Restricted Bipartite Graphs: Comparison and Hardness Results*

Tian Liu

Key Laboratory of High Confidence Software Technologies, Ministry of Education, Institute of Software, School of Electronic Engineering and Computer Science, Peking University, Beijing 100871, China
lt@pku.edu.cn

Abstract. Convex bipartite graphs are a subclass of circular convex bipartite graphs and chordal bipartite graphs. Chordal bipartite graphs are a subclass of perfect elimination bipartite graphs and tree convex bipartite graphs. No other inclusion among them is known. In this paper, we make a thorough comparison on them by showing the nonemptyness of each region in their Venn diagram. Thus no further inclusion among them is possible, and the known complexity results on them are incomparable. We also show the $\mathcal{NP}$-completeness of treewidth and feedback vertex set for perfect elimination bipartite graphs.

Keywords: Perfect elimination bipartite graphs, tree convex bipartite graphs, circular convex bipartite graphs, chordal bipartite graphs, convex bipartite graphs, $\mathcal{NP}$-completeness, treewidth, feedback vertex set.

1 Introduction

Some $\mathcal{NP}$-complete graph problems, such as treewidth and feedback vertex set, are still $\mathcal{NP}$-complete for bipartite graphs, but tractable for restricted bipartite graphs, such as convex bipartite graphs, chordal bipartite graphs, circular convex bipartite graphs, and so on. Exploring the properties of these restricted bipartite graphs and the boundary between $\mathcal{NP}$-completeness and tractability are well established research directions, see e.g. [2]. In this paper, we show some separation results for restricted bipartite graphs, including perfect elimination bipartite graphs, chordal bipartite graphs, convex bipartite graphs, tree convex bipartite graphs, and circular convex bipartite graphs. We also show the $\mathcal{NP}$-completeness of treewidth and feedback vertex set for perfect elimination bipartite graphs.

Perfect elimination bipartite graphs, chordal bipartite graphs, and convex bipartite graphs are well studied bipartite graph classes [2]. In a *convex bipartite* graph $G = (V_1, V_2, E)$, there is a linear ordering L defined on V_1, such that for each vertex in V_2, its neighborhood induces an interval under L [4]. Given a cycle,

* Partially supported by National 973 Program of China (Grant No. 2010CB328103) and Natural Science Foundation of China (Grant Nos. 61370052 and 61370156).

Q. Gu, P. Hell, and B. Yang (Eds.): AAIM 2014, LNCS 8546, pp. 241–252, 2014.

an edge with two endpoints nonconsecutive in the cycle is called a chord. In a *chordal bipartite* graph, each cycle of length at least six must have a chord [3]. An edge in a bipartite graph is *bisimplicial*, if its endpoint neighborhoods induce a complete bipartite subgraph. A perfect elimination ordering of a bipartite graph is a linear ordering on a subset of nonadjacent edges, such that each edge in this subset is bisimplicial in the remaining bipartite subgraphs when all endpoints of preceding edges are removed, and finally no edge is left in the graph. In a *perfect elimination bipartite* graph, there is a perfect elimination ordering [3].

Circular convex bipartite graphs and tree convex bipartite graphs are two natural generalizations to convex bipartite graphs [13,5,7,19,22,17,15,18,21,14]. In a *circular convex bipartite* graph $G = (V_1, V_2, E)$, there is a circular ordering R defined on V_1, such that for each vertex in V_2, its neighborhood induces a circular arc under R [13]. In a *tree convex bipartite* graph $G = (V_1, V_2, E)$, there is a tree T defined on V_1, such that for each vertex in V_2, its neighborhood induces a subtree on T [5]. When T is a path, G is just a convex bipartite graph. When T is a star, G is called a *star convex bipartite* graph [5]. When T is a triad, which is three paths with a common endpoint, G is called a *triad convex bipartite* graph [7].

It has been known that chordal bipartite graphs is sandwiched between convex bipartite graphs and perfect elimination bipartite graphs [3], and also between convex bipartite graphs and tree convex bipartite graphs [6]. Convex bipartite graphs are a subclass of circular convex bipartite graphs [13]. No other inclusion of them is known. So our first question is

- *Is there any other inclusion among perfect elimination bipartite graphs, tree convex bipartite graphs, circular convex bipartite graphs, chordal bipartite graphs, and convex bipartite graphs?*

In this paper, we give a negative answer by showing the nonemptyness of each region in their Venn diagram. Thus no further inclusion among them is possible, and the known complexity results on them in literatures are incomparable.

