol{a} \in A, \ i \in \mathcal{I} = \{1, \ldots, r\} \tag{7.29}$$

and for any $\boldsymbol{b} \in B$ there exists at least one $t \in \mathcal{I}$ such that

$$\min_{j \in \mathcal{J}_t} \{(\boldsymbol{x}_j)^T \boldsymbol{b}\rangle - y_j\} \geq 1. \tag{7.30}$$

Consequently we have

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{(\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1\} \leq 0, \quad \text{for all } \boldsymbol{a} \in A,$$

$$\min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \{-(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1\} \leq 0, \quad \text{for all } \boldsymbol{b} \in B.$$

Then from the definition of the error function we obtain that $f(\boldsymbol{x}, \boldsymbol{y}) = 0$.
Sufficiency. Assume that there exist a set of hyperplanes $\{\boldsymbol{x}_j, y_j\}, j \in \mathcal{J} = \{1, \ldots, l\}$ and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \ldots, \mathcal{J}_r\}$ of the set $\mathcal{J}$ such that $f(\boldsymbol{x}, \boldsymbol{y}) = 0$. Then from the definition of the error function f we immediately get that the inequalities (7.29) and (7.30) are satisfied, that is the sets A and B are max-min linearly separable. □

Proposition 7.5 *Assume that the sets A and B are max-min linearly separable with a set of hyperplanes $\{\boldsymbol{x}_j, y_j\}, j \in \mathcal{J} = \{1, \ldots, l\}$ and a partition $\mathcal{J}^r = \{\mathcal{J}_1, \ldots, \mathcal{J}_r\}$ of the set $\mathcal{J}$. Then*

(i) $\boldsymbol{x}_j = 0, \ j \in \mathcal{J}$ *cannot be an optimal solution to the problem (7.28);*
(ii) if

(a) for any $t \in \mathcal{I}$ there exists at least one $\boldsymbol{b} \in B$ such that

$$\max_{j \in \mathcal{J}_t} \left\{-(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1\right\} = \min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \left\{-(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1\right\}, \tag{7.31}$$

(b) there exists $\tilde{\mathcal{J}} = \{\tilde{\mathcal{J}}_1, \ldots, \tilde{\mathcal{J}}_r\}$ such that $\tilde{\mathcal{J}}_t \subset \mathcal{J}_t$, for all $t \in \mathcal{I}$, $\tilde{\mathcal{J}}_t$ is nonempty at least for one $t \in \mathcal{I}$ and $\boldsymbol{x}_j = 0$ for any $j \in \tilde{\mathcal{J}}_t, t \in \mathcal{I}$.

Then the sets A and B are max-min linearly separable with a set of hyperplanes $\{\boldsymbol{x}_j, y_j\}, j \in \mathcal{J}^0$ and a partition $\bar{\mathcal{J}} = \{\bar{\mathcal{J}}_1, \ldots, \bar{\mathcal{J}}_r\}$ of the set $\mathcal{J}^0$ where

$$\bar{\mathcal{J}}_t = \mathcal{J}_t \setminus \tilde{\mathcal{J}}_t, \ t \in \mathcal{I} \text{ and } \mathcal{J}^0 = \bigcup_{i=1}^{r} \bar{\mathcal{J}}_i.$$

Proof (i) Since the sets A and B are max-min linearly separable it follows from Proposition 7.4 that $f(\boldsymbol{x}, \boldsymbol{y}) = 0$. If $\boldsymbol{x}_j = 0,\ j \in \mathcal{J}$ then it follows from (7.27) that for any $\boldsymbol{y} \in \mathbb{R}^l$

$$f(\mathbf{0}, \boldsymbol{y}) = (1/|A|) \sum_{\boldsymbol{a} \in A} \max \left[0, \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{-y_j + 1\}\right] + (1/|B|) \sum_{\boldsymbol{b} \in B} \max \left[0, \min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \{y_j + 1\}\right].$$

Denote $R = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{-y_j\}$. Then we have

$$\min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} y_j = -\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \{-y_j\} = -R.$$

Thus

$$f(\mathbf{0}, \boldsymbol{y}) = \max\,[0, R + 1] + \max\,[0, -R + 1]\,.$$

It is clear that

$$\max\,[0, R+1] + \max\,[0, -R+1] = \begin{cases} -R+1 & \text{if } R \le -1, \\ 2 & \text{if } -1 < R < 1, \\ R+1 & \text{if } R \ge 1. \end{cases}$$

Thus $f(\mathbf{0}, \boldsymbol{y}) \ge 2$ for any $y \in \mathbb{R}^l$. On the other side $f(\boldsymbol{x}, \boldsymbol{y}) = 0$ for the optimal solution $(\boldsymbol{x}, \boldsymbol{y})$, that is $\boldsymbol{x}_j = 0,\ j \in \mathcal{J}$ cannot be the optimal solution.
(ii) Consider the following sets:

$$\mathcal{I}^1 = \{i \in \mathcal{I} : \bar{\mathcal{J}}_i \ne \emptyset\},$$

$$\mathcal{I}^2 = \{i \in \mathcal{I} : \tilde{\mathcal{J}}_i \ne \emptyset\},\ \ \mathcal{I}^3 = \mathcal{I}^1 \bigcap \mathcal{I}^2.$$

It is clear that $\tilde{\mathcal{J}}_i = \emptyset$ for any $i \in \mathcal{I}^1 \setminus \mathcal{I}^3$ and $\bar{\mathcal{J}}_i = \emptyset$ for any $i \in \mathcal{I}^2 \setminus \mathcal{I}^3$. It follows from the definition of the error function that

$$0 = f(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{|A|} \sum_{\boldsymbol{a} \in A} \max \left[0, \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{(\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1\right\}\right] + \frac{1}{|B|} \sum_{\boldsymbol{b} \in B} \max \left[0, \min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \left\{-(\boldsymbol{x}_j)^T \boldsymbol{b}\rangle + y_j + 1\right\}\right].$$

Since the function f is nonnegative we obtain

$$\max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \leq 0, \quad \text{for all } \boldsymbol{a} \in A, \tag{7.32}$$

$$\min_{i \in \mathcal{I}} \max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \leq 0, \quad \text{for all } \boldsymbol{b} \in B. \tag{7.33}$$

It follows from (7.31) and (7.33) that for any $i \in \mathcal{I}^2$ there exists a point $\boldsymbol{b} \in B$ such that

$$\max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \leq 0. \tag{7.34}$$

If $i \in \mathcal{I}^3 \subset \mathcal{I}^2$ then we have

$$\begin{aligned} 0 &\geq \max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \\ &= \max \left\{ \max_{j \in \bar{\mathcal{J}}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\}, \max_{i \in \tilde{\mathcal{J}}_i} \left\{ y_j + 1 \right\} \right\} \end{aligned}$$

which means that

$$\max_{j \in \bar{\mathcal{J}}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \leq 0 \tag{7.35}$$

and

$$\max_{j \in \tilde{\mathcal{J}}_i} \left\{ y_j + 1 \right\} \leq 0. \tag{7.36}$$

If $i \in \mathcal{I}^2 \setminus \mathcal{I}^3$ then from (7.34) we obtain

$$0 \geq \max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} = \max_{i \in \tilde{\mathcal{J}}_i} \left\{ y_j + 1 \right\}.$$

Thus we get that for all $i \in \mathcal{I}^2$ the inequality (7.36) is true. (7.36) can be rewritten as follows:

$$\max_{j \in \tilde{\mathcal{J}}_i} y_j \leq -1, \quad \text{for all } i \in \mathcal{I}^2. \tag{7.37}$$

Consequently for any $i \in \mathcal{I}^2$

$$\min_{j \in \tilde{\mathcal{J}}_i} \left\{ -y_j + 1 \right\} = -\max_{j \in \tilde{\mathcal{J}}_i} y_j + 1 \geq 2. \tag{7.38}$$

It follows from (7.32) that for any $i \in \mathcal{I}$ and $\boldsymbol{a} \in A$

$$\min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \leq 0. \tag{7.39}$$

Then for any $i \in \mathcal{I}^3$ we have

$$\begin{aligned} 0 &\geq \min_{j \in \mathcal{J}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \\ &= \min \left\{ \min_{j \in \bar{\mathcal{J}}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\}, \min_{j \in \tilde{\mathcal{J}}_i} \left\{ -y_j + 1 \right\} \right\}. \end{aligned}$$

Taking into account (7.38) we get that for any $i \in \mathcal{I}^3$ and $\boldsymbol{a} \in A$

$$\min_{j \in \bar{\mathcal{J}}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \leq 0. \tag{7.40}$$

If $i \in \mathcal{I}^2 \setminus \mathcal{I}^3$ then it follows from (7.39) that

$$\min_{j \in \tilde{\mathcal{J}}_i} \left\{ -y_j + 1 \right\} \leq 0$$

which contradicts (7.38). Thus we obtain that $\mathcal{I}^2 \setminus \mathcal{I}^3 \neq \emptyset$ cannot occur, $\mathcal{I}^2 \subset \mathcal{I}^1$ and $\mathcal{I}^3 = \mathcal{I}^2$. It is clear that $\bar{\mathcal{J}}_i = \mathcal{J}_i$ for any $i \in \mathcal{I}^1 \setminus \mathcal{I}^2$. Then it follows from (7.32) that for any $i \in \mathcal{I}^1 \setminus \mathcal{I}^2$ and $a \in A$

$$\min_{j \in \bar{\mathcal{J}}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \leq 0. \tag{7.41}$$

From (7.40) and (7.41) we can conclude that for any $i \in \mathcal{I}$ and $\boldsymbol{a} \in A$

$$\min_{j \in \bar{\mathcal{J}}_i} \left\{ (\boldsymbol{x}_j)^T \boldsymbol{a} - y_j + 1 \right\} \leq 0. \tag{7.42}$$

It follows from (7.33) that for any $\boldsymbol{b} \in B$ there exists at least one $i \in \mathcal{I}$ such that

$$\max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \leq 0.$$

Then from expression

$$\max_{j \in \mathcal{J}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} = \max \left\{ \max_{j \in \bar{\mathcal{J}}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\}, \max_{i \in \tilde{\mathcal{J}}_i} \left\{ y_j + 1 \right\} \right\}$$

we get that for any $\boldsymbol{b} \in B$ there exists at least one $i \in \mathcal{I}$ such that

$$\max_{j \in \bar{\mathcal{J}}_i} \left\{ -(\boldsymbol{x}_j)^T \boldsymbol{b} + y_j + 1 \right\} \leq 0. \tag{7.43}$$

Thus it follows from (7.42) and (7.43) that the sets A and B are max-min linearly separable with the set of hyperplanes $\{\boldsymbol{x}_j, y_j\},\ j \in \mathcal{J}^0$ and a partition $\bar{\mathcal{J}}$ of the set $\mathcal{J}^0$. □

The functions f_1 and f_2 are nonconvex piecewise linear. These functions are Lipschitz continuous and consequently subdifferentiable. Moreover, both functions are semismooth.

7.2.3 Piecewise Linear Approximations in Regression Analysis

One interesting application of nonsmooth analysis and optimization techniques is the problem of estimating a multivariate regression function using continuous piecewise linear functions. It is known that each continuous piecewise linear function can be represented as a maximum of minima of linear functions. Such representations are used to estimate regression functions.

In applications usually no a priori information about the regression function is known, therefore it is necessary to apply nonparametric methods for this estimation problem. The space of continuous piecewise linear functions provide rather a complex function space over which an empirical least squares risk is minimized. Since continuous piecewise linear functions are, in general, nonsmooth and nonconvex, the resulting least squares risk is nonconvex and nonsmooth function.

In *regression analysis* an $\mathbb{R}^p \times \mathbb{R}$-valued random vector (U, V) with $\mathbf{E}V^2 < \infty$ is considered and the dependency of V on the value of U is of interest. Here $\mathbf{E}$ stands for the mean value. More precisely, the goal is to find a function $\varphi : \mathbb{R}^p \to \mathbb{R}$ such that $\varphi(U)$ is a "good approximation" of V. In the sequel we assume that the main aim of the analysis is minimization of the *mean squared prediction error* or L_2 *risk*

$$\mathbf{E}\{|\varphi(U) - V|^2\}. \tag{7.44}$$

In this case the optimal function is the so-called *regression function* $m : \mathbb{R}^p \to \mathbb{R}$, $m(\boldsymbol{u}) = \mathbf{E}\{V|U = \boldsymbol{u}\}$. Indeed, let $\varphi : \mathbb{R}^p \to \mathbb{R}$ be an arbitrary (measurable) function and denote the distribution of U by μ. Then

$$\begin{aligned}
\mathbf{E}\{|\varphi(U) - V|^2\} &= \mathbf{E}\{((\varphi(U) - m(U)) + (m(U) - V))^2\} \\
&= \mathbf{E}\{|\varphi(U) - m(U)|^2\} + \mathbf{E}\{|m(U) - V|^2\} \\
&= \mathbf{E}\{|m(U) - V|^2\} + \int |\varphi(\boldsymbol{u}) - m(\boldsymbol{u})|^2 \mu(d\boldsymbol{u}). \qquad (7.45)
\end{aligned}$$

Since the integral on the right-hand side of (7.45) is always nonnegative, (7.45) implies that the regression function is the optimal predictor in view of minimization of the L_2 risk:

$$\mathbf{E}\{|m(U) - V|^2\} = \min_{\varphi:\mathbb{R}^p \to \mathbb{R}} \mathbf{E}\{|\varphi(U) - V|^2\}. \tag{7.46}$$

In addition, any function φ is a good predictor in the sense that its L_2 risk is close to the optimal value, if and only if the so-called L_2 *error*

$$\int |\varphi(\boldsymbol{u}) - m(\boldsymbol{u})|^2 \mu(d\boldsymbol{u}) \tag{7.47}$$

is small. This motivates to measure the error caused by using a function φ by the L_2 error (7.47) instead of the regression function.

In applications, usually the distribution of (U, V) (and hence also the regression function) is unknown. But often it is possible to observe a sample of the underlying distribution. This leads to *the regression estimation problem*. Here (U, V), (U_1, V_1), (U_2, V_2), ... are independent and identically distributed random vectors. The set of data

$$\mathcal{D}_l = \{(U_1, V_1), \dots, (U_l, V_l)\}$$

is given, and the goal is to construct an estimate

$$m_l(\cdot) = m_l(\cdot, \mathcal{D}_l) : \mathbb{R}^p \to \mathbb{R}$$

of the regression function such that the L_2 error

$$\int |m_l(\boldsymbol{u}) - m(\boldsymbol{u})|^2 \mu(d\boldsymbol{u})$$

is small.

The regression function minimizes the L_2 risk (7.44) over the set of all measurable functions, so in principle it can be computed by solving a minimization problem. However, in practical applications the term to be minimized is unknown, because it depends on the unknown distribution of (U, V). For *least squares estimates* the given data is used to estimate the L_2 risk by the so-called *empirical L_2 risk*

$$\frac{1}{l} \sum_{i=1}^{l} |\varphi(U_i) - V_i|^2, \tag{7.48}$$

and the regression estimate is defined by minimizing (7.48). Minimization of (7.48) with respect to all measurable functions [(as in (7.46)] leads to an estimate, which usually (at least if the values of U_1, ..., U_l are distinct) interpolates the given data.

Obviously, such an estimate is not a reasonable estimate for $m(\boldsymbol{u}) = \mathbf{E}\{V|U = \boldsymbol{u}\}$. In order to avoid this so-called overfitting, for least squares estimates, first a class $\mathcal{F}_l$ of functions $\varphi : \mathbb{R}^p \to \mathbb{R}$ is chosen and then the estimate is defined by minimizing the empirical L_2 risk over $\mathcal{F}_l$, that is,

$$m_l(\cdot) = \arg\min_{\varphi \in \mathcal{F}_l} \frac{1}{l} \sum_{i=1}^{l} |\varphi(U_i) - V_i|^2. \tag{7.49}$$

Here we assume that the minimum exists, however we do not require that it is unique.

We will use continuous piecewise linear functions to define $\mathcal{F}_l$. Since any continuous piecewise linear function can be represented as a max-min of finite number of linear functions we consider maxima of minima of linear functions. More precisely, let $K_l \in \mathbb{N}$ and $L_{1,l}, \ldots, L_{K_l,l} \in \mathbb{N}$ be parameters of the estimate and set

$$\mathcal{F}_l = \Big\{ \varphi : \mathbb{R}^p \to \mathbb{R} \mid \varphi(\boldsymbol{u}) = \max_{k=1,\ldots,K_l} \min_{j=1,\ldots,L_{k,l}} \left((\boldsymbol{x}^{k,j})^T \boldsymbol{u} + y_{k,j} \right), \\ \text{for some } \boldsymbol{x}^{k,j} \in \mathbb{R}^p,\ y_{k,j} \in \mathbb{R} \Big\}.$$

For this class of functions the estimate m_l is defined by (7.49).

It follows from (7.48) that the estimation of a regression function by continuous piecewise linear function can be formulated as the following minimization problem:

$$\begin{cases} \text{minimize} & F(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{l} \sum_{i=1}^{l} \left(\max_{k=1,\ldots,K_l} \min_{j=1,\ldots,L_{k,l}} \left((\boldsymbol{x}^{k,j})^T U_i + y_{k,j} \right) - V_i \right)^2 \\ \text{subject to} & \boldsymbol{x}^{k,j} \in \mathbb{R}^p,\ k = 1, \ldots, K_l, \\ & y_{k,j} \in \mathbb{R},\ j = 1, \ldots, L_{k,l}., \end{cases} \tag{7.50}$$

Here

$$\boldsymbol{x} = (x_1^{1,1}, \ldots, x_p^{1,1}, \ldots, x_1^{K_l, L_{K_l,l}}, \ldots, x_p^{K_l, L_{K_l,l}}) \in \mathbb{R}^{q \times p},$$
$$\boldsymbol{y} = (y_{1,1}, \ldots, \ldots, y_{K_l, L_{K_l,l}})^T \in \mathbb{R}^q$$

and

$$q = \sum_{k=1}^{K_l} L_{k,l}.$$

The objective function F in Problem (7.50) is locally Lipschitz continuous and semismooth. It is both nonsmooth and nonconvex, and the number of local minimizers is large whenever the numbers l and q are large.

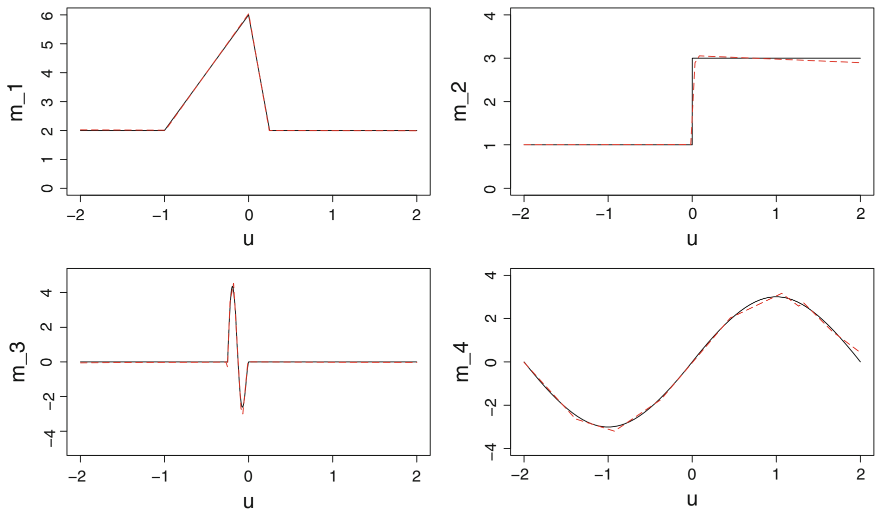


Fig. 7.10 Piecewise linear approximations to univariate functions

Remark 7.2 In general, nonconvex piecewise linear functions are not subdifferentially regular (see Definition 3.5 in Part I). Thus, the function F, in general, is not subdifferentially regular.

Remark 7.3 The function F is quasidifferentiable (see Definition 3.9 in Part I) and its subdifferential and superdifferential are polytopes. In addition, the function F is piecewise partially separable (see Definition 6.7 in Part I).

Figure 7.10 illustrates piecewise linear approximations (red dash lines) to univariate functions (black lines). The piecewise linear linear approximation (right) to the function of two variables (left) is illustrated in Fig. 7.11.

7.2.4 Clusterwise Linear Regression Problems

Clustering or unsupervised classification consists in finding subsets of similar points in a data set, in order to find patterns in the data. Regression analysis consists in fitting a function to the data to discover how one or more variables vary as a function of another. The aim of *clusterwise regression* is to combine both of these techniques, to discover trends within data, when more than one trend is likely to exist. Figure 7.12 illustrates one such situation where three linear functions are used to approximate the data.

Clusterwise regression has applications for instance in market segmentation, where it allows one to gather information on customer behaviors for several unknown groups of customers. It is also applied to investigate the stock-exchange data and the

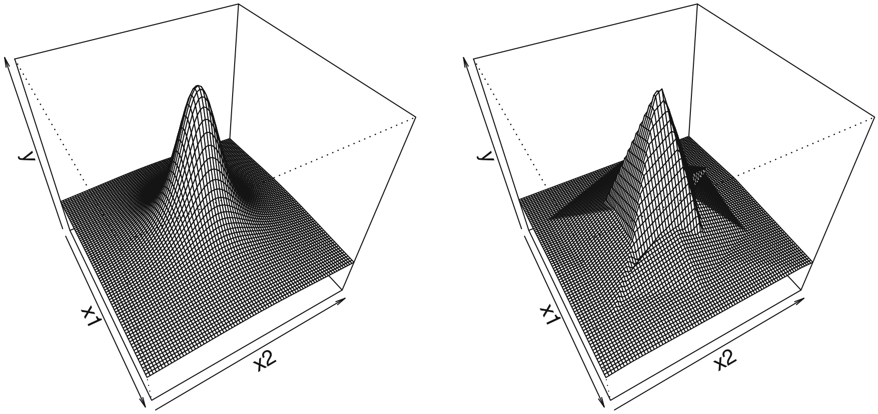


Fig. 7.11 Piecewise linear approximations to the function with two variables

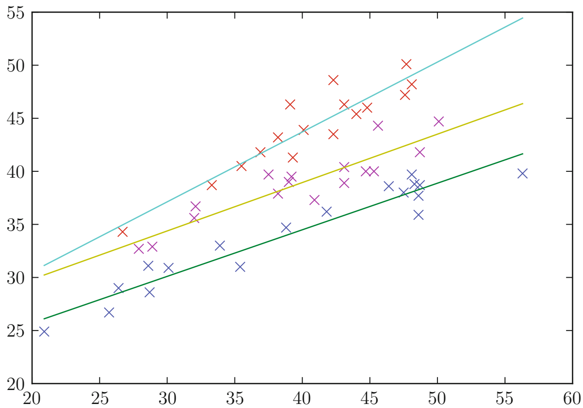


Fig. 7.12 Three linear functions provides good approximation

benefit segmentation. The presence of nonlinear relationships, heterogeneous subjects, or time series in these applications necessitate the use of two or more regression functions to best summarize the underlying structure of the data.

The simplest case in the clusterwise regression is the use of two or more linear regression functions to investigate the structure of the data. Such an approach is called *clusterwise linear regression* and it is widely used and studied better than other approaches. This problem can be formulated as an optimization problem.

Given a data set $A = \{(\boldsymbol{a}_i, b_i) \in \mathbb{R}^n \times \mathbb{R} : \ i = 1, \ldots, m\}$, the aim of the clusterwise linear regression is to find simultaneously an optimal partition of data in k clusters and regression coefficients $\{\boldsymbol{x}_j, y_j\}, \ \boldsymbol{x}^j \in \mathbb{R}^n, \ y_j \in \mathbb{R}, \ j = 1, \ldots, k$ within clusters in order to minimize the overall fit. Let $A^j, \ j = 1, \ldots, k$ be clusters such that

$$A^j \neq \emptyset, \quad A^j \bigcap A^t = \emptyset, \quad j, t = 1, \ldots, k, \quad t \neq j \quad \text{and} \quad A = \bigcup_{j=1}^{k} A^j.$$

Let $\{\boldsymbol{x}_j, y_j\}$ be linear regression coefficients computed using only data points from the cluster $A^j, \ j = 1, \ldots, k$. Then for the given data point $(\boldsymbol{a}_i, b_i)$ and coefficients $\{\boldsymbol{x}_j, y_j\}$ the regression error $h(\boldsymbol{x}_j, y_j, \boldsymbol{a}_i, b_i)$ is:

$$h(\boldsymbol{x}_j, y_j, \boldsymbol{a}_i, b_i) = \left|(\boldsymbol{x}_j)^T \boldsymbol{a}_i + y_j - b_i\right|^p.$$

Here $p > 0$. We associate a data point with the cluster whose regression error at this point is smallest. Then the overall fit function is:

$$f_k(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{m} \min_{j=1,\ldots,k} h(\boldsymbol{x}_j, y_j, \boldsymbol{a}_i, b_i), \tag{7.51}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \in \mathbb{R}^{k \times n}$ and $\boldsymbol{y} \in \mathbb{R}^k$. The function f_k is called the *k-th clusterwise linear regression function*. One can consider any positive values of p to define regression errors. However, most widely used values are $p = 1$ and $p = 2$.

If $p = 1$ then the function f_k is piecewise linear and nonsmooth for all $k \geq 1$. If $p = 2$ then it is piecewise quadratic. Moreover, if $k = 1$ then in both cases the objective function is convex and if $k > 1$ it becomes nonconvex.

The function f_k in can be represented as a difference of two convex functions as follows:

$$f_k(\boldsymbol{x}, \boldsymbol{y}) = f_k^1(\boldsymbol{x}, \boldsymbol{y}) - f_k^2(\boldsymbol{x}, \boldsymbol{y})$$

where

$$f_k^1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{m} \sum_{j=1}^{k} h(\boldsymbol{x}_j, y_j, \boldsymbol{a}_i, b_i),$$

$$f_k^2(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{m} \max_{j=1,\ldots,k} \sum_{t=1, t \neq j}^{k} h(\boldsymbol{x}_t, y_t, \boldsymbol{a}_i, b_i).$$

If $p = 2$ then the function $f_k^1(\boldsymbol{x}, \boldsymbol{y})$ is differentiable for any $\boldsymbol{x} \in \mathbb{R}^{k \times n}$ and $\boldsymbol{y} \in \mathbb{R}^k$. The function f_k^2 is nonsmooth for both $p = 1$ and $p = 2$.

Alternatively, one can define the overall fit function using membership coefficients of data points:

$$\psi_k(\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{m} \sum_{j=1}^{k} w_{ij} h(\boldsymbol{x}_j, y_j, \boldsymbol{a}_i, b_i), \tag{7.52}$$

where $w_{ij} = 0$ if the data point $(\boldsymbol{a}^i, b_i)$ does not belong to the cluster A^j and $w_{ij} = 1$ if this point belong to it.

Note that the number of variables in the function f_k is $(n + 1)k$ and this number in the function ψ_k is $(m + n + 1)k$. In many data sets the number of data points m is significantly larger than the number of attributes n, therefore the number of variables in nonsmooth function f_k is significantly smaller than that in the function ψ_k. Moreover, the number of variables in function f_k does not depend on the number of points in a data set. Therefore the nonsmooth nonconvex overall fit function (7.51) is preferable than the function ψ_k to design algorithms for clusterwise linear regression problems.

The *k-clusterwise linear regression problem* is formulated as follows:

$$\begin{cases} \text{minimize} & f_k(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to} & \boldsymbol{x} \in \mathbb{R}^{k \times n} \\ & \boldsymbol{y} \in \mathbb{R}^k. \end{cases} \tag{7.53}$$

The number of clusters k is not always known a priori and this number k should be specified before solving Problem (7.53).

7.3 Optimal Control Problems

This section is devoted to classical applications of NSO, namely optimal control problems, where the constraints are often determined via a complicated system of (partial) differential equations. Here we briefly describe three types of optimal control problems: optimal shape design problems, distributed parameter control problems, and hemivariational inequalities.

7.3.1 Optimal Shape Design

The primary problem commonly facing designers of structural systems is determining the shape of the structure. The goal in *optimal shape design* (structural optimization, or redesign) is to computerize the design process and therefore shorten the time that it takes to design new products or improve an existing design. Structural optimization is widely used in certain applications in the automobile, marine, and aerospace industries, and in designing truss and shell structures (with minimum weights).

The abstract setting of the optimal shape design problem is the following. Let $\Omega \in \mathcal{O}_{\text{ad}}$ (= set of admissible domains) be a domain for which we want to find an optimal design (an optimal geometrical layout). Our aim is to

$$\begin{cases} \text{minimize} & f(\Omega, y(\Omega)) \\ \text{subject to} & \Omega \in \mathcal{O}_{\text{ad}}, \end{cases}$$

where the control $\Omega \in \mathcal{O}_{\text{ad}}$ and the state $y(\Omega) \in V(\Omega)$ are related by some state problem (given by equations, inequalities etc. in Ω). Here $V(\Omega)$ denotes a Hilbert space of functions defined on Ω.

Depending on the structure of the state system, we obtain different types of nonsmooth problems that fit into this setting; for instance, the unilateral (Dirichlet–Signorini) boundary value problem, and the design of optimal covering (packaging) problem. The discretization of both of the above types of problems leads to solving NSO problems. In addition, in a multicriteria structural design problem (for instance, an Euler-Bernoulli beam with varying thickness), the nonsmoothness is caused by the eigenvalues.

7.3.2 Distributed Parameter Control Problems

In *distributed parameter control problems*, the state relation is given by equations or by inequalities, and the control variable appears both in the coefficients and on the right-hand side. Moreover, we impose additional constraints upon the state of the system, which are often of a technological nature and cause the problem to be nonsmooth.

The abstract setting of this problem class reads

$$\begin{cases} \text{minimize} & f(x, y(x)) \\ \text{subject to} & x \in X_{\text{ad}}, \end{cases}$$

where the control x and the state $y(x) \in Y$ are related by the state problem

$$A(x)y(x) + \partial\varphi(y(x)) \ni Bx + g$$

with a state constraint

$$y \in K.$$

Here X and Y are Banach spaces, Y' is a dual space of Y, $X_{\text{ad}} \subseteq X$, $K \subseteq Y$ are convex, closed and nonempty subsets, $A(x) : X \to Y'$ and $B : X \to Y'$ are linear continuous mappings, $g \in Y'$ and $\varphi : Y \to (-\infty, +\infty]$ is a convex, lower semicontinuous, proper function.

In [168] three practical examples fitting into this setting were presented. The problems were the axially loaded rod with stress constraints, the clamped beam with displacement constraints, and the clamped beam with obstacle. The exact penalty technique was utilized for handling the state constraints, which leads to an optimization problem with a nonsmooth objective function.

7.3.3 Hemivariational Inequalities

In an abstract setting our aim is to find $y \in V$ and $\Phi \in L^1(\Omega) \cap V'$ solving the nonlinear, nonsmooth elliptic equation

$$\begin{cases} a(y, z-y) + \langle \Phi, z-y \rangle + h(z) - h(y) \geq \langle g, z-y \rangle \quad \text{for all } z \in V, \\ \Phi(\boldsymbol{x}) \in \partial_y q(\boldsymbol{x}, y(\boldsymbol{x})) \quad \text{almost everywhere in } \Omega, \\ \langle \Phi, z \rangle = \int_\Omega \Phi(\boldsymbol{x}) z(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{for all } z \in V \cap C^\infty(\overline{\Omega}), \end{cases} \tag{7.54}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary, V is a real Hilbert space with a dual space of V', $a : V \times V \to \mathbb{R}$ is a continuous, symmetric and coercive bilinear form, $g \in V'$, $h : V \to \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous and proper functional, and $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying some conditions (see [61, 194]).

After some approximation and discretization procedures, the hemivariational inequality (7.54) can be formulated as finding the stationary points (i.e. $\boldsymbol{y} \in \mathbb{R}^n$ such that $\mathbf{0} \in \partial f(\boldsymbol{y})$) of

$$f(\boldsymbol{y}) = \frac{1}{2}\boldsymbol{y}^T A \boldsymbol{y} - \boldsymbol{b}^T \boldsymbol{y} + \Psi(\boldsymbol{y}) + h(\boldsymbol{y}), \tag{7.55}$$

where A is an $n \times n$ matrix, $\boldsymbol{b} \in \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Psi(\cdot) = \int_\Omega q(\cdot, \boldsymbol{y}(\cdot))\, d\boldsymbol{x}$$

is a nonconvex and nonsmooth function. Due to Theorem 5.17, the local minimum (and maximum) points are stationary points that can be found via NSO.

One practical example which fits into this setting is the problem of an elastic body subjected to body forces and surface tractions, and obeying a nonmonotone friction law on some part of the boundary.

7.4 Engineering and Industrial Applications

Next we consider some NSO problems arising in industrial applications.

7.4.1 Power Unit-Commitment Problem

The *power unit-commitment problem* is a very important practical application arising in short-term power production planning. The optimal scheduling of many power units (hydraulic valleys, nuclear plants, and classical thermal units) makes the problem mathematically very complicated. In the abstract form, the problem can be written as

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{D} \cap \mathcal{S}, \end{cases} \tag{7.56}$$

where $\boldsymbol{x} = \{\boldsymbol{x}_i^t\} \in \mathbb{R}^I \times \mathbb{R}^T$ represents the vector of productions, I is a set of power-generation units, t is a discrete time period $t \in \{1, 2, \ldots, T\}$, $\mathcal{D} = \prod_{i \in I} \mathcal{D}_i$ represents the operating dynamic constraints and $\mathcal{S} = \prod_{t=1}^{T} \mathcal{S}^t$ the static constraints. The cost function f is in the form

$$f(\boldsymbol{x}) = \sum_{i \in I} c_i(\boldsymbol{x}_i),$$

where $\boldsymbol{x}_i = (x_i^1, \ldots, x_i^T)$ is the production vector of unit i for the time period and c_i is the corresponding cost. The feasible set $\mathcal{S}$ is described by linear constraints, while $\mathcal{D}$ has a nonconvex and more complicated form.

The problem (7.56) is a large mixed integer nonlinear mathematical programming problem, but has a highly decomposable structure. By using some Lagrangian relaxation techniques (7.56) can be transformed to a nonsmooth unconstrained problem, which has considerably fewer variables than the original problem.

7.4.2 Continuous Casting of Steel

The main object in the *continuous casting* process of steel is to minimize the defects in the final product. The temperature distribution of the strand is calculated by solving a nonlinear heat transfer equation with free boundaries between the solid and liquid phases. Because of the piecewise linear approximation of nonlinear terms, the problem is nonsmooth.

The control variable x represents a heat transfer coefficient, which has an effect on the temperature distribution $y(x)$ (the state) of the steel strand. Molten steel is poured down from the tundish into the water-cooled mold, where the metal gains a solid shell. After the end of the mold (point z_0), the strand is supported by rollers and cooled down by water sprays, so that eventually the solidification is completed (the maximum length of the liquid pool is denoted by z_2). After the water sprays (point z_1), the strand is cooled down only by radiation. The strand is straightened at the unbending point z_3 and it is cut up in z_4.

Let $\Omega \subset \mathbb{R}^2$ denote the cross-section of the strand and $\Gamma = \text{bd}\ \Omega$ its boundary. We define t_i for $i = 0, \ldots, 3$ to be the time events when Ω passes the distances z_i and we consider the time period between t_0 and t_3, that is, $t_0 = 0$ and $t_3 = T$. Moreover, we denote $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \Gamma$, $Q_1 = (t_2, T) \times \Omega$ and $\Sigma_1 = (0, t_1) \times \Gamma$. The set of admissible controls is defined by

$$X_{\text{ad}} = \{x \in L^2(\Sigma) \mid 0 < \alpha(t) \le x(t, \boldsymbol{s}) \le \beta(t),\ t \in (0, T),\ \boldsymbol{s} \in \Gamma\}. \tag{7.57}$$

Then for $x \in X_{\text{ad}}$ the temperature distribution $y = y(x)$ is obtained by solving the state system

$$\begin{cases} \frac{\partial}{\partial t} H(y(x)) = \Delta K(y(x)) & \text{in } Q, \\ \frac{\partial}{\partial n} K(y(x)) = g(x, y(x)) & \text{on } \Sigma, \\ y(\boldsymbol{s}, 0; x) = y_0(\boldsymbol{s}) & \boldsymbol{s} \in \Omega, \end{cases}$$

where $\partial/\partial n$ is a derivative in a direction of normal vector,

$$g(x, y(x)) = \begin{cases} x \cdot (y_{wat} - y(x)) + c \cdot (y_{ext}^4 - y(x)^4) & \text{on } \Sigma_1 \\ c \cdot (y_{ext}^4 - y(x)^4) & \text{on } \Sigma \setminus \Sigma_1, \end{cases} \tag{7.58}$$

the enthalpy function H, and the Kirchhoff's transformation K are piecewise linear functions, and the constants y_0, y_{wat} and y_{ext} denote the initial, spray water, and surrounding environment temperatures, respectively, and c is a physical constant.

On the boundary of the strand, we define some temperature distribution $y_d = y_d(t, s)$, which in a technological sense is good, and we want the actual surface temperature to be as close to y_d as possible. Thus, our objective function reads

$$f(x, y(x)) = \varepsilon_1 \int_0^T \frac{1}{2} \|y(x) - y_d\|_{0,\Gamma}^2 \, dt. \tag{7.59}$$

Moreover, we have the following technological constraints

$$\begin{cases} y_{min} \le y(x) \le y_{max} & \text{on } \Sigma \\ y'_{min} \le \frac{\partial}{\partial t} y(x) \le y'_{max} & \text{on } \Sigma \\ 0 \le y(x) \le y_{sol} & \text{in } Q_1 \\ y_{duc} \le y(\cdot, T; x) \le y_{sol} & \text{in } \Omega, \end{cases} \tag{7.60}$$

where the constants y_{min}, y_{max}, y'_{min}, y'_{max}, y_{sol} and y_{duc} denote certain minimum and maximum bounds (also for derivatives), the solidus and the ductility temperatures, respectively.

