On P_3-Convexity of Graphs with Bounded Degree

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Abstract. Motivated by the large applicability as well as the hardness of P_3-convexity, we study new complexity aspects of such convexity restricted to graphs with bounded maximum degree. More specifically, we are interested in identifying either a minimum P_3-geodetic set or a minimum P_3-hull set of such graphs, from which the whole vertex set of G is obtained either after one or sufficiently many iterations, respectively. Each iteration adds to a set S all vertices of $V(G) \setminus S$ with at least two neighbors in S. We prove that: (i) a minimum P_3-hull set of a graph G can be found in polynomial time when $\delta(G) \geq \frac{n(G)}{c}$ (for some constant c); (ii) deciding if the size of a minimum P_3-hull set of a graph is at most k remains NP-complete even on planar graphs with maximum degree four; (iii) a minimum P_3-hull set of a cubic graph can be found in polynomial time; (iv) a minimum P_3-hull set can be found in polynomial time in graphs with minimum feedback vertex set of bounded size and no vertex of degree two; (v) deciding if the size of a minimum P_3-geodetic set of a planar graph with maximum degree three is at most k remains NP-complete.

Keywords: P_3-convexity, P_3-hull set, P_3-geodetic set, planar graphs, bounded degree, NP-hardness.

1 Introduction

Let $G = (V, E)$ be a graph. For $U \subseteq V$, let the interval $I[U]$ of U in G be the set $U \cup \{u \in V(G) \setminus U \mid |N_G(u) \cap U| \geq 2\}$. A set S of vertices of G is P_3-*geodetic* if $I[S]$ contains all vertices of G. The P_3-*geodetic number* $g_{P_3}(G)$ of a graph G is defined as the minimum cardinality of a P_3-geodetic set. The decision problem related to determining the P_3-geodetic number is known to be NP-complete for general graphs, and coincides with the well-studied 2-domination number [10,8,11,12,13].

A P_3-*hull* set U of G is a set of vertices such that:

- $U^0 = U$
- $U^k = I[U^{k-1}]$, for $k \geq 1$.
- $\exists\ k \geq 0 \mid U^k = V(G)$.

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We define $H_G(S) \subseteq V(G)$ as $I[S]^{k+1}$ where the non-negative integer k is such that $I[S]^{k+1} = I[S]^k$, $k \geq 0$. The cardinality of a minimum P_3-hull set of G is the P_3*-hull number* of G, denoted by $h_{p3}(G)$. Again, the decision problem related to determining the P_3-hull number of a graph is still a well known NP-complete problem [4].

According to [5], as one of the most elementary models of the spreading of a property within a network – like sharing an idea or disseminating a virus – one can consider a graph G, a set U of vertices of G that initially possesses the property, and an iterative process whereby new vertices u are added to U whenever sufficiently many neighbors of u are already in U. The simplest non-trivial choice leads to the *irreversible 2-threshold processes* by Dreyer and Roberts [6]. Similar models were studied in various contexts, such as statistical physics, social networks, marketing, and distributed computing under different names such as bootstrap percolation, influence dynamics, local majority processes, irreversible dynamic monopolies, catastrophic fault patterns, and many others [1,2,3,4,5,6].

In the next sections, we analyze the complexity of these problems when some parameters related to the maximum and minimum degree of a graph are known. In the following subsection we review some results on planar satisfiability problems. In Section 2 we present some results on finding a minimum P_3-hull set of graphs with bounded degree. Finally, in Section 3 we analyze complexity aspects of finding a minimum P_3-geodetic set on planar graphs with bounded degree.

1.1 Planar SAT-am3

SAT-am3 [9]
Instance: A set $F = \{C_1, C_2, \ldots, C_m\}$ of clauses, built on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ of boolean variables, such that each clause contains at most three literals, each variable appears at most three times, and each literal occurs at most twice.
Question: Is there a truth assignment to the variables in X that satisfies F?

SAT-am3 is an NP-complete problem [9]. In [9] the problem was not defined with the restriction of each literal occurs at most twice, but without loss of generality, if a literal l occurs three times, the clauses containing l can be considered satisfied and removed from the formula F to be analyzed. Another variant of SAT is described below.