Treewidth and feedback vertex set are two well studied $\mathcal{NP}$-complete problems. They are also $\mathcal{NP}$-complete for bipartite graphs [9,23] and for tree convex bipartite graphs [21,5,6], but tractable for chordal bipartite graphs [10,11]. Feedback vertex set is also tractable for circular convex bipartite graphs [18] and for triad convex bipartite graphs [7,6]. Our second question is

- *Where is the boundary between $\mathcal{NP}$-completeness and tractability to treewidth and feedback vertex set for these restricted bipartite graphs?*

In this paper, we give a partial answer by showing the $\mathcal{NP}$-completeness of treewidth and feedback vertex set for perfect elimination bipartite graphs. Therefore, the known tractability of them for chordal bipartite graphs [10,11] can not be extended to perfect elimination bipartite graphs, unless $\mathcal{NP} = \mathcal{P}$.

This paper is structured as follows. After introducing necessary definitions and facts in Section 2, separation results for restricted bipartite graph classes are shown in Section 3, $\mathcal{NP}$-completeness results for perfect elimination bipartite graphs are shown in Section 4, and finally are concluding remarks in Section 5.

2 Preliminaries

For a graph $G = (V, E)$, we denote the *neighborhood* of a vertex u by $N_G(u) = \{v|(u, v) \in E\}$. When G is clear from the context, we just write $N(u)$. A *complete bipartite* graph $G = (V_1, V_2, E)$ has $E = \{(u, v)|u \in V_1, v \in V_2\}$. For a bipartite graph $G = (V_1, V_2, E)$, a subset of pairwise nonadjacent edges $\{(u_1, v_1), (u_2, v_2), \cdots, (u_k, v_k)\}$ is a *perfect elimination ordering*, if each (u_i, v_i) is bisimplicial in G after removing $\{u_1, v_1, u_2, v_2, \cdots, u_{i-1}, v_{i-1}\}$, and there is no edge in G after removing $\{u_1, v_1, u_2, v_2, \cdots, u_k, v_k\}$. A *perfect elimination bipartite* graph has a perfect elimination ordering [3,2]. A hypergraph $H = (V, \mathcal{E})$ has the *Helly property*, if for every subset $\mathcal{E}' \subseteq \mathcal{E}$, if each pair of e_1, e_2 in $\mathcal{E}'$ has a nonempty intersection, then all the e's in $\mathcal{E}'$ have a nonempty intersection.

For a graph $G = (V, E)$, its *tree decomposition* is a tree $T = (B, F)$, with each vertex in B labeled by a subset of V, called *bag*, such that (1) each edge in E is contained in at least one bag; (2) for each vertex u in V, all bags containing u induce a subtree of T. The maximum size of bags minus one is the *width* of the tree decomposition. The minimum width over all tree decompositions of a graph is the *treewidth* of the graph [9,12]. The following Lemma is easy to prove by definition of treewidth [9].

Lemma 1. *Adding a new pendent vertex to a graph will not change its treewidth.*

A *feedback vertex set* is a subset of vertices whose removal renders the graph cycle-free. The minimum feedback vertex set problem is to decide whether a given graph has a feedback vertex set of size no more than a given integer [8]. The minimum size of feedback vertex sets is also called a *decycling number.*

3 Comparison Results

In this section, we make a thorough comparison on perfect elimination bipartite graphs, tree convex bipartite graphs, circular convex bipartite graphs, chordal bipartite graphs, and convex bipartite graphs, by showing the nonemptyness of each region in their Venn diagram, see Figure 1.

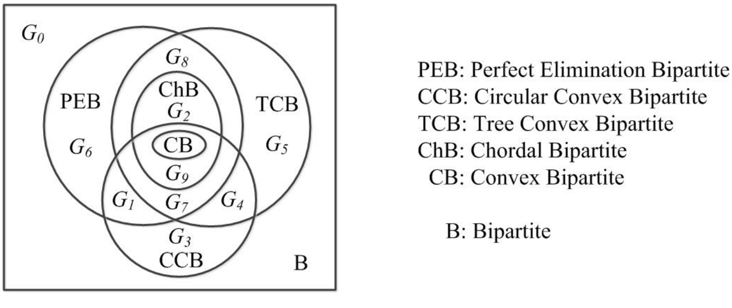

Fig. 1. Venn diagram of five restricted bipartite graph classes and graphs in each region

We use the following trick to deal with perfect elimination bipartite graphs. For a non-perfect elimination bipartite graph, we usually can add one pendent vertex or many pendent vertices to make it a perfect elimination bipartite graph, while to keep its other properties invariant.