7.5 Other Applications

Several other practical applications lead to NSO problems as well. Here we very briefly describe two of them, namely the image restoration (or denoising) problem, which is a fundamental task in image processing; and a nonlinear income tax problem, which is an important problem that arises in the field of economics.

7.5.1 Image Restoration

Image restoration (or denoising) is a fundamental task in image processing. In various applications of computer vision, image processing is usually started by removing noise and other distortions from an image taken by a digital camera or obtained using some other method, for instance, ultrasound scan or computer tomography. Variational, optimization based techniques have proven quite efficient in image denoising. The related optimization formulations are often nonsmooth.

In denoising, the aim is to recover an image $\boldsymbol{u}$ from an observed image $\boldsymbol{z}$, where the two are related by $\boldsymbol{z} = \boldsymbol{u} + noise$. When considering a variational approach, the denoising problem is formulated as an optimization problem consisting of fitting and regularization terms, the form of which vary according to noise, image, and application. Here, we introduce four different formulations, each of which has its own benefits.

The basic formulation consisting of the least-squares fit and so-called bounded variational (BV) regularization is written as

$$\min_{\boldsymbol{u}\in H_0^1(\Omega)} \frac{1}{2}\int_{\Omega} \|\boldsymbol{u}-\boldsymbol{z}\|^2 d\boldsymbol{x} + \frac{\mu}{2}\int_{\Omega} \|\nabla \boldsymbol{u}\|^2 d\boldsymbol{x} + g\int_{\Omega} \|\nabla \boldsymbol{u}\| d\boldsymbol{x}, \tag{7.61}$$

where $\boldsymbol{z}$ is the noisy data, $\mu, g > 0$ are regularization parameters and $\Omega \in \mathbb{R}^2$ is the image domain. The additional smooth regularization term is added to ensure the unique solvability of the problem in $H_0^1(\Omega)$. This kind of formulation is known to be very efficient in recovering the sharp edges of images. However, the obtained images have a staircase-like structure, which is not desirable if the true image contains smooth surfaces. The reduction of this staircase-like structure can be obtained by using a semi-adaptive formulation

$$\min_{\boldsymbol{u}\in H_0^1(\Omega)} \frac{1}{2}\int_{\Omega} \|\boldsymbol{u}-\boldsymbol{z}\|^2 d\boldsymbol{x} + \frac{\mu}{2}\int_{\Omega} \|\nabla \boldsymbol{u}\|^2 d\boldsymbol{x} + g\int_{\Omega} \frac{1}{p(\nabla \bar{\boldsymbol{u}})} \|\nabla \boldsymbol{u}\|^{p(\|\nabla \bar{\boldsymbol{u}}\|)} d\boldsymbol{x}, \tag{7.62}$$

where $\bar{\boldsymbol{u}}$ denotes the solution of the BV problem (7.61) and the exact form of function p is given in [121]. The idea of this formulation is to give BV regularization near the edges (that is, $p(\|\nabla \bar{\boldsymbol{u}}\|) = 1$, for large $\|\nabla \bar{\boldsymbol{u}}\|$, and smoother regularization on flat and smooth areas (that is, the smaller $\|\nabla \bar{\boldsymbol{u}}\|$, the larger value of $p(\|\nabla \bar{\boldsymbol{u}}\|) \leq 2$).

The least squares fit appearing in formulations (7.61) and (7.62) assumes Gaussian noise and is very sensitive to so-called outliers in the data (that is, large measurement errors in a small number of pixels). To obtain formulations that are less sensitive to outliers, one can use robust nonsmooth L^1 fitting and smooth regularization

$$\min_{\boldsymbol{u}\in H_0^1(\Omega)} \int_{\Omega} \|\boldsymbol{u}-\boldsymbol{z}\| d\boldsymbol{x} + \frac{g}{2}\int_{\Omega} \|\nabla \boldsymbol{u}\|^2 d\boldsymbol{x} \tag{7.63}$$

or L^1 fitting and BV regularization

$$\min_{\boldsymbol{u}\in H_0^1(\Omega)} \int_\Omega \|\boldsymbol{u}-\boldsymbol{z}\|d\boldsymbol{x} + g\int_\Omega \|\nabla \boldsymbol{u}\|d\boldsymbol{x}, \tag{7.64}$$

7.5.2 Nonlinear Income Tax Problem

Next, we model the *nonlinear income tax problem* for a two-dimensional population as an optimal control problem. We consider the case where individuals differ with respect to their productivity and work preferences. Thus, each individual is described by a vector $(t, s)^T$ of type parameters that varies among individuals. The distribution of these parameters in the population is given by a density function h such that $h(t, s) \geq 0$ on a rectangular domain $\Omega = [t_0, t_1] \times [s_0, s_1]$.

An economy comprises two commodities, namely a consumption good q and labor supply u. The government knows that when it offers a nonlinear income tax schedule $x : \mathbb{R}^+ \to \mathbb{R}$, each individual maximizes his utility function of the form

$$g(u, q, t, s)$$

subject to

$$q + x(y) = y, \quad y = tu$$

in choosing his labor supply behavior. We assume that $g \in C^2$,

$$\frac{dg(u, q, t, s)}{du} < 0 \qquad \text{and}$$
$$\frac{dg(u, q, t, s)}{dq} > 0$$

for all $u, q \geq 0$ and $u < 1$. Given the tax schedule x, the government can calculate the gross income $y(t, s)$ and the consumption (net income) $q(t, s)$ for an individual who possesses the personal characteristics (t, s). Now, the objective of the government in choosing the optimal income tax schedule can be described as follows. Our aim is to find (t, s), which solves the problem

$$\begin{cases} \text{maximize} & \iint_\Omega g(u(t, s), q(t, s), t, s)h(t, s)\,dt\,ds \\ \text{subject to} & \iint_\Omega x(y(t, s))h(t, s)\,dt\,ds = \\ & \iint_\Omega (y(t, s) - q(t, s))h(t, s)\,dt\,ds \geq 0. \end{cases}$$

But if the government chooses tax schedule x, the individuals react to this schedule and modify their labor supply behavior. Now we are looking for x, which solves the following optimization problem

$$\begin{cases} \text{maximize} & f(x, y(x)) = \int_{\Omega} g(y(x)/t, y(x) - x(y(x)), t, s)h\, dt\, ds \\ \text{subject to} & \int_{\Omega} x(y(x))h \geq 0, \end{cases} \tag{7.65}$$

where $y(x)$ is a solution of the following state problem

$$y(x)(t, s) = \text{argmax}_{\,y \geq 0}\, g(y/t, y - x(y), t, s) \quad \text{for all } (t, s) \in \Omega. \tag{7.66}$$

In order to solve the problem (7.65), we need the values of the state mapping $x \mapsto y(x)$ and thus each evaluation of y demands the solution of the state system (7.66). The state constraint $y \geq 0$ causes the mapping $x \mapsto y(x)$ to be nonsmooth.

Chapter 8
SemiAcademic Problems

Using certain important methodologies for solving difficult smooth problems lead directly to the need to solve nonsmooth problems, which are either smaller in dimension or simpler in structure. The examples of this kind of methodological nonsmoothness are Lagrange relaxation, different decompositions, dual formulations, and exact penalty functions. In this chapter, we briefly describe some of these formulations. In addition, we represent the maximum eigenvalue problem that is an important part of many engineering design problems and graph theoretical applications. The interested reader may find more details of each problem class in the Notes and References at the end of Part II.

8.1 Exact Penalty Formulation

In this section, we consider the *exact penalty function* formulation for solving the constrained optimization problems of the form

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \le 0, \quad \text{for } i = 1, \dots, p, \\ & h_j(\boldsymbol{x}) = 0, \quad \text{for } j = 1, \dots, q, \end{cases} \tag{8.1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}$ for each $i \in \mathcal{P} = \{1, \dots, p\}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ for each $j \in \mathcal{Q} = \{1, \dots, q\}$ are supposed to be locally Lipschitz continuous.

The idea in penalty function formulations is to convert the original constrained optimization problem (8.1) to a sequence of unconstrained problems by adding a penalty to the objective that punishes the exit from the feasible region. The new penalty formulation can then be solved with some unconstrained optimization method. Indeed, problem (8.1) may be solved by minimizing the penalty function

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$$F_{r,m}(\boldsymbol{x}) = f(\boldsymbol{x}) + r\left(\sum_{i=1}^{p} \max\{0, g_i(\boldsymbol{x})\}^m + \sum_{j=1}^{q} |h_j(\boldsymbol{x})|^m\right), \tag{8.2}$$

where $r > 0$ is a penalty parameter and m is an exponent for the penalties. When $m > 1$ function (8.2) is differentiable and it is smooth when $m \geq 2$. The weakness in these differentiable formulations is that the penalty parameter r has to tend towards infinity in order to guarantee the convergence to the minimum.

To avoid this difficulty the so-called *absolute value* or l_1 *exact penalty function* can be used; that is, $m = 1$ in (8.2). In what follows, we omit the index m whenever it equals to one. In other words, we denote the l_1 exact penalty function by

$$F_r(\boldsymbol{x}) = F_{r,1}(\boldsymbol{x}) = f(\boldsymbol{x}) + r\left(\sum_{i=1}^{p} \max\{0, g_i(\boldsymbol{x})\} + \sum_{j=1}^{q} |h_j(\boldsymbol{x})|\right). \tag{8.3}$$

Note that even if the original objective function f and the constraint functions g_i for all $i \in \mathcal{P}$ and h_j for all $j \in \mathcal{Q}$ were smooth, the l_1 exact penalty function (8.3) is nonsmooth.

The function (8.3) is exact in a sense that, under mild assumptions, there exist a finite lower bound $r_l > 0$ such that for all $r > r_l$ any local minimizer of problem (8.1) is also a local minimizer of function (8.3). Moreover, if $\boldsymbol{x}^*$ is a local minimizer of (8.3) for some r, such that it is feasible with respect to the original problem (8.1), then $\boldsymbol{x}^*$ is also a local minimizer of problem (8.1) assuming that the equality constraints h_j for all $j \in \mathcal{Q}$ are smooth.

When both the constrained problem (8.1) and the penalty function (8.3) are convex, it is enough to know that any local minimizer of problem (8.1) is also a local minimizer of function (8.3) with sufficiently large penalty parameter r, since every local minimum is also global. Thus, the minimizers of the original problem and the exact penalty formulation are identical. However, in the nonconvex case, it may happen that a point $\boldsymbol{x}^*$ is a local minimizer of (8.3) for all $r > 0$ large enough, while $\boldsymbol{x}^*$ is not feasible for the original problem (8.1) for any $r > 0$, even though feasible points do exist. As an example, consider the following problem [64]:

Example 8.1 (*Exact Penalty Function*).

$$\begin{cases} \text{minimize} & x \\ \text{subject to} & x^3 - x \leq 0, \\ & x \geq 0, \end{cases}$$

The corresponding penalty function is

$$F_r(\boldsymbol{x}) = x + r(\max\{0, x^3 - x\} + \max\{0, -x\}),$$

for which we can show that $x^* = -1$ is a local minimum for all $r \geq 1$. However, this local minimizer is not feasible for the original problem, although feasible points $x \in [0, 1]$ do exists.

Another difficulty may arise when one or more of the equality constraints h_j, $j \in \mathcal{Q}$, are nonsmooth. This is due to the fact that, in the presence of nonsmooth equality constraints, the stationary points of exact penalty function (8.3) that are feasible for the original problem (8.1) can not, in general, be shown to be stationary for problem (8.1) if they do not lie in an one dimensional subspace of $\mathbb{R}^n$. Moreover, by increasing the penalty parameter r, an arbitrary feasible point of (8.1) may become stationary for (8.3).

More details of the exact penalty function methods for solving the constrained optimization problems are given in Part III Sect. 16.1.

8.2 Integer Programming with Lagrange Relaxation

Next we consider some discrete optimization problems that can be reformulated as continuous nonsmooth problems. This is done via the so-called *Lagrange relaxation*.

8.2.1 Traveling Salesman Problem

The traveling salesman problem is a classical NP-complete combinatorial optimization problem. Starting from an arbitrary city we seek the shortest route, such that all the given n cities belong to the route and we return back to the first city. Mathematically this can be formulated as a linear programming problem

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in S, \end{cases} \tag{8.4}$$

where

$$S = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{a}, B\boldsymbol{x} \le \boldsymbol{b} \text{ and } x_i \in \{0, 1\}\}.$$

There do not exist any efficient algorithms for solving (8.4) exactly; therefore heuristic methods computing an approximate solution are widely used. In order to qualify the approximation, it is important to know some lower bound for the exact solution. Such a lower bound can be found by utilizing Lagrangian relaxation technique. By the duality theory we know that

$$\min_{\boldsymbol{x} \in S} f(\boldsymbol{x}) \ge \max_{\boldsymbol{y} \in \mathbb{R}^n} \phi(\boldsymbol{y}),$$

where ϕ is defined by

$$\phi(\boldsymbol{y}) = \min_{\boldsymbol{x} \in \hat{S}} f(\boldsymbol{x}) + \boldsymbol{y}^T(A\boldsymbol{x} - \boldsymbol{a})$$

and $S \subseteq \hat{S}$ such that

$$\hat{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid B\boldsymbol{x} \le \boldsymbol{b} \text{ and } x_i \in \{0, 1\}\}.$$

It is known from graph theory, that the value $\phi(\boldsymbol{y})$ can be calculated from a so-called minimum spanning tree. Thus we get a lower bound for the exact solution of (8.4) by looking for the unconstrained maximum of the ϕ, which is a nonsmooth, piecewise linear and concave function.

8.3 Maximum Eigenvalue Problem

Many engineering design problems and graph theoretical applications require the solution of the eigenvalue optimization problem

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) = \lambda_{\max}(A(\boldsymbol{x})) \\ \text{subject to} & \boldsymbol{x} \in \mathbb{R}^n, \end{cases} \tag{8.5}$$

where

$$\lambda_{\max}(A(\boldsymbol{x})) = \max_{1 \le i \le m} |\lambda_i(A(\boldsymbol{x}))|$$

denotes the eigenvalue of $A(\boldsymbol{x})$ with the largest absolute value, $A(\boldsymbol{x})$ is a real symmetric $m \times m$ -matrix-valued affine function of $\boldsymbol{x}$, and $\lambda_i(A(\boldsymbol{x}))$ for $i = 1, \ldots, m$ are its eigenvalues.

The problem (8.5) is convex as the largest eigenvalue of a matrix is a convex function of the matrix elements. Let $\boldsymbol{x}^*$ be a locally unique minimizer of $f(\boldsymbol{x})$. That is, $f(\boldsymbol{x}^*) < f(\boldsymbol{x})$ with all $\boldsymbol{x} \in B(\varepsilon; \boldsymbol{x}^*)$ and some $\varepsilon > 0$, If the eigenvalue $\lambda_{\max}(A(\boldsymbol{x}^*))$ is simple at $\boldsymbol{x}^*$, that is, it has the multiplicity of one, then the problem (8.5) is twice continuously differentiable in a neighborhood of $\boldsymbol{x}^*$. However, usually this is not the case but $A(\boldsymbol{x}^*)$ has multiple eigenvalues. Then the problem (8.5) is generally not differentiable at $\boldsymbol{x} = \boldsymbol{x}^*$. For example, consider the following problem [193]:

Example 8.2 (Maximum Eigenvalue Problem). Let $A(\boldsymbol{x})$ be a 2×2-matrix

$$A(\boldsymbol{x}) = \begin{bmatrix} 1 + x_1 & x_2 \\ x_2 & 1 + x_1 \end{bmatrix}.$$

The eigenvalues of $A(\boldsymbol{x})$ are

$$\lambda_1 = 1 + \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad \lambda_2 = 1 - \sqrt{x_1^2 + x_2^2}.$$

Thus, λ_1, the largest eigenvalue of $A(\boldsymbol{x})$, is not a smooth function of $\boldsymbol{x}$.

Chapter 9
Academic Problems

Many practical optimization problems involve nonsmooth functions. In this chapter, we give an extensive collection of problems for nonsmooth minimization which can be used to test nonsmooth optimization solvers. The general formula for these problems is written by

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in S, \end{cases} \tag{9.1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous on the feasible region $S \subseteq \mathbb{R}^n$. Note that no differentiability or convexity assumptions are made. All the problems given here are found in the literature and have been used in the past to develop, test, or compare NSO software.

We shall use a classification of test problems modified from that of [111]. That is we use a sequence of letters

O[O]-C-R-S,

where Table 9.1 give all possible abbreviations that could replace the letters O, C, R, and S (the brackets mean an optional abbreviation).

We first give a summary of all the problems in Table 9.2, where n denotes the number of variables and $f(\boldsymbol{x}^*)$ is the minimum value of the objective function. In addition, the classification and the references to the origin of the problems in each case are given in Table 9.2. Then, in Sects. 9.1 (small unconstrained problems), 9.2 (bound constraints), 9.3 (linearly constrained problems), 9.4 (large-scale unconstrained problems), and 9.5 (inequality constraints), we present the formulation of the objective function f, possible constraint functions g_j $(j = 1, \ldots, p)$, and the starting point $\boldsymbol{x}^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})^T$ for each problem. We also give the minimum point $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)^T$ for the problems with the precision of at least four decimals if possible (in larger cases this is not always practicable).

In what follows, we denote by div (i, j) the integer division for positive integers i and j, that is, the maximum integer not greater than i/j, and by mod (i, j) the remainder after integer division, that is, mod $(i, j) = j(i/j - \text{div}\,(i, j))$.

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Table 9.1 Classification of test problems

O	**Information about the objective**
L	Piecewise linear objective function
Q	Piecewise quadratic objective function
P	Generalized polynomial objective function
G	General objective function
D	Difference of two convex functions (DC)
M	Min-max- type objective function

C	**Information about constraint functions**
U	Unconstrained problem ($S = \mathbb{R}^n$ in (9.1))
B	Upper and lower bounds only
	($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^l \leq x_i \leq x_i^u$ for all $i = 1, \ldots, n\}$ in (9.1))
L	Linear constraint functions
Q	Quadratic constraint functions
G	General constraint functions
	($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \leq 0$ for all $j = 1, \ldots, p\}$ in (9.1))

R	**Regularity of the problem**
X	Convex problem
Z	Nonconvex problem

S	**Information about the solution**
E	Exact solution known
N	Only approximate numerical solution known

9.1 Small Unconstrained Problems

In this section we describe 40 small-scale nonsmooth unconstrained test problems. The number of variables varies from 2 to 50.

1 CB2 (Charalambous/Bandler) [59]

Classification: GM-U-X-N,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max \left\{ x_1^2 + x_2^4,\ (2 - x_1)^2 + (2 - x_2)^2,\ 2e^{x_2 - x_1} \right\}$,
Starting point 1: $\boldsymbol{x}^{(1)} = (2, 2)^T$,
Starting point 2: $\boldsymbol{x}^{(1)} = (1, -0.1)^T$,
Optimum point: $\boldsymbol{x}^* = (1.139286, 0.899365)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 1.9522245$.

Table 9.2 Condensed list of all test problems

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
1 CB2	GM-U-X-N	2	1.9522245	[59]
2 CB3	GM-U-X-E	2	2	[59]
3 DEM	QM-U-X-E	2	-3	[168]
4 QL	QM-U-X-E	2	7.2	[168]
5 LQ	QM-U-X-E	2	$-\sqrt{2}$	[168]
6 Mifflin 1	G-U-X-E	2	-1	[168]
7 Wolfe	G-U-X-E	2	-8	[159]
8 Rosen-Suzuki	QM-U-X-E	4	-44	[205]
9 Davidon 2	QM-U-X-N	4	115.70644	[159]
10 Shor	QM-U-X-N	5	22.600162	[168]
11 Maxquad	QM-U-X-N	10	-0.8414083	[146, 168]
12 Polak 2	GM-U-X-E	10	54.598150	[198]
13 Polak 3	GM-U-X-N	11	3.7034924	[198]
14 Wong 1	PM-U-X-N	7	680.63006	[7, 159]
15 Wong 2	QM-U-X-N	10	24.306209	[7, 159]
16 Wong 3	PM-U-X-N	20	133.72828	[7, 159]
17 Maxq	QM-U-X-E	20	0	[208]
18 Maxl	LM-U-X-E	20	0	[168]
19 TR48	LD-U-X-N	48	-638565.0	[146]
20 Goffin	LD-U-X-E	50	0	[168]
21 Crescent	QM-U-Z-E	2	0	[131]
22 Mifflin 2	G-U-Z-E	2	-1	[168]
23 WF	GM-U-Z-E	2	0	[159]
24 SPIRAL	GM-U-Z-E	2	0	[159]
25 EVD52	PM-U-Z-N	3	3.5997193	[159]
26 PBC3	GM-U-Z-N	3	$0.42021427 \cdot 10^{-2}$	[159]
27 Bard	GM-U-Z-N	3	$0.50816327 \cdot 10^{-1}$	[159]
28 Kowalik-Osborne	GM-U-Z-N	4	$0.80843684 \cdot 10^{-2}$	[159]
29 Polak 6	PM-U-Z-E	4	-44	[198]
30 OET5	QM-U-Z-N	4	$0.26359735 \cdot 10^{-2}$	[159]
31 OET6	GM-U-Z-N	4	$0.20160753 \cdot 10^{-2}$	[159]
32 EXP	GM-U-Z-N	5	$0.12237125 \cdot 10^{-3}$	[159]
33 PBC1	GM-U-Z-N	5	$0.22340496 \cdot 10^{-1}$	[159]
34 HS78	G-U-Z-N	5	-2.9197004	[111, 159]
35 El-Attar	G-U-Z-N	6	0.5598131	[159]
36 EVD61	GM-U-Z-N	6	$0.34904926 \cdot 10^{-1}$	[159]
37 Gill	PM-U-Z-N	10	9.7857721	[159]
38 Problem 1 in [21]	GD-U-Z-E	2	2	[21]
39 L1 Rosenbrock	LD-U-Z-E	2	0	[21]

(Continued)

Table 9.2 (Continued)

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
40 L1 Wood	LD-U-Z-E	4	0	[21]
41 Wong 2C	QM-L-X-N	10	24.306209	[159]
42 Wong 3C	PM-L-X-N	20	133.72828	[159]
43 MAD1	GM-L-Z-N	2	−0.38965952	[159]
44 MAD2	GM-L-Z-N	2	−0.33035714	[159]
45 MAD4	GM-L-Z-N	2	−0.44891079	[159]
46 MAD5	GM-L-Z-N	2	−0.42928061	[159]
47 PENTAGON	GM-L-Z-N	6	−1.85961870	[159]
48 MAD6	GM-L-Z-N	7	0.10183089	[159]
49 Dembo 3	GM-L-Z-N	7	1227.2260	[159]
50 Dembo 5	GM-L-Z-N	8	7049.2480	[159]
51 EQUIL	GM-L-Z-N	8	0	[146, 159]
52 HS114	GM-L-Z-N	10	−1768.8070	[159]
53 Dembo 7	GM-L-Z-N	16	174.78699	[159]
54 MAD8	QM-B-Z-N	20	0.50694799	[159]
55 Gen. of MAXL	LM-U-X-E	any	0	[155]
56 Gen. of L1HILB	L-U-X-E	any	0	[155]
57 Gen. of MAXQ	QM-U-X-E	any	0	[98]
58 Gen. of MXHILB	LM-U-X-E	any	0	[98]
59 Chained LQ	G-U-X-E	any	$-(n-1)2^{1/2}$	[98]
60 Chained CB3 I	G-U-X-E	any	$2(n-1)$	[98]
61 Chained CB3 II	GM-U-X-E	any	$2(n-1)$	[98]
62 Number of Active Faces	GM-U-Z-E	any	0	[95]
63 Gen. of Brown Function 2	G-U-Z-E	any	0	[98]
64 Chained Mifflin 2	G-U-Z-N	any	varies*	[98]
65 Chained Crescent I	QM-U-Z-E	any	0	[98]
66 Chained Crescent II	G-U-Z-E	any	0	[98]
67 Problem 6 in `test29`	QM-U-Z-N	any	0	[155]
68 Problem 17 in `test29`	GM-U-Z-E	any	0	[155]
69 Problem 19 in `test29`	GM-U-Z-N	any	0	[155]
70 Problem 20 in `test29`	GM-U-Z-N	any	0	[155]
71 Problem 22 in `test29`	GM-U-Z-N	any	0	[155]
72 Problem 24 in `test29`	GM-U-Z-N	any	0	[155]
73 DC Maxl	LD-U-Z-E	any	0	[21]
74 DC Maxq	QD-U-Z-E	any	0	[26]
75 Problem 6 in [26]	LD-U-Z-E	any	0	[26]
76 Problem 7 in [26]	LD-U-Z-E	any	0	[26]
57 + 77	QM-Q-Z-N	any	0.500065**	[122, 126]
58 + 77	LM-Q-Z-N	any	0.000163**	[122, 126]

(Continued)

Table 9.2 (Continued)

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
59 + 77	G-Q-Z-N	any	−1408.63**	[122, 126]
60 + 77	G-Q-Z-N	any	2003.24**	[122, 126]
61 + 77	GM-Q-Z-N	any	1998.36**	[122, 126]
62 + 77	GM-Q-Z-N	any	0.534851**	[122, 126]
63 + 77	G-Q-Z-N	any	5.00248**	[122, 126]
64 + 77	G-Q-Z-N	any	−680.628**	[122, 126]
65 + 77	QM-Q-Z-N	any	1.56604**	[122, 126]
66 + 77	G-Q-Z-N	any	5.99059**	[122, 126]
57 + 78	QM-Q-R-N	any	0.880569**	[122, 126]
58 + 78	LM-Q-Z-N	any	0.008487**	[122, 126]
59 + 78	G-Q-Z-N	any	−735.874**	[122, 126]
60 + 78	G-Q-Z-N	any	2808.45**	[122, 126]
61 + 78	GM-Q-Z-N	any	2796.35**	[122, 126]
62 + 78	GM-Q-Z-N	any	2.77674**	[122, 126]
63 + 78	G-Q-Z-N	any	not avail.	[122, 126]
64 + 78	G-Q-Z-N	any	4466.99**	[122, 126]
65 + 78	QM-Q-Z-N	any	483.441**	[122, 126]
66 + 78	G-Q-Z-N	any	not avail.	[122, 126]
57 + 79	QM-G-R-N	any	not avail.	[122, 126]
58 + 79	LM-G-Z-N	any	0.007981**	[122, 126]
59 + 79	G-G-Z-N	any	−1412.14**	[122, 126]
60 + 79	G-G-Z-N	any	2001.63**	[122, 126]
61 + 79	GM-G-Z-N	any	not avail.	[122, 126]
62 + 79	GM-G-Z-N	any	0.405473**	[122, 126]
63 + 79	G-G-Z-N	any	not avail.	[122, 126]
64 + 79	G-G-Z-N	any	−705.910**	[122, 126]
65 + 79	QM-G-Z-N	any	0.250063**	[122, 126]
66 + 79	G-G-Z-N	any	1.85396**	[122, 126]
57 + 80	QM-G-R-N	any	0.388891**	[122, 126]
58 + 80	LM-G-Z-N	any	0.007981**	[122, 126]
59 + 80	G-G-Z-N	any	−1412.13**	[122, 126]
60 + 80	G-G-Z-N	any	2001.72**	[122, 126]
61 + 80	GM-G-Z-N	any	not avail.	[122, 126]
62 + 80	GM-G-Z-N	any	0.405549**	[122, 126]
63 + 80	G-G-Z-N	any	not avail.	[122, 126]
64 + 80	G-G-Z-N	any	−705.926**	[122, 126]
65 + 80	QM-G-Z-N	any	0.250222**	[122, 126]
66 + 80	G-G-Z-N	any	1.39342**	[122, 126]
57 + 81	QM-Q-R-N	any	0.138009**	[122, 126]

(Continued)

Table 9.2 (Continued)

Problem	O[O]-C-R-S	n	$f(\boldsymbol{x}^*)$	Ref.
58 + 81	LM-Q-Z-N	any	0.600611**	[122, 126]
59 + 81	G-Q-Z-N	any	−1153.55**	[122, 126]
60 + 81	G-Q-Z-N	any	4043.82**	[122, 126]
61 + 81	GM-Q-Z-N	any	4043.82**	[122, 126]
62 + 81	GM-Q-Z-N	any	5.81129**	[122, 126]
63 + 81	G-Q-Z-N	any	589.469**	[122, 126]
64 + 81	G-Q-Z-N	any	−660.307**	[122, 126]
65 + 81	QM-Q-Z-N	any	490.173**	[122, 126]
66 + 81	G-Q-Z-N	any	not avail.	[122, 126]

* $f(\boldsymbol{x}^*) \approx -34.795$ for $n = 50$, $f(\boldsymbol{x}^*) \approx -140.86$ for $n = 200$, and $f(\boldsymbol{x}^*) \approx -706.55$ for $n = 1000$.
** $f(\boldsymbol{x}^*)$ for $n = 1000$.

Table 9.3 Values of vectors $\boldsymbol{s}$ and $\boldsymbol{d}$ for problem 19

i	s_i	d_i	i	s_i	d_i	i	s_i	d_i
1	22	61	17	95	32	33	30	52
2	53	67	18	34	21	34	88	66
3	64	24	19	59	61	35	74	89
4	15	84	20	36	21	36	59	65
5	66	13	21	22	51	37	93	63
6	37	86	22	94	14	38	54	47
7	16	89	23	28	89	39	89	7
8	23	46	24	34	79	40	30	61
9	67	48	25	36	38	41	79	87
10	18	50	26	38	20	42	46	19
11	52	74	27	55	97	43	35	36
12	69	75	28	77	19	44	41	43
13	17	88	29	45	10	45	99	9
14	29	40	30	34	73	46	52	12
15	50	29	31	32	59	47	76	8
16	13	45	32	58	92	48	93	67

2 CB3 (Charalambous/Bandler) [59]

Classification: GM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{x_1^4 + x_2^2,\ (2 - x_1)^2 + (2 - x_2)^2,\ 2e^{x_2 - x_1}\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (2, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 2$.

Table 9.4 Data for symmetric cost matrix A for problem 19

273	1272	744	1138	1972	1580	1878	1539	1457	429	1129	1251	1421	588	334	837
1364	229	961	754	1169	1488	720	1280	816	664	1178	939	1698	983	1119	1029
1815	721	1753	330	1499	1107	1576	942	484	617	896	1184	1030	1718	604	999
809	866	1722	1338	1640	1266	1185	440	894	992	1173	334	358	626	1124	358
847	533	915	1219	481	1009	543	937	915	667	1441	812	848	776	1560	526
1494	598	1244	1304	1306	685	668	444	1157	1359	1176	1475	335	1519	140	937
697	951	267	227	1229	587	369	554	721	1212	739	596	1291	1114	701	426
285	676	155	456	1936	319	337	604	907	214	424	748	817	666	1592	521
2172	356	467	1583	882	2139	2182	1961	781	678	1425	1861	1473	1713	1761	1617
370	1073	1304	1369	1092	453	798	1283	973	565	1315	1204	1796	846	1447	1143
959	1275	1213	2085	742	1309	1479	1760	703	1727	872	1479	686	1698	1057	387
1252	904	668	443	1600	930	1052	776	1049	402	361	1119	578	406	618	581
1095	670	641	1152	1060	567	433	374	579	235	325	1802	331	217	665	862
182	312	864	732	783	1456	608	2066	491	400	1466	744	2013	2082	1865	875
552	400	182	820	721	1735	851	740	551	1551	1769	1159	613	2072	1300	1605
807	1017	1251	818	1259	2596	826	1137	1255	1123	943	1359	188	1282	271	2300
483	2540	609	1038	2099	1766	2699	2493	2266	264	1398	304	699	538	1335	454
393	173	1198	1370	760	216	1692	919	1286	435	879	861	548	913	2198	483
803	1181	731	627	1086	292	883	279	1906	178	2156	490	662	1699	1430	2300
2117	1888	138	1023	884	755	1612	749	690	476	1501	1654	1049	516	1995	1149
1580	739	1079	1161	815	1214	2485	780	1100	1347	985	916	1361	260	1171	328
2202	445	2385	665	966	1969	1729	2568	2333	2108	177	1327	177	1486	757	506
609	981	1474	967	681	1552	1317	936	594	197	928	316	723	2203	500	604
482	1104	455	630	641	1058	562	1857	528	2425	220	704	1845	1122	2405	2428

(Continued)

Table 9.4 (Continued)

2204	738	945	1362	587	335	435	930	1358	819	504	1496	1153	927	428	341
803	180	649	2119	343	521	652	939	340	649	533	918	451	1783	362	2290
130	568	1727	1105	2301	2285	2059	595	853	891	1082	1199	726	96	583	1125
653	563	947	986	1493	560	1183	813	882	1033	902	1763	642	1032	1131	1604
463	1556	663	1298	947	1461	795	371	882	967	973	768	1472	588	252	308
803	920	309	238	1252	569	940	165	863	414	454	552	1745	269	482	1188
355	397	833	713	432	666	1453	410	1758	642	262	1260	1051	1858	1737	1508
592	598	222	814	1094	510	235	1335	820	892	100	626	541	219	524	1897
90	410	952	605	238	706	570	622	503	1581	257	1985	396	309	1453	1039
2043	1972	1744	514	661	1025	1227	617	90	1525	835	1114	263	770	700	400
740	2049	311	630	1087	630	459	924	405	739	360	1749	115	2055	428	492
1568	1256	2166	2026	1796	303	853	663	632	999	572	972	225	763	908	451
767	293	1240	726	420	1111	862	617	443	1374	586	1299	887	1070	1633	1057
547	999	252	1483	1681	1489	1326	236	610	1156	557	642	879	1000	1467	558
1178	780	831	1038	879	1726	700	1023	1082	1631	488	1579	586	1320	982	1463
796	371	802	949	1021	826	1508	550	546	983	397	821	411	1023	180	651
478	1438	476	485	1333	235	525	827	1022	123	973	1155	715	1475	902	273
953	882	1550	1467	1240	898	396	1479	745	1105	240	831	645	442	723	1983
316	623	1152	543	470	939	482	669	443	1690	205	1969	510	455	1492	1238
2091	1938	1709	354	813	1163	676	1264	1473	839	1326	847	801	1254	976	1643
1157	1169	983	1905	878	1836	346	1590	1286	1621	1034	689	503	995	1376	1239
1828	674	1183	725	1399	549	1004	869	1427	818	882	1716	214	902	1222	1210
390	1184	1225	949	1239	1210	660	863	1207	1446	1197	969	1042	741	865	821
644	790	388	1374	803	484	968	1056	665	318	1420	794	1341	1017	1137	1836

(Continued)

Table 9.4 (Continued)

1056	679	1200	189	1645	1891	1704	1403	442	699	453	290	483	1809	107	384
1024	511	251	712	646	525	585	1499	330	1885	495	231	1356	999	1949	1872
1644	567	591	950	410	690	2147	594	590	326	1191	499	504	838	1098	758
1794	703	2439	414	751	1837	1011	2374	2455	2237	928	921	624	325	1356	480
369	1241	413	473	680	1097	166	1038	1049	781	1497	905	238	925	702	1506
1506	1287	998	216	479	1941	188	350	736	792	161	547	632	745	552	1607
375	2115	296	392	1547	959	2121	2114	1890	641	676	1480	435	129	949	708
325	355	1081	492	1007	1137	779	1759	774	291	1148	516	1688	1785	1573	1038
231	1829	1603	2339	1524	1780	1673	2421	1315	2394	357	2136	825	2237	1589	579
1204	347	959	940	2336	1266	320	919	605	154	623	652	580	582	1508	344
1950	429	242	1402	949	1986	1943	1717	603	582	872	699	197	358	957	529
881	1263	660	1849	645	240	1250	631	1802	1867	1650	923	341	1511	815	669
1092	1397	1019	1982	1010	2708	695	1061	2089	1148	2594	2734	2520	1212	1176	697
1051	1018	290	985	1280	743	1427	996	466	987	1110	1584	1395	1166	861	626
469	761	607	685	1446	472	1969	457	254	1393	823	1963	1975	1752	739	515
1171	847	1089	1316	919	2063	776	598	1434	507	1926	2101	1898	1187	548	1144
83	2145	317	2445	426	875	1972	1584	2571	2408	2179	194	1231	1094	1036	836
1371	1008	354	833	828	1429	1369	1146	1021	352	2083	259	2412	345	811	1925
1507	2523	2380	2151	220	1163	1628	1005	1903	1272	504	849	653	1114	1019	2044
932	2165	330	559	1668	1291	2264	2138	1908	268	917	2377	1723	636	1720	534
145	290	2281	1531	667	1829	1235	2410	2367	2139	519	972	1162	792	1744	1724
1500	796	361	1087	600	701	550	1835	917	1490	1787	1614	1553	486	678	727
2435	1461	229	2238	1560	2010	1353	1157								

Matrix A is symmetric, that is $a(i, j) = a(j, i)$. Data is given for lower triangular

3 DEM [168]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{5x_1 + x_2,\ -5x_1 + x_2,\ x_1^2 + x_2^2 + 4x_2\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, -3)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -3$.

4 QL [168]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 3} f_i(\boldsymbol{x})$,
where $f_1(\boldsymbol{x}) = x_1^2 + x_2^2$,
$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(-4x_1 - x_2 + 4)$,
$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(-x_1 - 2x_2 + 6)$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, 5)^T$,
Optimum point: $\boldsymbol{x}^* = (1.2, 2.4)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 7.2$.

5 LQ [168]

Classification: QM-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\left\{-x_1 - x_2,\ -x_1 - x_2 + x_1^2 + x_2^2 - 1\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (-0.5, -0.5)^T$,
Optimum point: $\boldsymbol{x}^* = (1/\sqrt{2}, 1/\sqrt{2})^T$,
Optimum value: $f(\boldsymbol{x}^*) = -\sqrt{2}$.

