On the MAX MIN VERTEX COVER Problem*

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Abstract. We address the MAX MIN VERTEX COVER problem, which is the maximization version of the well studied MIN INDEPENDENT DOMINATING SET problem, known to be **NP**-hard and highly inapproximable in polynomial time. We present tight approximation results for this problem on general graphs, namely a polynomial approximation algorithm which guarantees an $n^{-1/2}$ approximation ratio, while showing that unless $\mathbf{P} = \mathbf{NP}$, the problem is inapproximable within ratio $n^{\varepsilon-(1/2)}$ for any strictly positive ε. We also analyze the problem on various restricted classes of graph, on which we show polynomiality or constant-approximability of the problem. Finally, we show that the problem is fixed-parameter tractable with respect to the size of an optimal solution, to treewidth and to the size of a maximum matching.

1 Introduction

In the MIN INDEPENDENT DOMINATING SET problem, also called MIN MAX INDEPENDENT SET, given a graph $G(V, E)$, we are asked to determine a minimum size vertex-subset that is simultaneously independent and dominating. This problem, although polynomially solvable in strongly chordal graphs [20], has been proved to be inapproximable within $n^{1-\varepsilon}$, for any $\epsilon > 0$, not only in general graphs [1] but also in restricted graph classes as, for instance, the circle graphs [2]. Also, and probably due to this fact, exact solution of MIN INDEPENDENT DOMINATING SET in general or in restricted classes of graphs by moderately exponential algorithms has received a growing attention in the past years [3–5]. This problem has also been tackled using exponential approximation techniques [5]. Finally, it is shown to be very hard from a parameterized complexity point of view since it is **W[2]**-hard [6].

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Surprisingly, to the best of our knowledge, the natural symmetric problem, the MAX MIN VERTEX COVER problem, where the goal is to compute a minimal (for exclusion) vertex cover of maximum size, has not been addressed yet. Knowing the direct applications of MIN INDEPENDENT DOMINATING SET in terms of ad-hoc wireless networks, it seems natural to study the symmetric version, where instead of minimizing the number of servers, one wishes to maximize the number of clients. This problem obviously has the same characteristics as its minimization counterpart in terms of **NP**-hardness and exact computation, but might have different behaviours in terms of approximability and parameterized complexity (as in the case of the pair MAX INDEPENDENT SET - MIN VERTEX COVER).

We show in this paper that, while also highly inapproximable, MAX MIN VERTEX COVER is better approximable than its mate, since it can be approximately solved in polynomial time within ratio $n^{-1/2}$, where n is the size of the input graph. This result is matched by an inapproximability bound of $n^{\varepsilon-(1/2)}$ that can be extended also to an $O(1/\Delta)$ inapproximability bound, where Δ is the maximum degree of the input graph. We also match it to an $O(3/2\Delta)$ approximation ratio achieved by a natural greedy algorithm. We also prove that, unlike MIN INDEPENDENT DOMINATING SET, MAX MIN VERTEX COVER is in **FPT**, the class of fixed-parameter tractable problems not only with respect to the standard parameter, i.e., the value of the optimum, but also with respect to the cardinality of a maximum matching (that is smaller than the value of the optimum). Let us note that both MIN WEIGHTED DOMINATING SET and MAX WEIGHTED INDEPENDENT SET are polynomially solvable in graphs with bounded treewidth [7–9] (and, actually, fixed parameter tractable with respect to the treewidth of the input graph [10]). With similar dynamic programming techniques, it can be shown that also both WEIGHTED MAX MIN VERTEX COVER and WEIGHTED MIN INDEPENDENT DOMINATING SET are fixed parameter tractable with respect to the treewidth. Since the techniques used for obtaining this result are quite similar to those in [10], the proof of the result is omitted.

2 Approximation of MAX MIN VERTEX COVER in General Graphs

We give in this section inapproximability upper bounds matched by lower bounds achieved in polynomial time for MAX MIN VERTEX COVER. We first study ratios functions of n and then functions of Δ.