Theorem 1. *There is a perfect elimination and circular convex bipartite graph G_1 which is not a tree convex bipartite graph.*

Proof. The graph $G_1 = (V_1, V_2, E)$, where $V_1 = \{x, y, z, u_a, u_b, u_c\}$, $V_2 = \{a, b, c, d_x, d_y, d_z\}$ and $E = \{(x, a), (a, y), (y, b), (b, z), (z, c), (c, x), (x, d_x), (y, d_y), (z, d_z), (a, u_a), (b, u_b), (c, u_c)\}$, is shown in Figure 2 (left).

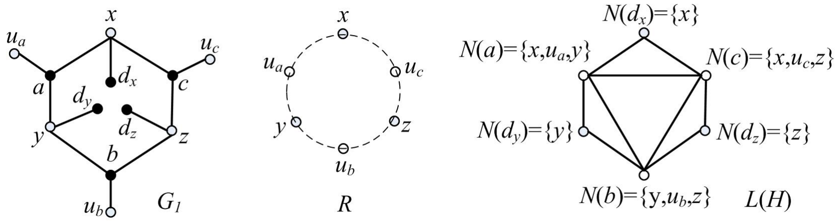

Fig. 2. A perfect elimination and circular convex bipartite graph G_1 which is not a tree convex bipartite graph

G_1 is a perfect elimination bipartite graph, since a perfect elimination ordering of G_1 is given by $\{(x, d_x), (y, d_y), (z, d_z), (a, u_a), (b, u_b), (c, u_c)\}$.

G_1 is a circular convex bipartite graph, since a circular ordering R on V_1 is given by $x \prec u_a \prec y \prec u_b \prec z \prec u_c \prec x$, as shown in Figure 2 (middle), such that the neighborhood of each vertex in V_2 induces a circular arc under R.

If G_1 is a tree convex bipartite graph with a tree associated on V_1, the hypergraph $H = (V_1, \mathcal{E})$ is a hypertree, where $\mathcal{E} = \{N(d) | d \in V_2\}$. Then $H = (V_1, \mathcal{E})$ has the Helly property and the line graph $L(H) = (\mathcal{E}, \mathcal{F})$ is chordal, where $\mathcal{F} = \{(N(d_1), N(d_2)) | N(d_1) \cap N(d_2) \neq \emptyset\}$ (Theorem 1.3.1, page 9, [2]). However, $H = (V_1, \mathcal{E})$ is not Helly, since $N(a), N(b), N(c)$ are pairwise intersect, but $N(a) \cap N(b) \cap N(c) = \emptyset$. This can be seen with the help of the line graph $L(H)$ shown in Figure 2 (right). The same holds for V_2, due to the symmetry of G_1. Therefore, G_1 is not a tree convex bipartite graph. □

Theorem 2. *There is a chordal bipartite graph G_2 which is not a circular convex bipartite graph.*

Proof. The graph $G_2 = (V_1, V_2, E)$, where $V_1 = \{x, y, z, u_1, u_2, u_3\}$, $V_2 = \{a_0, a_1, a_2, a_3\}$ and $E = \{(x, a_1), (a_1, u_1), (u_1, a_0), (y, a_2), (a_2, u_2), (u_2, a_0), (z, a_3), (a_3, u_3), (u_3, a_0)\}$, is shown in Figure 3 (left).

There is no cycle in G_2 at all, so G_2 is a chordal bipartite graph.

Since $N(a_0) = \{u_1, u_2, u_3\}$, u_1, u_2 and u_3 must be consecutive in any circular ordering on V_1 for G_2 to be circular convex bipartite. The same reasoning applies

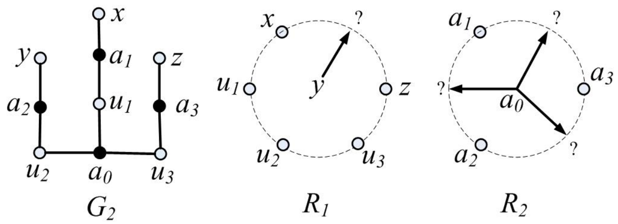

Fig. 3. A chordal bipartite graph G_2 which is not a circular convex bipartite graph

to $N(a_i)$ for $i = 1, 2, 3$. So x and u_1, y and u_2, z and u_3 respectively must be consecutive in any circular ordering, say R_1, on V_1, as shown in Figure 3 (middle). Due to the symmetry in G_2, without loss of generality, we can assume that $x \prec u_1 \prec u_2 \prec u_3 \prec z$ in R_1. Then the only possible place for y is at between y and z, but in this case, $N(a_2) = \{y, u_2\}$ is not a circular arc, since none of x, u_1, u_2, z is in $N(a_2)$. Thus, y can not be inserted into R_1 and G_2 is not circular convex bipartite with a circular ordering on V_1. A similar reasoning also applies to V_2, as shown in Figure 3 (right). Thus, G_2 is not a circular convex bipartite graph. □

Theorem 3. *There is a circular convex bipartite graph G_3 which is neither a perfect elimination bipartite graph nor a tree convex bipartite graph.*

Proof. The graph $G_3 = (V_1, V_2, E)$, where $V_1 = \{x, y, z\}$, $V_2 = \{a, b, c\}$, and $E = \{(x, a), (a, y), (y, b), (b, z), (z, c), (c, x)\}$, is shown in Figure 4 (left).

