Frames and Extension Problems II

Ole Christensen, Hong Oh Kim, and Rae Young Kim

Abstract This article is a follow-up on the article *Frames and Extension Problems I.* Here we will go into more recent progress on the topic and also present some open problems.

Keywords Frames • Gabor systems • Wavelet systems • Extension problems

1 Introduction

Based on the article *Frames and Extension Problems I,* see [3], we discuss recent progress and open problems concerning extension of Bessel sequences to frames and dual pairs of frames. We first consider the extension problem in general Hilbert spaces in Sect. 2. The special case of Gabor frames is discussed in Sect. 3. In Sect. 4 the similar (but much more complicated) problem for wavelet systems is considered, without use of any assumption of multiresolution structure. Finally, in Sect. 5, we present a few recent results about extension of wavelet Bessel systems to frames with two or three generators. These results use the multiresolution structure.

O. Christensen
Department of Mathematics and Computer Science, Technical University of Denmark, Building 303, 2800 Lyngby, Denmark
e-mail: ochr@dtu.dk

H.O. Kim (✉)
Division of General Studies, UNIST-gil 50, Ulsan 689-798, Republic of Korea
e-mail: kimhong@kaist.edu

R.Y. Kim
Department of Mathematics, Yeungnam University, 214-1, Dae-dong, Gyeongsan-si, Gyeongsangbuk-do 712-749, Republic of Korea
e-mail: rykim@ynu.ac.kr

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2 The Extension Problem in Hilbert Spaces

Extension problems have a long history in frame theory. It has been shown by several authors (see, e.g., [1,14]) that for any Bessel sequence $\{f_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$, there exists a sequence $\{g_k\}_{k=1}^{\infty}$ such that $\{f_k\}_{k=1}^{\infty} \cup \{g_k\}_{k=1}^{\infty}$ is a tight frame for $\mathcal{H}$. A natural generalization to construction of dual frame pairs appeared in [4]; we need to refer to the proof later, so we include it here as well.

Theorem 2.1. *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in $\mathcal{H}$. Then there exist Bessel sequences $\{p_j\}_{i\in J}$ and $\{q_j\}_{i\in J}$ in $\mathcal{H}$ such that $\{f_i\}_{i\in I} \cup \{p_j\}_{i\in J}$ and $\{g_i\}_{i\in I} \cup \{q_j\}_{i\in J}$ form a pair of dual frames for $\mathcal{H}$.*

Proof. Let T and U denote the preframe operators for $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively, i.e.,

$$T, U : \ell^2(I) \to \mathcal{H},\ T\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i,\ \ U\{c_i\}_{i\in I} = \sum_{i\in I} c_i g_i.$$

Let $\{a_j\}_{i\in J}, \{b_j\}_{i\in J}$ denote any pair of dual frames for $\mathcal{H}$. Then

$$\begin{aligned} f = UT^*f + (I - UT^*)f &= \sum_{i\in I}\langle f, f_i\rangle g_i + \sum_{j\in I}\langle (I - UT^*)f, a_j\rangle b_j \\ &= \sum_{i\in I}\langle f, f_i\rangle g_i + \sum_{j\in I}\langle f, (I - UT^*)^* a_j\rangle b_j \end{aligned}$$

The sequences $\{f_i\}_{i\in I}, \{g_i\}_{i\in I}$, and $\{b_j\}_{i\in J}$ are Bessel sequences by definition, and one can verify that $\{(I - UT^*)^* a_j\}_{j\in J}$ is a Bessel sequence as well. The result now follows from Lemma 2.2 in [3]. □

The reason for the interest in this more general version of the frame extension is that it often is possible to construct dual pairs of frames with properties that are impossible for tight frames. This is illustrated in the next section.

3 The Extension Problem for Gabor Frames

Li and Sun showed in [14] that if $ab \le 1$ and $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ is a Bessel sequences in $L^2(\mathbb{R})$, then there exists a Gabor systems $\{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ such that $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$. However, if we ask for extra properties of the functions g_1 and g_2 such an extension might be impossible. For example, if the given function g_1 has compact support, it is natural to ask for the function g_2 having compact support as well, but by Li and Sun [14] the existence of such a function is only guaranteed if $|\mathrm{supp} g_1| \le b^{-1}$. On the other hand, such an extension can always be obtained in the setting of dual frame pairs [4]:

Theorem 3.1. *Let* $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}}$ *be Bessel sequences in* $L^2(\mathbb{R})$, *and assume that* $ab \leq 1$. *Then the following hold:*

(i) There exist Gabor systems $\{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h_2\}_{m,n\in\mathbb{Z}}$ *in* $L^2(\mathbb{R})$ *such that*

$$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}} \text{ and } \{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}h_2\}_{m,n\in\mathbb{Z}}$$

form a pair of dual frames for $L^2(\mathbb{R})$.

