Shape-Topological Differentiability of Energy Functionals for Unilateral Problems in Domains with Cracks and Applications

Günter Leugering, Jan Sokołowski, and Antoni Żochowski

Abstract A review of results on first order shape-topological differentiability of energy functionals for a class of variational inequalities of elliptic type is presented.

The *velocity method in shape sensitivity analysis* for solutions of elliptic unilateral problems is established in the monograph (Sokołowski and Zolésio, Introduction to Shape Optimization: Shape Sensitivity Analysis, Springer, Berlin/Heidelberg/New York, 1992). The *shape and material derivatives* of solutions to frictionless contact problems in solid mechanics are obtained. In this way the *shape gradients* of the associated integral functionals are derived within the framework of nonsmooth analysis. In the case of the energy type functionals classical differentiability results can be obtained, because the shape differentiability of solutions is not required to obtain the shape gradient of the shape functional (Sokołowski and Zolésio, Introduction to Shape Optimization: Shape Sensitivity Analysis, Springer, Berlin/Heidelberg/New York, 1992). Therefore, for cracks the strong continuity of solutions with respect to boundary variations is sufficient in order to obtain first order shape differentiability of the associated energy functional. This simple observation which is used in Sokołowski and Zolésio (Introduction to Shape Optimization: Shape Sensitivity Analysis, Springer, Berlin/Heidelberg/New York, 1992) for the shape differentiability of multiple

G. Leugering
Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstr. 11 (03.322) 91058 Erlangen, Germany
e-mail: leugering@math.fau.de

J. Sokołowski (✉)
Laboratoire de Mathématiques, Institut Élie Cartan, UMR 7502
Nancy-Université-CNRS-INRIA, Université Henri Poincaré Nancy 1, B.P. 239, 54506 Vandoeuvre Lès Nancy Cedex, France

Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland
e-mail: Jan.Sokolowski@iecn.u-nancy.fr

A. Zochowski
Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland
e-mail: Antoni.Zochowski@ibspan.waw.pl

R. Hoppe (ed.), *Optimization with PDE Constraints*, Lecture Notes in Computational Science and Engineering 101,
DOI 10.1007/978-3-319-08025-3_8

eigenvalues is further applied in Khludnev and Sokołowski (Eur. J. Appl. Math. 10:379–394, 1999; Eur. J. Mech. A Solids 19:105–120, 2000) to derive the first order shape gradient of the energy functional with respect to perturbations of the crack tip. A domain decomposition technique in shape-topology sensitivity analysis for problems with unilateral constraints on the crack faces (lips) is presented for the shape functionals.

We introduce the *Griffith shape functional* as the distributed shape derivative of the elastic energy evaluated in a domain with a crack, with respect to the crack length. We are interested in the dependence of this functional on domain perturbations far from the crack. As a result, the directional shape and topological derivatives of the nonsmooth Griffith shape functional are obtained with respect to boundary variations of an inclusion.

Keywords Conical differential of metric projection • Dirichlet Sobolev space • Griffith criterium for crack propagation • Hadamard shape differentiability • Nonsmooth analysis • Shape gradient • Shape Hessian • Signorini variational inequality

Mathematics Subject Classification (2010). Primary 35J86; Secondary 35R35, 49J40, 74R99.

1 Introduction

First order shape sensitivity analysis of the energy functional for an elliptic boundary value problem with unilateral constraints defined in domains with cracks is of broad interest and, therefore, it is named *Griffith shape functional*. In order to introduce the Griffith shape functional we make use of

- the crack model within an elastic body, represented by an elliptic variational inequality with the unilateral constraints representing the first order linear approximation of the non-penetration condition;
- the energy shape functional defined for the solutions of the variational inequality depending on the *shape of the crack*;
- an abstract result on the directional differentiability of the optimal value for constrained optimization problems over convex sets with respect to a parameter $t \to 0$,

$$t \to j(t, v^*(t)) := \inf_{v \in K} j(t, v)$$

  which requires only the strong convergence of the minimizers $v^*(t) \to v^*(0)$ with respect to the parameter as well as the existence of the partial derivative of the mapping $\mathbb{R} \ni t \to j(t, v) \in \mathbb{R}$;
- a technical result on linear transformations of the displacement field in the elasticity model obtained in [25] which provides the convex cone K, invariant

under the change of variables of the velocity method; it means that in order to apply the abstract sensitivity result for optimal values, we have in hand the linear transformation of the unknown solution to the variational inequality such that we could analyze the variational inequality transformed to the fixed geometrical domain with the parameter independent convex cone K.

Therefore, the Griffith shape functional is the first order shape derivative of the elastic energy with respect to the perturbation of the crack tip for a given direction of the velocity vector field. In addition, the second order shape derivative of the energy functional, whenever it does exist, becomes the first order shape derivative of the Griffith shape functional. But it is not our primary concern, since we are more interested in the influence of elastic inclusions far from the crack on the behaviour of the Griffith shape functional. We believe that such an influence is possible and can be used for the control of crack propagation in elastic media. Indeed, the dependence of the Griffith functional with respect to shape changes of an elastic or rigid inclusion has been considered in [8, 17]. This research has been triggered by numerical studies on optimization an control of crack growth also for the case of cohesive crack theories in [18, 21, 22]. See also [7, 19].

We recall also that the second order shape differentiability of the energy functional with respect to the perturbations of the crack tip is known for the Signorini type variational inequalities which governs frictionless contact problems [6]. This result can be extended to the crack problems with non-penetration contact conditions on the crack faces (lips), but this is a subject of the forthcoming paper.

1.1 Interface Problems in Lipschitz Domains

In this paper a class of models with defects in solids is introduced. The defect takes the form of a cut in the geometrical domain. The cut is a part of a curve in two spatial dimensions, and the unilateral boundary conditions for displacements and the tractions are prescribed for the jumps from both sides of the cut. The variational formulation of the model include the unilateral conditions for the displacements imposed in the convex cone constraints for admissible displacements. The variational inequality for displacements is obtained for the minimization problem of the energy functional over a convex cone. In the specific case of our setting, the solution operator is Lipschitz continuous with respect to the right-hand side of the variational inequality. This property leads usually to the Lipschitz continuity of the solution with respect to the regular boundary variations in the framework of the velocity method of shape sensitivity analysis. On the other hand, the asymptotic analysis of solutions to singular perturbations of the geometrical domain can be performed for linear problems or a restricted class of nonlinear problems. Since the technique of compound asymptotic expansions cannot be directly applied to the variational inequalities under considerations, a domain decomposition technique is used in order to obtain the first order asymptotic expansion of the energy functional and to obtain the topological derivatives of the energy functionals for the variational inequalities.

In this section the framework is introduced for the crack problem in the bounded domain Ω in two spatial dimensions. It is assumed [8–17] that a crack in Ω is a part Σ_l of the Lipschitz interface Σ. By an interface we mean a Lipschitz, closed curve without intersections $\Sigma \Subset \Omega$ such that the jumps $[u]$ of values for traces of Sobolev functions u from both sides of the interface are allowed.

In addition, in our model the interface, thus, also the crack are supposed to be sufficiently smooth, say Σ is a $C^{1,1}$ closed curves without intersections. This regularity assumption is added in order to use the standard properties of traces of Sobolev functions on the interface.

However, the shape sensitivity analysis is performed in our framework by the bi-Lipschitz changes of variables, we refer to [25] for all details necessary for such a construction.

Let us consider the Lipschitz domain Ω with the boundary $\Gamma = \partial\Omega$ decomposed into two Lipschitz subdomains Ω', Ω'' and the interface $\Sigma \subset \Omega$, i.e., $\Omega := \Omega' \cup \Sigma \cup \Omega''$. For the decomposition of functions in $v \in H_0^1(\Omega)$, we use the notation for restrictions to subdomains $v' \in H_0^1(\Omega')$ and $v'' \in H_0^1(\Omega'')$. Thus, the traces on Σ are well defined

$$v|_\Sigma := v'|_\Sigma = v''|_\Sigma \in H^{1/2}(\Sigma)\,.$$

Now, we define a broader space $H_0^1(\Omega) \subset H_\Gamma^1(\Omega_\Sigma) \subset L^2(\Omega)$ of functions which admit the jump

$$[v] := v'|_\Sigma - v''|_\Sigma \in H^{1/2}(\Sigma)$$

over the interface Σ. This leads also to the boundary value problems in Ω with the prescribed jump over the interface, which is not our primary interest. We are interested in the cracks $\Sigma_l \subset \Sigma$ modeled by closed subsets of the interface, with $\Omega_l := \Omega \setminus \overline{\Sigma}_l$, thus, in solutions of the boundary value problems in the convex set

$$K(\Omega_l) := \{v \in H_\Gamma^1(\Omega_\Sigma) \,:\, [v] \geqslant 0 \quad \text{on } \Sigma_l\,, \quad [v] = 0 \quad \text{on } \Sigma \setminus \overline{\Sigma}_l\}\,.$$

The primary interest of such a function space setting for the crack problems with unilateral non-penetration conditions on the crack faces (lips) is the so-called polyhedricity of the set $K(\Omega_l)$. In other words, polyhedral convex sets admit the Hadamard differential of the metric projection [6, 25]. This property is inherited from the polyhedricity of the positive cone in the fractional Sobolev space $H^{1/2}(\Sigma)$, since the space $H^{1/2}(\Sigma)$ is the so-called Dirichlet space with respect to the natural order. Let us recall the known facts [6].

Proposition 1.1. *The scalar product* $(\cdot,\cdot)_\Sigma$ *in the Dirichlet space* $H^{1/2}(\Sigma)$ *satisfies the condition*

$$(v^+, v^-)_\Sigma \leqslant 0 \quad \forall v \in H^{1/2}(\Sigma),$$

therefore, the metric projection in $H^{1/2}(\Sigma)$ onto the positive cone of $H^{1/2}(\Sigma)$ is conically differentiable.

This implies

Corollary 1.2. *The metric projection in $H^1_\Gamma(\Omega_\Sigma)$ onto the closed, convex cone $K(\Omega_l)$ is conically differentiable.*

The above results lead to the first order shape derivatives of the Griffith shape functional for the cracks with the nonlinear non-penetration conditions prescribed on the crack lips (or faces in three spatial dimensions).

Remark 1.3. The Griffith shape functional of the crack $\Sigma_l := \{(x_1, 0) \in \mathbb{R}^2, \quad 0 < x_1 < l\}$ at the tip $P_l := (l, 0)$ is defined by the shape derivative which is denoted by

$$J(\Omega_l) := \frac{d\Pi(\Omega_l; u_l)}{dl}$$

of the energy functional

$$l \to \Pi(\Omega_l; u_l) = \inf_{v \in K(\Omega_l)} \int_{\Omega_l} \left(\frac{1}{2}|\nabla v|^2 - fv \right)$$

where $u_l \in K(\Omega_l)$ is the minimizer for a given length $l > 0$ of the crack, and $f \in L^2(\Omega)$ is a given element.

We are going to extend such results to elastic bodies Ω_l with cracks Σ_l and unilateral conditions on the crack lips (faces) $\Sigma_l^{\pm}$. Then, we consider the differentiability properties of the Griffith functional

- evaluation of the first order shape derivative with respect to the perturbations of the crack;
- asymptotic analysis of the Griffith functional with respect to singular perturbations of the geometrical domain far from the crack;

2 Modeling of Cracks in Elastic Bodies

2.1 Non-Penetration Conditions on the Crack Faces

It is well known that classical crack theory in elasticity is characterized by linear boundary conditions which leads to linear boundary value problems. This approach has a clear shortcoming from a mechanical standpoint, since opposite crack faces can penetrate each other. We consider nonlinear boundary conditions on crack faces, the so-called non-penetration conditions, written in terms of inequalities. From the standpoint of applications, these boundary conditions are preferable since they provide a mutual non-penetration between crack faces. As a result, a free boundary problem is obtained which means that a concrete boundary condition at a given point can be found provided that we have a solution of the problem.

The main attention in this paper is paid to dependence of solutions of the problem on domain perturbations, and in particular, on the crack shape.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\Gamma_c \subset \Omega$ be a smooth curve without self-intersections, $\Omega_c = \Omega \setminus \overline{\Gamma}_c$.

It is assumed that Γ_c can be extended in such a way that this extension crosses Γ at two points, and Ω_c is divided into two subdomains D_1 and D_2 with Lipschitz boundaries ∂D_1, ∂D_2, $meas(\Gamma \cap \partial D_i) > 0$, $i = 1, 2$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to Γ_c. We assume that Γ_c does not contain its tip points, i.e. $\Gamma_c = \overline{\Gamma}_c \setminus \partial\Gamma_c$.

The equilibrium problem for a linear elastic body occupying Ω_c is as follows. In the domain Ω_c we have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_c, \tag{1}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_c, \tag{2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{3}$$

$$[u]\nu \geqslant 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_c, \tag{4}$$

$$\sigma_\nu \leqslant 0, \quad \sigma_\tau = 0 \quad \text{on} \quad \Gamma_c^{\pm}. \tag{5}$$

Here $[v] = v^+ - v^-$ is a jump of v on Γ_c, and signs $\pm$ correspond to positive and negative crack faces with respect to ν, $f = (f_1, f_2) \in L^2(\Omega_c)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2),$$
$$\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),$$

the strain tensor components are denoted by $\varepsilon_{ij}(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad i, j = 1, 2.$$

Elasticity tensor $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is given and satisfies the usual properties of symmetry and positive definiteness

$$a_{ijkl}\xi_{kl}\xi_{ij} \geqslant c_0|\xi|^2, \quad \forall\, \xi_{ij},\ \xi_{ij} = \xi_{ji}, \quad c_0 = const,$$

$a_{ijkl} = a_{klij} = a_{jikl}$, $a_{ijkl} \in L^\infty(\Omega)$.