G_3 is a circular convex bipartite graph, since a circular ordering R on V_1 can be defined by $x \prec y \prec z \prec x$, as shown in Figure 4 (right).

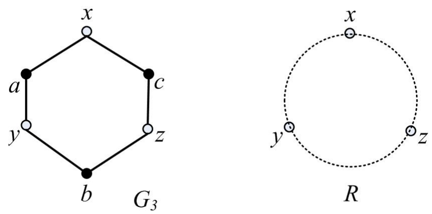

Fig. 4. A circular convex bipartite graph G_3 which is neither a perfect elimination bipartite graph nor a tree convex bipartite graph

G_3 is not a perfect elimination bipartite graph, since in any perfect elimination ordering of G_3, the first edge must be bisimplicial in G_3, but each edge of G_3 is not bisimplicial in G_3. For example, consider an edge (x, a). We have $N(x) = \{a, c\}$ and $N(a) = \{x, y\}$. Since there is no edge (c, y) in E, $N(x) \cup N(a)$ does not induce a biclique in G_3. Thus the edge (x, a) is not bisimplicial. The same holds for other five edges in E due to the symmetry of G_3.

G_3 is not a tree convex bipartite graph, since V_1 has only three vertices, any tree on V_1 is a path, say $x - y - z$. But then, the neighborhood of c, which is $N_{G_3}(c) = \{x, z\}$, does not induce a subtree. Same for V_2 by symmetry. □

Theorem 4. *There is a circular convex and tree convex bipartite graph G_4 which is not a perfect elimination bipartite graph.*

Proof. The graph $G_4 = (V_1, V_2, E)$, where $V_1 = \{x, y, z, u\}$, $V_2 = \{a, b, c\}$, and $E = \{(x, a), (a, y), (y, b), (b, z), (z, c), (c, x), (u, a), (u, b), (u, c)\}$, is shown in Figure 5 (left).

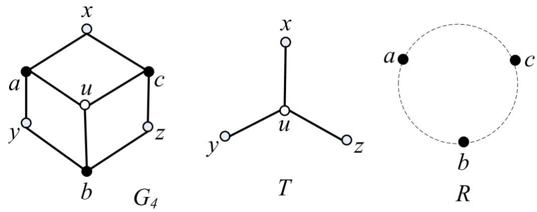

Fig. 5. A circular convex and tree convex bipartite graph G_4 which is not a perfect elimination bipartite graph

G_4 is a tree convex bipartite graph, since a tree $T = (V_1, F)$ on V_1 can be defined by $F = \{(x, u), (y, u), (z, u)\}$, as shown in Figure 5 (right), such that for each vertex in V_2, its neighborhood induces a subtree in T.

G_4 is a circular convex bipartite graph, since a circular ordering R on V_2 can be defined by $a \prec b \prec c \prec a$, as shown in Figure 5 (right), such that for each vertex in V_2, its neighborhood induces a circular arc in R.

G_4 is not a perfect elimination bipartite graph, similarly as G_3. □

Theorem 5. *There is a tree convex bipartite graph G_5 which is neither a perfect elimination bipartite graph nor a circular convex bipartite graph.*

Proof. The graph $G_5 = (V_1, V_2, E)$, where $V_1 = \{x_1, y_1, z_1, u_1, x_2, y_2, z_2, u_2\}$, $V_2 = \{a_1, b_1, c_1, d, a_2, b_2, c_2\}$, and $E = \{(x_1, a_1), (a_1, y_1), (y_1, b_1), (b_1, z_1), (z_1, c_1), (c_1, x_1), (u_1, a_1), (u_1, b_1), (u_1, c_1), (u_1, d), (d, u_2), (x_1, a_1), (a_1, y_1), (y_1, b_1), (b_1, z_1), (z_1, c_1), (c_1, x_1), (u_1, a_1), (u_1, b_1), (u_1, c_1)\}$, is shown in Figure 6 (left).

G_5 is not a circular convex bipartite graph, since G_5 is essentially two copies of G_4 linked by a vertex d. Though each copies of G_4 has a circular ordering, they can not be combined into a larger one for G_5, as readers can check it.

