Frames and Extension Problems I

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Abstract In this article we present a short survey of frame theory in Hilbert spaces. We discuss Gabor frames and wavelet frames and set the stage for a discussion of various extension principles; this will be presented in the article *Frames and extension problems II* (joint with H.O. Kim and R.Y. Kim).

Keywords Frames • Gabor systems • Wavelet systems • Extension problems

1 Introduction

Frames provide us with a convenient tool to obtain expansions in Hilbert spaces of a similar type as the one that arise via orthonormal bases. However, the frame conditions are significantly weaker, which makes frames much more flexible. For this reason frame theory has attracted much attention in recent years, especially in connection with its concrete manifestations within Gabor analysis and wavelet analysis.

In this article we give a short overview of the general theory for frames in Hilbert spaces, as well as its concrete realizations in Gabor analysis and wavelet analysis. We set the stage for a discussion of various extension principles to be presented in the article *Frames and extension problems II* (joint paper with H.O. Kim and R.Y. Kim).

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2 A Survey on Frames and Operators

General frames were introduced already in the paper [18] by Duffin and Schaeffer in 1952. Apparently it did not find much use at that time, until it got re-introduced by Young in his book [31] from 1982. After that, Daubechies, Grossmann and Morlet took the key step of connecting frames with wavelets and Gabor systems in the paper [15].

2.1 General Frame Theory

Let $\mathcal{H}$ be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. A countable family of elements $\{f_k\}_{k\in I}$ in $\mathcal{H}$ is a

1. *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B||f||^2, \ \forall f \in \mathcal{H};$$

2. *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}; \tag{1}$$

   The numbers A, B in (1) are called *frame bounds*.
3. *Riesz basis* for $\mathcal{H}$ if $\overline{\text{span}}\{f_k\}_{k\in I} = \mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A \sum |c_k|^2 \leq \left|\left|\sum c_k f_k\right|\right|^2 \leq B \sum |c_k|^2. \tag{2}$$

   for all finite sequences $\{c_k\}$.

Every orthonormal basis is a Riesz basis, and every Riesz basis is a frame [the bounds A, B in (2) are frame bounds]; a frame which is not a Riesz basis is said to be *overcomplete* or *redundant*. Riesz bases and frames are natural tools to gain more flexibility than possible with an orthonormal basis. For an overview of the general theory for frames and Riesz bases we refer to [2] and [3]; a deeper treatment is given in the books [4, 6]. Here, we just mention that the difference between a Riesz basis and a frame is that the elements in a frame might be dependent. More precisely, a frame $\{f_k\}_{k\in I}$ is a Riesz basis if and only if

$$\sum_{k\in I} c_k f_k = 0, \ \{c_k\} \in \ell^2(I) \Rightarrow c_k = 0, \ \forall k \in I.$$

Associated with a Bessel sequence $\{f_k\}_{k=1}^\infty$, the *pre-frame operator* or *synthesis operator* is

$$T : \ell^2(\mathbb{N}) \to \mathcal{H},\ T\{c_k\}_{k=1}^\infty = \sum_{k=1}^\infty c_k f_k.$$

The operator T is bounded for any Bessel sequence $\{f_k\}_{k=1}^\infty$. The adjoint operator of T is called the *analysis operator* and is given by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}),\ \ T^* f = \{\langle f, f_k\rangle\}_{k=1}^\infty.$$

Finally, the *frame operator* is defined by

$$S : \mathcal{H} \to \mathcal{H},\ \ Sf = TT^* f = \sum_{k=1}^\infty \langle f, f_k\rangle f_k.$$

The following classical result shows that any frame leads to an expansion of the elements in $\mathcal{H}$ as a (infinite) linear combinations of the frame elements. It also shows that the general expansion simplifies considerably for tight frames. Finally, the last part of the result shows that for frames that are not Riesz bases, the coefficients in the series expansion of an element $f \in \mathcal{H}$ are not unique:

Theorem 2.1. *Let $\{f_k\}_{k=1}^\infty$ be a frame with frame operator S. Then the following holds:*

(i) Each $f \in \mathcal{H}$ has the decompositions

$$f = \sum_{k=1}^\infty \langle f, S^{-1} f_k\rangle f_k = \sum_{k=1}^\infty \langle f, f_k\rangle S^{-1} f_k.$$