Proposition 1. *For any positive constant ε,* MAX MIN VERTEX COVER *is inapproximable within ratio $O(n^{\varepsilon-(1/2)})$ unless $\mathbf{P} = \mathbf{NP}$.*

Proof. First, recall that MAX INDEPENDENT SET has been proved to be inapproximable within ratio $n^{\varepsilon-1}$ for a given $\varepsilon \in (0,1)$ [12] unless $\mathbf{P} = \mathbf{NP}$.

Consider an unweighted instance of MAX INDEPENDENT SET given by a graph $G(V,E)$. Out of this instance of MAX INDEPENDENT SET, we build an instance $H(V \cup S, E')$ of MAX MIN VERTEX COVER in the following way: for each vertex v of V one adds $n+1$ vertices connected only to v in H, while the

inner edges of the set V are left unchanged. In other words, graph H is obtained by adding an independent set S, of order $n^2 + n$ to the initial graph G, and connecting $n + 1$ vertices of the independent set to each vertex v.

Note that the graph H can be built in polynomial time, and has precisely $n^2 + 2n$ vertices. Denote by $\text{opt}(G)$ an optimal independent set in G, and by $\text{opt}(H)$ an optimal vertex cover in H.

Figure 1 provides an example of the construction where $\text{opt}(G)$ is the set of circled vertices, and $\text{opt}(H)$ the set of black vertices.

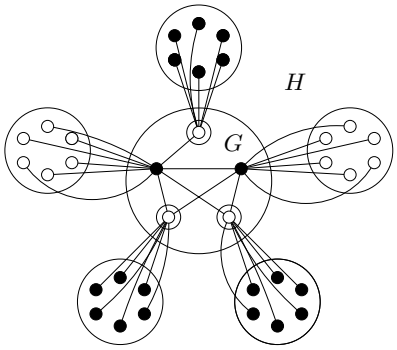

Fig. 1. An example of the reduction in Proposition 1

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Consider a vertex cover $\text{SOL}(H)$ that has cardinality $\text{sol}(H)$ in H. First, notice that H admits a maximal matching of n edges, that consists of taking, for each vertex of V, one edge linking this vertex to one of its neighbors in S. Hence, any vertex cover in H takes at least n vertices, i.e., $\text{sol}(H) \geqslant n$.

Notice also that, for any vertex v of V that does not belong to $\text{SOL}(H)$, then $\text{SOL}(H)$ must take all its neighbors in S, that is $n + 1$ vertices. Moreover the set $V \setminus \text{SOL}(H)$ of vertices of V that do not belong to $\text{SOL}(H)$ defines an independent set in G with $n - |\text{sol}(H) \cap V|$ vertices. In other words, one can assert that any solution $\text{SOL}(H)$ of cardinality $\text{sol}(H)$ in H can be easily transformed into an independent set $\text{SOL}(G)$ in G of cardinality:

$$\text{sol}(G) \geqslant \frac{\text{sol}(H) - n}{n} \tag{1}$$

Conversely, the existence of a maximal independent set of size h in G induces the existence of a minimal vertex cover of size $nh + n$ in H. It suffices to consider the following vertex cover: all vertices of V that do not belong in the independent set ($n - h$ vertices), and all vertices of S linked to a vertex of the independent set ($h(n + 1)$ vertices). Therefore, it holds that a minimal vertex cover of size $n \cdot \text{opt}(G) + n$ exists in H. In other words, $\text{opt}(H) \geqslant n \cdot \text{opt}(G) + n$.