G_5 is not a perfect elimination bipartite graph, by the same reasoning as G_4, as well as the fact that the edges (u_1, d) and (d, u_1) are not bisimplicial.

G_5 is a tree convex bipartite graph, since a tree T on V_1 can be defined as shown in Figure 6 (right), such that for each vertex in V_2, its neighborhood induces a subtree in T. □

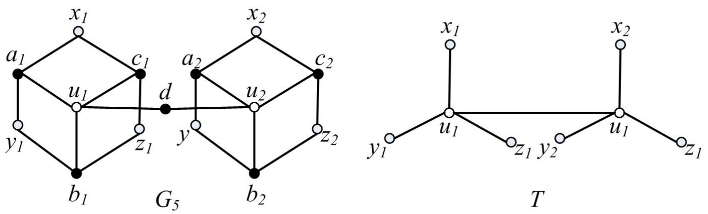

Fig. 6. A tree convex bipartite graph G_5 which is neither a perfect elimination bipartite graph nor a circular convex bipartite graph

Theorem 6. *(1) There is a bipartite graph G_0 which is neither a tree convex bipartite graph, a circular convex bipartite graph, nor a perfect elimination bipartite graph. (2) There is a perfect elimination bipartite graph G_6 which is neither a tree convex bipartite graph nor a circular convex bipartite graph.*

Proof. (1) The graph $G_0 = (V_1, V_2, E)$ is shown in Figure 7 (left).

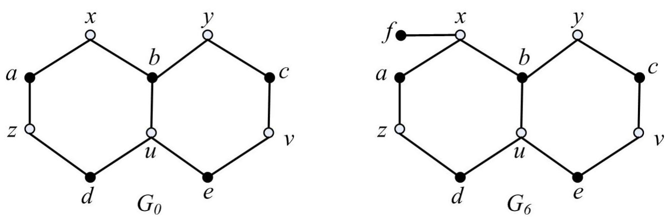

Fig. 7. A bipartite graph G_0 which is neither a tree convex bipartite graph, a circular convex bipartite graph, nor a perfect elimination bipartite graph, and a perfect elimination bipartite graph G_6 which is neither a tree convex bipartite graph nor a circular convex bipartite graph

G_0 is not a circular convex bipartite graph, since G_0 is essentially two copies of G_1 with a common edge (b, u). Though each copies of G_1 has a circular ordering, they can not be combined into a larger one for G_0.

G_0 is not a perfect elimination bipartite graph, by the same reasoning as G_1, as well as the fact that the edges (x, b) and (b, u) are not bisimplicial.

G_0 is not a tree convex bipartite graph, since any tree on V_1 must be a path $x-z-u-v-y$, due to the degree two vertices a, d, e, c. But then $N(b) = \{x, y, u\}$ does not induce a subtree. The same holds for V_2 due to symmetry.

(2) The graph $G_6 = (V_3, V_4, F)$ is shown in Figure 7 (right).

G_6 is neither a circular convex bipartite graph, nor a tree convex bipartite graph, by exactly the same reasoning as for G_0.

G_6 is a perfect elimination bipartite graph, since a perfect elimination ordering is given by $\{(f, x), (a, z), (d, u), (e, v), (c, y)\}$, as readers can check it. □

Theorem 7. *There is a circular convex and tree convex and perfect elimination bipartite graph G_7 which is not a chordal bipartite graph.*

Proof. The graph $G_7 = (V_1, V_2, E)$, where $V_1 = \{x, y, z, u, w\}$, $V_2 = \{a, b, c\}$, and $E = \{(x, a), (a, y), (y, b), (b, z), (z, c), (c, x), (u, a), (u, b), (u, c), (w, a)\}$, is shown in Figure 8 (left).

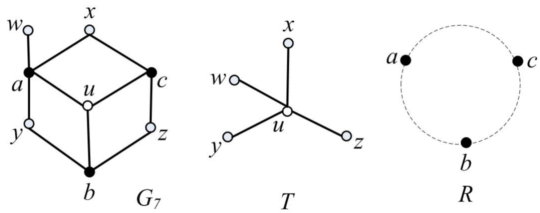

Fig. 8. A circular convex and tree convex and perfect elimination bipartite graph G_7 which is not a chordal bipartite graph

G_7 is a perfect elimination bipartite graph, since a perfect elimination ordering is given by $\{(w, a), (y, b), (z, c)\}$, as readers can check it.

G_7 is a tree convex bipartite graph, since a tree $T = (V_1, F)$ on V_1 can be defined by $F = \{(x, u), (y, u), (z, u), (w, u)\}$, as shown in Figure 8 (middle), such that for each vertex in V_2, its neighborhood induces a subtree in T.

