The Competitive Diffusion Game in Classes of Graphs

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Abstract. We study a game based on a model for the spread of influence through social networks. In game theory, a Nash-equilibrium is a strategy profile in which each player's strategy is optimized with respect to her opponents' strategies. Here we focus on a specific two player case of the game. We show that there always exists a Nash-equilibrium for paths, cycles, trees, and Cartesian grids. We use the centroid of trees to find a Nash-equilibrium for a tree with a novel approach, which is simpler compared to previous works. We also explore the existence of Nash-equilibriums for uni-cyclic graphs, and offer some open problems.

Keywords: Competitive information diffusion, Nash-equilibriums, Network game theory, Social networks.

1 Introduction

Social networks play an important role in society, and are actively studied in a number of different disciplines, including mathematics. Recent studies have concentrated on interactions and influence in a social network. Such studies can lead to better techniques for viral marketing. In viral marketing, different techniques are combined with the knowledge about the social network to achieve marketing objectives in a way which is analogous to the spread of viruses, where contagion occurs through the links of the network. Many of these studies try to find a model for the spread of an idea or innovation through a social network. Usually these models use a graph to show the structure of a network, in which every individual in the network is denoted by a vertex, and two vertices are adjacent if there exists a relation or link between them in the corresponding network.

In a very well studied point of view (look at [6] and [3]), the propagation process is modelled in a way that usually each node or vertex has two status, either active or inactive. The process starts by targeting (or setting active) a small subset of the nodes in the social network with the hope of getting a large number of the individuals at the end who become active, i.e., affected by the influence. These models are basically involved with optimization techniques. On the other hand, there are some other studies looking at the propagation process as a competition among the individuals in the network, see [5]. There exist also

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some Voronoi game models involving a game among the parties or agents out of the network with representatives inside the network, where the objective is achieving the largest number of the users (see [4] and [7]).

In 2009, Alon et al [1], introduced a new model for the competitive diffusion process in social networks. Their approach was a novel way of modelling the spread of influence as a game, where the aim of this game is to influence users in the network through "infection" with a particular brand, spreading through the links of the network. In other words, suppose that we have a set of firms that want to advertise their products. Initially they target a small group of people, which they hope will extend into a larger group of society. Any individual, who has learned about a product brand from one of these firms first, either directly or through a social link, will be biased in favour of that brand. However, if a node is getting the influence from different products, she becomes confused and we cancel her out of the game. The gain of each firm is the total number of users that, at the end of the diffusion process, are biased towards its brand.

In the language of mathematics, we can model this competitive propagation process as a game on an undirected finite graph, in which our users form the vertex set of the graph, and the product of each firm is denoted by a distinct colour. A game $\Gamma = \langle G, N \rangle$ is induced by a graph G, representing the underlying social network, and a set of N players corresponding to the set of agents (we identify each player with a number i, $1 \leq i \leq N$). The strategy space of each player is the set of vertices V of G. That is, each player i, $1 \leq i \leq N$ selects a single node that is coloured in colour i at round 0, and every other vertex is uncoloured. If two or more agents select the same vertex at round 0, then, that vertex becomes gray, and those players automatically leave the game. If S_t is the set of the coloured vertices at round $t \geq 0$, then at round $t+1$, every player i can colour an uncoloured vertex v in the neighbourhood of S_t by the following rule: If v has coloured neighbours only in colour i, then v gets colour i. If v has coloured neighbours with different colours, then it becomes gray. The players continue until no one can colour any uncoloured vertex. At the end, the pay-off of the i-th player is the number of the vertices in G which have colour i. Note that, in this game, after choosing the strategies of the players, every thing in the process is deterministic.

