

Deciding the Borel Complexity of Regular Tree Languages

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Abstract. We show that it is decidable whether a given a regular tree language belongs to the class Δ_2^0 of the Borel hierarchy, or equivalently whether the Wadge degree of a regular tree language is countable.

1 Introduction

In [14] there was given an algorithm which for a *deterministic* parity tree automaton \mathcal{A} decides whether the language $L(\mathcal{A})$ is Borel. This was further extended to a finer classification in [12] and finally to a full Wadge classification in [13]. The algorithms look for a pattern in the graph of the automaton and decide the Borel and Wadge classes upon finding of these special patterns.

Similar problems for *non-deterministic* parity tree automata seem to be much harder. Recently in [2] was provided an algorithm which decides for a given non-deterministic parity tree automaton \mathcal{A} , whether $L(\mathcal{A})$ is a Boolean combination of open sets. For other Borel classes there was no known algorithm. This paper provides a relatively simple extension of the result in [2] to the class of $\Delta_2^0 = \Sigma_2^0 \cap \Pi_2^0$ sets, that is the sets which are simultaneously presentable as countable unions of closed sets and countable intersections of open sets. This result is presented in Section 4 in Theorem 1. The proofs in [2] are based on an analysis of an algebraic structure computable from \mathcal{A} and the main result states that the language $L(\mathcal{A})$ is a Boolean combination of open sets if and only if a certain finite number of algebraic requirements hold. Since the class Δ_2^0 is bigger, in order to characterize this class, the set of algebraic requirements must be relaxed. In this paper we show that indeed this is the case. Our proofs closely follow the proofs from [2] with some necessary adjustments. In particular the crucial concept of the topological cutting game introduced in [2] is considered in this paper not only in the finite, but also in the infinite case.

The approach presented in [2] and in the present paper in a certain sense is a reminiscent of the approach applied to deterministic automata in [12,13,14]. Namely, the algebraic structure computed from a given automaton \mathcal{A} induces a graph with edges reflecting the algebraic properties. In the deterministic case it is possible to decide Borel

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and Wadge classes analyzing patterns in the graph of the automaton, in the present paper we are looking for patterns in the algebraic graph.

Finally let us mention results which provide information about the set-theoretical complexity of a language accepted by a non-deterministic automaton \mathcal{A} assuming some additional properties of \mathcal{A} :

- Rabin in [16] proved, that if L and its complement are accepted by a non-deterministic Büchi tree automata, then L is weakly definable, in particular it is Borel.
- Recently in [5] it was shown using decidability results about the cost functions, that for a given non-deterministic Büchi tree automaton it is decidable whether the language is weakly definable.
- In [9] the decidability results regarding deterministic automata were lifted to a more general context of game automata.

This paper consists of four Sections: the Introduction, a preliminary Section 2 introducing automata, set-theoretical and algebraic notations, a Section 3 introducing topological games and linking these games to the Wadge hierarchy and Section 4 containing the main result. All missing proofs can be found in the long version of the present paper [8].

2 Preliminaries

Trees and Contexts. Given a finite alphabet A , a *tree over A* is a partial function $t : \{0, 1\}^* \rightarrow A$ such that its domain $\text{dom}(t)$ is prefix closed. A node of a tree t is an element $v \in \text{dom}(t)$. A left child of a node v of t is the node $v0$, while its right child is $v1$. A leaf of a tree is a node without children. We denote by T_A the family of all trees over A . A set of trees over A is called a tree language, or simply a language. A *multi-context over A* is a tree c over $A \cup \{\star\}$, where

- $\star \notin A$, and
- \star only labels some leaves of c .

