Chapter 6
Beyond the Schrödinger Equation

The lack of a concrete picture [for the electron spin] is the most satisfactory state of affairs.
W. Pauli,quoted in van der Waerden (1960) .

... I should like to preserve this ideal of the past, to describe everything that happens in the world with distinct images. I am ready to accept other theories, on condition that one is able to re-express them in terms of clear and distinct images.
H. A. Lorentz, quoted in Bacciagaluppi and Valentini 2009.

In previous two chapters, the quantum behavior of matter has been shown to emerge as a result of the permanent interaction with the random zero-point field. Fundamental quantum results, such as the Schrödinger and the Heisenberg formalism, have emerged within this framework. In this chapter, the theory developed so far will take us beyond the realm of quantum mechanics, in two important directions.

On the one hand, arriving at the Schrödinger (or Heisenberg) description meant neglecting the radiative terms—the radiation reaction and the Lorentz force due to the fluctuating vacuum field—once they had played their main role in taking the system to the quantum regime. Due consideration of these neglected terms will now allow us to calculate the effects that they produce on the already quantized system. Specifically, the (nonrelativistic) formulas of quantum electrodynamics for the atomic lifetimes and the Lamb shift will thus be recovered. More generally, the mean evolution of any integral of motion under a breakdown of the balance equations will be analyzed.

Another most relevant quantum phenomenon that cannot be predicted by the Schrödinger theory is the spin of the electron. The second part of this chapter is therefore devoted to an inquiry about the genesis of the electron spin from the perspective of the present treatment. Just as it gives rise to position, momentum and energy fluctuations, the ZPF is seen to induce an angular momentum resulting from the instantaneous torque exerted by the Lorentz force on the particle. A close analysis based on the separation of the modes of the ZPF of given circular polarization reveals the existence of a spin angular momentum of value $\hbar/2$, as well as of a corresponding magnetic moment with a g-factor of value 2, associated with the particle. This leads

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us to the identification of the spin of the electron as a further emergent property, generated by the interaction of the particle with the ZPF.

6.1 Radiative Corrections. Contact with QED

It is today widely accepted that the fluctuations of the electromagnetic vacuum are responsible for important observable physical phenomena that pertain to the realm of quantum electrodynamics. Among their best known manifestations are the finite atomic lifetimes. Indeed, the vacuum fluctuations are known to contribute, along with radiation reaction, to the 'spontaneous' transitions of the excited states (see e.g. Davydov 1965; Dalibard et al. 1982; Milonni 1994). Moreover, both the atomic Lamb shift and the Casimir and van der Waals forces have been shown to be attributable to changes in the energy of the vacuum field due to the presence of matter (see e.g. Boyer 1968, 1969; Milonni 1994; Bordag et al. 2009).

Within the traditional framework of QM, the analysis of the radiative corrections implies introducing by hand the quantized electromagnetic field (including its vacuum component) and using perturbative methods for the calculation of its effects. In SED, by contrast, the field—both radiation reaction and the ZPF—is there from the very beginning. It is in fact an essential ingredient in any quantum system: both the Schrödinger and the Heisenberg description have been obtained by considering its influence to zero order in τ (or e^2). Additional effects of this field on matter show up explicitly in the equations for the averaged dynamical variables that ensue from the generalized Fokker-Planck equation (see Sect. 4.2.1). It is therefore pertinent to investigate how these neglected radiative terms can be brought back into the picture in a self-consistent approach, and to calculate the effects of such terms on the quantum-mechanical system.

Strictly speaking, in order to find the *exact* solution of the problem of the particle subject to the radiation field one should revert to the original SED Hamiltonian equations of motion, and study the evolution of the complete (particle plus field) system into the quantum regime, without making approximations along the way. This would indeed be the ideal way to proceed. However, as stated in Chap. 4, solving this problem is beyond present possibilities. The practical solution, therefore, is to use the present, more restricted approach: let the full system evolve into the quantum regime, take the corresponding (Schrödinger) zero-order solutions, and use them to calculate the effects of the radiative terms contained in the original equations. This procedure leads to closed formulas for the most important radiative corrections to the already quantized system, to lowest significative order in the fine-structure constant $\alpha = e^2/\hbar c$. The results obtained do not show any difference with respect to the (nonrelativistic) QED predictions to the same order of approximation. However, in contrast to QED where these corrections represent the *main* effects due to the vacuum fluctuations, in SED they represent *secondary* effects, since the central effect of the action of the vacuum radiation field is quantum mechanics itself.

The calculation of radiative corrections has constituted a central component of the program of SED. A number of significant results have been thus obtained in the past, although with varying degrees of success and mostly restricted to the harmonic oscillator.[1] The derivations presented in this chapter are more general and therefore applicable to the atomic case, where they lead to formulas that are directly comparable to those of QED. [2]

6.1.1 Radiative Transitions

In Chap. 4 we found that in the time-asymptotic, Markovian regime, the evolution of the mean value of the particle Hamiltonian $H(\boldsymbol{p},\boldsymbol{x}) = (1/2m)\boldsymbol{p}^2 + V(\boldsymbol{x})$ is determined by the Eq. (4.32), namely

$$\frac{d}{dt}\langle H\rangle = \tau\langle \dddot{\boldsymbol{x}}\cdot\boldsymbol{p}\rangle - \frac{e^2}{m}\left\langle \boldsymbol{p}\cdot\hat{\boldsymbol{\mathcal{D}}}\right\rangle, \tag{6.1}$$

where the first term on the right-hand side represents the average power lost by the particle through radiation reaction, and the second one represents the mean power exchanged between the particle and the background field. Energy balance exists if and only if $(d\langle H\rangle/dt) = 0$. Under this condition, we found that the ensuing description for the mechanical system is governed by a Schrödinger-like equation (4.78), namely

$$-\frac{2\eta^2}{m}\nabla^2\psi + V\psi = 2i\eta\frac{\partial\psi}{\partial t}. \tag{6.2}$$

The value of the parameter η was determined by imposing the balance condition

$$\tau\langle \dddot{\boldsymbol{x}}\cdot\boldsymbol{p}\rangle_0 = \frac{e^2}{m}\left\langle \boldsymbol{p}\cdot\hat{\boldsymbol{\mathcal{D}}}\right\rangle_0 \tag{6.3}$$

for the particle in the ground state. The value of the right-hand side term depends, through the diffusion operator $\hat{\boldsymbol{\mathcal{D}}}$, on the spectral energy density of the field. By introducing the value corresponding to the ZPF,

$$\rho_0(\omega) = \frac{\hbar\omega^3}{2\pi^2c^3}, \tag{6.4}$$

[1] A representative list of related works along the years is Kalitsin (1953), Sokolov and Tumanov (1956), Braffort et al. (1965), Braffort and Taroni (1967), Surdin (1970, 1974), Boyer (1968–1980), Santos (1974), de la Peña and Cetto (1976–1979), Moore (1977, 1984), Jáuregui and de la Peña (1981), Moore and Ramírez (1982), de la Peña and Jáuregui (1982) and Cetto and de la Peña (1988a, b, c). See also Davies (1982) .

