Chapter 8
Causality, Nonlocality, and Entanglement in Quantum Mechanics

> *Why is the pilot-wave picture ignored in the text books? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate theoretical choice?*
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> Bell (1987, page 160)

The material presented in previous chapters has dealt with some fundamental elements of QM such as (non)locality, (non)causality, (in)determinism and the (non)existence of trajectories. These elements have been at the core of some of the most longstanding controversies, and also of the most thorough attempts to develop alternative interpretations of the quantum formalism.

Among such reformulations, perhaps the best known one is the de Broglie-Bohm theory. This represented the first serious attempt to recover for QM two notions that are dear to (classical) physics, namely those of causality and of trajectory (or realism, in a by now extended usage)—although at the high price of nonlocality. It seems therefore appropriate to initiate the discussion on these subjects with a brief, critical review of the de Broglie-Bohm theory. This will set the framework for a fresh analysis of quantum (non)locality from the perspective of the SED theory as developed in the preceding chapters.

8.1 Causality at Stake

Quantum indeterminism, as represented by the Heisenberg inequalities or the inherently unpredictable specific outcome of a measurement, is for some a cause of discomfort with the theory. When the status of QM as a fundamental theory of physics is at stake, such trait is indeed hardly acceptable from a realist standpoint. The simplest solution to the problem of quantum indeterminism consists in assuming that there exist some variables that specify, along with the quantum-mechanical state

L. de la Peña et al., *The Emerging Quantum*,
DOI: 10.1007/978-3-319-07893-9_8,
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vector, a *dispersion-free state*, i.e. a state for which the specific outcomes of a measurement are fully determined. These hypothetical variables that restore determinism are the so-called 'hidden variables' or 'hidden parameters'. Of course, one must assume that their existence is compatible with the quantum laws. And here sprouts the problem...

8.1.1 Von Neumann's Theorem

In 1932 the mathematician J. von Neumann published what constituted the first formal textbook on quantum mechanics (English version: von Neumann 1932). The book contains a theorem that was quickly accepted, and which half a century ago still seemed to be in full force. This theorem—which stands even today as the archetype of the 'impossibility proofs', more popularly known as 'no-go' theorems— asserts that one cannot add hidden variables to render QM deterministic.[1] Since QM is supported by a vast amount of empirical evidence, a deterministic description of it happens to be refuted by such evidence.

In 1952 a second round started with the publication by D. Bohm of two now classical papers (Bohm 1952a, b; see also Bohm 1953) providing a specific counterexample to von Neumann's theorem. It was just a causal and deterministic description completely consistent with the quantum-mechanical formalism, built in terms of *nonlocal* hidden variables. One should have expected such a result to immediately draw the attention from physicists (and mathematicians and philosophers of science) eager to find the error in von Neumann's demonstration. However, this was not the case; the large majority simply ignored Bohm's results and took von Neumann's conclusion for granted.

In the proof of his theorem, von Neumann had introduced some postulates that he considered of general validity. Starting with the expression

$$\left\langle \hat{C} \right\rangle = \mathrm{tr}\,(\rho C) \tag{8.1}$$

that defines the mean value of an operator $\hat{C}$ (associated with the observable C) in the quantum state described by the density matrix ρ, it is straightforward to arrive at

$$\left\langle \hat{A} + \hat{B} \right\rangle = \left\langle \hat{A} \right\rangle + \left\langle \hat{B} \right\rangle, \tag{8.2}$$

irrespective of the operators $\hat{A}$ and $\hat{B}$. Von Neumann then assumed that this additive relation holds *also* when the mean values are calculated in dispersion-free states. And since such states must yield one (well-defined) result from among the possible ones predicted by the quantum state, the expectation values of $\hat{C}$ in a dispersion-free state

[1] It is much less known that almost simultaneously and independently, a similar result was published by Solomon (1933).

are just the eigenvalues of $\hat{C}$ (Bell 1987, articles 1 and 4). Yet for noncommuting $\hat{A}$ and $\hat{B}$ the eigenvalue of the sum $\hat{A} + \hat{B}$ is not necessarily equal to the sum of the corresponding eigenvalues,[2] whence the relation (8.2) ceases to be true for dispersion-free states.

Two years after the original publication of von Neumann's theorem, Hermann (1935) observed that this additivity assumption was too restrictive within the quantum domain.[3] Hermann's criticism was largely ignored, as were the few other criticisms raised much later, notably by Feyerabend (1956), and Mugur-Schächter (1964).[4] The widely accepted proof that von Neumann's theorem demanded revision came only with the work of Bell (1966), who again observed that the theorem was mathematically correct but not of general applicability, as the additivity postulate was too restrictive.

The generally accepted conclusion at present is that the theorem is not general enough to eliminate all kinds of hidden-variable theories (see e.g. Bub 2010); in particular, it does not exclude nonlocal hidden variables, as Bohm's work evinced. Alternative versions of von Neumann's theorem have been devised in the meantime that escape from the previous criticisms, by Gleason (1957) for Hilbert spaces of dimensionality greater than 2, Bell (1966), Kochen and Specker (1967), Belinfante (1973), and Peres (1996), among others. In their turn, Gudder (1970) and Santos (1975) have proved that QM does accept contextual hidden variables.

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[2] Take for example the spin projections along three different directions: $\hat{A} = \hat{S}_x$, $\hat{B} = \hat{S}_y$ and $\hat{C} = (\hat{A} + \hat{B})/\sqrt{2} = (\hat{S}_x + \hat{S}_y)/\sqrt{2}$. If the system possesses spin 1/2, the eigenvalues of each of these operators are the same and equal to ± 1; clearly the eigenvalues of $\hat{C}$ are not the linear combination $(\pm 1 \pm 1)/\sqrt{2}$.

[3] Since Hermann's argument is little known, and is just the same discovered by Bell 30 years later, it seems of interest to transcribe it here: "Suppose we have an ensemble of physical systems, with $\mathfrak{R}$ and $\mathfrak{S}$ physical quantities that can be measured on this ensemble; the expectation value of $\mathfrak{R}$ (Expt($\mathfrak{R}$)) is the average value of all measurement outcomes that will be obtained when measuring $\mathfrak{R}$ on all systems of the ensemble, and is also the value that is expected to be obtained when measuring $\mathfrak{R}$ on an arbitrary element of this ensemble. Von Neumann requires that for this expectation value-function Expt($\mathfrak{R}$), defined using an ensemble of physical systems and producing a number for every physical quantity, Expt($\mathfrak{R} + \mathfrak{S}$) = Expt($\mathfrak{R}$) + Expt($\mathfrak{S}$). In words: The expectation value of a sum of physical quantities is equal to the sum of the expectation values of both quantities. With this assumption the proof of von Neumann either succeeds or fails."
"For classical physics this requirement is trivial and also for those quantum mechanical observables that [commute]... Not trivial however is the relation for quantum mechanical quantities for which indeterminacy relations hold. In fact the sum of two such quantities is not even defined: Because a sharp measurement of one of them excludes sharp measurement of the other one and thus because both quantities cannot have sharp values at the same time, the commonly used definition of the sum of two quantities breaks down."

[4] Feyerabend noticed that the postulates used in von Neumann's derivation did not exclude dispersive hidden variables. Now if the hidden variables added to QM had an irreducible dispersion, the quantum variables themselves should continue to be dispersive and things remained essentially the same, except that the theorem needed some reformulation. Mugur-Schächter, on her part, argued that the demonstration was not as general as assumed, since it presupposes that the distribution of the hidden variables (once more, distributed variables) has properties similar to those of the quantum distribution.

8.1.2 Bohm's Counterexample

For historical fairness, the theory proposed by Bohm—*causal quantum mechanics*, as he called it—should be named after de Broglie and Bohm, since L. de Broglie proposed his *pilot-wave theory* already during the construction of quantum mechanics as an alternative to both Heisenberg's and Schrödinger's routes to the new theory (de Broglie 1926a, b, 1927a, b, c; see also de Broglie 1963).[5] The pillars on which de Broglie's and Bohm's theories rest are quite similar, although from the outset the former was more ambitious. The theory proposed by de Broglie was an attempt to arrive at QM through the study of the trajectories followed by electrons or photons 'guided' by the quantum field (hence the name of pilot-wave theory). De Broglie's initial work preceded the theories of Heisenberg and Schrödinger, and can therefore be considered to be the first attempt to find a formulation for QM. Unfortunately the author abandoned it around 1930, due to several unresolved difficulties, and with this the search for quantum trajectories was left aside for a long time. A detailed discussion of de Broglie's theory, covering both the technical and the historical aspects, can be seen in the excellent book by Bacciagaluppi and Valentini (2009); see also Cushing (1992) and Bohm and Hiley (1995). In 1956, de Broglie himself published a general discussion of his theory

As for Bohm's theory, it is derived from QM—just the converse of what de Broglie's theory was designed for—so one can say that it is QM supplemented with a guidance formula—a pilot formula that can be derived from within QM—which is the entry point for the hidden variables. Thus Bohm's theory represents an alternative reading of QM rather than a different theory. Bohm himself insisted on this point: his theory is totally consistent with QM, it is QM seen from a causal and deterministic stance. The theory was born precisely out of the drive to demonstrate that, in defiance of von Neumann's theorem, QM accepts a hidden-variable description that makes it causal and deterministic. Yet even though Bohm's formulation did provide at its time a real counterexample to von Neumann's theorem, it was largely left aside, as mentioned earlier. An eloquent testimony of the reaction generated by Bohm's theory is given by F. Bopp in his summary of the discussions at a 1957 Conference in Bristol (Jammer 1974, p. 256):

> '...we say that Bohm's theory cannot be refuted, adding... that we don't believe in it.'.

Hostility towards Bohm's theory continued and still continues in many circles, although much attenuated. An illustrative example is due to Pauli (1952), who in the book in homage to de Broglie dismissed the de Broglie-Bohm approach as 'artificial metaphysics' because this theory breaks the symmetric treatment of canonically conjugate variables. The theory developed in Chap. 4 shows that this violation is merely

[5] Without pretending to undermine de Broglie's credit for his seminal contribution, in most of this book we shall refer to *Bohm's theory*, for short, as is customary in present-day literature. An alternative form of QM, similar to Bohm's, had been proposed many years earlier by Madelung (1927).

apparent, since the description in configuration space is the result of a free choice rather than of necessity.

Since Bohm's formulation is derived just from QM, its novelty might be questioned; however, this question would be out of place. Bohm's approach has contributed in various important ways to our understanding of quantum theory. Firstly, it opened a door to the idea of the feasibility of hidden-variable theories. This stimulated work on the subject, which led to important results and to a weakening of the extended conviction that the quantum world is elusory. A most important contribution is the insistence on a *causal* interpretation of QM, openly confronting the orthodox interpretation in terms of quantum fluctuations and selections without an underlying cause. A further merit is that it does not resort to the observer, as orthodox QM does. If you want to perform a measurement, you should incorporate the instrument into the description just as you would do in a classical context (an example of this is given below in Sect. 8.2.3). Moreover, the variables used in Bohm's formulation allow for the introduction into QM of the notion of trajectory. In fact, the electron remains all the time a corpuscle following a trajectory and never becoming a wave. This additional touch of realism is another important virtue that is in want in the usual renderings of QM. In discussing Bohm's formulation more at depth in what follows, we will find a new opportunity to cogently establish the meaning of several of the most characteristic features of QM from a realist point of view, and to show the many coincidences and some divergences with the theory discussed in this book.

As Bohm's formulation gained acceptance with time, the emphasis shifted towards a full-fledged alternative to the orthodox interpretation, and the theory expanded its aims. From a counterexample to von Neumann's theorem it evolved into a causal and realistic interpretation of QM that acquired some popularity.[6] It has even been presented as a quantum theory by itself, not based on the Schrödinger equation but on its own principles [see e.g. Dürr and Teufel (2009)] and leading to the Schrödinger equation as one of its major results.[7] A very important extension of the theory is its generalization to the many-body problem. Here also de Broglie was the originator, having presented his many-body theory already at the 1927 Solvay Conference (de Broglie 1928). Also since its 1952 revival by Bohm, the theory was presented as a many-body corpus. There are works studying further possibilities, such as the introduction of spin, the extension to the relativistic domain, (see e.g. Nikolić 2007; Hernández-Zapata and Hernández-Zapata 2010), or the generalization to a statistical

[6] The well-known book by Bell containing the collection of his articles on the foundations of QM (Bell 1987) was very influential in the revival of Bohm's theory. Bell appreciated the objective, deterministic and causal aspects of the pilot-wave theory. It was the search for an answer to the question: Is it that any hidden-variables theory is by necessity nonlocal? what prompted Bell's work leading to his now famous inequalities.

[7] This is achieved by simply inverting the reasoning in the derivations. Two crucial postulates are needed: one is of course the guidance equation; the second demand serves to introduce the quantum potential V_Q into Eq. (8.9) on the basis of an appropriately contrived argument. The simplest procedure is to consider the quantum potential as an empirical—and thus phenomenological—expression, and to proceed from there on. There exist all sorts of interpretations and 'derivations' of the quantum potential, as commented in footnote 4.14.

situation (Bohm and Hiley 1996), usually treated by means of the density matrix, and into other modern topics (Oriols and Mompart 2012). Extensive monographs on the subject are Bohm and Hiley (1995), Holland (1993), Dürr and Teufel (2009), or the introductory course Towler (2009); see also Passon (2005) and Thiounn (1965). For another 'branch' of the theory see Floyd (2000). A related critical work on Bohm's theory is Dürr et al. (1992).