As an example, let G be a graph as shown in Figure 1, and take $N = 2$. If the first player with colour 1, and the second player with colour 2, choose the two

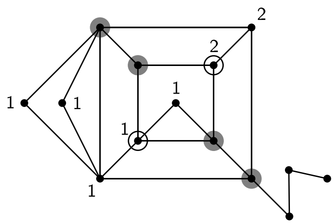

Fig. 1.

vertices with the circles around them at the beginning, then the pay-off of them will be the number of the vertices which are indexed by 1 and 2 in the figure, respectively. As we can see in Figure 1, there are four vertices that become gray by the rules of the game, and three vertices which are not reachable by any player and therefore, remain uncoloured at the end.

Note that, throughout this game, it is as if we delete all the gray vertices, so the metric of the graph is changing within the rounds of the game. This is unlike the Voronoi games [4], in which the gain of each agent is the number of individuals whose distance to the agent representative is less than the other agents.

In real networks finding a kind of stable situation in which every agent is satisfied is called a *Nash-equilibrium*. This is often of more interest than finding the winner of the game. A Nash-equilibrium is a strategy profile or a vector of strategies in the Cartesian product of the strategy sets of all players, such that the strategy of each player in such a vector is the best against the strategies of the others. In other words, in a Nash-equilibrium each player, by choosing that specific strategy, has maximized her pay-off with respect to the strategy of the other players. That is, no player can gain more by changing only her own strategy unilaterally. For further information about game theory concepts we refer the reader to [2]. Alon et al [1] in their paper, proved the existence of Nash-equilibriums for the game on graphs of diameter 2, and gave an example of a graph with diameter more than 2 which does not admit a Nash-equilibrium in the two-player case of the game. However, Takehara et al [10] provided a counter example with a graph of diameter 2 which does not admit a Nash-equilibrium, and presented a restatement of the theorem (about graphs with diameter at most 2) in [1] by putting some restrictions on the graph structure. Recently, Small and Mason [8] considered the existence of Nash-equilibriums for the two player game on trees, and also for the ILT model of online social networks [9], with focus on utility functions.

In this paper we will consider the special two player case of the above game for different families of graphs. However, we take a novel approach based on the graph properties of these families. In the second section, we prove the existence of Nash-equilibriums for trees, paths, cycles, and we consider the game for unicyclic graphs. Our proof for trees is much simpler and shorter compared to previous works [8]. In section 3, we show that Cartesian grids always admit a Nash-equilibrium, and we end by suggesting some open problems. In the paper we assume that the graphs are connected. We denote the vertex set and the edge set of a graph G by $V = V(G)$ and $E = E(G)$, respectively. For two vertices like $u, v \in G$, we call the length of the shortest path between u and v in G the *distance* between u and v, and we denote it by $d(u, v)$. If S is a subset of the vertices in G, by $G[S]$ we mean the subgraph induced by S. We denote a path and a cycle on n vertices by P_n and C_n, respectively. For two graphs G and H, we show the Cartesian product of G and H by $G \square H$. We refer the reader to [11] for graph-theoretic notation and terminology.

2 Trees, Paths, Cycles, and Uni-Cyclic Graphs

In this section, we consider some simple facts about the game, and use them to find Nash-equilibriums for different known families of graphs. The following definition help us to have a simpler language to describe the obtained results.

Assume that we are playing the game on a graph G, and, u and v are two distinct vertices of G. Suppose that in some round of the game there is a shortest path $P : u, v_1, \ldots, v_{n-1}, v$, and the vertices u and v have been coloured by two different players, such that no other vertex of P has been coloured yet. Then, we call path P a *blocked path* induced by the vertices u and v, or simply, a blocked path if $\min\{d(v_i, u), d(v_i, v)\} < d(v_i, w)$, for every $1 \leq i \leq n-1$, and for all vertices w $(w \neq u, v)$ which have been coloured so far throughout the game.

We need the following lemma to find a better understanding of the dynamic of a path between a pair of vertices with different colours throughout the game. We omit the proof, which follows immediately from the definition.