Planar 3-SAT [9]
Instance: A set $F = \{C_1, C_2, \ldots, C_m\}$ of clauses, built on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ of boolean variables, where each clause contains at most three literals, and the bipartite graph $H_F = (V, E)$ such that $V = \{w_{c_1}, w_{c_2}, \ldots, w_{c_m}\} \cup \{v_{x_1}, v_{x_2}, \ldots, v_{x_n}\}$ and E contains exactly those pairs (w_{c_i}, v_{x_j}) such that either x_j or $\neg x_j$ belongs to the clause C_i, is planar.
Question: Is there a truth assignment to the variables in X that satisfies F?

Note that not every instance of SAT-am3 is an instance of Planar 3-SAT. For example, $F = (\neg x_1 + x_2 + x_3)(x_2 + \neg x_3 + \neg x_5)(x_1 + \neg x_2 + x_4)(x_3 + \neg x_4)(\neg x_1 +$

x_5) is non-planar because it contains a subdivision of $K_{3,3}$. However, it is well known [9,14] that Planar 3-SAT is also an NP-complete problem.

At this point, we describe the intersection of these problems.

Planar SAT-am3

Instance: A set $F = \{C_1, C_2, \ldots, C_m\}$ of clauses, built on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ of boolean variables, where each clause contains at most three literals, each variable appears at most three times, each literal occurs at most twice, and the bipartite graph $H_F = (V, E)$ such that $V = \{w_{c_1}, w_{c_2}, \ldots, w_{c_m}\} \cup \{v_{x_1}, v_{x_2}, \ldots, v_{x_n}\}$ and E contains exactly those pairs (w_{c_i}, v_{x_j}) such that either x_j or $\neg x_j$ belongs to the clause C_i, is planar.

Question: Is there a truth assignment to the variables in X that satisfies F?

Lemma 1. Planar SAT-am3 *is NP-complete.*

Proof. It is easy to see that the problem is in NP. To prove the hardness, we perform a reduction from Planar 3-SAT. Consider a general Planar 3-SAT expression F in which x_i appears k_i times. Assign $F' = F$, and for each x_i in F' replace the first occurrence of x_i by x_i^1, the second by x_i^2, and so on, where $x_i^1, x_i^2, \ldots, x_i^{k_i}$ are new variables. Add $(\neg x_i^1, x_i^2), (\neg x_i^2, x_i^3), \ldots, (\neg x_i^{k_i}, x_i^1)$ to F'. Clearly, F' is satisfiable if and only if F is satisfiable.

By Kuratowski's theorem a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$. To show that $H_{F'}$ is planar, just observe that given a planar embedding of the bipartite graph corresponding to F, one can obtain a planar embedding of the graph corresponding to F' by replacing some vertices u of degree k with a cycle C of order k and a matching of k edges between $V(C)$ and the neighbors of u. It is easy to see that the constructed graph has a planar embedding. $\square$

2 P_3-Hull Set

In this section we consider both search and decision problems on P_3-hull sets.

P_3-Hull Set

Instance: A graph G.

Goal: Find a P_3-hull set of G with minimum cardinality.

P_3-Hull Number

Instance: A graph G; an integer k.

Goal: Decide if G has a P_3-hull set with cardinality at most k.

Note that P_3-Hull Number is clearly in NP. Moreover, it is easy to see that if P_3-Hull Number is NP-complete then P_3-Hull Set is NP-hard.

Let $n(G)$ be the number of vertices of G, $N_G(x)$ the neighborhood of a vertex x in G, $d_G(x) = |N_G(x)|$ the degree of vertex x in G, and $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of a vertex in G, respectively.

Lemma 2. *Let k be a positive integer. If G is a graph, then*

$$\Delta_k(G) := \max\left\{\left|\bigcap_{x\in U} N_G(x)\right| \mid U \in \binom{V(G)}{k}\right\} \geq n(G)\frac{\binom{\delta(G)}{k}}{\binom{n(G)}{k}}.$$

Proof. Let $\mathcal{R} = \left\{(u,U) : u \in V(G), U \in \binom{V(G)}{k}, u \in \bigcap_{x\in U} N_G(x)\right\}$. Since for every vertex v of G there are $\binom{d_G(v)}{k} \geq \binom{\delta(G)}{k}$ pairs (u,U) in $\mathcal{R}$ with $u = v$, we have $|\mathcal{R}| \geq n(G)\binom{\delta(G)}{k}$. Conversely, by the definition of $\Delta_k(G)$, for every set $V \in \binom{V(G)}{k}$, there are at most $\Delta_k(G)$ pairs (u,U) in $\mathcal{R}$ with $U = V$, which implies $|\mathcal{R}| \leq \Delta_k(G)\binom{n(G)}{k}$. □