(ii) If g_1 *and* h_1 *have compact support, the functions* g_2 *and* h_2 *can be chosen to have compact support.*

Proof. Let us give the proof of (i). Let T and U denote the preframe operators for $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}}$, respectively. Then

$$UT^*f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g_1\rangle E_{mb}T_{na}h_1.$$

Consider the operator $\Phi := I - UT^*$, and let $\{E_{mb}T_{na}r_1\}_{m,n\in\mathbb{Z}}$, $\{E_{mb}T_{na}r_2\}_{m,n\in\mathbb{Z}}$ denote any pair of dual frames for $L^2(\mathbb{R})$. By the proof of Theorem 2.1, $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{\Phi^* E_{mb}T_{na}r_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}r_2\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$. By Lemma 2.6 in [3] we know that Φ^* commutes with the time-frequency shift operators $E_{mb}T_{na}$. This concludes the proof. □

4 An Extension Problem for Wavelet Frames

It turns out that the extension problem for wavelet systems is considerably more involved than for Gabor systems. In order to explain this, consider the proof of Theorem 2.1 and assume that $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ have wavelet structure, i.e., $\{f_i\}_{i\in I} = \{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{g_i\}_{i\in I} = \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ for some $\psi_1, \widetilde{\psi_1} \in L^2(\mathbb{R})$. Assume further that these sequences are Bessel sequences, with preframe operators T, U, respectively. Then, still referring to the proof of Theorem 2.1, $(I - UT^*)^*a_j = (I - TU^*)a_j$. Unfortunately the operator TU^* in general does not commute with D^jT_k, so even if we choose $\{a_j\}_{i\in J}$ to have wavelet structure, the system $\{(I - TU^*)a_j\}_{j\in J}$ might not be a wavelet system. Thus, we cannot apply the proof technique from the Gabor case. The following partial result was obtained in [4].

Theorem 4.1. *Let* $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ *and* $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ *be Bessel sequences in* $L^2(\mathbb{R})$. *Assume that the Fourier transform of* $\widetilde{\psi_1}$ *satisfies*

$$supp\ \widehat{\widetilde{\psi_1}} \subseteq [-1, 1]. \tag{1}$$

Then there exist wavelet systems $\{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}}$ *and* $\{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$ *such that*

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$$

form dual frames for $L^2(\mathbb{R})$. *If we further assume that* $\widehat{\psi_1}$ *is compactly supported and that*

$$supp\ \widehat{\widetilde{\psi_1}} \subseteq [-1, 1] \setminus [-\epsilon, \epsilon]$$

for some $\epsilon > 0$, *the functions* ψ_2 *and* $\widetilde{\psi_2}$ *can be chosen to have compactly supported Fourier transforms as well.*

In the Gabor case, no assumption of compact support was necessary, neither for the given functions nor their Fourier transform. From this point of view it is natural to ask whether the assumption (1) is necessary in Theorem 4.1.

Question: Let $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$.

(i) Do there exist functions $\psi_2, \widetilde{\psi_2} \in L^2(\mathbb{R})$ such that

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}} \quad (2)$$

form dual frames for $L^2(\mathbb{R})$?

(ii) If $\widehat{\psi_1}$ and $\widehat{\widetilde{\psi_1}}$ are compactly supported, can we find compactly supported functions ψ_2 and $\widetilde{\psi_2} \in L^2(\mathbb{R})$ such that the functions in (2) form dual frames?

The problem (i) can also be formulated in the negative way: can we find just one example of a pair of Bessel sequences $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ that cannot be extended to a pair of dual wavelet frames, each with two generators? The open question is strongly connected to the following conjecture by Han [8]:

Conjecture by Deguang Han. *Let* $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ *be a wavelet frame with upper frame bound* B. *Then there exists* $D > B$ *such that for each* $K \geq D$, *there exists* $\widetilde{\psi_1} \in L^2(\mathbb{R})$ *such that* $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ *is a tight frame for* $L^2(\mathbb{R})$ *with bound* K.