Now, for some constant positive $\rho < 1$, suppose that there exists a polynomial time algorithm `A` for MAX MIN VERTEX COVER that guarantees an approximation ratio $n^{-\rho}$, and suppose that a solution $\text{SOL}(H)$ has been computed by this

algorithm on graph H. Reminding that graph H has $O(n^2)$ vertices, the approximation ratio guaranteed by $\mathtt{A}$ on H turns to be $n^{-2\rho}$. Then, it holds that:

$$\mathrm{sol}(H) \geqslant n^{-2\rho} \cdot \mathrm{opt}(H) \tag{2}$$

By combining (1) and (2), one can assert that $\mathrm{SOL}(H)$ can easily be transformed into an independent set $\mathrm{SOL}(G)$ in G of value:

$$\begin{aligned} \mathrm{sol}(G) &\geqslant \frac{n^{-2\rho} \cdot \mathrm{opt}(H) - n}{n} \;\geqslant\; n^{-2\rho-1} \cdot \mathrm{opt}(H) - 1 \\ &\geqslant n^{-2\rho} \cdot \mathrm{opt}(G) + n^{-2\rho} - 1 \;\geqslant\; n^{-2\rho} \cdot \mathrm{opt}(G) \end{aligned}$$

where the last inequality holds for n big enough.

Hence, the existence of an $n^{-\rho}$-approximation algorithm $\mathtt{A}$ for MAX MIN VERTEX COVER induces the existence of an $n^{-2\rho}$-approximation algorithm for MAX INDEPENDENT SET, that would consist of:

- building the instance H of MAX MIN VERTEX COVER out of the instance G of MAX INDEPENDENT SET;
- running the algorithm $\mathtt{A}$ on the instance H that outputs a solution $\mathrm{SOL}(H)$;
- returning the solution $V \setminus \mathrm{SOL}(H)$ for the initial problem.

Since, for any constant $0 < \varepsilon \leqslant 1$, the existence of an $n^{1-\varepsilon}$-approximation algorithm for MAX WEIGHTED INDEPENDENT SET induces $\mathbf{P} = \mathbf{NP}$ [12], an $n^{-\rho}$-approximation algorithm for MAX MIN VERTEX COVER can exist only subject to the condition that $n^{-2\rho} \leqslant n^{1-\varepsilon}$ for any $0 < \varepsilon \leqslant 1$. This leads to $\rho \leqslant \varepsilon - 1/2$, which concludes the proof.

Observe now that the order of graph H in the gap-reduction of Proposition 1 is $O(n^2)$, while the maximum degree of H is $O(n)$. Then, the following inapproximability bound also can be immediately derived.

Corollary 1. MAX MIN VERTEX COVER *is inapproximable in polynomial time within ratios* $O(\Delta^{\epsilon-1})$*, for any* $\epsilon > 0$.

Let us now recall the following very classical and obvious observation that will be used later.

Remark 1. Denoting by M a maximum matching of a graph G, any vertex cover (a fortiori a minimal one) of G uses at least $|M|$ vertices (since, at least one distinct vertex is needed per one edge of M).

Lemma 1. *Consider a graph* $G(V, E)$ *and an independent set* S *of* G*. Denote by* $\Gamma(S)$ *the set of neighbors of* S*, and* $V' = V \setminus (S \cup \Gamma(S))$*. Finally, denote by* $SOL(G')$ *a minimal vertex cover on the induced subgraph* $G[V']$ *and by* $\mathrm{sol}(G')$ *its cardinality. It holds that* $\Gamma(S) \cup SOL(G')$ *is a feasible solution for* MAX MIN VERTEX COVER.

Proof. First, let us prove that $\Gamma(S) \cup \mathrm{sol}\,(V')$ is a vertex cover: all edges of $S \times \Gamma(S)$ and $\Gamma(S) \times V'$ are covered by vertices of $\Gamma(S)$, and all edges inside

the induced subgraph $G[V']$ are covered by $\mathrm{SOL}(G')$. By hypothesis, $S \times S = \emptyset$, so that $\Gamma(S) \cup \mathrm{SOL}(G')$ is, indeed, a vertex cover.

Then, let us establish the minimality of such a vertex cover: on the one hand, no vertex of $\Gamma(S)$ can be removed, as they all cover an edge linked to a vertex of S (and no vertex of S is in the vertex cover), and on the other hand, $\mathrm{SOL}(G')$ is a minimal vertex cover on a subgraph of G, so that none of its vertices can be removed without uncovering an edge.