Theorem 3. *Let c be a positive integer.*
If G is a graph with $\delta(G) \geq \frac{n(G)}{c}$, then

$$h_{P_3}(G) \leq 2\left\lceil \frac{\log(2c)}{\log\left(\frac{2c^2}{2c^2-1}\right)} \right\rceil + 2c^3.$$

Proof. In order to construct a small P_3-hull set of G we describe an inductive construction of a sequence $G_1, \ldots, G_k$ of induced subgraphs of G such that

- $G_i = G - H_G(S_{i-1})$ for a set S_{i-1} of at most $2(i-1)$ vertices of G,
- $n(G_i) \leq n(G)\left(1 - \frac{1}{2c^2}\right)^{i-1}$, and
- $\delta(G_i) \geq \frac{n(G_i)}{c}$

for $i \in [k]$.

Let $G_1 = G$ and $S_0 = \emptyset$.

Now let i be such that G_i and S_{i-1} are defined. If G_i is complete or $n(G_i) < 2c^3$, then terminate the construction of the sequence and set k to i. Since

$$h_{P_3}(G) \leq |S_{k-1}| + h_{P_3}(G_k) \leq 2(k-1) + 2c^3,$$

it suffices to bound k in order to complete the proof.

Therefore, we may assume that G_i is not complete and that $n(G_i) \geq 2c^3$. By Lemma 2, there are two vertices u_i and v_i of G_i with at least

$$n(G_i)\frac{\binom{\delta(G_i)}{2}}{\binom{n(G_i)}{2}} \geq n(G_i)\frac{\binom{\frac{n(G_i)}{c}}{2}}{\binom{n(G_i)}{2}} = \frac{n(G_i)(n(G_i)-c)}{c^2(n(G_i)-1)} \geq \frac{n(G_i)}{2c^2}$$

common neighbors. Let $S_i = S_{i-1} \cup \{u_i, v_i\}$ and $G_{i+1} = G - H_G(S_i)$. We obtain

$$\begin{aligned} n(G_{i+1}) &= n(G) - |H_G(S_i)| \\ &\leq n(G) - |H_G(S_{i-1}) \cup H_{G_i}(\{u_i, v_i\})| \\ &\leq n(G) - |H_G(S_{i-1})| - |H_{G_i}(\{u_i, v_i\})| \\ &= n(G_i) - |H_{G_i}(\{u_i, v_i\})| \\ &\leq n(G_i) - \frac{n(G_i)}{2c^2} \\ &= n(G_i)\left(1 - \frac{1}{2c^2}\right) \\ &\leq n(G)\left(1 - \frac{1}{2c^2}\right)^i . \end{aligned}$$

Since $G_{i+1} = G - H_G(S_i)$, we have $\delta(G_{i+1}) \geq \delta(G) - 1 \geq \frac{n(G)}{c} - 1$. Therefore,

$$\frac{\delta(G_{i+1})}{n(G_{i+1})} \geq \frac{\frac{n(G)}{c} - 1}{n(G_i)\left(1 - \frac{1}{2c^2}\right)} \geq \frac{\frac{n(G_i)}{c} - 1}{n(G_i)\left(1 - \frac{1}{2c^2}\right)} \geq \frac{1}{c}.$$

Since the minimum degree of all graphs G_i in the sequence is at least $\delta - 1$, the value of k is less than or equal to the smallest integer r with

$$n(G)\left(1 - \frac{1}{2c^2}\right)^{r-1} \leq \frac{n(G)}{c} - 1.$$

Since $\frac{n(G)}{c} - 1 \geq \frac{n(G)}{2c}$, we obtain

$$k \leq \left\lceil \frac{\log(2c)}{\log\left(\frac{2c^2}{2c^2-1}\right)} \right\rceil + 1,$$

which completes the proof. □

Corollary 4. *A minimum P_3-hull set of a graph G with $\delta(G) \geq \frac{n(G)}{c}$ (for some constant c) can be found in polynomial time.*

Proof. The proof follows immediately from Theorem 3. □

Theorem 5. P_3-Hull Number *remains NP-complete on planar graphs with maximum degree four.*

Proof. To prove that deciding whether the P_3-hull number of a graph G is less than or equal k is NP-complete, we perform a reduction from Planar SAT-am3, proved to be NP-complete in Lemma 1. Here *cross edges* are meant in the usual sense of a planar graph: edges crossing other edges in a specific embedding of a graph in the plane.