A leaf of c labelled by \star is called a *port*. Notice that a multi-context may have infinitely many ports. For a multi-context c and a function η mapping each port of c to a tree t over A , by $c[\eta]$ we denote the tree given by inserting into every port x a tree $\eta(x)$. When $\eta(x) = t$ for each port x , we just write $c[t]$. We say that a tree t extends a multi-context c if there is a mapping η such that $c[\eta] = t$. Given a multi-context c and a language L , by $[c]^{-1}L$ we denote the language of trees $t \in L$ extending c . The class generated by c and all possible mappings η is denoted by $c[T_A]$. A finite multi-context is called a *prefix*. A multi-context with only one port is called a *context*.

Topology. For a finite alphabet A , we equip the class T_A of all trees over A with the prefix topology. That is the basic open sets are sets of the form $p[T_A]$, for a prefix p over A , and thus the open sets are of the form $\bigcup_{p \in P} p[T_A]$ for some set P of prefixes.

The class of Borel tree languages of T_A is the closure of the class of open sets of T_A with respect to countable unions and complementations. Given T_A , the initial finite levels of the Borel hierarchy are defined as follows:

- $\Sigma_1^0(T_A)$ is the class of open subsets of T_A ,
- $\Pi_n^0(T_A)$ consists of complements of sets from $\Sigma_n^0(T_A)$,
- $\Sigma_{n+1}^0(T_A)$ consists of countable unions of sets from $\Pi_n^0(T_A)$.

A much finer measure of the topological complexity is the *Wadge degree* (see [10, Chapter 21.E]). If $L \subseteq T_A$ and $M \subseteq T_B$, we say that L is *continuously (or Wadge) reducible* to M , if there exists a continuous function $f : T_A \rightarrow T_B$ such that $L = f^{-1}(M)$. We write $L \leq_W M$ iff L is continuously reducible to M . This pre-ordering is called the *Wadge ordering*. If $L \leq_W M$ and $M \leq_W L$, then we write $L \equiv_W M$. If $L \leq_W M$ but not $M \leq_W L$, then we write $L <_W M$. The Wadge hierarchy is the partial order induced by $<_W$ on the equivalence classes given by \equiv_W . A language L is called *self dual* if it is equivalent to its complement, otherwise it is called *non self dual*.

Given a certain family of sets \mathcal{C} , we say that M is \mathcal{C} -hard if $L \leq_W M$ for every $L \in \mathcal{C}$. A \mathcal{C} -hard set L is said to be \mathcal{C} -complete if moreover $L \in \mathcal{C}$.

Algebra. The Wadge hierarchy of the regular languages of infinite words is well understood thanks to a classification result by K. Wagner ([18]). In particular from Wagner's result one can derive an algorithm which decides whether a given regular language of infinite words is a Boolean combination of open sets. Being a Boolean combination of open sets is equivalent to being in Δ_2^0 class in the context of regular languages of infinite words. This is not true for the regular languages of infinite trees (see [2, Section 4.1] and Proposition 7 in [8] - the long version of the present paper - for an analysis of one special case). A natural algebraic interpretation of Wagner's result can be found in [15, Theorem V.6.2]. In the case of languages of infinite trees, the algebraic theory is not yet fully developed. As a general reference may serve papers [1,2,3,4]. For details of the approach applied in the present paper refer to [2, Section 3].

Following the approach presented in [2], the family of all trees T_A is divided into finitely many Myhill-Nerode equivalence classes H_L . Similarly, there are finitely many equivalence classes V_L of contexts. The same holds for multi-contexts with a fixed number of holes. Starting from an automaton accepting language L , one can compute families H_L and V_L . The equivalence class of a tree t or a context v is denoted $\alpha_L(t)$, $\alpha_L(v)$, respectively. For a given tree t and contexts v_1, v_2 multiplication of contexts and trees $v_1 t, v_1 v_2$ naturally induces multiplication between elements of H_L and V_L . Similarly, for a given context v the operation of infinite power $v^\infty = vv \dots$ induces a mapping from V_L to H_L .