G_7 is a circular convex bipartite graph, since a circular ordering R on V_2 can be defined by $a \prec b \prec c \prec a$, as shown in Figure 8 (right), such that for each vertex in V_2, its neighborhood induces a circular arc in R.

G_7 is not a chordal bipartite graph, since the cycle $x - a - y - b - z - c - x$ of length six has no chord. □

Theorem 8. *There is a tree convex and perfect elimination bipartite graph G_8 which is neither a chordal bipartite graph nor a circular convex bipartite graph.*

Proof. The graph $G_8 = (V_1, V_2, E)$ is shown in Figure 9 (left).

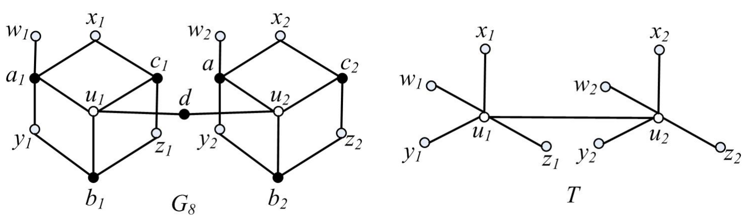

Fig. 9. A tree convex and perfect elimination bipartite graph G_8 which is neither a chordal bipartite graph nor a circular convex bipartite graph

G_8 is a perfect elimination bipartite graph, since a perfect elimination ordering is given by $\{(w_1, a_1), (y_1, b_1), (z_1, c_1), (w_2, a_2), (y_2, b_2), (z_2, c_2), (u_1, d_1)\}$, as readers can check it.

G_8 is a tree convex bipartite graph, similarly as G_5, see Figure 9 (right).

G_8 is not a circular convex bipartite graph, similarly as G_5.

G_8 is not a chordal bipartite graph, since the cycle $x_1 - a_1 - y_1 - b_1 - z_1 - c_1 - x_1$ of length six has no chord. □

Theorem 9. *There is a circular convex and chordal bipartite graph G_9 which is not a convex bipartite graph.*

Proof. The graph $G_9 = (V_1, V_2, E)$, where $V_1 = \{x, y, z, u, w\}$, $V_2 = \{a, b, c\}$, and $E = \{(x, a), (a, y), (y, b), (b, z), (z, c), (c, x), (u, a), (u, b), (u, c), (w, a)\}$, is shown in Figure 10 (left).

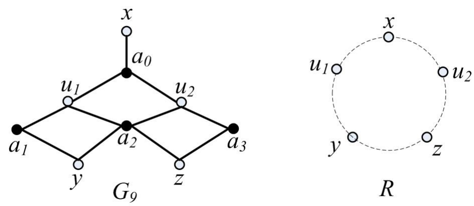

Fig. 10. A circular convex and chordal bipartite graph G_9 which is not a convex bipartite graph

G_9 is a chordal bipartite graph, since each cycle of length at least six has a chord, as readers can check it.

G_9 is a circular convex bipartite graph, since a circular ordering R on V_2 can be defined by $x \prec u_1 \prec y \prec z \prec u_2 \prec x$, as shown in Figure 10 (right), such that for each vertex in V_2, its neighborhood induces a circular arc in R.

G_9 is not a convex bipartite graph, since G_9 is a forbidden subgraph in Tucker's characterization of convex bipartite graphs [20]. □

4 Hardness Results

In this section, we show the $\mathcal{NP}$-completeness of treewidth and feedback vertex set for perfect elimination bipartite graphs. These two problems are known to be $\mathcal{NP}$-complete for bipartite graphs. We use a simple reduction from bipartite graphs to perfect elimination bipartite graphs, which keeps treewidth and decycling number invariant. The reduction just adds a different pendent vertex for each vertex in one side of the bipartite graph.

Theorem 10. *Treewidth is $\mathcal{NP}$-complete for perfect elimination bipartite graphs.*

Proof. Treewidth is well known in $\mathcal{NP}$ [1,9]. We reduce from Treewidth which is $\mathcal{NP}$-complete for bipartite graphs [9].

Reduction 1.

Input: A bipartite graph $G = (V_1, V_2, E)$ and a positive integer k, where $V_1 = \{x_1, x_2, \ldots, x_n\}$.

Output: A bipartite graph $G' = (V_1, V_2', E')$ and a positive integer k, where $V_2' = V_2 \cup \{a_1, a_2, \cdots, a_n\}$ and $E' = E \cup \{(x_k, a_k) | k = 1, 2, \cdots, n\}$.