Given an instance F of Planar SAT-am3, we construct an instance G of P_3-Hull Set as follows:

– For each variable x_i of F, create a gadget G_{x_i} composed of 62 vertices as illustrated in Figure 1. Note that G_{x_i} is composed of two subgadgets g_{x_i} and $g_{\bar{x}_i}$, which represent the literals x_i and $\bar{x}_i$, respectively.

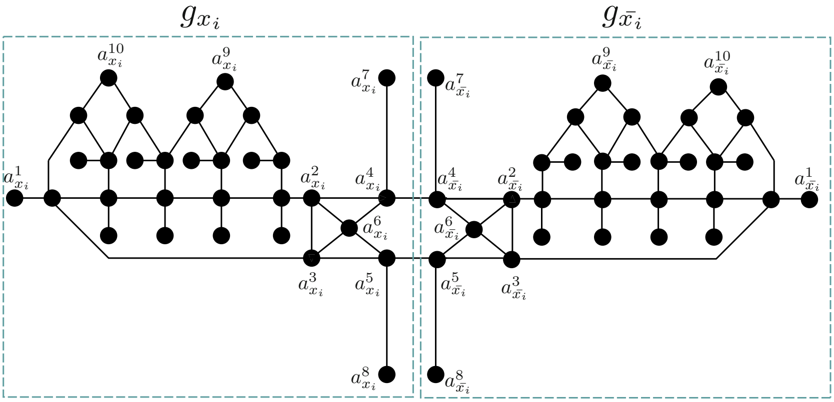

Fig. 1. Gadget G_{x_i}

– For each clause C_j of F, create a gadget G_{c_j} composed of the cycle $b^1_{c_j}$, $b^2_{c_j}$, $b^3_{c_j}$, $b^4_{c_j}$, $b^5_{c_j}$, $b^6_{c_j}$, $b^7_{c_j}$, $b^8_{c_j}$ plus the vertices $b^9_{c_j}$, $b^{10}_{c_j}$, $b^{11}_{c_j}$, $b^{12}_{c_j}$, $b^{13}_{c_j}$, $b^{14}_{c_j}$, $b^{15}_{c_j}$, $b^{16}_{c_j}$ and edges $(b^1_{c_j}, b^9_{c_j}), (b^2_{c_j}, b^{10}_{c_j}), (b^3_{c_j}, b^{11}_{c_j}), (b^4_{c_j}, b^{12}_{c_j}), (b^5_{c_j}, b^{13}_{c_j}), (b^6_{c_j}, b^{14}_{c_j}), (b^7_{c_j}, b^{15}_{c_j}), (b^8_{c_j}, b^{16}_{c_j})$. Figure 2 illustrates a gadget G_{c_j}.

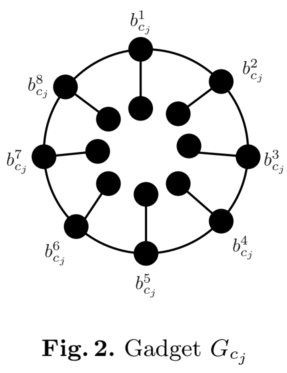

Fig. 2. Gadget G_{c_j}

– If the literal x_i occurs twice in F, then create the vertices $f^1_{x_i}$, $f^2_{x_i}$, and add edges $(f^1_{x_i}, a^7_{x_i}), (f^2_{x_i}, a^8_{x_i})$. Otherwise, create only $f^1_{x_i}$ and add $(f^1_{x_i}, a^7_{x_i})$.
– If the literal $\bar{x}_i$ occurs twice in F, then create the vertices $f^1_{\bar{x}_i}$, $f^2_{\bar{x}_i}$, and add edges $(f^1_{\bar{x}_i}, a^7_{\bar{x}_i}), (f^2_{\bar{x}_i}, a^8_{\bar{x}_i})$. Otherwise, create only $f^1_{\bar{x}_i}$ and add $(f^1_{\bar{x}_i}, a^7_{\bar{x}_i})$.
– For each clause C_j do:

1. if x_i is the first literal of C_j, then: if C_j contains the first occurrence of x_i then add edges $(a^7_{x_i}, b^1_{c_j}), (a^9_{x_i}, b^2_{c_j})$; else add edges $(a^{10}_{x_i}, b^1_{c_j}), (a^8_{x_i}, b^2_{c_j})$.
2. if x_i is the second literal of C_j, then: if C_j contains the first occurrence of x_i then add edges $(a^7_{x_i}, b^5_{c_j}), (a^9_{x_i}, b^6_{c_j})$; else add edges $(a^{10}_{x_i}, b^5_{c_j}), (a^8_{x_i}, b^6_{c_j})$.
3. if x_i is the third literal of C_j, then: if C_j contains the first occurrence of x_i then add edges $(a^7_{x_i}, b^7_{c_j}), (a^9_{x_i}, b^8_{c_j})$; else add edges $(a^{10}_{x_i}, b^7_{c_j}), (a^8_{x_i}, b^8_{c_j})$. If this step generates cross edges, remove the newly created edges, and repeat this step replacing $b^7_{c_j}$ and $b^8_{c_j}$ by $b^3_{c_j}$ and $b^4_{c_j}$, respectively. This operation keeps the graph planar, as one can check by verifying all possible configurations.
4. if $\bar{x}_i$ is the first literal of C_j, then: if C_j contains the first occurrence of $\bar{x}_i$ then add edges $(a^7_{\bar{x}_i}, b^2_{c_j}), (a^9_{\bar{x}_i}, b^1_{c_j})$; else add edges $(a^{10}_{\bar{x}_i}, b^2_{c_j}), (a^8_{\bar{x}_i}, b^1_{c_j})$.
5. if $\bar{x}_i$ is the second literal of C_j, then: if C_j contains the first occurrence of $\bar{x}_i$ then add edges $(a^7_{\bar{x}_i}, b^6_{c_j}), (a^9_{\bar{x}_i}, b^5_{c_j})$; else add edges $(a^{10}_{\bar{x}_i}, b^6_{c_j}), (a^8_{\bar{x}_i}, b^5_{c_j})$.
6. if $\bar{x}_i$ is the third literal of C_j, then: if C_j contains the first occurrence of $\bar{x}_i$ then add edges $(a^7_{\bar{x}_i}, b^8_{c_j}), (a^9_{\bar{x}_i}, b^7_{c_j})$; else add edges $(a^{10}_{\bar{x}_i}, b^8_{c_j}), (a^8_{\bar{x}_i}, b^7_{c_j})$. If this step generates cross edges, remove the newly created edges, and repeat this step replacing $b^7_{c_j}$ and $b^8_{c_j}$ by $b^3_{c_j}$ and $b^4_{c_j}$, respectively. As above, this operation keeps the graph planar, as one can check by verifying all possible configurations.

Let G be the graph obtained by the construction above from an instance F of Planar SAT-am3. At this point, we will prove that F is satisfiable if and only if G has a hull set of size $8m + 23n$, where m is the number of clauses, and n is the number of variables of F.

If F is satisfiable, then we can obtain a P_3-hull set S of G by first adding all the pendant vertices of G to S. Note that G has $8m + 22n$ pendant vertices. Let A be a truth assignment of F. If $x_i = true$ in A we add $a^2_{x_i}$ to S, else we add $a^2_{\bar{x}_i}$ to S. As A is a truth assignment of F, each gadget G_{c_j} will be contaminated, i.e. in $H_G(S)$, and consequently all vertices of G will be contaminated. Hence S is a P_3-hull set of size $8m + 23n$.

Conversely, if G has a P_3-hull set S of size $8m + 23n$, S contains $8m + 22n$ pendant vertices and n non-pendant vertices of G. As we can observe in each gadget G_{x_i} of G, there is a subgraph B_{x_i} such that every vertex v of B_{x_i} is not a pendant vertex and either it is adjacent to only one leaf and has no non-pendant neighbor outside B_{x_i}, or v has only one neighbor outside B_{x_i}. Figure 3 illustrates a gadget G_{x_i} and its subgraph B_{x_i}. Consequently, each subgraph B_{x_i} must have exactly one vertex in S, which is not a pendant vertex. Otherwise either S is not a P_3-hull set or S has size greater than $8m + 23n$. At this point we can construct an assignment A of F by setting $x_i = true$ if and only if $S \cap V(g_{x_i}) \cap V(B_{x_i}) \neq \emptyset$. By construction, we can see that A is a truth assignment of F. □

A *feedback vertex set* of a graph is a set of vertices whose removal leaves a graph without cycles. In other words, each feedback vertex set contains at least one vertex of any cycle in the graph.