(ii) If $\{f_k\}_{k=1}^\infty$ is a tight frame with frame bound A, then $S = AI$, and

$$f = \frac{1}{A}\sum_{k=1}^\infty \langle f, f_k\rangle f_k,\ \forall f \in \mathcal{H}. \tag{3}$$

(iii) If $\{f_k\}_{k=1}^\infty$ is an overcomplete frame, there exist frames $\{g_k\}_{k=1}^\infty \neq \{S^{-1} f_k\}_{k=1}^\infty$ for which

$$f = \sum_{k=1}^\infty \langle f, g_k\rangle f_k,\ \forall f \in \mathcal{H}. \tag{4}$$

Any Bessel sequence $\{g_k\}_{k=1}^\infty$ satisfying (4) for a given frame $\{f_k\}_{k=1}^\infty$ is called a *dual frame* of $\{f_k\}_{k=1}^\infty$. The special choice $\{g_k\}_{k=1}^\infty = \{S^{-1} f_k\}_{k=1}^\infty$ is called the *canonical dual frame.* In order to avoid confusion we note that if (4) holds for two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, they are automatically frames:

Lemma 2.2. *If $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences and (4) holds, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames.*

Note that duality between Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ can be expressed entirely in terms of operators. In fact, if T, U denote the pre-frame operators for $\{f_k\}_{k=1}^{\infty}$, respectively, $\{g_k\}_{k=1}^{\infty}$, the sequences are dual frames if and only if

$$TU^* = I.$$

2.2 Operators on $L^2(\mathbb{R})$

In order to construct concrete frames in the Hilbert space $L^2(\mathbb{R})$, we need to consider some important classes of operators.

Definition 2.3 (Translation, Modulation, Dilation). Consider the following classes of linear operators on $L^2(\mathbb{R})$:

(i) For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$(T_a f)(x) := f(x-a), \; x \in \mathbb{R}. \tag{5}$$

(ii) For $b \in \mathbb{R}$, the operator E_b, called modulation by b, is defined by

$$(E_b f)(x) := e^{2\pi i b x} f(x), \; x \in \mathbb{R}. \tag{6}$$

(iii) For $c > 0$, the operator D_c, called dilation by c, is defined by

$$(D_c f)(x) := \frac{1}{\sqrt{c}} f(\frac{x}{c}), \; x \in \mathbb{R}. \tag{7}$$

(iv) The dyadic dilation operator is defined by

$$(Df)(x) := 2^{1/2} f(2x), \; x \in \mathbb{R}.$$

All the above operators are linear, bounded, and unitary. We will also need the Fourier transform, for $f \in L^1(\mathbb{R})$ defined by

$$\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} dx.$$

The Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$ in the usual way.

The operators $T_a, E_b, D,$ and $\mathcal{F}$ are related by the following commutator relations:

$$T_a E_b = e^{-2\pi i b a} E_b T_a, \;\; T_b D = D T_{b/a}, \;\; D E_b = E_{b/a} D$$
$$\mathcal{F} T_a = E_{-a} \mathcal{F}, \;\; \mathcal{F} E_a = T_a \mathcal{F}, \;\; \mathcal{F} D = D^{-1} \mathcal{F}.$$

3 Gabor Systems

Gabor systems in $L^2(\mathbb{R})$ have the form

$$\{e^{2\pi i m b x} g(x - na)\}_{m,n\in\mathbb{Z}}$$

for some $g \in L^2(\mathbb{R}),\; a, b > 0$. Using operator notation, we can write a Gabor system as $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$.

We will not go into a general description of Gabor analysis and its role in time–frequency analysis, but just refer to the books [19–21].

Letting $\chi_{[0,1]}$ denote the characteristic function for the interval [0, 1], it is easy to show that $\{E_m T_n \chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. But the function $\chi_{[0,1]}$ is discontinuous and has very slow decay in the Fourier domain, so this function is not suitable for time–frequency analysis. For the sake of time–frequency analysis we want the frame generator g to be a continuous function with compact support. The following classical result shows that this more or less forces us to work with frames.

Lemma 3.1. *If g is be a continuous function with compact support, then*

(i) $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ *cannot be an ONB.*
(ii) $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ *cannot be a Riesz basis.*
(iii) $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ *can be a frame if* $0 < ab < 1$;

In addition to (iii), if $0 < ab < 1$, it is always possible to find a function $g \in C_c(\mathbb{R})$ such that $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. We also note that no matter whether g is continuous or not, Gabor frames $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ only exist if $ab \le 1$.