Given a regular language L , its *strategy graph* G_L is the pair $(V_L \times H_L, E)$ such that $((v, h), (v', h')) \in E$ iff there exists a tree t of type h such that t can be decomposed as the concatenation of a context of type v and another tree, and each prefix of t can be completed into a context of type v' . We thus say that the strategy graph is *recursive* if there exists a strongly connected component that contains two nodes (v, h) and (v', h') with $h \neq h'$. For a more formal approach to the strategy graph refer to [2, Section G]. We will need the following

Proposition 1 ([2]). *If there exists a path from (v, h) to (v', h') in G_L , then there exists an edge from (v, h) to (v', h') .*

3 Topological Complexity and Games

Topological Games. Let L and M be two languages. The *Wadge game* $\mathcal{W}(L, M)$ is an infinite two-player game between Player I and Player II. It is defined as follows. During a play Player I constructs a tree t and Player II a tree t' . At the first round Player I plays a root of t and Player II plays a root of t' , and at each consecutive round both players add a level to their corresponding tree (thus either Player adds some child to a leaf or Player signalizes that the node will be also a leaf of the final resulting tree of the play by not adding any children to it). Player I plays first and Player II is allowed to skip her turn but not forever. Player II wins the game iff $t \in L \Leftrightarrow t' \in M$. The game was designed precisely in order to obtain a characterisation of continuous reducibility.

Lemma 1 ([17]). *Let L, M be two languages. Then $L \leq_W M$ iff Player II has a winning strategy in the game $\mathcal{W}(L, M)$.*

From Borel determinacy ([11]), if both L and M are Borel, then $\mathcal{W}(L, M)$ is determined. The ordering $<_W$ restricted to the Borel sets is well-founded (see [10, Theorem 21.15]). The *Wadge degree* for sets of finite Borel rank can be defined inductively. First, we remark that since every self dual set A is Wadge equivalent to the disjoint union of a certain non self dual set B and its complement B^c , it is enough to start associating a Wadge degree only to non self dual sets and say that the Wadge degree of A equals the Wadge degree of B . For each degree there are exactly three equivalence classes with the same degree, represented by L , L^c and L^\pm — the disjoint union of L and L^c . Clearly $L, L^c <_W L^\pm$ and L^\pm is self dual.

In [6], J. Duparc showed that for non self dual sets, it is possible to determine its sign, $+$ or $-$, which specifies precisely the \equiv_W -class. For instance, \emptyset and complete open sets have sign $-$, while the whole space and complete closed sets have sign $+$. All self dual sets by definition have sign \pm . Let κ be the length of Wadge hierarchy of Borel sets of finite rank. Thus an ordinal $\alpha < \kappa$ determines a \equiv_W -class, denoted $[\alpha]^\epsilon$ for $\epsilon \in \{+, -, \pm\}$. In the same paper, in the context of Wadge degrees, Duparc defined set-theoretical counterparts of ordinal multiplication by a countable ordinal, and (quasi) exponentiation of base ω_1 . From now on $[\alpha]^\epsilon$ will also denote the canonical sets of Wadge degree generated with Duparc's operations.

Cutting Games. Below we define a family of two-player games of perfect information, called *cutting games*. These games were introduced in [2]. For the argument in [2] the most important was the finite version of the game. In the present paper we will consider both infinite and finite versions of the cutting game.

Let L_i ($i = 1, 2, \dots$) be languages over the alphabet A , and let p be a prefix over the alphabet A . The *simple cutting game* of length k , denoted $\mathcal{H}_k^p(L_1, \dots, L_k)$ is played by two players, Constrainer and Alternator. For each $i \in \{1, \dots, k\}$ the i -th round of the game is played as follows:

- Alternator chooses a tree $t_i \in L_i$ extending the prefix chosen in the previous round by the Constrainer; in the first round of the game Alternator must choose an extension of the given prefix p ,
- Constrainer chooses a prefix of the tree t_i .
- If Alternator cannot move, she loses, but if she survives k rounds then she wins.

The *infinite cutting game*, denoted by $\mathcal{H}_\infty^p(L_1, \dots)$, is played just like a simple game but without the restriction to a fixed given number of rounds. Alternator wins iff she can make infinitely many moves.