Lemma 1. *Suppose that we have a game on graph G. If P is a blocked path of length n induced by vertices v_1 and v_2 in G, by the end of the game, each player wins the first $\lfloor (n+1)/2 \rfloor$ nearest vertices in path P, and in the case that the length of P is even, one vertex in the middle becomes gray.*

A vertex v of a graph G is called a *cut vertex* if removing v from G results in a graph which is not connected. An edge uv is a *cut edge* if deletion of uv from G is a disconnected graph. The following lemma is quite useful for some of the results as we will see later on.

Lemma 2. *Assume that graph $G = G_1 \cup G_2$ is the union of two induced subgraphs G_1 and G_2 such that, for some cut vertex like v, $G_1 \cap G_2 = \{v\}$. Then, any possible Nash-equilibrium of the two player game on G consists of either two vertices in G_1 or two vertices in G_2.*

Proof. Assume that $\{u_1, u_2\}$ form a Nash-equilibrium such that $u_1 \in G_1 - G_2$ and $u_2 \in G_2 - G_1$. Then, each player changing her strategy to v can increase her pay-off. Because, this way, she can reach on the vertices in the other side earlier than before.

We now state and prove our first result on the competitive diffusion game for paths.

Theorem 1. *In a two-player game on a path of length n, the set of possible Nash-equilibriums is determined as below.*

(i) If n is odd, then the two adjacent vertices in the middle form the only possible Nash-equilibrium, and the equilibrium pay-offs are equal to $(n+1)/2$.

(ii) If n is even, then any two vertices in the middle (i.e., we have two possibilities, the central vertex and one of its neighbours) form a Nash-equilibrium, and the equilibrium pay-offs are both equal to $n/2$.

Proof. With a simple discussion using Lemma 1, we can show that if vertex v is the strategy of her opponent, then the best strategy for any of the players is to choose a neighbour of v which separates v from a larger number of the vertices in P. So in a possible Nash-equilibrium, the strategies of the players must be adjacent. However, if the players choose two adjacent vertices as their strategies which are not selected as in (i) or (ii), then the player who is closer to one of the end points can improve her pay-off by changing her strategy to another neighbour of her opponent. So, such a case is not a Nash-equilibrium. Finally, if they both have taken their strategies as in (i) or (ii), then no one can improve her pay-off by changing her strategy. Therefore, (i) and (ii) form the only possible Nash-equilibriums of this game.

Theorem 2. *In a two player game on cycle C_n of length n we have the following statements.*

(i) *If n is odd, then every two vertices on C_n selected by the players as their strategies, form a Nash-equilibrium, and the pay-offs are equal to $(n-1)/2$.*

(ii) *If n is even, then two vertices on C_n form a Nash-equilibrium if and only if they are of odd distance, and the equilibrium pay-offs are equal to $n/2$.*

Proof. When we have a two player game on a cycle C_n, the strategies of the players divide the cycle into two blocked paths. If n is odd, then one of the blocked paths is always of odd length and the other one is of even length. Obviously, by Lemma 1, every player wins $(n-1)/2$ vertices, and one vertex in the middle of the even path becomes gray. Since this happens for any selection of the vertices, any two vertices form a Nash-equilibrium when n is odd.

If n is even, then the two blocked paths are both even or odd. If they are both of odd length, then by Lemma 1, each player wins exactly half of the vertices on C_n, and no one can improve this. If the blocked paths are both even, then every player wins $(n/2)-1$, and one of the vertices in each blocked path becomes gray. Thus, each player can improve her pay-off by changing her strategy to an adjacent vertex. Hence, two vertices of C_n form a Nash-equilibrium if and only if they are of odd distance.

A maximal sub-tree which contains a vertex v of a tree T as a leaf is called a *branch* of T at v. The *weight* of a vertex v of T, denoted by $wt(v)$ is the maximum number of vertices in a branch at v (not including v). A vertex u is a *centroid vertex* of T if it has the minimum weight among all vertices. The *centroid* of T is the set of all centroid vertices of T.