8.2 Essentials of the de Broglie-Bohm Theory

8.2.1 The Guiding Field

A straightforward derivation of Bohm's theory starts with the introduction into the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{8.3}$$

of the wave function $\psi(\boldsymbol{x}, t)$ written in polar form

$$\psi(\boldsymbol{x}, t) = \sqrt{\rho(\boldsymbol{x}, t)}e^{iS(\boldsymbol{x},t)}, \tag{8.4}$$

with $\rho(\boldsymbol{x}, t)$ and $S(\boldsymbol{x}, t)$ real functions. By separating Eq. (8.3) into its real and imaginary parts one obtains the couple of equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\boldsymbol{v} = 0, \tag{8.5a}$$

$$\hbar\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{8.5b}$$

where $\rho\boldsymbol{v} = \boldsymbol{j}$ is the probability current or particle flux, with [see Eqs. (4.57) and (4.127)]

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\hbar}{m}\nabla S(\boldsymbol{x}, t). \tag{8.6}$$

Clearly the content of Eqs. (8.5a) and (8.5b) is the same as that of Schrödinger's equation. However, in the causal interpretation a formal analogy with classical mechanics is established, by taking $\boldsymbol{v}(\boldsymbol{x}, t)$ as the velocity field of a *single* particle located at $\boldsymbol{x}$,

$$\frac{d\boldsymbol{x}}{dt} = \frac{\hbar}{m}\,\nabla S(\boldsymbol{x}, t)|_{\boldsymbol{x}=\boldsymbol{x}(t)}\,, \tag{8.7}$$

and thus interpreting $\hbar S$ as the action function of the quantum problem. This action differs from the classical one (S_c) in that the latter solves a true (classical) Hamilton-Jacobi equation

$$\frac{\partial S_c}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla} S_c)^2 + V = 0, \tag{8.8}$$

whereas the (dimensionless) action S is governed by Eq. (8.5b), which can be rewritten as

$$\hbar\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}(\boldsymbol{\nabla} S)^2 + V_{\text{eff}} = 0, \quad V_{\text{eff}} = V + V_Q, \tag{8.9}$$

with V_Q the *quantum potential* or *Bohm's potential* [already found, for example, in Eqs. (2.80) and (4.86)],

$$V_Q = -\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}}. \tag{8.10}$$

In terms of the 'effective potential' $V_{\text{eff}} = V + V_Q$, Eq. (8.9) takes the *form* of a Hamilton-Jacobi equation for the principal function S (Goldstein et al. 2002). Equation (8.9) is thus interpreted as the quantum Hamilton-Jacobi equation, that is, the quantum version of Eq. (8.8). According to this interpretation, the difference between a classical problem and the corresponding quantum one is due the presence of the function V_Q (cf. the discussion in Sect. 4.4.1).[8]

Notice, however, that in contrast to the external (classical) potential V, V_Q is not a preestablished function of $\boldsymbol{x}$; it depends on the evolution of $\rho = \psi\psi^*$, the dynamics of which is in its turn determined by the quantum potential itself. Hence V_Q applies a kind of feedback on the particle, dependent on the distribution of particles, a feature that endows the theory with highly nonclassical properties. It should come as no surprise that the implications of this potential for the dynamics are far reaching, as we know is just the case. In particular, the dependence of V_Q on ρ endows Eq. (8.9)

[8] The kinetic origin of the quantum potential is discussed in de la Peña et al. (2011). To the varied proposals to derive the quantum potential cited in footnote 4.14, one should add those of Dürr et al. (1992), and Ván and Fülöp (2003), as well as.the thermodynamic approach of Grössing (2008, 2009). A somewhat bolder one is that of Floyd (2002), who proposes a trajectory description based on a peculiar quantum potential containing derivatives of third order. An interesting point of this theory is that it contains extra parameters that allow for a distribution of the velocity $\boldsymbol{v}$, resulting in a more realistic description. Salesi (1996) and Recami and Salesi (1998) propose that the quantum potential can be derived by considering the energy associated with the internal zitterbewegung (considered as the antecedent of the spin). A similar proposal is made by Esposito (1999), who associates the quantum potential with the (internal) kinetic energy due to a generalized spin; see also Yang (2006). For these authors, the notions of spin, zitterbewegung and quantum potential are intimately related. Garbaczewski (1992) offers a nice derivation of the quantum potential as due to the fluctuations of the momentum. In Carroll (2007, 2010), additional arguments are introduced about the origin of the quantum potential, related to Fisher information.

with a statistical content, and is responsible for the essential difference between (8.5b) and a true Hamilton-Jacobi equation, which by definition describes the motion of a congruency of (single) particles acted on by local potentials (a congruency refers to a single-valued trajectory field).

The fact that in Bohm's theory $\boldsymbol{v}$ stands for the velocity of a single particle, allows for the introduction of the notion of trajectory into the description. Such trajectory is described by the velocity field, determined by ψ according to Eq. (8.6) [see also Eq. (4.125)],

$$\boldsymbol{v}(\boldsymbol{x},t) = \frac{i\hbar}{2m}\left(\frac{\boldsymbol{\nabla}\psi^*}{\psi^*} - \frac{\boldsymbol{\nabla}\psi}{\psi}\right). \tag{8.11}$$

The wave function ψ is taken here as a physically real field—just as real as, say, the electromagnetic field—that pervades the entire available space and guides the particle according to (8.11), which is therefore known as the *guidance* (or *pilot*) equation.[9] The two basic physical elements of the theory are thus the wave (guiding or pilot) field, determined by the Schrödinger equation, and the particle, with its motion determined by the solution of Eq. (8.7), or rather by the solution of

$$m\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} = m\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}(V + V_Q), \tag{8.12}$$

obtained by applying the operator $\boldsymbol{\nabla}$ to Eq. (8.9). Notice that the operator

$$\mathcal{D}_c = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}, \tag{8.13}$$

frequently called co-moving derivative (introduced in Sect. 2.3.1), coincides with the the total time derivative when the latter is taken along the path determined by the velocity field $\boldsymbol{v}$ (whence in the Bohmian approach $\mathcal{D}_c\boldsymbol{v}$ is the actual acceleration of the particle).

The solution of Eq. (8.7) depends on the initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(t=0)$. When different values are assigned to $\boldsymbol{x}_0$, an ensemble of motions (for a given ψ) is determined; a statistical meaning is therefore ascribed to the field ψ by *postulating* that the probability for a particle to be in the vicinity $d\boldsymbol{x}$ around $\boldsymbol{x}$ is given by

$$|\psi(\boldsymbol{x},t)|^2\, d\boldsymbol{x}. \tag{8.14}$$

This allows to establish contact with the probability interpretation of the density $\rho(\boldsymbol{x},t)$. Moreover, it can be shown that given the initial distribution $\rho(\boldsymbol{x},0)$, Eq.

[9] The field ψ differs in essence from those known to classical physics. In contrast to the gravitational or the electromagnetic field, for example, it does not have a generating source. Moreover, it affects the particle (by guiding it) but is not affected by it. This lack of reciprocity in the field-particle influence led de Broglie (1956) (and afterwards Bohm himself) to regard the pilot-wave theory as just a step towards a necessarily more developed theory. [See item 15 in Bell (1987)].

(8.11) holds at any future time. The fact that the (ensemble of) initial positions $\boldsymbol{x}_0$ is the only information required by the theory that is not contained in $\psi(\boldsymbol{x}, t)$, has led to refer to such positions as the set of hidden variables of the theory. They are here, but remain hidden to usual QM.

The pilot equation was introduced early in de Broglie's work (1927, 1928) as a law that unifies the Maupertuis principle of least action for the path of a particle and Fermat's principle of least time for a ray in optics; pt thus represents a law that unifies mechanics and optics. That both principles led to the same Eq. (8.11) allowed de Broglie to bring together the mechanical and undulatory aspects of the behavior of a quantum particle, following his fundamental proposal of the dual behavior of both, particles and photons. Equation (8.11) was therefore the starting point of de Broglie's theory (the pilot-wave theory). For Bohm, on the other hand, the point of departure was Eq. (8.12), which represents the guidance principle applied to the acceleration, not to the velocity, and which can be identified with Newton's equation of motion for a particle subject to the potential V_{eff}. Of course, the presence of the quantum potential in this latter modifies drastically the strictly classical (Newtonian) interpretation of Eq. (8.12).

What is sometimes called *Bohmian mechanics* is the theory based on Eqs. (8.11) and (8.3); it is a kind of mixture of the two theories (de Broglie's and Bohm's) that leaves aside the initial motivations of both authors in favor of the formal aspects of the theory. Notwithstanding its shortcomings (some of which will be discussed below) Bohmian mechanics should be recognized for its success in restoring realism, objectivity, determinism and causality for QM.

8.2.2 Quantum Trajectories

In the de Broglie-Bohm theory of motion a particle possesses an exact position $\boldsymbol{x}$ and an exact velocity $\boldsymbol{v}$ at any given time. This endows the theory with one of its main virtues, namely that of allowing for a phase-space trajectory (Holland 1993, Vasudevan et al. 2008). Since the notion of trajectory is foreign to the usual quantum description, it seems appropriate to comment on it here.

Textbooks on QM usually appeal to physical intuition to introduce some concepts, yet very soon the physical intuition dissapears in favour of a dynamical description carried out in an appropriate Hilbert space. In this new (highly mathematical) context, the notion of trajectory in physical space is dispensed of, avoided, or even negated.[10]

[10] This assertion requires some qualification. It is not too difficult to find (both in orthodox textbooks and in research papers, and of course also in popular works), arguments that bear implicitly or explicitly on the notion of trajectory. For example, in discussions on van der Waals or molecular forces a drawing is sometimes made of atoms with well-localized orbiting (point) electrons, and the Hamiltonian is written accordingly. True, at some moment an average is taken, but nevertheless the discussion refers, or at least seems to refer, to orbiting point particles. Another example is an atom or a particle in a Stern-Gerlach experiment, which in every analysis is considered to follow a definite trajectory.

In general, the denial of quantum trajectories is rooted on a certain reading of the uncertainty relations, so this kind of assertions is interpretation-dependent; valid only within the Copenhagen interpretation broadly understood, and particularly embraced by Bohr, as revealed in the following:

> Bohr was also at the meeting. After I had tried many times to explain what I was doing and didn't succeed, I talked about trajectories, then I would swing back –I was being forced back all the time. I said that in quantum mechanics one could describe the amplitude of each particle in such and such a way. Bohr got up and said 'Already in 1925, 1926, we knew that the classical idea of a trajectory or a path is not legitimate in quantum mechanics: one could not talk about the trajectory of an electron in the atom, because it was something not observable.' In other words, he was telling me about the uncertainty principle. It became clear to me that there was no communication between what I was trying to say and what they were thinking. Bohr thought that I didn't know the uncertainty principle, and was actually not doing QM right either. He didn't understand at all what I was saying. I got a terrible feeling of resignation. R. P. Feynman, taken from Towler (2009), lecture 6.

In contrast to the Copenhagen interpretation (where the quantum description is taken to refer to a single particle), for the ensemble interpretation the Heisenberg inequality $\Delta x \Delta p \geq \hbar/2$ says nothing about the impossibility for a single particle to have a definite position and momentum. It represents, instead, a statistical expression relating the simultaneous dispersions of position and momentum in the ensemble. Further, acknowledging the statistical essence of QM, the issue is not whether both x and p are simultaneously distributed, but *why* their distributions are conditioned by the Heisenberg inequality. We have found in Chaps. 3–5 an answer to this question, involving the action of the zero-point field. Indeed, in the present approach the original equation of motion refers to a stochastic process with a well-defined trajectory for every realization of the field. However, when an ensemble of systems is considered, the possibility to *identify* the single trajectories is lost.[11] Our corollary is thus that QM is unable to explicitly allude to the notion of trajectory, without this however implying a negation of the existence of trajectories.

Historically speaking, the denial of quantum trajectories was not always the case. Quite the opposite, as stated in Sect. 8.1.2, in his very first attempts to construct modern quantum mechanics—some ten years after Bohr's model of the H-atom—de Broglie was trying to describe the quantum trajectories when he proposed the guidance formula $\boldsymbol{v} = \hbar \nabla S/m$. In the course of time, the notion of quantum trajectory has found invigoration with the emergence of novel ideas from the consistent-histories interpretation of quantum mechanics (see e.g. Griffiths 1993, Omnès 1994), the quantum-trajectory method (Lopreore and Wyatt 1999, 2000), quantum optics (see e.g. Brun 2002), and even from a study of the Dirac equation (Gull et al. 1993). The Workshops on Quantum Trajectories held in 2008 and 2010 attest to the current interest on the subject. Indeed, the possibility of tracing quantum trajectories has attracted the attention of a vast number of investigators due to the valuable information that can be effectively gained from their study. The related literature shows a growing trend,

[11] It is this statistical treatment what engenders 'indistinguishability', and this occurrs regardless of whether the system is classical or quantum. This, for instance, explains the use of the notion of indistinguishability to solve the Gibbs paradox in classical statistical physics (see e.g. Mandl 1988).

and eventually such endeavour should bear important fruits. Hence, even though the notion of trajectory in QM continues to meet opposition among strongly Copenhagen-minded physicists, it seems that we are still in a period of search and definition about the physical principles of quantum theory, the trajectory issue being only a small part of the picture.