Relations (1) are equilibrium equations, and (2) is Hooke's law, $u_{i,j} = \frac{\partial u_i}{\partial u_j}$, $(x_1, x_2) \in \Omega_c$. All functions with two below indices are symmetric in those indices, i.e. $\sigma_{ij} = \sigma_{ji}$ etc. Summation convention is assumed over repeated indices throughout the paper.

The first condition in (4) is called the non-penetration condition. It provides a mutual non-penetration between the crack faces $\Gamma_c^{\pm}$. The second condition of (5) provides zero friction on Γ_c. For simplicity we assume a clamping condition (3) at the external boundary Γ.

Note that a priori we do not know points on Γ_c where strict inequalities in (4), (5) are fulfilled. Due to this, the problem (1)–(5) is a free boundary value problem. If we have $\sigma_\nu = 0$ then, together with $\sigma_\tau = 0$, the classical boundary condition $\sigma\nu = 0$ follows which is used in linear crack theory. On the other hand, due to (4), the condition $\sigma_\nu < 0$ implies $[u]\nu = 0$, i.e. we have a contact between the crack faces at a given point. The strict inequality $[u]\nu > 0$ at a given point means that we have no contact between the crack faces.

Hence, the first difficulty in studying the problem (1)–(5) is concerned with boundary conditions (4)–(5). The second one is related to the general crack problem difficulty—a presence of nonsmooth boundaries. We refer the reader to [6] for related results on boundary value problems defined in domains with cracks.

2.2 Existence of Solutions

First of all we note that problem (1)–(5) admits several equivalent formulations. In particular, it corresponds to the minimization of the energy functional. To check this, introduce the Sobolev space

$$H^1_\Gamma(\Omega_c) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and the closed convex set of admissible displacements

$$K = \{v \in H^1_\Gamma(\Omega_c) \mid [v]\nu \geqslant 0 \quad \text{a.e. on} \quad \Gamma_c\}. \tag{6}$$

In this case, due to the Weierstrass theorem, the problem

$$\min_{v \in K} \left\{ \frac{1}{2} \int\limits_{\Omega_c} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_c} f_i v_i \right\}$$

has (a unique) solution u satisfying the variational inequality

$$u \in K, \tag{7}$$

$$\int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geqslant \int\limits_{\Omega_c} f_i(v_i - u_i), \quad \forall v \in K, \tag{8}$$

where $\sigma_{ij}(u) = \sigma_{ij}$ are defined from (2).

Problem formulations (1)–(5) and (7)–(8) are equivalent. We shall use in Sect. 47 the abstract form (144) of the variational inequality (7)–(8).

Remark 2.1. It follows from the coercivity on the energy space $H^1_\Gamma(\Omega_c)$ of the symmetric bilinear form

$$H^1_\Gamma(\Omega_c) \times H^1_\Gamma(\Omega_c) \ni (u,v) \to a(u,v) := \int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v) \in \mathbb{R}$$

that the solution u to (7)–(8) is Lipschitz continuous in the energy space with respect to the right-hand side f in the dual space $\left(H^1_\Gamma(\Omega_c)\right)^*$.

Any smooth solution of (1)–(5) satisfies (7)–(8) and, conversely, from (7)–(8) it follows (1)–(5).

Below we provide two more equivalent formulations for the problem (1)–(5), the so-called mixed and smooth domain formulations. To this end, we first discuss in what sense boundary conditions (4)–(5) are fulfilled. Denote by Σ a closed curve without self-intersections of the class $C^{1,1}$, which is an extension of Γ_c such that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1 and Ω_2. In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

Introduce the space $H^{\frac{1}{2}}(\Sigma)$ with the norm

$$\|v\|^2_{H^{\frac{1}{2}}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int\limits_{\Sigma}\int\limits_{\Sigma} \frac{|v(x)-v(y)|^2}{|x-y|^2} dxdy \tag{9}$$

and denote by $H^{-\frac{1}{2}}(\Sigma)$ a space dual of $H^{\frac{1}{2}}(\Sigma)$. Also, consider the space

$$H^{1/2}_{00}(\Gamma_c) = \left\{ v \in H^{\frac{1}{2}}(\Gamma_c) \mid \frac{v}{\sqrt{\rho}} \in L^2(\Gamma_c) \right\}$$

with the norm

$$\|v\|^2_{1/2,00} = \|v\|^2_{1/2} + \int\limits_{\Gamma_c} \rho^{-1} v^2,$$

where $\rho(x) = dist(x; \partial\Gamma_c)$ and $\|v\|_{1/2}$ is the norm in the space $H^{1/2}(\Gamma_c)$. It is known that functions from $H^{1/2}_{00}(\Gamma_c)$ can be extended to Σ by zero values, and moreover this extension belongs to $H^{1/2}(\Sigma)$. More precisely, let v be defined at Γ_c, and $\overline{v}$ be the extension of v by zero, i.e.

$$\overline{v}(x) = \begin{cases} v(x), & x \in \Gamma_c \\ 0, & x \in \Sigma \setminus \Gamma_c. \end{cases}$$

Then

$$v \in H_{00}^{1/2}(\Gamma_c) \quad \text{if and only if} \quad \overline{v} \in H^{1/2}(\Sigma).$$

With the above notations, it is possible to describe in what sense boundary conditions (4)–(5) are fulfilled. Namely, the condition $\sigma_\nu \leqslant 0$ in (5) means that

$$\langle \sigma_\nu, \phi \rangle_{1/2,00} \leqslant 0, \quad \forall\, \phi \in H_{00}^{1/2}(\Gamma_c), \quad \phi \geqslant 0 \text{ a.e. on } \Gamma_c,$$

where $\langle \cdot, \cdot \rangle_{1/2,00}$ is a duality pairing between $H_{00}^{-1/2}(\Gamma_c)$ and $H_{00}^{1/2}(\Gamma_c)$. The condition $\sigma_\tau = 0$ in (5) means that

$$\langle \sigma_\nu, \phi \rangle_{1/2,00} = 0, \quad \forall\, \phi = (\phi_1, \phi_2) \in H_{00}^{1/2}(\Gamma_c).$$

The last condition of (4) holds in the following sense

$$\langle \sigma_\nu, [u]\nu \rangle_{1/2,00} = 0.$$

2.3 Mixed Formulation of the Problem

Now we are interested to give a mixed formulation of the problem (1)–(5). Introduce the space for stresses

$$H(\text{div}) = \left\{ \sigma = \{\sigma_{ij}\} \mid \sigma \in L^2(\Omega_c), \text{div}\sigma \in L^2(\Omega_c) \right\}$$

with the norm

$$\|\sigma\|^2_{H(\text{div})} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\text{div}\sigma\|^2_{L^2(\Omega_c)}$$

and the set of admissible stresses

$$H(\text{div}; \Gamma_c) = \left\{ \sigma \in H(\text{div}) \mid [\sigma\nu] = 0 \text{ on } \Gamma_c; \quad \sigma_\nu \leqslant 0, \quad \sigma_\tau = 0 \text{ on } \Gamma_c^\pm \right\}.$$

We should note at this step that for $\sigma \in H(\text{div})$ the traces $(\sigma\nu)^\pm$ are correctly defined on $\Sigma^\pm$ as elements of $H^{-1/2}(\Sigma)$. The first condition in the definition of $H(\text{div}; \Gamma_c)$ is fulfilled in the following sense

$$(\sigma\nu)^+ = (\sigma\nu)^- \text{ on } \Sigma$$

for any curve Σ with the prescribed properties. Relations $\sigma \leqslant 0, \sigma_\tau = 0$ on $\Gamma_c^\pm$ also make sense. The values σ_ν, σ_τ are defined as elements of the space $H_{00}^{-1/2}(\Gamma_c)$.

The mixed formulation of the problem (1)–(5) is as follows. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega_c), \quad \sigma \in H(\mathrm{div}; \Gamma_c), \tag{10}$$

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_c, \tag{11}$$

$$\int\limits_{\Omega_c} C\sigma(\overline{\sigma} - \sigma) + \int\limits_{\Omega_c} u(\mathrm{div}\overline{\sigma} - \mathrm{div}\sigma) \geqslant 0 \quad \forall \overline{\sigma} \in H(\mathrm{div}; \Gamma_c). \tag{12}$$

The tensor C is obtained by inverting the Hooke's law (2), i.e.

$$C\sigma = \varepsilon(u).$$

It is possible to establish the existence of a solution to the problem (10)–(12) and check that (10)–(12) is formally equivalent to (1)–(5) (see [16]). Existence of solutions to (10)–(12) can be proved independently of (1)–(5). On the other hand, the solution exists due to the equivalence, and we already have the solution to the problem (1)–(5).

2.4 Smooth Domain Formulation

Along with the mixed formulation (10)–(12), the so-called smooth domain formulation of the problem (1)–(5) can be provided. In this case the solution of the problem is defined in the smooth domain Ω. To do this, we should notice that the solution of the problem (1)–(5) satisfies (7)–(8), thus, the condition

$$[\sigma\nu] = 0 \quad \text{on} \quad \Gamma_c$$

holds, and, therefore, it can be proved that in the distributional sense

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega.$$

Hence, the equilibrium equations (1) hold in the smooth domain Ω.

Introduce the space for stresses defined in Ω,

$$\mathcal{H}(\mathrm{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma,\ \mathrm{div}\sigma \in L^2(\Omega)\}$$

and the set of admissible stresses

$$\mathcal{H}(\mathrm{div}; \Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) \mid \sigma_\tau = 0, \quad \sigma_\nu \leqslant 0 \text{ on } \Gamma_c\}.$$

The norm in the space $\mathcal{H}(\text{div})$ is defined as follows

$$\|\sigma\|^2_{\mathcal{H}(\text{div})} = \|\sigma\|^2_{L^2(\Omega)} + \|\text{div}\sigma\|^2_{L^2(\Omega)}.$$

We see that for $\sigma \in \mathcal{H}(\text{div})$, the boundary condition $\sigma_\tau = 0$, $\sigma_\nu \leqslant 0$ on Γ_c are correctly defined in the sense $H^{-1/2}_{00}(\Gamma_c)$. Thus, we can provide the smooth domain formulation for the problem (1)–(5). It is necessary to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega), \quad \sigma \in \mathcal{H}(\text{div}; \Gamma_c), \tag{13}$$

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega, \tag{14}$$

$$\int\limits_\Omega C\sigma(\overline{\sigma} - \sigma) + \int\limits_\Omega u(\text{div}\overline{\sigma} - \text{div}\sigma) \geqslant 0 \quad \forall \overline{\sigma} \in \mathcal{H}(\text{div}; \Gamma_c). \tag{15}$$

It is possible to prove existence of a solution to the problem (13)–(15) (see [14]). Moreover, any smooth solution of (1)–(5) satisfies (13)–(15) and, conversely, from (13)–(15) it follows (1)–(5). Advantage of the formulation (13)–(15) is that it is given in the smooth domain. This formulation reminds contact problems with thin obstacle when restrictions are imposed on sets of small dimensions.

Numerical aspects for the problems like (1)–(5) can be found, for example, in [2, 3].

2.5 *Fictitious Domain Method*

In this section we provide a connection between the problem (1)–(5) and the Signorini contact problem. It turns out that the Signorini problem is a limit problem for a family of problems like (1)–(5). First we give a formulation of the Signorini problem. Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ_1, $\Gamma_1 = \Gamma_c \cup \Gamma_0$, $\Gamma_c \cap \Gamma_0 = \emptyset$, $meas\Gamma_0 > 0$.

For simplicity, we assume that Γ_c is a smooth curve (without its tip points). Denote by $\nu = (\nu_1, \nu_2)$ a unit normal inward vector to Γ_c. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_1, \tag{16}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_1, \tag{17}$$

$$u = 0 \quad \text{on} \quad \Gamma_0, \tag{18}$$

$$u\nu \geqslant 0,\ \sigma_\nu \leqslant 0,\ \sigma_\tau = 0,\ u\nu \cdot \sigma_\nu = 0 \quad \text{on} \quad \Gamma_c. \tag{19}$$

Here $f = (f_1, f_2) \in L^2_{loc}(\mathbb{R}^2)$ is a given function, $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$ is a given elasticity tensor, $a_{ijkl} \in L^\infty_{loc}(\mathbb{R}^2)$, with the usual properties of symmetry and positive definiteness.

It is well known (see [4, 5]) that the problem (16)–(19) has a variational formulation providing a solution existence. Namely, denote

$$H^1_{\Gamma_0}(\Omega_1) = \{v = (v_1, v_2) \in H^1(\Omega_1) \mid v_i = 0 \text{ on } \Gamma_0, \quad i = 1, 2\}$$

and introduce the set of admissible displacements

$$K_c = \{v = (v_1, v_2) \in H^1_{\Gamma_0}(\Omega_1) \mid v\nu \geqslant 0 \text{ a.e. on } \Gamma_c\}.$$

In this case the problem (16)–(19) is equivalent to minimization of the functional

$$\frac{1}{2} \int\limits_{\Omega_1} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_1} f_i v_i$$

over the set K_c and can be written in the form of the variational inequality

$$u \in K_c, \tag{20}$$

$$\int\limits_{\Omega_1} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geqslant \int\limits_{\Omega_1} f_i (v_i - u_i) \quad \forall v \in K_c. \tag{21}$$

Here $\sigma_{ij}(u) = \sigma_{ij}$ are defined from the Hooke's law (17). Variational inequality (20)–(21) is equivalent to (16)–(19) and, conversely, i.e., any smooth solution of (16)–(19) satisfies (20)–(21) and from (20)–(21) it follows (16)–(19). Along with variational formulation (20)–(21), the problem (16)–(19) admits a mixed formulation which is omitted here.