Theorem 3. *[12] If $C = C(T)$ is the centroid of a tree T of order n, then we have,*

(i) *C consists of either a single vertex or two adjacent vertices.*

(ii) *If $C = \{c_1, c_2\}$, then $wt(c_1) = wt(c_2) = n/2$.*

(iii) *$C = \{c\}$ if and only if, $wt(c) \leq (n-1)/2$.*

Note that, according to the above theorem, in both possible cases for the centroid of a tree T, if $v \notin C(T)$, then $wt(v) > n/2$.

The following theorem, using the centroid of a tree shows that there exists a Nash-equilibrium for any tree.

Theorem 4. *In a two-player game on a tree T of order n with centroid C, we have the following statements.*

(i) If $C = \{c_1, c_2\}$, then C is the unique Nash-equilibrium, and the equilibrium pay-offs are equal to $n/2$.

(ii) If $C = \{c\}$, then $\{c, v\}$ is an equilibrium, in which v is a neighbour of c in a branch with maximum weight attached at c, and any equilibrium for this game consists of such two vertices.

Proof. Assume that v_1 and v_2 are the strategies of the players, and g_1 and g_2 are their pay-offs, respectively. Since there exists a unique path between any two vertices in a tree, then we conclude that the gain of v_1 is a subset of a branch attached at v_2 like B_2 which contains this unique path. Thus, we have $g_1 \leq |B_2| \leq wt(v_2)$. Similarly, $g_2 \leq wt(v_1)$.

Now, if v_1 and v_2 are not adjacent, then the path between v_1 and v_2 is a blocked path of length more than one and therefore, by Lemma 1, either every player wins half of the vertices on it, or there is a gray vertex in the middle of this path which no one gains. Thus, in such a case, we have the strict inequalities $g_1 < wt(v_2)$ and $g_2 < wt(v_1)$.

On the other hand, we know that always one of the branches attached at v_1 (similarly v_2) has the maximum weight, and if the second player chooses the neighbour of v_1 on such a branch, then she gains exactly $wt(v_1)$ vertices. Similarly, the first player can gain $wt(v_2)$. Hence, for the first player we have $g_1 \leq wt(v_2)$, and the equality achieved if and only if she chooses a vertex adjacent to v_2 from a maximum branch attached at v_2 (we have a similar result for the second player). In other words, fixing the strategy of a player on a vertex like v, the best strategy for the other player is to select a neighbour of v on a maximum branch attached at v. Therefore, in a possible Nash-equilibrium v_1 and v_2 must be adjacent.

Now, assume that v_1 and v_2 are adjacent, and for example (without loss of generality), $g_1 = wt(v_2)$. We know $g_1 + g_2 \leq n$. Also, by Theorem 3, if v_1 and v_2 are not in C then, $wt(v_1) > \frac{n}{2}$, and $wt(v_2) > \frac{n}{2}$. Consequently, $g_2 < \frac{n}{2} < wt(v_1)$, and therefore, the second player can move to a vertex adjacent to v_1 which achieves the maximum weight and increases her pay-off. Hence, such a case is not a Nash-equilibrium. Therefore, in a possible Nash-equilibrium at least one of the players' strategies must be in C. Now, by the above discussion and by Theorem 3, we can easily see that, the best strategy for the other player is to choose the strategy in (i) or in (ii), depending on the structure of C.

Suppose that G is a *uni-cyclic* graph, that is, G has only one cycle C. We can easily see that, $G - C$ is a forest, such that each tree component of this forest is adjacent to exactly one vertex on C. For each vertex $v \in C$, if there are $t = d(v) - 2$ different tree components in $G - C$ that are connected to v, we denote the union of each of these trees together with v (which is like adding a leaf to a tree and making a new tree) by T_{iv}, for $1 \leq i \leq t$; that is, all T_{iv}s share v.