Bessel sequences of the form $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ will play a central role in some of the open problems to be considered in this article, so let us state a classical sufficient condition that is easy to verify.

Lemma 3.2. *Let g be a bounded function with compact support. Then $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for any $a, b > 0$.*

For a Gabor system $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$, the frame operator commutes with the operators $E_{mb}, T_{na},\; m, n \in \mathbb{Z}$. We will need the result below, which is almost identical to Lemma 9.3.1 in [6].

Lemma 3.3. *Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are Bessel sequences. Then the following holds:*

(i) Letting T and U denote the preframe operators for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$,

$$TUE_{mb}T_{na} = E_{mb}T_{na}TU, \ \forall m, n \in \mathbb{Z}.$$

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with frame operator $S = TT^*$, then

$$S^{-1}E_{mb}T_{na} = E_{mb}T_{na}S^{-1}, \ \forall m, n \in \mathbb{Z}.$$

Lemma 3.3 (ii) implies that for a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with associated frame operator S, the canonical dual frame also has Gabor structure, in contrast with the situation we encountered for wavelet frames. However, even for a nice frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ it is nontrivial to control the properties of the canonical dual frame $\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$, so often it is a better strategy to construct dual pairs $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$,$\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ such that g and h have required properties. Dual pairs of Gabor frames have been characterized by Ron and Shen [26] and Janssen [23]:

Theorem 3.4. *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)}h(x - ka) = b\delta_{n,0}, \ a.e. \ x \in [0, a].$$

One of the most important results in Gabor analysis is the so-called *duality principle*. It was discovered almost simultaneously by three groups of researchers, namely Daubechies et al. [16], Janssen [22], and Ron and Shen [26]. It concerns the relationship between frame properties for a function g with respect to the *lattice* $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ and with respect to the so-called *dual lattice* $\{(n/b, m/a)\}_{m,n\in\mathbb{Z}}$:

Theorem 3.5. *Given $g \in L^2(\mathbb{R})$ and $a, b > 0$, the following are equivalent:*

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B;
(ii) $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence with bounds A, B.

The intuition behind the duality principle is that if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then $ab \leq 1$, i.e., the sampling points $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ are "sufficiently dense." Therefore the points $\{(n/b, m/a)\}_{m,n\in\mathbb{Z}}$ are "sparse," in the sense that $\frac{1}{ab} \geq 1$. Technically, this implies that the functions $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ are linearly independent and only span a subspace of $L^2(\mathbb{R})$. The reason for the importance of the duality principle is that in general it is much easier to check that a system of vectors is a Riesz sequence than to check that it is a frame. The duality principle is clearly related with the *Wexler–Raz theorem* stated next, which was discovered in 1994.

Theorem 3.6. *If the Gabor systems* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *are Bessel sequences, then the following are equivalent:*

(i) The Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *are dual frames;*
(ii) The Gabor systems $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *and* $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}h\}_{m,n\in\mathbb{Z}}$ *are biorthogonal, i.e.,*

$$\langle\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g,\frac{1}{\sqrt{ab}}E_{m'/a}T_{n'/b}h\rangle=\delta_{m,m'}\delta_{n,n'}.$$

Theorem 3.4 characterizes pairs of dual Gabor frames, but it does not show how to construct convenient pairs of Gabor frames. A class of convenient dual pairs of frames are constructed in [5, 8]:

Theorem 3.7. *Let* $N\in\mathbb{N}$. *Let* $g\in L^2(\mathbb{R})$ *be a real-valued bounded function for which* $supp\ g\subseteq[0,N]$ *and*

$$\sum_{n\in\mathbb{Z}}g(x-n)=1. \tag{8}$$

Let $b\in]0,\frac{1}{2N-1}]$. *Define* $\tilde{g}\in L^2(\mathbb{R})$ *by*

$$h(x)=\sum_{n=-N+1}^{N-1}a_ng(x+n),$$

where

$$a_0=b,\ \ a_n+a_{-n}=2b,\ n=1,2,\cdots,N-1.$$

Then g *and* h *generate dual frames* $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Let us apply Theorem 3.7 to the classical B-splines $B_N,\ N\in\mathbb{N}$, given inductively by

$$B_1(x):=\chi_{[0,1]}(x),\ \ B_{N+1}(x):=B_N*B_1(x)=\int_0^1B_N(x-t)\,dt. \tag{9}$$

Example 3.8. The conditions in Theorem 3.7 are satisfied for any B-spline $B_N,\ N\in\mathbb{N}$. Some choices of the coefficients a_n are given by (Fig. 1):

1) Take

$$a_0=b,a_n=0\text{ for }n=-N+1,\ldots,-1,a_n=2b,n=1,\ldots N-1.$$

This choice gives the dual frame generated by the function with shortest support.