The aim of this section is to prove that the problem (16)–(19) is a limit problem for a family of problems like (1)–(5). In what follows we provide explanation of this statement.

First of all we extend the domain Ω_1 by adding a domain Ω_2 with smooth boundary Γ_2. An extended domain is denoted by Ω_c, and it has a crack (cut) Γ_c. Boundary of Ω_c is $\Gamma \cup \Gamma_c^\pm$. Denote $\Sigma_0 = \Gamma_1 \cap \Gamma_2$, $\Sigma = \Sigma_0 \setminus \Gamma$, thus Σ does not contain its tip points.

We introduce a family of elasticity tensors with a positive parameter λ,

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in } \Omega_1 \\ \lambda^{-1} a_{ijkl} & \text{in } \Omega_2. \end{cases}$$

Denote $A^\lambda = \{a^\lambda_{ijkl}\}$, and in the extended domain Ω_c, consider a family of the crack problems. Find a displacement field $u^\lambda = (u^\lambda_1, u^\lambda_2)$, and stress tensor components

$\sigma^\lambda = \{\sigma^\lambda_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma^\lambda = f \quad \text{in} \quad \Omega_c, \tag{22}$$

$$\sigma^\lambda = A^\lambda \varepsilon(u^\lambda) \quad \text{in} \quad \Omega_c, \tag{23}$$

$$u^\lambda = 0 \quad \text{on} \quad \Gamma, \tag{24}$$

$$[u^\lambda]\nu \geqslant 0,\ [\sigma^\lambda_\nu] = 0,\ \sigma^\lambda_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_c, \tag{25}$$

$$\sigma^\lambda_\nu \leqslant 0,\ \sigma^\lambda_\tau = 0 \quad \text{on} \quad \Gamma^\pm_c. \tag{26}$$

As before, $[v] = v^+ - v^-$ is the jump of v through Γ_c, where $\pm$ fit positive and negative crack faces $\Gamma^\pm_c$. All the remaining notations correspond to those of Sect. 1. We see that for any fixed $\lambda > 0$ the problem (22)–(26) describes an equilibrium state of linear elastic body with the crack Γ_c where non-penetration conditions are prescribed. Hence, the problem (22)–(26) is exactly the problem like (1)–(5), and we are interested in passage to the limit as $\lambda \to 0$. In particular, the problem (22)–(26) admits a variational formulation. Boundary conditions (25)–(26) are fulfilled in the form as it is explained in Sect. 1. It can be shown that the following convergence takes place as $\lambda \to 0$

$$u^\lambda \to u^0 \quad \text{strongly in} \quad H^1_\Gamma(\Omega_c), \tag{27}$$

$$\frac{u^\lambda}{\sqrt{\lambda}} \to 0 \quad \text{strongly in} \quad H^1(\Omega_2), \tag{28}$$

where $u^0 = u$ on Ω_1, i.e. a restriction of the limit function from (27) to Ω_1 coincides with the unique solution of the Signorini problem (16)–(19). From (27)–(28) it is seen that the limit function u^0 is zero in Ω_2. On the other hand, there is no limit passage for σ^λ in Ω_2 as $\lambda \to 0$. Thus, the domain Ω_2 can be understood as undeformable body, and the stresses are not defined in Ω_2. This means that the Signorini problem is, in fact, a crack problem with non-penetration condition between crack faces, where the crack Γ_c is located between the elastic body Ω_1 and non-deformable (rigid) body Ω_2. It is worth noting that, in fact, we can write the problem (22)–(26) in the equivalent form in the smooth domain $\Omega_c \cup \overline{\Gamma}_c$ by using the smooth domain formulation of Sect. 2.4.

3 Griffith Functionals Evaluation by the Shape Sensitivity Analysis of Energy Functionals

The velocity method [6, 25] is used in the shape sensitivity analysis of the energy functionals with respect to perturbations of a crack tip in two spatial dimensions. In Frémiot et al. [6] the Hadamard structure [25] theorem for the first and the second order shape derivatives of differentiable shape functionals in domains with cracks is given with full proof. We use the distributed form of the shape gradient for the energy functional with respect to the crack tip perturbations in order to define the Griffith shape functional which is further considered in Sect. 47. In applications, the Griffith functional can be used, it seems, to control the crack propagation in elastic body with elastic and/or rigid inclusions.

In the crack theory, the Griffith criterion can be used for the prediction of crack propagation. This criterion says that a crack propagates provided that the derivative of the energy functional with respect to the crack length reaches a critical value. In this section we discuss the Griffith criterion and the associated Griffith functional for the model (1)–(5).

The general point of view is that we should consider a perturbed problem with respect to (1)–(5). In particular, a crack length may be perturbed. Perturbation will be characterized by a small parameter t, and $t = 0$ corresponds to the unperturbed problem, i.e. to the problem (1)–(5). To describe properly a perturbation of the problem, we should define a perturbation of the domain Ω_c. This can be done in the framework of the sensitivity analysis by the so-called velocity method (see [25]). We briefly recall this method in a way useful for our purposes.

Let us consider a given velocity field V defined in $\mathbb{R}^2$ and describe a perturbation of Ω_c by solving a Cauchy problem for a system of ODE. Namely, let $V \in W^{1,\infty}(\mathbb{R}^2)^2$ be a given field, $V = (V_1, V_2)$. Consider a Cauchy problem for finding a function $\Phi = (\Phi_1, \Phi_2)$, with x the spatial variable,

$$\frac{d\Phi}{dt}(t,x) = V(\Phi(t,x)) \quad \text{for} \quad t \neq 0, \quad \Phi(0,x) = x. \tag{29}$$

There exists a unique solution Φ to (29) such that

$$\Phi = (\Phi_1, \Phi_2)(t,x) \in C^1([0,t_0]; W^{1,\infty}_{loc}(\mathbb{R}^2)^2), \quad |t_0| > 0. \tag{30}$$

Simultaneously, we can find a solution $\Psi = (\Psi_1, \Psi_2)$ to the following Cauchy problem

$$\frac{d\Psi}{dt}(t,y) = -V(\Psi(t,y)) \quad \text{for} \quad t \neq 0, \quad \Psi(0,y) = y \tag{31}$$

with the some regularity

$$\Psi = (\Psi_1, \Psi_2)(t,y) \in C^1([0,t_0]; W^{1,\infty}_{loc}(\mathbb{R}^2)^2), \quad |t_0| > 0. \tag{32}$$

It can be proved that for any fixed t, the inverse function of $\Phi(t,\cdot)$ is the function $\Psi(t,\cdot)$, thus

$$y = \Phi(t, \Psi(t,y)), \quad x = \Psi(t, \Phi(t,x)), \quad x, y \in \mathbb{R}^2.$$

Due to this, we have a one-to-one mapping between the domain Ω_c and a perturbed domain Ω_c^t, namely

$$y = \Phi(t,x) : \Omega_c \to \Omega_c^t,$$
$$x = \Psi(t,y) : \Omega_c^t \to \Omega_c.$$

Moreover, by (30), (32), we have the following asymptotic expansions (I denotes the identity operator)

$$\Phi(t,x) = x + tV(x) + r_1(t), \tag{33}$$

$$\Psi(t,y) = y - tV(y) + r_2(t), \tag{34}$$

$$\frac{\partial\Phi(t)}{\partial x} = I + t\frac{\partial V}{\partial x} + r_3(t), \tag{35}$$

$$\frac{\partial\Psi(t)}{\partial y} = I - t\frac{\partial V}{\partial y} + r_4(t), \tag{36}$$

$$\|r_i(t)\|_{W^{1,\infty}_{loc}(\mathbb{R}^2)^2} = o(t), \quad i = 1,2,$$

$$\|r_i(t)\|_{L^{\infty}_{loc}(\mathbb{R}^2)^{2\times 2}} = o(t), \quad i = 3,4.$$

Hence, in the domain Ω_c^t it is possible to consider the following boundary value problem (perturbed with respect to (1)–(5)). Find a displacement field $u^t = (u_1^t, u_2^t)$, and stress tensor components $\sigma^t = \{\sigma_{ij}^t\}$, $i,j = 1,2$, such that

$$-\mathrm{div}\sigma^t = f \quad \text{in} \quad \Omega_c^t, \tag{37}$$

$$\sigma^t = A\varepsilon(u^t) \quad \text{in} \quad \Omega_c^t, \tag{38}$$

$$u^t = 0 \quad \text{on} \quad \Gamma^t, \tag{39}$$

$$[u^t]\nu^t \geqslant 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on} \quad \Gamma_c^t, \tag{40}$$

$$\sigma_{\nu^t}^t \leqslant 0,\ \sigma_{\tau^t}^t = 0 \quad \text{on} \quad \Gamma_c^{t\pm}. \tag{41}$$

Here,

$$y = \Phi(t,x) : \Gamma \to \Gamma^t, \quad \Gamma_c \to \Gamma_c^t,$$

and we assume in this section that $f = (f_1, f_2) \in C^1(\mathbb{R}^2)$ and that $a_{ijkl} = const$, $i,j,k,l = 1,2$. All the rest notations in (37)–(41) remind those of (1)–(5), in particular, $\nu^t = (\nu_1^t, \nu_2^t)$ is a unit normal vector to Γ_c^t.

We can provide a variational formulation of the problem (37)–(41). Indeed, introduce the Sobolev space

$$H^1_{\Gamma^t}(\Omega_c^t) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c^t),\ v_i = 0 \text{ on } \Gamma^t, \quad i = 1,2\}$$

and the set of admissible displacements

$$K^t = \{v \in H^1_{\Gamma^t}(\Omega_c^t) \mid [v]\nu^t \geqslant 0 \quad \text{a.e. on} \quad \Gamma_c^t\}.$$

Consider the functional

$$\Pi(\Omega_c^t; v) = \frac{1}{2}\int\limits_{\Omega_c^t} \sigma_{ij}^t(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_c^t} f_i v_i$$

and the minimization problem

$$\min_{v \in K^t} \Pi(\Omega_c^t; v). \tag{42}$$

Here, $\sigma_{ij}^t(v)$ are defined from Hooke's law similar to (38). Solution of the problem (42) exists and it satisfies the variational inequality

$$u^t \in K^t, \tag{43}$$

$$\int\limits_{\Omega_c^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(v - u^t) \geqslant \int\limits_{\Omega_c^t} f_i(v_i - u_i^t) \quad \forall v \in K^t. \tag{44}$$

Having found a solution of the problem (43)–(44) we can define the energy functional

$$\Pi(\Omega_c^t; u^t) = \frac{1}{2}\int\limits_{\Omega_c^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int\limits_{\Omega_c^t} f_i u_i^t.$$

Note that for $t = 0$, we have $\Omega_c^0 = \Omega_c$ and $u^0 = u$, where u is the solution of the unperturbed problem (7), (8). The question arises whether the functional $t \to \Pi(\Omega_c^t; u^t)$ is differentiable at $t = 0$. Thus, we consider the existence of The question whether

$$\frac{d}{dt}\Pi(\Omega_c^t; u^t)|_{t=0} = \lim_{t \to 0} \frac{\Pi(\Omega_c^t; u^t) - \Pi(\Omega_c; u)}{t}.$$

The answer is positive in many practical situations. We consider two cases, where the derivative

$$I = \frac{d}{dt}\Pi(\Omega_c^t; u^t)|_{t=0} \tag{45}$$

can be evaluated.

3.1 Griffith Functionals for Rectilinear Cracks

Assume for simplicity that the normal vector ν to Γ_c keeps its value under the mapping $x \to \Phi(t, x)$, i.e. $\nu^t = \nu$. In this case,

$$I = \frac{1}{2} \int_{\Omega_c} \left\{ \mathrm{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u) \right\} \sigma_{ij}(u) - \int_{\Omega_c} \mathrm{div}(V f_i) u_i\,, \tag{46}$$

where

$$E_{ij}(U; v) = \frac{1}{2}(v_{i,k} U_{k,j} + v_{j,k} U_{k,i}), \quad U = \{U_{ij}\}, \quad i, j = 1, 2.$$

Note that the assumption concerning the normal vector ν holds for rectilinear cracks Γ_c and vector fields V tangential to Γ_c. In this situation, (46) provides a formula for the derivative of the energy functional with respect to the crack length what is practically needed for using the Griffith criterion.

- It will be the case when $V = 1$ in a vicinity of the right crack tip and the support denoted by supp$V$ belongs to a small neighborhood of this tip.
- Formula (46) for the shape derivative of the energy functional with respect to the crack length is called the *distributed shape gradient*. More precisely, by the shape gradient we understand the mapping

$$V \to \frac{1}{2} \int_{\Omega_c} \left\{ \mathrm{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u) \right\} \sigma_{ij}(u) - \int_{\Omega_c} \mathrm{div}(V f_i) u_i\,. \tag{47}$$

- In Sect. 7 the expression of the distributed gradient (47) is shown to be differentiable with respect to the perturbations of the linear boundary conditions for the displacement field. In this way the shape derivative of the Griffith functional with respect to the boundary variations of an inclusion far from the crack is determined.