[2] Previous versions of the material presented in this section can be found in de la Peña et al. (2010, 2012), Cetto and de la Peña (2012), and Cetto et al. (2012, 2013).

the correct result, namely $\eta = \hbar/2$, was obtained. This made it clear that the Schrödinger equation contains key information about the background field with which the particle is interacting. Additionally, the result indicates that detailed energy balance for a particle in its ground state takes place only when it is in equilibrium with a field of spectral energy density equal to ρ_0.

Now, for *any* state ψ that is solution of the Schrödinger equation, this latter predicts $d\langle H\rangle_\psi/dt = 0$ (provided that H does not depend explicitly on time)[3]. Yet, for a stationary state ψ_n the equation

$$\tau\langle \dddot{\boldsymbol{x}}\cdot\boldsymbol{p}\rangle_n = \frac{e^2}{m}\left\langle \boldsymbol{p}\cdot\hat{\boldsymbol{\mathcal{D}}}\right\rangle_n \tag{6.5}$$

does not hold in general for states other than the ground one (see Sect. 6.1.4). This could appear as contradicting the balance equation (6.1) applied to the n-state. However, the apparent contradiction is solved by observing that whereas $d\langle H\rangle_n/dt = 0$ is a strictly quantum-mechanical expression, consistent with the Schrödinger equation which has been derived neglecting terms of order τ (or e^2), the right-hand-side terms in (6.1) represent the *radiative* terms —precisely those that were neglected in the Schrödinger approximation. Now we shall take them into account. As a result, when the particle or the radiation field or both are in an excited state, the energy balance is broken in general, and radiative transitions take place. The finite lifetimes of the excited states are accounted for by these radiative terms. The corresponding rate of change is, therefore, not $d\langle H\rangle_n/dt$, but a (radiative) correction of the latter, which we denote as

$$\frac{d}{dt}\langle H\rangle_{\rm nr} = \tau\langle \dddot{\boldsymbol{x}}\cdot\boldsymbol{p}\rangle_n - \frac{e^2}{m}\left\langle \boldsymbol{p}\cdot\hat{\boldsymbol{\mathcal{D}}}\right\rangle_n . \tag{6.6}$$

For actual calculations it is simpler to use instead of Eq. (6.6) its alternative form, namely

$$\frac{d}{dt}\langle H\rangle_{\rm nr} = \tau\langle \dddot{\boldsymbol{x}}\cdot\boldsymbol{p}\rangle_n + \frac{1}{m}\left\langle \mathrm{Tr}\boldsymbol{D}^{pp}\right\rangle_n , \tag{6.7}$$

which ensues from Eq. (4.45) with $\mathcal{G} = H$.

In Sect. 6.1.4 a similar analysis will be shown to apply to more general 'classical' integrals of motion, i.e., dynamical variables that are conserved in the absence of the radiation terms. It is understood that all the calculations that follow involve the radiative terms. Therefore, and for simplicity in the writing, the additional index r will be dropped everywhere, except where it must be kept to avoid confusion.

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[3] That $d\langle H\rangle_\psi/dt = 0$ can be easily verified resorting to the general form of the solution

$$\psi(\boldsymbol{x},t) = \sum_n c_n e^{-i\mathcal{E}_n t/\hbar}\varphi_n(\boldsymbol{x}).$$

6.1.2 Breakdown of Energy Balance

Instead of considering the mechanical system in its ground state, as was done for the calculation of the parameter η in Sect. 4.4.4, we assume now that it is in an excited state n, the background field still being in its ground state (the ZPF). Then both terms on the right-hand side of Eq. (6.7) must be recalculated. Since the quantum regime has already been attained, the calculation of such terms is performed following the same procedure as in Sect. 4.4.4. In particular, the mean value $\tau \langle \boldsymbol{p} \cdot \ddot{\boldsymbol{x}} \rangle_n$ is given, in the one-dimensional case, for simplicity, by (see Eqs. (4.106) and (4.107))

$$\tau(\widehat{\ddot{x}}\,\hat{p})_{nn} = \tau(\hat{p}\,\widehat{\ddot{x}})_{nn} = -m\tau \sum_k \omega_{nk}^4 \, |x_{nk}|^2 \,, \tag{6.8}$$

with $\omega_{nk} = (\mathcal{E}_n - \mathcal{E}_k)/\hbar$.

For the second term on the right-hand side of (6.7), we proceed as in the Appendix 4D for the calculation for $\langle D^{pp} \rangle_0$, just noticing that for negative values of ω_{kn} it is the second integral on the right-hand side of equation (D.8) that contributes to the sum. One thus obtains

$$\frac{1}{m} \left\langle D^{pp} \right\rangle_n = -m\tau \sum_k \omega_{nk}^4 \, |x_{nk}|^2 \, \mathrm{sign}\omega_{nk} . \tag{6.9}$$

This expression contains a mixture of positive and negative terms, whilst in (6.8) all contributions have the same sign. As follows from equations (6.7)–(6.9), the net loss of average energy per unit time is given by (recall that we are dropping the additional index r that appears in (6.7))

$$\begin{aligned} \frac{d \langle H \rangle_n}{dt} &= -m\tau \sum_k \omega_{nk}^4 \, |x_{nk}|^2 \, (1 - \mathrm{sign}\omega_{kn}) \\ &= -2m\tau \sum_{k<n} \omega_{nk}^4 \, |x_{nk}|^2 \,. \end{aligned} \tag{6.10}$$