Let X be a language over the alphabet A . The X -*delayed cutting game*, denoted by $\mathcal{H}_\omega^X(L_1, \dots)$ is similar to a simple cutting game, except that a mini game is played to determine the prefix p and the length k of the match. The mini game goes as follows. Firstly, Alternator chooses a tree $t \in X$. Then Constrainer chooses a prefix p of t and a finite ordinal k . Finally the two players start to play the simple cutting game $\mathcal{H}_k^p(L_1, \dots, L_k)$.

When $L_{2i} = L$ and $L_{2i+1} = L^\complement$, then we simply write $\mathcal{H}_k^p(L, L^\complement)$, $\mathcal{H}_\infty^p(L, L^\complement)$ and $\mathcal{H}_\omega^X(L, L^\complement)$. It was verified in [2] that a given language M has a Wadge degree less than ω iff Constrainer has a winning strategy in $\mathcal{H}_k^\varepsilon(M, M^\complement)$, for all but finitely many $k < \omega$. In [2] it was also remarked that the language L described in [2, Section 4.1] and in Proposition 7 in [8], even if it is such that Alternator has a winning strategy in every corresponding finite cutting game, she loses the infinite one. In the next two propositions we establish a link between delayed cutting games and infinite Wadge degrees on the one hand, and infinite simple cutting games and uncountable Wadge degrees on the other hand.

Proposition 2. *Let L be a tree language, $[\omega]^+ \leq_W L$ iff Alternator has a winning strategy in $\mathcal{H}_\omega^L(L^\complement, L)$.*

Proposition 3. *Let L be a tree language. For every prefix p , $d_W([p]^{-1}L) \geq \omega_1$ iff Alternator has a winning strategy in $\mathcal{H}_\infty^p(L, L^\complement)$.*

4 A Characterization of Languages of Uncountable Degree

Games on Types and Strategy Trees. Following [2], for a given regular language of trees L , a prefix p and types $h_i \in H_L$ ($i = 1, 2, \dots$) we define games on types $\mathcal{H}_k^p(h_1, \dots, h_k)$ and $\mathcal{H}_\infty^p(h_1, h_2, \dots)$. The Constrainer plays as in the simple and infinite cutting games and the task of the Alternator is to play in the i -th round a tree of type h_i , that is an element of $\alpha_L^{-1}(h_i)$.

A type tree for L is a tree over the finite alphabet H_L . For a given tree t , there is a type tree σ_t induced by t such that for every node $w \in \text{dom}(\sigma_t)$,

$$\sigma_t(w) \text{ is the type of the tree } t.w. \quad (1)$$

Let σ be a type tree, and t a tree. We say that a type tree σ is *locally consistent* with a tree t if $\text{dom}(\sigma) = \text{dom}(t)$ and for every node $w \in \text{dom}(t)$ such that $t(w) = a$,

- if w is a leaf, then $\sigma(w)$ is the type of a ,
- if w has two children m_ℓ and m_r , then $\sigma(w)$ is the type obtained by applying a to the pair $(\sigma(m_\ell), \sigma(m_r))$.

Definition 1. *A finite strategy tree is a tuple $\mathfrak{s} = (t, \sigma_1, \dots, \sigma_k)$ where*

- t is a tree, the support of the strategy and $\sigma_1 = \sigma_t$,

- σ_ℓ is locally consistent with t , for each $\ell \leq k$,
- for each $w \in \text{dom}(t)$, Alternator has a winning strategy in $\mathcal{H}_k^\varepsilon(\sigma_1(w), \dots, \sigma_k(w))$.

An infinite strategy tree $\mathfrak{s} = (t, \sigma_1, \sigma_2, \dots)$ is defined analogously.