3.2 Griffith Functionals for Curvilinear Cracks

The formula for the derivative (45) can be derived for curvilinear cracks if the simplified assumption on the normal vector ν is not fulfilled by using an appropriate transformation of unknown functions i.e., of the displacement field [25]. We provide here the formula (45) for the crack Γ_c which is defined by a graph of a smooth function.

Let $\psi \in H^3(0, l_1)$ be a given function, $l_1 > 0$, and

$$\Sigma = \{(x_1, x_2) \mid x_2 = \psi(x_1), \quad 0 < x_1 < l_1\}.$$

Consider a crack Γ_l, $\Gamma_l \subset \Sigma$, as a graph of the function ψ,

$$\Gamma_l = \{(x_1, x_2) \mid x_2 = \psi(x_1), \quad 0 < x_1 < l\}, \quad 0 < l < l_1.$$

Here, l is a parameter that characterizes the length of the projection of the crack Γ_l onto x_1 axis. Consider a smooth cut-off function θ with a support in a vicinity of the crack tip $(l, \psi(l))$, moreover, we assume that $\theta = 1$ in a small neighborhood of $(l, \psi(l))$. We can consider a perturbation of the crack Γ_l along Σ via a small parameter t. Denote $\Omega_l = \Omega \setminus \overline{\Gamma}_l$. Perturbed crack Γ_l^t has a tip $(l + t, \psi(l + t))$, and we consider a perturbed domain $\Omega_l^t = \Omega \setminus \overline{\Gamma}_l^t$. It is possible to establish a one-to-one correspondence between Ω_l and Ω_l^t by formulas

$$\begin{aligned} y_1 &= x_1 + t\theta(x), \\ y_2 &= x_2 + \psi(x_1 + t\theta(x)) - \psi(x_1), \end{aligned} \qquad (x_1, x_2) \in \Omega_l,\ (y_1, y_2) \in \Omega_l^t. \tag{48}$$

Transformation (48) is equivalent to the following (cf. (33))

$$y = x + tV(x) + r(t, x)$$

with the velocity field

$$V(x) = (\theta(x), \psi'(x_1)\theta(x)). \tag{49}$$

In the domain Ω_l^t, we can consider a perturbed problem formulation. Namely, it is necessary to find a displacement field $u^t = (u_1^t, u_2^t)$ and the stress tensor components $\sigma^t = \{\sigma_{ij}^t\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma^t = f \quad \text{in} \quad \Omega_l^t, \tag{50}$$

$$\sigma^t = A\varepsilon(u^t) \quad \text{in} \quad \Omega_l^t, \tag{51}$$

$$u^t = 0 \quad \text{on} \quad \Gamma, \tag{52}$$

$$[u^t]\nu^t \geqslant 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on} \quad \Gamma_l^t, \tag{53}$$

$$\sigma_{\nu^t}^t \leqslant 0,\ \sigma_{\tau^t}^t = 0 \quad \text{on} \quad \Gamma_l^{t\pm}. \tag{54}$$

Here, $\nu^t = (\nu_1^t, \nu_2^t)$ is a unit normal vector to Γ_l^t. For a solution u^t of (50)–(54) it is possible to define the energy functional

$$\Pi(\Omega_l^t; u^t) = \frac{1}{2} \int\limits_{\Omega_l^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int\limits_{\Omega_l^t} f_i u_i^t$$

and to find the derivative

$$\Pi'(l) = \frac{d\Pi(\Omega_l^t; u^t)}{dt}|_{t=0}$$

with the formula

$$\Pi'(l) = \frac{1}{2}\int_{\Omega_l} \{\mathrm{div}V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V;u)\}\sigma_{ij}(u) - \int_{\Omega_l} \mathrm{div}(Vf_i)u_i + \int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(w) - \int_{\Omega_l} f_i w_i, \tag{55}$$

where the vector field V is defined in (49) and $w = (0, \theta\psi'' u_1)$ is a given function. Note that the formula (55) contains the function θ, but in fact there is no dependence of the right-hand side of (55) on θ. In particular, if $\psi'' = 0$, the formula (55) reduces to (46) with $\Omega_c = \Omega_l$. In this case we have a rectilinear crack and $\nu^t = \nu$. Formula (55) defines a derivative of the energy functional with respect to the length of the projection of the crack Γ_l onto the x_1 axis. Hence, the derivative of the energy functional with respect to the length of the curvilinear crack is as follows

$$\Pi'(s) = \Pi'(l)(\psi'(l)^2 + 1)^{-1/2},$$

where

$$s = \int_0^l \sqrt{\psi'(t)^2 + 1}$$

is the length of the crack Γ_l.

To conclude this section we briefly discuss the existence of so-called invariant integrals in crack theory. It is turned out that the formula (46) for the derivative of the energy functional can be rewritten as an integral over closed curve surrounding the crack tip.

Consider the most simple case of a rectilinear crack $\Gamma_c = (0, 1) \times \{0\}$ assuming that $\overline{\Gamma}_c \subset \Omega$. Let θ be a smooth cut-off function equal to 1 near the point $(1, 0)$, and suppθ belong to a small neighborhood of the point $(1, 0)$. Then we can take the vector field

$$V = (\theta, 0)$$

in (29), (31) which, according to (33), corresponds to the following change of independent variables

$$y_1 = x_1 + t\theta(x) + r_{11}(t),$$
$$y_2 = x_2.$$

In this case the formula (46) (or the formula (55) in a particular case $\psi = 0$) provides a derivative of the energy functional with respect to the crack length. This formula can be rewritten [13] as an integral over curve L surrounding the crack tip $(1, 0)$,

$$I = \int_L \left\{ \frac{1}{2} \nu_1 \sigma_{ij}(u) \varepsilon_{ij}(u) - \sigma_{ij}(u) u_{i,1} \nu_j \right\} \tag{56}$$

provided that f is equal to zero in a neighborhood of the point $(1, 0)$. We should underline two important points. First, the formula (56) is independent of L, and second, the right-hand side of (56) is equal to the derivative of the energy functional with respect to the crack length.

In fact, invariant integrals like (56) can be obtained in more complex situations. For example, we can assume that the crack Γ_c is situated on the interface between two media which means that the elasticity tensor $A = \{a_{ijkl}\}$ is as follows

$$a_{ijkl} = \begin{cases} a^1_{ijkl} \text{ for } x_2 > 0 \\ a^2_{ijkl} \text{ for } x_2 < 0. \end{cases}$$

Here, $a^1_{ijkl} = const$, $a^2_{ijkl} = const$, $i, j, k, l = 1, 2$, and $\{a^1_{ijkl}\}$, $\{a^2_{ijkl}\}$ satisfy the usual properties of symmetry and positive definiteness. In this case, formula (46) for the derivative of the energy functional holds true provided that V is tangential to Γ_c. This formula provides an existence of invariant integral of the form (56). We should remark at this point that while the integral (56) is calculated, the values $\sigma_{ij}(u)u_{i,1}\nu_j$ can be taken at Γ_c^+ or at Γ_c^-. It gives the same value of the integral (56) due to the equality

$$[\sigma_{ij}(u)u_{i,1}\nu_j] = 0 \text{ on } \Gamma_c.$$

On the other hand, we can analyze the case when a rigidity of the elastic body part $\Omega_c \cap \{x_2 < 0\}$ goes to infinity. Indeed, consider the following elasticity tensor for a positive parameter $\lambda > 0$,

$$a^\lambda_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for } x_2 > 0 \\ \lambda^{-1} a^2_{ijkl} & \text{for } x_2 < 0. \end{cases}$$

Then for any fixed $\lambda > 0$, the solution of the equilibrium problem like (1)–(5) exists, and a passage to the limit as $\lambda \to 0$ can be fulfilled. As we already noted in Sect. 3, in the limit the following contact Signorini problem is obtained. Find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_c \cap \{x_2 > 0\}, \tag{57}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_c \cap \{x_2 > 0\}, \tag{58}$$

$$u = 0 \quad \text{on} \quad \partial(\Omega_c \cap \{x_2 > 0\}) \setminus \Gamma_c, \tag{59}$$

$$u\nu \geqslant 0,\ \sigma_\nu \leqslant 0,\ \sigma_\tau = 0, \sigma_\nu \cdot u\nu = 0 \quad \text{on} \quad \Gamma_c. \tag{60}$$

For the problem (57)–(60) it is possible to differentiate the energy functional in the direction of the vector field $V = (\theta, 0)$, where the properties of θ are described above. The formula for the derivative has the following form (cf. (46))

$$I = \frac{1}{2} \int_{\Omega_1} \{\text{div} V \cdot \sigma_{ij}(u) - 2E_{ij}(V, u)\}\sigma_{ij}(u) - \int_{\Omega_1} \text{div}(V f_i) u_i. \tag{61}$$

Assume that $f = 0$ in a neighborhood of the point $(1, 0)$. In this case, formula (61) can be rewritten in the form of invariant integral

$$I = \int_{L_1} \left\{ \frac{1}{2} \nu_1 \sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u) u_{i,1} \nu_j \right\}, \tag{62}$$

where L_1 is a smooth curve "covering" the point $(1, 0)$. Like for invariant integrals in the crack problems, formula (62) is independent of a choice of L_1.

4 Domain Decomposition Technique for Singularly Perturbed Elliptic Boundary Value Problems

Our primary concern is the domain decomposition technique [20, 23, 24] in application to the shape sensitivity analysis of the Griffith shape functional. However, the precise results on the shape sensitivity analysis of the Griffith shape functional are given in a forthcoming paper. In this paper we collect all the results recently obtained for shape-topological sensitivity analysis of the broad class of variational inequalities for elastic bodies with cracks. The asymptotic analysis in singularly perturbed geometrical domain is performed by domain decomposition technique. The boundary variations are used far from the defect, and the influence of the domain perturbations is imposed on the variational inequality by means of the Steklov–Poincaré operator defined within the domain decomposition technique. In this way the conical differentiability of solutions to the variational inequality with respect to the regular perturbations of the boundary conditions can be employed for shape-topological sensitivity analysis of the specific functional defined in the subdomain which contains the crack. This is the case of the Griffith shape functional evaluated for a crack with nonlinear boundary conditions prescribed on the crack lips.

The reference domain $\Omega \setminus \overline{\Gamma_c}$ of the elastic body under considerations is divided into two subdomains Ω_c with a crack Γ_c inside and Ω_i with an elastic inclusion ω inside. The domains are coupled within the nonlinear elasticity boundary value problem with the nonlocal boundary conditions defined on the interface $\Gamma_{sp} := \overline{\Omega_i} \cap \overline{\Omega_c}$ by an appropriate Steklov–Poincaré operator. In this section, however, we introduce the domain decomposition technique for the evaluation of the topological derivatives [20, 23, 24].

Let us consider the linear elliptic boundary value problems, and describe the domain decomposition technique for asymptotic analysis of the energy functional in singularly perturbed geometrical domains. The method is presented for simplicity for circular holes and for the Laplacian with Neumann conditions on the hole, and the Dirichlet condition on the outer boundary. In such a case the function $f(\varepsilon) = \varepsilon^2$ is used in asymptotic analysis. The shape functional is defined by the associated energy functional to the boundary value problem.

The domain decomposition technique and the Steklov–Poincaré nonlocal boundary operators are used in the topological sensitivity analysis of nonlinear variational problems. We start with a scalar linear boundary value problem in order to present the outline of the method. Therefore, given domains Ω and $\Omega_\varepsilon(\hat{x}) = \Omega \setminus \overline{B_\varepsilon(\hat{x})} \subset \mathbb{R}^2$, where $B_\varepsilon(\hat{x})$ is a ball of radius $\varepsilon \to 0$ and center at a point $\hat{x} \in \Omega$ far from the boundary $\Gamma = \partial\Omega$, with $\overline{B_\varepsilon} \Subset \Omega$. By u_ε we denote a unique classical solution of the Poisson equation in singularly perturbed domain:

$$\begin{cases} \text{Find } u_\varepsilon \text{ such that} \\ -\Delta u_\varepsilon = b \ \text{ in } \ \Omega_\varepsilon \ , \\ \qquad u_\varepsilon = 0 \text{ on } \partial\Omega \ , \\ \quad \partial_n u_\varepsilon = 0 \text{ on } \partial B_\varepsilon \ , \end{cases} \tag{63}$$

where $b \in C^{0,\alpha}(\overline{\Omega})$, with $\alpha \in (0, 1)$, is a given element which vanishes in the vicinity of the point $\hat{x} \in \Omega$. The solution u_ε of the boundary value problem (63) is variational, since $u_\varepsilon \in \mathcal{V}_\varepsilon \subset H^1(\Omega_\varepsilon)$ minimizes the quadratic functional

$$\mathcal{I}_\varepsilon(\varphi) = \frac{1}{2}\int_{\Omega_\varepsilon} \|\nabla\varphi\|^2 - \int_{\Omega_\varepsilon} b\varphi \tag{64}$$

over the linear subspace $\mathcal{V}_\varepsilon \subset H^1(\Omega_\varepsilon)$, where $\mathcal{V}_\varepsilon$ is defined as

$$\mathcal{V}_\varepsilon := \{\varphi \in H^1(\Omega_\varepsilon) : \varphi_{|\Gamma} = 0\} \ . \tag{65}$$

The shape functional

$$\mathcal{J}(\Omega_\varepsilon) := \mathcal{J}(\Omega_\varepsilon; u_\varepsilon)\frac{1}{2}\int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2 - \int_{\Omega_\varepsilon} bu_\varepsilon = -\frac{1}{2}\int_{\Omega_\varepsilon} bu_\varepsilon \tag{66}$$

defined by the equality

$$\mathcal{J}(\Omega_\varepsilon; u_\varepsilon) := \mathcal{I}_\varepsilon(u_\varepsilon) \tag{67}$$

is the energy functional evaluated for the solution of the boundary value problem (63) posed in the singularly perturbed domain Ω_ε.