The upshot is that there cannot be energy balance between the ZPF and a particle in an excited state—as was to be expected, since the ZPF is the background radiation field in its ground state. Only for $n = 0$ (hence all $k > n$) Eq. (6.10) gives zero, which means that only the ground state of the particle is sustained by the ZPF. Since in (6.10) the transitions from state n to (lower-energy) states k take place without the intervention of an external radiation field, we speak of 'spontaneous' transitions (more on this in the following section). The corresponding average energy loss per unit time in each such transition, W_{nk}, is obtained by writing the total average energy loss as a sum of contributions from the various possible transitions,

$$\frac{d\langle H\rangle_n}{dt} = \sum_k W_{nk}, \tag{6.11}$$

whence

$$W_{nk} = -2m\tau\omega_{nk}^4 |x_{nk}|^2 . \tag{6.12}$$

It is clear that expressions such as (6.10) have a meaning only in a statistical sense. To understand this meaning we have to think of an ensemble of systems all prepared initially in the same excited state n, and subject to the action of the ZPF. Then according to (6.10), the members of the ensemble have a certain probability per unit time to make a transition to a lower-energy state k, which is determined by the values of ω_{nk} and $|x_{nk}|$, i.e., by the specific properties of the system. However, which transition will take place in every instance is impossible to predict with the present statistical description.

Let us now inquire whether there is *any* (excited or external) background field with which a mechanical system in an excited state n can be in equilibrium. The excited background field is defined by its spectral energy density $\rho(\omega) = \rho_0(\omega)g(\omega)$, with $g(\omega) > 1$ an even function, so that

$$\frac{d\langle H\rangle_n}{dt} = -m\tau \sum_k \omega_{nk}^4 |x_{nk}|^2 (1 - g(\omega_{kn})\text{sign}\omega_{kn}). \tag{6.13}$$

This expression is the generalization of (6.10) for the general density $\rho(\omega)$. To find the answer to the above question we observe that the terms within the parentheses have different signs, depending on whether ω_{kn} refers to an upward or a downward transition (i.e., $k > n$ or $k < n$). Therefore, there is no way that detailed balance can be satisfied in general.

Nevertheless, there is a particular system that can coexist with the field in an excited state, namely the harmonic oscillator. In this case, all $|\omega_{nk}|$ that contribute to the sum in (6.13) are equal in value and coincide with the oscillator frequency ω_0. Since for the harmonic oscillator,

$$|x_{nk}|^2 = \frac{\hbar}{2m\omega_0}\left[\delta_{k,n+1}(n+1) + \delta_{k,n-1}n\right], \tag{6.14}$$

the first term on the right-hand side of Eq. (6.13) gives

$$-m\tau \sum_k \omega_{nk}^4 |x_{nk}|^2 = -\tfrac{1}{2}\hbar\tau\omega_0^3(2n+1) \tag{6.15}$$

and the second terms gives

$$m\tau \sum_k \omega_{nk}^4 |x_{nk}|^2 g(\omega_{kn})\text{sign}\omega_{kn} = \tfrac{1}{2}\hbar\tau\omega_0^3 g(\omega_{n+1n}) = \tfrac{1}{2}\hbar\tau\omega_0^3 g_n(\omega_0), \tag{6.16}$$

because $g(\omega_{n+1,n}) = g_n(\omega_0)$. Therefore, detailed balance exists if $g_n(\omega_0) = 2n+1$; in other words, if the harmonic oscillator in its excited state n is embedded in a background field with spectral energy density

$$\rho(\omega) = \rho_0(\omega)(2n+1), \tag{6.17}$$

there are as many absorptions as there are emissions per unit time, all with the same frequency ω_0, so that the average energy of the oscillator does not change. This result should not come as a surprise, since this field has precisely an energy per normal mode $\hbar\omega_0(2n+1)/2$, equal to the energy of the mechanical oscillator with which it is in equilibrium.

6.1.3 Atomic Lifetimes: Einstein's A and B Coefficients

We now investigate further implications of the absence of detailed balance. This can be conveniently done by using Eq. (6.13) to calculate the average energy lost (or gained) per unit time by the atom (or a mechanical system in general) prepared in an arbitrary state n when it is subject to the action of a radiation field with an arbitrary spectral energy density $\rho(\omega) = \rho_0(\omega)g(\omega)$. It is convenient to write the adimensional function g as $g(\omega) = 1 + g_a(\omega)$, in order to separate the contribution coming from the additional background field, so that

$$\begin{aligned} \rho(\omega) &= \rho_0(\omega)g(\omega) = \rho_0(\omega) + \rho_a(\omega), \\ &\text{with} \quad \rho_a(\omega) = \rho_0(\omega)g_a(\omega). \end{aligned} \tag{6.18}$$

Equation (6.13) thus writes as

$$\begin{aligned} \frac{d\langle H\rangle_n}{dt} &= -m\tau \sum_k \omega_{nk}^4 |x_{nk}|^2 \left[1 - (1 + g_a(\omega_{nk}))\text{sign}\omega_{kn}\right] \\ &= m\tau \sum_k \omega_{nk}^4 |x_{nk}|^2 \left[(g_a)_{\omega_{kn}>0} - (2 + g_a)_{\omega_{kn}<0}\right]. \end{aligned} \tag{6.19}$$

The first term within the brackets in the second line of this equation, proportional to g_a, represents the absorptions ($k > n$) and the second one, proportional to $2 + g_a$, the emissions ($k < n$). It is clear from this expression that there can be absorptions only when the background field is excited so that there is an external component $\rho_a \neq 0$. This additional field can therefore be identified with the *photonic* radiation field. The emissions, on the other hand, may be either 'spontaneous' (in presence of just the ZPF, as in the previous section) or else stimulated by the additional field, represented by g_a.