Suppose that we have a two-player game on a uni-cyclic graph G with cycle C. By the above definition, we can assume that every vertex v on C has a weight $wt_C(v) := | \cup_{i=1}^{d(v)-2} T_{iv}|$. As we will see, sometimes we play the game on the weighted cycle C (instead of G) by the regular rules. The only difference here is that, the gain of each player after taking vertex v is increased by the weight of v. In such cases, we denote C by C_W (when we are playing the game only on C with weighted vertices).

We use the above notations for the results on uni-cyclic graphs. We use the following lemma, which is an immediate result of Theorem 3, to prove the next theorem (we omit the proof here).

Lemma 3. *Assume that T is a tree with centroid C. Then, for any vertex v which is not in C the maximum branch attached at v is the one that contains C (which is the only branch attached at v with weight more than $\frac{n}{2}$).*

In general, we have two possibilities for a uni-cyclic graph G with cycle C; either there is a vertex v on C with $|T_{iv}| \geq \frac{n}{2} + 1$, for some i, $1 \leq i \leq d(v) - 2$, or $|T_{iv}| \leq \frac{n}{2}$ for all $v \in C$, and $1 \leq i \leq d(v) - 2$. So, we have the following theorem.

Theorem 5. *Suppose that G is a uni-cyclic graph with cycle C. If there is a vertex v on C with $|T_{iv}| \geq \frac{n}{2} + 1$, for some $1 \leq i \leq d(v) - 2$, then there exists a Nash-equilibrium by playing on T_{iv}. Otherwise, if there exists a Nash-equilibrium for this game, then it must consist of a set of two vertices either on C_W or on a T_{iv}, for some $v \in C$ and $1 \leq i \leq d(v) - 2$.*

Proof. If there is a vertex v on C with $|T_{iv}| \geq \frac{n}{2} + 1$, for some i, then the players' strategies must be somewhere on T_{iv}. Because, first, if no one selected her strategy on T_{iv}, then the player with the smaller gain by moving to v can improve her pay-off (because this way she wins more than half of the vertices in G). So, in a possible Nash-equilibrium, at least one of the players must choose her strategy on a vertex in T_{iv}. Moreover, since v is a cut vertex, by Lemma 2, both of them should choose their equilibrium strategies in T_{iv}.

Now, we show that in such a case, we always have a Nash-equilibrium. In fact, in this case, we can replace $G - T_{iv}$ by a path P consisting of $|G - T_{iv}|$ vertices. If we take $T = T_{iv} \cup P$ (obviously, T is a tree) and $C(T)$ to be the centroid of T, then for the neighbour of v on P, called u, we have,

$$wt_T(u) = |T_{iv}| \geq \frac{n}{2} + 1 > \frac{n}{2}.$$

Thus, by Lemma 3, the centroid of T is in T_{iv}. Moreover, can easily see that, playing in a Nash-equilibrium of T is like playing in a Nash-equilibrium of G. Because, no one can increase her pay-off unilaterally. Therefore, by Theorem 4, we know that T always has a Nash-equilibrium.

Now, assume that for every $v \in C$ and each $1 \leq i \leq d(v) - 2$, $|T_{iv}| \leq \frac{n}{2}$, and there exists a Nash-equilibrium for this game. If the equilibrium vertices both are not included simultaneously in any T_{iv}, for a vertex v on C, and some $1 \leq i \leq d(v) - 2$, then, since every vertex in C of weight greater than one is a cut vertex, by Lemma 2, the strategies must be selected on C_W.

If for every vertex $v \in C$, $wt_C(v) \leq \frac{n}{2}$, then the following lemma could be helpful.

Lemma 4. *Assume that G is a uni-cyclic graph with weighted cycle C_W such that $wt(v) \leq \frac{n}{2}$ for all $v \in C$. Then, in a two player game on C_W with Nash-equilibrium $\{u, v\}$ we have,*

(i) either $\{u, v\}$ is a Nash-equilibrium for the regular game on G, or

(ii) one of the neighbours of u (or v) together with v (or u) form a Nash-equilibrium for G.