Proposition 4.1. *The energy admits the expansion with respect to the small parameter $\varepsilon \to 0$ of the following form:*

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}_\Omega(u) - \pi\varepsilon^2 \|\nabla u(\hat{x})\|^2 + o(\varepsilon^2) \ , \tag{68}$$

where $\|\nabla u(\hat{x})\|^2$ is the bulk energy density at the point $\hat{x} \in \Omega$ and u is a solution to (63) for $\varepsilon = 0$.

Remark 4.2. The bulk energy density functional $H^1(\Omega) \ni \varphi \mapsto \|\nabla\varphi(\hat{x})\|^2 \in \mathbb{R}$, in general, is not continuous at a point $\hat{x} \in \Omega$. Therefore, the bulk energy density is replaced by a continuous bilinear form $H^1(\Omega) \ni \varphi \mapsto \langle \mathcal{B}(\varphi), \varphi\rangle_{\Gamma_R} \in \mathbb{R}$. For the Laplacian in two spatial dimensions and the solution of unperturbed problem u which is harmonic in a neighborhood of $\hat{x}$, the appropriate continuous bilinear form with respect to $H^1(\Omega)$ norm, such that there is equality for u,

$$\|\nabla u(\hat{x})\|^2 = \langle \mathcal{B}(u), u\rangle_{\Gamma_R}$$

is given by (72) or (74). This replacement of $\|\nabla\varphi(\hat{x})\|^2$ by $\langle \mathcal{B}(\varphi), \varphi\rangle_{\Gamma_R}$ in the energy functional for problem (63) has been introduced in [23, 24] for the purposes of topological derivatives evaluation in the framework of domain decomposition method.

Note 4.1. If we combine (64) with (68), we arrive at the conclusion that the modified energy functional

$$H^1(\Omega) \ni \varphi \to \frac{1}{2}\int_\Omega \|\nabla\varphi\|^2 - \int_\Omega b\varphi - \pi\varepsilon^2 \langle \mathcal{B}(\varphi), \varphi\rangle_{\Gamma_R} \in \mathbb{R}$$

is an approximation of (64) which furnishes the topological derivative (68) but with the minimization over unperturbed space $H^1(\Omega)$. This observation is in fact used in the domain decomposition method for unilateral problems.

4.1 Domain Decomposition Technique

Now, we are going to decompose the linear elliptic problem (63) into two parts, defined in two disjoint domains Ω_R and $C(R, \varepsilon) := B_R \setminus \overline{B_\varepsilon} \subset \Omega$, $R > \varepsilon > 0$. Two non-overlapping subdomains $\Omega_R, C(R, \varepsilon)$ of Ω_ε are selected $\Omega_\varepsilon = \Omega_R \cup \Gamma_R \cup$

$C(R,\varepsilon)$, where we assume that $R > \varepsilon_0$, $\varepsilon \in (0,\varepsilon_0]$ and Γ_R stands for the exterior boundary ∂B_R of $C(R,\varepsilon)$. Since the gradient of Sobolev functions is not continuous for test functions in $H^1(\Omega)$, but it is the case for harmonic functions, we replace the pointwise values of the gradient of test functions by a representation formula valid only for the pointwise values of the gradient of a harmonic function.

Proposition 4.3. *If the function u is harmonic in a ball $B_R \subset \mathbb{R}^2$, of radius $R > 0$ and center at $\hat{x} \in \Omega$, then the gradient of u evaluated at $\hat{x}$ is given by*

$$\nabla u(\hat{x}) = \frac{1}{\pi R^3} \int_{\Gamma_R} (x - \hat{x}) u(x) \; . \tag{69}$$

Proof. The proof of this result we leave as an exercise. □

In view of (69), since $b \equiv 0$ in B_R for sufficiently small $R > \varepsilon_0$, expansion (68) can be rewritten in the equivalent form

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega) - \frac{\varepsilon^2}{\pi R^6} \left[\left(\int_{\Gamma_R} u\, x_1 \right)^2 + \left(\int_{\Gamma_R} u\, x_2 \right)^2 \right] + o(\varepsilon^2) \; , \tag{70}$$

where $x - \hat{x} = (x_1, x_2)$. As observed in [23, 24], it is interesting to note that (70) can be rewritten as follows

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega) - \varepsilon^2 \langle \mathcal{B}(u), u \rangle_{\Gamma_R} + o(\varepsilon^2) \; . \tag{71}$$

with the nonlocal, positive and self-adjoint boundary operator $\mathcal{B}$ uniquely determined by its bilinear form

$$\langle \mathcal{B}(u), u \rangle_{\Gamma_R} = \frac{1}{\pi R^6} \left[\left(\int_{\Gamma_R} u\, x_1 \right)^2 + \left(\int_{\Gamma_R} u\, x_2 \right)^2 \right] . \tag{72}$$

From the above representation, since the line integrals on Γ_R are well defined for functions in $L^1(\Gamma_R)$, it follows that the operator $\mathcal{B}$ can be extended e.g., to a bounded operator on $L^2(\Gamma_R)$, namely

$$\mathcal{B} \in \mathcal{L}(L^2(\Gamma_R); L^2(\Gamma_R)) \; , \tag{73}$$

with the same symmetric bilinear form

$$\langle \mathcal{B}(\varphi), \phi \rangle_{\Gamma_R} = \frac{1}{\pi R^6} \left[\int_{\Gamma_R} \varphi\, x_1 \int_{\Gamma_R} \phi\, x_1 + \int_{\Gamma_R} \varphi\, x_2 \int_{\Gamma_R} \phi\, x_2 \right] , \tag{74}$$

which is continuous for all $\varphi, \phi \in L^2(\Gamma_R)$. We observe that the bilinear form

$$L^2(\Gamma_R) \times L^2(\Gamma_R) \ni (\varphi, \phi) \mapsto \langle \mathcal{B}(\varphi), \phi \rangle_{\Gamma_R} \in \mathbb{R} \tag{75}$$

is continuous with respect to the weak convergence since it has the simple structure

$$\langle \mathcal{B}(\varphi), \phi \rangle_{\Gamma_R} = L_1(\varphi) L_1(\phi) + L_2(\varphi) L_2(\phi) \quad \varphi, \phi \in L^1(\Gamma_R) \tag{76}$$

with two linear forms $\varphi \mapsto L_1(\varphi)$ and $\phi \mapsto L_2(\phi)$, given by the line integrals on Γ_R.

4.2 Steklov–Poincaré Pseudodifferential Boundary Operators

Note 4.2. We determine the family Steklov–Poincaré boundary operators on the *outer boundary* Γ_R of the domain $C(R, \varepsilon)$, if there is a hole B_ε inside of $C(R, \varepsilon)$.

We select $R > 0$ such that the circle (or the ball for $d = 3$) B_R contains the hole B_ε and introduce the truncated domain Ω_R. For the boundary value problem defined in Ω_ε, we introduce its approximation in Ω_R. The singular perturbation Ω_ε of the geometrical domain Ω is replaced by a regular perturbation of the Steklov–Poincaré boundary operator living on the interface, which coincides with the interior boundary Γ_R of Ω_R.

Definition 4.4. The Steklov–Poincaré boundary operator

$$\mathcal{A}_\varepsilon : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R) \tag{77}$$

is defined for the Poisson equation in the domain $C(R, \varepsilon)$. For a fixed parameter $\varepsilon > 0$ and a given element $v \in H^{1/2}(\Gamma_R)$, the corresponding element in the range of the operator $\mathcal{A}_\varepsilon$ is given by the Neumann trace of a unique solution to the boundary value problem

$$\begin{cases} \text{Find } w_\varepsilon \text{ such that} \\ -\Delta w_\varepsilon = 0 \text{ in } C(R, \varepsilon) \,, \\ \quad w_\varepsilon = v \text{ on } \Gamma_R \,, \\ \partial_n w_\varepsilon = 0 \text{ on } \partial B_\varepsilon \,. \end{cases} \tag{78}$$

Then we set

$$\mathcal{A}_\varepsilon(v) = \partial_n w_\varepsilon \quad \text{on} \quad \Gamma_R \,, \tag{79}$$

where n is the unit exterior normal vector on $\partial C(R, \varepsilon)$.

Remark 4.5. Let us note that, in absence of the source term b, the energy shape functional in $C(R, \varepsilon)$ evaluated for the harmonic function w_ε coincides with the boundary energy of the Steklov–Poincaré operator on Γ_R evaluated for the Dirichlet trace of the solution w_ε, namely

$$\int_{C(R,\varepsilon)} \|\nabla w_\varepsilon\|^2 = \langle \mathcal{A}_\varepsilon(v), v\rangle_{\Gamma_R} \ . \tag{80}$$

Therefore, the asymptotics of the energy shape functional in $C(R,\varepsilon)$ for $\varepsilon \to 0$, gives rise to the regular expansion of the Steklov–Poincaré operator:

$$\mathcal{A}_\varepsilon = \mathcal{A} - 2\varepsilon^2\mathcal{B} + \mathcal{R}_\varepsilon \ , \tag{81}$$

where the remainder denoted by $\mathcal{R}_\varepsilon$ in the above expansion is of order $o(\varepsilon^2)$ in the operator norm $\mathcal{L}(H^{1/2}(\Gamma_R); H^{-1/2}(\Gamma_R))$.

By Remark 4.5 we obtain the strong convergence of solutions in the truncated domain. In fact, let us state the following important result:

Proposition 4.6. *The sequence of solutions u_ε converges as $\varepsilon \to 0$ in the following sense. For any $R > 0$,*

$$u_\varepsilon^R \to u^R \quad \text{strongly in} \quad H^1(\Omega_R) \ , \tag{82}$$

where $\Omega_R := \Omega \setminus \overline{B_R}$, $\varepsilon \in (0, \varepsilon_0]$, and $R > \varepsilon_0 > 0$, where B_R is a ball of radius R and center at $\hat{x} \in \Omega$.

Proof. Let u_ε^R be the restriction to Ω_R of the solution u_ε to (63), namely

$$u_\varepsilon^R \in H^1_\Gamma(\Omega_R) : \int_{\Omega_R} \nabla u_\varepsilon^R \cdot \nabla\eta + \int_{\Gamma_R} \mathcal{A}_\varepsilon(u_\varepsilon^R)\eta = \int_{\Omega_R} b\eta \quad \forall \eta \in H^1_\Gamma(\Omega_R) \ . \tag{83}$$

In the same way, for $\varepsilon = 0$ we have

$$u^R \in H^1_\Gamma(\Omega_R) : \int_{\Omega_R} \nabla u^R \cdot \nabla\eta + \int_{\Gamma_R} \mathcal{A}(u^R)\eta = \int_{\Omega_R} b\eta \quad \forall \eta \in H^1_\Gamma(\Omega_R) \ , \tag{84}$$

where u^R is the restriction to Ω_R of the solution to (63) for $\varepsilon = 0$. In addition, $H^1_\Gamma(\Omega_R)$ is a subset of $H^1(\Omega_R)$, which is defined as

$$H^1_\Gamma(\Omega_R) := \{\varphi \in H^1(\Omega_R) : \varphi_{|\Gamma} = 0\} \ . \tag{85}$$

By taking $\eta = u_\varepsilon^R - u^R$ and after subtracting the second equation from the first one we get

$$\int_{\Omega_R} \|\nabla(u_\varepsilon^R - u^R)\|^2 + \int_{\Gamma_R} (\mathcal{A}_\varepsilon(u_\varepsilon^R) - \mathcal{A}(u^R))(u_\varepsilon^R - u^R) = 0 \ . \tag{86}$$

By taking into account the expansion (81) we observe that

$$\int_{\Omega_R} \|\nabla(u_\varepsilon^R - u^R)\|^2 = \int_{\Gamma_R} (2\varepsilon^2 \mathcal{B}(u^R) - \mathcal{R}_\varepsilon(u^R))(u_\varepsilon^R - u^R) \ . \tag{87}$$

From the *Cauchy–Schwarz inequality* we obtain

$$\int_{\Omega_R} \|\nabla(u_\varepsilon^R - u^R)\|^2 \leqslant 2\varepsilon^2 \|\mathcal{B}(u^R)\|_{H^{-1/2}(\Gamma_R)} \|u_\varepsilon^R - u^R\|_{H^{1/2}(\Gamma_R)} + \|\mathcal{R}_\varepsilon(u^R)\|_{H^{-1/2}(\Gamma_R)} \|u_\varepsilon^R - u^R\|_{H^{1/2}(\Gamma_R)} \ . \tag{88}$$

Taking into account the *trace theorem* and the compactness of the remainder $\mathcal{R}_\varepsilon$, we have

$$\int_{\Omega_R} \|\nabla(u_\varepsilon^R - u^R)\|^2 \leqslant \varepsilon^2 C_1 \|u_\varepsilon^R - u^R\|_{H^1(\Omega_R)} \ . \tag{89}$$

Finally, from the *coercivity* of the bilinear form on the left hand side of the above inequality, namely,

$$c\|u_\varepsilon^R - u^R\|^2_{H^1(\Omega_R)} \leqslant \int_{\Omega_R} \|\nabla(u_\varepsilon^R - u^R)\|^2 \ , \tag{90}$$

we obtain

$$\|u_\varepsilon^R - u^R\|_{H^1(\Omega_R)} \leqslant C\varepsilon^2 \ , \tag{91}$$

which leads to the result, with $C = C_1/c$. □

Now, we make use of the Steklov–Poincaré operator defined above for the annulus $C(R, \varepsilon)$ in order to rewrite the energy shape functional in Ω_ε as a sum of integrals over Ω_R and of the boundary bilinear form on Γ_R,

$$\mathcal{J}(\Omega_\varepsilon) = \frac{1}{2}\int_{\Omega_R} \|\nabla u_\varepsilon\|^2 - \int_{\Omega_R} b u_\varepsilon + \frac{1}{2}\langle \mathcal{A}_\varepsilon(u_\varepsilon), u_\varepsilon\rangle_{\Gamma_R} \ , \tag{92}$$

which is possible since the source term b vanishes in the small ball B_R around the point $\hat{x} \in \Omega$.