Proposition 2. MAX MIN VERTEX COVER *is approximable within ratio* $n^{-1/2}$ *in polynomial time.*

Proof. Consider a graph $G(V, E)$, with $|V| = n$. Let $\Gamma(x)$ be the set of neighbors of a given vertex x and, given $V' \subseteq V$, let $G[V']$ be the subgraph of G induced by the set of vertices V'. Consider the following approximation algorithm for MAX MIN VERTEX COVER:

- compute a maximum matching M;
- among the matched vertices, let x be the one with the maximal number of exposed neighbors;
- compute a minimal vertex cover on $G[V']$ with a greedy algorithm, where $V' = V \setminus (\{x\} \cup \Gamma(x))$, and denote it by $\mathrm{SOL}(G')$;
- output $\mathrm{SOL}(G) = \Gamma(x) \cup \mathrm{SOL}(G')$.

First, by Lemma 1, we can assert that the solution returned by our approximation algorithm is feasible. Then notice that the algorithm runs in polynomial time, all steps of the algorithm are so: step 1 can be performed in $O(n^{2.376})$ time by the algorithm presented in [13], identifying vertex x is done in $O(n)$, and building a minimal vertex cover is done in $O(n^2)$ (starting from the whole set of vertices, the greedy algorithm deletes them one by one as long as the solution remains a vertex cover, when no vertex can be deleted, the remaining set is a minimal vertex cover). Finally, without loss of generality, let us suppose that the graph has no isolated vertices, since such vertices obviously make the problem easier to approximate.

Let us now analyze the approximation guarantee of this algorithm. Given the maximum matching M computed at the first step of the algorithm, denote by P the set of unmatched vertices of V with respect to M (i.e., the set of vertices of V that are not endpoints of M), which obviously forms an independent set. Finally, set $p = |P|$ and $m = |M|$. Our analysis is based upon the following *maximality argument* that will be used also in Proposition 3.

Notice that each edge (v_i, v_j) of M is linked to a set of vertices $P_{ij} \subseteq P$, so that $P_{ij} \subset \Gamma(v_i)$, or $P_{ij} \subset \Gamma(v_j)$. Indeed, suppose v_i has some neighbor v_k in P not linked to v_j, and v_j some neighbor v_l in P not linked to v_i. Then, by deleting (v_i, v_j) from M and adding (v_i, v_k) and (v_j, v_l) to it, one could produce a matching with $m + 1$ edges, so that M would not be a maximum matching. In other words, there exists a covering of P by m sets P_{ij}, each of them been included in the neighborhood of a single matched vertex. For the algorithm, this implies that the vertex x picked at step 2 has at least p/m neighbors in P.

The algorithm includes the whole neighborhood of this vertex x, which might also include some matched vertices. Suppose that $\Gamma(x)$ contains exactly h matched vertices. Then it holds that:

$$|\Gamma(x)| \geqslant h + \frac{p}{m} = h + \frac{n-2m}{m} \tag{3}$$

We now bound the value of $\mathrm{SOL}(G')$. Among the edges of the matching M, at least $m-h$ still exist in the subgraph $G[V']$. Indeed, this subgraph is obtained by deleting from G the vertex x together with all its neighbors, and all edges incident to these vertices. It is clear that, by deleting h vertices from M, only h edges are deleted from it. Thus, $G[V']$ contains a matching with $m-h$ edges, so that any vertex cover in $G[V']$ has at least $m-h$ vertices. A fortiori, this also holds for the vertex cover computed at step 3 of the algorithm; so:

$$\mathrm{sol}\,(G')\,| \geqslant m-h \tag{4}$$

Combining (3) and (4), we finally get the following bound on the value of the solution computed by the algorithm: $\mathrm{sol}(G) \geqslant m + \frac{n}{m} - 2 \geqslant \sqrt{n}$, where the last inequality results from a simple case analysis on the value of m with respect to $\sqrt{n}$: if $m \geqslant \sqrt{n}$, then the first term of the sum is at least $\sqrt{n}$ and the second at least 2 (as $m \leqslant n/2$). In the opposite case ($m < \sqrt{n}$), the second term of the sum is at least $\sqrt{n}$, where $m \geqslant 2$ (if $m = 1$, the graph is a star, and the problem is polynomial).