6 Mifflin 1 [168]

Classification: G-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = -x_1 + 20 \max\left\{x_1^2 + x_2^2 - 1,\ 0\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (0.8, 0.6)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -1$.

7 Wolfe [159]

Classification: G-U-X-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = 5\sqrt{9x_1^2 + 16x_2^2}$, when $x_1 \geq |x_2|$,
$f(\boldsymbol{x}) = 9x_1 + 16|x_2|$, when $0 < x_1 \leq |x_2|$,
$f(\boldsymbol{x}) = 9x_1 + 16|x_2| - x_1^9$, when $x_1 \leq 0$,
Starting point: $\boldsymbol{x}^{(1)} = (3, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (-1, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -8$.

8 Rosen-Suzuki [205]

Classification: QM-U-X-E,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 4} f_i(\boldsymbol{x})$,
where $f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$,
$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3,$
$-x_4 - 8)$,
$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10)$,
$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5)$,
Starting point: $\boldsymbol{x}^{(1)} = (0, 0, 0, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 1, 2, -1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -44$.

9 Davidon 2 [159]

Classification: QM-U-X-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 20} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = \left(x_1 + x_2 t_i - e^{t_i}\right)^2 + (x_3 + x_4 \sin(t_i) - \cos(t_i))^2$ and
$t_i = 0.2i$, for $i = 1, \ldots, 20$,
Starting point: $\boldsymbol{x}^{(1)} = (25, 5, -5, -1)^T$,
Optimum point: $\boldsymbol{x}^* = (-12.2437, 14.0218, -0.4515, -0.0105)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 115.70644$.

10 Shor [168]

Classification: QM-U-X-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 10} \left\{ b_i \sum_{j=1}^{5} (x_j - a_{ij})^2 \right\}$,

where

$$A = \begin{pmatrix} 0&0&0&0&0\\ 2&1&1&1&3\\ 1&2&1&1&2\\ 1&4&1&2&2\\ 3&2&1&0&1\\ 0&2&1&0&1\\ 1&1&1&1&1\\ 1&0&1&2&1\\ 0&0&2&1&0\\ 1&1&2&0&0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1\\ 5\\ 10\\ 2\\ 4\\ 3\\ 1.7\\ 2.5\\ 6\\ 3.5 \end{pmatrix},$$

Starting point: $\boldsymbol{x}^{(1)} = (0, 0, 0, 0, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (1.1244, 0.9795, 1.4777, 0.9202, 1.1243)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 22.600162$.

11 **Maxquad** [146,168]

Classification: QM-U-X-N,
Dimension: 10,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 5} \boldsymbol{x}^T A_i \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{b}_i$,
where $A_{ijk} = A_{ikj} = e^{j/k}\cos(jk)\sin(i)$, $j < k$,
$A_{ijj} = \frac{j}{10}|\sin(i)| + \sum_{k\neq j} |A_{ijk}|$, and
$b_{ij} = e^{i/j}\sin(ij)$,
Starting point: $\boldsymbol{x}^{(1)} = (0, 0, \ldots, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (-0.1263, -0.0344, -0.0069, 0.0264, 0.0673, -0.2784, 0.0742, 0.1385, 0.0840, 0.0386)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -0.8414083$.

12 **Polak 2** [198]

Classification: GM-U-X-E,
Dimension: 10,
Objective function: $f(\boldsymbol{x}) = \max\{\, g(\boldsymbol{x} + 2\boldsymbol{e}_2),\ g(\boldsymbol{x} - 2\boldsymbol{e}_2)\,\}$,
where $g(\boldsymbol{x}) = e^{10^{-8}x_1^2+x_2^2+x_3^2+4x_4^2+x_5^2+x_6^2+x_7^2+x_8^2+x_9^2+x_{10}^2}$,
$\boldsymbol{e}_2$ = second column of the identity matrix,
Starting point: $x_1^{(1)} = 100.0$ and
$x_i^{(1)} = 0.1$ for $i = 2, \ldots, 10$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 54.598150 = e^4$.

13 **Polak 3** [198]

Classification: GM-U-X-N,
Dimension: 11,

Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 10} f_i(\boldsymbol{x})$,
where	$f_i(\boldsymbol{x}) = \sum_{j=0}^{10} \frac{1}{i+j} e^{(x_{j+1} - \sin(i-1+2j))^2}$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (0.0124, 0.2904, -0.3347, -0.1265, 0.2331, -0.2766, -0.1666, 0.2291, -0.1858, -0.1704, 0.2402)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 3.7034924$.
Note:	A different optimal value $f(\boldsymbol{x}^*) = 261.08258$ has been given in [159]. This is due to erroneous code used in calculations.

14 Wong 1 [7,159]

Classification:	PM-U-X-N,
Dimension:	7,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 5} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$,
	$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127)$,
	$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282)$,
	$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196)$,
	$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7)$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 2, 0, 4, 0, 1, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (2.3305, 1.9514, -0.4775, 4.3657, -0.6245, 1.0381, 1.5942)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 680.63006$.

15 Wong 2 [7,159]

Classification:	QM-U-X-N,
Dimension:	10,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le 9} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45$,
	$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120\right)$,
	$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40\right)$,
	$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6, -30\right)$,
	$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6\right)$,
	$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\,(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105)$,
	$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\,(10x_1 - 8x_2 - 17x_7 + 2x_8)$,
	$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\left(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}\right)$,
	$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10\,(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12)$,

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1720, 2.3637, 8.7739, 5.0960, 0.9907, 1.4306, 1.3217, 9.8287, 8.2801, 8.3759)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 24.306209$.

16 Wong 3 [7,159]

Classification: PM-U-X-N,
Dimension: 20,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 18} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 \\
&\quad + 4(x_4-5)^2 + (x_5-3)^2 + 2(x_6-1)^2 \\
&\quad + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 \\
&\quad + (x_{11}-9)^2 + 10(x_{12}-1)^2 + 5(x_{13}-7)^2 \\
&\quad + 4(x_{14}-14)^2 + 27(x_{15}-1)^2 + x_{16}^4 + (x_{17}-2)^2 \\
&\quad + 13(x_{18}-2)^2 + (x_{19}-3)^2 + x_{20}^2 + 95, \\
f_2(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 \right.\\
&\qquad\left. -120\right), \\
f_3(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40\right), \\
f_4(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 \right.\\
&\qquad\left. -30\right), \\
f_5(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6\right), \\
f_6(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105), \\
f_7(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(10x_1 - 8x_2 - 17x_7 + 2x_8), \\
f_8(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(-3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10}\right), \\
f_9(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12), \\
f_{10}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\,(x_1 + x_2 + 4x_{11} - 21x_{12}), \\
f_{11}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 + 15x_{11} - 8x_{12} - 28\right), \\
f_{12}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(4x_1 + 9x_2 + 5x_{13}^2 - 9x_{14} - 87\right), \\
f_{13}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(3x_1 + 4x_2 + 3(x_{13}-6)^2 - 14x_{14} - 10\right), \\
f_{14}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(14x_1^2 + 35x_{15} - 79x_{16} - 92\right), \\
f_{15}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(15x_2^2 + 11x_{15} - 61x_{16} - 54\right), \\
f_{16}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(5x_1^2 + 2x_2 + 9x_{17}^4 - x_{18} - 68\right), \\
f_{17}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 - x_2 + 19x_{19} - 20x_{20} + 19\right), \\
f_{18}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10\left(x_1^2 + 5x_2^2 + x_{19}^2 - 30x_{20}\right),
\end{aligned}$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10, 2, 2, 6, 15, 1, 2, 1, 2, 1, 3)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1752, 2.3529, 8.7665, 5.0669, 0.9887, 1.4310, 1.3295, 9.8359, 8.2873, 8.3702, 2.2758, 1.3586, 6.0772, 14.1708, 0.9962, 0.6557, 1.4666, 2.0004, 1.0466, 2.0632)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 133.72828$.

17 MAXQ [208]

Classification: QM-U-X-E,

Dimension:	20,	
Objective function:	$f(\boldsymbol{x}) = \max_{1\le i\le 20}\ x_i^2$,	
Starting point:	$x_i^{(1)} = i$	for $i = 1, \ldots, 10$ and
	$x_i^{(1)} = -i$	for $i = 11, \ldots, 20$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

18 MAXL [168]

Classification:	LM-U-X-E,	
Dimension:	20,	
Objective function:	$f(\boldsymbol{x}) = \max_{1\le i\le 20}\ \lvert x_i\rvert$,	
Starting point:	$x_i^{(1)} = i$	for $i = 1, \ldots, 10$ and
	$x_i^{(1)} = -i$	for $i = 11, \ldots, 20$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

19 TR48 [146]

Classification:	LD-U-X-N,
Dimension:	48,
Objective function:	$f(\boldsymbol{x}) = \sum_{j=1}^{48} d_j \max_{1\le i\le 48} (x_i - a_{ij}) - \sum_{i=1}^{48} s_i x_i$,
Starting point:	See Table 9.5,
Optimum point:	See Table 9.5,
Optimum value:	$f(\boldsymbol{x}^*) = -638565.0$.

20 Goffin [168]

Classification:	LD-U-X-E,	
Dimension:	50,	
Objective function:	$f(\boldsymbol{x}) = \max_{1\le i\le 50}\ x_i + \sum_{i=1}^{50} x_i$,	
Starting point:	$x_i^{(1)} = i - 25.5$	for $i = 1, \ldots, 50$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

21 Crescent [131]

Classification:	QM-U-Z-E,
Dimension:	2,
Objective function:	$f(\boldsymbol{x}) = \max\big\{x_1^2 + (x_2 - 1)^2 + x_2 - 1, -x_1^2 - (x_2 - 1)^2 + x_2 + 1\big\}$,
Starting point:	$\boldsymbol{x}^{(1)} = (-1.5, 2)$,

Table 9.5 Initialization and optimal point for 19

i	Starting point 1	Starting point 2	Optimum point $\boldsymbol{x}^*$
1	0.0	11.19	144.0
2	0.0	127.20	257.0
3	0.0	−129.70	0.0
4	0.0	344.50	483.0
5	0.0	−40.72	89.0
6	0.0	−295.30	−165.0
7	0.0	−202.30	−72.0
8	0.0	−382.30	−252.0
9	0.0	−217.70	−88.0
10	0.0	−307.70	−178.0
11	0.0	178.10	−311.0
12	0.0	−4.36	126.0
13	0.0	−123.30	7.0
14	0.0	−265.30	−135.0
15	0.0	28.28	158.0
16	0.0	70.57	209.0
17	0.0	−31.81	101.0
18	0.0	−222.30	−92.0
19	0.0	96.19	229.0
20	0.0	−52.79	80.0
21	0.0	−34.71	95.0
22	0.0	−59.16	71.0
23	0.0	−373.70	−244.0
24	0.0	−28.35	102.0
25	0.0	−141.70	−12.0
26	0.0	2.28	132.0
27	0.0	198.50	337.0
28	0.0	−69.16	61.0
29	0.0	−26.35	104.0
30	0.0	−88.72	41.0
31	0.0	130.80	261.0
32	0.0	−12.35	118.0
33	0.0	−30.70	99.0
34	0.0	−376.30	−246.0
35	0.0	23.18	156.0
36	0.0	−400.30	−270.0
37	0.0	197.10	330.0
38	0.0	−260.30	−130.0
39	0.0	813.50	952.0

(Continued)

Table 9.5 (Continued)

i	Starting point 1	Starting point 2	Optimum point $\boldsymbol{x}^*$
40	0.0	−191.70	−62.0
41	0.0	31.29	161.0
42	0.0	345.50	484.0
43	0.0	−7.72	122.0
44	0.0	335.50	474.0
45	0.0	947.50	1086.0
46	0.0	722.50	861.0
47	0.0	−300.30	−170.0
48	0.0	73.20	206.0

Optimum point: $\boldsymbol{x}^* = (0, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

22 Mifflin 2 [168]

Classification: G-U-Z-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = -x_1 + 2(x_1^2 + x_2^2 - 1) + 1.75|x_1^2 + x_2^2 - 1|$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -1$.

23 WF [159]

Classification: GM-U-Z-E,
Dimension: 2,
Objective function:
$$f(\boldsymbol{x}) = \max\left\{\frac{1}{2}\left(x_1 + \frac{10x_1}{x_1+0.1} + 2x_2^2\right),\ \frac{1}{2}\left(-x_1 + \frac{10x_1}{x_1+0.1} + 2x_2^2\right),\ \frac{1}{2}\left(x_1 - \frac{10x_1}{x_1+0.1} + 2x_2^2\right)\right\},$$
Starting point: $\boldsymbol{x}^{(1)} = (3, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

24 SPIRAL [159]

Classification: GM-U-Z-E,
Dimension: 2,
Objective function: $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})\}$,
where
$$f_1(\boldsymbol{x}) = \left(x_1 - \sqrt{x_1^2 + x_2^2}\cos\sqrt{x_1^2 + x_2^2}\right)^2 + 0.005(x_1^2 + x_2^2),$$
$$f_2(\boldsymbol{x}) = \left(x_2 - \sqrt{x_1^2 + x_2^2}\sin\sqrt{x_1^2 + x_2^2}\right)^2 + 0.005(x_1^2 + x_2^2),$$

Starting point: $\boldsymbol{x}^{(1)} = (1.411831, -4.79462)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

25 EVD52 [159]

Classification: PM-U-Z-N,
Dimension: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 6} f_i(\boldsymbol{x})$,
where
$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 - 1$,
$f_2(\boldsymbol{x}) = x_1^2 + x_2^2 + (x_3 - 2)^2$,
$f_3(\boldsymbol{x}) = x_1 + x_2 + x_3 - 1$,
$f_4(\boldsymbol{x}) = x_1 + x_2 - x_3 - 1$,
$f_5(\boldsymbol{x}) = 2x_1^3 + 6x_2^2 + 2(5x_3 - x_1 + 1)^2$,
$f_6(\boldsymbol{x}) = x_1^2 - 9x_3$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.3283, 0.0000, 0.1313)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 3.5997193$.

26 PBC3 [159]

Classification: GM-U-Z-N,
Dimension: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} f_i(\boldsymbol{x})$,
where
$f_i(\boldsymbol{x}) = \frac{x_3}{x_2} e^{-t_i x_1} \sin(t_i x_2) - y_i$,
$y_i = \frac{3}{20} e^{-t_i} + \frac{1}{52} e^{-5t_i} - \frac{1}{65} e^{-2t_i} (3 \sin(2t_i) + 11 \cos(2t_i))$,
and $t_i = 10(i - 1)/20$, for $i = 1, \ldots, 21$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.9516, 0.8761, 0.1623)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.42021427 \cdot 10^{-2}$.

27 Bard [159]

Classification: PM-U-Z-N,
Dimension: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 15} |f_i(\boldsymbol{x})|$,
where
$f_i(\boldsymbol{x}) = x_1 + \frac{i}{(16-i)x_2 + u_i x_3}$, for $i = 1, \ldots, 15$,
$\boldsymbol{u} = (1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1)^T$, and
$\boldsymbol{y} = (0.14, 0.18, 0.22, 0.25, 0.29, 0.32, 0.35, 0.39, 0.37, 0.58, 0.73, 0.96, 1.34, 2.10, 4.39)^T$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.0535, 1.5106, 1.9894)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.50816327 \cdot 10^{-1}$.

28 Kowalik-Osborne [159]

Classification: GM-U-Z-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 11} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = \frac{x_1(u_i^2 + x_2 u_i)}{u_i^2 + x_3 u_i + x_4} - y_i$,
$\boldsymbol{u} = (4.0000, 2.0000, 1.0000, 0.5000, 0.2500, 0.1670, 0.1250, 0.1000, 0.0833, 0.0714, 0.0625)^T$,
$\boldsymbol{y} = (0.1957, 0.1947, 0.1735, 0.1600, 0.0844, 0.0627, 0.0456, 0.0342, 0.0323, 0.0235, 0.0246)^T$,
Starting point: $\boldsymbol{x}^{(1)} = (0.250, 0.390, 0.415, 0.390)^T$,
Optimum point: $\boldsymbol{x}^* = (0.1846, 0.1052, 0.0196, 0.1118)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.80843684 \cdot 10^{-2}$.

29 Polak 6 [198]

Classification: PM-U-Z-E,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 4} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) = {} & (x_1 - (x_4+1)^4)^2 + \left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 + 2x_3^2 \\
& + x_4^2 - 5(x_1 - (x_4+1)^4) \\
& - 5\left(x_2 - (x_1 - (x_4+1)^4)^4\right) - 21x_3 + 7x_4, \\
f_2(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10\left((x_1 - (x_4+1)^4)^2\right. \\
& + \left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 \\
& + x_3^2 + x_4^2 + (x_1 - (x_4+1)^4) \\
& - \left(x_2 - (x_1 - (x_4+1)^4)^4\right) \\
& \left. + x_3 - x_4 - 8\right), \\
f_3(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10\left((x_1 - (x_4+1)^4)^2\right. \\
& + 2\left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 \\
& \left. + x_3^2 + 2x_4^2 - (x_1 - (x_4+1)^4) - x_4 - 10\right), \\
f_4(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10\left((x_1 - (x_4+1)^4)^2\right. \\
& + \left(x_2 - (x_1 - (x_4+1)^4)^4\right)^2 \\
& + x_3^2 + 2(x_1 - (x_4+1)^4) - \left(x_2 - (x_1 - (x_4+1)^4)^4\right) \\
& \left. - x_4 - 5\right),
\end{aligned}$$

Starting point: $\boldsymbol{x}^{(1)} = (0, 0, 0, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 1, 2, -1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -44$.

30 OET5 [159]

Classification: QM-U-Z-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = x_4 - (x_1 t_i^2 + x_2 t_i + x_3)^2 - \sqrt{t_i}$ and
$t_i = 0.25 + 0.75(i-1)/20 \quad \text{for } i = 1, \ldots, 21$,

Starting point: $\boldsymbol{x}^{(1)} = (1, 1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.0876, -0.497, 1.1155, 1.4963,)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.26359735 \cdot 10^{-2}$.

31 **OET6** [159]

Classification: GM-U-Z-N,
Dimension: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = x_1 e^{x_3 t_i} + x_2 e^{x_4 t_i} - \frac{1}{1+t_i}$ and
$t_i = -0.5 + (i-1)/20, \quad$ for $i = 1, \dots, 21$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, -3, -1)^T$,
Optimum point: $\boldsymbol{x}^* = (0.0987, 0.9009, -4.0619, -0.6477)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.20160753 \cdot 10^{-2}$.

32 **EXP** [159]

Classification: GM-U-Z-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 21} f_i(\boldsymbol{x})$,
where $f_i(\boldsymbol{x}) = \frac{x_1 + x_2 t_i}{1 + x_3 t_i + x_4 t_i^2 + x_5 t_i^3} - e^{t_i}$ and
$t_i = -1 + (i-1)/10, \quad i = 1, \dots, 21$,
Starting point: $\boldsymbol{x}^{(1)} = (0.5, 0, 0, 0, 0)^T$,
Optimum point: $\boldsymbol{x}^* = (0.9999, 0.2536, -0.7466, 0.2452, -0.0375)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.12237125 \cdot 10^{-3}$.

33 **PBC1** [159]

Classification: GM-U-Z-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 30} |f_i(\boldsymbol{x})|$,
where $f_i(\boldsymbol{x}) = \frac{x_1 + x_2 t_i + x_3 t_i^2}{1 + x_4 t_i + x_5 t_i^2} - \frac{\sqrt{(8t_i - 1)^2 + 1} \arctan(8t_i)}{8t_i}$ and
$t_i = -1 + 2(i-1)/29, \quad$ for $i = 1, \dots, 30$,
Starting point: $\boldsymbol{x}^{(1)} = (0, -1, 10, 1, 10)^T$,
Optimum point: $\boldsymbol{x}^* = (1.4136, -10.5797, 40.7117, -4.0213, 27.6150)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.22340496 \cdot 10^{-1}$.

34 **HS78** [111,159]

Classification: G-U-Z-N,
Dimension: 5,
Objective function: $f(\boldsymbol{x}) = x_1 x_2 x_3 x_4 x_5 + 10 \sum_{i=1}^{3} |f_i(\boldsymbol{x})|$,
where $f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10$,

$f_2(\boldsymbol{x}) = x_2x_3 - 5x_4x_5,$
$f_3(\boldsymbol{x}) = x_1^3 + x_2^3 + 1,$

Starting point: $\boldsymbol{x}^{(1)} = (-2.0, 1.5, 2.0, -1.0, -1.0)^T,$
Optimum point: $\boldsymbol{x}^* = (-1.7171, 1.5957, 1.8272, -0.7636, -0.7636)^T,$
Optimum value: $f(\boldsymbol{x}^*) = -2.9197004.$
Note: Function HS78 is unbounded from below. The reported minimum is a local one.

35 **El-Attar** [159]

Classification: G-U-Z-N,
Dimension: 6,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{50} \left| x_1 e^{-x_2 t_i} \cos(x_3 3 t_i + x_4) + x_5 e^{-x_6 t_i} - y_i \right|,$
where $y_i = 0.5e^{-t_i} - e^{-2t_i} + 0.5e^{-3t_i} + 1.5e^{-1.5t_i}\sin(7t_i) + e^{-2.5t_i}\sin(5t_i),$
$t_i = (i-1)/10, \quad \text{for } i = 1, \ldots, 51,$
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, 7, 0, -2, 1)^T,$
Optimum point: $\boldsymbol{x}^* = (2.2407, 1.8577, 6.7701, -1.6449, 0.1659, 0.7423)^T,$
Optimum value: $f(\boldsymbol{x}^*) = 0.5598131.$

36 **EVD61** [159]

Classification: GM-U-Z-N,
Dimension: 6,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 51} |f_i(\boldsymbol{x})|,$
where $f_i(\boldsymbol{x}) = x_1 e^{-x_2 t_i} \cos(x_3 t_i + x_4) + x_5 e^{-x_6 t_i} - y_i,$
$y_i = 0.5e^{-t_i} - e^{-2t_i} + 0.5e^{-3t_i} + 1.5e^{-1.5t_i}\sin(7t_i) + e^{-2.5t_i}\sin(5t_i),$
$t_i = (i-1)/10, \quad \text{for } i = 1, \ldots, 51,$
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, 7, 0, -2, 1)^T,$
Optimum point: $\boldsymbol{x}^* = (2.2759, 1.8993, 6.8482, -1.6503, 0.1457, 0.5170)^T,$
Optimum value: $f(\boldsymbol{x}^*) = 0.34904926 \cdot 10^{-1}.$

37 **Gill** [159]

Classification: PM-U-Z-N,
Dimension: 10,
Objective function: $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x})\},$
where $f_1(\boldsymbol{x}) = \sum_{i=1}^{10}(x_i - 1)^2 + 10^{-3}\sum_{i=1}^{10}\left(x_i^2 - \frac{1}{4}\right)^2,$

$$f_2(\boldsymbol{x}) = \sum_{i=2}^{30}\left[\sum_{j=2}^{10} x_j (j-1)\left(\frac{i-1}{29}\right)^{j-2} - \left(\sum_{j=1}^{10} x_j \left(\frac{i-1}{29}\right)^{j-1}\right)^2 - 1\right]^2 + x_1^2 + (x_2 - x_1^2 - 1)^2,$$

	$f_3(\boldsymbol{x}) = \sum_{i=2}^{10} \left[100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right]$,
Starting point:	$\boldsymbol{x}^{(1)} = (-0.1, -0.1, \ldots, -0.1)^T$,
Optimum point:	$\boldsymbol{x}^* = (-0.6022, 0.4907, 0.3096, 0.1416, 0.0542,$ $0.0287, 0.0197, 0.0137, 0.0087, 0.0045)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 9.7857721$.

38 Problem 1 in [21]

Classification:	GD-U-Z-E,
Dimension:	2,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x}) + \min_{4 \leq i \leq 6} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = x_1^4 + x_2^2$, and
	$f_2(\boldsymbol{x}) = (2 - x_1)^2 + (2 - x_2)^2$,
	$f_3(\boldsymbol{x}) = 2e^{-x_1 + x_2}$,
	$f_4(\boldsymbol{x}) = x_1^2 - 2x_1 + x_2^2 - 4x_2 + 4$,
	$f_5(\boldsymbol{x}) = 2x_1^2 - 5x_1 + x_2^2 - 2x_2 + 4$,
	$f_6(\boldsymbol{x}) = x_1^2 + 2x_2^2 - 4x_2 + 1$,
Starting point:	$\boldsymbol{x}^{(1)} = (2, 2)^T$,
Optimum point:	$\boldsymbol{x}^* = (1, 1)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 2$.

39 L1 version of Rosenbrock function [21]

Classification:	LD-U-Z-E,
Dimension:	2,
Objective function:	$f(\boldsymbol{x}) = \lvert x_1 - 1\rvert + 100\lvert x_2 - \lvert x_1\rvert\rvert$,
Starting point:	$\boldsymbol{x}^{(1)} = (-1.2, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (1, 1)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 0$.
Note:	DC representation can be found in [21].

40 L1 version of Wood function [21]

Classification:	LD-U-Z-E,
Dimension:	4,
Objective function:	$f(\boldsymbol{x}) = \lvert x_1 - 1\rvert + 100\lvert x_2 - \lvert x_1\rvert\rvert + 90\lvert x_4 - \lvert x - 3\rvert\rvert$ $+ \lvert x_3 - 1\rvert + 10.1(\lvert x_2 - 1\rvert + \lvert x_4 - 1\rvert)$ $+ 4.95(\lvert x_2 + x_4 - 2\rvert - \lvert x_2 - x_4\rvert)$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 3, 3, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (1, 1, 1, 1)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 0$.
Note:	DC representation can be found in [21].

9.2 Bound Constrained Problems

Bound constrained problems ($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i^l \leq x_i \leq x_i^u$ for all $i = 1, \dots, n\}$ in (9.1)) are easily constructed from the problems given above, for instance, by inclosing the bounds

$$x_i^* + 0.1 \leq x_i \leq x_i^* + 1.1 \qquad \text{for all odd } i.$$

If the starting point $\boldsymbol{x}^{(1)}$ is not feasible, it can simply be projected to the feasible region (if a strictly feasible starting point is needed an additional safeguard of 0.0001 may be added). The classification of the bound constrained problems is the same as that of unconstrained problems (see Sect. 9.1 and also Sect. 9.4) but, naturally, the information about constraint functions should be replaced with B (see Table 9.1).

9.3 Linearly Constrained Problems

In this section we present small-scale nonsmooth linearly constrained test problems ($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$ with the inequality taken component-wise in (9.1)). The number of variables varies from 2 to 20 and there are up to 15 constraint functions.

41 Wong 2C [159]

Classification: QM-L-X-N,
Dimension: 10,
No. of constraints: 3,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq 6} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) = {} & x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 \\
& + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 \\
& + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45, \\
f_2(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120), \\
f_3(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40), \\
f_4(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30), \\
f_5(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6), \\
f_6(\boldsymbol{x}) = {} & f_1(\boldsymbol{x}) + 10(-3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10}),
\end{aligned}$$

Constraint function:

$$\begin{aligned}
g_1(\boldsymbol{x}) &= 4x_1 + 5x_2 - 3x_7 + 9x_8 \leq 105, \\
g_2(\boldsymbol{x}) &= 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0, \\
g_3(\boldsymbol{x}) &= -8x_1 + 2x_2 + 5x_9 - 2x_{10} \leq 12,
\end{aligned}$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1722, 2.3634, 8.7737, 5.0959, 0.9906, 1.4307, 1.3219, 9.8289, 8.2803, 8.3756)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 24.306209$.

42 Wong 3C [159]

Classification: PM-L-X-N,
Dimension: 20,
No. of constraints: 4,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 14} f_i(\boldsymbol{x})$,
where

$$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 + 4(x_4-5)^2 + (x_5-3)^2 + 2(x_6-1)^2 + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 + (x_{11}-9)^2 + 10(x_{12}-1)^2 + 5(x_{13}-7)^2 + 4(x_{14}-14)^2 + 27(x_{15}-1)^2 + x_{16}^4 + (x_{17}-2)^2 + 13(x_{18}-2)^2 + (x_{19}-3)^2 - x_{20}^2 + 95,$$
$$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120),$$
$$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40),$$
$$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 - 30),$$
$$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6),$$
$$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(-3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10}),$$
$$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 + 5x_{11} - 8x_{12} - 28),$$
$$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(4x_1 + 9x_2 + 5x_{13}^2 - 9x_{14} - 87),$$
$$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(3x_1 + 4x_2 + 3(x_{13}-6)^2 - 14x_{14} - 10),$$
$$f_{10}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(14x_1^2 + 35x_{15} - 79x_{16} - 92),$$
$$f_{11}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(15x_2^2 + 11x_{15} - 61x_{16} - 54),$$
$$f_{12}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(5x_1^2 + 2x_2 + 9x_{17}^4 - x_{18} - 68),$$
$$f_{13}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(x_1^2 - x_9 + 19x_{19} - 20x_{20} + 19),$$
$$f_{14}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10(7x_1^2 + 5x_2^2 + x_{19}^2 - 30x_{20}),$$

Constraint function:

$$g_1(\boldsymbol{x}) = 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 105,$$
$$g_2(\boldsymbol{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0,$$
$$g_3(\boldsymbol{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} \le 12,$$
$$g_4(\boldsymbol{x}) = x_1 + x_2 + 4x_{11} - 21x_{21} \le 0,$$

Starting point: $\boldsymbol{x}^{(1)} = (2, 3, 5, 5, 1, 2, 7, 3, 6, 10, 2, 2, 6, 15, 1, 2, 1, 2, 1, 3)^T$,
Optimum point: $\boldsymbol{x}^* = (2.1749, 2.3537, 8.7666, 5.0669, 0.9888, 1.4309, 1.3288, 9.8354, 8.2867, 8.3709, 2.2759, 1.3586, 6.0771, 14.1708, 0.9962, 0.6566, 1.4666, 2.0004, 1.0471, 2.0636)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 24.306209$.

43 MAD1 [159]

Classification: GM-L-Z-N,
Dimension: 2,
No. of constraints: 1,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 3} f_i(\boldsymbol{x})$,
where

$$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 1,$$
$$f_2(\boldsymbol{x}) = \sin x_1,$$

	$f_3(\boldsymbol{x}) = -\cos x_2$,
Constraint function:	$g_1(\boldsymbol{x}) = x_1 + x_2 - 0.5 \geq 0$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 2)^T$,
Optimum point:	$\boldsymbol{x}^* = (-0.4003, 0.9003)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -0.38965952$.

44 MAD2 [159]

Classification:	GM-L-Z-N,
Dimension:	2,
No. of constraints:	1,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1 x_2 - 1$,
	$f_2(\boldsymbol{x}) = \sin x_1$,
	$f_3(\boldsymbol{x}) = -\cos x_2$,
Constraint function:	$g_1(\boldsymbol{x}) = -3x_1 - x_2 + 2.5 \geq 0$,
Starting point:	$\boldsymbol{x}^{(1)} = (-2, -1)^T$,
Optimum point:	$\boldsymbol{x}^* = (-0.8929, 0.1786)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -0.33035714$.

45 MAD4 [159]

Classification:	GM-L-Z-N,
Dimension:	2,
No. of constraints:	1,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = -\exp(x_1 - x_2)$,
	$f_2(\boldsymbol{x}) = \sinh(x_1 - 1) - 1$,
	$f_3(\boldsymbol{x}) = -\log(x_2) - 1$,
Constraint function:	$g_1(\boldsymbol{x}) = 0.05x_1 - x_2 + 0.5 \geq 0$,
Starting point:	$\boldsymbol{x}^{(1)} = (-1, 0.01)^T$,
Optimum point:	$\boldsymbol{x}^* = (1.5264, 0.5763)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -0.44891079$.

46 MAD5 [159]

Classification:	GM-L-Z-N,
Dimension:	2,
No. of constraints:	1,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \leq i \leq 3} f_i(\boldsymbol{x})$,
where	$f_1(\boldsymbol{x}) = -\exp(x_1 - x_2)$,
	$f_2(\boldsymbol{x}) = \sinh(x_1 - 1) - 1$,
	$f_3(\boldsymbol{x}) = -\log(x_2) - 1$,
Constraint function:	$g_1(\boldsymbol{x}) = -0.9x_1 + x_2 - 1 \geq 0$,

Starting point: $\boldsymbol{x}^{(1)} = (-1, 3)^T$,
Optimum point: $\boldsymbol{x}^* = (1.5436, 2.3892)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -0.42928061$.

47 PENTAGON [159]

Classification: GM-L-Z-N,
Dimension: 6,
No. of constraints: 15,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 3} f_i(\boldsymbol{x})$,
where $f_1(\boldsymbol{x}) = -\sqrt{(x_1 - x_3)^2 + (x_2 - x_4)^2}$,
$f_2(\boldsymbol{x}) = -\sqrt{(x_3 - x_5)^2 + (x_4 - x_6)^2}$,
$f_3(\boldsymbol{x}) = -\sqrt{(x_5 - x_1)^2 + (x_6 - x_2)^2}$,
Constraint function: $g_{ij}(\boldsymbol{x}) = x_i \cos\frac{2\pi j}{5} + x_{i+1}\sin\frac{2\pi j}{5} \le 1$,
where $i = 1, 3, 5,\ j = 0, 1, 2, 3, 4$
Starting point: $\boldsymbol{x}^{(1)} = (-1, 0, 0, -1, 1, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (-0.9723, 0.2436, 0.5322, -0.8494, 0.7265, 1.0000)^T$,
Optimum value: $f(\boldsymbol{x}^*) = -1.85961870$.

48 MAD6 [159]

Classification: GM-L-Z-N,
Dimension: 7,
No. of constraints: 9,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 163} f_i(\boldsymbol{x})$,
where $f_i(\boldsymbol{x}) = \frac{1}{15} + \frac{2}{15}\sum_{i=1}^{7}\cos(2\pi x_j \sin\vartheta_i)$,
$\vartheta_i = \frac{\pi}{180}(8.5 + i0.5),\ 1 \le i \le 163$,
Constraint function: $g_1(\boldsymbol{x}) = x_1 \ge 0.4$,
$g_2(\boldsymbol{x}) = -x_1 + x_2 \ge 0.4$,
$g_3(\boldsymbol{x}) = -x_2 + x_3 \ge 0.4$,
$g_4(\boldsymbol{x}) = -x_3 + x_4 \ge 0.4$,
$g_5(\boldsymbol{x}) = -x_4 + x_5 \ge 0.4$,
$g_6(\boldsymbol{x}) = -x_5 + x_6 \ge 0.4$,
$g_7(\boldsymbol{x}) = -x_6 + x_7 \ge 0.4$,
$g_8(\boldsymbol{x}) = -x_4 + x_6 = 1.0$,
$g_9(\boldsymbol{x}) = x_7 = 3.5$,
Starting point: $\boldsymbol{x}^{(1)} = (0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5)^T$,
Optimum point: $\boldsymbol{x}^* = (0.4000, 0.8198, 1.2198, 1.6940, 2.0940, 2.6940, 3.5000)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.10183089$.

49 Dembo 3 [159]

Classification: GM-L-Z-N,
Dimension: 7,
No. of constraints: 2 (+ bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 13} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) &= a_1x_1 + a_2x_1x_6 + a_3x_3 + a_4x_2 + a_5 + a_6x_3x_5,\\
f_2(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_7x_6^2 + a_8x_1^{-1}x_3 + a_9x_6 - 1),\\
f_3(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{10}x_1x_3^{-1} + a_{11}x_1x_3^{-1}x_6\\
&\quad + a_{12}x_1x_3^{-1}x_6^2 - 1),\\
f_4(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{13}x_6^2 + a_{14}x_5 + a_{15}x_4 + a_{16}x_6 - 1),\\
f_5(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{17}x_5^{-1} + a_{18}x_5^{-1}x_6 + a_{19}x_4x_5^{-1}\\
&\quad + a_{20}x_5^{-1}x_6^2 - 1),\\
f_6(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{21}x_7 + a_{22}x_2x_3^{-1}x_4^{-1} + a_{23}x_2x_3^{-1}\\
&\quad - 1),\\
f_7(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{24}x_7^{-1} + a_{25}x_2x_3^{-1}x_7^{-1}\\
&\quad + a_{26}x_2x_3^{-1}x_4^{-1}x_7^{-1} - 1),\\
f_8(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{27}x_5^{-1} + a_{28}x_5^{-1}x_7 - 1),\\
f_9(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{33}x_1x_3^{-1} + a_{34}x_3^{-1} - 1),\\
f_{10}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{35}x_2x_3^{-1}x_4^{-1} + a_{36}x_2x_3^{-1} - 1),\\
f_{11}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{37}x_4 + a_{38}x_3^{-1}x_3x_4 - 1),\\
f_{12}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{39}x_1x_6 + a_{40}x_1 + a_{41}x_3 - 1),\\
f_{13}(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(a_{42}x_1^{-1}x_3 + a_{43}x_1 + a_{44}x_6 - 1),
\end{aligned}$$

Constraint function:

$$\begin{aligned}
&g_1(\boldsymbol{x}) = a_{29}x_5 + a_{30}x_7 \le 1,\\
&g_2(\boldsymbol{x}) = a_{31}x_3 + a_{32}x_1 \le 1,\\
&1 \le x_1 \le 2000, \quad 1 \le x_2 \le 120, \quad 1 \le x_3 \le 5000,\\
&85 \le x_4 \le 93, \quad 90 \le x_5 \le 95, \quad 3 \le x_6 \le 12,\\
&145 \le x_7 \le 162,
\end{aligned}$$

Starting point: $\boldsymbol{x}^{(1)} = (1745, 110, 3048, 89, 92, 8, 145)^T$,
Optimum point: $\boldsymbol{x}^* = (1698.0025, 53.7482, 3031.1493, 90.1212, 95.0000, 10.4870, 153.5354)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 1227.2260$.