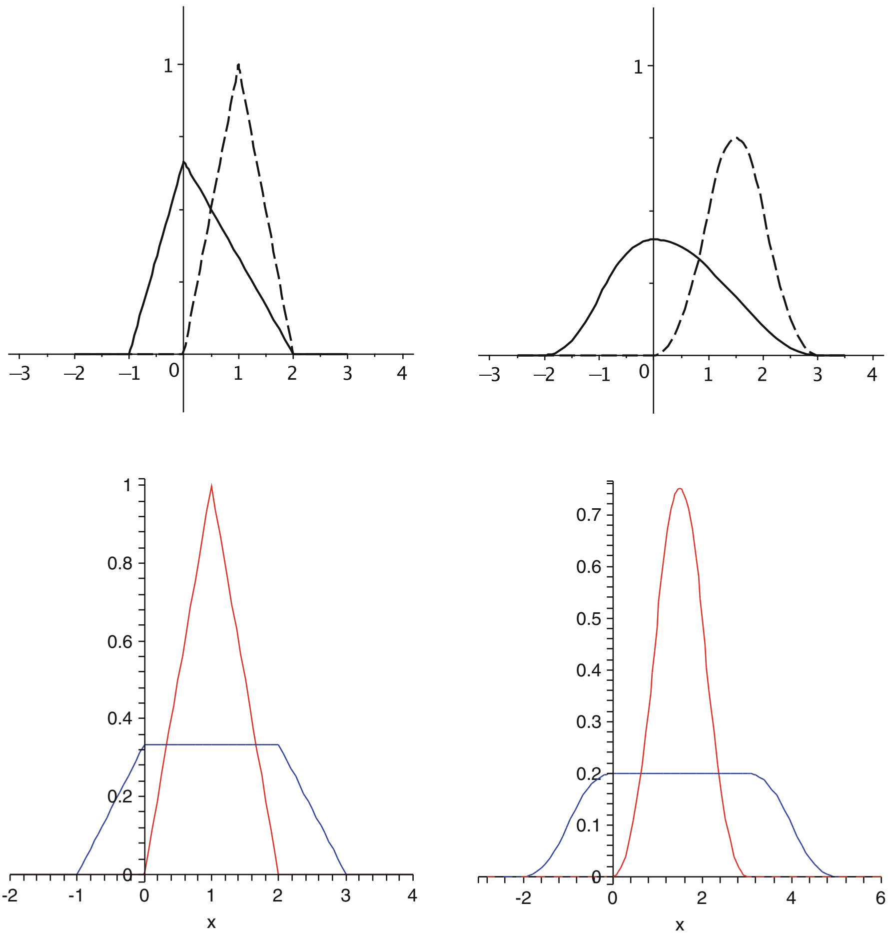

Fig. 1 The generators B_2 and B_3 and some dual generators

2) Take

$$a_{-N+1} = a_{-N+2} = \cdots = a_{N-1} = b :$$

if g is symmetric, this leads to a symmetric dual generator

$$\tilde{g}(x) = b \sum_{n=-N+1}^{N-1} g(x+n).$$

□

4 Wavelet Systems in $L^2(\mathbb{R})$

A *wavelet system* in $L^2(\mathbb{R})$ has the form $\{a^{j/2}\psi(a^j x - kb)\}_{j,k\in\mathbb{Z}}$ for some parameters $a > 1, b > 0$ and a given function $\psi \in L^2(\mathbb{R})$. Introducing the scaling operators and the translation operators, the wavelet system can be written as $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$.