Lemma 6. *Let G be a cubic graph. $S \subseteq V(G)$ is a P_3-hull set of G if and only if S is also a feedback vertex set of G.*

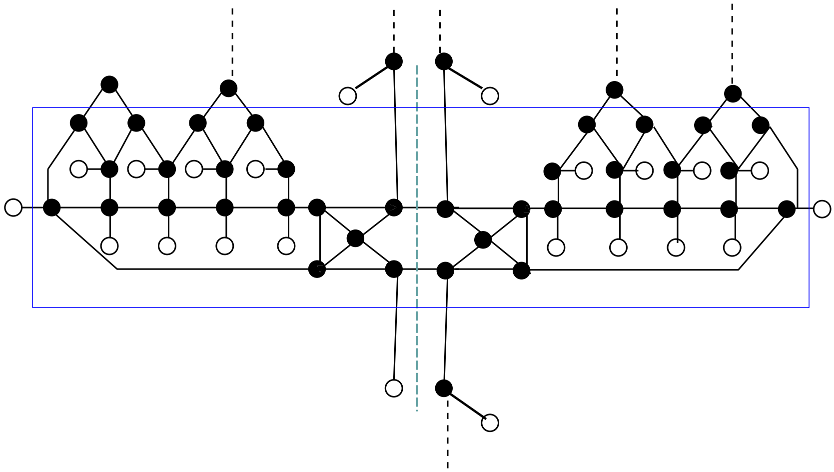

Fig. 3. Gadget G_{x_i} and its subgraph B_{x_i} inside the rectangle. The white vertices are pendant vertices in G and are not contained in B_{x_i}.

Proof. Let G be a cubic graph and S be a P_3-hull set of G. If $G[V \setminus S]$ has a cycle C, then each vertex $v \in C$ has at most one neighbor outside C, and consequently C is not in the hull of S, which is a contradiction because S is a P_3-hull set of G.

Conversely, let B be a feedback vertex set of G. As $G[V \setminus B]$ is a forest and G is cubic, all pendant vertices of $G[V \setminus B]$ are in $H_G(B)$; by removing these pendant vertices of $G[V \setminus B]$, we obtain a forest T where each leaf of T has two neighbors in $H_G(B)$. Applying this step recursively, we can see that all vertices of $G[V \setminus B]$ are in $H_G(B)$. □

Proposition 1. [15] *A minimum feedback vertex set of a graph G with maximum degree at most three can be found in polynomial time.*

Corollary 7. *A minimum P_3-hull set of a cubic graph can be found in polynomial time.*

Proof. The proof follows immediately from Lemma 6 and Proposition 1. □

Theorem 8. *Let $\mathscr{F}$ be the class of graphs with no vertex of degree two and with a minimum feedback vertex set of size bounded by a constant c. Then P_3-*Hull Set *on $\mathscr{F}$ can be solved in polynomial time.*

Proof. Let $G \in \mathscr{F}$. As G has a minimum feedback vertex set of size bounded by a constant c, we can find a minimum feedback vertex set B of G in polynomial time. Let L be the set of pendant vertices in G, and let $T = G \setminus \{B \cup L\}$. Since G has no vertex of degree two, each leaf of T has at least two neighbors in $\{B \cup L\}$ and just as in the proof of Lemma 6, $\{B \cup L\}$ is a hull set of G. As L is in any hull set of G, it is sufficient to examine all subsets of vertices in $V(G) \setminus L$ of size at most c to find a minimum P_3-hull set of G. □

3 P_3-Geodetic Set

Now we consider the following decision problem:

P_3-Geodetic Number
Instance: A graph G; an integer k.
Goal: Decide if G has a P_3-geodetic set with cardinality at most k.

Note that P_3-Geodetic Number is clearly in NP.

As Dominating Set is NP-complete even restricted to planar graphs with maximum degree three [9], it is easy to see that P_3-geodetic Number problem remains NP-complete on planar graphs with maximum degree four. Just take an instance G of such restricted Dominating Set problem and construct a graph G' by adding a new vertex w_v and a new edge (v, w_v) for each vertex v of G. Note that G has a dominating set of size k if and only if G' has a P_3-geodetic set of size $n + k$. As G is a planar graph with maximum degree 3, G' is a planar graph with maximum degree 4.