50 Dembo 5 [159]

Classification: GM-L-Z-N,
Dimension: 8,
No. of constraints: 3 (+ bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 4} f_i(\boldsymbol{x})$,
where

$$\begin{aligned}
f_1(\boldsymbol{x}) &= x_1 + x_2 + x_3,\\
f_2(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(ax_1^{-1}x_4x_6^{-1} + 100x_6^{-1} + bx_1^{-1}x_6^{-1} - 1),\\
f_3(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(x_4x_7^{-1} + 1250(x_5 - x_4)x_2^{-1}x_7^{-1} - 1),\\
f_4(\boldsymbol{x}) &= f_1(\boldsymbol{x}) + 10^5(cx_3^{-1}x_8^{-1} + x_5x_8^{-1} - 2500x_3^{-1}x_5x_8^{-1} - 1),
\end{aligned}$$

$a = 833.33252$, $b = -83333.333$, and $c = 1250000.0$,

Constraint function: $g_1(\boldsymbol{x}) = 0.0025(x_4 + x_5) \le 1$,

$$g_2(\boldsymbol{x}) = 0.0025(x_5 - x_7 - x_4) \le 1,$$
$$g_3(\boldsymbol{x}) = 0.01(x_8 - x_5) \le 1,$$
$$100 \le x_1 \le 10000, \quad 1000 \le x_2 \le 10000,$$
$$1000 \le x_3 \le 10000, \quad 10 \le x_4 \le 1000,$$
$$10 \le x_5 \le 1000, \quad 10 \le x_6 \le 1000,$$
$$10 \le x_7 \le 1000, \quad 10 \le x_8 \le 1000,$$

Starting point: $\boldsymbol{x}^{(1)} = (5000, 5000, 5000, 200, 359, 150, 225, 425)^T$,

Optimum point: $\boldsymbol{x}^* = (581.1358, 1358.8591, 5109.2561, 182.1702, 295.6298, 217.8298, 286.5404, 395.6298)^T$,

Optimum value: $f(\boldsymbol{x}^*) = 7049.2480$.

51 EQUIL [146,159]

Classification: GM-L-Z-N,

Dimension: 8,

No. of constraints: 1 (+ bound constraints),

Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 8} f_i(\boldsymbol{x})$,

where
$$f_i(\boldsymbol{x}) = \sum_{j=1}^{5}\left(\frac{a_{ji}\sum_{k=1}^{8} w_{jk}x_k}{x_i^{b_j}\sum_{k=1}^{8} a_{jk}x_k^{1-b_j}} - w_{ji}\right), \quad \text{for } i = 1, \ldots, 8,$$

$$W = [w_{jk}] = \begin{pmatrix} 3 & 1 & 0.1 & 0.1 & 5 & 0.1 & 0.1 & 6 \\ 0.1 & 10 & 0.1 & 0.1 & 5 & 0.1 & 0.1 & 0.1 \\ 0.1 & 9 & 10 & 0.1 & 4 & 0.1 & 7 & 0.1 \\ 0.1 & 0.1 & 0.1 & 10 & 0.1 & 3 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 11 \end{pmatrix},$$

$$A = [a_{jk}] = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0.8 & 1 & 0.5 & 1 & 1 & 1 & 1 \\ 1 & 1.2 & 0.8 & 1.2 & 1.6 & 2 & 0.6 & 0.1 \\ 2 & 0.1 & 0.6 & 2 & 1 & 1 & 1 & 2 \\ 1.2 & 1.2 & 0.8 & 1 & 1.2 & 0.1 & 3 & 4 \end{pmatrix},$$

$$b = [b_j] = (0.5, 1.2, 0.8, 2.0, 1.5)^T,$$

Constraint function: $g_1(\boldsymbol{x}) = \sum_{i=1}^{8} x_i = 1$,

$x_i \ge 0, \quad$ for $i = 1, \ldots, 8$,

Starting point: $x_i^{(1)} = 0.125, \quad$ for $i = 1, \ldots, 8$,

Optimum point: $\boldsymbol{x}^* = (0.2712, 0.0296, 0.0629, 0.0931, 0.0672, 0.3059, 0.1044, 0.0657)^T$,

Optimum value: $f(\boldsymbol{x}^*) = 0$.

52 HS114 [159]

Classification: GM-L-Z-N,

Dimension: 10,

No. of constraints: 5 (+ bound constraints),

Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 9} f_i(\boldsymbol{x})$,

where $f_1(\boldsymbol{x}) = 5.04x_1 + 0.035x_2 + 10x_3 + 3.36x_5 - 0.063x_4x_7$,

$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(1.12x_1 + 013167x_1x_8 - 0.00667x_1x_8^2$
$- \frac{1}{a}x_4)$,
$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(1.12x_1 + 013167x_1x_8 - 0.00667x_1x_8^2$
$-ax_4)$,
$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(1.098x_8 - 0.038x_8^2 + 0.352x_6 - \frac{1}{a}x_7$
$+ 57.425)$,
$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(1.098x_8 - 0.038x_8^2 + 0.352x_6 - ax_7$
$+ 57.425)$,
$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(\frac{98000x_3}{x_4x_9+1000x_3} - x_6)$,

$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(\frac{98000x_3}{x_4x_9+1000x_3} - x_6)$,

$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 500(\frac{x_2+x_5}{x_1} - x_8)$,

$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) - 500(\frac{x_2+x_5}{x_1} - x_8)$,

$a = 0.99$, $b = 0.90$,

Constraint function: $g_1(\boldsymbol{x}) = 0.222x_{10} + bx_9 \le 35.82$,
$g_2(\boldsymbol{x}) = 0.222x_{10}\frac{1}{b}x_9 \ge 35.82$,
$g_3(\boldsymbol{x}) = 3x_7 - ax_{10} \ge 133$,
$g_4(\boldsymbol{x}) = 3x_7 - \frac{1}{a}x_{10} \le 133$,
$g_5(\boldsymbol{x}) = 1.22x_4 - x_1 - x_5 = 0$,
$10^{-5} \le x_1 \le 2000$, $10^{-5} \le x_2 \le 16000$,
$10^{-5} \le x_3 \le 120$, $10^{-5} \le x_4 \le 5000$,
$10^{-5} \le x_5 \le 2000$, $85 \le x_6 \le 93$,
$90 \le x_7 \le 95$, $3 \le x_8 \le 12$,
$1.2 \le x_9 \le 4$, $145 \le x_{10} \le 162$,

Starting point: $\boldsymbol{x}^{(1)} = (1745, 12000, 110, 3048, 1974, 89.2, 92.8, 8.0, 3.6, 145)^T$,

Optimum point: $\boldsymbol{x}^* = (1697.8253, 15782.8314, 54.2333, 3031.0044, 2000.0000, 90.1334, 95.0000, 10.4739, 1.5616, 153.5354)^T$,

Optimum value: $f(\boldsymbol{x}^*) = -1768.8070$.

53 Dembo 7 [159]

Classification: GM-L-Z-N,
Dimension: 16,
No. of constraints: 1 (+ bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le 19} f_i(\boldsymbol{x})$,
where
$f_1(\boldsymbol{x}) = a(x_{12} + x_{13} + x_{14} + x_{15} + x_{16})$
$+ b(x_1x_{12} + x_2x_{13} + x_3x_{14} + x_4x_{14} + x_5x_{16})$
$f_2(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_1x_6^{-1} + 100dx_1 - dx_1^2x_6^{-1} - 1)$,
$f_3(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_2x_7^{-1} + 100dx_2 - dx_2^2x_7^{-1} - 1))$,
$f_4(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_3x_8^{-1} + 100dx_3 - dx_3^2x_8^{-1} - 1)$,

$f_5(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_4x_9^{-1} + 100dx_4 - dx_4^2x_9^{-1} - 1),$
$f_6(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(cx_5x_{10}^{-1} + 100dx_5 - dx_5^2x_{10}^{-1} - 1),$
$f_7(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_6x_7^{-1} + x_1x_7^{-1}x_{11}^{-1}x_{12}$
$\quad - x_6x_7^{-1}x_{11}^{-1}x_{12} - 1),$
$f_8(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_7x_8^{-1} + 0.002(x_7 - x_1)x_8^{-1}x_{12}$
$\quad + 0.002(x_2x_8^{-1} - 1)x_{13} - 1),$
$f_9(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_8 + 0.002(x_8 - x_2)x_{13}$
$\quad + 0.002(x_3 - x_9)x_{14} + x_9 - 1),$
$f_{10}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_3^{-1}x_9 + (x_4 - x_8)x_3^{-1}x_{14}^{-1}x_{15}$
$\quad + 500(x_10 - x_9)x_3^{-1}x_{14}^{-1} - 1),$
$f_{11}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3((x_4^{-1}x_5 - 1)x_{15}^{-1}x_{16} + x_4^{-1}x_{10}$
$+ 500(1 - x_4^{-1}x_{10})x_{15}^{-1} - 1),$
$f_{12}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(0.9x_4^{-1} + 0.002(1 - x_4^{-1}x_5)x_{16} - 1),$
$f_{13}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_{11}^{-1}x_{12} - 1),$
$f_{14}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_4x_5^{-1} - 1),$
$f_{15}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_3x_4^{-1} - 1),$
$f_{16}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_2x_3^{-1} - 1),$
$f_{17}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_1x_2^{-1} - 1),$
$f_{18}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_9x_{10}^{-1} - 1),$
$f_{19}(\boldsymbol{x}) = f_1(\boldsymbol{x}) + 10^3(x_8x_9^{-1} - 1),$
$a = 1.262626,$ $\quad b = -1.231060,$
$c = 0.034750,$ $\quad d = 0.009750,$

Constraint function: $g_1(\boldsymbol{x}) = 0.002(x_{11} - x_{12}) \le 1,$
$0.1 \le x_1 \le 0.9,$ $\quad 0.1 \le x_2 \le 0.9,$
$0.1 \le x_3 \le 0.9,$ $\quad 0.1 \le x_4 \le 0.9,$
$0.9 \le x_5 \le 1.0,$ $\quad 10^{-4} \le x_6 \le 0.1,$
$0.1 \le x_7 \le 0.9,$ $\quad 0.1 \le x_8 \le 0.9,$
$0.1 \le x_9 \le 0.9,$ $\quad 0.1 \le x_{10} \le 0.9,$
$1 \le x_{11} \le 1000,$ $\quad 10^{-6} \le x_{12} \le 500,$
$1 \le x_{13} \le 500,$ $\quad 500 \le x_{14} \le 1000,$
$500 \le x_{15} \le 1000,$ $\quad 10^{-6} \le x_{16} \le 500,$

Starting point: $\boldsymbol{x}^{(1)} = (0.80, 0.83, 0.85, 0.87, 0.90, 0.10, 0.12, 0.19, 0.25, 0.29, 512, 13.1, 71.8, 640, 650, 5.7)^T$

Optimum point: $\boldsymbol{x}^* = (0.8038, 0.8161, 0.9000, 0.9000, 0.9000, 0.1000, 0.1070, 0.1908, 0.1908, 0.1908, 505.0526, 5.0526, 72.6358, 500.0000, 500.0000, 0.0000)^T,$

Optimum value: $f(\boldsymbol{x}^*) = 174.78699.$

54 MAD8 [159]

Classification: QM-B-Z-N,
Dimension: 20,
No. of constraints: 0 (only bound constraints),
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le 38} |f_i(\boldsymbol{x})|,$
where $f_1(\boldsymbol{x}) = -1 + x_1^2 + \sum_{j=2}^{20} x_j,$

Table 9.6 Values of $\boldsymbol{a}$ for problem 49

i	a_i	i	a_i	i	a_i
1	1.715	16	$-0.19120592 \cdot 10^{-1}$	31	0.00061000
2	0.035	17	$0.56850750 \cdot 10^{2}$	32	−0.0005
3	4.0565	18	1.08702000	33	0.81967200
4	10.0	19	0.32175000	34	0.81967200
5	3000.0	20	−0.03762000	35	24500.0
6	−0.063	21	0.00619800	36	−250.0
7	$0.59553571 \cdot 10^{-2}$	22	$0.24623121 \cdot 10^{4}$	37	$0.10204082 \cdot 10^{-1}$
8	0.88392857	23	$-0.25125634 \cdot 10^{2}$	38	$0.12244898 \cdot 10^{-4}$
9	−0.11756250	24	$0.16118996 \cdot 10^{3}$	39	0.00006250
10	1.10880000	25	5000.0	40	0.00006250
11	0.13035330	26	$-0.48951000 \cdot 10^{6}$	41	-0.00007625
12	−0.00660330	27	$0.44333333 \cdot 10^{2}$	42	1.22
13	$0.66173269 \cdot 10^{-3}$	28	0.33000000	43	1.0
14	$0.17239878 \cdot 10^{-1}$	29	0.02255600	44	−1.0
15	$-0.56595559 \cdot 10^{-2}$	30	−0.00759500		

$f_i(\boldsymbol{x}) = -1 + c_i x_k^2 + \sum_{j=1, j\neq k}^{20} x_j$ for $1 < i < 38$,
$f_{38} = -1 + x_{20}^2 + \sum_{j=1}^{19} x_j$,
$k = (i+2)/2$, $c_i = 1$, for $i = 2, 4, ..., 36$,
$k = (i+1)/2$, $c_i = 2$, for $i = 3, 5, ..., 37$,

Constraint function: $x_j \geq 0.5$ for $1 \leq j \leq 10$,
Starting point: $x_j^{(1)} = 100$ for $j = 1, \ldots, 20$,
Optimum point: $\boldsymbol{x}^* = (0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, 0.5000, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.4167, -0.5069)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0.50694799$.

9.4 Large Problems

In this section we present 21 large-scale nonsmooth unconstrained test problems. The problems can be formulated with any number of variables.

55 Generalization of MAXL [155]

Classification: LM-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \leq i \leq n} |x_i|$,

Starting point:	$x_i^{(1)} = i/n$	for $i = 1, \ldots, n/2$ and
	$x_i^{(1)} = -i/n$	for $i = n/2+1, \ldots, n$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

56 Generalization of L1HILB [155]

Classification:	L-U-X-E,
Dimension:	any,
Objective function:	$f(\boldsymbol{x}) = \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} \frac{x_j}{i+j-1} \right\|$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 0$.

57 Generalization of MAXQ [98]

Classification:	QM-U-X-E,	
Dimension:	any,	
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le n} x_i^2$,	
Starting point:	$x_i^{(1)} = i$	for $i = 1, \ldots, n/2$ and
	$x_i^{(1)} = -i$	for $i = n/2+1, \ldots, n$.
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,	
Optimum value:	$f(\boldsymbol{x}^*) = 0$.	

58 Generalization of MXHILB [98]

Classification:	LM-U-X-E,
Dimension:	any,
Objective function:	$f(\boldsymbol{x}) = \max_{1 \le i \le n} \left\| \sum_{j=1}^{n} \frac{x_j}{i+j-1} \right\|$,
Starting point:	$\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point:	$\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value:	$f(\boldsymbol{x}^*) = 0$.

59 Chained LQ [98]

Classification:	G-U-X-E,
Dimension:	any,
Objective function:	$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \{ -x_i - x_{i+1}, -x_i - x_{i+1} + (x_i^2 + x_{i+1}^2 - 1) \}$,
Starting point:	$\boldsymbol{x}^{(1)} = (-0.5, -0.5, \ldots, -0.5)^T$,
Optimum point:	$\boldsymbol{x}^* = (1/\sqrt{2}, 1/\sqrt{2}, \ldots, 1/\sqrt{2})^T$,
Optimum value:	$f(\boldsymbol{x}^*) = -(n-1)\sqrt{2}$.

60 Chained CB3 I [98]

Classification: G-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max \left\{x_i^4 + x_{i+1}^2, (2-x_i)^2 + (2-x_{i+1})^2, 2e^{-x_i+x_{i+1}}\right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 1, \ldots, 1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 2(n-1)$.

61 Chained CB3 II [98]

Classification: GM-U-X-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max \left\{ \sum_{i=1}^{n-1} \left(x_i^4 + x_{i+1}^2\right), \sum_{i=1}^{n-1} \left((2-x_i)^2 + (2-x_{i+1})^2\right), \sum_{i=1}^{n-1} \left(2e^{-x_i+x_{i+1}}\right) \right\}$,
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$,
Optimum point: $\boldsymbol{x}^* = (1, 1, \ldots, 1)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 2(n-1)$.

62 Number of active faces [95]

Classification: GM-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \left\{ g\left(-\sum_{i=1}^n x_i\right), \ g(x_i) \right\}$,
where $g(y) = \ln(|y| + 1)$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

63 Nonsmooth generalization of Brown function 2 [98]

Classification: G-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(|x_i|^{x_{i+1}^2+1} + |x_{i+1}|^{x_i^2+1}\right)$,
Starting point: $x_i^{(1)} = 1.0$, when $\mathrm{mod}\,(i, 2) = 0$ and
$x_i^{(1)} = -1.0$, when $\mathrm{mod}\,(i, 2) = 1$, $i = 1, \ldots, n$.
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

64 Chained Mifflin 2 [98]

Classification: G-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \big(-x_i + 2\left(x_i^2 + x_{i+1}^2 - 1\right) + 1.75\left|x_i^2 + x_{i+1}^2 - 1\right|\big)$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*)$ varies:
$f(\boldsymbol{x}^*) \approx -34.795$, when $n = 50$,
$f(\boldsymbol{x}^*) \approx -140.86$, when $n = 200$, and
$f(\boldsymbol{x}^*) \approx -706.55$, when $n = 1000$.

65 Chained crescent I [98]

Classification: QM-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max\Big\{\sum_{i=1}^{n-1}\left(x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1\right), \sum_{i=1}^{n-1}\left(-x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1\right)\Big\}$.
Starting point: $x_i^{(1)} = 2.0$, when $\operatorname{mod}(i, 2) = 0$ and
$x_i^{(1)} = -1.5$, when $\operatorname{mod}(i, 2) = 1$, $i = 1, \ldots, n$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

66 Chained crescent II [98]

Classification: G-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \max\left\{x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1, -x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1\right\}$,
Starting point: $x_i^{(1)} = 2.0$, when $\operatorname{mod}(i, 2) = 0$ and
$x_i^{(1)} = -1.5$, when $\operatorname{mod}(i, 2) = 1$, $i = 1, \ldots, n$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

67 Problem 6 in `TEST29` [155]

Classification: QM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} |(3 - 2x_i)x_i + 1 - x_{i-1} - x_{i+1}|$,
where $x_0 = x_{n+1} = 0$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1, \ldots, -1)^T$,

Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

68 Problem 17 in TEST29 [155]

Classification: GM-U-Z-E,
Dimension: any, divisible by five
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \left| 5 - (j+1)(1 - \cos x_i) - \sin x_i - \sum_{k=5j+1}^{5j+5} \cos x_k \right|$,
where $j = \text{div}\,(i-1, 5)$,
Starting point: $\boldsymbol{x}^{(1)} = (1/n, 1/n, \ldots, 1/n)^T$,
Optimum point: $\boldsymbol{x}^* = (0, 0, \ldots, 0)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

69 Problem 19 in TEST29 [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \left((3 - 2x_i)x_i - x_{i-1} - 2x_{i+1} + 1\right)^2$,
where $x_0 = x_{n+1} = 0$.
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1, \ldots, -1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

70 Problem 20 in TEST29 [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} |(0.5x_i - 3)x_i - 1 + x_{i-1} + 2x_{i+1}|$,
where $x_0 = x_{n+1} = 0$,
Starting point: $\boldsymbol{x}^{(1)} = (-1, -1, \ldots, -1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

71 Problem 22 in TEST29 [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1 \le i \le n} \left| 2x_i + \frac{1}{2(n+1)^2}\left(x_i + \frac{i}{n+1} + 1\right)^3 - x_{i-1} - x_{i+1} \right|$,
where $x_0 = x_{n+1} = 0$,

Starting point: $x_i^{(1)} = \frac{i}{n+1}\left(\frac{i}{n+1} - 1\right)$ for all $i = 1, \ldots, n$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

72 Problem 24 in TEST29 [155]

Classification: GM-U-Z-N,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = \max_{1\le i\le n} \left| 2x_i + \frac{10}{(n+1)^2}\sinh(10x_i) - x_{i-1} - x_{i+1} \right|$,
where $x_0 = 0$ and $x_{n+1} = 1$,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$,
Optimum point: $\boldsymbol{x}^*$ not available,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

73 DC Maxl [21]

Classification: LD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = n\ \max_{1\le i\le n} |x_i| - \sum_{i=1}^n |x_i|$,
Starting point: $x_i^{(1)} = i$ for $i = 1, \ldots, n/2$,
$x_i^{(1)} = -i$ for $i = n/2 + 1, \ldots, n$,
Optimum point: $\boldsymbol{x}^* = (\alpha, \alpha, \ldots, \alpha)$, $\alpha \in \mathbb{R}$
Optimum value: $f(\boldsymbol{x}^*) = 0$.

74 DC Maxq [26]

Classification: QD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = (n+1)\ \max_{1\le i\le n} x_i^2 - \sum_{i=1}^n x_i^2$,
Starting point: $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ not specified,
Optimum point: $\boldsymbol{x}^* = 0$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

75 Problem 6 in [26]

Classification: LD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = 10\ \max_{1\le j\le 10} \left\{ |\sum_{i=1}^n (x_i - x_i^*) t_j{}^{i-1}| \right\} - \sum_{j=1}^{10} |\sum_{i=1}^n (x_i - x_i^*) t_j{}^{i-1}|$,
$t_j = \max_{1\le j\le 10} \{0.001, 0.1j\}$
Starting point: $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ not specified,

Optimum point: $\boldsymbol{x}^* = (1/n, ..., 1/n)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

76 Problem 7 in [26]

Classification: LD-U-Z-E,
Dimension: any,
Objective function: $f(\boldsymbol{x}) = 10 \max_{1 \le j \le 10} \left\{ | \sum_{i=1}^{n} |x_i - x_i^*| {t_j}^{i-1} \right\}$
$- \sum_{j=1}^{10} | \sum_{i=1}^{n} |x_i - x_i^*| {t_j}^{i-1}$,
$t_j = \max_{1 \le j \le 10} \{0.001, 0.1j\}$
Starting point: $\boldsymbol{x}^{(1)} \in \mathbb{R}^n$ not specified,
Optimum point: $\boldsymbol{x}^* = (1/n, ..., 1/n)^T$,
Optimum value: $f(\boldsymbol{x}^*) = 0$.

Similarly to small-scale problems these problems can be turned to bound constrained ones, for instance, by *inclosing* the additional bounds

$$x_i^* + 0.1 \le x_i \le x_i^* + 1.1 \qquad \text{for all odd } i.$$

9.5 Inequality Constrained Problems

In this section, we describe five nonlinear or nonsmooth inequality constraints (or constraint combinations). The constraints can be combined with the problems 57–66 given in Sect. 9.4 to obtain 50 inequality constrained problems ($S = \{\boldsymbol{x} \in \mathbb{R}^n \mid g_j(\boldsymbol{x}) \le 0$ for all $j = 1, \dots, p\}$ in (9.1)).

The constraints are selected such that the original unconstrained minimizers of problems in Sect. 9.4 are not feasible. Note that, due to nonconvexity of the constraints, all the inequality constrained problems formed this way are nonconvex.

The starting points $\boldsymbol{x}^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})^T$ for inequality constrained problems are chosen to be strictly feasible. In what follows, *the starting points for the problems with constraints are the same as those for problems without constraints* (see Sect. 9.4) unless stated otherwise. The optimum values for the problems with different objective functions and $n = 1000$ are given in Table 9.2.

77 Modification of Broyden tridiagonal constraint I [122,126]

Classification: O[O]-Q-Z-N,
No. of constraints: $n - 2$,
Dimension: any,
Objective functions: 57, 58, 62, 63, 65, and 66,
Constraint function: $g_j(\boldsymbol{x}) = (3.0 - 2.0x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 1.0$,
$j = 1, \dots, n - 2$,

Objective functions: 59, 60, 61, and 64,
Constraint function: $g_j(\boldsymbol{x}) = (3.0 - 2.0x_{j+1})x_{j+1} - x_j - 2.0x_{j+2} + 2.5,$
$j = 1, \ldots, n-2,$
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$ for objectives 59 and 64,
Starting point: $\boldsymbol{x}^{(1)} = (1, 1, \ldots, 1)^T$ for objectives 65 and 66,
Starting point: $x_i^{(1)} = -1, i \leq n$ and
mod $(i, 2) = 0$ for objective 63.

78 Modification of Broyden tridiagonal constraint II [122,126]

Classification: O[O]-Q-Z-N,
No. of constraints: 1,
Dimension: any,
Objective functions: 57, 58, 62, 63, 65, and 66,
Constraint function: $g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} ((3.0 - 2.0x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 1.0),$
Objective functions: 59, 60, 61, and 64,
Constraint function: $g_1(\boldsymbol{x}) = \sum_{i=1}^{n-2} ((3.0 - 2.0x_{i+1})x_{i+1} - x_i - 2.0x_{i+2} + 2.5),$
Starting point: $\boldsymbol{x}^{(1)} = (2, 2, \ldots, 2)^T$ for objectives 59 and 64.

79 Modification of MAD1 I [122, 126]

Classification: O[O]-G-Z-N,
No. of constraints: 2,
Dimension: any,
Objective functions: 57–66,
Constraint function: $g_1(\boldsymbol{x}) = \max\left\{x_1^2 + x_2^2 + x_1x_2 - 1.0,\ \sin x_1,\ -\cos x_2\right\},$
$g_2(\boldsymbol{x}) = -x_1 - x_2 + 0.5,$
Starting point: $x_1^{(1)} = -0.5$ and $x_2^{(1)} = 1.1$ for all objectives,
otherwise, the starting points given in Sect. 9.4 are used.

80 Modification of MAD1 II [122, 126]

Classification: O[O]-G-Z-N,
No. of constraints: 4,
Dimension: any,
Objective functions: 57–66,
Constraint function: $g_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_1x_2 - 1.0,$
$g_2(\boldsymbol{x}) = \sin x_1,$
$g_3(\boldsymbol{x}) = -\cos x_2,$
$g_4(\boldsymbol{x}) = -x_1 - x_2 + 0.5,$
Starting point: $x_1^{(1)} = -0.5$ and $x_2^{(1)} = 1.1,$ for all objectives,
otherwise the starting points given in Sect. 9.4 are used.

81 Simple modification of MAD1 [122,126]

Classification:	O[O]-Q-Z-N,
No. of constraints:	1,
Dimension:	any,
Objective functions:	57, 58, 62, 63, 65, and 66,
Constraint function:	$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(x_i^2 + x_{i+1}^2 + x_i x_{i+1} - 2.0x_i - 2.0x_{i+1} + 1.0\right)$,
Objective functions:	59, 60, 61, and 64,
Constraint function:	$g_1(\boldsymbol{x}) = \sum_{i=1}^{n-1} \left(x_i^2 + x_{i+1}^2 + x_i x_{i+1} - 1.0\right)$,
Starting point:	$\boldsymbol{x}^{(1)} = (0.5, 0.5, \dots, 0.5)^T$ for objectives 57, 58, 62, 63, 65, and 66,
Starting point:	$\boldsymbol{x}^{(1)} = (0, 0, \dots, 0)^T$ for objectives 60, 61, and 64.

Part II
Notes and References

Practical Problems

Computational Chemistry and Biology

A comprehensive review of mathematics in molecular modeling can be found, for instance, in [141, 187].

Polyatomic clustering problem: In [153], a smooth penalized modification for the Lennard-Jones potential function (7.1) and (7.2) was introduced that allows a local search method to escape from the enormous number of local minima found in the Lennard-Jones energy landscape. The local minimum of the modified objective function was then used as a starting point for the local optimization of the Lennard-Jones potential function (7.1) and (7.2). This procedure was reported to result in convergence to the global minimum with much greater success than when starting local optimization with random points [153]. The idea was further modified in [31], resulting in a nonsmooth penalized Lennard-Jones potential (7.3) with $p = 6$.

Molecular distance geometry problem: Reviews and background of the molecular distance geometry problem (MDGP) in protein structure determination can be found, for instance, in [65, 140, 184]. The exact MDGP (7.4) can be solved by a linear time algorithm, as shown in [80], when all distances between all atom pairs of a molecule are known. However, in [183], it has been shown that when the upper and lower bounds are close to each other, an MDGP with relaxed distances belongs to the NP-hard class.

Most approaches to solving a nonsmooth MDGP rely on some kind of smoothing of the model (see e.g., [3, 183, 211]). In [4], the MDGP problems (7.4) and (7.5) were modeled as a D.C. (difference of two convex functions) programming problem and solved by D.C. programming approach.

Although we have approached the MDGP mainly via a protein folding problem, this kind of formulation can also be used in other molecular modeling. For instance, in [171], a nonsmooth MDGP was applied to the modeling of polymer and composite systems.

Protein structural alignment: In 1960, Perutz et. al. [196] showed, using structural alignment, that myoglobin and hemoglobin have similar three-dimensional structures even though their sequences (e.g., the order of residues) differ. Indeed, these two proteins are functionally similar, being involved with the storage and transport of oxygen, respectively. Since 1960, the structural similarity of proteins has been studied intensely in the hope of finding shared functionalities.

In [135], a method for globally solving the protein structural alignment problem in polynomial time was introduced. However, this algorithm is not computationally affordable and, in practice, an heuristic procedure called the *Structal Method* [216] is generally used. In [134], the Structal Method was reported to be the best practical method for solving the protein alignment problem. Later, in [5, 169], different kinds of convergent algorithms for LOVO formulation [6] of the protein alignment problem were developed and analyzed.

Molecular docking: Finding a solution to the molecular docking problem involves two critical elements: a good scoring function, and an efficient algorithm for searching conformation and orientation spaces. In [226], 11 different scoring functions for molecular docking were compared and evaluated. The above-mentioned PLP [92] was evaluated as one of the best.

The algorithms used for solving molecular docking problems involve genetic and evolutionary algorithms (see e.g., [88, 92, 220, 232]), Monte Carlo simulations (see e.g., [224]), and L-BFGS (see e.g., [87]) to mention but a few. In all of the references given above, PLP or its modification has been used as a scoring function. In [87, 88], the energy barrier caused by two nonbonded ligand atoms being too close to each other was avoided by using the same term for internal ligand–ligand interactions as for ligand–protein interactions.

A review and comparison of different molecular docking methods can be found, for instance, in [66, 218].

Data Analysis

Data analysis is a process of gathering, modeling, and transforming data with the goal of highlighting useful information, suggesting conclusions, and supporting decision-making. Some elementary books of data and regression analysis include [229] and [96].

Clustering problem: The clustering problem is considered, for instance, in [39, 40, 213]. There exist different approaches to clustering, including agglomerative and divisive hierarchical clustering algorithms, as well as algorithms based on mathematical programming techniques. A survey of some existing approaches is provided in [101, 116], and a comprehensive list of literature on clustering algorithms is available in these papers.

The clustering problem (7.11) is an optimization problem. Various optimization techniques have been applied to solve this problem. These techniques include branch and bound [77], cutting plane [101], interior point methods [81], the variable neighborhood search algorithm [102], and metaheuristics like simulated annealing [48] and tabu search [2]. Algorithms based on nonsmooth nonconvex optimization formulation include the modified global k-means [14], incremental discrete gradient [17, 20], and hyperbolic smoothing-based clustering [230] algorithms. A unified approach to cluster analysis problems was studied in [219]. The nonsmooth formulation of the clustering was used, for example, in [25, 39, 40].

Supervised data classification: The problems of supervised data classification arise in many areas, including management science, medicine, and chemistry [117]. Many algorithms for solving data classification problems have been proposed and studied. These algorithms are mainly based on statistical, machine learning, and neural networks approaches (see e.g., [89, 172, 170]). Algorithms based on mathematical programming techniques were developed, for example, in [8, 9, 10, 33, 45, 49, 90, 221, 223].

Regression analysis: It is known that each continuous piecewise linear function can be represented as a maxima of minima of linear functions [28, 93]. For a detailed introduction to nonparametric regression, we refer the reader to the monograph [96].

There are several established methods for nonparametric regression, including regression trees like CART [47], adaptive spline fitting like MARS [85] and least squares neural network estimates [105]. All these methods minimize a kind of least squares risk of the regression estimate, either heuristically over a fixed and very complex function space as for neural networks, or over a stepwise defined data dependent space of piecewise constant functions or piecewise polynomials as for CART or MARS. Results presented in this book on piecewise linear estimators are given in [15, 16]

Clusterwise regression: Clusterwise regression techniques have been applied to investigate the market segmentation, the stock-exchange [200], and the benefit segmentation data [228].

Optimization models for clusterwise linear regression problems include the mixed integer nonlinear programming [54, 55, 74] and nonsmooth nonconvex optimization models [19].

Algorithms for solving the clusterwise linear regression problem are based on generalizations of classical clustering algorithms such as k-means [212], a partial least square regression [200], a conditional mixture, maximum likelihood methodology [74], a simulated annealing methodology [75], a logistic regression clustering [201], mixed logical-quadratic programming [54], the repetitive branch-and-bound [55], and NSO algorithms [19].

Optimal Control Problems

Optimal shape design: In [167, 168], two types of optimal shape design problems were considered: the unilateral (Dirichlet–Signorini) boundary value problem, and the design of optimal covering (packaging) problem. Several practical examples were solved by the proximal bundle method. A comparison with the results obtained by the regularization technique and smooth sequential quadratic programming method was also presented.

A multicriteria structural design problem (Euler-Bernoulli beam with varying thickness) was considered in [174]. The problem was solved by an interactive multiobjective optimization method utilizing the proximal bundle method as a nonsmooth single optimization method.

Distributed parameter control problems: In [168], there were presented three practical examples of distributed parameter control problems: the axially loaded rod with stress constraints problem, the clamped beam with displacement constraints problem and the clamped beam with obstacle problem. The exact penalty technique was utilized for handling the state constraints, which led to an optimization problem with nonsmooth objective functions. These problems were solved using the proximal bundle method. By comparing the results with those in [104] obtained by the exterior penalty technique and smooth sequential quadratic programming method, we can demonstrate the superiority of the nonsmooth approach. In [209], the same clamped beam problem with obstacle as in [168] was solved using the bundle trust region method.

Hemivariational inequalities: The problem of an elastic body subjected to body forces and surface tractions and obeying a nonmonotone friction law on some part of the boundary was considered in [176–179]. The NSO problems were successfully solved using the proximal bundle method. The bundle-Newton method was also utilized in [103, 166].

Engineering and Industrial Applications

Power unit-commitment problem: In [148], the so-called space-time decomposition scheme proposed by [29] was utilized for the power unit-commitment problem, and the resulting nonsmooth problem was solved by the diagonal quasi-Newton method of [145]. Six different test problems with about 10,000 (dual)variables were successfully solved. In [46], three types of test problems (two of them from [27, 235]) with 20–96 variables were solved using two versions of the level bundle method. More recently, ten test problems with 240–16 800 variables were solved in [83] using the proximal bundle method.

Continuous casting of steel: The optimization problem arising in the continuous casting process of steel has been solved by the proximal bundle method in many publications with different kind of strategy to handle the state constraints (7.60). The constraints are penalized with quadratic penalty functions in [165], and with exact penalty functions in [164, 168]. In [173, 175], the constraints are handled as objective functions, and an interactive multiobjective optimization method is then applied.

Other Applications

Image restoration: Much effort has been applied in order to develop efficient solution methods for the basic formulation (7.61) of image denoising [114, 120, 206]. In addition, the reduction of the staircasing effect caused by formulation (7.61) has been studied in several papers [56, 115, 150], and the semi-adaptive formulation (7.62) was introduced in [121].

Robust nonsmooth L^1 fitting has been studied, for instance, in [188, 189]. L^1 fitting with smooth regularization (7.63) was studied in [119], while the more complex formulation (7.64) with L^1 fitting and BV regularization was studied, for instance, in [188, 57]. However, in both of these articles, only the smoothed version of formulation (7.64) was solved numerically. In [160], the formulation (7.64) was finally solved in its nonsmooth form.

Semiacademic Problems

Exact Penalty Function

Nonsmooth exact penalty functions were first introduced in [234] and have been widely investigated after that point (see e.g. [76, 100, 130, 133, 199, 217]). The exactness of the exact penalty function is proved, for instance, in [34, 58, 61, 199, 217].

Integer Programming

In [209] several traveling salesman test problems with 100–2116 variables were solved by the bundle trust region method and the results were compared with those obtained using the classical subgradient approach of [210]. In [46], test problems with 6–442 variables were solved by two versions of the level bundle method. In [147], three test problems up to 3,038 variables were solved with the reversal quasi-Newton method.

Eigenvalue Optimization

An excellent overview of eigenvalue optimization is given in [152]. The mathematics behind the problem is considered, for example, in [50, 151, 193] and the different methods for solving the problem are given, for instance, in [46, 106, 192, 209].

Part III
Nonsmooth Optimization Methods

Introduction

In this part, we give short descriptions of the most commonly used methods for NSO. We consider the NSO problem of the form

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \end{cases}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is not required to have continuous derivatives. We suppose only that f is a locally Lipschitz continuous function on the feasible set $S \subseteq \mathbb{R}^n$.

There are several approaches for solving NSO problems. The direct application of *smooth gradient-based methods* to nonsmooth problems is a simple approach, but it may lead to a failure in convergence, in optimality conditions, or in gradient approximation. All of these difficulties arise from the fact that the objective function fails to have a derivative for some values of the variables. Figure 1 demonstrates the difficulties caused by nonsmoothness.