Proof. Assume that u and v are the strategies of the first and the second player in a Nash-equilibrium for the game on C_W. Also, suppose that g_x denotes the pay-off of a player who takes a vertex like x as her strategy. By definition, we know that no one can increase her pay-off by changing her strategy to another vertex on C_W. So, we have $g_u \geq g_z$, for any $z \in C_W$. Now, we consider the changes in the pay-off of the first player after moving to any vertex like w on a tree attached at a vertex like z on C, with $w \neq z$. We can easily see that, if the first player changes her strategy to vertex $w \neq z$, then $g_w < g_z$. Because, this way she gains the vertices on C_W at a later time ($d(w, v) > d(z, v)$). Therefore, she loses at least one of the vertices that she was able to take by choosing z. Thus, we have, $g_u \geq g_z > g_w$. Hence, the only way for the first player to increase her pay-off is to move to a vertex on T_{iv}, for some $1 \leq i \leq d(v) - 2$. Now, we can take a path P of length $|G - \cup_{i=1}^{d(v)-2} T_{iv}|$ and let T be the tree obtained from connecting P to $\cup_{i=1}^{d(v)-2} T_{iv}$ via v. We can see that, finding the best strategy with respect to v among the vertices in $\cup_{i=1}^{d(v)-2} T_{iv}$ in the game on G, is equivalent to finding such a strategy in the game on T. By proof of Theorem 4, in a game on a tree always the best strategy against an opponent is to play in her neighbourhood. So, if for any neighbour of v, like $w \in T_{iv}$, $g_w > g_u$ and g_w is the maximum over such neighbours of v, then the best strategy for the first player (against v) is to move to w. Moreover, in this case, $\{w, v\}$ forms a Nash-equilibrium for G. Because, in one side, w is the best strategy against v, and in the other side, the second player, moving to a vertex $z \neq w$ in T_{iv}, will gain $g_z < |T_{iv}| \leq \frac{n}{2} \leq |G - T_{iv}| = g_v$. Also, moving to a vertex in $G - T_{iv}$, she will lose some of the vertices (by getting further with respect to her opponent). Thus, v is also the best strategy against w.

However, if for every neighbour of v, like w, $g_w \leq g_u$, then, u is the best strategy against v in G. We can do the same discussion for the second player, and conclude that, either u together with one of its neighbours form a Nash-equilibrium for G, or otherwise, v is the best strategy against u. Therefore, $\{u, v\}$ forms a Nash-equilibrium for G.

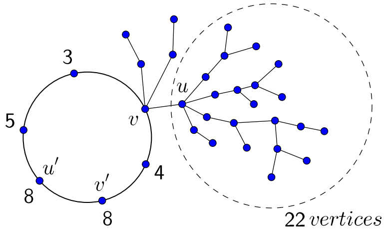

Fig. 2.

In reverse, if we have a Nash-equilibrium for G on cycle C, then it is also a Nash-equilibrium for C_W. But, if we find a Nash-equilibrium for G such that one of the strategies chosen by the players is out of C, then this case does not necessarily help to find a Nash-equilibrium for C_W. In Figure 2 we see a uni-cyclic graph G, in which $\{u, v\}$ form a Nash-equilibrium (nobody can increase her pay-off moving to another vertex). But, if we try to play the game on the weighted cycle C_W, then the only possible Nash-equilibrium is $\{u', v'\}$, which has no intersection with $\{u, v\}$ and is obtained independently.

As a consequence of Lemma 4, if we find a Nash-equilibrium for the game on C_W, then we can find a Nash-equilibrium for the game on G. This conclusion shows the importance of the following theorem as the last result of this section. We omit the proof here which is a long technical one, and will be published in a future paper.