The coefficients appearing in the various terms determine the respective rates of energy gain and energy loss; therefore, they must be directly related to Einstein's A

and B coefficientsfor the transition probabilities. We recall that the coefficient A is defined as determining the time rate for spontaneous emissions (the term independent of g_a in Eq. (6.19)),

$$\frac{d}{dt}\langle H\rangle_n^{\text{sp em}} = -\hbar\sum_{k<n}\omega_{nk}A_{nk}. \tag{6.20}$$

Thus, A_{nk} is the probability that the atom realizes a spontaneous transition from state n to a lower state k during a unit of time, with a consequent loss of energy given by $\hbar\omega_{nk}$. In its turn, the coefficient B_{nk}^{em}, associated with the rate of energy loss due to transitions induced (stimulated) by the external field, is defined through

$$\frac{d}{dt}\langle H\rangle_n^{\text{ind em}} = -\hbar\sum_{k<n}\omega_{nk}B_{nk}^{\text{em}}\rho_a(\omega_{nk}), \tag{6.21}$$

whereas the coefficient B_{kn}^{abs}, associated with the rate of energy gain due to absorptions induced by the external field, is defined according to

$$\frac{d}{dt}\langle H\rangle_n^{\text{ind abs}} = \hbar\sum_{k>n}\omega_{kn}B_{kn}^{\text{abs}}\rho_a(\omega_{nk}). \tag{6.22}$$

The B coefficients are thus transition probabilities in presence of a photonic field with spectral energy density $\rho_a(\omega_{nk})$. The total rate of energy change can therefore be rewritten in the more transparent form

$$\begin{aligned}\frac{d\langle H\rangle_n}{dt} &= \frac{d}{dt}\langle H\rangle_n^{\text{sp em}} + \frac{d}{dt}\langle H\rangle_n^{\text{ind em}} + \frac{d}{dt}\langle H\rangle_n^{\text{ind abs}}\\ &= \sum_{k>n}\hbar\omega_{kn}B_{kn}^{\text{abs}}\rho_a(\omega_{nk})\\ &\quad -\sum_{k<n}\hbar\omega_{nk}\left[A_{nk}+B_{nk}^{\text{em}}\rho_a(\omega_{nk})\right],\end{aligned} \tag{6.23}$$

which can be recast, using (6.18), as

$$\frac{d\langle H\rangle_n}{dt} = \sum_k \hbar|\omega_{nk}|\left[\left(B_{kn}^{\text{abs}}\rho_0 g_a\right)_{\omega_{kn}>0} - \left(A_{nk}+B_{nk}^{\text{em}}\rho_0 g_a\right)_{\omega_{kn}<0}\right]. \tag{6.24}$$

Comparison of this expression with Eq. (6.19) gives for the spontaneous emissions coefficient

$$A_{nk} = \frac{4e^2\omega_{nk}^3}{3\hbar c^3}|x_{nk}|^2, \quad (n>k), \tag{6.25}$$

which is just the respective QED formula (see e.g. Louisell 1973). In its turn, the coefficients B are found to be

$$B_{kn}^{\text{abs}} = B_{nk}^{\text{em}} = \frac{4\pi^2 e^2}{3\hbar^2} |x_{nk}|^2 \equiv B_{nk}. \tag{6.26}$$

These results coincide with the respective formula of QED (or QM) (Louisell 1973). It is important to note that the expressions for the coefficients A_{nk}, and B_{nk} involve each one the single frequency ω_{nk}, which means that the transition between states involves a resonance with a field component of that frequency—in line with the results of Chap. 5—and a consequent exchange of energy given by $\Delta\mathcal{E}_{nk} = \hbar\omega_{nk}$.

The ratio of the A to the B coefficients is

$$\frac{A_{nk}}{B_{nk}} = \frac{\hbar |\omega_{nk}|^3}{\pi^2 c^3} = 2\rho_0(\omega_{nk}). \tag{6.27}$$

Notice in particular the factor 2 in this equation. Given the definition of the coefficients, one could have expected this ratio to correspond exactly to the spectral density of the ZPF, which would have meant a factor of 1 instead of the factor 2. This latter seems to suggest that the ZPF has double the ability of the rest of the electromagnetic field to induce transitions. The correct explanation, however, is another: the structure of Eq. (6.19) indicates that one should actually write $2\rho_0 = \rho_0 + \rho_0$. One of these two equal contributions to spontaneous decay is due to the effect of the fluctuations impressed on the particle by the field; the second one is the expected contribution due to Larmor radiation. Not surprisingly, they turn out to be equal: it is precisely their equality what leads to the exact balance between the two contributions when the system is in its ground state, guaranteeing the stability of this state. Yet one can frequently find in the literature that all the spontaneous decay is attributed to one or the other of these two causes, more frequently to Larmor radiation. It is an important result of both the present theory and quantum electrodynamics (provided the symmetric operator ordering is used) that the two effects contribute equal shares. Interesting related discussions can be seen in Davydov (1965), Fain (1966), Fain and Khanin (1969), Dalibard et al. (1982), Milonni (1994).

The relation (6.27) and the equality of both coefficients $B_{kn}^{\text{abs}} = B_{nk}^{\text{em}}$, were predicted by Einstein on the basis of statistical considerations in his 1916 paper on the theory of radiation. It is pertinent to ask here at which point the quantization enters in Einstein's paper—a somewhat confused issue that comes to surface every now and then.[4] A current answer to this question is that quantization is introduced by assuming discrete atomic levels. However, this is wrong, as Einstein and Ehrenfest demonstrated some time after the initial paper, by redoing the calculations with a continuous distribution of atomic levels (Einstein 1917) and recovering the old results. The correct answer is that quantization enters through the assumption of a physical

[4] The derivation of the Planck distribution from the A and B coefficients goes back to Einstein's 1916 work. In Chap. 3 the same law was derived by considering the existence of the ZPF. A strong relation should therefore exist between these two different forms of arriving at Planck's law. The remark made reveals this relation: the A coefficient is proportional to the spectral density of the ZPF. Therefore, postulating the existence of $\rho_0 \neq 0$ or of $A \neq 0$ are two different but equivalent forms of treating the problem.

source that can generate 'spontaneous' transitions. This can be easily verified by redoing the Einsteinian calculation but omitting the term that allows for spontaneous emissions, which leads to absurd results, such as atomic coefficients that depend on the temperature. On the other hand, it is interesting to observe that the omission of the term associated with stimulated emissions in Eq. (6.19) (after introducing appropriate populations) leads to the approximate expression for Planck's law that was proposed by Wien (Eq. 3.63), which is a fair approximation for low temperatures, so it already contains a quantum seed.