The root sequence of a strategy tree $\mathfrak{s} = (t, \sigma_1, \sigma_2, \dots)$ is the sequence of types $(\sigma_1(\varepsilon), \sigma_2(\varepsilon), \dots)$. We define the *alternation* of a sequence (h_1, \dots, h_ℓ) of types as the cardinality of the set $\{i : h_i \neq h_{i+1}\}$. The same definition applies to infinite sequences of types. Let \mathfrak{s} be a finite strategy tree. The *root alternation* of \mathfrak{s} is the alternation of the root sequence, while the *limit alternation* of \mathfrak{s} is the maximal number k such that infinitely many subtrees of \mathfrak{s} have root alternation at least k . We say that a set \mathfrak{S} of finite strategy trees has *bounded root alternation* if there is a k such that the root alternation of each $\mathfrak{s} \in \mathfrak{S}$ is at most k , unbounded otherwise. Analogously for limit alternation.

A finite or infinite strategy tree $\mathfrak{s} = (t, \sigma_1, \dots)$ is *locally optimal* if for every strategy tree $\mathfrak{s}' = (t, \sigma'_1, \dots)$ with same root sequence, and every $i > 1$, the depth at which σ_i and σ_{i+1} first differ is greater than or equal to the depth at which σ'_i and σ'_{i+1} first differ. The next Proposition is a very important technical point of [2].

Proposition 4 (Lemma G.2 in Appendix of [2]). *For a regular tree language L , if \mathfrak{S} is a set of locally optimal finite strategy trees with both root and unbounded limit alternation, then the strategy graph G_L is recursive.*

The next Proposition establishes an important link between infinite cutting games and strategy trees.

Proposition 5. *Assume Alternator has a winning strategy in $\mathcal{H}_\infty^\varepsilon(L, L^{\mathbb{G}})$. Then there is an infinite strategy tree \mathfrak{s}^∞ with infinite root alternation.*

Proof. Assume Alternator has a winning strategy f in $\mathcal{H}_\infty^\varepsilon(L, L^{\mathbb{G}})$. The infinite strategy tree \mathfrak{s}^∞ is constructed as follows. First of all, we can represent f as a tree satisfying the following properties:

- the root is labelled by ε , and its unique child is labelled by Alternator’s move obtained by applying the winning strategy f at the first round of the game,
- if a node v is labelled with a tree t , then for every prefix p of t there is a unique child of v labelled by p ,
- if a node v is labelled with a prefix p , then v has a unique child, and such a child is labelled by the answer obtained by applying the winning strategy f to the position in the cutting game given by the labels of the path from the root to v .

Notice that nodes at odd depth represent Alternator’s moves (according to f) and are therefore labelled by trees, while nodes at even depth represent Constrainer’s move and are thus labelled by prefixes. From now on, we always identify f and the aforementioned tree.

Claim. For every node v of f labelled by a prefix p , there is an infinite sequence of strategy trees $(\mathfrak{s}_\ell^v : \ell < \omega)$ such that for each ℓ

1. $\mathfrak{s}_\ell^v = (t, \sigma_1, \dots, \sigma_\ell)$, with the type $\sigma_{2k+1}(\varepsilon)$ included in L and the type $\sigma_{2k}(\varepsilon)$ included $L^{\mathbb{G}}$ if v is at depth $2i$ with i even, else dually. In particular this means that $\sigma_{2k+1}(\varepsilon) \neq \sigma_{2k}(\varepsilon)$;

2. $\mathfrak{s}_{\ell+1}^v$ extends \mathfrak{s}_ℓ^v , that is $\mathfrak{s}_{\ell+1}^v = (t, \sigma_1, \dots, \sigma_\ell, \sigma_{\ell+1})$ and $\mathfrak{s}_\ell^v = (t, \sigma_1, \dots, \sigma_\ell)$.