Considering that $\mathrm{opt}(G) \leqslant n$, the algorithm clearly guarantees an $n^{-1/2}$ approximation ratio, and the proof is concluded.

The following proposition, provides a lower bound, function of the maximum degree of the input graph, for the approximation ratio of MAX MIN VERTEX COVER.

Proposition 3. MAX MIN VERTEX COVER *is polynomially approximable within ratio* $3/2\Delta$*, where* Δ *is the maximum degree of the input graph. Furthermore, in bounded-degree graphs, regular graphs and graphs admitting a perfect matching* MAX MIN VERTEX COVER *is in* ***APX***.

Proof. Denote by d_i the degree of a vertex $v_i \in V$, by d the average degree of G and, as previously, by M a maximum matching of G, by m the cardinality of M and by p the cardinality of the set $P = V \setminus V(M)$ of the exposed vertices of V with respect to M.

The *maximality argument* stated in the proof of Proposition 2, has the following consequence for sets M and P:

> *for an edge* $(v_i, v_j) \in M$*, if one, say* v_i*, of its endpoints has more than one exposed neighbor, then* v_j *has no exposed neighbour at all; in the opposite case, an augmenting path would occur; in other words, in the case that a matching edge* (v_i, v_j) *is incident to some edge* (v, u) *with either* $v_i = v$*, or* $v_j = v$*, and* $u \in P$*, it holds that* $|(\Gamma(v_i) \cup \Gamma(v_j)) \cap P| \leqslant 1$.

Suppose that there exist a set M' of m' edges of M, whose one endpoint is adjacent to some exposed vertex of G with respect to M. Obviously, $p \leqslant m' \cdot \Delta$. Since $n = 2m + p$, we get, using the quoted consequence above, $n = 2m + p \leqslant 2(m - m') + (\Delta + 1)m' = 2m + (\Delta - 1)m' \leqslant (\Delta + 1)m$; hence:

$$m \geqslant \frac{n}{\Delta + 1} \tag{5}$$

By the seminal Turán's Theorem, every maximal (for inclusion) independent set of G has size at least $n/(d + 1)$; consequently:

$$\mathrm{opt}(G) \leqslant \frac{dn}{d + 1} \tag{6}$$

Combining (5) and (6) and taking into account Remark 1, the following holds for the approximation of every minimal vertex cover:

$$\frac{\mathrm{sol}(G)}{\mathrm{opt}(G)} \geqslant \frac{d + 1}{d(\Delta + 1)} \simeq \frac{d + 1}{d\Delta} \tag{7}$$

for arbitrarily large values of Δ. Also the following fact holds for any vertex cover of a graph $G(V, E)$ of order n.

> *Any vertex cover (a fortiori a minimal one) C guarantees approximation ratio at least $(d + 1)/2\Delta$ for* MAX MIN VERTEX COVER.

In fact, since C covers E, it holds that $\sum_{v_i \in C} d_i \geqslant |E|$. Also, $\sum_{v_i \in C} d_i \leqslant \Delta|C|$ and $|E| = nd/2$. Putting all this together, we get:

$$\mathrm{sol}(G) = |C| \geqslant \frac{nd}{2\Delta} \tag{8}$$

Combining (8) and (6), we derive:

$$\frac{\mathrm{sol}(G)}{\mathrm{opt}(G)} \geqslant \frac{d + 1}{2\Delta} \tag{9}$$

Ratio in (9) is increasing with d, while in (7) is decreasing with d. Equality holds for $d = 2$, which derives ratio $3/2\Delta$.

3 Parameterized Analysis

We prove in this section that, continuing the asymmetry between MIN INDEPENDENT DOMINATING SET and MAX MIN VERTEX COVER, the later is fixed parameter tractable when parameterized by the standard parameter, i.e., the cardinality opt of a maximum minimal vertex cover.