In conclusion, another method of evaluation of the topological derivative for the energy shape functional is now available. We have the energy shape functional in the form

$$\mathcal{J}(\Omega_\varepsilon) = \inf_{\varphi \in H^1_\Gamma(\Omega_R)} \left\{ \frac{1}{2}\int_{\Omega_R} \|\nabla\varphi\|^2 - \int_{\Omega_R} b\varphi + \frac{1}{2}\langle \mathcal{A}_\varepsilon(\varphi), \varphi\rangle_{\Gamma_R} \right\} \ , \tag{93}$$

where $H^1_\Gamma(\Omega_R)$ is defined trough (85). Taking into account expansion (81), from (93) it follows by an elementary argument that

$$\mathcal{J}(\Omega_\varepsilon) = \inf_{\varphi \in H^1_\Gamma(\Omega_R)} \left\{ \frac{1}{2} \int_{\Omega_R} \|\nabla \varphi\|^2 - \int_{\Omega_R} b\varphi + \frac{1}{2} \langle \mathcal{A}(\varphi), \varphi \rangle_{\Gamma_R} \right\} - \varepsilon^2 \langle \mathcal{B}(u), u \rangle_{\Gamma_R} + o(\varepsilon^2) \,, \qquad (94)$$

where (94) coincides with (71). The range of applications of the presented method is not limited to linear problems only. In fact, this is the only available method without any strict complementarity type assumptions on the unknown solution of the variational inequality, for evaluation of topological derivatives of the energy shape functional for unilateral problems.

5 Domain Decomposition Technique for Topological Derivatives Evaluation

The method of compound asymptotic expansions is usually used for the purposes of asymptotic analysis of elliptic boundary value problems in singularly perturbed geometrical domains. The application of this method requires the linearization of the boundary value problem under considerations which becomes quite involved in the case of variational inequalities [1]. Therefore, the domain decomposition technique was proposed and used in [23, 24], as well as used in [20] for the purposes of *topological derivation* for variational inequalities which describe the static frictionless contact between an elastic body and a rigid foundation as well as for cracks with the unilateral non-penetration condition.

We recall that the Sobolev space $H^1(\Omega)$ is the Dirichlet space for the natural order, we refer the reader e.g. to Frémiot et al. [6] for further details in the case of contact problems in linear elasticity. By the Dirichlet-Sobolev space we mean the ordered Sobolev spaces e.g., $H^1(\Omega)$ or $H^{1/2}(\partial\Omega)$ with the following property for the natural order. If the function $x \mapsto u(x)$ is in the Sobolev space, then the function $x \mapsto u^+(x) := \max\{u(x), 0\}$ belongs to the Sobolev space.

5.1 Problem Formulation

Let us consider the new boundary value problem, with nonlinear boundary conditions on $\Gamma_c \subset \Omega$. For the domain with a hole $B_\varepsilon(\hat{x})$, where $\hat{x} \in \Omega$, the *boundary value problem* takes the following form:

$$\begin{cases} \text{Find } u_\varepsilon \text{ such that} \\ -\Delta u_\varepsilon = b \quad \text{in } \Omega_\varepsilon \ , \\ u_\varepsilon = 0 \quad \text{on } \Gamma \ , \\ \partial_n u_\varepsilon = 0 \quad \text{on } \partial B_\varepsilon \ , \\ \left. \begin{array}{r} u_\varepsilon \geqslant 0 \\ \partial_n u_\varepsilon \leqslant 0 \\ u_\varepsilon \, \partial_n u_\varepsilon = 0 \end{array} \right\} \text{ on } \Gamma_c \ , \end{cases} \tag{95}$$

where the source term $b \in C^{0,\alpha}(\overline{\Omega})$ vanishes in the neighborhood of the point $\hat{x} \in \Omega$. A weak solution u_ε of problem (95) minimizes the energy functional (64) over a cone in the Sobolev space, and the shape energy functional takes the form

$$\mathcal{J}(\Omega_\varepsilon) = \inf_{\varphi \in \{\mathcal{V}_\varepsilon : \varphi_{|\Gamma_c} \geqslant 0\}} \left\{ \frac{1}{2} \int_{\Omega_\varepsilon} \|\nabla \varphi\|^2 - \int_{\Omega_\varepsilon} b\varphi \right\} \ , \tag{96}$$

where the liner space $\mathcal{V}_\varepsilon$ is defined by (65).

Now, let us consider the domain decomposition method for (95), assuming that $\Gamma_c \subset \Omega_R$. In particular, this means that the linear space $H^1_\Gamma(\Omega_R)$ defined through (85) is replaced in (93) by the *convex and closed subset*

$$\mathcal{K} := \{\varphi \in H^1_\Gamma(\Omega_R) : \varphi_{|\Gamma_c} \geqslant 0\} \ , \tag{97}$$

and the functional including the Steklov–Poincaré operator is as follows

$$\mathcal{I}^R_\varepsilon(u^R_\varepsilon) = \inf_{\varphi \in \mathcal{K}} \left\{ \frac{1}{2} \int_{\Omega_R} \|\nabla \varphi\|^2 - \int_{\Omega_R} b\varphi + \frac{1}{2} \langle \mathcal{A}_\varepsilon(\varphi), \varphi \rangle_{\Gamma_R} \right\} \ . \tag{98}$$

In order to establish the equality

$$\mathcal{I}^R_\varepsilon(u^R_\varepsilon) \equiv \mathcal{J}(\Omega_\varepsilon) \ , \tag{99}$$

it is sufficient to show that the minimizer u^R_ε in (98) coincides with the restriction to Ω_R of the minimizer u_ε of the corresponding quadratic functional defined in the whole singularly perturbed domain Ω_ε, which is left as an exercise. In this way we obtain

$$\begin{aligned} \mathcal{J}(\Omega_\varepsilon) &= \frac{1}{2} \int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2 - \int_{\Omega_\varepsilon} b u_\varepsilon \\ &= \frac{1}{2} \int_{\Omega_R} \|\nabla u_\varepsilon\|^2 - \int_{\Omega_R} b u_\varepsilon + \frac{1}{2} \langle \mathcal{A}_\varepsilon(u_\varepsilon), u_\varepsilon \rangle_{\Gamma_R} \\ &= \mathcal{I}^R_\varepsilon(u^R_\varepsilon) \\ &= \inf_{\varphi \in \mathcal{K}} \left\{ \frac{1}{2} \int_{\Omega_R} \|\nabla \varphi\|^2 - \int_{\Omega_R} b\varphi + \frac{1}{2} \langle \mathcal{A}_\varepsilon(\varphi), \varphi \rangle_{\Gamma_R} \right\} \ , \end{aligned} \tag{100}$$

thus, the topological derivative of $\mathcal{J}(\Omega)$ can be evaluated by using the expansion of $\mathcal{I}_\varepsilon^R(u_\varepsilon^R)$. The assumption required for the derivation of $\mathcal{I}_\varepsilon^R(u_\varepsilon^R)$ with respect to the parameter ε at $\varepsilon = 0^+$ is only the strong convergence as $\varepsilon \to 0$ for fixed $R > 0$, namely $u_\varepsilon^R \to u^R$ strongly in $H^1(\Omega_R)$, i.e., there is no need for differentiability properties of the minimizer $u_\varepsilon^R \in H^1(\Omega_R)$ with respect to ε (see the proof of Proposition 4.6).

5.2 *Hadamard Differentiability of Minimizer for Parametric Programming in Function Spaces*

The existence of the conical differential for the mapping

$$[0, \varepsilon_0) \ni \varepsilon \mapsto u_\varepsilon^R \in H^1(\Omega_R) \tag{101}$$

is established.

We introduce:

- The quadratic functional

$$\mathcal{G}^R(\varphi) := \frac{1}{2}a^R(\varphi,\varphi) - l^R(\varphi) + \frac{1}{2}\langle \mathcal{A}(\varphi), \varphi\rangle_{\Gamma_R} - \varepsilon^2\langle \mathcal{B}(\varphi), \varphi\rangle_{\Gamma_R} \, , \tag{102}$$

  where

$$a^R(\varphi,\varphi) = \int_{\Omega_R} \|\nabla\varphi\|^2 \quad \text{and} \quad l^R(\varphi) = \int_{\Omega_R} b\varphi \, . \tag{103}$$

- The coincidence set

$$\Xi := \{x \in \Gamma_c : u^R = 0\} \, . \tag{104}$$

- The linear form (non-negative measure)

$$\langle \mu_c, \varphi\rangle := a^R(u^R, \varphi) - l^R(\varphi) + \langle \mathcal{A}(u^R), \varphi\rangle_{\Gamma_R} \, . \tag{105}$$

- The convex cone

$$\mathcal{S}_K(u^R) = \{\varphi \in H^1_\Gamma(\Omega_R) : \varphi \geqslant 0 \quad \text{q.e. on} \quad \Xi, \quad \langle \mu_c, \varphi\rangle = 0\} \, . \tag{106}$$

We recall that the symbol q.e. reads "quasi everywhere" and it means, everywhere, with possible exception on a set of *null capacity*.

Theorem 5.1. *For fixed $R > 0$ we have*

$$\|u_\varepsilon^R - u^R\|_{H^1_\Gamma(\Omega_R)} \leqslant C_R \varepsilon^2 \ . \tag{107}$$

Furthermore, there is an expansion with respect to $\varepsilon \to 0^+$,

$$u_\varepsilon^R = u^R + \varepsilon^2 v^R + o^R(\varepsilon^2) \quad in \quad H^1(\Omega_R) \ . \tag{108}$$

The element $v^R \in H^1(\Omega_R)$ is uniquely determined by a solution to the following quadratic minimization problem

$$\mathcal{G}^R(v^R) = \inf_{\varphi \in \mathcal{S}_K(u^R)} \mathcal{G}^R(\varphi) \ . \tag{109}$$

Remark 5.2. The result established in Theorem 5.1 can be obtained as well for a class of contact problems by an application of general results given in [6, 25].

5.3 Topological Derivatives

In this section the outline of the domain decomposition method for variational inequalities is given. The topological derivative can be evaluated for the energy shape functional. The scalar elliptic equation as well as the linear elasticity system in two spatial dimensions with the unilateral conditions far from the hole are considered. The case of three spatial dimensions can be described in the same manner. The unilateral conditions are imposed for the weak solutions of elliptic boundary value problems by a cone constraint for the minimization of the quadratic energy functional. We recall that the cone of admissible displacements in contact problems of linear elasticity is defined by the non-penetration condition. The unilateral condition is only an approximation of the real condition and it is prescribed for normal displacements in the contact zone. Thus the normal displacements in the contact zone belong to a positive cone in the space of traces.

In this part we restrict ourselves to the circular holes. Let us recall the notation for the domain decomposition technique. Given a domain $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon} \subset \mathbb{R}^2$, with a small hole $B_\varepsilon \subset B_R$ of radius $\varepsilon \to 0$ and center at $\hat{x} \in \Omega$, we denote by $\Omega_R = \Omega \setminus \overline{B_R}$ the domain without the hole B_ε, and by $C(R, \varepsilon) = B_R \setminus \overline{B_\varepsilon}$ the ring with the small hole B_ε inside. It means that the domain Ω_ε is decomposed into two subdomains, the truncated one Ω_R and the ring $C(R, \varepsilon)$. The main idea which is employed here is to perform the asymptotic analysis for a linear problem and then apply the result to the nonlinear problem in a smaller domain called truncated domain. This is possible for unilateral conditions prescribed on $\Gamma_c \subset \Omega_R$, where the set Γ_c is far from the hole B_ε, and therefore far from the ball B_R.