There are also characterizing equations for dual wavelet frames; see [11]. They are formulated in terms of the Fourier transform:

Theorem 4.1. *Given $a > 1$, $b > 0$, two Bessel sequences $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j} T_{kb}\tilde{\psi}\}_{j,k\in\mathbb{Z}}$, where $\psi, \tilde{\psi} \in L^2(\mathbb{R})$, form dual wavelet frames for $L^2(\mathbb{R})$ if and only if the following two conditions hold:*

(i) $\sum_{j\in\mathbb{Z}} \overline{\hat{\psi}(a^j\gamma)}\hat{\tilde{\psi}}(a^j\gamma) = b$ *for a.e.* $\gamma \in \mathbb{R}$.
(ii) For any number $\alpha \neq 0$ of the form $\alpha = m/a^j$, $m, j \in \mathbb{Z}$,

$$\sum_{(j,m)\in I_\alpha} \overline{\hat{\psi}(a^j\gamma)}\hat{\tilde{\psi}}(a^j\gamma + m/b) = 0, \ a.e.\ \gamma \in \mathbb{R},$$

where $I_\alpha := \{(j,m) \in \mathbb{Z}^2 \mid \alpha = m/a^j\}$.

We will present a few aspects of wavelet theory, beginning with the dyadic wavelet systems and classical multiresolution analysis.

4.1 Dyadic Wavelet Systems

A systems of functions of the form $\{D^j T_k\psi\}_{j,k\in\mathbb{Z}}$, where $\psi \in L^2(\mathbb{R})$ is a fixed function, is called a *dyadic wavelet system.* Note that $D^j T_k\psi(x) = 2^{j/2}\psi(2^j x - k)$, $x \in \mathbb{R}$. Given a frame $\{D^j T_k\psi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, the associated frame operator is

$$S : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Sf = \sum_{j,k\in\mathbb{Z}} \langle f, D^j T_k\psi\rangle D^j T_k\psi,$$

and the frame decomposition reads

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, S^{-1}D^j T_k\psi\rangle D^j T_k\psi,\ f \in L^2(\mathbb{R}).$$

In order to use the frame decomposition we need to calculate the numbers $\langle f, S^{-1}D^j T_k\psi\rangle$ for all $j, k \in \mathbb{Z}$, i.e., a double-infinite sequence of numbers. One can show that

$$S^{-1}D^j T_k\psi = D^j S^{-1}T_k\psi,$$

so in practice it is enough to calculate the action of S^{-1} on the functions $T_k\psi$, and then apply the scaling D^j. Unfortunately, in general

$$D^j S^{-1} T_k \psi \neq D^j T_k S^{-1} \psi.$$

Thus, we cannot expect the canonical dual frame of a wavelet frame to have wavelet structure. As a concrete example (taken from [10, 13]), let $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ be a wavelet orthonormal basis for $L^2(\mathbb{R})$. Given $\epsilon \in]0,1[$, let $\theta = \psi + \epsilon D\psi$. Then $\{D^j T_k \theta\}_{j,k\in\mathbb{Z}}$ is a Riesz basis, but the canonical dual frame of $\{D^j T_k \theta\}_{j,k\in\mathbb{Z}}$ does *not* have the wavelet structure. Since the dual is unique for a Riesz basis, this example demonstrates that there are wavelet frames where no dual with wavelet structure exists. On the other hand, Bownik and Weber [1] have given an interesting example of a wavelet frame $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ for which the canonical dual does not have the wavelet structure, but other dual frames with wavelet structure exist.

4.2 Classical Multiresolution Analysis

Multiresolution analysis is a tool to construct orthonormal bases for $L^2(\mathbb{R})$ of the form $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ for a suitably chosen function $\psi \in L^2(\mathbb{R})$. Such a function ψ is called a *wavelet*. Its original definition of a multiresolution analysis was given by Mallat and Meyer [24, 25] is as follows:

Definition 4.2. A multiresolution analysis for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$ such that

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$
(ii) $\cap_j V_j = \{0\}$ and $\overline{\cup_j V_j} = L^2(\mathbb{R})$
(iii) $f \in V_j \Leftrightarrow Df \in V_{j+1}$.
(iv) $f \in V_0 \Rightarrow T_k f \in V_0, \ \forall k \in \mathbb{Z}$.
(v) $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

A multiresolution analysis is in fact generated just by a suitable choice of the function ϕ : if the conditions in Definition 4.2 are satisfied, then necessarily

$$V_j = \overline{\text{span}}\{D^j T_k \phi\}_{k\in\mathbb{Z}}, \ \forall j \in \mathbb{Z}.$$

The following result, due to Mallat and Meyer [24, 25], shows how to construct a wavelet based on a multiresolution analysis. Other proofs can be found in [7, 14, 30].