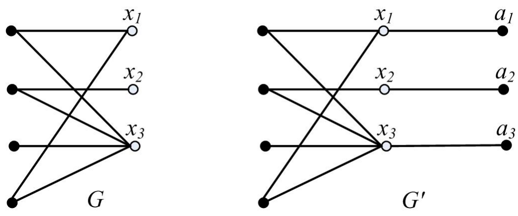

Fig. 11. An example of Reduction 1

Clearly, G' is bipartite and is computable from G in polynomial time. An example of G and G' is shown in Figure 11.

The graph G' is a perfect elimination bipartite graph, since a perfect elimination ordering of G' is given by $\{(x_1, a_1), (x_2, a_2), \cdots, (x_n, a_n)\}$. Indeed, these edges are pairwise nonadjacent. Each edge in them has a degree one endpoint b_i, thus are bisimplicial. These edges contain all the vertices in V_1, no edge in G' will be left after removing these edges and their endpoints.

By repeatedly applying Lemma 1 in the construction of G' from G, G has treewidth k if and only if G' has treewidth k. □

Theorem 11. *Feedback vertex set is $\mathcal{NP}$-complete for perfect elimination bipartite graphs.*

Proof. Feedback vertex set problem is well known in $\mathcal{NP}$ [8]. We reduce from feedback vertex set which is $\mathcal{NP}$-complete for bipartite graphs [23]. The reduction is exact the same as Reduction 1 in proof of Theorem 10. The correctness of this reduction is shown as follows.

First, for any feedback vertex set D' of G', there is a feedback vertex set D'' of G', such that D'' only contains vertices in $V_1 \cup V_2$ and D'' is not larger than D'. Indeed, if there is a vertex a_i in D', then we can replace a_i by x_i, since a_i is a pendent vertex not on any cycle.

Second, for any $D \subseteq V_1 \cup V_2$, D is a feedback vertex set in G if and only if it is a feedback vertex set in G'. Therefore, G has a feedback vertex set of size at most k if and only if G' has a feedback vertex set of size at most k. □

5 Conclusions

We have made a thorough comparison for perfect elimination bipartite graphs, chordal bipartite graphs, convex bipartite graphs, tree convex bipartite graphs, and circular convex bipartite graphs, showing the nonemptyness of each region in their Venn diagram (Figure 1), thus ruling out any further inclusion among them. We also show the $\mathcal{NP}$-completeness of treewidth and feedback vertex set for perfect elimination bipartite graphs.

A trick we used to obtain these results is that, for a bipartite graph, we usually can add one pendent vertex or many pendent vertices to make it a perfect elimination bipartite graph, while to keep its other properties invariant. This trick may be useful to obtain further results for perfect elimination bipartite graphs.

The complexity of feedback vertex set for restricted bipartite graphs is shown in Figure 12. The complexity of treewidth for triad convex bipartite graphs or circular convex bipartite graphs is unknown. We conjecture that treewidth is also tractable for these two classes of bipartite graphs, and thus the same picture as Figure 12 also holds for treewidth.

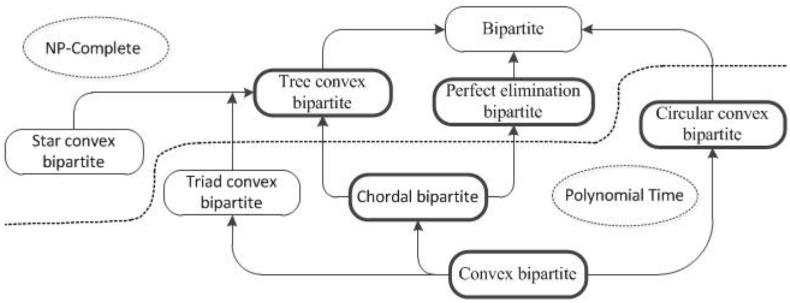

Fig. 12. The known inclusion among some restricted bipartite graphs and complexity classification of feedback vertex set for these bipartite graphs

A set system $(U, \mathcal{S})$ contains a universe set U and a family $\mathcal{S}$ of subsets of U. A set system $(U, \mathcal{S})$ can be represented by a bipartite graph $(U, \mathcal{S}, E)$, where $E = \{(x, Y) | x \in U, Y \in \mathcal{S}\}$. When the bipartite graphs are restricted, we also get the corresponding restricted set systems. Our separation results for the restricted bipartite graphs are also applicable to the restricted set systems. Recently, some complexity results on set cover, set packing and hitting set for tree convex, circular convex, tree-like and circular-like set systems are obtained in [16]. We can also define perfect elimination set systems and chordal set systems, and the complexity results for them is largely unknown.

Acknowledgments. The help of anonymous reviewers has improved our presentation greatly.