Under this geometrical assumption it is possible to restrict the asymptotic analysis to the ring $C(R, \varepsilon)$. Then the obtained result on the asymptotic behavior of the associated solution to the boundary value problem defined in the ring is applied to the variational inequality considered in the truncated domain Ω_R. In this way the singular domain perturbation in the ring influences, by a regular perturbation, the boundary conditions on the interface for variational inequality. The regular perturbation is governed by a nonlocal, pseudodifferential, self-adjoint boundary operator of Steklov–Poincaré type. The nonlocal Steklov–Poincaré operator is introduced on the interface between two subdomains, it is the exterior boundary Γ_R of the ring, which is exactly the interior boundary of the truncated domain Ω_R. The subproblem to be solved in the truncated domain is a variational inequality associated to the constrained minimization problem over a closed convex cone $\mathcal{K} \subset H^1(\Omega_R)$:

Find a unique minimizer $u_\varepsilon \in \mathcal{K}$ of the quadratic energy functional

$$\mathcal{I}^R_\varepsilon(\varphi) = \frac{1}{2}a^R(\varphi,\varphi) - l^R(\varphi) + \frac{1}{2}\langle \mathcal{A}_\varepsilon(\varphi),\varphi\rangle_{\Gamma_R} , \tag{110}$$

where $\mathcal{A}_\varepsilon$ stands for the Steklov–Poincaré operator for the ring $C(R,\varepsilon)$ and $\langle\cdot,\cdot\rangle_{\Gamma_R}$ is the duality pairing defined for the fractional Sobolev spaces $H^{-1/2}(\Gamma_R)\times H^{1/2}(\Gamma_R)$ on the interface Γ_R, associated with the corresponding Steklov–Poincaré operator $\mathcal{A}_\varepsilon : H^{1/2}(\Gamma_R) \mapsto H^{-1/2}(\Gamma_R)$. We need an assumption on its asymptotic behavior, which is:

Condition 5.3 *The Steklov–Poincaré operator for the ring $C(R,\varepsilon)$ admits the expansion for $\varepsilon > 0$, ε small enough,*

$$\mathcal{A}_\varepsilon = \mathcal{A} - 2f(\varepsilon)\mathcal{B} + \mathcal{R}_\varepsilon , \tag{111}$$

with an appropriate function $f(\varepsilon) \to 0$, when $\varepsilon \to 0$, depending on the boundary conditions on the hole, where the remainder $\mathcal{R}_\varepsilon$ is of order $o(f(\varepsilon))$ in the operator norm $\mathcal{L}(H^{1/2}(\Gamma_R); H^{-1/2}(\Gamma_R))$.

Remark 5.4. In the scalar case the operator $\mathcal{B}$ is defined by the bilinear form (74). From (81) it follows that $f(\varepsilon) = \varepsilon^2$ for the Neumann boundary conditions on the hole B_ε. For our specific applications, expansion (111) results from the asymptotics of the shape energy functional in the ring $C(R,\varepsilon)$, as it is for the scalar problem. If the form of operator $\mathcal{B}$ in (111) is known, in order to apply the general scheme the only assumption to check is the compactness condition for the remainder in the operator norm $\mathcal{L}(H^{1/2}(\Gamma_R); H^{-1/2}(\Gamma_R))$.

Therefore, the original variational inequality defined in the domain Ω_ε is replaced by the variational inequality defined in the truncated domain Ω_R. In this way, for the purposes of asymptotic analysis the original quadratic functional defined in the domain of integration Ω_ε, namely $\mathcal{J}(\Omega_\varepsilon;\varphi)$, is replaced by the functional $\mathcal{I}^R_\varepsilon(\varphi)$ defined in the truncated domain without any hole. Two problems are equivalent under the following assumption on the minimizers u_ε and u^R_ε of $\mathcal{J}(\Omega_\varepsilon;\varphi)$ and $\mathcal{I}^R_\varepsilon(\varphi)$, respectively.

Condition 5.5 *For $\varepsilon > 0$, with ε small enough, the minimizer u_ε^R in the truncated domain coincides with the restriction to the truncated domain Ω_R of the minimizer u_ε in the singularly perturbed domain Ω_ε.*

If Conditions 5.3 and 5.5 are fulfilled, then the topological asymptotic expansion of the energy functional

$$\mathcal{J}(\Omega_\varepsilon; u_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2 - \int_{\Omega_\varepsilon} b u_\varepsilon \tag{112}$$

can be determined from the expansion of the energy functional in the truncated domain, namely

$$\mathcal{I}_\varepsilon^R(u_\varepsilon^R) = \frac{1}{2} a^R(u_\varepsilon^R, u_\varepsilon^R) - l^R(u_\varepsilon^R) + \frac{1}{2}\langle \mathcal{A}_\varepsilon(u_\varepsilon^R), u_\varepsilon^R \rangle_{\Gamma_R} , \tag{113}$$

where u_ε^R is the restriction to the truncated domain Ω_R of the solution u_ε to the variational inequality in the perturbed domain Ω_ε. Under our assumptions, the solution u_ε coincides with the solution obtained by the domain decomposition method.

The evaluation of the topological asymptotic expansion for the energy functional (112) is based on the equality (99), so we have $\mathcal{J}(\Omega_\varepsilon; u_\varepsilon) = \mathcal{I}_\varepsilon^R(u_\varepsilon^R)$, combined with the following characterization of the energy functional

$$\mathcal{I}_\varepsilon^R(u_\varepsilon^R) = \inf_{\varphi \in \mathcal{K}} \left\{ \frac{1}{2} a^R(\varphi, \varphi) - l^R(\varphi) + \frac{1}{2}\langle \mathcal{A}_\varepsilon(\varphi), \varphi \rangle_{\Gamma_R} \right\} . \tag{114}$$

The quadratic term $\varphi \mapsto \frac{1}{2}\langle \mathcal{A}_\varepsilon(\varphi), \varphi \rangle_{\Gamma_R}$ of the functional $\mathcal{I}_\varepsilon^R(\varphi)$ is, in view of assumption (111) or of Condition 5.3, the regular perturbation of the bilinear form in the quadratic functional $\mathcal{I}_\varepsilon^R(\varphi)$. Therefore, we obtain the result on the differentiability of the optimal value in (113) with respect to the parameter ε.

Proposition 5.6. *Assume that:*

- *The Condition 5.3 given by (111) holds in the operator norm.*
- *The strong convergence takes place $u_\varepsilon^R \to u^R$ in the norm of the space $H^1(\Omega_R)$, which also defines the energy norm for the functional (114).*

Then, the energy in the truncated domain Ω^R has the following topological asymptotic expansion

$$\mathcal{I}_\varepsilon^R(u_\varepsilon^R) = \mathcal{I}^R(u^R) - f(\varepsilon)\langle \mathcal{B}(u^R), u^R \rangle_{\Gamma_R} + o(f(\varepsilon)) , \tag{115}$$

where u^R is the restriction to the truncated domain Ω_R of the solution u to the original variational inequality in the unperturbed domain Ω. Therefore, the *topological derivative of the energy shape functional is obtained from the asymptotic expansion*

$$\mathcal{J}(\Omega_\varepsilon; u_\varepsilon) = \mathcal{J}(\Omega; u) - f(\varepsilon)\langle\mathcal{B}(u), u\rangle_{\Gamma_R} + o(f(\varepsilon)) \ . \tag{116}$$

Proof. There are inequalities

$$\frac{\mathcal{I}_\varepsilon^R(u_\varepsilon^R) - \mathcal{I}^R(u_\varepsilon^R)}{f(\varepsilon)} \leqslant \frac{\mathcal{I}_\varepsilon^R(u_\varepsilon^R) - \mathcal{I}^R(u^R)}{f(\varepsilon)} \leqslant \frac{\mathcal{I}_\varepsilon^R(u^R) - \mathcal{I}^R(u^R)}{f(\varepsilon)} \ , \tag{117}$$

which imply the existence of the limit

$$\begin{aligned} &\limsup_{f(\varepsilon)\to 0} \frac{\mathcal{I}_\varepsilon^R(u_\varepsilon^R) - \mathcal{I}^R(u_\varepsilon^R)}{f(\varepsilon)} = \\ &\lim_{f(\varepsilon)\to 0} \frac{\mathcal{I}_\varepsilon^R(u_\varepsilon^R) - \mathcal{I}^R(u^R)}{f(\varepsilon)} = \\ &\liminf_{f(\varepsilon)\to 0} \frac{\mathcal{I}_\varepsilon^R(u^R) - \mathcal{I}^R(u^R)}{f(\varepsilon)} = \langle\mathcal{B}(u^R), u^R\rangle_{\Gamma_R} \ . \end{aligned} \tag{118}$$

From (115), in view of (99), it follows (116). □

We can conclude the analysis for the Signorini problem, and confirm that the topological derivative of the energy shape functional is given by the same formula as it is in the linear case.

Theorem 5.7. *The energy functional for the Signorini problem admits the expansion*

$$\mathcal{J}(\Omega_\varepsilon; u_\varepsilon) = \mathcal{J}(\Omega; u) - \pi\varepsilon^2\|\nabla u\|^2 + o(\varepsilon^2) \ , \tag{119}$$

where the topological derivative $\mathcal{T}(\hat{x}) = -\|\nabla u(\hat{x})\|^2$ is the negative bulk energy density at the point $\hat{x} \in \Omega$. Since the solution of the Signorini problem is harmonic in a vicinity of $\hat{x}$, the expansion is well defined. Therefore, the topological derivative of the energy shape functional is given by the same expression as it is in the case of linear problem.

6 Conical Differentiability of Metric Projections in Dirichlet Spaces onto Positive Cones and Applications to the Shape Sensitivity Analysis of Variational Inequalities

The conical differentiability of the metric projection onto the positive cone in the Dirichlet space is considered in [6, 25] with applications to the sensitivity analysis of variational inequalities. There are numerous applications of such results for the shape sensitivity analysis of the Signorini problem and frictionless contact problems in elasticity [25], crack models with unilateral non-penetration condition [6]. We recall that the shape differentiability of the energy functional for cracks with unilateral non-penetration condition which is established in [12], does require only the appropriate strong shape continuity of solutions to variational inequalities and can be obtained under mild regularity assumptions on the governing variational inequality [6]. In Sect. 6.3 the topological derivative of the energy functional is given for the elastic body with a rigid inclusion, weakened by a crack on the boundary of the inclusion. It is assumed that on the crack the unilateral non-penetration condition is prescribed which makes the analysis more involved [20] compared to the linear case.

For the convenience of the reader we recall here the abstract result [25] which is a generalization of the implicit function theorem for variational inequalities. We use the result on the Hadamard differentiability of the metric projection on polyhedral convex sets in Hilbert spaces due to Mignot and Haraux, we refer the reader to [6] for a simple proof of such a result.

6.1 Generalization of Implicit Function Theorem for Variational Inequalities. Hadamard Differentiability of Solutions to Variational Inequalities.

Let $\mathcal{K} \subset \mathcal{V}$ be a convex and closed subset of a Hilbert space $\mathcal{V}$, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$, where $\mathcal{V}'$ denotes the dual of $\mathcal{V}$. We shall consider the following family of variational inequalities depending on a parameter $t \in [0, t_0)$, $t_0 > 0$,

$$u_t \in \mathcal{K} \ : \ a_t(u_t, \varphi - u_t) \geq \langle b_t, \varphi - u_t \rangle \qquad \forall \varphi \in \mathcal{K} \ . \tag{120}$$

Moreover, let $u_t = \mathcal{P}_t(b_t)$ be a solution to (120). For $t = 0$ we denote

$$u \in \mathcal{K} \ : \ a(u, \varphi - u) \geq \langle b, \varphi - u \rangle \qquad \forall \varphi \in \mathcal{K} \ , \tag{121}$$

with $u = \mathcal{P}(b)$ solution to (121).

Theorem 6.1. *Let us assume that:*

- *The bilinear form $a_t(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is coercive and continuous uniformly with respect to $t \in [0, t_0)$. Let $\mathcal{Q}_t \in \mathcal{L}(\mathcal{V}; \mathcal{V}')$ be the linear operator defined as follows $a_t(\phi, \varphi) = \langle \mathcal{Q}_t(\phi), \varphi \rangle$ $\forall \phi, \varphi \in \mathcal{V}$; it is supposed that there exists $\mathcal{Q}' \in \mathcal{L}(\mathcal{V}; \mathcal{V}')$ such that*

$$\mathcal{Q}_t = \mathcal{Q} + t\mathcal{Q}' + o(t) \quad in \quad \mathcal{L}(\mathcal{V}; \mathcal{V}') \ . \tag{122}$$

- *For $t > 0$, t small enough, the following equality holds*

$$b_t = b + tb' + o(t) \quad in \quad \mathcal{V}' , \tag{123}$$

where $b_t, b, b' \in \mathcal{V}'$.

- *The set $\mathcal{K} \subset \mathcal{V}$ is convex and closed, and for the solutions to the variational inequality*

$$\Pi b = \mathcal{P}(b) \in \mathcal{K} : \quad a(\Pi b, \varphi - \Pi b) \geq \langle b, \varphi - \Pi b \rangle \quad \forall \varphi \in \mathcal{K} \tag{124}$$

the following differential stability result holds

$$\forall h \in \mathcal{V}' : \quad \Pi(b + sh) = \Pi b + s\Pi' h + o(s) \quad in \quad \mathcal{V} \tag{125}$$

for $s > 0$, s small enough, where the mapping $\Pi' : \mathcal{V}' \to \mathcal{V}$ is continuous and positively homogeneous and $o(s)$ is uniform, with respect to $h \in \mathcal{V}'$, on compact subsets of $\mathcal{V}'$.

Then, the solutions to the variational inequality (120) *are right-differentiable with respect to t at $t = 0$, i.e. for $t > 0$, t small enough,*

$$u_t = u + tu' + o(t) \quad in \quad \mathcal{V} , \tag{126}$$

where

$$u' = \Pi'(b' - \mathcal{Q}' u) . \tag{127}$$

Let us note, that for $b_t = 0$ and $u_t = \mathcal{P}_t(0)$ we obtain $u' = \Pi'(-\mathcal{Q}' u)$.