The paper [8] contains an example showing that (again in contrast with the Gabor setting) it might not be possible to extend the Bessel system $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ to a tight frame without enlarging the upper bound; hence it is essential that the conjecture includes the option that the extended wavelet system has a strictly larger frame bound than the upper frame bound B for $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$. We also note that Han's conjecture is based on an example where supp $\widehat{\psi_1} \subseteq [-1, 1]$, i.e., a case that is covered by Theorem 4.1.

Observe that a pair of wavelet Bessel sequences always can be extended to dual wavelet frame pairs by adding *two* pairs of wavelet systems. In fact, we can always add one pair of wavelet systems that cancels the action of the given wavelet system, and another one that yields a dual pair of wavelet frames by itself. Thus, the issue is really whether it is enough to add one pair of wavelet systems, as stated in the formulation of the open problem.

Note that extension problems have a long history in frame theory. Most of the results deal with the unitary extension principle (UEP) [16, 17] and its variants, and are thus based on the assumption of an underlying refinable function. The open problems formulated in this section are not based on such an assumption.

5 Extension Problems via the UEP

In this section we present recent results from [5]; more information and examples can be found there. We will consider the extension problem for wavelet systems in $L^2(\mathbb{R})$ that are generated from the UEP by Ron and Shen. That is, we consider wavelet system $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ generated from a given scaling function and characterize the existence of a UEP-type wavelet system $\{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}}$ generated by the same scaling function, such that the system $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}}$ forms a Parseval frame for $L^2(\mathbb{R})$, i.e., a tight frame with frame bound 1. In the process of doing so, we identify two conditions on the filters associated with the scaling function and with ψ_1, which are necessary for any extension of $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ to a tight UEP-type frame with any number of generators. Interestingly, we are able to show that these conditions imply that we can always construct a Parseval frame by adding *at most two* wavelet systems.

Let $\mathbb{T}$ denote the unit circle which will be identified with $[-1/2, 1/2]$. Also, for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we denote the Fourier transform by $\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx$. As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$.

In the rest of the paper we will use the following setup.

General Setup. Consider a *scaling function* $\varphi \in L^2(\mathbb{R})$, i.e., a function such that $\hat{\varphi}$ is continuous at the origin and $\hat{\varphi}(0) = 1$, and there exists a function $m_0 \in L^\infty(\mathbb{T})$ (called a *refinement mask*) such that $\hat{\varphi}(2\gamma) = m_0(\gamma)\hat{\varphi}(\gamma), \quad$ a.e.$\gamma \in \mathbb{R}$. Given functions $m_1, m_2, \ldots, m_n \in L^\infty(\mathbb{T})$, consider the functions $\psi_\ell \in L^2(\mathbb{R})$ defined by

$$\widehat{\psi_\ell}(2\gamma) = m_\ell(\gamma)\hat{\varphi}(\gamma), \ \ell = 1, \ldots, n. \tag{3}$$

In the classical UEP-setup by Ron and Shen, one search for functions $m_1, m_2, \ldots, m_n \in L^\infty(\mathbb{T})$ such that

$$\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}} \cup \cdots \cup \{D^j T_k \psi_n\}_{j,k\in\mathbb{Z}}$$

is a Parseval frame. We will modify this slightly. In fact, we will consider a given refinement mask m_0 and a given filter $m_1 \in L^\infty(\mathbb{T})$, and derive equivalent conditions for the existence of appropriate functions $m_2, \ldots, m_n \in L^\infty(\mathbb{T})$ for the cases $n = 2$ and $n = 3$.

We will base the analysis on the *UEP*, which is formulated in terms of the $(n+1)\times 2$ matrix-valued function M defined by

$$M(\gamma) = \begin{pmatrix} m_0(\gamma) & m_0\left(\gamma+\frac{1}{2}\right) \\ m_1(\gamma) & m_1\left(\gamma+\frac{1}{2}\right) \\ \vdots & \vdots \\ m_n(\gamma) & m_n\left(\gamma+\frac{1}{2}\right) \end{pmatrix}. \tag{4}$$

Proposition 5.1 (UEP by Ron and Shen [16]). *Let $\varphi \in L^2(\mathbb{R})$ be a scaling function and $m_0 \in L^\infty(\mathbb{T})$ the corresponding refinement mask. For each $\ell = 1,\cdots,n$, let $m_\ell \in L^\infty(\mathbb{T})$, and define $\psi_\ell \in L^2(\mathbb{R})$ by (3). If the corresponding matrix-valued function M satisfies*

$$M(\gamma)^* M(\gamma) = I,\ a.e.\ \gamma \in \mathbb{T}, \tag{5}$$

then $\{D^j T_k \psi_i : j \in \mathbb{Z}, k \in \mathbb{Z}, 1 \le i \le n\}$ is a Parseval frame for $L^2(\mathbb{R})$.