Proposition 4. MAX MIN VERTEX COVER *can be solved in* $O^*(4^{\mathrm{opt}/3})$.

Proof. Let, as previously, $\Gamma(v_i)$ be the neighborhood of vertex v_i. First, notice that if all vertices have degree $\leqslant 2$, then the problem becomes straightforwardly polynomially solvable by dynamic programming. Then, we assume that there exists a vertex v_j such that $d_j \geqslant 3$. Notice also that for each vertex v_i at least one vertex among the set $\Gamma(v_i) \cup \{v_i\}$ cannot be part of the vertex cover or else that vertex cover would not be minimal. For this, just observe that if $\Gamma(v_i) \cup \{v_i\}$ is included in the solution, v_i can be removed, since its incident edges are covered by the vertices of $\Gamma(v_i)$. We consider a branch and reduce approach where in each branch a vertex is excluded from the vertex cover and its neighbors are then necessarily included. We point out that such branch guarantees that all vertex covers generated will be minimal. We branch on vertex v_j according to the following exhaustive cases.

Case 1. $d_j \geqslant 3$ and all $v_i \in \Gamma(v_j)$ have degree $d_i \geqslant d_j$. We generate $|\Gamma(v_j)| + 1$ branches as follows: in one branch, vertex v_j is excluded from the vertex cover and correspondingly all its neighbors are included; in all other branches one of the vertices $v_i \in \Gamma(v_j)$ is excluded while all $v_k \in \Gamma(v_i)$ are included. This corresponds to $|d_j + 1|$ branches where in each branch at least $|d_j|$ vertices are included in the vertex cover. The worst-case occurs for $|d_j| = 3$, where we have four branches each including 3 vertices in the vertex set. Correspondingly, the complexity is $O^*(4^{\mathrm{opt}/3}) = O^*(1.5874^{\mathrm{opt}})$.

Case 2. $d_j \geqslant 3$ and there exists $v_i \in \Gamma(v_j)$ with $d_i = 2$. Three subcases occur with respect to the degree of the other neighbor v_k of v_i.

Subcase 2 (a). If $d_k \geqslant 3$, then either v_i or v_j or v_k are excluded from the vertex cover and correspondingly their neighbors are included in the vertex cover. Then, the recursion is at least $T(\mathrm{opt}) \leqslant T(\mathrm{opt} - 2) + 2T(\mathrm{opt} - 3)$ and the worst-case complexity is $O^*(1.5214^{\mathrm{opt}})$.

Subcase 2 (b). If $d_k = 2$, v_j may or may not be adiacent to v_k. If v_j and v_k are adjacent, then a branch on v_j can be performed: either v_j is excluded from the vertex cover and correspondingly its neighbors (at least three) are included in the vertex cover, or v_j is included in the vertex cover and arbitrarily v_i (v_k) is excluded from the vertex cover and v_k (v_i) is included in the vertex cover. Then, the recursion is at least $T(\mathrm{opt}) \leqslant T(\mathrm{opt} - 2) + T(\mathrm{opt} - 3)$ and the worst-case complexity is $O^*(1.3248^{\mathrm{opt}})$. Alternatively, v_j and v_k are non adjacent and v_k is adjacent to another vertex v_l. Then, either v_i is excluded from the vertex cover (and its two neighbors are included), or v_k is excluded from the vertex cover (and again its two neighbors are included), or both v_i and v_k are included in the vertex cover and correspondingly v_j and v_l are excluded from the vertex cover. In this last case, the other neighbors of v_j and v_l (that may possibly coincide) must be included in the vertex cover and globally at least four vertices must be included in the vertex cover. Correspondingly, the recursion is at least $T(\mathrm{opt}) \leqslant 2T(\mathrm{opt}-2)+T(\mathrm{opt}-4)$ and the worst-case complexity is $O^*(1.5538^{\mathrm{opt}})$ (notice that this last branch does not even occur if v_j and v_l are adjacent).