From within SED, refined studies have been carried out to determine the trajectories as predicted by the theory. Such studies demand a detailed simulation of the ZPF, instead of the hazard being introduced by hand through the use of distributed initial velocities (Bohm's theory) or of a white noise (stochastic mechanics). For this purpose a background noise with spectrum proportional to ω^3 has to be generated and the full equation of motion (4.2) must be used. A most important example of this kind of calculations is the work of Cole and Zou (2003, 2004a, b, c, d, 2009) on the H-atom, which (nearly) reproduces the stationary quantum predictions as the result of a statistical analysis of a long run of the ground-state orbit. It should be stressed that these results are obtained by averaging the orbital motions, that is, by tracking individual trajectories of electrons for a long time interval, in correspondence with the discussions in Chaps. 4 and 5. Another most interesting example is given in Huang and Batelaan (2012); here the authors study the details and statistics of the trajectories followed by a classical oscillator immersed in the ZPF, and exhibit the mechanism by which the classical distribution of positions gets transformed into the corresponding quantum distribution.

That quantum trajectories may differ greatly from classical trajectories should be expected in advance. Popular illustrations such as the stylised atoms used in postage stamps, logos and comic strips are of course very rude representations, far from reality. The atomic orbits do not resemble planetary orbits; they are something much more complex due to the complex dynamics they obey. The best-known graphs of quantum trajectories are perhaps those obtained within Bohm's theory.[12] Even if they cannot provide a detailed description, the studies of electron trajectories from Bohm's perspective satisfactorily reproduce some central features of the experimental results and offer an intuitive picture of the underlying dynamics. For some physicists the notion of quantum trajectory is natural and is an interesting consequence of the quantum potential (see e.g. Dewdney et al. 1993, Dürr et al. 1993); for others, their weirdness makes them unacceptable, or even surrealistic (Englert et al. 1992, Aharonov and Vaidman 1996).

The trajectories predicted by Bohm's theory are in general very complex, since the quantum potential is an intricate function of ψ and ψ^* that can vary very fast with the coordinates, and thus along the path followed by the particle. Moreover, the trajectories are normally quite sensitive to the initial conditions and to the full distribution of particles. A most popular example of a Bohmian trajectory is presented in Fig. 8.1, which shows the paths followed by electrons diffracted by two Gaussian slits (taken from Philippidis et al. 1979). Another example is given in Fig. 8.2 (taken

[12] Also Nelson's theory and more generally the stochastic description of QM have been successfully used to investigate quantum trajectories, as shown by the examples in Chap. 2.

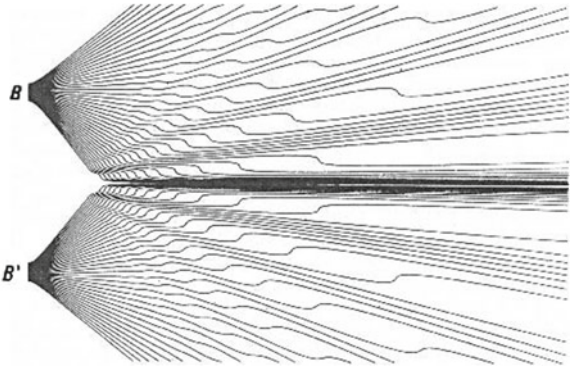

Fig. 8.1 Trajectories scattered by two Gaussian slits, according to Bohm's theory. The initial beams of particles have a uniform distribution at each slit. Reprinted from Philippidis et al. (1979) with permission from Springer

from Dewdney and Hiley 1982), which shows the trajectories of electrons from a Gaussian packet incident on a semitransparent barrier with a transmission ratio of one-half. The pattern shows clearly how the outer particles are reflected *before* reaching the barrier, whilst the ones closer to the barrier are mostly transmitted. In the next section we will have opportunity to discuss this (nonlocal) feature in more detail.[13] Notice that *individual* particles do not conserve their *mechanical* energy, but change speed before reaching the barrier, as shown also in Chap. 2. The total energy is of course conserved, but since QM does not consider the energy of interaction with the ZPF, it loses accountability in instances as the present one.

As mentioned earlier, the Bohmian trajectories are described by individual particles having an instantaneous velocity $\boldsymbol{v}(\boldsymbol{x})$ as given by Eq. (8.6). Such interpretation differs from the one developed in this book, according to which (8.6) refers to a local *mean* velocity, obtained as a result of a (partial) averaging over the momentum space, and hence containing statistical information about all the individual instantaneous velocities at $\boldsymbol{x}$. From the perspective adopted here, the stochastic field is responsible for fluctuations around the mean motion, as shown for example in Figs. 2.1 and 2.2, so that the actual velocity of a single particle varies at random from case to case and should be expressed as[14]

$$\boldsymbol{v}_{\text{actual}}(\boldsymbol{x}, t) = \frac{\hbar}{m} \nabla S(\boldsymbol{x}, t) + \Delta \boldsymbol{v}, \tag{8.15}$$

with $\Delta \boldsymbol{v}$ a stochastic deviation that averages to zero. Consequently the SED approach admits (random) trajectories that can mutually cross, an effect that the de Broglie-Bohm theory, by its deterministic nature, cannot take into account. Indeed, a well-

[13] Further examples can be seen in Holland (1993), Lopreore and Wyatt (1999, 2000)—who have generated what they call the 'quantum trajectory method'—; Suñé and Oriols (2000), Matzkin and Nurock (2008), Sanz et al. (2002), Philippidis et al. (1982), and Kumar Chattaraj (2010).

[14] In a variant of Bohm's theory the idea of a fluid *à la* Madelung is entertained. By considering this fluid to be subject to fluctuations, a random element is then added; see Bohm and Vigier (1954). See also Wang (2006) for related work.

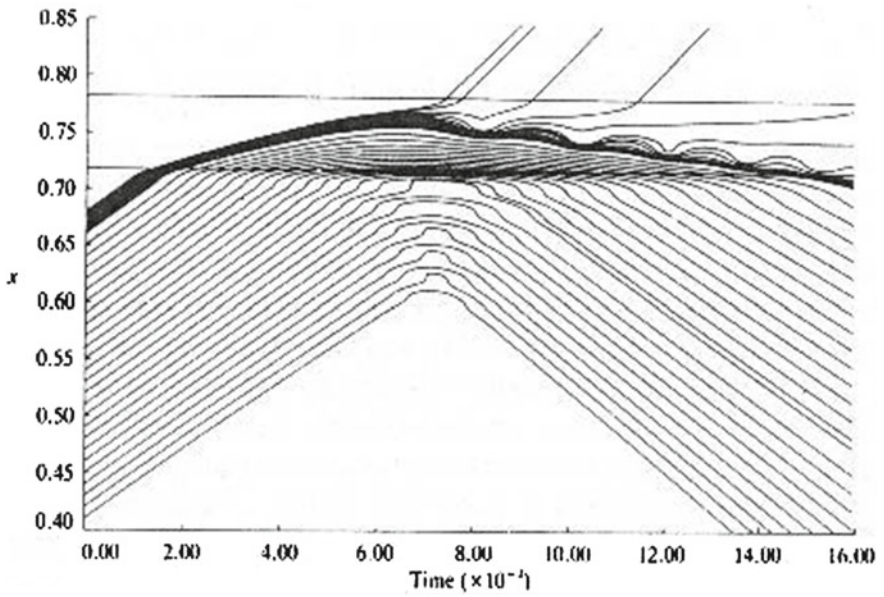

Fig. 8.2 A collimated beam of particles scattered by a barrier according to Bohm's theory. Notice that many particles are scattered before reaching the barrier. Reprinted from Dewdney and Hiley (1982) with permission from Springer

known prediction of Bohm's theory is that the (deterministic) trajectories never cross. This is foreseeable, for particles that follow the (by form) classical rule, Eq. (8.12): the crossing of classical trajectories at a point $\boldsymbol{x}$ would mean that at that point the velocity (the local tangent) would not be a single-valued function. Illustrative examples of crossing trajectories that go in line with the stochastic perspective can be seen in the cited figures of Chap. 2.

The aforementioned difference between the meanings ascribed to $\boldsymbol{v}$ is reflected also in the way Eq. (8.9) is read. Whereas in Bohm's theory the statistical content of this equation is encoded in the quantum potential V_Q (via its dependence on the probability density ρ), within our approach the analogy with the classical case is more distant, due to the intrinsic statistical sense of the kinetic terms involved. In addition, the fact that in Bohm's theory the velocity $\boldsymbol{v}$ is identified with $d\boldsymbol{x}/dt$ runs into a problem for stationary states with real (spatial) wave function, since in such case $\nabla S = 0$ and there is no flow of particles, as argued by Heisenberg (1955). This sounds unconvincing, especially considering that in general (even for stationary states) a nontrivial distribution of momenta is to be expected, so that a static image seems incongruent with the theory. Of course, there still exists the diffusive velocity $\boldsymbol{u}$ (see Sect. 4.5.1), although it is not recognized as a velocity in the usual quantum account, nor in the causal interpretation. By contrast, if $\boldsymbol{v}$ refers to a mean

local velocity, the result $\boldsymbol{v} = 0$ only indicates that there are equally many particles travelling in opposite directions. This restores consistency with the distribution of the momenta.[15]

8.2.3 The Measurement Task in the Pilot Theory

The causal and realistic posture that characterizes the pilot theory can be extended to cope with the measurement problem in a natural, objective and rational way: if a measurement is to be performed, that means the system has been prepared by inserting the measuring instrument, which has thus become an integral part of it. The next step is therefore to solve the Schrödinger equation with the apparatus and its gear included; the solution should give the statistical answer, and so the measurement problem should dissolve.

Let us illustrate how this is achieved, with the help of a simple example proposed by Bell (1987, article 17) and derived from the original theory presented by de Broglie (1928) in the Fifth Solvay Conference 1927 (other, more elaborate and realistic examples can be seen in the suggested literature). Suppose that a spin component is to be measured, and that the 'measurement' is represented by an interaction Hamiltonian similar to the one proposed by von Neumann (1932) in his famous textbook (one-dimensional notation is used for simplicity),

$$-i\hbar g\hat{O}\frac{\partial}{\partial x}. \tag{8.16}$$

Here g is a coupling constant and $\hat{O}$ an appropriate spin operator (the observable to be measured). Assume also that the particle is sufficiently massive so that the complete Hamiltonian is very nearly represented by its interaction part. The Schrödinger equation for the state $|\psi\rangle$ reads then

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = -i\hbar g\hat{O}\frac{\partial}{\partial x}|\psi\rangle\,. \tag{8.17}$$

Introduction of the spin eigenvectors $|\alpha_n\rangle$ such that $\hat{O}|\alpha_n\rangle = O_n|\alpha_n\rangle$, allows to express the state function in the form

[15] Here it is in place to recall the argument against the Copenhagen interpretation raised by Einstein (1953), considering the stationary states of an infinite one-dimensional square well potential. The spatial part of the wave function can be written in the form $\varphi = N\sin kx$. From Eq. (8.11) it follows that $v = 0$, hence there is no flow velocity. However, by writing the wave function in the form $\psi = (N/2i)(e^{ikx} - e^{-ikx})$, it can be interpreted as referring to two similar subensembles of particles, traveling to the right and to the left, with velocities $\pm\hbar k/m$. Thus, it is the net (mean) velocity that is null. Einstein used this example to argue that the statistical reading is the single one that can be made in the limit of high energies. Since passing to this limit does not change the nature of the problem, Einstein concluded that one should consider the wave function as describing an ensemble, not an individual particle.

$$|\psi(x,t)\rangle = \sum_n \Phi_n(x,t)\,|\alpha_n\rangle\,. \tag{8.18}$$

Substituting into Eq. (8.17) leads to

$$\frac{\partial \Phi_n}{\partial t} + gO_n \frac{\partial \Phi_n}{\partial x} = 0, \tag{8.19}$$

with solution

$$\Phi_n(x,t) = \Phi_n(x - v_n t) \tag{8.20}$$

with $v_n = gO_n$. Therefore the state is given by

$$|\psi(x,t)\rangle = \sum_n \Phi_n(x - v_n t)\,|\alpha_n\rangle\,. \tag{8.21}$$

The result speaks for itself: the wavepackets $\Phi_n(x - v_n t)$ travel each with its own velocity gO_n, and thus move apart with time. Eventually the overlap between packets will be negligible so that they can be considered isolated from one another. A position measurement corresponds then to a specific O_n and thus to a particular spin state $|\alpha_n\rangle$, which contributes with the probability $|\Phi_n|^2$. For example, when x is taken to represent a radial direction, different positions correspond to different angles and the example becomes an elementary model of a Stern-Gerlach experiment.

When the overlap between the different Φ_n becomes negligible, there is no appreciable interference between them. It thus appears as if the state has suffered an effective, practically irreversible collapse, although subject at all times to a unitary evolution. It is the approximation of treating the final packets as truly independent which breaks the unitary dynamics, thus evoking a kind of 'collapse' (introduced by hand) that does not correspond to any physical process, yet is useful as a practical procedure to deal with the final state.[16] One merit of the theory becomes thus clear: there is no observer that 'induces' the wave collapse; no need for partial tracings; no two laws of evolution but only the one that governs the dynamics all along the process. The notion of collapse dissolves and no measurement theory is needed, since now the measurement becomes a normal experiment, softly merged into the conventional theory of quantum evolution. The conclusion is in full agreement with the stance promoted by van Kampen (1988) and others: "The measuring act is fully described by the Schrödinger equation for object and apparatus together..." In plain words, quantum mechanics is more fitting without the addendum of the weighty theory of measurement.