On the other hand, using a *derivative free method* may be another approach, but standard derivative-free methods such as *genetic algorithms* or *Powell's method* may be unreliable and become inefficient as the dimension of the problem increases. Moreover, the convergence of such methods has been proved only for smooth functions. In addition, different kinds of *smoothing* and *regularization techniques* may give satisfactory results in some cases but are not, in general, as efficient as the direct nonsmooth approach. Thus, special tools for solving NSO problems are needed.

Methods for solving NSO problems, and described in this part, include *subgradient methods* (see Chap. 10), *cutting plane methods* (see Chap. 11), *bundle methods* (see Chap. 12), and *gradient sampling methods* (see Chap. 13). All of these are based on the assumption that only the objective function value and one arbitrary subgradient at each point are available. In addition, there exist some *hybrid methods* that combine the features of the previously mentioned methods,

Smooth problem:

- *Descent direction is obtained at the opposite direction of the gradient* $\nabla f(\boldsymbol{x})$.
- *The necessary optimality condition* $\nabla f(\boldsymbol{x})=0$.
- *Difference approximation can be used to approximate the gradient.*

Nonsmooth problem:

- *The gradient does not exist at every point, leading to difficulties in defining the descent direction.*
- *Gradient usually does not exist at the optimal point.*
- *Difference approximation is not useful and may lead to serious failures*
- *The (smooth) algorithm does not converge or it converges to a non–optimal point.*

Fig. 1 Difficulties caused by nonsmoothness

and *discrete gradient methods* that can be considered as semi-derivative free methods for NSO problems. These are described in Chaps. 14 and 15, respectively. Note that NSO techniques can be successfully applied to smooth problems but not vice versa, and thus we can state that NSO deals with a broader class of problems than smooth optimization. Although using a smooth method may be desirable when all the functions involved are known to be smooth, it is often hard to confirm the smoothness in practical applications, for instance, if function values are calculated via simulation. Moreover, as already mentioned at the beginning of Part II, the problem may be analytically smooth but still behave numerically nonsmoothly, or certain important methodologies for solving difficult smooth problems may lead directly to the need to solve NSO problems, in which cases NSO methods are also needed.

In addition to the methods for solving NSO problems, we introduce some common ways to deal with constraints in NSO in Chap. 16 and, at the end of the part, in Chap. 17, we test and compare implementations of different NSO methods for solving unconstrained NSO problems.

In the text that follows (if not stated otherwise), we assume that at every point $\boldsymbol{x}$ we can evaluate the value of the objective function $f(\boldsymbol{x})$ and an arbitrary subgradient ξ from the subdifferential $\partial f(\boldsymbol{x})$. In iterative optimization methods, we try to generate a sequence $(\boldsymbol{x}_k)$ that converges to a minimum point $\boldsymbol{x}^*$ of the objective function; that is, $(\boldsymbol{x}_k) \to \boldsymbol{x}^*$ whenever $k \to \infty$. If an iterative method converges to a (local) minimum $\boldsymbol{x}^*$ from any arbitrary starting point $\boldsymbol{x}_1$, it is said to be globally convergent. If it converges to a (local) minimum in some neighborhood of $\boldsymbol{x}^*$, it is said to be locally convergent. Note that the methods described in this book are local methods; that is, they do not attempt to find the global minimum of the objective function.

Chapter 10
Subgradient Methods

The history of subgradient methods (Kiev methods) starts in the 1960s and they were mainly developed in the Soviet Union. The basic idea behind subgradient methods is to generalize the smooth methods by replacing the gradient with an arbitrary subgradient. Due to this simple structure, they are widely used methods in nonsmooth optimization although they suffer from some serious drawbacks. Firstly, a nondescent search direction may occur and thus a standard line search operation can not be applied. For this reason the step sizes have to be chosen a priori. Secondly, the lack of an implementable stopping criterion and the poor rate of the convergence speed (less than linear) are also disadvantages of the subgradient methods. To overcome the last handicap, the variable metric ideas were adopted to subgradient context in [210] by introducing two space dilation methods (ellipsoid method and r-algorithm). In addition, some modified ideas have been proposed in [222], where two adaptive variable metric methods, deviating in step size control, were derived. For an excellent overview of the different subgradient methods we refer to [210]. The first method to be considered here is the cornerstone of NSO, the *standard subgradient method* [210]. Then the ideas of the more sophisticated subgradient method, the well-known, *Shor's r-algorithm* are introduced.

10.1 Standard Subgradient Method

As already said the idea behind subgradient methods is to generalize smooth methods (e.g. the steepest descent method) by replacing the gradient with an arbitrary subgradient. Therefore, the iteration formula for these methods is

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - t_k \frac{\boldsymbol{\xi}_k}{\|\boldsymbol{\xi}_k\|},$$

where $\boldsymbol{\xi}_k \in \partial f(\boldsymbol{x}_k)$ is any subgradient and $t_k > 0$ is a predetermined step size.

Due to this simple structure and low storage requirements, subgradient methods are widely used methods in NSO. However, basic subgradient methods suffer from

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three serious disadvantages: a nondescent search direction may occur and thus, the selection of step size is difficult; there exists no implementable subgradient based stopping criterion; and the convergence speed is poor (not even linear) (see e.g. [144]).

The standard subgradient method is proved to be globally convergent if the objective function is convex and step sizes satisfy

$$\lim_{k\to\infty} t_k = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} t_j = \infty.$$

An extensive overview of various subgradient methods can be found in [210].

10.2 Shor's r-Algorithm (Space Dilation Method)

Now we shortly describe the ideas of the Shor's r-algorithm with space dilations along the difference of two successive subgradients. The basic idea of the Shor's r-algorithm is to interpolate between the steepest descent and conjugate gradient methods.

Let f be a convex function on $\mathbb{R}^n$. The Shor's r-algorithm is given as follows:

PROGRAM Shor's r-algorithm
  INITIALIZE $x_0 \in \mathbb{R}^n$, $\beta \in (0,1)$, $B_1 = I$, and $t_1 > 0$;
  Compute $\xi_0 \in \partial f(x_0)$ and $x_1 = x_0 - t_1\xi_0$;
  Set $\bar{\xi}_1 = \xi_0$ and $k = 1$;
  WHILE the termination condition is not met
    Compute $\xi_k \in \partial f(x_k)$ and $\xi_k^* = B_k^T \xi_k$;
    Calculate $r_k = \xi_k^* - \bar{\xi}_k$ and $s_k = r_k/\|r_k\|$;
    Compute $B_{k+1} = B_k R_\beta(s_k)$, where $R_\beta(s_k) = I + (\beta - 1)s_k s_k^T$
      is the space dilation matrix;
    Calculate $\bar{\xi}_{k+1} = B_{k+1}^T \xi_k$;
    Choose a step size t_{k+1};
    Set $x_{k+1} = x_k - t_k B_{k+1}\bar{\xi}_{k+1}$ and $k = k+1$;
  END WHILE
  RETURN final solution x_k;
END Shor's r-algorithm

In order to turn the above r-algorithm into an efficient optimization routine, one has to find a solution to the following problems: how to choose the step sizes t_k (including the initial step size t_1) and how to design a stopping criterion which does not need information on subgradients.

If the objective function is twice continuously differentiable, its Hessian is Lipschitz, and the starting point is chosen from some neighborhood of the optimal solution, then the n-step quadratic rate convergence can be proved for the Shor's

r-algorithm [210]. In the nonconvex case, if the objective function is coercive under some additional assumptions, the r-algorithm is convergent to isolated local minimizers [210].

Chapter 11
Cutting Plane Methods

Subgradient methods described in the previous chapter use only one arbitrary subgradient at a time, without memory of past iterations. If the information from previous iterations is kept, it is possible to define a model—the so-called cutting plane model—of the objective function. In this way, more information about the local behavior of the function is obtained than what an individual arbitrary subgradient can yield. The cutting plane idea was first developed independently in [60, 129]. In this chapter, we first introduce the basic ideas of the *standard cutting plane method* (CP) and then the more advanced *cutting plane method with proximity control* (CPPC) [86]. In addition, the history of the so-called bundle methods (see Chap. 12) originates from the cutting plane idea.

11.1 Standard Cutting Plane Method

In this section we describe the ideas of the standard cutting plane method (CP) by Kelley for convex nonsmooth minimization [129] (see also [60]). Due to the Theorem 2.30 in Part I, a convex function f has the representation

$$f(\boldsymbol{x}) = \max\,\{f(\boldsymbol{y}) + \boldsymbol{\xi}^T(\boldsymbol{x} - \boldsymbol{y}) \mid \boldsymbol{\xi} \in \partial f(\boldsymbol{y}),\ \boldsymbol{y} \in \mathbb{R}^n\} \text{ for all } \boldsymbol{x} \in \mathbb{R}^n. \quad (11.1)$$

However, for this representation we need the whole subdifferential $\partial f(\boldsymbol{y})$, which, in practice, is too big a requirement. For this reason we have to approximate it somehow. We now suppose that in addition to the current iteration point $\boldsymbol{x}_k$ we have some auxiliary points $\boldsymbol{x}_j \in \mathbb{R}^n$ and subgradients $\boldsymbol{\xi}_j \in \partial f(\boldsymbol{x}_j)$ for $j \in \mathcal{J}_k$, where the index set $\mathcal{J}_k$ is such that $\emptyset \neq \mathcal{J}_k \subset \{1, \ldots, k\}$. Now instead of Eq. (11.1) we can define a finite piecewise affine approximation of f at the current iteration k by

$$\hat{f}^k(\boldsymbol{x}) := \max\,\{f(\boldsymbol{x}_j) + \boldsymbol{\xi}_j^T(\boldsymbol{x} - \boldsymbol{x}_j) \mid j \in \mathcal{J}_k\} \qquad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n. \quad (11.2)$$

The minimization of the approximation $\hat{f}^k$ on a convex compact set S containing the minimum point of f gives a new iterate $\boldsymbol{x}_{k+1}$. By the definition of the approximation we have for all k

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$$\hat{f}^k(\boldsymbol{x}) \le f(\boldsymbol{x}), \qquad \hat{f}^k(\boldsymbol{x}_j) = f(\boldsymbol{x}_j), \qquad \text{and} \quad \hat{f}^k(\boldsymbol{x}) \le \hat{f}^{k+1}(\boldsymbol{x}).$$

The minimization of (11.2) can be transformed to a problem of finding a solution $(\boldsymbol{d}, v) \in \mathbb{R}^{n+1}$ to a linearly constrained smooth cutting plane problem

$$\begin{cases} \text{minimize} & v \\ \text{subject to} & -\alpha_j + \boldsymbol{\xi}^T \boldsymbol{d}_k \le v \text{ for all } j \in \mathcal{J}_k \\ & \boldsymbol{x}_k + \boldsymbol{d} \in S \text{ and } v \in \mathbb{R}, \end{cases}$$

where α_j is the so-called *linearization error* between the actual value of the objective function at $\boldsymbol{x}_k$ and the linearization generated at $\boldsymbol{x}_j$ and evaluated at $\boldsymbol{x}_k$, that is,

$$\alpha_j := f(\boldsymbol{x}_k) - f(\boldsymbol{x}_j) + \boldsymbol{\xi}_j^T(\boldsymbol{x}_k - \boldsymbol{x}_j) \quad \text{for all } j \in \mathcal{J}_k.$$

and $s_i = \|\boldsymbol{x}_k - \boldsymbol{x}_j\|$. The cutting planes for some iterations of the CP are illustrated in Fig. 11.1.

Let us now suppose that we have a convex compact set S containing the minimum point of f available. The pseudo-code of the CP is the following:

```
PROGRAM CP
  INITIALIZE x_1 ∈ S, J_1 = {1}, and ε > 0;
  Set f̂^0(x) = −∞, α_1 = ∞ and k = 1;
  WHILE the termination condition α_k ≤ ε is not met
    Generate the search direction
      d_k = argmin_{x_k+d∈S} {f̂_k(x_k + d)};
    Find step size t^k;
    Set x_{k+1} = x_k + t^k d_k and compute α_{k+1} = f(x_{k+1}) − f̂_k(x_{k+1});
    Update J_k according some updating rules;
    Set k = k + 1;
  END WHILE
  RETURN final solution x_k;
END PROGRAM CP
```

Like in subgradient methods (see Chap. 10) the sequence $(\boldsymbol{x}_k)$ generated by the CP does not necessarily have decreasing objective values $f(\boldsymbol{x}_k)$. The step size t^k can be selected by using some kind of line-search procedure or just use a constant step size (e.g. $t^k = 1$). At the initial iterations the minimization of the cutting plane model $\hat{f}_k$ may be unbounded from below unless a suitable set S is introduced. Thus, the choice of set S is a key element to overcome the instability of cutting planes.

By the convexity of f, the graph of the cutting plane model $\hat{f}_k$ approaches the graph of f from below with increasing accuracy as k grows. This guarantees the global convergence of the method (for more details, see e.g. [42] Chap. 9). Furthermore, this property provides an implementable stopping criterion (note the decreasing values of the distances α_k in Fig. 11.1)

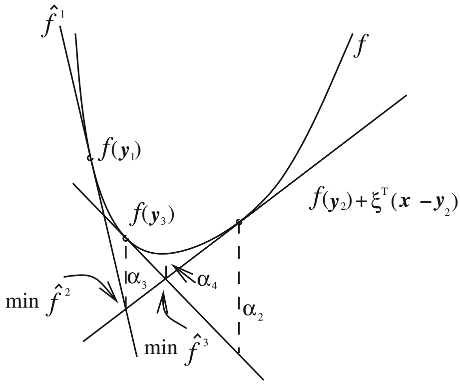

Fig. 11.1 Cutting planes

There are two main disadvantages in the CP: first, the choice of set S such that the minimization problem has a solution in the set and, second, the method generally attains rather poor convergence results in practice. However, it is also obvious, that if the original objective function f is piecewise linear or almost piecewise linear, then the cutting plane method may convergence in a reliable way and rapidly—in the piecewise linear case the convergence is finite—to the exact global minimum.

11.2 Cutting Plane Method with Proximity Control

The extension of the cutting plane method for nonconvex functions is not straightforward. A basic observation is that, in nonconvex case, the first order information does not necessarily provide the lower approximation of the objective function any longer. In this section, we briefly introduce the cutting plane method with proximity control (CPPC) for nonconvex NSO developed by Fuduli, Gaudioso, and Giallombardo. For more details, see [86].

Let us denote the set of available information—*the bundle*—as

$$(\boldsymbol{x}_j, f(\boldsymbol{x}_j), \boldsymbol{\xi}_j, \alpha_j, s_j), \quad j \in \mathcal{J}_k,$$

where again $\boldsymbol{x}_j \in \mathbb{R}^n$ are auxiliary points, $\boldsymbol{\xi}_j \in \partial f(\boldsymbol{x}_j)$ and α_j is a linearization error. Note that the linearization error α_j can be negative in nonconvex case.

The CPPC is based on the construction of both a lower and an upper polyhedral approximation of the objective function. That is, instead of just one index set $\mathcal{J}_k$ (cf. standard cutting plane method in Sect. 11.1), we have two sets $\mathcal{J}_+$ and $\mathcal{J}_-$ defined as follows:

$$\mathcal{J}_+ = \{j \mid \alpha_j \geq 0\} \quad \text{and} \quad \mathcal{J}_- = \{j \mid \alpha_j < 0\}.$$

The bundles defined by index sets $\mathcal{J}_+$ and $\mathcal{J}_-$ are characterized by points that somehow exhibit, respectively, the "convex behavior" and the "nonconvex behavior" of the objective function relative to point $\boldsymbol{x}_k$. Notice that the set $\mathcal{J}_+$ is never empty since at least the element $(\boldsymbol{x}_j, f(\boldsymbol{x}_j), \boldsymbol{\xi}_j, 0, 0)$ belongs to the bundle. The basic idea of the CPPC is to treat differently the two bundles in the construction of a piecewise affine model.

The proximity control [132] is introduced by defining the *proximal trajectory* $\boldsymbol{d}^\gamma$ of the piecewise affine function $\max_{j\in\mathcal{J}_+}\{\boldsymbol{\xi}_j^T\boldsymbol{d}-\alpha_j\}$. The optimal proximal trajectory $\boldsymbol{d}^\gamma$ is computed by solving a quadratic direction finding problem ($v \in \mathbb{R}$ and $\boldsymbol{d} \in \mathbb{R}^n$ are variables) parametrized by scalar $\gamma > 0$ (see the pseudo-code given below):

$$\begin{cases} \text{minimize} & \gamma v + \frac{1}{2}\|\boldsymbol{d}\|^2 \\ \text{subject to} & v \geq \boldsymbol{\xi}_j^T\boldsymbol{d} - \alpha_j, \quad j \in \mathcal{J}_+, \\ & v \leq \boldsymbol{\xi}_j^T\boldsymbol{d} - \alpha_j, \quad j \in \mathcal{J}_-. \end{cases} \tag{11.3}$$

In what follows we denote by $\boldsymbol{\xi}_t$ the subgradient computed at $\boldsymbol{x}_k + t\boldsymbol{d}_{\hat{\gamma}}$ and by α_t the corresponding linearization error, that is, $\alpha_t = f(\boldsymbol{x}_k) - f(\boldsymbol{x}_k + t\boldsymbol{d}_{\hat{\gamma}}) + t\boldsymbol{\xi}_t^T\boldsymbol{d}_{\hat{\gamma}}$. The pseudo-code of the CPPC is the following:

```
PROGRAM CPPC
  INITIALIZE x_1 ∈ R^n, ε > 0, δ > 0, m ∈ (0,1), ρ ∈ (m,1), and r ∈ (0,1);
  COMPUTE f(x_1) and ξ_1 ∈ ∂f(x_1) and set k = 1;
  SET the bundle (x_1, f(x_1), ξ_1, 0, 0), so that J_- = ∅ and J_+ = {1};
  MAIN ITERATION
    Initialize θ > 0, γ_min > 0 and γ_max > γ_min;
    WHILE the termination condition ||ξ_k|| ≤ ε is not met
      Solve (11.3) for increasing values of γ to obtain (v_k^γ, d_k^γ);
      Choose γ̂ = min{γ | γ ∈ (γ_min, γ_max) and f(x_k + d_k^γ) > f(x_k) + m v_k^γ}
        if it exists, otherwise, set γ̂ := γ_max;
      IF ||d_k^γ̂|| > θ THEN
        Set x_γ̂ = x_k + d_γ̂;
        Compute ξ_γ̂ ∈ ∂f(x_γ̂) and α_γ̂ = f(x_k) - f(x_γ̂) + ξ_γ̂^T d_γ̂;
        BUNDLE INSERTION
          IF α_γ̂ < 0 and ||d_γ̂|| > δ THEN
            Insert (x_γ̂, f(γ̂), ξ_γ̂, α_γ̂, ||d_γ̂||) in the bundle with j ∈ J_-;
            Set γ̂ = γ̂ - r(γ̂ - γ_min);
          ELSE IF ξ_γ̂^T d_γ̂ ≥ ρ v_γ̂ THEN
            Insert (x_γ̂, f(x_γ̂), ξ_γ̂, max{0, α_γ̂}, ||d_γ̂||) in the bundle
              with j ∈ J_+;
          ELSE
            Find step size t ∈ (0,1) such that ξ_t^T d_γ̂ ≥ ρ v_γ̂;
```

```
          Insert (x_k + t d_γ̂, f(x_k + t d_γ̂), ξ_t, max{0, α_t}, t‖d_γ̂‖) in the
            bundle with j ∈ J_+;
        END IF
      END BUNDLE INSERTION
      IF ‖d_γ̂‖ ≤ θ go to BUNDLE DELETION;
      IF f(x_γ̂) ≤ f(y) + m v_γ̂ THEN
        Set the new stability center x_{k+1} = x_γ̂;
      ELSE
        Solve (11.3) with γ = γ̂ to obtain (v_k^γ̂, d_k^γ̂);
        Go to BUNDLE INSERTION;
      END IF
    ELSE
      BUNDLE DELETION
      Set J_+ := J_+ \ {j ∈ J_+ | s_j > δ} and J_- := J_- \ {j ∈ J_- | s_j > δ};
      Compute ξ* = min_{ξ ∈ {ξ_j | j ∈ J_+}} ‖ξ‖;
      IF ‖ξ*‖ ≤ ε THEN
        STOP with the solution x_k + d_k^γ̂;
      ELSE
        Set γ_max := γ_max − r(γ_max − γ_min);
      END IF
    END IF
  END WHILE
  Update J_+ and J_- with respect to x_{k+1};
  Set k = k + 1 and go to next MAIN ITERATION;
END MAIN ITERATION
RETURN final solution x_k;
END PROGRAM CPPC
```

The global convergence of the CPPC to a stationary point is proved for weakly semi-smooth objective functions [86].

Chapter 12
Bundle Methods

At the moment, bundle methods are regarded as the most effective and reliable methods for NSO. They are based on the subdifferential theory developed by Rockafellar [204] and Clarke [61], where the classical differential theory is generalized for convex and locally Lipschitz continuous functions, respectively. The basic idea of bundle methods is to approximate the subdifferential of the objective function by gathering subgradients from previous iterations into a bundle. In this way, more information about the local behavior of the function is obtained than what an individual arbitrary subgradient can yield (cf. subgradient methods). In this chapter, we first introduce the most frequently used bundle methods, that is, the *proximal bundle* (PBM) and the *bundle trust methods* (BT), and then we describe the basic ideas of the second order *bundle-Newton method* (BNEW).

12.1 Proximal Bundle and Bundle Trust Methods

In this section we describe the ideas of the proximal bundle (PBM) and the bundle trust (BT) methods for nonsmooth and nonconvex minimization. For more details we refer to [132, 168, 209].

The basic idea of bundle methods is to approximate the whole subdifferential of the objective function instead of using only one arbitrary subgradient at each point. In practice, this is done by gathering subgradients from the previous iterations into a bundle. Suppose that at the kth iteration of the algorithm we have the current iteration point $\boldsymbol{x}_k$ and some trial points $\boldsymbol{y}_j \in \mathbb{R}^n$ (from past iterations) and subgradients $\boldsymbol{\xi}_j \in \partial f(\boldsymbol{y}_j)$ for $j \in \mathcal{J}_k$, where the index set $\mathcal{J}_k$ is a nonempty subset of $\{1, \ldots, k\}$.

The idea behind the PBM and the BT is to approximate the objective function f below by a piecewise linear function, that is, f is replaced by so called *cutting-plane model*

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$$\hat{f}_k(\boldsymbol{x}) = \max_{j \in \mathcal{J}_k} \{f(\boldsymbol{y}_j) + \boldsymbol{\xi}_j^T(\boldsymbol{x} - \boldsymbol{y}_j)\}. \tag{12.1}$$

This model can be written in an equivalent form

$$\hat{f}_k(\boldsymbol{x}) = \max_{j \in \mathcal{J}_k} \{f(\boldsymbol{x}_k) + \boldsymbol{\xi}_j^T(\boldsymbol{x} - \boldsymbol{x}_k) - \alpha_j^k\},$$

where

$$\alpha_j^k = f(\boldsymbol{x}_k) - f(\boldsymbol{y}_j) - \boldsymbol{\xi}_j^T(\boldsymbol{x}_k - \boldsymbol{y}_j) \quad \text{for all} \quad j \in \mathcal{J}_k.$$

is a so-called *linearization error*. If f is convex, then $\alpha_j^k \geq 0$ for all $j \in \mathcal{J}_k$ and $\hat{f}_k(\boldsymbol{x}) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. In other words, the cutting-plane model $\hat{f}_k$ is an underestimate for f, and the nonnegative linearization error α_j^k measures how good an approximation the model is to the original problem. In the nonconvex case, these facts are not valid anymore and thus the linearization error α_j^k can be replaced by the so-called *subgradient locality measure* (cf. [131])

$$\beta_j^k = \max \{|\alpha_j^k|, \gamma \|\boldsymbol{x}_k - \boldsymbol{y}_j\|^2\}, \tag{12.2}$$

where $\gamma \geq 0$ is the *distance measure parameter* ($\gamma = 0$, if f is convex). Then obviously $\beta_j^k \geq 0$ for all $j \in \mathcal{J}_k$ and $\min_{\boldsymbol{x} \in K} \hat{f}_k(\boldsymbol{x}) \leq f(\boldsymbol{x}_k)$.

The descent direction in the PBM is calculated by

$$\boldsymbol{d}_k = \operatorname{argmin}_{\boldsymbol{d} \in \mathbb{R}^n} \{\hat{f}_k(\boldsymbol{x}_k + \boldsymbol{d}) + \frac{1}{2} u_k \boldsymbol{d}^T \boldsymbol{d}\}, \tag{12.3}$$

where the stabilizing term $\frac{1}{2} u_k \boldsymbol{d}^T \boldsymbol{d}$ guarantees the existence of the solution $\boldsymbol{d}_k$ and keeps the approximation local enough. The weighting parameter $u_k > 0$ improves the convergence rate and accumulates some second order information about the curvature of f around $\boldsymbol{x}_k$.

Instead of adding the stabilizing term $\frac{1}{2} u_k \boldsymbol{d}^T \boldsymbol{d}$ to the objective function in (12.3), the BT utilizes the idea of classical trust region methods keeping it as a constraint for the cutting plane model. Let $\sigma_k > 0$ be the ray of the trust region. Then the descent direction in the BT is calculated by

$$\boldsymbol{d}_k = \operatorname{argmin}_{\boldsymbol{d} \in \mathbb{R}^n} \{\hat{f}_k(\boldsymbol{x}_k + \boldsymbol{d}) \mid \frac{1}{2} \boldsymbol{d}^T \boldsymbol{d} \leq \sigma_k\}. \tag{12.4}$$

However, for numerical reasons it is usefull to utilize the exact penalty function method (see Sect. 16.1) and move the quadratic constraint of the problem (12.4) to the objective function. Then we end up to the same problem (12.3) as in the PBM. The main differences between the PBM and the BT consist in strategies for updating the weight u_k.

In order to determine the step size into the search direction $\boldsymbol{d}_k$, we use so-called *line search procedure*: Assume that $m_L \in (0, \frac{1}{2})$, $m_R \in (m_L, 1)$ and $\bar{t} \in (0, 1]$ are some fixed line search parameters. We first search for the largest number $t_L^k \in [0, 1]$ such that $t_L^k \geq \bar{t}$ and

$$f(\boldsymbol{x}_k + t_L^k \boldsymbol{d}_k) \leq f(\boldsymbol{x}_k) + m_L t_L^k v_k, \tag{12.5}$$

where v_k is the predicted amount of descent

$$v_k = \hat{f}_k(\boldsymbol{x}_k + \boldsymbol{d}_k) - f(\boldsymbol{x}_k) < 0.$$

If such a parameter exists we take a *long serious step*

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + t_L^k \boldsymbol{d}_k \quad \text{and} \quad \boldsymbol{y}_{k+1} = \boldsymbol{x}_{k+1}. \tag{12.6}$$

Otherwise, if (12.5) holds but $0 < t_L^k < \bar{t}$, we take a *short serious step*

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + t_L^k \boldsymbol{d}_k \quad \text{and} \quad \boldsymbol{y}_{k+1} = \boldsymbol{x}_k + t_R^k \boldsymbol{d}_k$$

and, if $t_L^k = 0$, we take a *null step*

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k \quad \text{and} \quad \boldsymbol{y}_{k+1} = \boldsymbol{x}_k + t_R^k \boldsymbol{d}_k, \tag{12.7}$$

where $t_R^k > t_L^k$ is such that

$$-\beta_{k+1}^{k+1} + \boldsymbol{\xi}_{k+1}^T \boldsymbol{d}_k \geq m_R v_k. \tag{12.8}$$

In short serious steps and null steps there exists discontinuity in the gradient of f. Then the requirement (12.8) ensures that $\boldsymbol{x}_k$ and $\boldsymbol{y}_{k+1}$ lie on the opposite sides of this discontinuity and the new subgradient $\boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{y}_{k+1})$ will force a remarkable modification of the next search direction finding problem.

The iteration is terminated if

$$v_k \geq -\varepsilon_s,$$

where $\varepsilon_s > 0$ is a final accuracy tolerance supplied by the user.

Under the upper semi-smoothness assumption [38], the PBM and the BT can be proved to be globally convergent for locally Lipschitz continuous objective functions, which are not necessarily differentiable or convex (see e.g. [132, 209]). In addition, in order to implement the above algorithm one has to bound somehow the number of stored subgradient and trial points, that is, the cardinality of index set $\mathcal{J}_k$. The global convergence of bundle methods with a limited number of stored subgradients can be guaranteed by using a *subgradient aggregation strategy* [131], which accumulates information from the previous iterations. The convergence rates of the PBM and the BT are linear for convex functions [203] and for piecewise linear problems they achieve a finite convergence [209].

The pseudo-code of bundle methods is the following:

```
PROGRAM PBM and BT
  INITIALIZE x_1 ∈ ℝ^n, 𝒥_1 = {1}, t̄ ∈ (0,1], m_L ∈ (0, 1/2), u_1 > 0
    ε > 0 and v_0 ≤ −ε;
  Set k = 1;
  Evaluate f(x_1) and ξ_1 ∈ ∂f(x_1);
  WHILE the termination condition |v_{k−1}| ≤ ε is not met
    Generate the search direction
      d_k = argmin_{d∈ℝ^n} { f̂_k(x_k + d) + 1/2 u_k d^T d };
    Compute v_k = f̂_k(x_k + d_k) − f(x_k);
    Find step sizes t_L^k and t_R^k;
    IF f(x_k + t_L^k d_k) ≤ f(x_k) + m_L t_L^k v_k THEN
      IF t_L^k > t̄ THEN
        LONG SERIOUS STEP
          Set x_{k+1} = x_k + t_L^k d_k and y_{k+1} = x_{k+1};
          Evaluate ξ_{k+1} ∈ ∂f(y_{k+1});
        END LONG SERIOUS STEP
      ELSE
        SHORT SERIOUS STEP
          Set x_{k+1} = x_k + t_L^k d_k and y_{k+1} = x_k + t_R^k d_k;
          Evaluate ξ_{k+1} ∈ ∂f(y_{k+1});
        END SHORT SERIOUS STEP
      END IF
    ELSE
      NULL STEP
        Set x_{k+1} = x_k and y_{k+1} = x_k + t_R^k d_k;
        Evaluate ξ_{k+1} ∈ ∂f(y_{k+1});
      END NULL STEP
    END IF
    Update 𝒥_k and u_k according some updating rules;
    Set k = k + 1;
  END WHILE
  RETURN final solution x_k;
END PROGRAM PBM and BT
```

12.2 Bundle Newton Method

Next we describe the main ideas of the second order bundle-Newton method (BNEW) by Lukšan and Vlček [156]. We suppose that at each $\boldsymbol{x} \in \mathbb{R}^n$ we can evaluate, in addition to the function value and an arbitrary subgradient $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$, also an $n \times n$ symmetric matrix $G(\boldsymbol{x})$ approximating the Hessian matrix $\nabla^2 f(\boldsymbol{x})$. Now, instead of piecewise linear cutting-plane model (12.1), we introduce a piecewise quadratic model of the form

$$\tilde{f}_k(\boldsymbol{x}) = \max_{j \in \mathcal{J}_k} \{f(\boldsymbol{y}_j) + \boldsymbol{\xi}_j^T(\boldsymbol{x} - \boldsymbol{y}_j) + \frac{1}{2}\varrho_j(\boldsymbol{x} - \boldsymbol{y}_j)^T G_j(\boldsymbol{x} - \boldsymbol{y}_j)\}, \tag{12.9}$$

where $G_j = G(\boldsymbol{y}_j)$ and $\varrho_j \in [0, 1]$ is some damping parameter. The model (12.9) can be written equivalently as

$$\tilde{f}_k(\boldsymbol{x}) = \max_{j \in \mathcal{J}_k} \{f(\boldsymbol{x}_k) + \boldsymbol{\xi}_j^T(\boldsymbol{x} - \boldsymbol{x}_k) + \frac{1}{2}\varrho_j(\boldsymbol{x} - \boldsymbol{x}_k)^T G_j(\boldsymbol{x} - \boldsymbol{x}_k) - \alpha_j^k\}$$

and for all $j \in \mathcal{J}_k$ the linearization error takes the form

$$\alpha_j^k = f(\boldsymbol{x}_k) - f(\boldsymbol{y}_j) - \boldsymbol{\xi}_j^T(\boldsymbol{x}_k - \boldsymbol{y}_j) - \frac{1}{2}\varrho_j(\boldsymbol{x}_k - \boldsymbol{y}_j)^T G_j(\boldsymbol{x}_k - \boldsymbol{y}_j). \tag{12.10}$$

Note that now, even in the convex case, α_j^k might be negative. Therefore we replace the linearization error (12.10) by the subgradient locality measure (12.2) and we remain the property $\min_{\boldsymbol{x} \in \mathbb{R}^n} \tilde{f}_k(\boldsymbol{x}) \leq f(\boldsymbol{x}_k)$.

The search direction $d_k \in \mathbb{R}^n$ is now calculated as the solution of

$$\boldsymbol{d}_k = \operatorname{argmin}_{\boldsymbol{d} \in \mathbb{R}^n}\{\tilde{f}_k(\boldsymbol{x}_k + \boldsymbol{d})\}. \tag{12.11}$$

The line search procedure of the BNEW is similar to that in the PBM (see Sect. 12.1). The only remarkable difference occurs in the termination condition for short and null steps. In other words, (12.8) is replaced by two conditions

$$-\beta_{k+1}^{k+1} + (\boldsymbol{\xi}_{k+1}^{k+1})^T \boldsymbol{d}_k \geq m_R v_k \quad \text{and} \quad \|\boldsymbol{x}_{k+1} - \boldsymbol{y}_{k+1}\| \leq C_S,$$

where $C_S > 0$ is a parameter supplied by the user.

The pseudo-code of the BNEW is the following:

```
PROGRAM BNEW
  INITIALIZE x_1 ∈ ℝ^n, J_1 = {1}, t̄ ∈ (0,1], m_L ∈ (0, 1/2), ρ_1 ∈ [0,1]
    C_S > 0, ε > 0 and v_0 ≤ −ε;
  Set k = 1;
  Evaluate f(x_1), ξ_1 ∈ ∂f(x_1) and G_1 = G(x_i);
  WHILE the termination condition |v_{k−1}| ≤ ε is not met
    Generate the search direction d_k = argmin_{d∈ℝ^n} {f̄_k(x_k + d)};
    Compute v_k = f̄_k(x_k + d_k) − f(x_k);
    Find step sizes t_L^k and t_R^k;
    IF f(x_k + t_L^k d_k) ≤ f(x_k) + m_L t_L^k v_k THEN
      IF t_L^k > t̄ THEN
        LONG SERIOUS STEP
          Set x_{k+1} = x_k + t_L^k d_k and y_{k+1} = x_{k+1};
          Evaluate ξ_{k+1} ∈ ∂f(y_{k+1}) and G_{k+1} = G(y_{k+1});
        END LONG SERIOUS STEP
      ELSE
        SHORT SERIOUS STEP
          Set x_{k+1} = x_k + t_L^k d_k and y_{k+1} = x_k + t_R^k d_k;
          Evaluate ξ_{k+1} ∈ ∂f(y_{k+1}) and G_{k+1} = G(y_{k+1});
        END SHORT SERIOUS STEP
      END IF
    ELSE
      NULL STEP
        Set x_{k+1} = x_k and y_{k+1} = x_k + t_R^k d_k;
        Evaluate ξ_{k+1} ∈ ∂f(y_{k+1}) and G_{k+1} = G(y_{k+1});
      END NULL STEP
    END IF
    Update J_k according some updating rules;
    Set k = k + 1;
  END WHILE
  RETURN final solution x_k;
END PROGRAM BNEW
```

Under the upper semi-smoothness assumption [38] the BNEW has been proved to be globally convergent for locally Lipschitz continuous objective functions. For strongly convex functions, the convergence rate of the BNEW is superlinear.

Chapter 13
Gradient Sampling Methods

One of the newest approaches in general NSO is to use gradient sampling algorithms developed by Burke et al. [51, 52]. The *gradient sampling method* (GS) is a method for minimizing an objective function that is locally Lipschitz continuous and smooth on an open dense subset $D \subset \mathbb{R}^n$. The objective may be nonsmooth and/or nonconvex. The GS may be considered as a stabilized steepest descent algorithm. The central idea behind these techniques is to approximate the subdifferential of the objective function through random sampling of gradients near the current iteration point. The ongoing progress in the development of gradient sampling algorithms (see e.g. [67]) suggests that they may have potential to rival bundle methods in the terms of theoretical might and practical performance. However, here we introduce only the original GS [51, 52].

13.1 Gradient Sampling Method

Let f be a locally Lipschitz continuous function on $\mathbb{R}^n$, and suppose that f is smooth on an open dense subset $D \subset \mathbb{R}^n$. In addition, assume that there exists a point $\bar{\boldsymbol{x}}$ such that the level set $\text{lev}_{f(\bar{\boldsymbol{x}})} = \{\boldsymbol{x} \mid f(\boldsymbol{x}) \leq f(\bar{\boldsymbol{x}})\}$ is compact.

At a given iterate $\boldsymbol{x}_k$ the gradient of the objective function is computed on a set of randomly generated nearby points $\boldsymbol{u}_{kj}$ with $j \in \{1, 2, \ldots, m\}$ and $m > n + 1$. This information is utilized to construct a search direction as a vector in the convex hull of these gradients with the shortest norm. A standard line search is then used to obtain a point with lower objective function value. The stabilization of the method is controlled by the *sampling radius* ε_k used to sample the gradients.