References

1. Arnborg, S., Corneil, D.G., Proskurowski, A.: Complexity of finding embeddings in a k-tree. SIAM J. Algebraic Discrete Methods 8, 277–284 (1987)
2. Brandstad, A., Le, V.B., Spinrad, J.P.: Graph Classes - A Survey. Society for Industrial and Applied Mathematics, Philadelphia (1999)

3. Golumbic, M.C., Goss, C.F.: Perfect elimination and chordal bipartite graphs. J. Graph Theory 2, 155–163 (1978)
4. Grover, F.: Maximum matching in a convex bipartite graph. Nav. Res. Logist. Q. 14, 313–316 (1967)
5. Jiang, W., Liu, T., Ren, T., Xu, K.: Two hardness results on feedback vertex sets. In: Atallah, M., Li, X.-Y., Zhu, B. (eds.) FAW-AAIM 2011. LNCS, vol. 6681, pp. 233–243. Springer, Heidelberg (2011)
6. Jiang, W., Liu, T., Wang, C., Xu, K.: Feedback vertex sets on restricted bipartite graphs. Theor. Comput. Sci. 507, 41–51 (2013)
7. Jiang, W., Liu, T., Xu, K.: Tractable feedback vertex sets in restricted bipartite graphs. In: Wang, W., Zhu, X., Du, D.-Z. (eds.) COCOA 2011. LNCS, vol. 6831, pp. 424–434. Springer, Heidelberg (2011)
8. Karp, R.: Reducibility among combinatorial problems. In: Complexity of Computer Computations, pp. 85–103. Plenum Press, New York (1972)
9. Kloks, T.: Treewidth: Computations and Approximations. Springer (1994)
10. Kloks, T., Kratsch, D.: Treewidth of chordal bipartite graphs. J. Algorithms 19, 266–281 (1995)
11. Kloks, T., Liu, C.H., Pon, S.H.: Feedback vertex set on chordal bipartite graphs. arXiv:1104.3915 (2011)
12. Kloks, T., Wang, Y.L.: Advances in Graph Algorithms (2013) (manuscript)
13. Liang, Y.D., Blum, N.: Circular convex bipartite graphs: Maximum matching and Hamiltonian circuits. Inf. Process. Lett. 56, 215–219 (1995)
14. Liu, T., Lu, Z., Xu, K.: Tractable connected domination for restricted bipartite graphs. J. Comb. Optim. (2014), , doi 10.1007/s10878-014-9729-x
15. Lu, M., Liu, T., Xu, K.: Independent domination: Reductions from circular- and triad-convex bipartite graphs to convex bipartite graphs. In: Fellows, M., Tan, X., Zhu, B. (eds.) FAW-AAIM 2013. LNCS, vol. 7924, pp. 142–152. Springer, Heidelberg (2013)
16. Lu, M., Liu, T., Tong, W., Lin, G., Xu, K.: Set cover, set packing and hitting set for tree convex and tree-like set systems. In: Gopal, T.V., Agrawal, M., Li, A., Cooper, S.B. (eds.) TAMC 2014. LNCS, vol. 8402, pp. 248–258. Springer, Heidelberg (2014)
17. Lu, Z., Liu, T., Xu, K.: Tractable connected domination for restricted bipartite graphs (extended abstract). In: Du, D.-Z., Zhang, G. (eds.) COCOON 2013. LNCS, vol. 7936, pp. 721–728. Springer, Heidelberg (2013)
18. Lu, Z., Lu, M., Liu, T., Xu, K.: Circular convex bipartite graphs: Feedback vertex set. In: Widmayer, P., Xu, Y., Zhu, B. (eds.) COCOA 2013. LNCS, vol. 8287, pp. 272–283. Springer, Heidelberg (2013)
19. Song, Y., Liu, T., Xu, K.: Independent domination on tree convex bipartite graphs. In: Snoeyink, J., Lu, P., Su, K., Wang, L. (eds.) AAIM 2012 and FAW 2012. LNCS, vol. 7285, pp. 129–138. Springer, Heidelberg (2012)
20. Tucker, A.: A structure theorem for the consecutive 1's property. Journal of Combinatorial Theory 12(B), 153–162 (1972)
21. Wang, C., Chen, H., Lei, Z., Tang, Z., Liu, T., Xu, K.: Tree convex bipartite graphs: $\mathcal{NP}$-complete domination, hamiltonicity and treewidth. In: Proc. of FAW (2014)
22. Wang, C., Liu, T., Jiang, W., Xu, K.: Feedback vertex sets on tree convex bipartite graphs. In: Lin, G. (ed.) COCOA 2012. LNCS, vol. 7402, pp. 95–102. Springer, Heidelberg (2012)
23. Yannakakis, M.: Node-deletion problem on bipartite graphs. SIAM J. Comput. 10, 310–327 (1981)