The rather elementary model just discussed illustrates well various fundamental aspects of the meaning of 'measurement of an observable'. The first one, remarkable

[16] In the usual quantum theory of measurement, the process of extracting from (8.21) the observed result Φ_n (the reduction or collapse of the after-measurement state) is referred to as the problem of objectification. See e.g. Mittelstaedt (2009).

enough, is that every measurement (here of a spin component) ends up being a position measurement. Thus, the breaking of the initial wave packet into several ones as a product of the interaction, which separate in the course of time (until becoming almost noninterfering) allows to identify the presence or absence of any component in the initial wave packet. A second, fundamental aspect is that the description is entirely objective and avoids dividing the world into undefined observed and observer, object and subject—which is one of the most discomforting aspects of the usual quantum theory of measurement (see e.g. Bell 1987; Wick 1995).

The amplitude of the nth component in Eq. (8.21) depends on the instrument through the factor gO_n. Thus one is tempted to conclude that the result of the measurement is determined by both the system and the measuring apparatus (which here is part of the enlarged system). In this direct sense, it is true. The probability with which the packet Φ_n contributes to $|\psi(x,t)\rangle$ is $|\Phi_n(x - gO_n t)|^2$, which for $t \neq 0$ is instrument-dependent. However, it is only in the relative weights of the states that there is a dependence on the measuring device: if initially $|\Phi_n(x)|^2$ is zero for a given component, this component will never be registered in the output. Only those components that contribute at $t = 0$ will have a chance to show up at later times. In this sense, the end result depends exclusively on the system itself. In other words, only predefined values are eventually observed, assuming the measuring instrument does not directly affect the system itself.[17]

One of the reasons for the success of Bohm's approach with the measurement problem is the fact that it contains and uses the notion of quantum trajectory, as in the above example, where the packets representing particles tend to separate. We have here a nice and important instance of both the possibility of introducing this notion into conventional QM , as already discussed, and the usefulness of such endeavor. Even if, according to our perspective, a more detailed description exists that contains fluctuations that are absent in Bohm's' theory, for many purposes such local mean description suffices to give an approximate idea of how the quantum system behaves.

8.3 The Quantum Potential

As stated in Sect. 8.2.1, a key element in Eq. (8.5b) is the term $V_Q(\boldsymbol{x}, t)$. It stands as a sui generis potential, essentially different from any classical one in many ways, on some of which we comment in the present section. Our purpose is both to show

[17] The possibility that the result of a measurement depends on both the system under observation and the measuring apparatus is also present at the classical level. A common example of the class of nondisturbing classical procedures is a photocell detector that checks the presence or absence of somebody before closing the door of an elevator. An example of the second class could be a 'tail or head' detector for tossed coins which operates by inserting a card to stop and receive the coins. Of course this second mechanism can be replaced with more elaborate optical procedures that do not disturb the observed coins. This is a matter of the measurer's skills and of the existing technical possibilities.

that essential features of the quantum system are linked to V_Q, and to provide new insights into its physical origin.[18]

Rather than starting from Eq. (8.4), we resort to the more general Eq. (2.81),

$$ia\frac{\partial\psi}{\partial t} = -\frac{a^2}{2m}\nabla^2\psi + V\psi + (1-\lambda)\,\frac{a^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\psi, \text{ with } \psi = \sqrt{\rho}e^{iS}, \tag{8.22}$$

which is valid provided the flux momentum of the system has the structure $m\boldsymbol{v}(\boldsymbol{x}, t) = a\nabla S(\boldsymbol{x}, t)$, with a constant. The procedure used at the beginning of Sect. 8.2.1 leads then to a couple of equations involving the variables S and ρ: one is the continuity Eq. (8.5a), the other is a generalized form of Eq. (8.5b),

$$a\frac{\partial S}{\partial t} + \frac{a^2}{2m}(\nabla S)^2 + V - \lambda\frac{a^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = 0. \tag{8.23}$$

The case of interest here corresponds evidently to $a = \hbar$; however, the specific value of a is irrelevant for the present purposes. As discussed in Sect. 2.4, what is important is that Eq. (8.22) is linear in ψ if and only if $\lambda = 1$, that is, if Eq. (8.23) involves an additional 'potential' given precisely by $-\left(a^2/2m\right)\left(\nabla^2\sqrt{\rho}/\sqrt{\rho}\right)$ (which for $a = \hbar$ is the quantum potential). This establishes the following concomitance: for a given purpose we can resort to the (Schrödinger) equation for ψ, or to the equivalent (Hamilton-Jacobi-type) equation for (ρ, S)—the continuity equation is merely a constraint, not a dynamical equation—, and the effects that arise from the linearity of the former will be attributable to the term V_Q in the latter. In the next two sections we apply this parallelism—exploiting the linearity of the Schrödinger equation—to draw conclusions about the role played by V_Q in the dynamics of the system. We shall thereafter inquire further into the physical meaning of V_Q.

8.3.1 Linearity and Nonlocality

The most recognized feature of Bohm's theory is its essential nonlocality. However, almost every analysis on nonlocality that one finds in the physical literature is carried out within the context of composite systems, the discussions on nonlocal effects in single-particle systems being rather scarce. A possible explanation for this rests on the

[18] A first peculiarity of V_Q is that it is independent of the field's strength, or rather of the intensity ($\sim \rho$) of the wave. This follows from the fact that $V_Q(\rho) = V_Q(A\rho)$ for any constant A, and indicates that the effects due to the particles do not depend on the number of particles present; but on their distribution. That there are forces within the classical realm, particularly in the hydrodynamical analogy, with similar peculiarities does not suffice to surmount the problem, since in the hydrodynamical case there is a medium that supports and transmits the presure and the stresses. By contrast, in the quantum single-particle problem we are not dealing with a collective system; the 'collection' may be a conceptual ensemble, devoid of physical existence.

fact that since the EPR paper, and mostly from Bell's theorem on, the debates regarding nonlocality have centered on bipartite (or multi-partite) quantum systems, to the extent that the notion of nonlocality has become widely understood as synonymous with violation of Bell's inequalities,[19] or as a manifestation of some sort of action at a distance between the constituents of the system. Another reason may be that for single-particle systems the notion of nonlocality is somewhat more blurred and subtle. In particular, there is apparently no room for actions at a distance simply because a single particle has no partner to interact with! However, a look at Fig. 8.2—which shows that as an effect of the barrier some particles are reflected before reaching it—indicates that it does make sense to talk about nonlocal effects in single-particle systems. As the alien element in the otherwise classical (single-particle) Eq. (8.5b), V_Q is to be blamed for the characteristic quantum properties of the system: hence it must account also for its nonlocal properties.

The fact that V_Q depends explicitly on the spatial distribution of particles could be considered indeed as a sufficiently strong argument to ascribe to it the nonlocal properties characteristic of quantum systems. However, a more detailed argument goes as follows. Expressing the Schrödinger equation in the form $\mathcal{L}\psi = 0$, with $\mathcal{L}$ a linear operator, the determination of the (causal) Green function for $\mathcal{L}$, call it $K(\boldsymbol{x}, t|\boldsymbol{x}', t')$, suffices to express the solution in the form

$$\psi(\boldsymbol{x}, t) = \int K(\boldsymbol{x}, t|\boldsymbol{x}', t')\psi(\boldsymbol{x}', t')\, d^3x', \quad t \geq t', \tag{8.24}$$

with $K = 0$ for $t < t'$. Equation (8.24) shows that ψ at any point $\boldsymbol{x}$ and time t carries information regarding its previous value at *all* points of the available space. A consequence of this is that every quantity that is determined by ψ bears in general information about the whole setup. Therefore, the trajectory defined by the guidance Eq. (8.11) reflects the presence of boundary conditions and of (possibly distant) external potentials, for example. This accounts for some of the nonclassical effects manifested in numerous Bohm trajectories; in particular, it explains why the particles begin to gain or lose energy or deviate before reaching the barrier, as in Fig. 8.2, as if 'perceiving' its effect in advance.[20] This kind of behavior is the one we identify as the quantum single-particle nonlocality, where the term 'nonlocal' means that the dynamics of the particle in some region is affected by what happens in regions that may be far away from it, without an intermediate recognized (external) agent.

In the de Broglie-Bohm interpretation the underlying cause for such nonlocal behaviour is found in the quantum potential, which transmits, to each point, information about the wave field in the entire space. Thus, a particle at a point where V is constant (e.g., located at $x < a$ in the barrier example) is not a free particle, but

[19] Bell inequalities is a collective name referring to a number of inequalities (such as the CHSH-type inequalities) that involve correlations between variables of the constituents of a composite system and are violated by QM, reputedly due to the nonlocal properties of the quantum description.

[20] The transmitted particles are among those that gain enough energy to travel not *through* the barrier, but *over* it (see e.g. Lopreore and Wyatt 1999).

is acted upon by a *quantum force* $-\nabla V_Q(\boldsymbol{x}, t)$. This force, being dependent on the point $\boldsymbol{x}$ where it is exerted, acts of course locally; what is nonlocal is the information it carries. There is no room for superluminal action at a distance. Moreover, since in Bohm's interpretation the field ψ is considered to be a physical entity, the nonlocality appears as entirely admissible: the anticipated reflection of the particles is as natural (and local) as it would be if there existed a *real* fluid, as was conceived by Madelung. In this scenario some incident particles are reflected before reaching the barrier simply because the reflected wave acts upon them (Holland 1993).

This explanation does not hold in the SED framework, where the correct description of the entire (field plus particle) system has a local structure and quantum nonlocality appears as a feature of the reduced quantum-mechanical description, rather than an ontological property. At the end of Sect. 8.3.3 we comment on this point in more detail.

8.3.2 Linearity and Fluctuations

Equation (8.24) is a direct consequence of the linearity of the Schrödinger equation. Another most important property of the solutions of linear equations, is that they satisfy the superposition principle. Let us apply it here to a free-particle system, by superposing plane waves of different momenta to construct a Gaussian wave packet, and use it to show that V_Q is also linked to the presence of quantum fluctuations, which appear as irreducible.

Consider the following one-dimensional packet of free particles of mass m,

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int \phi(k) e^{-i(\hbar k^2 t/2m) + ikx} dk, \tag{8.25}$$

where $\hbar k = p$. If the initial wave function is

$$\psi(x, 0) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4} \exp(-x^2/4\sigma_0^2), \tag{8.26}$$

the distribution $\rho(x, t)$ is a Gaussian centered at the origin,

$$\rho(x, t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-x^2/2\sigma^2\right) \tag{8.27}$$

with

$$\sigma^2(t) = \sigma_x^2(t) = \sigma_0^2 + \left(\frac{\hbar t}{2m\sigma_0}\right)^2. \tag{8.28}$$

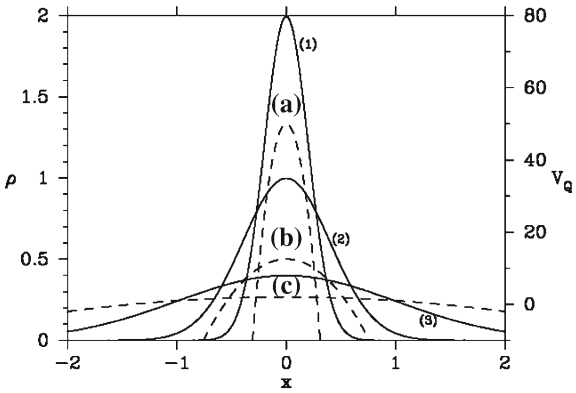

Fig. 8.3 Distribution of particles ρ (*solid lines*, marked with numbers) and quantum potential V_Q (*dotted lines*, marked with letters) of Eqs. (8.27) and (8.29) for three different times $t_1 < t_2 < t_3$. Curves (1) and (a) correspond to t_1, (2) and (b) to t_2, and (3) and (c) to t_3. Arbitrary units

A direct calculation gives for the quantum potential

$$V_Q = \frac{\hbar^2}{8m\sigma^4}\left(2\sigma^2 - x^2\right). \tag{8.29}$$

Figure 8.3 shows ρ (solid line) and V_Q (dotted line) for three different times, as the spread of the packet is seen to increase. A first conspicuous property of V_Q is that it does not approach a constant value at the boundaries ($x \to \pm\infty$), where the field intensity, or rather the distribution of particles ρ, tends to zero.[21] On the contrary, the effect of the potential (8.29), as measured by the quantum force $-\partial V_Q/\partial x$, increases in those regions where the particles have a smaller probability to be. Indeed, it follows from Eq. (8.29) that the exerted quantum force is repulsive and linear in x; those particles in the region $x > 0$ are 'pushed' to the right by such force, and those in $x < 0$ are 'pushed' to the left, so the packet spreads continuously. In the frame of the Hamilton-Jacobi-type equation, the dispersion is explained in mechanistic terms as an effect of the quantum potential.

Further, from Eq. (8.29) it follows that

$$2m\left\langle x^2\right\rangle\langle V_Q\rangle = \frac{\hbar^2}{4}, \tag{8.30}$$

since $\sigma^2 = \left\langle x^2\right\rangle$. This result (to be analyzed later for the general case) suggests a strong relationship between the mean quantum potential and the irreducible momentum fluctuations that lie at the root of the Heisenberg inequalities. It also exhibits quantitatively what was said above in qualitative terms about the dispersive effect of the quantum potential. Thus, in Bohm's approach, the inherent dispersive nature of quantum systems becomes causal, the quantum potential being the physical element that causes it by exerting a (here) repulsive force on the particles.

[21] This feature is not exclusive of the present example, but rather the general rule: V_Q does not decay as $x \to \pm\infty$, i.e., at far distances from the particles.