Given the Claim, from Property 1 we have that for each node v labelled by a prefix p , and each $\ell = 1, 2, \dots$, \mathfrak{s}_ℓ^v has root alternation ℓ and defines a winning strategy for Alternator in $\mathcal{H}_\ell^p(L, L^{\text{G}})$ if v is at depth $2i$ with i even, in $\mathcal{H}_\ell^p(L^{\text{G}}, L)$ otherwise. Let

$$\mathfrak{s}_\ell^\varepsilon = (t, \sigma_1, \dots, \sigma_\ell) \text{ for } \ell = 1, 2, \dots$$

The required infinite strategy tree is defined as $\mathfrak{s}^\infty = (t, \sigma_1, \dots)$. It remains to prove the Claim. Firstly, by induction with respect to $\ell = 1, 2, \dots$ we will assign a strategy tree \mathfrak{s}_ℓ^v to each node v of f labelled by a prefix. In the process of inductive construction we will also verify that Property 1 of the Claim is satisfied. Verification of Property 2 will be done later. Let us start from a remark that given an infinite sequence of type trees (σ_1, \dots) , by compactness there is a converging subsequence (σ'_1, \dots) . We assume that every time we have to choose a converging subsequence (σ'_1, \dots) of a given sequence (σ_1, \dots) , we always choose the same subsequence and denote its limit as $\text{limit}(\sigma_1, \dots)$. We also assume that given a tree t , we have fixed an enumeration (p_1, \dots) of all its prefixes such that sequence $(p_k)_{k=1,2,\dots}$ converge to the tree t . For $\ell = 1$, it is enough to take for each node v

$$\mathfrak{s}_1^v = (t, \sigma_t),$$

where t is given by applying f to the considered position and σ_t is defined by formula (1) at the beginning of this section. By choice of σ_t , Property 1 is satisfied. For $\ell > 1$ we proceed as follows. We assume the construction performed for $\ell - 1$. Fix any node v labelled by a prefix p . Assume that a tree t is the answer given by f at the position in the game given by the path from the root to the node v . To every prefix p of t corresponds a child w of v to which we already associated a strategy tree $\mathfrak{s}_{\ell-1}^w = (t^p, \sigma_2^p, \dots, \sigma_\ell^p)$. Let us thence consider the sequence (p_1, \dots) , with limit t and the sequences (t^{p_1}, \dots) , $(\sigma_2^{p_1}, \dots)$, \dots , $(\sigma_\ell^{p_1}, \dots)$. The limits $\text{limit}(\sigma_2^{p_k}), \dots, \text{limit}(\sigma_\ell^{p_k})$ were chosen in advance and are equal $\sigma_2^*, \dots, \sigma_\ell^*$. Since each t^{p_k} extends p_k , the limit t^* of $(t^{p_1}, t^{p_2}, \dots)$ is t . Now, for each p , the type trees $(\sigma_2^p, \dots, \sigma_\ell^p)$ are locally consistent with t^p . Furthermore, given a sequence of trees (t_1, \dots) that converges to t^* and a sequence of type trees (σ_1, \dots) that converges to σ^* , if σ_k is locally consistent with t_k for every k , then σ^* is locally consistent with t^* . From this fact follows that the limits $\sigma_2^*, \dots, \sigma_\ell^*$ are locally consistent with t . Finally, define σ_1^* to be σ_t as in formula (1). We have just proved that $\mathfrak{s}_\ell^v = (t, \sigma_1^*, \dots, \sigma_\ell^*)$ is a strategy tree. From induction hypothesis together with definition of σ_t and preservation of Property 1 under limits follows that \mathfrak{s}_ℓ^v also satisfies Property 1.