Theorem 6. *In a two player game on a uni-cyclic graph G with cycle C (or weighted cycle C_W) of lengths* 3, 4, *and* 5 *always there exists a Nash-equilibrium.*

The uni-cyclic graph G in Figure 3 is an example of a weighted 6-cycle that does not admit any Nash-equilibrium. First, the weight of each tree attached at a vertex on the cycle C is less than half of the whole number of the vertices. Hence, using Lemma 2, we can easily consider all different possibilities to conclude that there can not be any Nash-equilibrium in which one of the players chooses a strategy out of C. So, by Theorem 5, it is enough to consider the game on C_W. Now, we have the following bimatrix as the pay-off matrix (see [2]) of the players in C_W (note that it is a symmetric game and the columns are corresponding to vertices v_1, v_2, v_3, v_4, v_5, and v_6, as well as the rows, respectively):

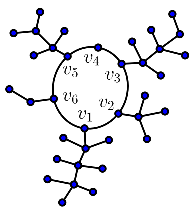

Fig. 3.

$$\begin{bmatrix} (0,0) & (21^*,15) & (14,10) & (19,17^*) & (16,8) & (25^*,11) \\ (15,21^*) & (0,0) & (19,17) & (16,8) & (25^*,11) & (14,10) \\ (10,14) & (17,19) & (0,0) & (25^*,11) & (14,10) & (15,21^*) \\ (17^*,19) & (8,16) & (11,25^*) & (0,0) & (15,21) & (10,14) \\ (8,16) & (11,25^*) & (10,14) & (21,15) & (0,0) & (17,19) \\ (11,25^*) & (10,14) & (21^*,15) & (14,10) & (19,17) & (0,0) \end{bmatrix}$$

From game theory (see [2]), we know that a possible Nash-equilibrium for such a game is determined by an entry of this matrix, in which the first component is the largest in the same column and the second component is the largest in the row. Here, for each column and each row we determine such components with a star. As we can see, there is no entry with a star on both components. Thus, there is no Nash-equilibrium for this game.

As another example, assume that G is a uni-cyclic graph with trees of equal order attached at the vertices of the cycle. Then, the two-player game on G is like playing on a weighted cycle with equal weight on all vertices. So, we can easily see that the set of Nash-equilibriums is determined exactly as for a regular cycle. The only difference is that here the pay-off of the players is a multiple (a constant multiple, which is equal to the weight of the vertices on C) of the pay-off in the regular game on a cycle without weights.

3 Cartesian Grids

In this section we investigate the existence of Nash-equilibriums for the *Cartesian grids*. In graph theory, a *grid* (or Cartesian grid) is the Cartesian product of two paths. If $G = P_n \square P_m$, then we call such a grid a $m \times n$ *grid* [11]. We call a subgraph of G which is also a grid by itself, a *subgrid* of G. If A and B are two vertices of a grid G, then G_{AB} is the maximal subgrid of G which contains A and B as the corner points and consists of all the shortest paths between A and B in G.

We need the following concepts to reach the result on grids. Assume that G is a graph and v is a vertex of G. Then, the *eccentricity* of v is defined to be $\max\{d(v,u) : u \in G\}$. The *center* of G is the set of the vertices in G which have the minimum eccentricity [11]. We have the following fact about the center of a grid, which is quite easy to prove only using the definition.

Theorem 7. *Assume that G is a $m \times n$ grid with center C, in which m and n are positive integers. Then, depending on the parity of m and n, we have the following possibilities for C.*

(*i*) *If m and n are odd, then C consists of a single point in the middle.*

(*ii*) *If one of m and n is odd and the other one is even, then C consists of two adjacent vertices in the middle.*

(*iii*) *If m and n are even, then C consists of a 1×1 subgrid in the middle.*

Using the center of a grid, we can always find a Nash-equilibrium for the two player competitive diffusion game on grids.

Theorem 8. *Assume that we have a two player game on a $m \times n$ grid G, in which m and n are positive integers, and $m \leq n$. Let C be the center of G. Then,*

(i) If m and n are odd, then the single vertex in $C = \{c\}$ together with one of the neighbours of c like v which is placed in the same row as c, form a Nash-equilibrium.