If at variance with Bohm's interpretation, ψ is taken as a mathematical entity that bears statistical information, but is unable to produce direct physical effects on the particles, the spread of the wave packet cannot be understood in mechanistic terms. It becomes explained instead in statistical terms: the ensemble is composed of subensembles of particles traveling with different velocities [each contributing with a probability $|\phi(k)|^2$, as follows from (8.25)]; the faster ones move forward whereas the slower ones are left behind, so the packet spreads. In this scenario each particle of the ensemble is always a true free particle, since no physical force acts upon it. The quantum potential is the bearer of the statistical information regarding the momentum dispersion, so that not only the spread of the packet becomes natural, but so does Eq. (8.30), which implies a relation between the quantum potential and the momentum fluctuations.

The spread of the packet induced by V_Q given by Eq. (8.29) can be compensated by an external harmonic potential, so as to ensure that the net force $-\partial V_{\text{eff}}/\partial x$ vanishes. Writing the external potential as $V = m\omega^2 x^2/2$, such condition is guaranteed provided the oscillator frequency is exactly $\omega = \hbar/\left(2m\sigma^2\right)$. The effective potential becomes thus

$$V_{\text{eff}} = V + V_Q = \tfrac{1}{2}\hbar\omega. \tag{8.31}$$

Under this condition there is no net force on the particle, the packet (8.5b) does not spread and a stationary state is reached. This is the simplest example of a coherent state, and serves also to explain the stability of the distribution of the ground state of the harmonic oscillator (which is a Gaussian): it is an effectively free particle, in the sense the no *net* force acts upon it. Moreover, Eq. (8.31) fixes a natural reference energy level—an observation that can be used to infer the existence of the ZPF underlying the Schrödinger (or Hamilton-Jacobi-type) description.

8.3.3 The Quantum Potential as a Kinetic Term

In Bohm's theory the quantum potential is accepted as a natural entity that needs no further explanation. Just as is the case with fundamental laws of nature, it is taken as an expression of the structure and workings of the physical world, which physics has the duty to discover and describe. Yet clarification of the deeper meaning of V_Q *does* have importance because in it resides a fundamental ingredient of the quantum description. All quantum problems involve the quantum potential, even if normally it remains concealed behind the veil of the Schrödinger equation.

The discussion in Chap. 4 helps us elucidate the nature of the quantum potential and understand, from a more fundamental perspective, why it is so intimately related with the dispersive and nonlocal effects studied in the previous section. In particular, by disclosing the origin of V_Q it will become clear that when dealing with a quantum problem one is (knowingly or unknowingly) taking into account much more

information about the behavior of the system in momentum space than is explicitly acknowledged.

Let us for this purpose recall an alternative derivation of Eqs. (8.5a) and (8.5b) that serves to emphasize the significance of both velocities $\boldsymbol{v}$ and $\boldsymbol{u}$. We start from Eq. (4.128), namely

$$\hat{\boldsymbol{p}}\psi = -i\hbar\boldsymbol{\nabla}\psi = m(\boldsymbol{v} - i\boldsymbol{u})\psi, \tag{8.32}$$

with $\boldsymbol{u}$ the diffusive velocity given by

$$\boldsymbol{u} = \frac{\hbar}{2m}\boldsymbol{\nabla}\ln\rho. \tag{8.33}$$

From Eq. (8.32) it follows that the kinetic energy operator $(1/2m)\,\hat{\boldsymbol{p}}^2$ applied to ψ results in

$$\frac{\hat{\boldsymbol{p}}^2}{2m}\psi = \frac{1}{2}\left[\left(m\boldsymbol{v}^2 - m\boldsymbol{u}^2 - \hbar\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) - i\left(\hbar\boldsymbol{\nabla}\cdot\boldsymbol{v} + 2m\boldsymbol{v}\cdot\boldsymbol{u}\right)\right]\psi. \tag{8.34}$$

On the other hand, Eq. (8.4) gives

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\hbar\frac{\partial S}{\partial t} + i\frac{\hbar}{2}\frac{\partial\ln\rho}{\partial t}\right)\psi. \tag{8.35}$$

Combining the last two expressions with Schrödinger's equation

$$\frac{\hat{\boldsymbol{p}}^2}{2m}\psi + V(\boldsymbol{x})\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{8.36}$$

gives an expression whose imaginary and real parts are, respectively, the continuity Eq. (8.5a) and the dynamical law

$$\hbar\frac{\partial S}{\partial t} + \frac{1}{2}\left(m\boldsymbol{v}^2 - m\boldsymbol{u}^2 - \hbar\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + V = 0. \tag{8.37}$$

A comparison with Eq. (8.5b) (with $\left(\hbar^2/2m\right)(\boldsymbol{\nabla}S)^2 = (m/2)\,\boldsymbol{v}^2$) allows to write the quantum potential in a form that reveals its kinetic nature,

$$V_Q = -\tfrac{1}{2}\left(m\boldsymbol{u}^2 + \hbar\boldsymbol{\nabla}\cdot\boldsymbol{u}\right). \tag{8.38}$$

In Bohm's theory the term $(m/2)\,\boldsymbol{v}^2$ alone is identified with the total kinetic energy, whereas the remaining terms in Eq. (8.5b) are taken as a 'potential' energy. This latter form of separating the kinetic and potential terms is a consequence of reading (8.5b) in a classical fashion, as if it were a true Hamilton-Jacobi equation. Equations

(8.34) and (8.38), by contrast, identify V_Q with a contribution to the kinetic energy due to the diffusive velocity $\boldsymbol{u}$. These equations give for the mean kinetic energy

$$\frac{1}{2m}\left\langle \hat{\boldsymbol{p}}^2 \right\rangle = \int \left(\frac{1}{2} m \boldsymbol{v}^2 + V_Q \right) \rho d^3x = \frac{1}{2} m \left\langle \boldsymbol{v}^2 \right\rangle + \left\langle V_Q \right\rangle, \tag{8.39}$$

so with the help of Eq. (4.120), namely

$$\left\langle \hat{\boldsymbol{p}}^2 \right\rangle = m^2 \left\langle \boldsymbol{v}^2 + \boldsymbol{u}^2 \right\rangle, \tag{8.40}$$

one gets

$$\left\langle V_Q \right\rangle = \tfrac{1}{2} m \left\langle \boldsymbol{u}^2 \right\rangle = \tfrac{1}{2} m \sigma_{\boldsymbol{u}}^2. \tag{8.41}$$

This is an interesting result: the mean quantum potential coincides with the mean kinetic energy of diffusion. Since $\boldsymbol{u}(\boldsymbol{x}) \neq 0$ whenever $\rho(\boldsymbol{x})$ is not constant (which happens in all cases of interest, when there is finite spatial dispersion), Eq. (8.41) implies that $\langle V_Q \rangle$ is strictly positive. From Eq. (8.40) we find $\sigma_{\hat{\boldsymbol{p}}}^2 = m^2 \sigma_{\boldsymbol{v}}^2 + m^2 \sigma_{\boldsymbol{u}}^2$ [which is Eq. (4.122)], whence

$$\sigma_{\hat{\boldsymbol{p}}}^2 = \sigma_{\boldsymbol{p}}^2 = \sigma_{m\boldsymbol{v}}^2 + 2m \left\langle V_Q \right\rangle > \sigma_{m\boldsymbol{v}}^2, \tag{8.42}$$

where $\sigma_{m\boldsymbol{v}}^2$ stands for the variance of the flux momentum $m\boldsymbol{v}$. Classically, this latter coincides with the total momentum dispersion, $\sigma_{\boldsymbol{p}}^2 = \sigma_{m\boldsymbol{v}}^2$. Therefore, (8.42) states that the quantum momentum dispersion (normally) exceeds the classical one. The result $\sigma_{\boldsymbol{p}}^2 > \sigma_{m\boldsymbol{v}}^2$ is immediate from (4.122), yet the inequality (8.42) is expressed in terms of the (mean) quantum potential, thus confirming that the 'quantumness' of the system is indeed encoded in V_Q. From (8.42) it follows that the minimum value of $\sigma_{\boldsymbol{p}}^2$ is

$$(\sigma_{\hat{\boldsymbol{p}}}^2)_{\min} = 2m \left\langle V_Q \right\rangle_{\min}, \tag{8.43}$$

a result that exhibits the existence of irreducible momentum fluctuations of value $2m \langle V_Q \rangle_{\min}$. Equation (8.30) corresponds to the particular case $\boldsymbol{v} = 0$ (a stationary bounded s state), when $\sigma_{m\boldsymbol{v}}^2$ vanishes.

From the previous results and the Schwartz inequality it follows that

$$\sigma_{\boldsymbol{x}}^2 \sigma_{\boldsymbol{p}}^2 \geq m^2 \sigma_{\boldsymbol{x}}^2 \sigma_{\boldsymbol{u}}^2 \geq \tfrac{1}{4} \hbar^2. \tag{8.44}$$

The diffusive velocity is therefore the one that determines the Heisenberg inequality, by expressing the presence of diffusion in the quantum system.

With Eq. (8.40) rewritten in terms of local mean values (with $\langle \boldsymbol{p} \rangle_{\boldsymbol{x}} = m\boldsymbol{v}$) one obtains, after rearrangements,

$$\int V_Q \rho d^3x = \frac{1}{2m} \int \left\langle \left(\boldsymbol{p} - \langle \boldsymbol{p} \rangle_{\boldsymbol{x}} \right)^2 \right\rangle_{\boldsymbol{x}} \rho d^3x. \tag{8.45}$$

From here it follows that up to an arbitrary term that averages to zero, $2mV_Q$ plays the role of $\langle \left(\boldsymbol{p} - \langle \boldsymbol{p} \rangle_{\boldsymbol{x}} \right)^2 \rangle_{\boldsymbol{x}} = \langle \boldsymbol{p}^2 \rangle_{\boldsymbol{x}} - \langle \boldsymbol{p} \rangle_{\boldsymbol{x}}^2$, the local mean deviation of the momentum from its local mean value. This assigns a more fundamental meaning to V_Q: it bears information, at each point $\boldsymbol{x}$, of the local fluctuations impressed upon the momentum of the particle.

With this, Eq. (8.43) becomes natural from the point of view of SED. Indeed, unavoidable fluctuations exist due to the ZPF , the information about which is contained in V_Q. The quantum potential is the element that (re)incorporates the momentum fluctuations impressed by the ZPF into the dynamics governed by the Hamilton-Jacobi-type equation.

In addition, the fact that V_Q plays the role of a partially averaged quantity that results from restricting the description to the configuration subspace of the particle, explains the origin of its nonlocal effects, since at each point $\boldsymbol{x}$, V_Q bears statistical information about the entire momentum space. The single-particle quantum nonlocality, rather than an ontological property, appears thus as a semblance, an artifact of the reduced statistical description, which would dissolve by going back to the full, original phase-space description.

8.4 Nonlocality in Bipartite Systems

As mentioned at the beginning of Sect. 8.3.1, nonlocality is normally discussed in relation with composite systems. Nonlocality in such context has gained so much attention in the last decades, that huge numbers of papers and entire volumes have been devoted to its study with different purposes and in many directions. The issue has evidently not been exhausted, and it therefore seems pertinent to contribute to its clarification from the perspective of the present theory. In this section we focus on a two-particle system as the simplest example that can be used to study the complexities arising in composite systems, and resort to the tools developed so far to gain further insight into (bipartite) entanglement and nonlocality.

The two-particle system is described by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m_1} \nabla_1^2 - \frac{\hbar^2}{2m_2} \nabla_2^2 + V \right) \psi, \tag{8.46}$$

with a general (time-independent) external potential of the form $V = V(\boldsymbol{x}_1, \boldsymbol{x}_2)$. Substitution of

$$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \sqrt{\rho(\boldsymbol{x}_1, \boldsymbol{x}_2, t)} e^{iS(\boldsymbol{x}_1, \boldsymbol{x}_2, t)} \tag{8.47}$$

in (8.46) leads again to a couple of equations (corresponding to its real and its imaginary part). The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\hbar}{m_1}\nabla_1 \cdot (\rho \nabla_1 S) + \frac{\hbar}{m_2}\nabla_2 \cdot (\rho \nabla_2 S) = 0, \tag{8.48a}$$

whereas the dynamical law reads

$$\hbar\frac{\partial S}{\partial t} + \frac{\hbar^2}{2m_1}(\nabla_1 S)^2 + \frac{\hbar^2}{2m_2}(\nabla_2 S)^2 - \frac{\hbar^2}{2m_1}\frac{\nabla_1^2\sqrt{\rho}}{\sqrt{\rho}} - \frac{\hbar^2}{2m_2}\frac{\nabla_2^2\sqrt{\rho}}{\sqrt{\rho}} + V = 0. \tag{8.48b}$$

Comparison of Eq. (8.48a) with its one-particle version (8.5a), allows to identify the flow velocity associated to particle i ($i = 1, 2$) with

$$\boldsymbol{v}_i = \frac{\hbar}{m_i}\nabla_i S, \tag{8.49}$$

so that the continuity equation for the bipartite case reads

$$\frac{\partial \rho}{\partial t} + \nabla_1 \cdot (\rho \boldsymbol{v}_1) + \nabla_2 \cdot (\rho \boldsymbol{v}_2) = 0, \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{8.50}$$

where the last equation is written in the six-dimensional configuration space. On comparing Eq. (8.48b) with Eq. (8.5b) one obtains for the quantum potential V_{Qi} associated with particle i

$$V_{Qi} = -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m_i}\left[\frac{\nabla_i^2\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla_i \rho}{\rho}\right)^2\right]. \tag{8.51}$$

Equation (8.48b) takes thus the form

$$\hbar\frac{\partial S}{\partial t} + \frac{1}{2}m_1\boldsymbol{v}_1^2 + \frac{1}{2}m_2\boldsymbol{v}_2^2 + V_{Q1} + V_{Q2} + V = 0. \tag{8.52}$$