With the additional constraint that the generating functions should be symmetric, the issue of constructing Parseval wavelet frames with two or three generators has attracted quite some attention in the literature, see, e.g., the papers [15] by Petukhov, [13] by Jiang, [18] by Selesnick and Abdelnour, and the papers [11, 12] by Han and Mo. For example, in the paper [11] B-splines were used as scaling functions, while a more general approach, valid for real-valued, compactly supported, and symmetric scaling functions, was provided in [12]. Other cases where a UEP-based construction with n generators can be modified to a Parseval frame with two or three generators have been considered in [6, 7]. These papers are based on the so-called oblique extension principle, which is known to be equivalent to the UEP. However, a characterization of the conditions that ensure the possibility of extension with two or three generators, as provided in the current paper, has not been available before.

Note that the analysis in the current paper is complementary to the one in Sect. 4, where the key condition for obtaining an extension of a (general) wavelet system $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ to a tight frame of the same form is that $\widehat{\psi_1}$ is compactly supported. The extension principle applied in the current paper usually involves functions that are compactly supported in time (even though this is not strictly necessary).

In the current paper we have restricted our attention to wavelet systems in $L^2(\mathbb{R})$. An interesting discussion of the complexity of the extension problem for wavelet systems in higher dimensions, together with several deep results, recently appeared in [2].

In the rest of the paper we assume that we have given functions $m_0, m_1 \in L^\infty(\mathbb{R})$ as described in the general setup. Associated with functions $m_2,\cdots,m_n \in L^\infty(\mathbb{T})$, we consider the $(n-1)\times 2$ matrix-valued function $M_{2,n}$ defined by

$$M_{2,n}(\gamma) = \begin{pmatrix} m_2(\gamma) & m_2\left(\gamma + \frac{1}{2}\right) \\ \vdots & \vdots \\ m_n(\gamma) & m_n\left(\gamma + \frac{1}{2}\right) \end{pmatrix}.$$

Note that

$$\begin{aligned}
&M_{2,n}(\gamma)^* M_{2,n}(\gamma) \\
&= \begin{pmatrix} \overline{m}_2(\gamma) & \cdots & \overline{m}_n(\gamma) \\ \overline{m}_2(\gamma + 1/2) & \cdots & \overline{m}_n(\gamma + 1/2) \end{pmatrix} \begin{pmatrix} m_2(\gamma) & m_2\left(\gamma + \frac{1}{2}\right) \\ \vdots & \vdots \\ m_n(\gamma) & m_n\left(\gamma + \frac{1}{2}\right) \end{pmatrix} \\
&= M(\gamma)^* M(\gamma) - \begin{pmatrix} \overline{m}_0(\gamma) & \overline{m}_1(\gamma) \\ \overline{m}_0(\gamma + 1/2) & \overline{m}_1(\gamma + 1/2) \end{pmatrix} \begin{pmatrix} m_0(\gamma) & m_0(\gamma + 1/2) \\ m_1(\gamma) & m_1(\gamma + 1/2) \end{pmatrix} \qquad (6) \\
&= M(\gamma)^* M(\gamma) - \\
&\begin{pmatrix} |m_0(\gamma)|^2 + |m_1(\gamma)|^2 & \overline{m_0(\gamma)} m_0(\gamma + 1/2) + \overline{m_1(\gamma)} m_1(\gamma + 1/2) \\ \overline{m_0(\gamma + 1/2)} m_0(\gamma) + \overline{m_1(\gamma + 1/2)} m_1(\gamma) & |m_0(\gamma + 1/2)|^2 + |m_1(\gamma + 1/2)|^2 \end{pmatrix}
\end{aligned}$$