As P_3-Geodetic Number is NP-complete on planar graphs with maximum degree four, and trivially solvable in polynomial time on graphs with maximum degree two, it is natural to ask about the complexity of P_3-Geodetic Number on planar graphs with maximum degree 3.

Theorem 9. P_3-Geodetic Number *remains NP-complete on planar graphs with maximum degree three.*

Proof. Deciding whether the P_3-geodetic number of a graph G is less than or equal to k is clearly a problem in NP. To prove the NP-hardness we perform a reduction from Planar SAT-am3, proved to be NP-complete in Lemma 1. Given an instance F of Planar SAT-am3 we construct an instance G of P_3-Geodetic Set as follows:

- for each variable x_i do: create in G a gadget g_{x_i} composed of a cycle $f^1_{x_i}, t^1_{x_i}, a^1_{x_i}, a^2_{x_i}, f^2_{x_i}, t^2_{x_i}, a^3_{x_i}, a^4_{x_i}$;
- for each clause C_i containing at most two literals do: create in G a gadget g_{c_i} composed of the vertices c^1_i, c^2_i and edge (c^1_i, c^2_i);
- for each clause C_j containing exactly three literals do: create in G a gadget g_{c_j} composed of the vertices $c^1_j, c^2_j, c^3_j, l^1_j, l^2_j$ and the edges $(c^1_j, c^2_j), (c^1_j, c^3_j), (c^1_j, l^1_j), (c^3_j, l^2_j)$;
- for each clause C_j of F do:
  1. add an edge $(c^2_j, t^p_{x_i})$ if x_i is the first or second literal of C_j and it is the p-th occurrency of x_i $(1 \le p \le 2)$;
  2. add an edge $(c^2_j, f^p_{x_i})$ if $\neg x_i$ is the first or second literal of C_j and it is the p-th occurrency of $\neg x_i$ $(1 \le p \le 2)$;
  3. add an edge $(c^3_j, t^p_{x_i})$ if x_i is the third literal of C_j and it is the p-th occurrency of x_i $(1 \le p \le 2)$;
  4. add an edge $(c^3_j, f^p_{x_i})$ if $\neg x_i$ is the third literal of C_j and it is the p-th occurrency of $\neg x_i$ $(1 \le p \le 2)$.

At this point, we show that given an instance F of SAT-AM3, where n is the number of variables, m_1 the number of clauses with at most two literals, and m_2 the number of clauses with three literals, by the construction above we obtain a graph G such that: F is satisfiable if and only if G has a P_3-geodetic set S of size k, where $k = 4n + m_1 + 3m_2$.

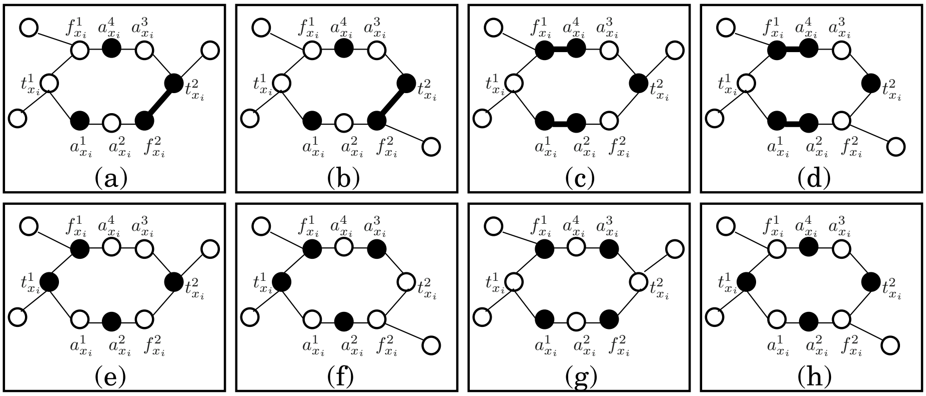

Fig. 4. $(a)-(d)$ Choices of vertices in S_A that imply in at least 5 vertices to be added to S_A; thicker edges mean that one of its endpoints must be added to S_A; $(e)-(h)$ Choices of vertices in S_A that imply in exactly 4 vertices to be added to S_A