Application of the operator ∇_i to Eq. (8.52) gives, with the aid of (8.49) (unless explicitly stated, from now on we assume $i, j = 1, 2$ with $i \neq j$ when both indices appear in the same expression),

$$m_i \mathcal{D}_c^{(i)} \boldsymbol{v}_i = -\nabla_i V - \nabla_i \left(V_{Qi} + V_{Qj}\right) - \tfrac{1}{2}m_j \nabla_i \boldsymbol{v}_j^2, \tag{8.53}$$

where $\mathcal{D}_c^{(i)}$ stands for the co-moving derivative of particle i [see Eq. (8.13)],

$$\mathcal{D}_c^{(i)} = \frac{\partial}{\partial t} + (\boldsymbol{v}_i \cdot \nabla_i)\,. \tag{8.54}$$

In the Bohmian approach, where $\boldsymbol{v}_i = d\boldsymbol{x}_i/dt$, $\mathcal{D}_c^{(i)}\boldsymbol{v}_i$ equals the total derivative $d\boldsymbol{v}_i/dt$ and is taken as the actual acceleration of the particle describing the trajectory $\boldsymbol{x}_i(t)$. From the present perspective instead, according to which $\boldsymbol{v}_i$ stands for a (partially) averaged velocity, $\mathcal{D}_c^{(i)}\boldsymbol{v}_i$ constitutes a statistical acceleration characterizing the local flow. Hence $m_i\mathcal{D}_c^{(i)}\boldsymbol{v}_i$ stands for a (local mean) representative force, already much smoother than the actual (stochastic) one. Let us denote this force by $\boldsymbol{F}_i$; then Eq. (8.53) reads

$$\begin{aligned}\boldsymbol{F}_i &= -\boldsymbol{\nabla}_i V - \boldsymbol{\nabla}_i V_{Qi} - \boldsymbol{\nabla}_i V_{Qj} - \tfrac{1}{2}m_j\boldsymbol{\nabla}_i\boldsymbol{v}_j^2 \\ &= \boldsymbol{f}_i^{\mathrm{c}} + \boldsymbol{f}_{ii}^{\mathrm{q}} + \boldsymbol{f}_{ij}^{\mathrm{q}} + \boldsymbol{f}_{ij}^{\mathrm{f}}.\end{aligned} \tag{8.55}$$

The second equality identifies the different forces that contribute to $\boldsymbol{F}_i$. The first one, $\boldsymbol{f}_i^{\mathrm{c}} = -\boldsymbol{\nabla}_i V$, is the classical force due to the external potential V, whereas the three remaining terms

$$\begin{aligned}\boldsymbol{f}_{ii}^{\mathrm{q}} &= -\boldsymbol{\nabla}_i V_{Qi}, \\ \boldsymbol{f}_{ij}^{\mathrm{q}} &= -\boldsymbol{\nabla}_i V_{Qj}, \\ \boldsymbol{f}_{ij}^{\mathrm{f}} &= -\frac{1}{2}m_j\boldsymbol{\nabla}_i\boldsymbol{v}_j^2,\end{aligned} \tag{8.56}$$

are of kinetic origin. This is obvious for $\boldsymbol{f}_{ij}^{\mathrm{f}}$—which originates in the flux kinetic energy proportional to $\boldsymbol{v}_j^2$—and becomes clear for the 'quantum' forces $\boldsymbol{f}_{ii}^{\mathrm{q}}$ and $\boldsymbol{f}_{ij}^{\mathrm{q}}$ once the quantum potential (8.51) is written in the form

$$V_{Qi} = -\tfrac{1}{2}\left(m_i\boldsymbol{u}_i^2 + \hbar\boldsymbol{\nabla}_i\cdot\boldsymbol{u}_i\right), \tag{8.57}$$

with $\boldsymbol{u}_i$ the diffusive velocity associated wtih particle i,

$$\boldsymbol{u}_i = \frac{\hbar}{2m_i}\boldsymbol{\nabla}_i\ln\rho. \tag{8.58}$$

Equations (8.57) and (8.58) generalize the single-particle expressions (8.38) and (8.33), and show that both $\boldsymbol{f}_{ii}^{\mathrm{q}}$ and $\boldsymbol{f}_{ij}^{\mathrm{q}}$ are due to a diffusive velocity.

The fact that $\boldsymbol{f}_{ij}^{\mathrm{q}}$ and $\boldsymbol{f}_{ij}^{\mathrm{f}}$ represent forces that are exerted at $\boldsymbol{x}_i$ but originate in quantities (velocities) associated with an (arbitrarily distant) point $\boldsymbol{x}_j$, suggests the emergence of further nonlocal effects that add to the nonlocal features characteristic of the single-particle case (which ensued from the term $\boldsymbol{f}_{ii}^{\mathrm{q}}$). In the following we shall investigate some aspects of this new kind of nonlocality.

8.4.1 Nonlocality and Entanglement

For simplicity, in what follows we shall assume that each particle is restricted to a one-dimensional motion. The above equations thus give for the two contributions to F_i due to the presence of particle j,

$$f_{ij}^{\mathrm{q}} = -\frac{\partial}{\partial x_i} V_{Qj} = \left(m_j u_j + \frac{\hbar}{2} \frac{\partial}{\partial x_j} \right) \frac{\partial u_j}{\partial x_i}, \tag{8.59}$$

$$f_{ij}^{\mathrm{f}} = -m_j v_j \frac{\partial v_j}{\partial x_i}. \tag{8.60}$$

Clearly, a necessary condition for the force f_{ij}^{q} to exist is that $\partial_i u_j \neq 0$, and similarly for f_{ij}^{f}, that $\partial_i v_j \neq 0$. Let us determine the properties of those states $\psi(x_1, x_2, t)$ for which these conditions hold.

We start by writing the density $\rho(x_1, x_2, t)$ and the phase $S(x_1, x_2, t)$ of the wave function (8.47) in the general form

$$\rho(x_1, x_2, t) = r_1(x_1, t) r_2(x_2, t) r(x_1, x_2, t), \tag{8.61}$$

$$S(x_1, x_2, t) = s_1(x_1, t) + s_2(x_2, t) + s(x_1, x_2, t). \tag{8.62}$$

With this, and using the one-dimensional version of Eqs. (8.49) and (8.58), we arrive at

$$\frac{\partial u_j}{\partial x_i} = \frac{\hbar}{2m_j} \frac{\partial^2}{\partial x_j \partial x_i} \ln r(x_1, x_2, t), \tag{8.63a}$$

$$\frac{\partial v_j}{\partial x_i} = \frac{\hbar}{m_j} \frac{\partial^2}{\partial x_j \partial x_i} s(x_1, x_2, t). \tag{8.63b}$$

Notice that $m_j \partial_i u_j = m_i \partial_j u_i$ and $m_j \partial_i v_j = m_i \partial_j v_i$, two properties that will be (and had been) freely used without explicit mention. According to Eq. (8.63a), $\partial_i u_j = 0$ if and only if r has the form $r(x_1, x_2, t) = R_1(x_1, t) R_2(x_2, t)$, i.e., if and only if ρ factorizes as

$$\rho(x_1, x_2, t) = \rho_1(x_1, t) \rho_2(x_2, t), \tag{8.64}$$

with ρ_i the marginal distribution functions,

$$\rho_i(x_i, t) = \int \rho(x_i, x_j, t) dx_j. \tag{8.65}$$

Analogously, Eq. (8.63b) implies that $\partial_i v_j = 0$ if and only if s decomposes as $s(x_1, x_2, t) = \Sigma_1(x_1, t) + \Sigma_2(x_2, t)$, which in its turn means that S has the additive structure

$$S(x_1, x_2, t) = S_1(x_1, t) + S_2(x_2, t). \tag{8.66}$$

It is evident that with ρ and S given by (8.64) and (8.66), respectively, $\psi = \sqrt{\rho}\exp(iS)$ factorizes as

$$\psi(x_1, x_2, t) = \psi_1(x_1, t)\psi_2(x_2, t), \tag{8.67}$$

with $\psi_i(x_i, t) = \sqrt{\rho_i}\exp(iS_i)$. We are thus led to conclude that

$$\psi = \psi_i\psi_j \Longleftrightarrow \frac{\partial}{\partial x_i}u_j = 0 \text{ and } \frac{\partial}{\partial x_i}v_j = 0, \tag{8.68}$$

and consequently, that for a factorizable (separable) state the forces (8.59) and (8.60) exerted at x_i due to the presence of particle j vanish.

On the other hand, with $\partial_i u_j = 0$, V_{Qi} reduces to

$$V_{Qi} = -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2\sqrt{\rho_i}}{\sqrt{\rho_i}} = V_{Qi}(x_i, t). \tag{8.69}$$

If in addition $\partial_i v_j = 0$, consistency with Eq. (8.52) demands the external potential to be of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$. This shows that a separable state is a consistent solution only for a system of noninteracting particles, and allows us to write F_i as

$$F_i(x_i, t) = -\frac{\partial}{\partial x_i}V_i(x_i) - \frac{\partial}{\partial x_i}V_{Qi}(x_i, t). \tag{8.70}$$

Therefore, whenever ψ factorizes as in (8.67), no force arises in the composite system additional to those found in the single-particle case. In other words, in a factorizable state the dynamics corresponds to that of a couple of independent particles, each following its own laws. The nonlocalities are, then, those that correspond to single-particle systems.

However, if at least one of the conditions for the velocities in (8.68) fails, the wave function can no longer be factorized and hence it describes a nonseparable or entangled state. According to the statement following Eq. (8.60), only in this case the forces f_{ij}^{q} and f_{ij}^{f} may be different from zero. In other words, f_{ij}^{q} and f_{ij}^{f} are conditioned by the existence of a nonfactorizable ρ or a nonfactorizable $\exp(iS)$, respectively. This serves to identify the separate physical effects of the magnitude and the phase of the wave function on the dynamics of the bipartite system. The entanglement may be encoded either in the amplitude of ψ (if $\partial_i u_j \neq 0$), or in its phase (if $\partial_i v_j \neq 0$), or in both; occasionally we will refer to amplitude entanglement or phase-entanglement, respectively. The fact that either u_i, or v_i, or both, may depend on x_i *and* x_j, precludes the possibility of determining such velocities by focusing on the subsystem i only. Instead, these dynamical variables pertain to the bipartite system as a whole; they are associated with one of the subsystems but cannot

be defined separately from the second one. We will come back to this point in Sect. 8.4.3.

Equations (8.61) and (8.62) allow us to write (for simplicity in what follows we omit the time dependence)

$$v_i = \frac{\hbar}{m_i}\left[\frac{\partial s_i(x_i)}{\partial x_i} + \frac{\partial s(x_1, x_2)}{\partial x_i}\right], \tag{8.71a}$$

$$u_i = \frac{\hbar}{2m_i}\left[\frac{\partial \ln r_i(x_i)}{\partial x_i} + \frac{\partial \ln r(x_1, x_2)}{\partial x_i}\right]. \tag{8.71b}$$

These expressions display the extra contributions to v_i and u_i that are exclusively due to entanglement, and which introduce the two-point dependence. Whenever there is amplitude-entanglement ($\partial_i \ln r \neq 0$) the quantum potential (8.57) takes the form $V_{Q_i} = V_{Q_i}(x_1, x_2)$, the (total) quantum potential $V_Q = V_{Q_1} + V_{Q_2}$ acquires an additional contribution that formally plays the role of an interaction potential, and a generally nonzero term $f_{ij}^{\mathrm{q}} = -\partial_i V_{Qj}$ depending on both x_1 *and* x_2 adds to F_i, which becomes a function of the form $F_i(x_1, x_2, t)$.[22] Analogouity, when there is phase entanglement ($\partial_i s \neq 0$), the flux kinetic energy associated with particle j affects the particle located at x_i, and a force f_{ij}^{f} results which also depends on the position of both particles, leading again to a total force $F_i(x_1, x_2, t)$. While giving rise to the forces f_{ij}^{f} and f_{ij}^{q}, the entanglement may also modify the force $f_{ii}^{\mathrm{q}} = -\partial_i V_{Qi}$, which for a nonfactorizable ρ will depend in general on the position variables of both particles.

The ensuing two-point dependence of F_i brings out naturally the notion of nonlocal effects, particularly when there is no (external) interaction between the particles. This is the case, for example, if the particles interacted (got entangled) in the past, or if the system is composed of identical noninteracting parties in a state described by the superposition

$$\psi(x_1, x_2, t) = \tfrac{1}{\sqrt{2}}\left[\phi_n(x_1, t)\phi_m(x_2, t) \pm \phi_m(x_1, t)\phi_n(x_2, t)\right], \tag{8.72}$$

despite the fact that $V = V_1 + V_2$. In the absence of an external interaction potential, the issue of nonlocal effects due to entanglement has become so befogged that even 'spooky' actions at a distance—of unexplained physical origin, of course—have been invoked. Nevertheless, considering for example that the Bohm particle that follows the trajectory $x_i(t)$ with a velocity $v_i = dx_i/dt$ is not the real, physical particle, but rather a representative particle that at each point reproduces the statistical dynamics of the appropriate subensemble, the spooky-action problem disappears. The nonlocalities due to the two-point dependence of v_i are not the result of a direct

[22] The effective interaction potential introduced via $V_Q(\boldsymbol{x}_1, \boldsymbol{x}_2)$, which remains 'hidden' in the depths of the Schrödinger equation, formally transforms the original noninteracting system into an interacting one [see Eq. (8.52)]. By contrast, the possible nonfactorizability of $\exp(iS)$ does not manifest itself as a formal interaction potential in Eq. (8.52). The nonlocal effect of this kind of entanglement is manifested when a description in terms of forces is made, as we have seen.

physical action between the real particles, but only an imprint (in configuration space) of the presence of correlated fluxes. In the identical-particle case, as follows from the results of Chap. 7, it is clear that such correlations are rooted in the coupling of both particles through common modes of the field. Of course, a similar mechanism takes place, although less effectively, for nonidentical particles provided they have common relevant frequencies.