Subcase 2 (c). If $d_k = 1$, then v_i and v_j cannot be both included in the vertex cover as such solution is not better than the one with v_j and v_k included in the vertex cover and v_i excluded from the vertex cover. Then either v_i is

excluded from the vertex and two vertices (v_i andv_j) are included in the vertex cover, or v_j is excluded from the vertex cover and all its neighbors (at least three vertices) are included in the vertex cover. Correspondingly, the recursion is at least $T(\text{opt}) \leqslant T(\text{opt}-2)+T(\text{opt}-3)$ and the worst-case complexity is $O^*(1.3248^{\text{opt}})$.

Case 3. $d_j \geqslant 3$ and there exists $v_i \in \Gamma(v_j)$ with $d_i = 1$. We generate 2 branches where either v_i is excluded from the vertex cover and v_j is included, or v_j is excluded from the vertex and all its neighbors are included. Correspondingly, the recursion is at least $T(\text{opt}) \leqslant T(\text{opt}-1)+T(\text{opt}-3)$ and the worst-case complexity is $O^*(1.4656^{\text{opt}})$.

Overall, the worst-case is attained in case 1 with complexity $O^*(4^{\text{opt}/3}) = O^*(1.5874^{\text{opt}})$.

In what follows, we further strengthen the result of Proposition 4, showing that MAX MIN VERTEX COVER is FPT even when parameterized by the cardinality of a maximum matching M of the input graph (recall that $m \leqslant \text{opt}(G)$).

Proposition 5. MAX MIN VERTEX COVER *can be solved in* $O^*(3^m)$ *where* m *is the cardinality of a maximum matching of the input graph.*

Proof. Consider a general graph $G(V,E)$, and a maximum matching $M \subseteq E$ on G. All exposed vertices obviously form an independent set, that we denote by S. We also denote, as previously, by $V(M)$ the set of matched vertices.

We show that any feasible solution $\text{SOL}(G)$ for MAX MIN VERTEX COVER can be unequivocally characterized by its subset of matched vertices. Consider any subset $\text{SOL}(G) \cap V(M)$ of $V(M)$ known to be the subset of a unknown feasible solution $\text{SOL}(G)$. There actually exists a single solution $\text{SOL}(G)$ which admits $\text{SOL}(G) \cap V(M)$ as subset of matched vertices. Indeed denote by $\hat{S}$ the subset of S containing all exposed vertices incident to a matched vertex that does not belong in $\text{SOL}(G) \cap V(M)$. Then, the whole set $\hat{S}$ must be part of $\text{SOL}(G)$ in order to make it feasible. Conversely, all exposed vertices that do not belong to $\hat{S}$ cannot belong to $\text{SOL}(G)$, because they would make the solution non minimal: by definition, all of their neighbors already belong in the vertex cover.

Therefore, by identifying the subset $\text{OPT} \cap V(M)$, where OPT denotes a maximum minimal vertex cover, one would be able to reconstruct the whole solution OPT by simply adding to $\text{OPT} \cap V(M)$ all exposed vertices incident to a matched vertex not in the vertex cover.

Finally, notice that for each edge of the matching M, any vertex cover (a fortiori the optimal one) must take at least one endpoint of this edge. So, for each edge, any solution can take one endpoint, or the other, or both endpoints, that is three possibilities. Hence, there are at most 3^m vertex covers in the subgraph induced by the matched vertices.

Consider the following algorithm:

- compute a maximum matching M;
- build all 3^m possible vertex-covers $V_i \subseteq V(M)$ among the matched vertices;

– complete each of these vertex-covers by adding all vertices of S incident to a vertex not in the vertex cover;
– output the maximal feasible solution.

It is clear that through the exhaustive search performed at step 2 of the algorithm, the subset $\mathrm{OPT} \cap V(M)$ will be found, and when completed by exposed vertices at step 3, the optimal solution will be produced. Hence, an optimal solution can be computed in $O^*(3^m)$, and the proof is concluded.