It is easy to follow the procedure used by Einstein to arrive at the Planck distribution from the A and B coefficients by focusing on just two states n and m, with $\mathcal{E}_n - \mathcal{E}_m = \hbar\omega_{nm} > 0$ and respective populations N_n, N_m. For a system in thermal equilibrium at temperature T, Einstein used the relation (k_B stands for the Bolztmann constant)

$$N_m/N_n = \exp(\mathcal{E}_n - \mathcal{E}_m)/k_B T, \tag{6.28}$$

leaving aside possible but inconsequential degeneracies. In line with Eq. (6.19) (first with $n = m$ and then with n) the number of absorptions ($m \to n$ transitions) is proportional to $N_m g_a(\omega_{nm})$, and the number of emissions ($n \to m$ transitions) is proportional to $N_n[2 + g_a(\omega_{nm})]$. From the equilibrium condition

$$N_m g_a = N_n\,(2 + g_a) \tag{6.29}$$

and Eq. (6.28) one thus obtains indeed Planck's law (for the thermal field)

$$g_a(\omega_{nm}) = \frac{2}{e^{(\mathcal{E}_n - \mathcal{E}_m)/k_B T} - 1}. \tag{6.30}$$

Notice that the equilibrium condition (6.29) implies detailed energy balance, since the individual emissions and absorptions involve a same amount of energy, $\hbar\omega_{nk}$. Notice also that this result implies the Bohr rule $(\mathcal{E}_n - \mathcal{E}_m)/\hbar = \omega_{nm}$, which was *derived* for the first time with this procedure.

6.1.4 A More General Equation for the Balance Breakdown

The procedure just used allows to calculate the rate of change of other dynamical quantities that correspond to 'classical' conserved variables. For this purpose we go back to Eq. (4.33), which holds for any dynamical quantity $\xi(\boldsymbol{x}, \boldsymbol{p})$ that is a 'classical' integral of motion (i.e., an integral of motion when the action of the radiation field is turned off),

$$\frac{d}{dt}\langle\xi\rangle = m\tau\left\langle \dddot{x}_i \frac{\partial\xi}{\partial p_i}\right\rangle - e^2\left\langle \frac{\partial\xi}{\partial p_i}\hat{\mathcal{D}}_i\right\rangle. \tag{6.31}$$

In one-dimensional problems the only integral of motion is H, so that in order to study the balance equation for $\xi \neq H$ it is necessary to analyze the more general, multidimensional case. Since both the Hamiltonian H and ξ are constants of motion, the corresponding operators, $\hat{H}$ and $\hat{\xi}$, commute, which means that in the energy representation the matrix associated with $\hat{\xi}$ is diagonal, i.e.,

$$\xi_{nk} = \xi_n \delta_{nk}, \tag{6.32}$$

where the index $n = (n'_H, n''_\xi)$ includes the quantum numbers corresponding to eigenvalues for both $\hat{H}$ and $\hat{\xi}$. If the system is in a state n, the (radiative) time evolution of $\langle \xi \rangle_n$ is, according to Eq. (6.31),

$$\frac{d}{dt} \langle \xi \rangle_{\mathrm{nr}} = m\tau \left\langle \dddot{x}_i \frac{\partial \xi}{\partial p_i} \right\rangle_n - e^2 \left\langle \frac{\partial \xi}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle_n . \tag{6.33}$$

As before, we omit the subindex r in the following. For the calculation of both terms on the right-hand side we use the relation

$$i\hbar \frac{\partial \hat{\xi}}{\partial p_i} = [\hat{x}_i, \hat{\xi}], \tag{6.34}$$

whence

$$\left(\frac{\partial \xi}{\partial p_i} \right)_{kn} \to \frac{1}{i\hbar} [\hat{x}_i, \hat{\xi}]_{kn} = \frac{1}{i\hbar} x_{ikn} (\xi_n - \xi_k). \tag{6.35}$$

This expression, together with $\dddot{x}_{ink} = -i\omega_{nk}^3 x_{ink}$, leads to

$$[\hat{x}_i, \frac{\partial \hat{\xi}}{\partial p_i}]_{nn} = \frac{1}{\hbar} \sum_k |x_{ink}|^2 (\xi_k - \xi_n) \left(\omega_{nk}^3 + \omega_{kn}^3 \right) = 0, \tag{6.36}$$

due to the antisymmetry of ω_{nk}. This means that

$$\left(\hat{x}_i \frac{\partial \hat{\xi}}{\partial p_i} \right)_{nn} = \left(\frac{\partial \hat{\xi}}{\partial p_i} \hat{x}_i \right)_{nn}, \tag{6.37}$$

and hence no ambiguity in the operator ordering arises when calculating the first term in Eq. (6.33), which reduces to

$$m\tau \left\langle \dddot{x}_i \frac{\partial \xi}{\partial p_i} \right\rangle_n \to m\tau \left(\hat{x}_i \frac{\partial \hat{\xi}}{\partial p_i} \right)_{nn} = -\frac{m\tau}{\hbar} \sum_k \omega_{nk}^3 |\boldsymbol{x}_{nk}|^2 (\xi_n - \xi_k). \tag{6.38}$$

For the second term in (6.33) we resort again to the Markovian approximation and follow the same procedure as in appendix 4D, thus arriving at

$$e^2\left\langle\frac{\partial\xi}{\partial p_i}\hat{\mathcal{D}}_i\right\rangle_n=\frac{4\pi e^2}{3\hbar^2}\int_0^\infty d\omega\rho(\omega)\int_{-\infty}^t dt'\cos\omega(t-t')\left[\hat{x}_i',\left[\hat{x}_i,\hat{\xi}\right]\right]_{nn}. \quad (6.39)$$

Allowing for a possible additional (external) background field so that $\rho(\omega) = \rho_0(\omega)g(\omega)$, we get

$$\begin{aligned}e^2\left\langle\frac{\partial\xi}{\partial p_i}\hat{\mathcal{D}}_i\right\rangle_n&=\frac{2m\tau}{\pi\hbar}\sum_k|\boldsymbol{x}_{nk}|^2(\xi_n-\xi_k)\int_0^\infty d\omega\,\omega^3 g(\omega)\times\\&\quad\times\int_{-\infty}^t dt'\cos\omega(t-t')\cos\omega_{nk}(t-t')\\&=\frac{m\tau}{\hbar}\sum_k|\boldsymbol{x}_{nk}|^2(\xi_n-\xi_k)\int_0^\infty d\omega\,\omega^3 g(\omega)[\delta(\omega+\omega_{nk})+\delta(\omega-\omega_{nk})]\\&=\frac{m\tau}{\hbar}\sum_k|\boldsymbol{x}_{nk}|^2(\xi_n-\xi_k)\,|\omega_{nk}|^3\,g(\omega_{nk}). \quad (6.40)\end{aligned}$$