6.2 Applications to Unilateral Contact Problems

We recall a result on the topological derivatives of the energy functional for elastic bodies with rigid inclusions with cracks on the interfaces. We refer to [20] for the proof.

Let us introduce the description of the convex cone $\mathcal{S}_K(u)$,

$$\mathcal{S}_K(u) = \left\{ \varphi \in H^{1,\omega}_{\Gamma}(\Omega_\Upsilon) : [\![\varphi]\!] \cdot n \geqslant 0 \text{ on } \Upsilon_0; \int_{\Omega\setminus\overline{\omega}} \sigma(u) \cdot \nabla\varphi^s = \int_{\Omega_\Upsilon} b \cdot \varphi \right\} \tag{128}$$

where $\Upsilon_0 = \{x \in \Upsilon : (u - \rho_0) \cdot n = 0\}$, where $\rho_0 := u_{|_\omega}$. We have the following result:

Theorem 6.2. *Let there be given the right hand side $b_t = b + th$ of the variational inequality which governs the unilateral contact problem under investigations, then* *the unique solution $u_t \in \mathcal{K}_\omega$ is Lipschitz continuous*

$$\|u_t - u\|_{H^1(\Omega_\Upsilon;\mathbb{R}^2)} \leqslant Ct \tag{129}$$

and conically differentiable in $H^1(\Omega_\Upsilon;\mathbb{R}^2)$, that is, for $t > 0$, t small enough,

$$u_t = u + tv + o(t) \ , \tag{130}$$

where the conical differential solves the variational inequality

$$v \in \mathcal{S}_K(u) : \int_{\Omega\setminus\overline{\omega}} \sigma(v) \cdot \nabla(\eta - v)^s \geqslant \int_{\Omega_\Upsilon} h \cdot (\eta - v) \quad \forall \eta \in \mathcal{S}_K(u) \ . \tag{131}$$

The remainder converges to zero

$$\frac{1}{t}\|o(t)\|_{H^1(\Omega_\Upsilon;\mathbb{R}^2)} \underset{t\to 0}{\to} 0 \tag{132}$$

uniformly with respect to the direction h on the compact sets of the dual space $(H^{1,\omega}_\Gamma(\Omega_\Upsilon))'$. Thus, v is the Hadamard directional derivative of the solution to the variational inequality with respect to the right hand side.

6.3 Example: Topological Derivative of Energy Functional for the Crack on Boundaries of Rigid Inclusions

We present an example of shape-topological sensitivity analysis for a crack located on the boundary of a rigid inclusion. The rigid inclusion can be considered as the limit case of elastic inclusions. In this particular case the general theory applies and we are able to present the topological derivative of the energy functional following [20].

Let us now consider a singularly perturbed domain $\Omega_\varepsilon(\hat{x}) = \Omega \setminus \overline{B_\varepsilon(\hat{x})}$, where $B_\varepsilon(\hat{x})$ is a ball of radius $\varepsilon > 0$, $\varepsilon \to 0$, and center at $\hat{x} \in \Omega \setminus \overline{\omega}$. We assume that the hole B_ε do not touch the rigid inclusion ω, namely $\overline{B_\varepsilon} \Subset \Omega \setminus \overline{\omega}$.

We are interested in the topological asymptotic expansion of the energy shape functional of the form

$$\mathcal{J}(\Omega_\varepsilon;\varphi) = \frac{1}{2}\int_{\Omega_\varepsilon\setminus\overline{\omega}} \sigma(\varphi) \cdot \nabla\varphi^s - \int_{\Omega_\Upsilon} b \cdot \varphi \ , \tag{133}$$

with $\varphi = u_\varepsilon$ solution to the following *nonlinear system*:

$$\begin{cases} \text{Find } u_\varepsilon \text{ such that} \\ \qquad -\text{div}\sigma(u_\varepsilon) = b \quad \text{in } \Omega_\varepsilon \setminus \overline{\omega}\ , \\ \qquad \sigma(u_\varepsilon) = \mathbb{C}\nabla u_\varepsilon^s\ , \\ \qquad u_\varepsilon = 0 \quad \text{on } \Gamma\ , \\ \qquad \sigma(u_\varepsilon)n = 0 \quad \text{on } \partial B_\varepsilon\ , \\ \left.\begin{aligned} (u_\varepsilon - \rho_0)\cdot n &\geqslant 0 \\ \sigma^\tau(u_\varepsilon) &= 0 \\ \sigma^{nn}(u_\varepsilon) &\leqslant 0 \\ \sigma^{nn}(u_\varepsilon)(u_\varepsilon - \rho_0)\cdot n &= 0 \end{aligned}\right\} \quad \text{on } \Upsilon^+\ , \\ \qquad -\int_{\partial\omega} \sigma(u_\varepsilon)n\cdot\rho = \int_\omega b\cdot\rho \ \ \forall\ \ \rho \in \mathcal{R}(\omega)\ . \end{cases} \tag{134}$$

Since the problem is nonlinear, let us introduce two disjoint domains Ω_R and $C(R,\varepsilon)$, with $\Omega_R = \Omega \setminus \overline{B_R(\hat{x})}$ and $C(R,\varepsilon) = B_R \setminus \overline{B_\varepsilon} \Subset \Omega \setminus \overline{\omega}$, where $B_R(\hat{x})$ is a ball of radius $R > \varepsilon$ and center at $\hat{x} \in \Omega \setminus \overline{\omega}$. For the sake of simplicity, we assume that $b = 0$ in $B_R(\hat{x})$, that is, the source term b vanishes in the neighborhood of the point $\hat{x} \in \Omega \setminus \overline{\omega}$. Thus, we have the following linear elasticity system defined in the ring $C(R,\varepsilon)$:

$$\begin{cases} \text{Find } w_\varepsilon \text{ such that} \\ -\text{div}\sigma(w_\varepsilon) = 0 \quad \text{in } C(R,\varepsilon)\ , \\ \qquad \sigma(w_\varepsilon) = \mathbb{C}\nabla w_\varepsilon^s\ , \\ \qquad w_\varepsilon = v \quad \text{on } \Gamma_R\ , \\ \qquad \sigma(u_\varepsilon)n = 0 \quad \text{on } \partial B_\varepsilon\ , \end{cases} \tag{135}$$

where Γ_R is used to denote the exterior boundary ∂B_R of the ring $C(R,\varepsilon)$. We are interested in the Steklov–Poincaré operator on Γ_R, that is

$$\mathcal{A}_\varepsilon : v \in H^{1/2}(\Gamma_R;\mathbb{R}^2) \to \sigma(w_\varepsilon)n \in H^{-1/2}(\Gamma_R;\mathbb{R}^2)\ . \tag{136}$$

Then we have $\sigma(u_\varepsilon^R)n = \mathcal{A}_\varepsilon(u_\varepsilon^R)$ on Γ_R, where u_ε^R is solution of the variational inequality in Ω_R, that is

$$u_\varepsilon^R \in \mathcal{K}_\omega : \int_{\Omega_R} \sigma(u_\varepsilon^R)\cdot\nabla(\eta - u_\varepsilon^R) + \int_{\Gamma_R} \mathcal{A}_\varepsilon(u_\varepsilon^R)\cdot(\eta - u_\varepsilon^R)$$
$$\geqslant \int_{\Omega_\Upsilon \setminus \overline{B_R}} b\cdot(\eta - u_\varepsilon^R) \quad \forall \eta \in \mathcal{K}_\omega\ . \tag{137}$$

Finally, in the ring $C(R,\varepsilon)$ we have

$$\int_{C(R,\varepsilon)} \sigma(w_\varepsilon)\cdot\nabla w_\varepsilon^s = \int_{\Gamma_R} \mathcal{A}_\varepsilon(w_\varepsilon)\cdot w_\varepsilon\ , \tag{138}$$

where w_ε is the solution of the elasticity system in the ring (135). Therefore the solutions u_ε^R and w_ε are defined as restriction of u_ε to the truncated domain Ω_R and to the ring $C(R,\varepsilon)$, respectively.

In particular, in the neighborhood of $\hat{x} \in \Omega \setminus \overline{\omega}$, the energy in the ring $C(R,\varepsilon)$ admits the following topological asymptotic expansion

$$\int_{C(R,\varepsilon)} \sigma(w_\varepsilon)\cdot\nabla w_\varepsilon^s = \int_{B_R} \sigma(w)\cdot\nabla w^s - 2\pi\varepsilon^2 \mathbb{P}\sigma(w(\hat{x}))\cdot\nabla w^s(\hat{x}) + o(\varepsilon^2)\ . \tag{139}$$

where w is solution to (135) for $\varepsilon = 0$ and $\mathbb{P}$ is the polarization tensor. It means that w is the restriction to the disk B_R of the solution u to the nonlinear system defined in the unperturbed domain Ω_Υ. Therefore, we have that the Steklov–Poincaré operator defined by (136) admits the expansion for $\varepsilon > 0$, with ε small enough,

$$\mathcal{A}_\varepsilon = \mathcal{A} - 2\varepsilon^2\mathcal{B} + o(\varepsilon^2)\ , \tag{140}$$

where the operator $\mathcal{B}$ is determined by its bilinear form

$$\langle \mathcal{B}(w), w\rangle_{\Gamma_R} = \pi\mathbb{P}\sigma(w(\hat{x}))\cdot\nabla w^s(\hat{x})\ . \tag{141}$$

From the above results, we have that the energy shape functional associated to the cracks on boundaries of rigid inclusions embedded in elastic bodies has the following topological asymptotic expansion

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega) - \pi\varepsilon^2\mathbb{P}\sigma(u(\hat{x}))\cdot\nabla u^s(\hat{x}) + o(\varepsilon^2)\ , \tag{142}$$

with the *topological derivative* $\mathcal{T}(\hat{x})$ given by

$$\mathcal{T}(\hat{x}) = -\mathbb{P}\sigma(u(\hat{x}))\cdot\nabla u^s(\hat{x})\ , \tag{143}$$

where u is solution of the variational inequality in the unperturbed domain Ω_Υ and $\mathbb{P}$ is the Pólya–Szegö polarization tensor.

Remark 6.3. From equality (138) we observe that the bilinear form (141) represents the topological derivative of the Steklov–Poincaré operator (136). In addition, since solution $u \in \mathcal{K}_\omega$ of the variational inequality is a $H^1(\Omega_\Upsilon;\mathbb{R}^2)$ function, then it is convenient to compute the topological derivative from quantities evaluated on the boundary Γ_R in similar way as for the scalar case.

7 Shape Sensitivity Analysis of the Griffith Functional

In a forthcoming paper the first order shape-topological sensitivity analysis of energy functionals is used to establish the shape differentiability of the so-called Griffith shape functional. We are going to describe briefly a result of this sort.

Example 7.1. Let $\Omega := \Omega_c \cup \Gamma \cup \Omega_i$ be an elastic body with the rectilinear crack $\Gamma_c \subset \Sigma \subset \Omega_c$, thus $\partial\Omega := \Gamma_c \cup \partial\Omega$. We consider the shape functional defined by (46) which is called the Griffith functional

$$J(\Omega) := \frac{1}{2}\int_{\Omega_c} \left\{\mathrm{div}V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V;u)\right\}\sigma_{ij}(u) - \int_{\Omega_c} \mathrm{div}(Vf_i)u_i\,,$$

where the displacement field u is given by the unique solution of the variational inequality

$$u \in K \;:\; a(u, v-u) \geqslant (f, v-u) \quad \forall u \in K\,, \tag{144}$$

and the velocity vector field V is compactly supported in Ω_c. We need the decomposition of Ω into Ω_c and Ω_i for the purposes of the domain decomposition technique to our problem. Let $\omega \subset \Omega_i$ be an elastic inclusion.

Proposition 7.2. *Assume that the energy shape functional $\mathcal{E}(\Omega_i)$ is shape differentiable in the direction of the velocity field W compactly supported in a neighborhood of the inclusion $\omega \subset \Omega_i$, then the Griffith functional is directionally differentiable in the direction of the velocity field W.*

The result is proved by the domain decomposition technique with a linear problem in Ω_i which is used to determine the expansion of the energy functional with respect to the boundary variations of an inclusion and the nonlinear problem in cracked subdomain Ω_c which is used to obtain the conical differentiability of the solution with respect to the variations of the Steklov–Poincaré operator:

- the differentiability of the energy functional in the subdomain Ω_i implies the differentiability of the associated Steklov–Poincaré operator defined on the Lipschitz curve given by the interface $\overline{\Omega}_i \cap \overline{\Omega}_c$ with respect to the scalar parameter $t \to 0$ which governs the boundary variations of the inclusion ω;
- the expansion of the Steklov–Poincaré nonlocal boundary pseudodifferential operator obtained in the subdomain Ω_i is used in the boundary conditions for the variational inequality defined in the cracked subdomain Ω_c and leads to the conical differential of the solution to the unilateral problem in the subdomain;
- the one term expansion of the solution to the unilateral problem is used in the Griffith functional in order to obtain the directional derivative with respect to the boundary variations of the inclusion.

Acknowledgements The authors acknowledge support by the European Science Foundation within the Programme 'Optimization with PDE Constraints (OPTPDE)'.

Jan Sokolowski is supported by the Brazilian Research Council (CNPq), through the Special Visitor Researcher Framework of the Science Without Borders Programme, under grant 400273/2012-8.

This work has been supported by the DFG-EC315 "Engineering of Advanced Materials" and by DFG-SFB 814 "Additive Manufacturing".

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