Taking into account that $m \leqslant \tau$, the cardinality of a minimum vertex cover of the input graph, the following corollary immediately holds.

Corollary 2. MAX MIN VERTEX COVER *parameterized by* τ *is FPT.*

Let us now quickly point out how combination of Propositions 4 and 5 allows us to handle interesting trade-offs between parameterization and approximation. Indeed, we shall show that approximation ratios for MAX MIN VERTEX COVER, unachievable in polynomial time (unless an unlikely complexity condition holds), can be achieved in parameterized time. This issue has been already studied in [14–18], etc., for several problems, as MIN VERTEX COVER, STEINER TREE, MIN EDGE DOMINATING SET, several restricted versions of MIN HITTING SET, etc.

Revisit Proposition 5 and remark that if $m < (\log 1.5874/\log 3)\mathrm{opt}(G) \approx 0.42\mathrm{opt}(G)$, then the parameterized algorithm of Proposition 5 runs faster than that of Proposition 4 while, if $m \geqslant 0.42\mathrm{opt}(G)$, any minimal vertex cover (a fortiori the one of Proposition 3) achieves ratio greater than, or equal to, 0.42, ratio "forbidden" in polynomial time. For instance, we can guarantee ratio 0.1 in parameterized time less than $O^*(1162^{\mathrm{opt}})$, much smaller than $O^*(1.5874^{\mathrm{opt}})$, or even, approximation ratio 0,4 in time less than $O^*(1.552^{\mathrm{opt}})$ that always remains less than $O^*(1.5874^{\mathrm{opt}})$.

We conclude the section by showing that the same kind of trade-off can me made combining Proposition 4 and Proposition 1 [5] in order to get approximation results unachievable in polynomial time through exponential algorithms running faster than the currently best known exact algorithms. Recall that Proposition 1 in [5] claims that *for any positive* $\epsilon \leqslant 5$, MIN MAX INDEPENDENT SET *is* $(1+\epsilon)$*-approximable in time* $O^*(1.3351^{(1-(\epsilon/168))n})$.

If $\mathrm{opt}(G) < 0.626(1-(\epsilon/168))n$, then the algorithm of Proposition 4 computes a maximum minimal vertex cover of G in time smaller than $O^*(1.3351^n)$, which is the best worst-case complexity known for MIN MAX INDEPENDENT SET and, consequently, for MAX MIN VERTEX COVER. Suppose now that $\mathrm{opt}(G) \geqslant 0.626(1-(\epsilon/168))n$. In this case the $(1+\epsilon)$-approximation algorithm of Proposition 1 in [5] (indeed this algorithm can be seen as a kind of *moderately exponential approximation schema*) can be transformed a moderately exponential approximation schema for MAX MIN VERTEX COVER.

Denote by $\mathrm{opt}'(G)$ the size of a minimum dominating set in G and use the algorithm of Proposition 1 [5] in order to get an $(1+\epsilon)$-approximate independent

dominating set S. Obviously, the set $C = V \setminus S$ is a minimal vertex cover of G. The approximation ratio of C is:

$$\frac{|C|}{\mathrm{opt}(G)} = \frac{n - |S|}{n - \mathrm{opt}'(G)} \geqslant \frac{n - (1+\epsilon)\mathrm{opt}'(G)}{n - \mathrm{opt}'(G)} \tag{10}$$

The last expression in (10) decreases with $\mathrm{opt}'(G)$; since $\mathrm{opt}(G) \geqslant 0.626(1 - (\epsilon/168))n$, $\mathrm{opt}'(G) \leqslant (0.374 + (0.626\epsilon/168))n$ and setting it in the last term of (10) we get after some easy but tedious algebra that $|C|/\mathrm{opt}(G) \geqslant 1 + \epsilon'$ for some ϵ' that only depends on ϵ.

Let us finally note, that the same reasoning can be applied even with respect to future improved exact algorithms (just do the same analysis simply parameterizing the bases of the exponentials, i.e., using, for instance γ^n and δ^{opt} instead of 1.3351^n and 1.5874^{opt}).

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