With (6.38) and (6.40), Eq. (6.33) becomes (recall that $g(\omega_{kn}) = g(|\omega_{kn}|)$)

$$\frac{d}{dt}\langle\xi\rangle_n=-\frac{m\tau}{\hbar}\sum_k\omega_{nk}^3|\boldsymbol{x}_{nk}|^2(\xi_n-\xi_k)[1-g(\omega_{kn})\mathrm{sign}\omega_{kn}]. \quad (6.41)$$

This means that the ZPF with spectral energy density $\rho_0(\omega)$ (i.e. $g(\omega) = 1$) guarantees not only detailed energy balance, but more generally, detailed balance of *any* ('classical') integral of motion of the form here considered, for a mechanical system in its ground state (i.e., $\mathrm{sign}\omega_{kn} = +1$ for all ω_{kn}).

In analogy with Eq. (6.19), (6.41) can be rewritten as

$$\frac{d}{dt}\langle\xi\rangle_n=\frac{m\tau}{\hbar}\sum_k\omega_{nk}^3|\boldsymbol{x}_{nk}|^2(\xi_n-\xi_k)[(g_a)_{\omega_{kn}>0}-(2+g_a)_{\omega_{kn}<0}]. \quad (6.42)$$

In terms of the Einstein coefficients given by (6.25) and (6.26) (expressed in three-dimensional notation), this equation gives

$$\frac{d}{dt}\langle\xi\rangle_n=\sum_{\omega_{kn}>0}(\xi_k-\xi_n)\rho_a(\omega_{nk})B_{kn}-\sum_{\omega_{kn}<0}(\xi_n-\xi_k)[A_{nk}+\rho_a(\omega_{nk})B_{nk}] \quad (6.43)$$

for the net change of $\langle\xi\rangle$ per unit time due to (upward and downward) radiative transitions from state n to states k.

6.1.5 Radiative Corrections to the Energy: The Lamb Shift

The calculations presented in previous pages confirm that the radiative terms neglected in the process of deriving QM give rise to corrections to the solutions of the (unperturbed, i.e. radiationless) Schrödinger equation. The Einstein A and B coefficients for the lifetimes of atomic states pertain to this category. A further important—even if smaller—radiative correction, one that represents a major success of QED, is the shift of the atomic levels due to another residual effect of the ZPF. Indeed, the effective work realized by the fluctuating motions of the bound particle gives rise to a tiny modification of the mean kinetic energy, as is shown in the following by means of a direct approach to the subject.

To calculate the radiative energy shift let us go back to the one-dimensional version of Eq. (4.29) (where the subindex Q reminds us that the calculations are carried out according to the quantum rules),

$$\frac{d}{dt}\langle xp\rangle_{\mathrm{Q}} = \frac{1}{m}\left\langle p^2\right\rangle_{\mathrm{Q}} + \langle xf\rangle_{\mathrm{Q}} + m\tau\,\langle x\,\dddot{x}\rangle_{\mathrm{Q}} - e^2\left\langle x\hat{\mathcal{D}}\right\rangle_{\mathrm{Q}}. \tag{6.44}$$

As explained is Sect. 4.2.1, this equation is a time-dependent version of the virial theorem, with radiative corrections included and the average values taken over the ensemble instead of over time, as is customarily done. In line with the discussion in Sect. 6.1.1, the correction $\langle\delta T\rangle_{nr}$ (to lowest order in $\tau \sim e^2$) to the mean kinetic energy in the quantum state n, is given by

$$\langle\delta T\rangle_{nr} = -\frac{m\tau}{2}\,\langle x\,\dddot{x}\rangle_n + \frac{e^2}{2}\left\langle x\hat{\mathcal{D}}\right\rangle_n, \tag{6.45}$$

where the two average values on the right-hand side are calculated using the solutions of the Schrödinger equation. We shall again dispose of the additional index r, whenever this does not lead to confusion.

The first term on the right-hand side of (6.45), associated with the Larmor radiation, can be approximated to lowest order in τ, by

$$-\frac{m\tau}{2}\,\langle x\,\dddot{x}\rangle_n = \frac{m\tau}{2}\,\langle\dot{x}\,\ddot{x}\rangle_n = \frac{m\tau}{4}\frac{d}{dt}\left\langle\dot{x}^2\right\rangle_n = 0, \tag{6.46}$$

and hence it does not contribute to the energy shift in the mean. Now, it is interesting to note that when $\langle x\,\dddot{x}\rangle$ is calculated in quantum terms, there is an ambiguity in the order of the factors (since $(\hat{x}\,\widehat{\dddot{x}})_{nn} \neq (\widehat{\dddot{x}}\hat{x})_{nn}$), and only the symmetrized operator

$$(\hat{x}\,\widehat{\dddot{x}})^S = \tfrac{1}{2}(\hat{x}\,\widehat{\dddot{x}} + \widehat{\dddot{x}}\hat{x}) \tag{6.47}$$

has a real and null mean value, in accordance with (6.46), which defines the rule of correspondence in this case. The anti-Hermitian, antisymmetric combination

$$(\hat{x}\,\hat{\dot{x}})^A = \tfrac{1}{2}(\hat{x}\,\hat{\dot{x}} - \hat{\dot{x}}\hat{x}) \tag{6.48}$$

is not devoid of physical meaning, however. To identify this, we write

$$\langle \delta T\rangle_n^A \equiv -i\frac{m\tau}{2}(\hat{x}\,\hat{\dot{x}})_{nn}^A = \frac{1}{2}m\tau\sum_k |x_{nk}|^2\,\omega_{nk}^3. \tag{6.49}$$

In terms of the spontaneous-emission coefficient given by Eq. (6.25) we have (for $\mathcal{E}_n > \mathcal{E}_k$)

$$\langle \delta T\rangle_n^A = \frac{\hbar}{4}\sum_k A_{nk} = \frac{\hbar}{4T_n}, \tag{6.50}$$

where $T_n = (\sum_k A_{nk})^{-1}$ is the lifetime of state n against spontaneous decay. This result shows that the expectation value of $(\hat{x}\,\hat{\dot{x}})^A$ furnishes a measure of the (inverse) lifetime of the corresponding excited state—whereas the mean value of $(\hat{x}\,\hat{\dot{x}})^S$ represents a contribution to the energy shift (which turns out to be zero) for the same state.