(ii) If one of m and n is odd and the other one is even, then the two vertices in $C = \{c_1, c_2\}$ form the unique Nash-equilibrium.

(iii) If m and n are even, then any pair of adjacent vertices in C form a Nash-equilibrium.

Proof. Assume that A and B are the strategies of the players, with g_1 and g_2 as their pay-offs, respectively. Then, there is a vertical as well as a horizontal line

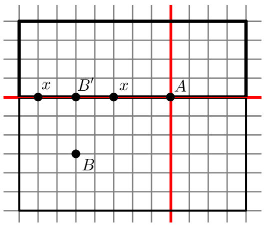

Fig. 4.

which passes through point A in the grid plane, and forms part of the perimeter of some rectangles created by A (in total, there are at most four possible such rectangles as we see in Figure 4, depending on the position of A). We observe that B is always inside of one of those rectangular regions created by A. Now, if G_{AB} is a square, then the distance between A and B is even. Thus, by Lemma 1, there must be some gray vertices appeared on the diagonal points of G_{AB} through out the game, and obviously, one of the players by changing her position and making these gray vertices vanish can gain more. So, this can not be a Nash-equilibrium.

If G_{AB} is not a square, then B is further with respect to one of these rectangles like R_{Ai} than the others. Thus, assuming that B' is the closest point of R_{Ai} with respect to B, for any point like x on the perimeter of R_{Ai}, we have,

$$d(x, A) \leq d(x, B') + d(B', B) = d(x, B).$$

Therefore, through the rounds of the game, the first player (choosing A) gets x before the second player. Thus, the first player wins at least all the vertices in R_{Ai}. Hence, in a possible Nash-equilibrium we have,

$$g_2 \leq mn - |R_{Ai}| \leq mn - \min\{|R_{Aj}| : R_{Aj} \text{ is a rectangle created by A}\}. \quad (1)$$

But, this bound can be achieved only when B is the neighbour of A opposite to the smallest rectangle created by A. Otherwise, the first player wins all the vertices in the smallest rectangle created by A, plus at least the neighbour of A opposite to this rectangle. Thus, the best strategy for the second player is to achieve this bound as discussed. Similarly, we can consider the rectangles created by B, and again we have,

$$g_1 \leq mn - |R_{Bi}| \leq mn - \min\{|R_{Bj}| : R_{Bj} \text{ is a rectangle created by B}\}, \quad (2)$$

which can be achieved only when A is the neighbour of B opposite to the smallest rectangle created by B. Hence, in a possible Nash-equilibrium, the strategies of the players should be adjacent. If the players do not choose their strategies as in (i), (ii), or (iii), then using the above discussion and inequities (1) and (2), we can see that one of the players can increase her pay-off. Thus, such a case is not a Nash-equilibrium.

Now, assume that players choose their strategies like in (i), (ii), or (iii). Then, no one can increase her pay-off, since no one can enlarge the smallest rectangle created by her strategy. Therefore, (i), (ii), and (iii) form the Nash-equilibriums of this game.

Although for the two player game on grids there exists a Nash-equilibrium, it seems that for the three player case the existence of Nash-equilibriums is not certain. For example, discussing around different possibilities, we can easily see that for the three player game on $P_2 \square P_n$, or $P_3 \square P_n$, there is no Nash-equilibrium. In general, we have the following conjecture.

Conjecture 1. There exist no Nash-equilibrium for a three player game on a Cartesian grid.

Another family of graphs that we often consider for a graph theoretic problem are bipartite graphs. We can simply discuss that for a complete bipartite graph, a Nash-equilibrium is to choose a vertex as the strategy of the first player from the first part and a vertex for the second player from the second part. This way, each player wins all the vertices in the opposite part except for the strategy of her opponent. But, finding a Nash-equilibrium for an arbitrary bipartite graph in general seems challenging.

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