It is clear from the above results that for a noninteracting system ($V = V_1 + V_2$), the force F_i depends on both x_i and x_j only if at least one of the conditions (8.68) fails. In such case the correlations that ensue from the two-point dependence of F_i are ascribed to the entanglement of ψ. In particular, correlations between the diffusive and flux velocities denote entanglement. To see this, observe that given a function $h(x_1, x_2)$, an integration by parts leads to

$$\left\langle \frac{\partial h}{\partial x_i} \right\rangle = \int \frac{\partial h}{\partial x_i} \rho dx_1 dx_2 = -\int h \frac{\partial \rho}{\partial x_i} dx_1 dx_2 = -\frac{2m_i}{\hbar} \langle h u_i \rangle , \tag{8.73}$$

under the assumption that $h\rho$ vanishes at infinity. This result is particularly useful when h is one of the velocities v_j or u_j, since in such case, according to Eq. (8.68) a nonzero value of any of the covariances[23]

$$\langle u_i u_j \rangle , \quad \langle u_i v_j \rangle \tag{8.74}$$

implies entanglement of the state ψ. Moreover, depending on which of the covariances is nonzero, one can determine whether there is entanglement encoded in either the modulus of the wave function ($\langle u_i u_j \rangle \neq 0$), or its phase ($\langle u_i v_j \rangle \neq 0$), or both.

Quantum mechanics does not recognize in u a quantity with a particular physical meaning (let alone a velocity with a diffusive connotation!), and something similar goes frequently for the flux velocity v. Yet the above results, particularly (8.68), indicate that these velocities play a significant role not only for the understanding of several properties of single-particle quantum systems (as follows, for example, from Chap. 4), but also in the bipartite case in connection with entanglement. In the following sections the velocities v and u are used for an analysis of certain aspects of entanglement and nonlocality. Even though some of the conclusions are well known, the method used to reach them is not, and this may help to get a fresh look at them.

8.4.2 Momentum Correlations

The entry point for nonlocality in the present description has been the two-point dependence of u_i and v_i. Now, these velocities are connected with the momentum

[23] We use here the term covariance to refer to a two-point momentum $\langle \hat{F}_1 \hat{G}_2 \rangle$, even if the product $\langle \hat{F}_1 \rangle \langle \hat{G}_2 \rangle$ differs from zero. In the literature the term 'correlation' is frequently used for $\langle \hat{F}_1 \hat{G}_2 \rangle$, so we use it here when convenient.

operator $\hat{p}_i$ by means of Eq. (8.32), which in the one-dimensional case reads

$$\hat{p}_i\psi = -i\hbar\frac{\partial}{\partial x_i}\psi = m_i(v_i - iu_i)\psi = \pi_i\psi, \tag{8.75}$$

where π_i stands for the complex variable [cf. Eq. (4.130)]

$$\pi_i = m_i(v_i - iu_i). \tag{8.76}$$

This suggests that the correlations (8.74) may be contained in the expectation value of $\hat{p}_i\hat{p}_j$. Applying the operator $\hat{p}_j$ to Eq. (8.75) one obtains

$$\hat{p}_j\hat{p}_i\psi = \left(\pi_j\pi_i - i\hbar\partial_j\pi_i\right)\psi. \tag{8.77}$$

The properties $m_j\partial_i u_j = m_i\partial_j u_i$ and $m_j\partial_i v_j = m_i\partial_j v_i$ imply that $\partial_i\pi_j = \partial_j\pi_i$, whence Eq. (8.77) is symmetrical in i, j, as expected. Multiplying this equation from the left by ψ^* and integrating leads to

$$\left\langle\hat{p}_i\hat{p}_j\right\rangle = m_i m_j\left\langle v_i v_j + u_i u_j\right\rangle. \tag{8.78}$$

So even though the imaginary part of π_i, u_i, does not contribute to the mean value of $\hat{p}_i$, it plays a central role in higher-order moments. This has already been pointed out when calculating $\left\langle\hat{p}_i^2\right\rangle$ [see also Eq. (8.40), or Sect. 4.5.2],

$$\left\langle\hat{p}_i^2\right\rangle = m_i^2\left\langle v_i^2 + u_i^2\right\rangle. \tag{8.79}$$

As discussed in Sect. 8.3.3, a nonzero value of $\left\langle u_i^2\right\rangle$ indicates the presence of irreducible momentum fluctuations; on the other hand, according to the discussion following Eq. (8.73), a nonzero value of $\left\langle u_i u_j\right\rangle$ reflects entanglement and hence a nonzero correlation $\left\langle\hat{p}_i\hat{p}_j\right\rangle$. Thus, the same physical entity, namely the velocity u, bears information about two of the most characteristic quantum features, which now appear as intimately related.[24]

Coming back to Eq. (8.78), the fact that for a factorizable ψ the mean value $\langle u_1 u_2\rangle$ vanishes implies that any discrepancy between $\left\langle\hat{p}_i\hat{p}_j\right\rangle$ and $m_i m_j\left\langle v_i v_j\right\rangle$ is due to entanglement, specifically due to the nonfactorizability of ρ. Moreover, (8.78) leads to

$$\frac{1}{m_i m_j}\left|\left\langle\hat{p}_i\hat{p}_j\right\rangle\right| \leq \left|\left\langle v_i v_j\right\rangle\right| + \left|\left\langle u_i u_j\right\rangle\right|, \tag{8.80}$$

which shows that a nonfactorizable ρ increases the upper limit of $\left|\left\langle\hat{p}_i\hat{p}_j\right\rangle\right|$. This result is in line with one of the main conclusions related with the violation of Bell's

[24] From this perspective, the conclusions reached regarding the dispersive and nonlocal features of the quantum potential (a quantity that depends on u only) become evident.

inequalitites, namely that entanglement introduces extra contributions to the correlations, with no classical analogue. Here the source of such additional terms are fluctuation-related velocities.

Before ending this section, it seems in place to add that the correspondence

$$\hat{p}_i \rightarrow \pi_i = m_i \left(v_i - i u_i\right) \tag{8.81}$$

discussed in Sect. 4.5.2, besides establishing the equivalence between the average of the quantum operator $\hat{p}_i$ and the average of the complex scalar function π_i, works also for the mean value $\left\langle \hat{p}_i \hat{p}_j \right\rangle$. Indeed, direct calculation shows that the covariance of π_i and π_j, defined as

$$C(\pi_i, \pi_j) = \tfrac{1}{2} \left\langle \pi_i^* \pi_j + \pi_i \pi_j^* \right\rangle, \tag{8.82}$$

is just Eq. (8.78), that is,

$$C(\pi_i, \pi_j) = \left\langle \hat{p}_i \hat{p}_j \right\rangle = m_i m_j \left\langle v_i v_j + u_i u_j \right\rangle. \tag{8.83}$$

This shows that the local mean value of $p_i p_j$ coincides, up to a term that averages to zero, with $m_i m_j \left(v_i v_j + u_i u_j\right)$.

8.4.3 The Whole and the Parts

The procedure that led to Eq. (8.77), introduced in Sect. 4.5.2, can be used to write the mean value of the operator $\hat{p}_i^n$ as an average of a scalar function, in the form

$$\left\langle \hat{p}_i^n \right\rangle = \left\langle p_i^n \right\rangle = \int P_i(\pi_i, \partial_i^{k_n} \pi_i) \rho(x_1, x_2) dx_1 dx_2, \tag{8.84}$$

where $P_i(\pi_i, \partial_i^{k_n} \pi_i)$, a real function of π_i and its first n derivatives ($k_n = 0, 1, \ldots n$), plays the role of the local—at point (x_1, x_2)—mean value of $\hat{p}_i^n$. It follows from Eq. (8.84) that for any operator $g_i(\hat{p}_i)$ that can be expanded as a power series of its argument, the quantum average $\left\langle g_i(\hat{p}_i) \right\rangle$ can also be obtained by averaging a c-number $G_i(\pi_i, \partial_i^k \pi_i)$, the local mean value of the variable $g_i(p_i)$, as

$$\left\langle g_i(\hat{p}_i) \right\rangle = \langle g_i(p_i) \rangle = \int G_i(\pi_i, \partial_i^{k_n} \pi_i) \rho(x_1, x_2) dx_1 dx_2. \tag{8.85}$$

It is clear that if $\pi_i = \pi_i(x_1, x_2)$, G_i will in general be also a two-point function; consequently any dynamical variable defined through it will in general display nonlocal features.

According to Eq. (8.68), if the state is factorizable $\partial_j \pi_i = 0$, i.e. $\pi_i = \pi_i(x_i)$ and Eq. (8.85) reduces to

$$\left\langle g_i(\hat{p}_i) \right\rangle = \langle g_i(p_i) \rangle = \int G_i(\pi_i, \partial_i^{k_n} \pi_i) \rho_i dx_i, \tag{8.86}$$

that is, an average weighted with the marginal probability distribution $\rho_i(x_i)$. For (8.85) to reduce to (8.86) irrespective of $g_i(\hat{p}_i)$, necessarily $\partial_j \pi_i = 0$, which means that the state is nonentangled. Therefore, the mean value of an arbitrary $g_i(\hat{p}_i)$ is just the average of the local mean value of $g_i(p_i)$ at point x_i weighted with the marginal distribution $\rho_i(x_i)$, if and only if the state is nonentangled. Otherwise stated, all the information required to determine $\left\langle g_i(\hat{p}_i) \right\rangle$ can be obtained from observations on system i *only*, if and only if the state is separable.

If, on the contrary, the state ψ is an entangled one, then the dependence of G_i on x_j precludes the possibility of expressing $\left\langle g_i(\hat{p}_i) \right\rangle$ in the form (8.86). In this case, determining the mean value of an arbitrary variable $g_i(p_i)$ of *one* of the particles requires information about the *whole* system; in particular, the marginal distribution ρ_i is not enough and we must resort to the joint distribution function $\rho(x_1, x_2)$.

The above observations provide an alternative way of looking at nonlocality in composite systems without the notion of action at a distance: *nonlocality, as a property encoded in the entanglement of the state, reflects the impossibility of considering each of the constituents of the system separately one from another, i.e., as nonseparability*. It forces us to consider the system as a whole, rather than as composed of two separate parts, well within the spirit of Bohm's interpretation. By considering the presence of the ZPF, as was done in Chap. 7, this point of view is the single natural one: there is a unique system, composed of field and particles, with certain field modes playing a correlating function between particles.

8.4.4 Nonlocality and Noncommutativity

At variance with what occurred with $\langle g_i(\hat{p}_i) \rangle$, the mean value of a function of the position operator only, $f_i(\hat{x}_i)$, is *blind* to the nonfactorizability of $\psi(x_1, x_2, t)$. This follows from the fact that any $f_i(\hat{x}_i)$ has a local mean value that is just the function $f_i(x_i)$, and hence the average

$$\left\langle f_i(\hat{x}_i) \right\rangle = \langle f_i(x_i) \rangle = \int f_i(x_i) \rho(x_1, x_2) dx_1 dx_2 = \int f_i(x_i) \rho_i(x_i) dx_i \tag{8.87}$$

does not exhibit nonlocal features, irrespective of the state. Therefore, a variable A_i proper to a *single* particle exhibits nonlocality only when such variable is momentum-dependent.[25]

[25] When the operator $\hat{A}$ does not correspond to a *single* particle, this statement ceases to be true. For example, for $\hat{A} = A_1(\hat{x}_1) A_2(\hat{x}_2)$, the entanglement is revealed in the covariance $\langle A_1 A_2 \rangle$ even

This particular asymmetry between momentum and position variables is due to the fact that the description is carried out in the configuration-space representation. If the momentum representation is used instead, with

$$\psi_p(p_1, p_2, t) = \sqrt{\rho_p(p_1, p_2, t)}e^{iS_p(p_1,p_2,t)}, \tag{8.88}$$

the p-local mean value associated with $g_i(\hat{p}_i)$ becomes the real variable $g_i(p_i)$,[26] whereas the p-local mean value for $f_i(\hat{x}_i)$ is defined, in analogy with (8.75), via the equation

$$\hat{x}_i\psi_p = i\hbar\frac{\partial}{\partial p_i}\psi_p = (\xi_i - i\zeta_i)\psi_p = \chi_i\psi_p, \tag{8.89}$$

where $\xi_i = \hbar(\partial S_p/\partial p_i)$ and $\zeta_i = (\hbar/2)(\partial \ln \rho_p/\partial p_i)$. An analysis entirely analogous to the one carried out above leads to conclude that the complex function χ_i, which now plays the role of the previous π_i, depends on both momenta p_1 and p_2 if and only if ψ_p does *not* factorize as $\psi_p = \psi_1(p_1, t)\psi_2(p_2, t)$. Under these circumstances, conclusions entirely similar to the previous ones apply, *mutatis mutandi*: nonlocal effects in momentum space are manifested in connection with the p-local mean value associated with $f(\hat{x}_i)$.