From Eq. (6.45) and the previous results, it follows that the correction to the energy is due solely to the coupling of the (instantaneous) electric dipole moment of the atom $d = ex$ to the electric component of the background field, represented by the second term in Eq. (6.45),

$$\langle \delta T\rangle_n = \frac{e^2}{2}\left\langle x\hat{\mathcal{D}}\right\rangle_n = \frac{e}{2}\left\langle d\hat{\mathcal{D}}\right\rangle_n. \tag{6.51}$$

The calculation of this term is carried out in appendix A, and gives the result

$$\frac{e^2}{2}\left\langle x\hat{\mathcal{D}}\right\rangle_n = -\frac{2e^2}{3\pi c^3}\sum_k |x_{nk}|^2\,\omega_{kn}\int_0^\infty d\omega\,\frac{\omega^3}{\omega_{kn}^2-\omega^2}. \tag{6.52}$$

The radiative correction to the mean energy is thus (writing $\langle \delta T\rangle_n = \delta\mathcal{E}_n$ and in three dimensions, for comparison purposes)

$$\delta\mathcal{E}_n = \frac{e^2}{2}\left\langle \boldsymbol{x}\cdot\hat{\boldsymbol{\mathcal{D}}}\right\rangle_n = -\frac{2e^2}{3\pi c^3}\sum_k |\boldsymbol{x}_{nk}|^2\,\omega_{kn}\int_0^\infty d\omega\,\frac{\omega^3}{\omega_{kn}^2-\omega^2}. \tag{6.53}$$

This coincides with the formula derived by Power in 1966 for the Lamb shift on the basis of Feynman's argument of 1961. We recall that according to Feynman, the presence of the atom creates a weak perturbation on the nearby field, thereby acting as a refracting medium. The effect of this perturbation is to change the frequencies of the background field in the vicinity of the atom from ω to $\omega/n(\omega)$, n being the refractive index. The shift of the ZPF energy due to the presence of the atom is then (Power 1966, see also Milonni 1994, Chap. 3)

$$\Delta\mathcal{E}_n = \sum_{\mathbf{k},\lambda} \frac{1}{2}\frac{\hbar\omega_k}{n(\omega_k)} - \sum_{\mathbf{k},\lambda} \frac{1}{2}\hbar\omega_k \simeq -\sum_{\mathbf{k},\lambda}[n(\omega_k)-1]\frac{1}{2}\hbar\omega_k, \tag{6.54}$$

and the refractive index is given in this approximation by (Davydov 1965, Chap. 9)

$$n(\omega_k) \simeq 1 + \frac{4\pi}{3\hbar V}\sum_m \frac{|\boldsymbol{d}_{mn}|^2\,\omega_{mn}}{\omega_{mn}^2 - \omega_k^2}, \tag{6.55}$$

where $\boldsymbol{d}_{mn} = e\boldsymbol{x}_{mn}$ is the electric dipole transition moment. After an integration over the solid angle $\hat{\boldsymbol{k}}$ and summation over the polarizations $\lambda = 1, 2$, Power obtains in the continuum limit for ω_k the formula

$$\Delta\mathcal{E}_n = -\frac{2}{3\pi c^3}\sum_m |\boldsymbol{d}_{mn}|^2\,\omega_{mn}\int_0^\infty d\omega\,\frac{\omega^3}{\omega_{mn}^2 - \omega^2}, \tag{6.56}$$

which coincides with the previous result, Eq. (6.53).

The observable Lamb shift (called also Lamb shift proper) is obtained by subtracting from the total energy shift given by Eq. (6.53), the free-particle contribution, $\delta\mathcal{E}_{\rm fp}$, represented by this same expression in the limit of continuous electron energies (when ω_{kn} can be ignored compared with ω in the denominator),

$$\delta\mathcal{E}_{\rm fp} = \frac{2e^2}{3\pi c^3}\sum_m |\boldsymbol{x}_{nm}|^2\,\omega_{mn}\int_0^\infty d\omega\,\omega = \frac{e^2\hbar}{\pi m c^3}\int_0^\infty d\omega\,\omega. \tag{6.57}$$

The last equality follows from the sum rule $\Sigma_m\,|\boldsymbol{x}_{nm}|^2\,\omega_{mn} = 3\hbar/2m$. The Lamb shift proper of the energy level n is therefore given by

$$\delta\mathcal{E}_{{\rm L}n} = \delta\mathcal{E}_n - \delta\mathcal{E}_{\rm fp} = -\frac{2e^2}{3\pi c^3}\sum_k |\boldsymbol{x}_{nk}|^2\,\omega_{kn}^3\int_0^\infty d\omega\,\frac{\omega}{\omega_{kn}^2 - \omega^2}, \tag{6.58}$$

which again agrees with the nonrelativistic QED formula.[5] The logarithmic divergence of the integral calls for the introduction of the usual (nonrelativistic) regularizing cutoff $\omega_C = mc^2/\hbar$, which gives thus

$$\delta\mathcal{E}_{{\rm L}n} = \frac{2e^2}{3\pi c^3}\sum_k |\boldsymbol{x}_{nk}|^2\,\omega_{kn}^3\,\ln\left|\frac{mc^2}{\hbar\omega_{kn}}\right|. \tag{6.59}$$

[5] In the denominator of Eq. (6.58) the term $\tau^2\omega^4$ due to Larmor radiation is missing (it has been neglected because the calculation is performed to lowest order in e^2). Its introduction is important for some applications, in particular to get a valid expression for the refractive index in Eqs. (6.55) and (6.58) (Sokolov and Tumanov 1956; de la Peña and Cetto 1977). This is a quite natural term in both QED and SED.

This is Bethe (1947) well known result. Note, however, that in the present approach (as in Power's) no mass renormalization was required.