We now verify that the described procedure preserves Property 2. For $\ell = 1$ there is nothing to check. For the induction step, we reason as follows. Assume the Property holds for each node and for each $k < \ell$. Now, let us consider an arbitrary node v . We have to prove that $\mathfrak{s}_{\ell+1}^v$ extends \mathfrak{s}_ℓ^v . By induction hypothesis, $\mathfrak{s}_{\ell-1}^w = (t^p, \sigma_2^p, \dots, \sigma_\ell^p)$ and $\mathfrak{s}_\ell^w = (t^p, \sigma_2^p, \dots, \sigma_\ell^p, \sigma_{\ell+1}^p)$, for every node w in the described procedure. Since the limits have been fixed in advance, we have that $\mathfrak{s}_\ell^w = (t, \sigma_t, \sigma_2^*, \dots, \sigma_\ell^*)$ and $\mathfrak{s}_{\ell+1}^w = (t, \sigma_t, \sigma_2^*, \dots, \sigma_\ell^*, \sigma_{\ell+1}^*)$, meaning that the latter extends the former. This concludes the proof of the Claim.

Using the above Proposition, we can generalize to infinite games Proposition 5.2 from [2]:

Proposition 6. *For a regular language L the following conditions are equivalent.*

1. *Alternator wins the game $\mathcal{H}_\infty^\varepsilon(L, L^{\text{G}})$,*
2. *There are tree types $h, g \in H_L$, such that $h \neq g$ and Alternator wins $\mathcal{H}_\infty^\varepsilon(h, g)$.*

We will use the following Lemma, presented in [2] for finite strategy trees, with proof extending straightforwardly to infinite strategy trees.

Lemma 2. *For every finite or infinite strategy tree, there is a locally optimal strategy tree with same root sequence.*

The next Lemma follows immediately from the definition of a strategy tree.

Lemma 3. *Let $\mathfrak{s} = (t, \sigma_1, \dots, \sigma_\ell)$ be a strategy tree. For the game $\mathcal{H}_\ell^\varepsilon(\sigma_1(\varepsilon), \dots, \sigma_\ell(\varepsilon))$ and a strategy of Constrainer given by always cutting at level i , Alternator wins by playing as follows:*

- *at first, Alternator plays t , then*
- *for each port w at level i of the multi context given by Constrainer's move, Alternator plugs in the tree given by her winning strategy $\mathcal{H}_\ell^\varepsilon(\sigma_1(w), \dots, \sigma_\ell(w))$.*

In particular, if from a certain $j < \ell$ on $\sigma_k(w) = \sigma_{k+1}(w)$, $j \leq k < \ell$, then for each round k such that $j < k < \ell$ Alternator always plugs in the same tree of type $\sigma_j(w)$ chosen at round j .

An Effective Characterization. Everything now is ready to prove the main result of this paper.

Theorem 1. *Let L be a regular tree language given by a non-deterministic tree automaton \mathcal{A} . The following conditions are equivalent:*

1. *The strategy graph G_L is recursive.*
2. *$d_W(L) \geq \omega_1$*

In particular, since the graph G_L is computable from the automaton \mathcal{A} , it is decidable whether the language accepted by \mathcal{A} is of Wadge degree greater than or equal to ω_1 .

Proof. (1) \Rightarrow (2). Assume the strategy graph is recursive. This means that there exists a strongly connected component that contains two nodes (v, h) and (v', h') with $h \neq h'$. Thanks to Proposition 1, if there exists a path between (v, h) and (v', h') , there is also an edge between (v, h) and (v', h') . Moreover, for vertices $(v_1, h_1), (v_2, h_2), \dots$, if for every $i = 1, 2, \dots$ there is an edge from (v_i, h_i) to (v_{i+1}, h_{i+1}) , this means that Alternator has a winning strategy in $\mathcal{H}_\infty^\varepsilon(h_1, h_2, \dots)$. So, take $(v_i, h_i) = (v, h)$ for i even, and $(v_i, h_i) = (v', h')$ for i odd. This shows that Alternator has a winning strategy in $\mathcal{H}_\infty^\varepsilon(h, h')$. By Proposition 6 Alternator has a winning strategy in $\mathcal{H}_\infty^\varepsilon(L, L^{\text{G}})$.