The pseudo-code of the GS is the following:

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```
PROGRAM GS
  INITIALIZE x_0 ∈ lev_{f(x̄)} ∩ D,  ε_0 > 0,  m > n+1,  ν_0 ≥ 0,  θ, μ ∈ (0, 1]
    and α, β ∈ (0, 1);
  Set k = 0;
  WHILE the termination condition is not met
    GRADIENT SAMPLING
      Sample u_{k1}, ... u_{km} from B̄(x; 1);
      Set x_{k0} = x_k and x_{kj} = x_k + ε_k u_{kj} for j = 1, ..., m;
      IF x_{kj} ∉ D for some j STOP;
      Set G_k = {∇f(x_{k1}), ∇f(x_{k2}), ..., ∇f(x_{km})};
    END GRADIENT SAMPLING
    Compute g_k = argmin_{g∈G_k} ||g||^2;
    IF ν_k = ||g_k|| = 0 STOP with the final solution x_k;
    IF ||g_k|| > ν_k THEN
      Set ν_{k+1} = ν_k and ε_{k+1} = ε_k;
      Compute the search direction d_k = −g_k/||g_k||;
      Find the step size t_k = max α^p such that
        f(x_k + α^p d_k) < f(x_k) − βα^p ||g_k|| and p ∈ {1, 2, ...};
    ELSE
      Set t_k = 0,  ν_{k+1} = θν_k,  and ε_{k+1} = με_k;
    END IF
    IF x_k + t_k d_k ∈ D THEN        Set x_{k+1} = x_k + t_k d_k;
    ELSE
      Let x̂^k be any point on B̄(x; ε_k) satisfying x̂_k + t_k d_k ∈ D
        and f(x̂_k + t_k d_k) < f(x̂_k) − βt_k ||g_k|| (such a point exists
        due to continuity of f);
      Set x_{k+1} = x̂_k + t_k d_k;
    END IF
    Set k = k + 1;
  END WHILE
  RETURN final solution x_k;
END PROGRAM GS
```

Note that the probability to obtain a point $\boldsymbol{x}_{kj} \notin D$ is zero in the above algorithm. In addition, it is reported in [52] that it is highly unlikely to have $\boldsymbol{x}_k + t_k\boldsymbol{d}_k \notin D$.

The GS algorithm may be applied to any function $f : \mathbb{R}^n \to \mathbb{R}$ that is continuous on $\mathbb{R}^n$ and differentiable almost everywhere. Furthermore, it has been shown that when f is locally Lipschitz continuous, smooth on an open dense subset D of $\mathbb{R}^n$, and has bounded level sets, the cluster point $\bar{\boldsymbol{x}}$ of the sequence generated by the GS with fixed ε is ε-stationary with probability 1 (that is, $\mathbf{0} \in \partial_\varepsilon^G f(\bar{\boldsymbol{x}})$, see also Definition 3.3 in Part I). In addition, if f has a unique ε-stationary point $\bar{\boldsymbol{x}}$, then the set of all cluster points generated by the algorithm converges to $\bar{\boldsymbol{x}}$ as ε is reduced to zero.

Chapter 14
Hybrid Methods

In this chapter, we describe some methods that can be considered as the hybrids of the methods described before. They are the *variable metric bundle method* (VMBM) and the *quasi-secant method* (QSM) for solving general small- and medium-scale NSO problems; the modification of the VMBM for solving large-scale NSO problems, that is, the *limited memory bundle method* (LMBM); and the *non-Euclidean restricted memory level method* (NERML) for extremely large-scale convex NSO problems.

14.1 Variable Metric Bundle Method

The development of a second order method has been fascinating the researchers in NSO during its whole history. Already in his pioneering work [142] Lemaréchal derived a version of the variable metric bundle method utilizing the classical BFGS secant updating formula from smooth optimization (see [84]). Due to the disappointing numerical results in [143] this idea was buried nearly for two decades. In [91] the space dilation updating scheme of [210] was adopted from the subgradient method context. More recently, a reversal quasi-Newton method based on the Moreau-Yosida regularization, BFGS update and the curved search technique, was proposed in [41, 145, 147, 181].

In this section, we present the variable metric bundle method (VMBM) by Lukšan and Vlček [157, 225] for general possible nonconvex, NSO problems. The method is a hybrid of the variable metric (quasi-Newton) methods for smooth optimization and the bundle methods described in Chap. 12. The idea of the method is to use only three subgradients (two calculated at $\boldsymbol{x}_k$ and $\boldsymbol{y}_{k+1}$, and one aggregated containing information from past iterations). This means, that the dimension of the normally time consuming quadratic programming subproblem (12.3) is only three and it can be solved with simple calculations.

The VMBM uses some properties of bundle methods to improve the robustness and the efficiency of variable metric methods in nonsmooth settings. The main

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differences when comparing the VMBM with the standard variable metric methods are the usage of null steps together with the aggregation of subgradients and the application of subgradient locality measures. Nevertheless, the search direction $\boldsymbol{d}_k$ is calculated using the variable metric approach. That is,

$$\boldsymbol{d}_k = -D_k \tilde{\boldsymbol{\xi}}_k,$$

where $\tilde{\boldsymbol{\xi}}_k$ is (an aggregate) subgradient and D_k is the variable metric update that, in the smooth case, represents the approximation of the inverse of the Hessian matrix. The role of matrix D_k is to accumulate information about previous subgradients.

In NSO the search direction is not necessarily a descent one. Using null steps gives more information about the nonsmooth objective function in the case the current search direction is "not good enough", that is, some descent condition is not satisfied. On the other hand, a simple aggregation of subgradients and the application of locality measures guarantee the convergence of the aggregate subgradients to zero and make it possible to evaluate a termination criterion.

In order to determine a new step into the search direction $\boldsymbol{d}_k$, the VMBM uses the so-called *line search procedure*: a new iteration point $\boldsymbol{x}_{k+1}$ and a new auxiliary point $\boldsymbol{y}_{k+1}$ are produced such that

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{x}_k + t_L^k \boldsymbol{d}_k \quad \text{and} \\ \boldsymbol{y}_{k+1} &= \boldsymbol{x}_k + t_R^k \boldsymbol{d}_k, \quad \text{for } k \geq 1 \end{aligned} \tag{14.1}$$

with $\boldsymbol{y}_1 = \boldsymbol{x}_1$, where $t_R^k \in (0, t_{max}]$ and $t_L^k \in [0, t_R^k]$ are step sizes, and $t_{max} > 1$ is an upper bound for the step size. Notice that this line search procedure is a little bit different from that used with the proximal bundle method in Sect. 12.1.

A necessary condition for a *serious step* to be taken is to have

$$t_R^k = t_L^k > 0 \quad \text{and} \quad f(\boldsymbol{y}_{k+1}) \leq f(\boldsymbol{x}_k) - \varepsilon_L^k t_R^k w_k, \tag{14.2}$$

where $\varepsilon_L^k \in (0, 1/2)$ is a fixed line search parameter and $w_k > 0$ represents the desirable amount of descent of f at $\boldsymbol{x}_k$. If condition (14.2) is satisfied, we have $\boldsymbol{x}_{k+1} = \boldsymbol{y}_{k+1}$. On the other hand, a *null step* is taken if

$$t_R^k > t_L^k = 0 \quad \text{and} \quad -\beta_{k+1} + \boldsymbol{d}_k^T \boldsymbol{\xi}_{k+1} \geq -\varepsilon_R^k w_k, \tag{14.3}$$

where $\varepsilon_R^k \in (\varepsilon_L^k, 1/2)$ is a fixed line search parameter, $\boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{y}_{k+1})$, and β_{k+1} is the subgradient locality measure [149, 180] similar to standard bundle methods. That is,

$$\beta_{k+1} = \max\{|f(\boldsymbol{x}_k) - f(\boldsymbol{y}_{k+1}) + (\boldsymbol{y}_{k+1} - \boldsymbol{x}_k)^T \boldsymbol{\xi}_{k+1})|, \gamma \|\boldsymbol{y}_{k+1} - \boldsymbol{x}_k\|^2\}. \tag{14.4}$$

Here, as before, $\gamma \geq 0$ is a distance measure parameter supplied by the user and it can be set to zero when f is convex. In the case of a null step, we set $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k$ but information about the objective function is increased because we store the auxiliary point $\boldsymbol{y}_{k+1}$ and the corresponding auxiliary subgradient $\boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{y}_{k+1})$.

In the convex case there is no need to use any two-point line search procedure. The procedure can be replaced by a simple step size selection and the resulting step size is either accepted (serious step) or not (null step).

The *aggregation procedure* used in the VMBM utilizes only three subgradients and two locality measures. The procedure is carried out by determining multipliers λ_i^k satisfying $\lambda_i^k \geq 0$ for all $i \in \{1, 2, 3\}$, and $\sum_{i=1}^3 \lambda_i^k = 1$ that minimize the function

$$\begin{aligned}\varphi(\lambda_1, \lambda_2, \lambda_3) = {} & [\lambda_1 \boldsymbol{\xi}_m + \lambda_2 \boldsymbol{\xi}_{k+1} + \lambda_3 \tilde{\boldsymbol{\xi}}_k\,]^T D_k [\lambda_1 \boldsymbol{\xi}_m + \lambda_2 \boldsymbol{\xi}_{k+1} + \lambda_3 \tilde{\boldsymbol{\xi}}_k\,] \\ & + 2(\lambda_2 \beta_{k+1} + \lambda_3 \tilde{\beta}_k).\end{aligned} \tag{14.5}$$

Here $\boldsymbol{\xi}_m \in \partial f(\boldsymbol{x}_k)$ is the current subgradient (m denotes the index of the iteration after the latest serious step, i.e. $\boldsymbol{x}_k = \boldsymbol{x}_m$), $\boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{y}_{k+1})$ is the auxiliary subgradient, and $\tilde{\boldsymbol{\xi}}_k$ is the current aggregate subgradient from the previous iteration ($\tilde{\boldsymbol{\xi}}_1 = \boldsymbol{\xi}_1$). In addition, β_{k+1} is the current subgradient locality measure and $\tilde{\beta}_k$ is the current aggregate subgradient locality measure ($\tilde{\beta}_1 = 0$). The resulting aggregate subgradient $\tilde{\boldsymbol{\xi}}_{k+1}$ and aggregate subgradient locality measure $\tilde{\beta}_{k+1}$ are computed by the formulae

$$\tilde{\boldsymbol{\xi}}_{k+1} = \lambda_1^k \boldsymbol{\xi}_m + \lambda_2^k \boldsymbol{\xi}_{k+1} + \lambda_3^k \tilde{\boldsymbol{\xi}}_k \quad \text{and} \quad \tilde{\beta}_{k+1} = \lambda_2^k \beta_{k+1} + \lambda_3^k \tilde{\beta}_k. \tag{14.6}$$

Due to this simple aggregation procedure only one trial point $\boldsymbol{y}_{k+1}$ and the corresponding subgradient $\boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{y}_{k+1})$ need to be stored. Note that the aggregate values are computed only if the last step was a null step. Otherwise, we set $\tilde{\boldsymbol{\xi}}_{k+1} = \boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{x}_{k+1})$ and $\tilde{\beta}_{k+1} = 0$.

As mentioned before, the matrices D_k are formed by using the usual variable metric updates. After a null step, the symmetric rank-one (SR1) update is used, since this update formula gives us a possibility to preserve the boundedness of the matrices generated as required for the proof of global convergence. A new SR1 matrix is given by the formula

$$D_{k+1} = D_k - \frac{\boldsymbol{v}_k \boldsymbol{v}_k^T}{\boldsymbol{u}_k^T \boldsymbol{v}_k},$$

where $\boldsymbol{v}_k = D_k \boldsymbol{u}_k - t_R^k \boldsymbol{d}_k$ and $\boldsymbol{u}_k = \boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_m$.

Because the boundedness of the generated matrices is not required after a serious step, the more efficient Broyden-Fletcher-Goldfarb-Shanno (BFGS) update is used. This update formula is given by

$$D_{k+1} = D_k + \left(t_L^k + \frac{\boldsymbol{u}_k^T D_k \boldsymbol{u}_k}{\boldsymbol{u}_k^T \boldsymbol{d}_k} \right) \frac{\boldsymbol{d}_k \boldsymbol{d}_k^T}{\boldsymbol{u}_k^T \boldsymbol{d}_k} - \frac{D_k \boldsymbol{u}_k \boldsymbol{d}_k^T + \boldsymbol{d}_k \boldsymbol{u}_k^T D_k}{\boldsymbol{u}_k^T \boldsymbol{d}_k},$$

where $\boldsymbol{u}_k = \boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_k$.

The condition

$$\tilde{\boldsymbol{\xi}}_k^T \boldsymbol{v}_k < 0$$

(or equivalently $\boldsymbol{u}_k^T \boldsymbol{d}_k > t_R^k \boldsymbol{d}_k^T D_k^{-1} \boldsymbol{d}_k$), which implies that $\boldsymbol{u}_k^T \boldsymbol{v}_k > 0$, ensures the positive definiteness of the matrices D_{k+1} obtained by the SR1 update. Similarly, the condition

$$\boldsymbol{u}_k^T \boldsymbol{d}_k > 0$$

ensures the positive definiteness of the matrices D_{k+1} obtained by the BFGS update (note that $\boldsymbol{u}_k^T \boldsymbol{d}_k > 0$ holds whenever f is convex). If a corresponding condition is not satisfied at the beginning of the updating procedure, the update is simply skipped (i.e. $D_{k+1} = D_k$). Therefore, all the matrices generated by the VMBM are positive definite.

As a stopping parameter, the VMBM uses the value

$$w_k = -\tilde{\boldsymbol{\xi}}_k^T \boldsymbol{d}_k + 2\tilde{\beta}_k$$

and it stops if $w_k \leq \varepsilon$ for some user specified $\varepsilon > 0$. The parameter w_k is also used during the line search procedure to represent the desirable amount of descent (cf. v_k in 12.1).

We now present a pseudo-code for the VMBM. The algorithm is suitable for minimizing both convex and nonconvex but locally Lipschitz continuous objective functions.

PROGRAM VMBM
  INITIALIZE $\boldsymbol{x}_1 \in \mathbb{R}^n$, $\boldsymbol{\xi}_1 \in \partial f(\boldsymbol{x}_1)$, $D_1 = I \in \mathbb{R}^{n \times n}$, and $\varepsilon > 0$;
  Set $k = m = 1$ and $\boldsymbol{d}_1 = -\boldsymbol{\xi}_1$;
  WHILE the termination condition $w_k \leq \varepsilon$ is not met
    Find step sizes t_L^k and t_R^k;
    Set $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + t_L^k \boldsymbol{d}_k$;
    Evaluate $f(\boldsymbol{x}_{k+1})$ and $\boldsymbol{\xi}_{k+1} \in \partial f(\boldsymbol{x}_k + t_R^k \boldsymbol{d}_k)$;
    IF $t_L^k > 0$ THEN
      SERIOUS STEP
        Compute the variable metric BFGS update

$$D_{k+1} = D_k + \left(t_L^k + \frac{\boldsymbol{u}_k^T D_k \boldsymbol{u}_k}{\boldsymbol{u}_k^T \boldsymbol{d}_k} \right) \frac{\boldsymbol{d}_k \boldsymbol{d}_k^T}{\boldsymbol{u}_k^T \boldsymbol{d}_k} - \frac{D_k \boldsymbol{u}_k \boldsymbol{d}_k^T + \boldsymbol{d}_k \boldsymbol{u}_k^T D_k}{\boldsymbol{u}_k^T \boldsymbol{d}_k},$$

        where $\boldsymbol{u}_k = \boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_k$;
        Compute the search direction

```
                $d_{k+1} = -D_{k+1}\xi_{k+1}$;
          Update the counter for serious steps $m = k+1$;
        END SERIOUS STEP
      ELSE
        NULL STEP
          Compute the variable metric SR1 update
                $D_{k+1} = D_k - \frac{v_k v_k^T}{u_k^T v_k}$,
          where $v_k = D_k u_k - t_R^k d_k$ and $u_k = \xi_{k+1} - \xi_m$;
          Compute the aggregate subgradient $\tilde{\xi}_{k+1}$;
          Compute the search direction
                $d_{k+1} = -D_{k+1}\tilde{\xi}_{k+1}$;
        END NULL STEP
      END IF
      Set $k = k+1$;
    END WHILE
    RETURN final solution $x_k$;
  END PROGRAM VMBM
```

Under mild assumptions, it can be proved that every accumulation point of the sequence $(\boldsymbol{x}_k)$ generated by the VMBM is a stationary point of the objective function.

14.2 Limited Memory Bundle Method

In this section, we shortly describe the limited memory bundle algorithm (LMBM) by Karmitsa (née Haarala) et al. [97, 98, 99, 126] for solving general, possibly nonconvex, large-scale NSO problems. The method is a hybrid of the VMBM [157, 225] (see Sect. 14.1) and the limited memory variable metric methods (see e.g. [53]), where the first one has been developed for small- and medium-scale NSO and the latter ones, on the contrary, for smooth large-scale optimization.

The LMBM combines the ideas of the VMBM with the search direction calculation of limited memory approach. Therefore, the time-consuming quadratic direction finding problem appearing in the standard bundle methods [see Eq. (12.3)] does not need to be solved, nor the number of stored subgradients needs to grow with the dimension of the problem. Furthermore, the method uses only a few vectors to represent the variable metric approximation of the Hessian matrix and, thus, it avoids storing and manipulating large matrices as is the case in the VMBM (see Sect. 14.1). These improvements make the LMBM suitable for large-scale optimization. Namely, the number of operations needed for the calculation of the search direction and the aggregate values is only linearly dependent on the number of variables while, for example, with the original VMBM this dependence is quadratic.

So, the LMBM exploits the ideas of the VMBM, namely the utilization of null steps, the simple aggregation of subgradients, and the subgradient locality measures, but the search direction $\boldsymbol{d}_k$ is calculated using a limited memory approach. That is,

$$\boldsymbol{d}_k = -D_k \tilde{\boldsymbol{\xi}}_k,$$

where $\tilde{\boldsymbol{\xi}}_k$ is (an aggregate) subgradient and D_k is the limited memory variable metric update that, in the smooth case, represents the approximation of the inverse of the Hessian matrix. Note that here the matrix D_k is not formed explicitly but the search direction $\boldsymbol{d}_k$ is calculated using the limited memory approach (to be described later).

The LMBM uses the original subgradient $\boldsymbol{\xi}_k \in \partial f(\boldsymbol{x}_k)$ after the serious step and the aggregate subgradient $\tilde{\boldsymbol{\xi}}_k$ [cf. Eq. (14.6)] after the null step for direction finding (i.e. we set $\tilde{\boldsymbol{\xi}}_k = \boldsymbol{\xi}_k$ if the previous step was a serious step). The aggregation procedure is similar to that of the VMBM [see Eqs. (14.5) and (14.6)] but, naturally, instead of using the explicitly formed matrix D_k we use its limited memory formulation. In addition, the line search procedure used in the LMBM is similar to that used in the VMBM.

In the LMBM both the limited memory BFGS (L-BFGS) and the limited memory SR1 (L-SR1) update formulae are used in calculations of the search direction and the aggregate values. The idea of limited memory matrix updating is that instead of storing large $n \times n$ matrices D_k, one stores a certain (usually small) number of vectors $\boldsymbol{s}_k = \boldsymbol{y}_{k+1} - \boldsymbol{x}_k$ and $\boldsymbol{u}_k = \boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_m$ obtained at the previous iterations of the algorithm, and uses these vectors to implicitly define the variable metric matrices.

Let us denote by $\hat{m}_c$ the user-specified maximum number of stored correction vectors ($3 \le \hat{m}_c$) and by $\hat{m}_k = \min\{k-1, \hat{m}_c\}$ the current number of stored correction vectors. Then the $n \times \hat{m}_k$ dimensional correction matrices S_k and U_k are defined by

$$\begin{aligned} S_k &= \begin{bmatrix} \boldsymbol{s}_{k-\hat{m}_k} & \ldots & \boldsymbol{s}_{k-1} \end{bmatrix} \quad \text{and} \\ U_k &= \begin{bmatrix} \boldsymbol{u}_{k-\hat{m}_k} & \ldots & \boldsymbol{u}_{k-1} \end{bmatrix}. \end{aligned}$$

The inverse L-BFGS update is defined by the formula

$$D_k = \vartheta_k I + \begin{bmatrix} S_k & \vartheta_k U_k \end{bmatrix} \begin{bmatrix} (R_k^{-1})^T (C_k + \vartheta_k U_k^T U_k) R_k^{-1} & -(R_k^{-1})^T \\ -R_k^{-1} & 0 \end{bmatrix} \begin{bmatrix} S_k^T \\ \vartheta_k U_k^T \end{bmatrix},$$

where R_k is an upper triangular matrix of order $\hat{m}_k$ given by the form

$$(R_k)_{ij} = \begin{cases} (\boldsymbol{s}_{k-\hat{m}_k-1+i})^T (\boldsymbol{u}_{k-\hat{m}_k-1+j}), & \text{if } i \le j \\ 0, & \text{otherwise,} \end{cases}$$

C_k is a diagonal matrix of order $\hat{m}_k$ such that

$$C_k = \operatorname{diag}[\boldsymbol{s}_{k-\hat{m}_k}^T \boldsymbol{u}_{k-\hat{m}_k}, \ldots, \boldsymbol{s}_{k-1}^T \boldsymbol{u}_{k-1}],$$

and ϑ_k is a positive scaling parameter.

In addition, the inverse L-SR1 update is defined by

$$D_k = \vartheta_k \boldsymbol{I} - (\vartheta_k U_k - S_k)(\vartheta_k U_k^T U_k - R_k - R_k^T + C_k)^{-1}(\vartheta_k U_k - S_k)^T.$$

The similar representations for the direct limited memory BFGS and SR1 updates can be given but the implementation of the LMBM only needs the inverse update formulae to be used.

In the case of a null step, the LMBM uses the L-SR1 update formula, since this formula allows to preserve the boundedness and some other properties of generated matrices which guarantee the global convergence of the method. Otherwise, since these properties are not required after a serious step, the more efficient L-BFGS update is employed. In the LMBM, the individual updates that would violate positive definiteness are skipped.

As a stopping parameter, the LMBM uses the value

$$w_k = -\tilde{\boldsymbol{\xi}}_k^T \boldsymbol{d}_k + 2\tilde{\beta}_k$$

and it stops if $w_k \le \varepsilon$ for some user specified $\varepsilon > 0$.

The pseudo-code of the LMBM is the following:

```
PROGRAM LMBM
  INITIALIZE x_1 ∈ ℝ^n, ξ_1 ∈ ∂f(x_1), and ε > 0;
  Set k = 1 and d_1 = −ξ_1;
  WHILE the termination condition w_k ≤ ε is not met
    Find step sizes t_L^k and t_R^k;
    Set x_{k+1} = x_k + t_L^k d_k;
    Evaluate f(x_{k+1}) and ξ_{k+1} ∈ ∂f(x_k + t_R^k d_k);
    IF t_L^k > 0 THEN
      SERIOUS STEP
      Compute the search direction d_{k+1} using ξ_{k+1} and L-BFGS
          update;
      END SERIOUS STEP
    ELSE
      NULL STEP
        Compute the aggregate subgradient ξ̃_{k+1};
        Compute the search direction d_{k+1} using ξ̃_{k+1} and L-SR1
          update;
      END NULL STEP
    END IF
    Set k = k + 1;
  END WHILE
  RETURN final solution x_k;
END PROGRAM LMBM
```

Under the upper semi-smoothness assumption [38] the LMBM can be proved to be globally convergent for locally Lipschitz continuous objective functions.

14.3 Quasi-Secant Method

In this section we briefly describe the quasi-secant method (QSM) by Bagirov and Ganjehlou. More details can be found in [22, 23]. The QSM can be considered as a hybrid of bundle methods and the gradient sampling method (see Chaps. 12 and 13). The method builds up information about the approximation of the subdifferential using bundling idea, which makes it similar to bundle methods, while subgradients are computed from a given neighborhood of a current iteration point, which makes the method similar to the gradient sampling method.

We start with the definition of a quasi-secant for locally Lipschitz continuous functions. Let us again denote by $S_1 = \{\boldsymbol{g} \in \mathbb{R}^n \mid \|\boldsymbol{g}\| = 1\}$ the sphere of the unit ball.

Definition 14.1 A vector $\boldsymbol{v} \in \mathbb{R}^n$ is called a *quasi-secant* of the function f at the point $\boldsymbol{x} \in \mathbb{R}^n$ in the direction $\boldsymbol{g} \in S_1$ with the length $h > 0$ if

$$f(\boldsymbol{x} + h\boldsymbol{g}) - f(\boldsymbol{x}) \leq h\boldsymbol{v}^T\boldsymbol{g}.$$

We will denote this quasi-secant by $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{g}, h)$.

For a given $h > 0$ let us consider the set of quasi-secants at a point $\boldsymbol{x} \in \mathbb{R}^n$

$$QSec(\boldsymbol{x}, h) = \big\{\boldsymbol{w} \in \mathbb{R}^n \mid \exists\, \boldsymbol{g} \in S_1 \quad \text{s.t. } \boldsymbol{w} = \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{g}, h)\big\}$$

and the set of limit points of quasi-secants as $h \downarrow 0$:

$$QSL(\boldsymbol{x}) = \{\boldsymbol{w} \in \mathbb{R}^n \mid \exists\, \boldsymbol{g} \in S_1,\ h_k > 0,\ h_k \downarrow 0 \text{ when } k \to \infty \\ \text{s.t. } \boldsymbol{w} = \lim_{k\to\infty} \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{g}, h_k)\}.$$

A mapping $\boldsymbol{x} \mapsto QSec(\boldsymbol{x}, h)$ is called a *subgradient-related (SR)-quasi-secant mapping* if the corresponding set $QSL(\boldsymbol{x}) \subseteq \partial f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. In this case elements of $QSec(\boldsymbol{x}, h)$ are called *SR-quasi-secants*. In the sequel, we will consider sets $QSec(\boldsymbol{x}, h)$ which contain only SR-quasi-secants.

It has been shown in [23] that the closed convex set of quasi-secants

$$W_0(\boldsymbol{x}, h) = \text{cl conv } QSec(\boldsymbol{x}, h)$$

can be used to find a descent direction for the objective with any $h > 0$. However, it is not easy to compute the entire set $W_0(\boldsymbol{x}, h)$, and therefore we use only a few quasi-secants from the set to calculate the descent direction in the QSM.

Let us denote by l the index of the subiteration in the direction finding procedure, by k the index of the outer iteration and by s the index of inner iteration. We start by describing the direction finding procedure. In what follows we use only the iteration counter l whenever possible without confusion. At every iteration k_s we first compute the vector

$$\boldsymbol{w}_l = \text{argmin}_{\boldsymbol{w} \in \bar{V}_l(\boldsymbol{x})} \|\boldsymbol{w}\|^2,$$

where $\bar{V}_l(\boldsymbol{x})$ is a set of all quasi-secants computed so far. If $\|\boldsymbol{w}_l\| < \delta$ with a given tolerance $\delta > 0$, we accept the point $\boldsymbol{x}$ as an approximate stationary point, the so-called (h, δ)-*stationary point* [23], and we go to the next outer iteration. Otherwise, we compute another search direction

$$\boldsymbol{g}_{l+1} = -\frac{\boldsymbol{w}_l}{\|\boldsymbol{w}_l\|}$$

and we check whether this direction is descent or not. If it is, we set $\boldsymbol{d}_{k_s} = \boldsymbol{g}_{l+1}, \boldsymbol{v}_{k_s} = \boldsymbol{w}_l$, and stop the direction finding procedure. Otherwise, we compute another quasi-secant $\boldsymbol{v}_{l+1}(\boldsymbol{x}, \boldsymbol{g}_{l+1}, h)$ in the direction $\boldsymbol{g}_{l+1}$, update the bundle of quasi-secants

$$\bar{V}_{l+1}(\boldsymbol{x}) = \text{conv}\{\bar{V}_l(\boldsymbol{x}) \cup \{\boldsymbol{v}_{l+1}(\boldsymbol{x}, \boldsymbol{g}_{l+1}, h)\}\}$$

and continue the direction finding procedure with $l = l + 1$. It has been proved in [23] that the direction finding procedure is terminating.

When the descent direction $\boldsymbol{d}_{k_s}$ has been found, we need to compute the next (inner) iteration point

$$\boldsymbol{x}_{k_{s+1}} = \boldsymbol{x}_{k_s} + t_{k_s}\boldsymbol{d}_{k_s},$$

where the step size t_{k_s} is defined as

$$t_{k_s} = \text{argmax} \left\{t \geq 0 \mid f(\boldsymbol{x}_{k_s} + t\boldsymbol{d}_{k_s}) - f(\boldsymbol{x}_{k_s}) \leq -c_2 t\|\boldsymbol{v}_{k_s}\|\right\},$$

with given $c_2 \in (0, c_1]$.

The pseudo-code of the QSM is the following:

```
PROGRAM QSM
  INITIALIZE x_1 ∈ R^n, d_{1_1} ∈ S_1 and k = 1;
  OUTER ITERATION
    Set s = 1 and x_{k_s} = x_k;
    Compute the first quasi-secant v_{k_1}(x_{k_1}, d_{k_1}, h_k) with h_k > 0
      s.t. h_k ↓ 0 when k → ∞;
    Set V̄(x_{k_1}) = {v_{k_1}};
    WHILE the termination condition is not met
      INNER ITERATION
        Compute the vector v̄_{k_s} = argmin_{v ∈ V̄(x_{k_s})} ||v||^2;
        IF ||v̄_{k_s}|| ≤ δ_k with δ_k > 0 s.t. δ_k ↓ 0 when k → ∞ THEN
```

```
            Set x_{k+1} = x_{k_s} and k = k+1;
            Go to the next OUTER ITERATION;
          ELSE
            Compute the search direction d_{k_s} = -v̄_{k_s}/||v̄_{k_s}||;
            Find a step size t_{k_s};
            IF Descent condition holds THEN
            Construct the following iteration x_{k_{s+1}} = x_{k_s} + t_{k_s} d_{k_s};
            Compute a new quasi-secant v_{k_{s+1}} at x_{k_{s+1}};
              Set V̄(x_{k_{s+1}}) = {v_{k_{s+1}}};
            ELSE
              Compute a new quasi-secant v_{k_{s+1}} at x_{k_s}
                in direction d_{k_s};
              Update the set V̄(x_{k_{s+1}}) = conv{V̄(x_{k_s}) ∪ v_{k_{s+1}}};
              Set x_{k_{s+1}} = x_{k_s};
            END IF
            Set s = s+1 and go to the next INNER ITERATION;
          END IF
        END INNER ITERATION
      END WHILE
    END OUTER ITERATION
    RETURN final solution x_k;
  END PROGRAM QSM
```

The QSM is globally convergent for locally Lipschitz continuous functions under the assumption that the set $QSec(\boldsymbol{x}, h)$ is a SR-quasi-secant mapping, that is, quasi-secants can be computed using subgradients.

14.4 Non-Euclidean Restricted Memory Level Method

The non-Euclidean restricted memory level method (NERML) by Ben-Tal and Nemirovski [32] is developed for solving extremely large-scale convex nonsmooth problems over simple domains. The NERML is a subgradient type technique that is adjustable, to some extent, to the geometry of the feasible set and, also, it is capable to utilize information gathered about the function in previous iterations. Therefore, the NERML may be considered as a mirror descent method [30, 185] with memory, which on the other hand, makes it a hybrid of subgradient and bundle types methods.

The NERML solves the problems of the form

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in X, \end{cases}$$

where $X \subset \mathbb{R}^n$ is a convex compact set with a nonempty interior and f is a Lipschitz continuous convex function on X.

The iteration formula of the NERML is given by

$$\boldsymbol{x}_{k+1} = \text{argmin}_{\boldsymbol{x}\in X}\{\omega(\boldsymbol{x}) - \boldsymbol{x}^T\nabla\omega(\boldsymbol{p}_k) \mid A_k\boldsymbol{x} \leq \boldsymbol{b}_k\},$$

where $\omega : X \to \mathbb{R}$ is a smooth strongly convex function on X, $\boldsymbol{p}_k \in X$ is the current *prox-center*, and the linear inequalities $A_k\boldsymbol{x} \leq \boldsymbol{b}_k$, with $[A_k, \boldsymbol{b}_k]$ the current *bundle* (c.f. bundle methods in Chap. 12), are such that outside of the set $X_k = \{\boldsymbol{x} \in X \mid A_k\boldsymbol{x} \leq \boldsymbol{b}_k\}$ the value of the objective f is lower or equal to the current *level* $l_k \in \mathbb{R}$. The level may be defined as

$$l_k = f_k + \lambda(f^k - f_k), \tag{14.7}$$

where f^k is the best value of the objective known so far, $f_k < f^k$ is a current lower bound for the objective and $\lambda \in (0, 1)$ is a parameter of the method.

The execution of the NERML is divided to the outer (index k) and inner (index s) iterations. To initiate the very first outer iteration we set

$$f^1 = f(\boldsymbol{p}_1) \quad \text{and} \quad f_1 = \min_{\boldsymbol{x}\in X}\{f(\boldsymbol{p}_1) + (\boldsymbol{x} - \boldsymbol{p}_1)^T\boldsymbol{\xi}_{p_1}\},$$

where the first prox-center $\boldsymbol{p}_1 \in X$ can be chosen in an arbitrary fashion and $\boldsymbol{\xi}_{p_1} \in \partial f(\boldsymbol{p}_1)$.

The search points $\boldsymbol{x}_s := \boldsymbol{x}_{k,s}$, $s = 1, 2, \ldots$ are generated as follows. At the outer iteration we first compute the current level l_k by (14.7) and then we initialize the inner iteration. Hence, at the beginning of the each inner iteration s we always have in our disposal the $(s-1)$th search point $\boldsymbol{x}_{s-1}$ ($\boldsymbol{x}_0 = \boldsymbol{p}_k$), a valid lower bound $\tilde{f}_{s-1}$ of the objective ($\tilde{f}_0 = f_k$) and a *localizer* $X_{s-1} \subseteq X$ ($X_0 = X$, for more details see [32] and the pseudo-code given below).

The NERML then solves an auxiliary problem

$$\tilde{f} = \min_{\boldsymbol{x}\in X_{s-1}}\{f(\boldsymbol{x}_{s-1}) + (\boldsymbol{x} - \boldsymbol{x}_{s-1})^T\boldsymbol{\xi}_{\boldsymbol{x}_{s-1}}\},$$

to compute a new candidate for lower bound

$$\tilde{f}_s = \max\{\tilde{f}_{s-1}, \min l_k, \tilde{f}\}.$$

To update $\boldsymbol{x}_{s-1}$ to $\boldsymbol{x}_s$, the NERML solves the auxiliary optimization problem

$$\boldsymbol{x}_s = \text{argmin}_{\boldsymbol{x}\in X_{s-1}}\{\omega(\boldsymbol{x}) - (\boldsymbol{x} - \boldsymbol{p}_k)^T\nabla\omega(\boldsymbol{p}_k) \mid f(\boldsymbol{x}_{s-1}) + (\boldsymbol{x} - \boldsymbol{x}_{s-1})^T\boldsymbol{\xi}_{\boldsymbol{x}_{s-1}} \leq l_k\}.$$

There are two different reasons to terminate the inner iteration. First, if there is significant decrease in the lower bound f_k or, then, if there is essential progress in the best objective value f^k (see the pseudo-code given below). Otherwise, the localizer X_{s-1} is updated to X_s according to given rules and the next inner iteration is started.

On the other hand, the outer iteration (and, thus, the NERML) is terminated when there is no significant difference between the current best solution and the current lower bound. That is,

$$f^k - f_k \leq \varepsilon$$

with some predefined $\varepsilon > 0$. The NERML returns the solution $\boldsymbol{x}^k$ that corresponds to the best value f^k of the objective obtained so far.

Various versions of the NERML differ from each other mainly by the choice of $\omega(\cdot)$, as well as rules for updating the prox-center, the bundle and the level. The choice of $\omega(\cdot)$ allows to adjust the NERML to the geometry of X For example, it has turned out that in the case of $X = B(\boldsymbol{x}; 1)$ (unit Euclidean ball in $\mathbb{R}^n$), a good choice of function ω is $\omega(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}$, in which case the NERML becomes similar to usual bundle methods (see Chap. 12).

The pseudo-code of the NERML is the following:

PROGRAM NERML
INITIALIZE $k = 1$, $\boldsymbol{p}_1 = \boldsymbol{x}^1 \in X \subset \mathbb{R}^n$, $\lambda \in (0,1)$, $\theta \in (0,1)$, and $\varepsilon > 0$;
COMPUTE $f(\boldsymbol{p}_1)$ and $\boldsymbol{\xi}_{p_1} \in \partial f(\boldsymbol{p}_1)$;
  SET $f^1 = f(\boldsymbol{p}_1)$ and $f_1 = \min_{\boldsymbol{x}\in X}\{f(\boldsymbol{p}_1) + (\boldsymbol{x} - \boldsymbol{p}_1)^T\boldsymbol{\xi}_{p_1}\}$;
  OUTER ITERATION
    WHILE the termination condition $f^k - f_k \leq \varepsilon$ is not met
      Set $l_k = f_k + \lambda(f^k - f_k)$;
      Initialize $\boldsymbol{x}_0 = \boldsymbol{p}_k$, $\tilde{f}_0 = f_k$, $X_0 = X$, and $s = 1$;
      INNER ITERATION
        Compute
            $\tilde{f} = \min_{\boldsymbol{x}\in X_{s-1}}\{f(\boldsymbol{x}_{s-1}) + (\boldsymbol{x} - \boldsymbol{x}_{s-1})^T\boldsymbol{\xi}_{\boldsymbol{x}_{s-1}}\}$,
            $\tilde{f}_s = \max\{\tilde{f}_{s-1}, \min l_k, \tilde{f}\}$;
        IF ($\tilde{f}_s \geq l_k - \theta(l_k - f_k)$) THEN (*progress in the lower bound*)
          Set
            $f^{k+1} = \min\{f^k, \min_{0\leq\tau\leq s-1} f(\boldsymbol{x}_\tau)\}$ and
            $f_{k+1} = \tilde{f}_s$;
          Choose $\boldsymbol{p}_{k+1} \in X$ and set $k = k+1$;
          Go to the next OUTER ITERATION;
        ELSE
          Compute
            $\boldsymbol{x}_s = \operatorname{argmin}_{\boldsymbol{x}\in X_{s-1}}\{\omega(\boldsymbol{x}) - (\boldsymbol{x} - \boldsymbol{p}_k)^T\nabla\omega(\boldsymbol{p}_k) \mid$
                $f(\boldsymbol{x}_{s-1}) + (\boldsymbol{x} - \boldsymbol{x}_{s-1})^T\boldsymbol{\xi}_{\boldsymbol{x}_{s-1}} \leq l_k\}$
          and $f(\boldsymbol{x}_s)$ and $\boldsymbol{\xi}_{\boldsymbol{x}_s} \in \partial f(\boldsymbol{x}_s)$;
          IF ($f(\boldsymbol{x}_s) - l_k \leq \theta(f^k - l_k)$) THEN (*progress in the objective*)
            Set
              $f^{k+1} = \min\{f^k, \min_{0\leq\tau\leq s} f(\boldsymbol{x}_\tau)\}$ and $f_{k+1} = \tilde{f}_s$;
            Choose $\boldsymbol{p}_{k+1} \in X$ and set $k = k+1$;
            Go to the next OUTER ITERATION;
          ELSE
            Choose X_s as any convex compact set satisfying
              $\{\boldsymbol{x} \in X_{s-1} \mid g_{s-1}(\boldsymbol{x}) \leq l_k\} \subseteq X_s$
                $\subseteq \{\boldsymbol{x} \in X \mid (\boldsymbol{x} - \boldsymbol{x}_k)^T\nabla\omega_k(\boldsymbol{x}_s) \geq 0\}$,