Theorem 4.3. *Assume that the function $\phi \in L^2(\mathbb{R})$ generates a multiresolution analysis. Then the following holds:*

(i) There exists a 1-periodic function $H_0 \in L^2(0,1)$ such that

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma), \ \gamma \in \mathbb{R}. \tag{10}$$

(ii) Define the 1-periodic function H_1 by

$$H_1(\gamma) := \overline{H_0(\gamma + \frac{1}{2})}\, e^{-2\pi i \gamma}. \tag{11}$$

Also, define the function ψ via

$$\hat{\psi}(2\gamma) := H_1(\gamma)\hat{\phi}(\gamma). \tag{12}$$

Then ψ is a wavelet.

The definition in (12) is quite indirect: it defines the function ψ in terms of its Fourier transform, so we have to apply the inverse Fourier transform in order to obtain an expression for ψ. This actually leads to an explicit expression of the function ψ in terms of the given function ϕ:

Proposition 4.4. *Assume that (12) holds for a 1-periodic function $H_1 \in L^2(0,1)$,*

$$H_1(\gamma) = \sum_{k\in\mathbb{Z}} d_k e^{2\pi i k\gamma}. \tag{13}$$

Then

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} d_k DT_{-k}\phi(x) = 2\sum_{k\in\mathbb{Z}} d_k \phi(2x+k), \; x \in \mathbb{R}. \tag{14}$$

The classical example of a wavelet generated by a multiresolution analysis is the *Haar wavelet*,

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}[\\ -1 & \text{if } x \in [\frac{1}{2}, 1[\\ 0 & \text{otherwise} \end{cases}$$

It is generated by the function $\phi = \chi_{[0,1]}$. In 1989 Daubechies managed to construct an important class of compactly supported wavelets with very good approximation properties. We will not go into a detailed discussion of these, but just refer to, e.g., [14, 30].

4.3 The Unitary Extension Principle

In this section we present results by Ron and Shen, which enables us to construct tight wavelet frames generated by a collection of functions $\psi_1, \ldots, \psi_n$. Our presentation is based on the papers [27–29]. Note also that a more flexible tool, the oblique extension principle, has later been introduced by two groups of researchers, see [12, 17].

The generators $\psi_1, \ldots, \psi_n$ will be constructed on the basis of a function which satisfy a refinement equation, and since we will work with all those functions simultaneously it is convenient to change our previous notation slightly and denote the refinable function by ψ_0.

General Setup: Let $\psi_0 \in L^2(\mathbb{R})$. Assume that $\lim_{\gamma\to 0} \hat{\psi}_0(\gamma) = 1$ and that there exists a function $H_0 \in L^\infty(\mathbb{T})$ such that

$$\hat{\psi}_0(2\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma). \tag{15}$$

Let $H_1, \ldots, H_n \in L^\infty(\mathbb{T})$, and define $\psi_1, \ldots, \psi_n \in L^2(\mathbb{R})$ by

$$\hat{\psi}_\ell(2\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \ \ell = 1, \ldots, n. \tag{16}$$

Finally, let H denote the $(n+1) \times 2$ matrix-valued function defined by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix}. \tag{17}$$

We will frequently suppress the dependence on γ and simply speak about the matrix H. The purpose is to find $H_1, \ldots, H_n$ such that

$$\{D^j T_k \psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^j T_k \psi_2\}_{j,k\in\mathbb{Z}} \cup \cdots \cup \{D^j T_k \psi_n\}_{j,k\in\mathbb{Z}} \tag{18}$$

constitute a tight frame. The *unitary extension principle* by Ron and Shen shows that a condition on the matrix H will imply this:

Theorem 4.5. *Let $\{\psi_\ell, H_\ell\}_{\ell=0,\ldots,n}$ be as in the general setup, and assume that the 2×2 matrix $H(\gamma)^* H(\gamma)$ is the identity for a.e. γ. Then the multi-wavelet system $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to one.*

As an application, we show how one can construct compactly supported tight spline frames.