Let F be a satisfiable formula and A be a truth assignment of F. We obtain a P_3-geodetic set S_A of G from A as follows: (i) every vertex with degree one is added to S_A; (ii) if $x_i = true$ in A then $t^1_{x_i}, t^2_{x_i}, a^2_{x_i}, a^4_{x_i}$ are added to S_A; (iii) if $x_i = false$ in A then $f^1_{x_i}, f^2_{x_i}, a^1_{x_i}, a^3_{x_i}$ are added to S_A; (iv) for each clause C_i with three literals, if c^3_i has two neighbors in S_A then c^2_i is added to S_A, otherwise c^1_i is added to S_A. As A is a truth assignment of F, each gadget g_{c_i} of G has at least one neighbor in $S_A \cap \{\bigcup_1^n V(g_{x_i})\}$; consequently, S_A is a P_3-geodetic set of G of size $k = 4n + m_1 + 3m_2$.

Conversely, Let S_A be a P_3-geodetic set of G of size $k = 4n + m_1 + 3m_2$. We construct a truth assignment A for the variables $x_1, x_2, \ldots, x_n$ that satisfies all the clauses in F as follows. Any P_3-geodetic set of G contains: (i) at least one vertex of each gadget g_{c_i} if C_i has at most two literals; (ii) at least three vertices of each gadget g_{c_i} if C_i has three literals; (iii) at least four vertices of each gadget g_{x_i}. As S_A has size k, each gadget g_{x_i} has exactly four vertices in S_A, and at most two of these vertices has degree three in G: either $t^1_{x_i}$ and $t^2_{x_i}\}$, or $f^1_{x_i}$ and $f^2_{x_i}\}$. See Figure 4. At this point, we can construct a truth assignment A of F by assigning $x_i = true$ if and only if $t^1_{x_i} \in S_A$ or $t^2_{x_i} \in S_A$ and $t^2_{x_i}$ has degree three in G. By (i) and (ii), each gadget g_{c_i} must have at least one neighbor in S_A, otherwise either S_A would not be a P_3-geodetic set or we would have $|S_A| > k$. Consequently, by the construction of G and A, if S_A is a P_3-geodetic set of G of size k then A is a truth assignment of F.

Figure 5 illustrates a boolean formula F and the graph G obtained from F by the construction above. A possible P_3-geodetic set S_A is colored red.

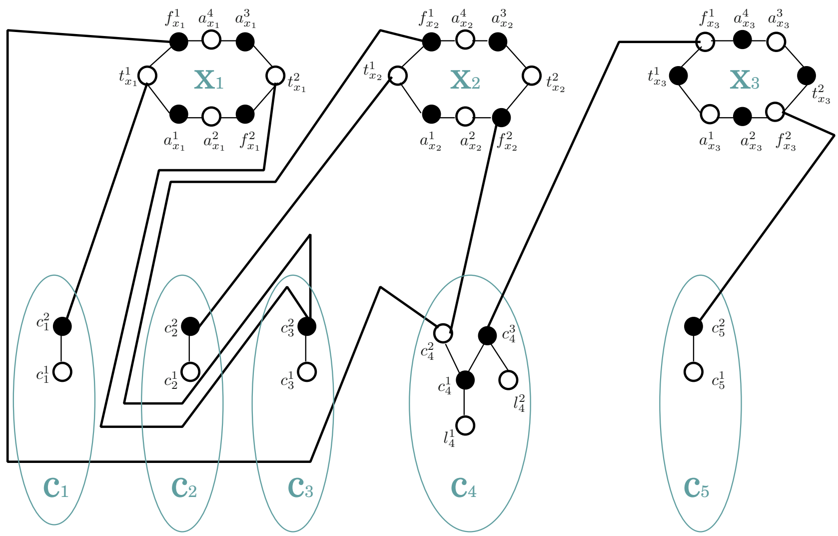

Fig. 5. (a) Satisfiable boolean formula $F = (x_1)(x_2)(x_1 + \neg x_2)(\neg x_1 + \neg x_2 + \neg x_3)(\neg x_3)$; ($b$) Graph G constructed from F

It is easy to see that G has maximum degree three. To show that G is planar, we can split G in two subgraphs $G_x = \{\bigcup_1^n g_{x_i}\}$ and $G_c = \{\bigcup_1^m g_{c_j}\}$. Note that G_x and G_c are both planar graphs. By contracting each graph g_{x_i} and each gadget g_{c_j} of G into a single vertex, we obtain the bipartite graph H_F which by assumption is planar. Hence, G is also a planar graph. □

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