```
                where $g_{s-1}(\boldsymbol{x}) = f(\boldsymbol{x}_{s-1}) + (\boldsymbol{x} - \boldsymbol{x}_{s-1})^T \boldsymbol{\xi}_{\boldsymbol{x}_{s-1}}$;
                Set $s = s+1$ and go to the next INNER ITERATION;
              END IF
            END IF
          END INNER ITERATION
        END WHILE
      END OUTER ITERATION
      RETURN final solution $\boldsymbol{x}^k$ that corresponds to $f^k$;
    END PROGRAM NERML
```

It has been proved that for every $\varepsilon > 0$ the NERML founds a solution in a finite number of function (and subgradient) evaluations.

Chapter 15
Discrete Gradient Methods

In this chapter, we introduce two discrete gradient methods that can be considered as semi-derivative free methods in a sense that they do not use subgradient information and they do not approximate the subgradient but at the end of the solution process (i.e., near the optimal point). The introduced methods are the original Discrete Gradient Method (DGM) for small-scale nonsmooth optimization and its limited memory bundle version Limited Memory Discrete Gradient Bundle Method (LDGB) for medium- and semi-large problems.

15.1 Discrete Gradient Method

The idea of the original discrete gradient method (DGM) by Bagirov et al. [24] is to hybridize derivative free methods with bundle methods. In contrast with bundle methods, which require the computation of a single subgradient of the objective function at each trial point, the DGM approximates subgradients by discrete gradients (see Part I, Sect. 6.2) using function values only. Similarly to bundle methods the previous values of discrete gradients are gathered into a bundle and the null step is used if the current search direction is not good enough.

In what follows we again denote by S_1 the sphere of the unit ball and by

$$P = \{z \mid z : \mathbb{R}_+ \to \mathbb{R}_+,\ \lambda > 0,\ \lambda^{-1} z(\lambda) \to 0,\ \lambda \to 0\}$$

the set of univariate positive infinitesimal functions. In addition, let

$$G = \{\boldsymbol{e} \in \mathbb{R}^n \mid \boldsymbol{e} = (e_1, \ldots, e_n), |e_j| = 1,\ j = 1, \ldots, n\}$$

be a set of all vertices of the unit hypercube in $\mathbb{R}^n$.

It has been proved in Sect. 6.2 that the closed convex set of discrete gradients

$$D_0(\boldsymbol{x}, \lambda) = \text{cl conv}\,\{\boldsymbol{v} \in \mathbb{R}^n \mid \exists\, \boldsymbol{g} \in S_1,\ \boldsymbol{e} \in G,\ z \in P \quad \text{such that } \boldsymbol{v} = \Gamma^i(\boldsymbol{x}, \boldsymbol{g}, \boldsymbol{e}, z, \lambda, \alpha)\}$$

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is an approximation to the subdifferential $\partial f(\boldsymbol{x})$ for sufficiently small $\lambda > 0$ and $\alpha > 0$. Thus, it can be used to compute the descent direction for the objective (see Corollary 6.5 and Theorem 6.9). However, the computation of the whole set $D_0(\boldsymbol{x}, \lambda)$ is not easy, and therefore, in the DGM we use only a few discrete gradients from the set to calculate the descent direction.

The procedures used in the DGM are pretty similar to those in the QSM (see Sect. 14.3) but we use here the discrete gradient instead of the quasi-secant. Thus, the DGM consists of outer and inner iterations. In turn the inner iteration consists of serious and null steps.

Let us denote by l the index of the subiteration in the direction finding procedure, by k the index of the outer iteration and by s the index of inner iteration. In what follows we use only the iteration counter l whenever possible without confusion. At every iteration k_s we first compute the discrete gradient $\boldsymbol{v}_1 = \Gamma^i(\boldsymbol{x}, \boldsymbol{g}_1, \boldsymbol{e}, z, \lambda, \alpha)$ (see Definition 6.5 and Remark 6.1) with respect to any initial direction $\boldsymbol{g}_1 \in S_1$ and we set the initial bundle of discrete gradients $\bar{D}_1(\boldsymbol{x}) = \{\boldsymbol{v}_1\}$. Then we compute the vector

$$\boldsymbol{w}_l = \text{argmin}_{\boldsymbol{w} \in \bar{D}_l(\boldsymbol{x})} \|\boldsymbol{w}\|^2,$$

that is the distance between the convex hull $\bar{D}_l(\boldsymbol{x})$ of all computed discrete gradients and the origin. If this distance is less than a given tolerance $\delta > 0$ we accept the point $\boldsymbol{x}$ as an approximate stationary point and go to the next outer iteration. Otherwise, we compute another search direction

$$\boldsymbol{g}_{l+1} = -\frac{\boldsymbol{w}_l}{\|\boldsymbol{w}_l\|}$$

and we check whether this direction is descent. If it is, we have

$$f(\boldsymbol{x} + \lambda \boldsymbol{g}_{l+1}) - f(\boldsymbol{x}) \le -c_1 \lambda \|\boldsymbol{w}_l\|,$$

with the given numbers $c_1 \in (0, 1)$ and $\lambda > 0$. Then we set $\boldsymbol{d}_{k_s} = \boldsymbol{g}_{l+1}$, $\boldsymbol{v}_{k_s} = \boldsymbol{w}_l$ and stop the direction finding procedure. Otherwise, we compute another discrete gradient $\boldsymbol{v}_{l+1} = \Gamma^i(\boldsymbol{x}, \boldsymbol{g}_{l+1}, \boldsymbol{e}, z, \lambda, \alpha)$ into the direction $\boldsymbol{g}_{l+1}$, update the bundle of discrete gradients

$$\bar{D}_{l+1}(\boldsymbol{x}) = \text{conv}\,\{\bar{D}_l(\boldsymbol{x}) \cup \{\boldsymbol{v}_{l+1}\}\}.$$

and continue the direction finding procedure with $l = l + 1$. Note that, at each subiteration the approximation of the subdifferential $\partial f(\boldsymbol{x})$ is improved. It has been proved in [24] that the direction finding procedure is terminating.

When the descent direction $\boldsymbol{d}_{k_s}$ has been found, we need to compute the next (inner) iteration point similarly to that in the QSM (see Sect. 14.3 and the pseudo-code below).

Let sequences (δ_k), (λ_k) such that $\delta_k \downarrow 0$, $\lambda_k \downarrow 0$ as $k \to \infty$ be given. These parameters are updated in every outer iterations. In the inner iteration parameters δ_k and λ_k are fixed. In addition, let a function $z \in P$, sufficiently small numbers

$\varepsilon > 0$, $\alpha > 0$ and a number $c_1 \in (0, 1)$ be given. The pseudo-code of the DGM is as follows:

```
PROGRAM DGM
  INITIALIZE x_1 ∈ R^n, g_{1_1} ∈ S_1, ε > 0, δ_1 > ε, λ_1 > ε, α > 0, and
  z ∈ P;
  Set k = 1;
  OUTER ITERATION
    Set s = 1 and x_{k_1} = x_k;
    Compute i = argmax{|g_{k_1,j}| | j = 1,...,n} and
      the discrete gradient v_1 = Γ^i(x_{k_1}, g_{k_1}, e, z, λ_k, α);
    Set D̄(x_{k_1}) = {v_1};
    WHILE the termination conditions δ_k ≤ ε and λ_k ≤ ε are
      not met
      INNER ITERATION
        LEAST DISTANCE COMPUTATION
          Compute the vector v̄_{k_s} = argmin_{v ∈ D̄(x_{k_s})} ||v||^2;
        END LEAST DISTANCE COMPUTATION
        INNER ITERATION TERMINATION
          IF ||v̄_{k_s}|| ≤ δ_k THEN
            Set x_{k+1} = x_{k_s} and k = k + 1;
            Update λ_k and δ_k;
            Go to the next OUTER ITERATION;
          END IF
        END INNER ITERATION TERMINATION
        Compute the search direction g_{k_s} = −v̄_{k_s}/||v̄_{k_s}||
          and i = argmax{|g_{k_s,j}| | j = 1,...,n};
        IF Descent condition holds THEN
          SERIOUS STEP
            Find a step size t_{k_s};
            Construct the following iteration x_{k_{s+1}} = x_{k_s} + t_{k_s} g_{k_s};
            Compute a new discrete gradient
              v_{k_{s+1}} = Γ^i(x_{k_{s+1}}, g_{k_{s+1}}, e, z, λ_k, α);
            Set D̄(x_{k_{s+1}}) = {v_{k_{s+1}}};
          END SERIOUS STEP
        ELSE
          NULL STEP
            Compute a new discrete gradient
              v_{k_{s+1}} = Γ^i(x_{k_s}, g_{k_{s+1}}, e, z, λ_k, α);
            Update the set D̄(x_{k_{s+1}}) = conv{ D̄(x_{k_s}) ∪ v_{k_{s+1}} };
            Set x_{k_{s+1}} = x_{k_s};
          END NULL STEP
        END IF
        Set s = s + 1 and go to the next INNER ITERATION;
      END INNER ITERATION
    END WHILE
  END OUTER ITERATION
  RETURN final solution x_k;
END PROGRAM DGM
```

In [24] it is proved that the DGM is globally convergent for locally Lipschitz continuous functions under assumption that the set of discrete gradients uniformly approximates the subdifferential (see Assumption 6.1). Since in the DGM the descent direction can be computed for any values of $\lambda > 0$, one can take $\lambda_1 \in (0, 1)$, some $\beta \in (0, 1)$ and update λ_k by the formula $\lambda_k = \beta^k \lambda_1,\ k > 1$. Thus, in this method approximations to subgradients are used only at the final stage which guarantees convergence. In most of iterations such approximations are not used. Therefore the DGM is a derivative-free method.

In a case of piecewise partially separable objective function (see Definition 6.7 in Sect. 6.3 of Part I) the discrete gradients can be computed very efficiently as shown in Sect. 6.3.4. For more details, we refer to [18].

15.2 Limited Memory Discrete Gradient Bundle Method

In this section, we introduce the derivative free limited memory discrete gradient bundle method (LDGB) by Karmitsa and Bagirov [124]. The LDGB is a hybrid of the LMBM and the DGM (see Sects. 14.2 and 15.1).

The LMBM and the DGM have some similarities in their structures. For instance, both of these methods wipe out the old information whenever the serious step occurs. This property is different from standard bundle methods (see Chap. 12) where the old information is collected near the prevailing iteration point and stored to be used in the next iterations nonetheless of the step in question. In practice, storing all the old information may have several disadvantages: first, it needs storage space; second, it adds computational costs; and, what is the worst, it may store and use information that is no longer relevant due to fact that it might have been collected far away from the current iteration point. The last point may be especially problematic in nonconvex cases.

The LMBM bundles the subgradients that are computed in a small neighborhood of the iteration point of the moment. This is similar to standard bundle methods although the LMBM uses this information only after null steps and, at the most, three subgradients are needed. On the other hand, the DGM computes and gathers discrete gradients into a bundle only at the current iteration point but in different directions (see Definition 6.5 in Part I and Sect. 15.1). In the LDGB these ideas are combined and discrete gradients are computed in a small neighborhood of the prevailing iteration point and in the different directions.

In the DGM a quadratic subproblem similar to standard bundle methods needs to be solved to find the discrete gradient with the shortest norm and, as a consequence, to calculate the search direction. In the LDGB, instead of bundling an unlimited number of discrete gradients in null steps and computing the shortest norm, the convex combination of at most three discrete gradients is computed and the search direction is calculated using the limited memory approach. Thus, a possibly time consuming quadratic direction finding problem needs not to be solved and also the difficulty with the unbounded amount of storage needed in the DGM has been dealt with.

The obvious difference between the LDGB and the LMBM is that the LDGB uses discrete gradients instead of subgradients of the objective function. In addition, both inner and outer iterations are used in order to avoid too tight approximations to the subgradients at the beginning of computation. The inner iteration of the LDGB is essentially the same as the LMBM. That is, the search direction is computed by the formula

$$\boldsymbol{d}_{k_s} = -D_{k_s}\tilde{\boldsymbol{v}}_{k_s},$$

where s and k are the indices of inner and outer iterations, $\tilde{\boldsymbol{v}}_{k_s}$ is an aggregate discrete gradient and D_{k_s} is a limited memory variable metric update. In addition, the line search procedure [cf. (14.1)–(14.3)] is used to determine a new iteration and auxiliary points $\boldsymbol{x}_{k_{s+1}}$ and $\boldsymbol{y}_{k_{s+1}}$, and the aggregation procedure [cf. (14.5) and (14.6)] is used to compute a new aggregate discrete gradient $\tilde{\boldsymbol{v}}_{k_{s+1}}$ and a new aggregate subgradient locality measure $\tilde{\beta}_{k_{s+1}}$.

The first discrete gradient

$$\boldsymbol{v}_{1_1} = \Gamma^i(\boldsymbol{x}, \boldsymbol{g}_{1_1}, \boldsymbol{e}, z, \zeta, \alpha),$$

where $i = \text{argmax}\,\{|g_j| \mid j = 1, \ldots, n\}$, is computed to an arbitrary initial direction $\boldsymbol{g}_{1_1} \in S_1$. After that we always use the previous normalized search direction $\boldsymbol{g}_{k_{s+1}} = \boldsymbol{d}_{k_s}/\|\boldsymbol{d}_{k_s}\|$ to compute the next discrete gradient $\boldsymbol{v}_{k_{s+1}}$. Parameters $z \in P$, $\zeta > 0$, and $\alpha > 0$ are selected similarly to the DGM (see Sect. 15.1).

The inner iteration is terminated if we have

$$\frac{1}{2}\|\tilde{\boldsymbol{v}}_{k_s}\|^2 + \tilde{\beta}_{k_s} \le \delta_k$$

for some outer iteration parameter $\delta_k > 0$.

The LDGB uses an adaptive updating strategy for the selection of outer iteration parameter δ_k. At the beginning, the outer iteration parameter δ_1 is set to a large number. Each time the inner iteration is terminated we set

$$\delta_{k+1} = \min\{\sigma\delta_k, w_{k_s}\},$$

where $\sigma \in (0, 1)$ and $w_{k_s} = -\tilde{\boldsymbol{v}}_{k_s}^T\boldsymbol{d}_{k_s} + 2\tilde{\beta}_{k_s}$. Similarly to the LMBM, the parameter w_{k_s} is used also during the line search procedure to represent the desirable amount of descent [cf. (14.2) and (14.3)].

Let us assume that the sequences $z_k \in P$, $\zeta_k > 0$, $z_k \downarrow 0$, $\zeta_k \downarrow 0$, $k \to \infty$, a sufficiently small number $\alpha > 0$ and the line search parameters $\varepsilon_L^{k_s} \in (0, 1/2)$ and $\varepsilon_R^{k_s} \in (\varepsilon_L^{k_s}, 1/2)$ are given. The pseudo-code of the LDGB is the following:

PROGRAM LDGB
  INITIALIZE $\boldsymbol{x}_1 \in \mathbb{R}^n$, $\boldsymbol{g}_{1_1} \in S_1$, $\varepsilon > 0$, $\delta_1 > \varepsilon$, $\zeta_1 > 0$, and $\sigma, \epsilon \in (0,1)$;
  Set $k = 1$;
  OUTER ITERATION
    Set $s = 1$ and $\boldsymbol{x}_{k_1} = \boldsymbol{x}_k$;
    WHILE the termination condition $\delta_k \le \varepsilon$ is not met
      INNER ITERATION
        SERIOUS STEP 1
          Compute the discrete gradient $\boldsymbol{v}_{k_s} \in V_0(\boldsymbol{x}_{k_s}, \zeta_k)$ in
            direction $\boldsymbol{g}_{k_s}$;
          Set $m = s$, $\tilde{\boldsymbol{v}}_{k_s} = \boldsymbol{v}_{k_s}$, and $\tilde{\beta}_{k_s} = 0$;
          Compute the search direction $\boldsymbol{d}_{k_s}$ using $\tilde{\boldsymbol{v}}_{k_s}$ and L-BFGS
            update;
        END SERIOUS STEP 1
        INNER ITERATION TERMINATION
          IF $1/2\|\tilde{\boldsymbol{v}}_{k_s}\|^2 + \tilde{\beta}_{k_s} \le \delta_k$ THEN;
            Set $\boldsymbol{x}_{k+1} = \boldsymbol{x}_{k_s}$, $\boldsymbol{g}_{k+1_1} = \boldsymbol{d}_{k_s}/\|\boldsymbol{d}_{k_s}\|$, $\zeta_{k+1} = \epsilon\zeta_k$,
              $\delta_{k+1} = \min\{\sigma\delta_k, -\tilde{\boldsymbol{v}}_{k_s}^T \boldsymbol{d}_{k_s} + 2\tilde{\beta}_{k_s}\}$, and $k = k+1$;
            Go to the next OUTER ITERATION;
          END IF
        END INNER ITERATION TERMINATION
        Find step sizes $t_L^{k_s}$ and $t_R^{k_s}$, and the subgradient
          locality measure $\beta_{k_{s+1}}$;
        IF $t_L^{k_s} > 0$ THEN
          SERIOUS STEP 2
            Construct the iteration $\boldsymbol{x}_{k_{s+1}} = \boldsymbol{x}_{k_s} + t_L^{k_s}\boldsymbol{d}_{k_s}$;
            Set $\boldsymbol{g}_{k_{s+1}} = \boldsymbol{d}_{k_s}/\|\boldsymbol{d}_{k_s}\|$;
            Set $s = s+1$ and go to the next SERIOUS STEP 1;
          END SERIOUS STEP 2
        ELSE
          NULL STEP
            Construct the trial point $\boldsymbol{y}_{k_{s+1}} = \boldsymbol{x}_{k_s} + t_R^{k_s}\boldsymbol{d}_{k_s}$;
            Compute new discrete gradient $\boldsymbol{v}_{k_{s+1}} \in V_0(\boldsymbol{y}_{k_{s+1}}, \zeta_k)$
              at point $\boldsymbol{y}_{k_{s+1}}$ in direction $\boldsymbol{g}_{k_{s+1}} = \boldsymbol{d}_{k_s}/\|\boldsymbol{d}_{k_s}\|$;
            Compute the aggregate values
              $\tilde{\boldsymbol{v}}_{k_{s+1}} = \lambda_1^{k_s}\boldsymbol{v}_{k_m} + \lambda_2^{k_s}\boldsymbol{v}_{k_{s+1}} + \lambda_3^{k_s}\tilde{\boldsymbol{v}}_{k_s}$ and
              $\tilde{\beta}_{k_{s+1}} = \lambda_2^{k_s}\beta_{k_{s+1}} + \lambda_3^{k_s}\tilde{\beta}_{k_s}$;
            Compute the new search direction $\boldsymbol{d}_{k_{s+1}}$ using $\tilde{\boldsymbol{v}}_{k_{s+1}}$
              and L-SR1 update;
            Set $\boldsymbol{x}_{k_{s+1}} = \boldsymbol{x}_{k_s}$ and $s = s+1$;
            Go to the INNER ITERATION TERMINATION;
          END NULL STEP
        END IF
      END INNER ITERATION
    END WHILE
  END OUTER ITERATION
  RETURN final solution $\boldsymbol{x}_k$;
END LDGB

The discrete gradient is computed according to Definition 6.5 and Remark 6.1. Similarly to the LMBM the search direction and the aggregate values are computed by using the L-BFGS update after serious steps and L-SR1 update otherwise (see Sect. 14.2). The step sizes $t_R^{k_s} \in (0, t_{max}]$ and $t_L^{k_s} \in [0, t_R^{k_s}$ with $t_{max} > 1$ are computed such that either condition (14.2) for serious steps or condition (14.3) for null steps is satisfied. In addition, the subgradient locality measure $\beta_{k_{s+1}}$ as well as the multipliers $\lambda_i^{k_s}$ satisfying $\lambda_i^{k_s} \geq 0$ for all $i \in \{1, 2, 3\}$, and $\sum_{i=1}^{3} \lambda_i^{k_s} = 1$ utilized in the aggregation procedure are computed similarly to the LMBM (see Eqs. (14.4) and (14.5) in Sect. 14.1).

It can be proved that the LDGB is globally convergent for locally Lipschitz continuous semi-smooth functions under assumption that the set of discrete gradients uniformly approximates the subdifferential (see Assumption 6.1 in Part I).

Chapter 16
Constraint Handling

Until now we have mainly considered unconstrained optimization problems. However, many practical optimization problems have restrictions of some kind that can be modeled as constraint functions to the cost function. In this chapter we introduce some common ways of dealing with nonsmooth constrained optimization. That is, exact penalty formulation, and linearization.

16.1 Exact Penalty

As stated in Part II Sect. 8.1 the l_1 exact penalty function formulation

$$F_r(\boldsymbol{x}) = f(\boldsymbol{x}) + r\left(\sum_{i=1}^{p} \max\{0, g_i(\boldsymbol{x})\} + \sum_{j=1}^{q} |h_j(\boldsymbol{x})|\right) \tag{16.1}$$

with $r > 0$ may be used to solve the constrained optimization problems of the form

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leq 0, \quad \text{for } i = 1, \ldots, p, \\ & h_j(\boldsymbol{x}) = 0, \quad \text{for } j = 1, \ldots, q, \end{cases} \tag{16.2}$$

where all the functions involved are supposed to be locally Lipschitz continuous. For the simplicity of the presentation, we have used a single scalar penalty parameter r for all the constraints. In practice, different penalty parameters can be used to accommodate different scalings for constraints.

The pseudo-code for exact penalty method is given below.

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```
PROGRAM Exact Penalty
  INITIALIZE $x_1^s \in \mathbb{R}^n$, $r_1 > 0$, $\varepsilon_r > 0$, and $k = 1$;
  WHILE the termination condition is not met
    Find an approximate minimizer $x_k$ of
      $F_{r_k}(x) = f(x) + r_k \left(\sum_{i=1}^p \max\{0, g_i(x)\} + \sum_{j=1}^q |h_j(x)|\right)$
    starting at $x_k^s$;
    IF $F_{r_k}(x_k) - f(x_k) < \varepsilon_r$, Then
      STOP with the approximate solution $x_k$;
    ELSE
      Choose new penalty parameter $r_{k+1} > r_k$;
      Choose new starting point $x_{k+1}^s$ (e.g. set $x_{k+1}^s = x_k$);
    END IF
  END WHILE
  RETURN final solution $x_k$;
END PROGRAM Exact Penalty
```

Note that the approximative minimizer $\boldsymbol{x}_k$ of the function $F_{r_k}(\boldsymbol{x})$ can be computed using any of the nonsmooth solvers introduced in previous chapters. Usually, the approximative minimizer $\boldsymbol{x}_k$ is also used as a new starting point $\boldsymbol{x}_{k+1}^s$.

The main difficulty of the penalty function methods lies in choosing the initial value and the updating strategy for the penalty parameter. If the value remains too small, the unconstrained problem (16.1) may produce a solution that is not feasible for the original problem. On the other hand, if the value becomes too large, the unconstrained problem (16.1) will be ill-conditioned and special techniques are then required to obtain an accurate solution. Moreover, both the too large and the too small penalty parameter could present numerical difficulties. From another point of view, the advantages of the penalty approach are that it does not require a feasible starting point and, thus, the difficulty of finding an initial feasible point is avoided. Moreover, the "best" solutions can be determined even when no feasible point exists (best in a sense that the l_1 norm of the infeasibilities is minimized), and the solution can be found with a finite r_k.

16.2 Linearization

For simplicity we formulate the linearization method only for inequality constraints. In other words, we consider the problem (16.2) (see Sect. 16.1) with $q = 0$. In addition, we suppose that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, p$ are convex.

As in Chap. 4 of Part I we define the *total constraint function* $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(\boldsymbol{x}) := \max \{g_i(\boldsymbol{x}) \mid i = 1, \ldots, p\}.$$

Now we can get rid of the constraints by replacing the objective function f by so called *improvement function* defined at $\boldsymbol{y} \in \mathbb{R}^n$ by

$$H(\boldsymbol{x}; \boldsymbol{y}) := \max \{f(\boldsymbol{x}) - f(\boldsymbol{y}), g(\boldsymbol{x})\}.$$

Like in bundle methods (see Chap. 12) we now form the cutting plane model of the improvement function by linearizing both the objective and the constraint functions. That is, we define

$$\hat{H}_k(\boldsymbol{x}) := \max \{\hat{f}_k(\boldsymbol{x}) - f(\boldsymbol{x}_k), \hat{g}_k(\boldsymbol{x})\}, \tag{16.3}$$

where $\hat{f}_k$ is defined like in (12.1),

$$\hat{g}_k(\boldsymbol{x}) = \max_{j \in \mathcal{J}_k} \{g(\boldsymbol{y}_j) + \boldsymbol{\xi}_j^T(\boldsymbol{x} - \boldsymbol{y}_j)\},$$

and $\boldsymbol{\xi}_j \in \partial g(\boldsymbol{y}_j)$.

The linearized improvement function (16.3) can be used, for instance, in bundle methods instead of the usual cutting plane model (12.1). In other words, the search direction finding problem (12.3) can be replaced by

$$\boldsymbol{d}_k = \operatorname{argmin}_{\boldsymbol{d} \in \mathbb{R}^n} \{\hat{H}_k(\boldsymbol{x}_k + \boldsymbol{d}) + \frac{1}{2} u_k \boldsymbol{d}^T \boldsymbol{d}\}$$

causing the inequality constrained problem to be solved.

Chapter 17
Numerical Comparison of NSO Softwares

In this chapter, we compare implementations of different NSO methods for solving unconstrained optimization problems of the form

$$\begin{cases} \text{minimize} & f(\boldsymbol{x}) \\ \text{such that} & \boldsymbol{x} \in \mathbb{R}^n, \end{cases}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous. Note that no differentiability or convexity assumptions are made.

Most of the NSO methods can be divided into two main groups: subgradient methods and bundle methods. Usually when developing new algorithms and testing them, the comparison is made between similar methods. In this chapter, we test and compare different methods from both groups as well as some methods which may be considered as hybrids of these two and/or others, and the two discrete gradient methods, all of which have been described in previous chapters. A broad set of nonsmooth optimization test problems is used for this purpose. A relatable comparison is also made in [125, 163], but here we have used a larger and partially different set of test problems and have involved more solvers in the comparison.

The methods included in our tests are the following:

- *Subgradient methods*:
  - standard subgradient method [210] (Sect. 10.1),
  - Shor's r-algorithm [118, 139, 210] (Sect. 10.2),
- *Bundle methods*:
  - proximal bundle method [168] (Sect. 12.1),
  - bundle-Newton method [156] (Sect. 12.2),
- *Hybrid methods*:
  - limited memory bundle method [98, 99] (Sect. 14.2),
  - quasi-secant method [22, 23] (Sect. 14.3),
- *Discrete gradient methods*:
  - discrete gradient method [24] (Sect. 15.1) and
  - limited memory discrete gradient bundle method [123] (Sect. 15.2).

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All of the solvers tested are so-called general black box methods and, naturally, cannot beat codes designed especially for a particular class of problem (say, for example, for piecewise linear, min–max, or partially separable problems). Losing in comparison to codes designed for specific problems is a weakness, but only if the problem is known to be of that type. The strength of the general methods is that they require minimal information on the objective function for their implementation. Namely, the value of the objective function and, possibly, one arbitrary subgradient are required at each point.

The aim of this chapter is not to foreground one particular method over the others—it is a well-known fact that different methods work well for different types of problems, and none of them is good for all types of problems—but to gain some kind of insight into which kind of method should be selected for certain types of problems. Suppose, for instance, that you want to minimize a problem known to be nonconvex and nonsmooth, with 200 variables. In this chapter, we analyze which is the best method to use.

Furthermore, in practice, no one will apply approximate (sub)gradients (i.e. a derivative free method) if (sub)gradient information is available. However, the situation does arise when obtaining (sub)gradient information requires some effort. In such a situation, comparison of the performance of an algorithm with exact and approximate (sub)gradients is important to help users to understand what they lose (or gain) if they apply only approximate (sub)gradients.

The chapter is organized as follows. Section 17.1 introduces the implementations of the NSO methods that are tested and compared. Section 17.2 describes the problems used in our experiments. In Sect. 17.3, we say a few words about the parameters and termination criteria used. The results of the numerical experiments are presented and discussed in Sect. 17.4, and Sect. 17.5 concludes the chapter and gives our credentials for high-performing algorithms for different classes of problems.

17.1 Solvers

Benchmarking results are dependent on the quality of the method and implementation. Most methods have a variety of implementations. We have chosen the tested implementations for their accessibility.[1] The tested optimization codes with references to their more detailed descriptions are presented in Table 17.1. Then, we say a few words on each implementation. In Table 17.2 we recall the basic assumptions needed for the solvers.

`SubG` is a crude implementation of the basic subgradient algorithm (see Sect. 10.1). The step length is chosen to be to some extent constant. Let us denote by l the largest integer, smaller than or equal to it_{max}/c, where it_{max} is the maximum number of iterations and $c > 0$ is the user-specified maximum number of different step sizes.

[1] Most of the solvers used here (i.e. `DGM`, `LDGB`, `LMBM`, `PBNCGC`, and `QSM`) are available for downloading from http://napsu.karmitsa.fi/nsosoftware/. Links to some other NSO solvers (including `PNEW` and `SolvOpt` used here) can also be found there.

Table 17.1 Tested pieces of software

Software	Author(s)	Method	Referencer
SubG	Karmitsa	Subgradient	[210]
SolvOpt	Kuntsevich and Kappel	Shor's r-algorithm	[118, 139, 210]
PBNCGC	Mäkelä	Proximal bundle	[161, 168]
PNEW	Lukšan and Vlček	Bundle-Newton	[156]
LMBM	Karmitsa	Limited memory bundle	[98, 99]
QSM	Bagirov and Ganjehlou	Quasi-Secant	[22, 23]
DGM	Bagirov et al.	Discrete gradient	[24]
LDGB	Karmitsa	L-discrete gradient Bundle	[123]

We take $t_k = t_{init}$ in the first l iterations and

$$t_k = \frac{t_{j\times l}}{10(j+1)} \qquad \text{for } k = j \times l + 1, \ldots, (j+1) \times l \text{ and } j = 1, \ldots, c.$$

We use the following three criteria as a stopping rule for SubG: the number of function evaluations (and iterations) is restricted by parameter it_{max} and also the algorithm stops if either it cannot decrease the value of the objective function within m_1 successive iterations (i.e. $f(\boldsymbol{x}_l) > f_{best}$ for all $l = k, \ldots, k + m_1$, where f_{best} is the smallest value of the objective function obtained so far and $k \geq 1$), or it cannot find a descent direction within m_2 successive iterations (i.e. $f(\boldsymbol{x}_{l+1}) > f(\boldsymbol{x}_l)$ for all $l = k, \ldots, k + m_2$, $k \geq 1$). Since a subgradient method is not a descent method, we store the best value f_{best} of the objective function and the corresponding point $\boldsymbol{x}_{best}$ and return them as a solution if any of the stopping rules above is met.

SolvOpt (Solver for local nonlinear optimization problems) is an implementation of Shor's r-algorithm (see Sect. 10.2). The approaches used to handle difficulties with step size selection and termination criteria in Shor's r-algorithm are heuristic (for details see [118]). In SolvOpt, one can choose to use either original subgradients or difference approximations of them (i.e. the user does not have to code difference approximations, but selects one parameter to do this automatically). In our experiments, we have used both analytically and numerically calculated subgradients. In the following, we denote SolvOptA and SolvOptN, respectively, as the corresponding solvers.

The MatLab, C and Fortran source codes for SolvOpt are available. In our experiments, we used SolvOpt v.1.1 HP-UX FORTRAN-90 sources.

PBNCGC is an implementation of the most frequently used bundle method in NSO; that is, the proximal bundle method (see Sect. 12.1). The code includes constraint handling (bound constraints, linear constraints, and nonlinear/nonsmooth constraints) and a possibility to optimize multiobjective problems. The quadratic direction finding problem (Sect. 12.3) is solved by the PLQDF1 subroutine implementing the dual projected gradient method proposed in [154].

Table 17.2 Assumptions needed for software

Software	Assumptions on objective	Needed information
SubG	Convex	$f(\boldsymbol{x})$, arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$
SolvOptA	Convex	$f(\boldsymbol{x})$, arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$
SolvOptN	Convex	$f(\boldsymbol{x})$
PBNCGC	Semi-smooth	$f(\boldsymbol{x})$, arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$
PNEW	Semi-smooth	$f(\boldsymbol{x})$, arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$, (approximated Hessian)
LMBM	Semi-smooth	$f(\boldsymbol{x})$, arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$
QSMA	Quasi-differentiable, semi-smooth	$f(\boldsymbol{x})$, arbitrary $\boldsymbol{\xi} \in \partial f(\boldsymbol{x})$
QSMN	Quasi-differentiable, semi-smooth	$f(\boldsymbol{x})$
DGM	Quasi-differentiable, semi-smooth	$f(\boldsymbol{x})$
LDGB	Semi-smooth	$f(\boldsymbol{x})$

PNEW is a bundle-Newton solver for unconstrained and linearly constrained NSO (see Sect. 12.2). We used the numerical calculation of the Hessian matrix in our experiments (this can be done automatically). The quadratic direction finding problem (Sect. 12.11) is solved by the subroutine PLQDF1 [154].

LMBM is an implementation of a limited memory bundle method specifically developed for large-scale NSO (see Sect. 14.2). In our experiments, we used the adaptive version of the code with the initial number of stored correction pairs used to form the variable metric update equal to 7 and the maximum number of stored correction pairs equal to 15. The Fortran 77 source code and the mex-driver (for MatLab users) are available.

QSM is a quasi-secant solver for nonsmooth possibly nonconvex minimization (see Sect. 14.3). We have used both analytically calculated subgradients and approximated subgradients in our experiments (this can be done automatically by selecting one parameter). In the following, we denote QSMA and QSMN, respectively, as the corresponding solvers.

DGM is a discrete gradient solver for derivative free optimization (see Sect. 15.1). To apply DGM, one only needs to be able to compute the value of the objective function at every point, and the subgradient will be approximated.

LDGB is a Fortran 95 implementation of the derivative free limited memory discrete gradient bundle method for general, possible nonconvex, nonsmooth minimization (see Sect. 15.2). Similarly to DGM, one only needs to compute the value of the objective function at every point. One can also use this code as a Fortran 95 version of LMBM (because of some implementational issues, it might use fewer subgradient evaluations than the previous version).

All of the algorithms except for LDGB and SolvOpt were implemented in Fortran 77 using double-precision arithmetic. To compile the codes, we used gfortran, the GNU Fortran compiler. The experiments were performed on an Intel® Core™ 2 CPU 1.80 GHz.

17.2 Problems

The test set used in our experiments consists of classical academic nonsmooth minimization problems from the literature and their extensions (see Part II). This includes, for instance, minmax, piecewise linear, piecewise quadratic, and sparse problems, as well as highly nonlinear nonsmooth problems. The test set was grouped into ten subclasses according to problems convexity and size:

XSC: Extra-small convex problems, $n \leq 20$, Problems 1–18 in Part II Chap. 9;

XSNC: Extra-small nonconvex problems, Problems 21–33, and 35–37 in Part II Chap. 9;

SC: Small-scale convex problems, $n = 50$ Problems 57–61 in Part II Chap. 9. We ran each problem five times, the first time using a fixed initial point $\boldsymbol{x}^{(1)}$ given in Part II Chap. 9 and the remaining four times using a starting point generated randomly from a ball centered at $\boldsymbol{x}^{(1)}$ with radius $\|\boldsymbol{x}^{(1)}\|/n$. Note that $\boldsymbol{x}^{(1)} \neq \mathbf{0}$ for all the problems, so the starting points for each run were unique;

SNC: Small-scale nonconvex problems Problems 62–63 in Part II Chap. 9. Similarly to convex problems, we ran each problem five times;

MC and MNC: Medium-scale convex and nonconvex problems, $n = 200$ (see SC and SNC problems);

LC and LNC: Large-scale convex and nonconvex problems, $n = 1{,}000$ (see SC and SNC problems);

XLC and XLNC: Extra-large-scale convex and nonconvex problems, $n = 4{,}000$ (see SC and SNC problems);

All of the solvers tested are so-called local methods; that is, they do not attempt to find the global minimum of the nonconvex objective function. To enable fair comparison of different solvers, the nonconvex test problems were selected such that all of the solvers converged to the same local minimum of a problem. However, it is worth mentioning that when solvers converged to different local minima (i.e. in some nonconvex problems omitted from the test set), solvers `LMBM`, `LDGB`, `SubG` and `SolvOpt(A+N)` usually converged to one local minimum, while `PBNCGC`, `DGM`, and `QSM(A+N)` converged to another. Solver `PNEW` sometimes converged with the first group and other times with the second. In addition, `DGM` and `QSM(A+N)` seem to have an aptitude for finding global or at least smaller local minima than the other solvers. For example, in Problem 34 (Part II, Chap. 9) all of the other solvers converged to the reported minimum, but `DGM` and `QSM(A+N)` "converged" to minus infinity.