We define

$$M^{\alpha,\beta}(\gamma) := \begin{pmatrix} M_\alpha(\gamma) & \overline{M}_\beta(\gamma) \\ M_\beta(\gamma) & M_\alpha(\gamma + 1/2) \end{pmatrix}, \qquad (7)$$

where

$$\begin{aligned}
M_\alpha(\gamma) &:= 1 - |m_0(\gamma)|^2 - |m_1(\gamma)|^2; \\
M_\beta(\gamma) &:= -m_0(\gamma)\overline{m}_0(\gamma + 1/2) - m_1(\gamma)\overline{m}_1(\gamma + 1/2).
\end{aligned}$$

Then the above calculation shows that

$$M(\gamma)^* M(\gamma) = I \Leftrightarrow M_{2,n}(\gamma)^* M_{2,n}(\gamma) = M^{\alpha,\beta}(\gamma). \qquad (8)$$

The following lemma gives two necessary conditions for the existence of $m_2, \cdots, m_n$ such that the equivalent conditions in (8) hold.

Lemma 5.2. *Suppose that $m_0, m_1, \cdots, m_n \in L^\infty(\mathbb{T})$ satisfy that $M(\gamma)^* M(\gamma) = I$ for a.e. $\gamma \in \mathbb{T}$, then the Hermitian matrix $M^{\alpha,\beta}(\gamma)$ is positive semidefinite and*

(a) $|m_0(\gamma)|^2 + |m_1(\gamma)|^2 \leq 1,\ a.e.\ \gamma \in \mathbb{T}$;
(b) $M_\alpha(\gamma) M_\alpha(\gamma + 1/2) \geq \left|M_\beta(\gamma)\right|^2,\ a.e.\ \gamma \in \mathbb{T}$.

On the other hand, if (a) and (b) are satisfied then $M^{\alpha,\beta}(\gamma)$ is positive semidefinite.

We are now ready to state the condition for extension to a UEP-type wavelet system $\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}}$ to a Parseval frame by adding just one UEP-type wavelet system.

Theorem 5.3. *Let $\varphi \in L^2(\mathbb{R})$ be a scaling function and $m_0 \in L^\infty(\mathbb{T})$ the corresponding refinement mask. Let $m_1 \in L^\infty(\mathbb{T})$, and define $\psi_1 \in L^2(\mathbb{R})$ by* (3)*. Assume that condition (a) in Lemma 5.2 is satisfied. Then the following are equivalent:*

(a) *There exists a 1-periodic function m_2 such that the matrix-valued function M in* (4) *with $n = 2$ satisfies that*

$$M(\gamma)^* M(\gamma) = I, \; a.e. \; \gamma \in \mathbb{T}; \tag{9}$$

(b) $M_\alpha(\gamma) M_\alpha(\gamma + 1/2) = M_\beta(\gamma) M_\beta(\gamma + 1/2)$.

In the affirmative case, the multi-wavelet system $\{D^j T_k \psi_l\}_{l=1,2;j,k\in\mathbb{Z}}$, with ψ_2 defined by (3)*, forms a Parseval frame for $L^2(\mathbb{R})$.*

If the necessary conditions in Lemma 5.2 are satisfied, then we can always extend $\{D^j T_k \psi_l\}_{j,k\in\mathbb{Z}}$ to a Parseval wavelet frame by adding *two* wavelet systems:

Theorem 5.4. *Let $\varphi \in L^2(\mathbb{R})$ be a scaling function and $m_0 \in L^\infty(\mathbb{T})$ the corresponding refinement mask. Let $m_1 \in L^\infty(\mathbb{T})$, and define $\psi_1 \in L^2(\mathbb{R})$ by* (3)*. Assume that the functions m_0, m_1 satisfy (a) and (b) in Lemma 5.2. Then there exist $m_2, m_3 \in L^\infty(\mathbb{T})$ such that $\{D^j T_k \psi_l\}_{l=1,2,3;j,k\in\mathbb{Z}}$, with ψ_2, ψ_3 defined by* (3)*, forms a Parseval frame.*

Note that Theorem 5.4 is related with Theorem 1.2 in [12], where it is shown that certain conditions on a scaling function imply the existence of three functions that generate a Parseval wavelet frame. However, the spirit of these two results is different: while the goal of Theorem 1.2 in [12] is to provide sufficient conditions for wavelet constructions that have attractive properties from the point of applications (i.e., symmetry properties and a high number of vanishing moments), the purpose of our result is to guarantee the existence of three functions generating a Parseval frame under the weakest possible conditions. We also note that for the case where the refinement mask m_0 is a trigonometric polynomial, the problem of characterizing associated Parseval frames generated by two or three symmetric functions has been solved in [9, 10].

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