The above observations lead us to assert that once a representation in terms of the eigenvalues of the operator $\hat{x}_i$ is chosen to describe the evolution of the system, the nonlocal features become manifest through those variables $\hat{A}_i$ that are functions of the corresponding noncommuting operator $\hat{p}_i$, and vice versa. Thus, considering that (for a spinless system) any dynamical variable is a function of the fundamental variables x and p, it follows that the x-local mean value of A_i will exhibit nonlocal features when $[\hat{x}_i, \hat{A}_i] \neq 0$, whereas its p-local mean value will exhibit nonlocal features when $[\hat{p}_i, \hat{A}_i] \neq 0$.[27]

The requirement of noncommutativity of operators for the disclosure of nonlocality can alternatively be shown as follows. Let $\{|\alpha\beta\rangle \equiv |\alpha\rangle_i \otimes |\beta\rangle_j\}$ be an orthonormal

(Footnote 25 continued)
though none of the variables is momentum-dependent. In fact, the point here is to show that the present approach allows to reach conclusions about entanglement by focusing on single-particle variables, rather than on correlations between variables of the two subsystems, as is customarily done [see discussion following Eq. (8.96)].

[26] The p-local mean value of a dynamic variable g is defined, in analogy with Eq. (4.50), as its partial average over the configuration space, using the distribution Q,

$$\langle g\rangle(p_1, p_2) = \langle g\rangle_p = \frac{1}{\rho_p}\int gQ(x_1, x_2, p_1, p_2)dx_1dx_2.$$

[27] The fact that the kind of variables that may exhibit nonlocality is representation-dependent does not mean that the very existence of nonlocality is representation-dependent. Indeed, for any entangled state $|\psi\rangle$ there will always be some variable exhibiting nonlocal features; which one depends on the representation used to project $|\psi\rangle$.

basis of the product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. One may write the expectation value of $\hat{A}_i$ in the pure state $|\psi\rangle$ as

$$\begin{aligned}\langle\psi|\,\hat{A}_i\,|\psi\rangle &= \int \langle\alpha\beta|\,\psi\rangle\,\langle\psi|\,\alpha'\beta'\rangle\langle\alpha'\beta'|\,\hat{A}_i\,|\alpha\beta\rangle\,d\alpha d\beta d\alpha' d\beta' \\ &= \int \langle\alpha\beta|\,\psi\rangle\,\langle\psi|\,\alpha'\beta\rangle\langle\alpha'|\,\hat{A}_i\,|\alpha\rangle\,d\alpha d\beta d\alpha'. \end{aligned} \tag{8.90}$$

If the basis $\{|\alpha\rangle_i\}$ is selected such that its elements are the eigenvectors of an operator $\hat{a}_i$ satisfying $[\hat{a}_i, \hat{A}_i] = 0$, then $\langle\alpha'|\,\hat{A}_i\,|\alpha\rangle \sim \delta(\alpha - \alpha')$, and (8.90) reduces to

$$\langle\psi|\,\hat{A}_i\,|\psi\rangle = \int \rho\,(\alpha, \beta)\,\langle\alpha|\,\hat{A}_i\,|\alpha\rangle\,d\alpha d\beta, \tag{8.91}$$

where $\rho\,(\alpha, \beta) = |\psi\,(\alpha, \beta)|^2$ and $\psi(\alpha, \beta) = \langle\alpha\beta|\,\psi\rangle$ is the wave function in the (α, β)-representation. Integration over β of the joint probability $\rho\,(\alpha, \beta)$ gives the marginal probability $\rho_i(\alpha)$, so that Eq. (8.91) simplifies into

$$\left\langle \hat{A}_i \right\rangle = \int \langle\alpha|\,\hat{A}_i\,|\alpha\rangle\,\rho_i(\alpha)d\alpha. \tag{8.92}$$

Equation (8.92) is usually read as saying that when projective measurements are performed corresponding to a physical variable a such that $\hat{a}_i\,|\alpha\rangle = \alpha\,|\alpha\rangle$ with $[\hat{a}_i, \hat{A}_i] = 0$, $\langle\hat{A}_i\rangle$ can be obtained from the sole (local) inspection of system i, and any nonlocality due to the possible entanglement of $\psi(\alpha, \beta)$ remains hidden.[28] By contrast, if a different basis $\{|\gamma\rangle\}$ is chosen for the representation, such that $[\hat{\gamma}_i, \hat{A}_i] \neq 0$, the above reduction cannot be made. Now, it is always possible to find a basis $\{|\alpha\rangle_1\}$ in which $\hat{A}_1$, say, is diagonal; but then the basis in which $\hat{B}_2$ is diagonal will not be $\{|\alpha\rangle_2\}$ unless $[\hat{A}_1, \hat{B}_1] = 0$; therefore, entanglement shows up only when dynamical variables $\hat{A}_1$, $\hat{B}_2$ are considered such that $[\hat{A}_i, \hat{B}_i] \neq 0$.[29,30]

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[28] Equation (8.87) is just Eq. (8.92) with $\hat{A}_i = f(\hat{x}_i)$, yet Eq. (8.86) differs from the structure of (8.92). To see this consider in particular Eq. (8.84) with $n = 1$; then $P_i(\pi_i, \partial_i^k \pi_i) = m_i v_i(x_i) \neq \langle x_i|\,\hat{p}_i\,|x_i\rangle$, so indeed (8.86) is not Eq. (8.92) for $\hat{A} = \hat{g}_i$ and $\alpha = x_i$.

[29] Notice that the use of a fixed representation for both elements of the composite system, i.e. $\{|\alpha\rangle_1\}$, $\{|\beta\rangle_2\}$, is a matter of necessity when discussing entanglement. The same applies when considering measurements on a system. In fact, given an (α, β)-representation, the distribution function $\rho\,(\alpha, \beta)$ is defined as the joint probability density that determines the probability of obtaining the values α and β when performing the projective measurements corresponding to the proyectors $\Pi^{\alpha}_{A_1} = |\alpha\rangle\,\langle\alpha| \in \mathcal{H}_1$ and $\Pi^{\beta}_{B_2} = |\beta\rangle\,\langle\beta| \in \mathcal{H}_2$, respectively. Thus the representation used is linked with the variables that are measured in a certain experiment.

[30] This conclusion is in line with the results obtained in Sect. 7.2.5. Specifically, the discussion following Eq. (7.63) tells us that for entanglement to become manifest through a correlation, both dynamical variables involved (i.e., F, G, the equivalent of A_1, B_2 in the present case) must have nondiagonal elements in a given representation (the energy representation, in that case).

The results obtained above disclose the tight relation between noncommutativity and nonlocality that is well recognized in the literature (see e.g. Tsirelson 1980, Landau 1987, Revzen et al. 1997), though generally limited to the context of dichotomic operators. In particular, Landau shows that given two operators $\hat{a}$ and $\hat{A}$ in $\mathcal{H}_1$, and two operators $\hat{b}$ and $\hat{B}$ in $\mathcal{H}_2$, the (CHSH) operator defined as

$$\begin{aligned}\hat{C} &= \hat{a}\hat{b} + \hat{a}\hat{B} + \hat{A}\hat{b} - \hat{A}\hat{B} \\ &= \hat{a}(\hat{b} + \hat{B}) + \hat{A}(\hat{b} - \hat{B}),\end{aligned} \tag{8.93}$$

satisfies the following inequality,[31]

$$\begin{aligned}\langle\hat{C}\rangle^2 \leq \left\langle \left(\hat{a}^2 + \hat{A}^2\right)(\hat{b}^2 + \hat{B}^2)\right\rangle + \left\langle \{\hat{b}, \hat{B}\}\left(\hat{a}^2 - \hat{A}^2\right)\right\rangle \\ + \left\langle \{\hat{a}, \hat{A}\}(\hat{b}^2 - \hat{B}^2)\right\rangle + \left\langle \left[\hat{a}, \hat{A}\right][\hat{B}, \hat{b}]\right\rangle.\end{aligned} \tag{8.94}$$

This result ensues exclusively from the fact that $\sigma_C^2 = \langle\hat{C}^2\rangle - \langle\hat{C}\rangle^2 \geq 0$. Landau, in line with the usual treatments, considered operators $\hat{a}$, $\hat{A}$, $\hat{b}$ and $\hat{B}$ such that

$$\hat{a}^2 = \hat{A}^2 = \mathbb{I}_1, \quad \hat{b}^2 = \hat{B}^2 = \mathbb{I}_2 \tag{8.95}$$

(the typical example being the Pauli matrices). In such case Eq. (8.94) reduces to

$$\langle\hat{C}\rangle^2 \leq 4 + \left\langle \left[\hat{a}, \hat{A}\right][\hat{B}, \hat{b}]\right\rangle, \tag{8.96}$$

an expression that exhibits the significant role of (the covariance of) the commutators in determining the maximum possible value of $\langle\hat{C}\rangle^2$. For commuting $\hat{a}$ and $\hat{A}$, or $\hat{b}$ and $\hat{B}$, the inequality gives $\left|\langle\hat{C}\rangle\right| \leq 2$, which is the limit established in Bell's theorem (see e.g. Bell 1966, 1987). Thus, according to Eq. (8.96), the noncommutativity of the operators involved is a necessary condition for the violation of Bell's inequalities (in the form of the CHSH inequality, Clauser et al. 1969), as well as the nonnull correlation between both commutators.

This observation that noncommutativity and covariance unequivocally signal the nonlocal feature of a state goes nicely with our previous exposition. Our conclusions above were drawn by focusing not on covariances between operators of the form $\hat{F}_i\hat{G}_j$, but on the single-particle operators $\hat{A}_i$, thus showing that some aspects of

[31] Note that, as already remarked in connection with von Neumann's theorem [see Eq. (8.2)], the equality

$$\langle\hat{C}\rangle = \langle\hat{a}\hat{b}\rangle + \langle\hat{a}\hat{B}\rangle + \langle\hat{A}\hat{b}\rangle - \langle\hat{A}\hat{B}\rangle$$

does not hold in general if the operators in the terms of the sum do not commute. This important restriction needs to be borne in mind when attempting to apply (8.94) [or (8.96)] to draw conclusions about correlations. See e.g. Accardi (1984).

nonlocality can be unveiled without resorting to nonlocal (i.e. i and j-dependent) operators. In addition, our exposition applies to continuous-variable systems, contrary to the more usual approach that focuses on dichotomic variables, such as spin projections, when discussing these matters. The natural question arises as to why noncommutativity is required for entanglement (or nonlocality) to become evident. To give an answer, let us put $[\hat{a}, \hat{A}] = i\hat{F}_1$, and $[\hat{B}, \hat{b}] = -i\hat{G}_2$, with $\hat{F}_1$ and $\hat{G}_2$ two Hermitian operators. Equation (8.96) thus reads

$$\langle \hat{C} \rangle^2 \leq 4 + \langle \hat{F}_1 \hat{G}_2 \rangle, \tag{8.97}$$

and the inequality now states that $\left| \langle \hat{C} \rangle \right|$ will exceed the value 2 only if the variables $\hat{F}_1$ and $\hat{G}_2$ are correlated. If one of the commutators in (8.96) vanishes, then $\langle \hat{F}_1 \hat{G}_2 \rangle$ is trivially zero, no information at all is obtained regarding any possible correlation between the systems, and no conclusion can be drawn about entanglement. In other words, (non)commutativity *per se* says nothing about nonlocality; it is required merely as a useful way to specify which are the operators that may give evidence of entanglement, via the correlation between $\hat{F}_1$ and $\hat{G}_2$. As stated above, in Chap. 7 a similar definition of the appropriate operators was made, by specifying that only those variables F_1 and G_2 that share relevant frequencies become correlated in such a way as to disclose entanglement [see the discussion following Eq. (7.63)].

8.5 Final Remarks

Nonlocality is a weird trait of quantum mechanics that is considered to have been revealed (and has been made popular) by the Bell inequalities. Bohm, on his side, used it to develop an entire line of philosophy of nature based on a holistic picture of the world. As a result, dominant voices today affirm that 'Nature *is* nonlocal'. But, is it? Or is it merely the (quantum) *description* that is nonlocal? There are of course physicists (and even a few philosophers of science, see e.g. Brown and Harré 1988) who cannot accept nonlocality as a trait of fundamental science, arguing that it is an artifact of the formalism and our reading of it. The derivations presented in previous chapters and lines above, add their own share.

Let us briefly elaborate on this point. There exist today several derivations of the Bell inequalities; one that has become standard with time started with the famous paper by Clauser et al. (1969) cited above and based on the expression (8.93). A careful consideration of the known derivations allows one to verify that the Bell inequalities are merely statistical relations: no physics is involved in their derivation, just as no physics is involved in Eq. (8.96). It suffices to take the mean value of (8.93) for a series of trials[32] and determine its bounds, to arrive at the CHSH inequality.

[32] With an eye put on note (31), making sure that the average is taken over the same distribution in each term.

In short, the Bell (or CHSH) inequalities by themselves say nothing about Nature. The physics enters when the theorem is applied to a given physical system. Any experimental violation means that at least some postulate used for the derivation, is not satisfied by that system. It is usual to blame the physical demand of locality for the violation of the inequalities.

Let us consider the case of a pair of noninteracting particles, discussed in Chap. 7. The results obtained there indicate that the nonlocality of the description—which is made evident in the case of an entangled state—arises from the neglect of the ZPF.[33] Since quantum mechanics lacks of a fundamental explanation for the origin of entanglement—it is the result of a basic postulate—, the physics behind the associated nonlocality remains hidden. Even if our results on this problem are still limited, our analysis suggests there is a real chance that a more refined description—in phase space, for example— would allow to recover locality.

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[33] The extra correlations that lead to the violation of a Bell inequality exist also in the case of photons, due to the correlations between the excitations of the field and the corresponding modes of the ZPF; see e.g. Casado et al. (1998).

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