17.3 Termination, Parameters, and Acceptance of Results

Each implementation of an optimization method involves a variety of tunable parameters. The improper selection of these parameters can skew any benchmarking result. In our experiments, we used the default settings of the codes as far as possible. However, some parameters naturally have to be tuned, in order to obtain reasonable results.

The determination of stopping criteria for different solvers, such that the comparison of different methods is fair, is not a trivial task. We say that a solver finds the solution with respect to a tolerance $\varepsilon > 0$ if

$$\frac{f_{best} - f_{opt}}{1 + \|f_{opt}\|} \leq \varepsilon,$$

where f_{best} is a solution obtained with the solver and f_{opt} is the best known (or optimal) solution.

We fixed the stopping criteria and parameters for the solvers using three different problems: problems 25 and 35 (XSNC), and problem 59 with $n = 50$ (SC). We sought the loosest termination parameters with all of the solvers, such that the results for all three test problems were still acceptable with respect to the tolerance $\varepsilon = 10^{-4}$.

In addition to the usual stopping criteria, we terminated the experiments if the elapsed CPU time exceeded half an hour for XS, S, M, and L problems, and an hour for XL problems. With XS, S, and M problems this never happened. For other problems, the appropriate discussion is given with the results of that problem class.

We accepted the results for XS and X problems ($n \leq 50$) with respect to the tolerance $\varepsilon = 5 \times 10^{-4}$. With larger problems ($n \geq 200$), we accepted the results with the tolerance $\varepsilon = 10^{-3}$. In the following, we also report the results for all problem classes with respect to the relaxed tolerance $\varepsilon = 10^{-2}$ to gain an insight into the reliability of the solvers (i.e. is a failure a real one or is it just an inaccurate result which could possibly be prevented with a more tight stopping parameter?).

With all of the bundle based solvers, the distance measure parameter value $\gamma = 0.5$ was used with nonconvex problems. With `PBNCGC` and `LMBM`, the value $\gamma = 0$ was used with convex problems and, since with `PNEW` γ has to be positive, $\gamma = 10^{-10}$ was used with `PNEW`. For those solvers storing subgradients (or approximations of subgradients)—that is, `PBNCGC`, `PNEW`, `LMBM`, `QSM(A+N)`, `DGM`, and `LDGB`—the maximum size of the bundle was set to $\min\{n + 3, 100\}$. For all other parameters, we used the default settings of the codes.

17.4 Results

The results are summarized in Figs. 17.1–17.11 and in Table 17.3. The results are analyzed using the performance profiles introduced in [79]. We compare the efficiency of the solvers both in terms of computational times and numbers of function and subgradient evaluations (evaluations for short). In the performance profiles, the value of $\rho_s(\tau)$ at $\tau = 0$ gives the percentage of test problems for which the corresponding solver is the best; that is, it uses least computational time or evaluations. On the other hand, the value of $\rho_s(\tau)$ at the rightmost abscissa gives the percentage of test problems that the corresponding solver can solve; in other words, the reliability of the solver. Note that the reliability of the solver does not depend on its measured performance. In addition, the relative efficiency of each solver can be directly seen from the performance profiles: the higher the particular curve, the better the corresponding solver.

17.4.1 Extra-Small Problems

There was not a large difference in computational times of the different solvers when solving the extra small problems. Thus, only the numbers of function and subgradient evaluations are reported in Fig. 17.1.

PBNCGC was usually the most efficient solver when comparing the numbers of evaluations. This is, in fact, true for all sizes of problems. Thus, PBNCGC should be a good choice as a solver in a case where the objective function value and/or the subgradient are expensive to compute. In convex cases, the accuracy of PBNCGC was one of the best. However, PBNCGC failed to achieve the desired accuracy in 50 % of the XSNC problems, which means that it had the worst degree of success in solving these problems (see Fig. 17.1b).

In addition, with all of the other solvers, there was a slight drop in the success rate when solving nonconvex problems. However, most of these failures are, in fact, inaccurate results: all of the solvers except PBNCGC, PNEW, and SubG succeeded in solving all of the XSNC problems with respect to the relaxed tolerance $\varepsilon = 10^{-2}$, and PBNCGC and SubG succeeded in solving more than 80 % of them. The reason

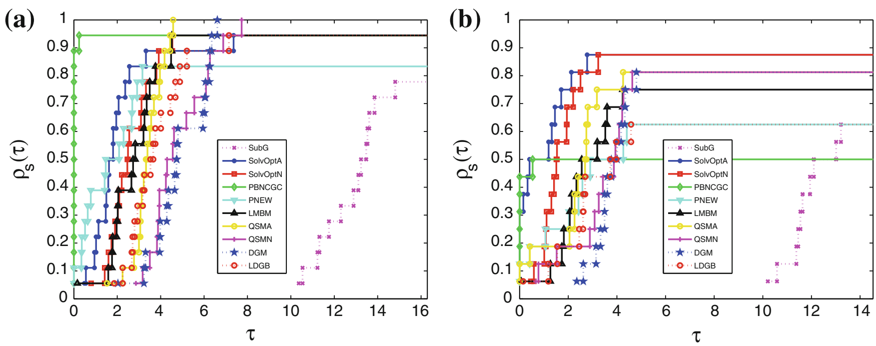

Fig. 17.1 Evaluations for XS problems ($n \leq 20, \varepsilon = 5 \times 10^{-4}$). (**a**) Convex (18 pcs.). (**b**) Nonconvex (16 pcs.)

for the relatively large number of failures with PNEW lies in its sensitivity to internal parameter XMAX (RPAR(9) in the code) which is also noted in [158]. If we used a selected value for this parameter, instead of only one (default) value, the solver PNEW also solved more than 80 % of the XSNC problems.

In XSC problems, PNEW was the second most efficient solver (see Fig. 17.1a). However, as already pointed out, it did not do so well in nonconvex cases. On the other hand, SolvOpt(A+N) were among the most reliable solvers in both convex and nonconvex settings, although, theoretically, Shor's r-algorithm is not supposed to solve nonconvex problems. SolvOptA was also the most efficient method apart from PBNCGC in nonconvex cases and, when compared to PBNCGC, it was more reliable.

As was stated at the beginning of this chapter, it is well known that solvers using the subderivative information usually beat those solvers that do not. However, with these XS problems, the differences in the efficiency of the solvers with and without subgradients were insignificant and, in fact, SolvOptN was one of the most efficient solvers tested (see Fig. 17.1).

17.4.2 Small-Scale Problems

Already with the small-scale problems, there was a wide diversity in the computational times of different codes. Moreover, the numbers of evaluations used with solvers were no longer directly comparable with the elapsed computational times. For instance, PBNCGC was clearly the winner when comparing the numbers of evaluations (see Figs. 17.2b and 17.3b). However, when comparing computational times, SolvOptA, QSMA, and LMBM were equally as efficient as PBNCGC for SC problems (see Fig. 17.2a), and LMBM was the most efficient solver for SNC problems (see Fig. 17.3a).

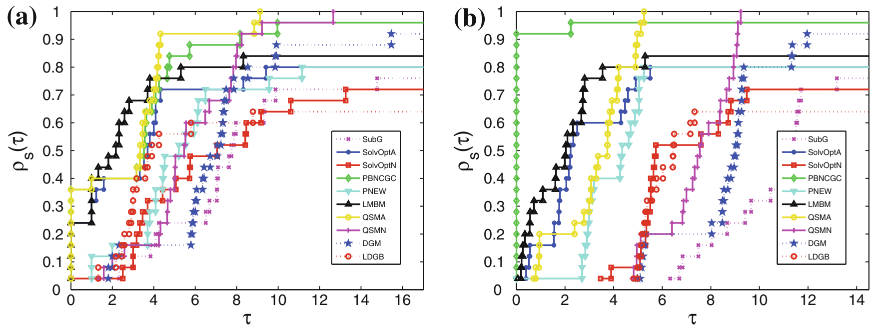

Fig. 17.2 Time and evaluations for SC problems (25 pcs., $n = 50$, $\varepsilon = 5 \times 10^{-4}$). (**a**) CPU-time. (**b**) Evaluations

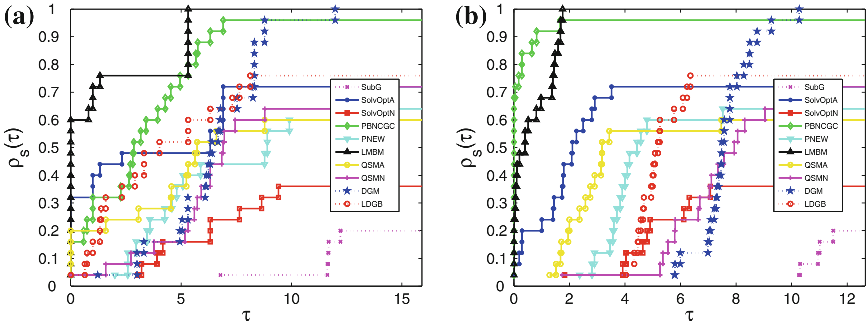

Fig. 17.3 Time and evaluations for SNC problems (25 pcs., $n = 50$, $\varepsilon = 5 \times 10^{-4}$). (**a**) CPU-time. (**b**) Evaluations

Solver LDGB was the most efficient of the derivative free solvers. It also competed successfully with solvers using subderivative information both in SC and SNC settings when comparing computational times. With respect to evaluations, LDGB was not as competitive, because it needs $n + 2$ function evaluations per iteration. Furthermore, LDGB had some serious difficulties with accuracy, especially in the convex case: it solved only 64 % of SC problems with the desired accuracy, thus making it the most unreliable solver tested for this problem class. With the relaxed tolerance $\varepsilon = 10^{-2}$, LDGB succeed in solving 96 % of the SC problems. In SNC problems it did a little better, achieving 76 % with $\varepsilon = 5 \times 10^{-4}$ and 100 % with the relaxed tolerance. Also, LMBM and DGM succeeded in solving more nonconvex than convex problems (see Figs. 17.2 and 17.3).

On the other hand, all of the subgradient solvers SubG and SolvOpt(A+N) had some difficulties in the nonconvex case. With SolvOpt(A+N), these difficulties arose mainly from accuracy: SolvOptN solved only 36 % of the SNC problems with respect to tolerance $\varepsilon = 5 \times 10^{-4}$ but as many as 84 % with $\varepsilon = 10^{-2}$. For SolvOptA, the corresponding values were 72 and 96 %. Note, however, that with XS problems SolvOpt(A+N) were also the most accurate solvers in the nonconvex settings. The standard subgradient solver SubG solved only 20 % of the SNC problems with respect to tolerance $\varepsilon = 5 \times 10^{-4}$, and even with the relaxed tolerance SubG, succeeded in solving only 40 % of the SNC problems.

The quasi-secant solvers QSM(A+N) also had some difficulties with SNC problems. While solving all of the SC problems successfully, QSMA and QSMN solved only 60 and 72 % (respectively) of the SNC problems with the desired accuracy. With the relaxed tolerance, both of these solvers also succeeded in solving all of the SNC problems.

The most reliable solvers for SC problems were QSMA and QSMN. They were the only solvers that succeeded in solving all of the SC problems with the desired accuracy. With the relaxed tolerance, DGM managed as well, and all the other solvers but SubG succeeded in solving at least 84 % of the problems. In the nonconvex case, LMBM and DGM were the most reliable solvers, as they solved all of the problems

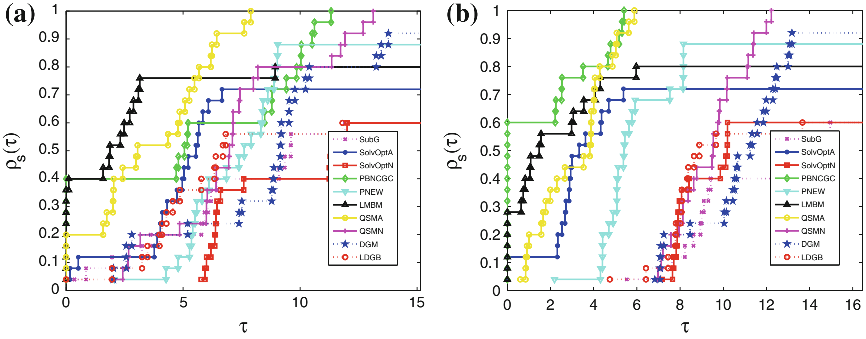

Fig. 17.4 Time and evaluations for MC problems (25 pcs., $n = 200$, $\varepsilon = 10^{-3}$). (**a**) CPU-time. (**b**) Evaluations

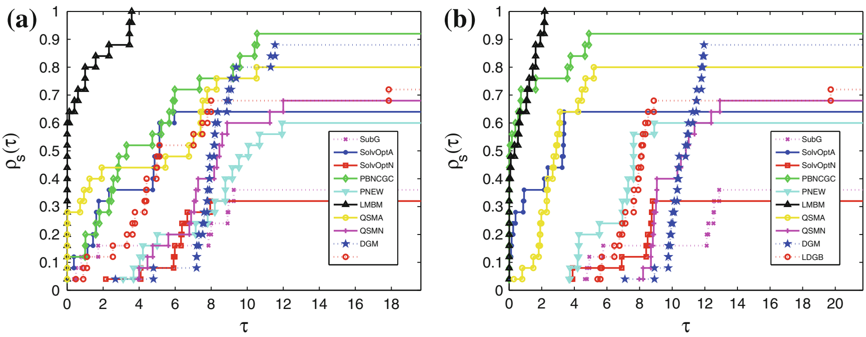

Fig. 17.5 Time and evaluations for MNC problems (25 pcs., $n = 200$, $\varepsilon = 10^{-3}$). (**a**) CPU-time. (**b**) Evaluations

successfully. With the relaxed tolerance, PBNCGC, QSMA, QSMN and LDGB also succeeded, and all of the solvers except PNEW and SubG managed to solve at least 84 % of the problems. Again, the reason for the failures with PNEW lie in its sensitivity to internal parameters XMAX.

17.4.3 Medium-Scale Problems

The results for medium-scale problems reveal similar trends to those for small problems (see Figs. 17.4 and 17.5). Nevertheless, here the efficiency of LMBM may be better seen, and QSMA also did better than before, especially with MNC problems.

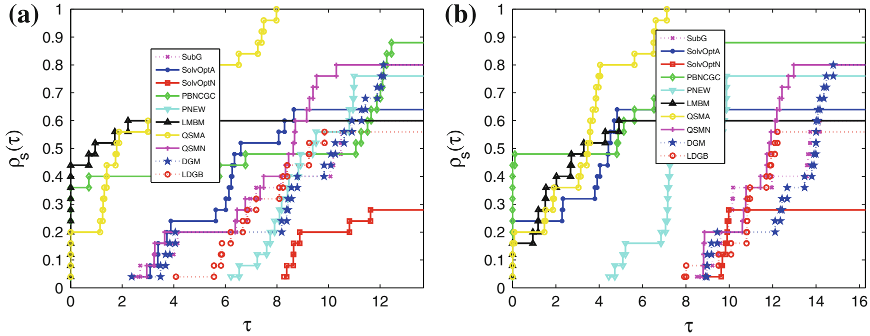

Fig. 17.6 Time and evaluations for LC problems (25 pcs., $n = 1{,}000$, $\varepsilon = 10^{-3}$). (**a**) CPU-time. (**b**) Evaluations

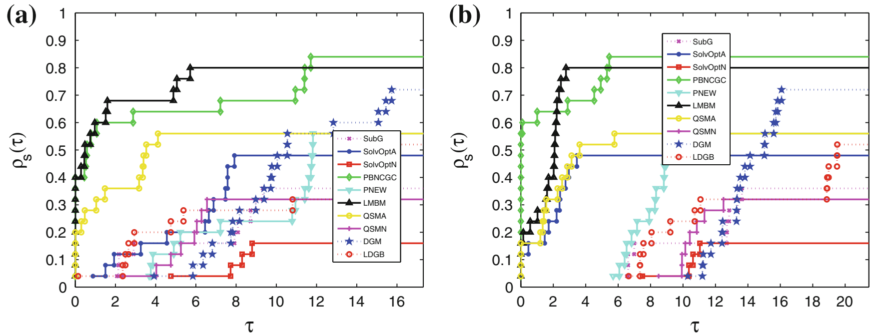

Fig. 17.7 Time and evaluations for LNC problems (25 pcs., $n = 1{,}000$, $\varepsilon = 10^{-3}$). (**a**) CPU-time. (**b**) Evaluations

17.4.4 Large Problems

When solving L problems, the solvers divided into two groups in terms of their efficiency (see Figs. 17.6, 17.7 and 17.8): the first group consists of more efficient solvers; `LMBM`, `PBNCGC`, `QSMA`, and `SolvOptA` (`SolvOptA` with respect to evaluations). The second group consists of solvers using some kind of approximation for subgradients or Hessian, and `SubG`. In the nonconvex case (see Fig. 17.7), the inaccuracy of `QSMA` almost caused it to slide into the group of less efficient solvers. In Fig. 17.8, which illustrates the results with the relaxed tolerance, `QSMA` is clearly among the more efficient solvers.

`LMBM` was usually the most efficient method tested when comparing computational times: for 44 % of LC problems and 40 % of LNC problems (see Figs. 17.6a and 17.7a). However, in the LC case, the overall efficiency of `LMBM` was ruined, because it could not solve piecewise linear and sparse problems (problems 57 and 58 in Part II, Chap. 9); that is, ten problems from the convex test set. These difficulties

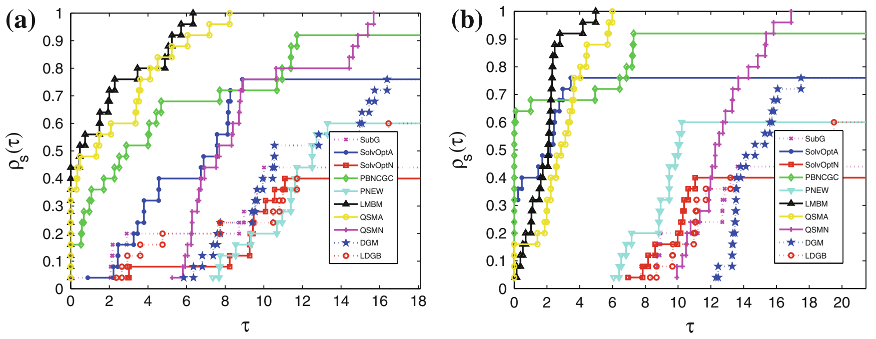

Fig. 17.8 Time and evaluations for LNC problems with relaxed tolerance (25 pcs., n = 1,000, $\varepsilon = 10^{-2}$). (**a**) CPU-time. (**b**) Evaluations

experienced by LMBM are easy to explain: the approximation of the Hessian formed during the calculations is dense and, naturally, not even close to the real Hessian in sparse problems. It has been reported [98] that LMBM is best suited for problems with a dense subgradient vector whose component depends on the current iteration point. This result is in line with the noted result that LMBM solves nonconvex problems very efficiently. As a derivative of LMBM, the derivative free solver LDGB seems to share both the weaknesses and strengths of LMBM.

In addition to LMBM, PBNCGC was also often the most efficient solver tested: in 36 % of LC and 40 % of LNC problems). However, PBNCGC was also the solver that needed the longest time to compute some of the problems. Indeed, PBNCGC used the whole time limit in all 59, 60 and 64 problems, regardless of the starting point (the results obtained were within the desired accuracy). The efficiency of PBNCGC is mostly due to its efficiency in piecewise linear (LC) and piecewise quadratic (LNC) problems: PBNCGC was superior in solving piecewise linear problems and the most efficient solver in almost all quadratic problems, in terms of both computational times and numbers of evaluations.

In addition, solver QSMA solved all of the problems quite efficiently but, as already stated, it had some difficulties with accuracy in the LNC case. However, in the LC case, QSMA was the only solver that succeeded in solving all problems with the desired accuracy. In fact, this also remains true with the relaxed tolerance. In the LNC case, none of the solvers succeeded in solving all of the problems with the desired accuracy. However, with the relaxed tolerance LMBM, QSMA and QSMN managed to solve all of the problems.

Solvers PBNCGC, QSM(A+N) and DGM were the only solvers that solved those LC problems in which there is only one nonzero element in the subgradient vector (i.e. Problem 57 Part II, Chap. 9). With the other methods, there were already some difficulties with n = 200 (note that for S, M, L and XL settings, the problems are the same, only the number of variables and the starting point changes). Furthermore, solvers DGM, LMBM, SubG and QSMN failed to solve (possibly in addition to the five

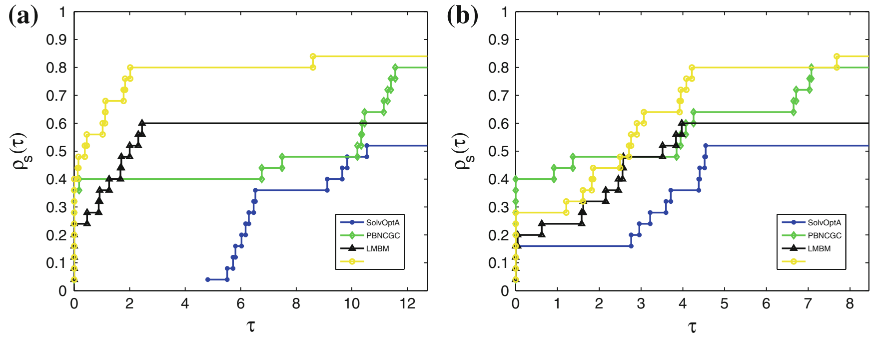

Fig. 17.9 Time and evaluations for XLC problems (25 pcs., $n = 4{,}000$, $\varepsilon = 10^{-3}$). (**a**) CPU-time. (**b**) Evaluations

abovementioned problems) five piecewise linear problems (Problem 58 in Part II, Chap. 9) and also SolvOptN failed to solve some of them.

Naturally, for the solvers using difference approximation, or some other approximation based on the calculation of the function or subgradient values, the number of evaluations and, thus, also the computational time, grows enormously when the number of variables increases. In particular, in L problems, the time limit was exceeded quite often by all of these solvers. Therefore, the number of failures with these solvers is probably a little larger than it would be without the time limit. However, in almost all cases, the results obtained were still far from the optima, indicating very long computational times.

17.4.5 Extra Large Problems

Finally we tested the most efficient solvers so far, that is SolvOptA, PBNCGC, LMBM, and QSMA, with the problems with $n = 4{,}000$.

In the convex case, solver QSMA, which had kept a rather low profile until that point, was clearly the most efficient method, although PBNCGC still usually used the fewest evaluations (see Fig. 17.9). QSMA was also the most reliable of the solvers tested. In the nonconvex case. LMBM and QSMA were approximately as good in terms of computational times, evaluations and reliability (see Fig. 17.10). Here, PBNCGC was the most reliable solver, although with the tolerance $\varepsilon = 10^{-2}$. QSMA was the only solver that solved all of the problems (both XLC and XLNC).

Both QSMA and LMBM solved all of the XL problems that they could solve in a relatively short time, while with the other solvers there was a wide variation in the computational times that elapsed for different problems. However, the efficiency of LMBM was again ruined by its unreliability in the piecewise linear and sparse problems.

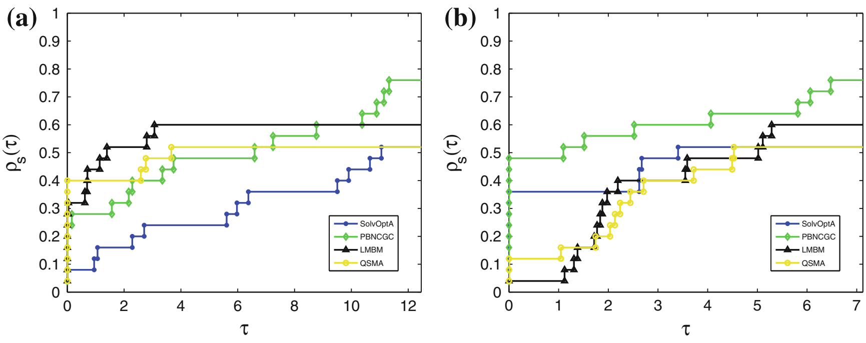

Fig. 17.10 Time and evaluations for XLNC problems (25 pcs., n = 4,000, $\varepsilon = 10^{-3}$). (**a**) CPU-time. (**b**) Evaluations

In XL problems (50 problems in total), `SolvOptA` stopped five times (with very far from optimum results), `QSMA` stopped four times (all results were a little inaccurate), and `LMBM` stopped two times (in piecewise linear problems with both results far from the optima), because of the time limit. `PBNCGC` stopped as many as 15 times. However, ten of these terminations were within the desired accuracy, indicating that `PBNCGC` could be more efficient in terms of the computational time used if it knew when to stop.

17.4.6 Convergence Speed and Iteration Path

In this subsection, we study (experimentally) the convergence speed and iteration paths of the solvers. The convergence speed is studied using one small-scale convex problem (Problem 59 in Part II, Chap. 9). The exact minimum value for this function (with $n = 50$) is $-49 \times 2^{1/2} \approx -69.296$.

The rate of convergence for the limited memory bundle method has not been studied theoretically. However, at least in this particular problem, solvers `LMBM` and `PBNCGC` converged at approximately the same rate. Moreover, if we study the number of evaluations, `PBNCGC` and `LMBM` seem to have the fastest convergence speed of the solvers tested (see Fig. 17.11b), although, theoretically, the proximal bundle method is at most linearly convergent.

`SubG` converged linearly but extremely slowly. With `PNEW`, a large number of subgradient evaluations is needed to compute the approximate Hessian. Although `PNEW` finally found the minimum, it did not decrease the value of the function in the first 200 evaluations. Solvers `SolvOptA`, `SolvOptN`, `DGM`, `QSMA`, and `QSMN` already took a very big downwards step in iteration two (see Fig. 17.11a). However, they, as well as `LDGB`, used quite a few function evaluations per iteration. In Fig. 17.11, it is easy to see that Shor's r-algorithm (i.e. solvers `SolvOptA` and `SolvOptN`) is not a descent method.

In order to see how quickly the solvers reach some specific level, we studied the value of the function equal to -69. With PBNCGC, it took only 8 iterations to go below that level. The corresponding values for other solvers were 17 with QSMA and QSMN, 20 with LMBM and PNEW, and more than 20 with all of the other solvers. In terms of function and subgradient evaluations, the values were 18 with PBNCGC, 64 with LMBM, and 133 with SolvOptA. Other solvers needed more than 200 evaluations to go below -69. The worst of the solvers were SubG and SolvOptN. SubG took 7,382 iterations and 14,764 evaluations to reach the desired accuracy and stop, while SolvOptN never reached the desired accuracy. The final value obtained after 42 iterations and 2,342 evaluations was -68.915.

The iteration paths are studied using an extra small nonconvex problem *crescent* (Problem 21 in Part II, Chap. 9). The exact minimum value for this function is zero at the point $(0, 0)$ and the starting point used is $(-1.5, 2)$. In Figs. 17.12 and 17.13 we have drawn the contour plot of the crescent function and the iteration paths of different solvers.

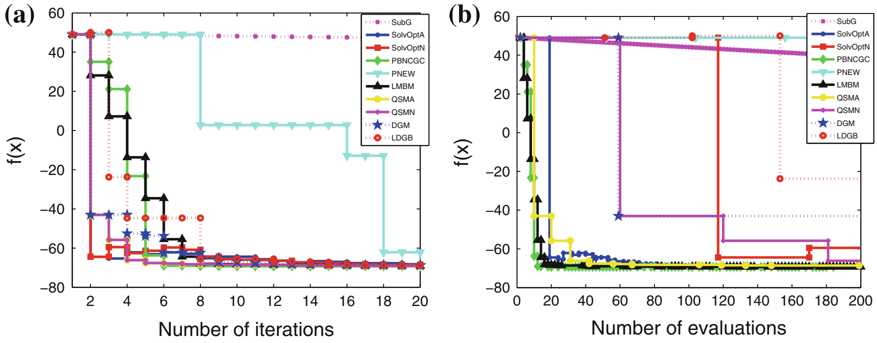

Fig. 17.11 (**a**) function values versus 20 first iterations, (**b**) function values versus 200 first function and subgradient evaluations

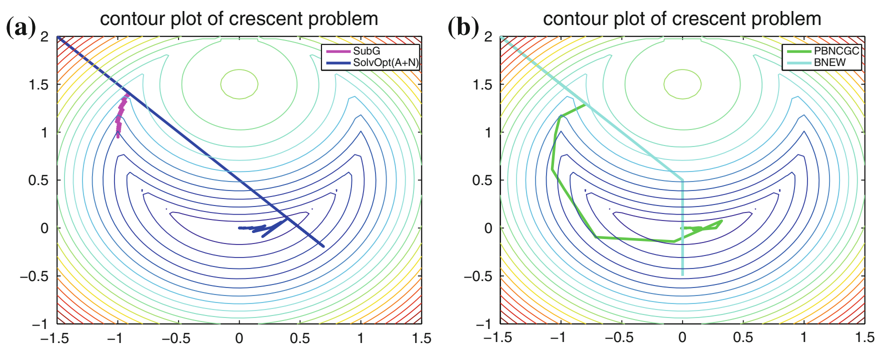

Fig. 17.12 Iteration paths for (**a**) subgradient methods subG and SolvOpt(A+N), and (**b**) bundle methods PBNCGC and PNEW

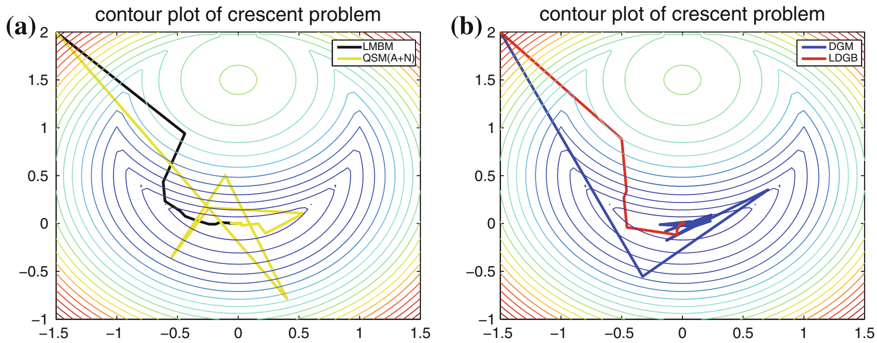

Fig. 17.13 Iteration paths for (**a**) hybrid methods LMBM and QSM(A+N), and (**b**) discrete gradient methods DGM and LDGB

All of the solvers have different iteration paths, except for those that are different versions of the same method: that is, SolvOptA and SolvOptN, and QSMA and QSMN. However, the first search direction is the same for subgradient methods, bundle methods and LMBM. The differences occur at the step sizes and then in the next search directions. All of the solvers except SubG converged to the minimum of the problem. SubG was terminated after 500 iterations while the current iteration point was still very far from optimum (see Fig. 17.12a). The long step sizes taken by QSM(A+N) and DGM (see Fig. 17.13a, b) may explain the tendency of these solvers to jump over the narrow local minima. However, a more precise line search could result in faster convergence.

17.5 Conclusions

We tested the performance of different NSO solvers in solving different nonsmooth problems. The results are summarized in Table 17.3, where we give our recommendations for the "best" solver for different classes of problems. Since the best result might be ambiguous, we give credentials in both cases where the most efficient, in terms of used CPU time, or the most reliable, solver is sought. When there is more than one recommendation in Table 17.3, the solvers are given in alphabetical order. The parentheses in the table mean that the solver is not exactly as good as the first one but is still a possible solver for using with that class of problems.

Although we obtained some extremely good results with the proximal bundle solver PBNCGC, we cannot say that it is clearly the best method tested. The inaccuracy in XSNC problems, great variations in computational time in larger problems, and the earlier results make us believe that our test set favored this solver over the others to a certain extent. PBNCGC is especially efficient for piecewise quadratic and piecewise linear problems. Moreover, PBNCGC usually used the fewest number of evaluations

Table 17.3 Summation of the results

Problem type	Problem size	Seeking for efficiency	Seeking for reliability
Convex	XS	PBNCGC, PNEW[a], (SolvOpt(A+N))	DGM, QSM(A+N)
	S, M, L	LMBM[b], PBNCGC, (QSMA, SolvOptA)	PBNCGC, QSMA
	XL	LMBM[b], QSMA	QSMA, (PBNCGC)
Nonconvex	XS	PBNCGC, SolvOpt(A+N), (QSMA)	SolvOpt(A+N), (DGM, QSM(A+N))
	S	LMBM, (PBNCGC,SolvOptA)	DGM, LMBM, (PBNCGC)
	M, L	LMBM, (PBNCGC, QSMA)	LMBM, PBNCGC, (DGM)
	XL	LMBM, QSMA	PBNCGC
Piecewise linear or sparse	XS, S	PBNCGC, QSMA, SolvOptA	PBNCGC, QSM(A+N), SolvOptA
	M, L, XL	PBNCGC, QSMA[c]	PBNCGC, QSMA
Piecewise quadratic	XS	PBNCGC, PNEW[a], (LMBM, SolvOptA)	LMBM, PBNCGC, PNEW[a], SolvOptA
	S, M, L, XL	LMBM[b], PBNCGC, (QSMA)	DGM, LMBM, PBNCGC, QSMA[d]
Highly nonlinear	XS	LMBM, PBNCGC, SolvOptA	LMBM, QSMA, SolvOptA
	S	LMBM, PBNCGC	LMBM, PBNCGC, QSMA[d]
	M, L, XL	LMBM	LMBM, QSMA[d]
Function evaluations are expensive	XS	PBNCGC, (PNEW[a], SolvOptA)	QSMA, SolvOptA
	S, M, L, XL	PBNCGC, (LMBM[e], SolvOptA)	PBNCGC, (LMBM[e], QSMA[d])
Subgradients are not available	XS	LDGB, SolvOptN	DGM, QSMN, SolvOptN
	S, M	LDGB, (SolvOptN, QSMN)	DGM, (LDGB[e], QSMN[f])
	L	LDGB[e], QSMN[f], DGM	DGM, QSMN

[a] PNEW may require tuning of internal parameter XMAX
[b] LMBM, if not a piecewise linear or sparse problem
[c] PBNCGC in piecewise linear problems, QSMA in other sparse problems
[d] QSMA in the convex case
[e] LMBM especially in the nonconvex case
[f] QSMN in the convex case

for problems of any size. Thus, it should be a good choice for a solver when the objective function value and/or the subgradient are expensive to compute.

The limited memory bundle solver LMBM suffered from ill-fitting test problems in the convex S, M, L and XL cases. In the test set, LMBM was known to have difficulties in 10 problems (out of 25). LMBM was quite reliable in the nonconvex cases with all numbers of variables tested, and it solved all of the problems—even the largest ones—in a relatively short time. LMBM works best for (highly) nonlinear functions.

In convex XS problems, the bundle-Newton solver PNEW was the second most efficient solver tested. However, PNEW suffers greatly from the fact that it is very sensitive to the choice of internal parameter XMAX. A light tuning of this parameter (e.g. using a default value XMAX $=$ 1,000 and subsequently the smallest recommended value XMAX $=$ 2 and then choosing the better result) would have yielded better results and, in particular, the degree of success would have been much higher. In [166], PNEW is reported to be very efficient in quadratic problems. Furthermore, in our experiments it solved (nonconvex) piecewise quadratic problems faster than non-quadratic ones. However, apart from some XS problems, it did not beat the other solvers in these problems due to the large approximation of the Hessian matrix required.

The standard subgradient solver SubG is usable only for XSC problems. In addition, the implementations of Shor's r-algorithm SolvOptA and SolvOptN did their best in XS problems both in convex and nonconvex settings. Nevertheless, SolvOptA also solved S, M, L, and even XL problems (convex) rather efficiently. In larger nonconvex problems, both of these methods suffered from inaccuracy.

The quasi-secant solver QSMA was uniformly efficient with all sized problems. Moreover, it was clearly the most reliable solver tested in convex settings: it failed to achieve the desired accuracy only in XLC problems. With nonconvex problems, QSMA had some difficulties with accuracy (with almost all sizes of problems). Thus, when comparing the reliability in M, L, and XL settings, it seems that one should select QSMA for convex problems, while LMBM is good for nonconvex problems. On the other hand, PBNCGC is rather reliable for both convex and nonconvex problems.

The solvers using discrete gradients, that is the discrete gradient solver DGM, the limited memory discrete gradient bundle solver LDGB, and the quasi-secant solver with discrete gradients QSMN, usually lost out in efficiency to the solvers that used analytical subgradients. However, in XS and S problems, the differences were not significant and, indeed, LDGB competed successfully with the solvers using subderivative information when comparing computational times. Furthermore, the reliability of DGM and QSMN seems to be very good with both convex and nonconvex XS and S problems. In larger cases, the usage of a method employing subgradients is, naturally, recommended. Nevertheless, if one needs to solve a problem where the subgradient is not available, the best solver would probably be LDGB or QSMN (convex case) because of their efficiency, or DGM because of its reliability. Moreover in the case of highly nonconvex functions (supposing that you seek for global optimum), DGM or QSM (either with or without subgradients) would be a good choice, since these methods tend to jump over the narrow local minima.

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