Example 4.6. Fix any $m = 1, 2, \ldots,$ and consider the function

$$\psi_0 = \chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]} * \cdots * \chi_{[-\frac{1}{2},\frac{1}{2}]} \quad (2m \text{ factors}).$$

The function ψ_0 is known as a B-spline of order $2m$, although it is defined using the function $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ rather than $\chi_{[0,1]}$ as we did in (9). Note that

$$\hat{\psi}_0(\gamma) = \frac{\sin^{2m}(\pi\gamma)}{(\pi\gamma)^{2m}}.$$

It is clear that $\lim_{\gamma\to 0}\hat{\psi}_0(\gamma) = 1$, and by direct calculation,

$$\hat{\psi}_0(2\pi\gamma) = \cos^{2m}(\pi\gamma)\hat{\psi}_0(\gamma).$$

Thus ψ_0 satisfies the refinement equation with

$$H_0(\gamma) = \cos^{2m}(\pi\gamma).$$

Let $\binom{2m}{\ell}$ denote the binomial coefficients $\frac{(2m)!}{(2m-\ell)!\ell!}$ and define the 1-periodic bounded functions $H_1, H_2, \ldots, H_{2m}$ by

$$H_\ell(\gamma) = \sqrt{\binom{2m}{\ell}}\sin^{\ell}(\pi\gamma)\cos^{2m-\ell}(\pi\gamma).$$

Then

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} \cos^{2m}(\pi\gamma) & \sin^{2m}(\pi\gamma) \\ \sqrt{\binom{2m}{1}}\sin(\pi\gamma)\cos^{2m-1}(\pi\gamma) & -\sqrt{\binom{2m}{1}}\cos(\pi\gamma)\sin^{2m-1}(\pi\gamma) \\ \sqrt{\binom{2m}{2}}\sin^{2}(\pi\gamma)\cos^{2m-2}(\pi\gamma) & \sqrt{\binom{2m}{2}}\cos^{2}(\pi\gamma)\sin^{2m-2}(\pi\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ \sqrt{\binom{2m}{2m}}\sin^{2m}(\pi\gamma) & \sqrt{\binom{2m}{2m}}\cos^{2m}(\pi\gamma) \end{pmatrix}.$$

Now consider the 2×2 matrix $M := H(\gamma)^* H(\gamma)$. Using the binomial formula

$$(x+y)^{2m} = \sum_{\ell=0}^{2m} \binom{2m}{\ell} x^\ell y^{2m-\ell}$$

we see that the first entry in the first row of M is

$$M_{1,1} = \sum_{\ell=0}^{2m} \binom{2m}{\ell} \sin^{2\ell}(\pi\gamma) \cos^{2(2m-\ell)}(\pi\gamma) = 1.$$

A similar argument gives that $M_{2,2} = 1$. Also,

$$\begin{aligned} M_{1,2} &= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma)\left(1 - \binom{2m}{1} + \binom{2m}{2} - \cdots + \binom{2m}{2m}\right) \\ &= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma)(1-1)^{2m} = 0. \end{aligned}$$

Thus M is the identity on $\mathbb{C}^2$ for all γ; by Theorem 4.5 this implies that the $2m$ functions $\psi_1, \ldots, \psi_{2m}$ defined by

$$\begin{aligned} \hat{\psi}_\ell(\gamma) &= H_\ell(\gamma/2)\hat{\psi}_0(\gamma/2) \\ &= \sqrt{\binom{2m}{\ell}} \frac{\sin^{2m+\ell}(\pi\gamma/2)\cos^{2m-\ell}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}} \end{aligned}$$

generate a multiwavelet frame for $L^2(\mathbb{R})$.

Frequently one takes a slightly different choice of H_ℓ, namely,

$$H_\ell(\gamma) = i^\ell \sqrt{\binom{2m}{\ell}} \sin^\ell(\pi\gamma)\cos^{2m-\ell}(\pi\gamma).$$

Inserting this expression in $\hat{\psi}_\ell(\gamma) = H_\ell(\gamma/2)\hat{\psi}_\ell(\gamma/2)$ and using the commutator relations for the operators $\mathcal{F}, D, T_k$ shows that ψ_ℓ is a finite linear combination with real coefficients of the functions

$$DT_k\psi_0, \; k = -m, \ldots, m.$$

It follows that ψ_ℓ is a real-valued spline with support in $[-m, m]$, degree $2m-1$, smoothness class C^{2m-2}, and knots at $\mathbb{Z}/2$. Note in particular that we obtain smoother generators by starting with higher order splines, but that the price to pay is that the number of generators increases as well. □

Note that the unitary extension principle has a more convenient (but mathematically equivalent) formulation in the *oblique extension principle,* which was discovered independently and simultaneously by Daubechies et al. [17] and Chui et al. [12]. We will not go into a discussion of this, but just refer to the original articles, as well as the compressed presentation in [4, 6] for a quick overview.

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