(2) \Rightarrow (1). By Propositions 3 and 4, it is enough to verify that if Alternator has a winning strategy in $\mathcal{H}_\infty^\varepsilon(L, L^{\text{G}})$ then there is a set \mathfrak{S} of locally optimal finite strategy

trees with both root and limit unbounded alternation. Assume Alternator has a winning strategy f in $\mathcal{H}_\infty^\varepsilon(L, L^c)$. From Proposition 5 there is a strategy tree $\mathfrak{s}^\infty = (t, \sigma_1, \dots)$ with infinite root alternation. By Lemma 2 we can assume that \mathfrak{s}^∞ is locally optimal. Let us define

$$\mathfrak{S} = \{(t, \sigma_1, \dots, \sigma_k) : k = 1, 2, \dots\}.$$

Note that each element of \mathfrak{S} is locally optimal. Now, assume limit alternation of \mathfrak{S} is bounded. From this fact and since every element of \mathfrak{S} is a prefix of \mathfrak{s}^∞ , it holds that with respect to \mathfrak{s}^∞ , the set of subtrees of t with infinite root alternation has to be finite. This means that \mathfrak{s}^∞ satisfies the following property:

- (*) there is a finite set X of nodes of t satisfying the following properties:
- the root is included in X , and each node of X is at most at depth i in t ,
 - $\sigma_k(v) \neq \sigma_{k'}(v)$, for every node v in the set X , and $\sigma_k(w) = \sigma_{k'}(w)$, for every node w of t of depth $i + 1$, for some k, k' , with $k < k' \leq j$.

The strategy tree $\mathfrak{s} = (t, \sigma_1, \dots, \sigma_j)$ from \mathfrak{S} , where j is given by the previous property, also satisfies the property (*) above (for the same X and the same k, k').

Let us consider the game $\mathcal{H}_j^\varepsilon(\sigma_1(\varepsilon), \dots, \sigma_j(\varepsilon))$ where at first Alternator plays t and then Constrainer uses the strategy given by cutting always at level $i + 1$. We can therefore apply Lemma 3 and assume that Alternator plays the winning strategy described there. This implies that the trees played at round k and k' are the same, say t' (from the root to level i they are the same, because the Constrainer insists on this and below they are the same, because the Alternator plays the same answers in rounds k and k'). But by local consistency, since $\sigma_k(\varepsilon) \neq \sigma_{k'}(\varepsilon)$, the two trees should have two different types, a contradiction. We therefore conclude that limit alternation of \mathfrak{S} is unbounded.

5 Conclusion

The algorithm provided in [2] decides whether a given non-deterministic automaton accepts a language which is a Boolean combination of open sets or equivalently is of a Wadge degree smaller than ω . By the same approach we showed an algorithm which decides whether a given non-deterministic automaton accepts a language in Δ_2^0 or equivalently, a language of a Wadge degree smaller than ω_1 . We propose for further research the following three generalizations of the result presented in this paper:

1. For a given $n = 1, 2, \dots$ there are natural topological games which characterize languages of Wadge degrees smaller than ω^n . Moreover, there are known examples of regular languages of degree ω^n . It would be a desirable and perhaps more involved extension of results in [2] if for a given n one can provide an algorithm deciding whether a given non-deterministic automaton accepts a language of degree smaller than ω^n .
2. In the absence of examples of regular languages between Wadge degree ω^ω and Wadge degree ω_1 , one could reasonably expect, that the decidability result in the present paper should show that indeed any regular language of countable Wadge degree is of Wadge degree smaller than ω^ω . However, this question still remains open.
3. Regarding higher Borel classes, in particular regular languages which are Boolean combinations of Σ_2^0 sets, the following extension of the method in [2] seems to be

plausible. The cutting game is based around restrictions of moves by prefixes, that is by languages in Δ_1^0 . Its topological counterpart on the next Borel level is a game, where the Constrainer is allowed to play constraints which are regular languages in Δ_2^0 . This leads to a natural topological characterization similar to the results in Section 3, but the algebraic counterpart of this generalized cutting game is not yet fully understood.

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