The interpretation of the Lamb shift as a change of the atomic energy levels due to the interaction with the surrounding ZPF is fully in line with the general approach of the present theory. It constitutes one more manifestation of the influence of the particle on the near field, which is then fed back to the particle. An alternative way of looking at this reciprocal influence is by considering the general relation between the atomic polarizability α and the refractive index of the medium affected by it (for $n(\omega) \simeq 1$),

$$n(\omega) = 1 + 2\pi\alpha(\omega). \tag{6.60}$$

A comparison of this expression with Eq. (6.55) shows that

$$\alpha_n(\omega) = \frac{2}{3\hbar} \sum_m \frac{|\boldsymbol{d}_{mn}|^2 \, \omega_{mn}}{\omega_{mn}^2 - \omega^2}, \tag{6.61}$$

which is the Kramers-Heisenberg formula (see Davydov 1965). This indicates that the Lamb shift can also be viewed as a Stark shift associated with the dipole moment $\boldsymbol{d}(\omega) = \alpha(\omega)\boldsymbol{E}$ induced by the electric component of the ZPF on the atom.

Let us recast Eq. (6.58) in a more familiar form, the one that is usual to find in textbooks and more adapted to direct calculation. In doing so it is convenient to introduce the energy $\mathcal{E}_n = \hbar\omega_n$ and the abbreviated notation

$$I_{nk} = \int_0^\infty d\mathcal{E} \, \frac{\mathcal{E}}{(\mathcal{E}_k - \mathcal{E}_n)^2 - \mathcal{E}^2}, \tag{6.62}$$

so that (6.58) rewrites as (with $\alpha = 3mc^2\tau/2\hbar$)

$$\begin{aligned} \delta\mathcal{E}_{\mathrm{L}n} &= -\frac{2\alpha}{3\pi c^2} \sum_k I_{nk} \, |\omega_{kn}\boldsymbol{x}_{nk}|^2 \, (\mathcal{E}_k - \mathcal{E}_n) \\ &= -\frac{2\alpha I_n}{3\pi m^2 c^2} \sum_k \left|\boldsymbol{p}_{nk}\right|^2 (\mathcal{E}_k - \mathcal{E}_n) \\ &= -\frac{2\alpha I_n}{3\pi m^2 c^2} i\hbar \sum_k \boldsymbol{f}_{nk} \cdot \boldsymbol{p}_{kn}, \end{aligned} \tag{6.63}$$

since $\boldsymbol{p}_{nk} = i\omega_{nk}\boldsymbol{x}_{nk}$ and $\boldsymbol{f}_{nk} = i\omega_{nk}\boldsymbol{p}_{nk}$. We have assumed that I_{nk} depends so weakly on the index k that such dependence can be ignored; $\delta\mathcal{E}_{\mathrm{L}n}$ becomes, with $\hat{\boldsymbol{f}} = -\nabla V$,

$$\begin{aligned} \delta\mathcal{E}_{\mathrm{L}n} &= -\frac{2\alpha I_n}{3\pi m^2 c^2} i\hbar \, \langle n| \, \boldsymbol{f} \cdot \boldsymbol{p} \, |n\rangle = \frac{\alpha I_n}{3\pi m^2 c^2} i\hbar \, \langle n| \left[\nabla V, \boldsymbol{p}\right] |n\rangle \\ &= -\frac{\hbar^2 \alpha I_n}{3\pi m^2 c^2} \, \langle n| \, \nabla^2 V \, |n\rangle \, . \end{aligned} \tag{6.64}$$

For the Coulomb potential, $\nabla^2 V = 4\pi Z e^2 \delta^3(\boldsymbol{r})$, so that only the wave function at the origin contributes to the Lamb shift in the present approximation.

The main interest of (6.64), apart from convenience of calculation, lies in the interpretation of the correction to the energy as due to fluctuations of the value of the potential V resulting from assumed fluctuations $\delta\boldsymbol{x}$ of the instantaneous position of the electron. Thus

$$V(\boldsymbol{x} + \delta(\boldsymbol{x})) = V(\boldsymbol{x}) + \delta x_i \frac{\partial V}{\partial x_i} + \frac{1}{2}\delta x_i \delta x_j \frac{\partial^2}{\partial x_i \partial x_j} V + \cdots \tag{6.65}$$

Assuming that the fluctuations are spherically symmetric on the average and statistically independent in orthogonal directions, so that in the mean $\overline{\delta x_i} = 0$, $\overline{\delta x_i \delta x_j} = \frac{1}{3}\overline{(\delta \boldsymbol{x})^2}\delta_{ij}$, the average (over the set of fluctuations) of the above equation becomes

$$\overline{V(\boldsymbol{x} + \delta(\boldsymbol{x}))} = V(\boldsymbol{x}) + \tfrac{1}{6}\overline{(\delta \boldsymbol{x})^2}\nabla^2 V + \cdots \tag{6.66}$$

Writing the energy shift (6.64) in terms of the expectation value of the deviation $\overline{V(\boldsymbol{x} + \delta(\boldsymbol{x}))} - V(\boldsymbol{x})$, and neglecting higher-order terms,

$$\delta\mathcal{E}_{\mathrm{L}n} = \left\langle \overline{V(\boldsymbol{x} + \delta(\boldsymbol{x}))} - V(\boldsymbol{x}) \right\rangle = \tfrac{1}{6}\overline{(\delta \boldsymbol{x})^2} \left\langle \nabla^2 V \right\rangle = -\frac{\hbar^2 \alpha I_n}{3\pi m^2 c^2} \left\langle \nabla^2 V \right\rangle \tag{6.67}$$

gives for the mean square displacement of $\boldsymbol{x}$

$$\overline{(\delta \boldsymbol{x})^2} = -\frac{2\hbar^2 \alpha I_n}{\pi m^2 c^2} = -\frac{2\alpha I_n}{\pi}\lambda_C^2, \quad \lambda_C = \frac{\hbar}{mc}, \tag{6.68}$$

where λ_C stands for the Compton wavelength. Thus

$$\frac{\sqrt{\overline{(\delta \boldsymbol{x})^2}}}{\lambda_C} = \sqrt{-\frac{4\alpha I_n}{\pi}} \sim 10^{-1}. \tag{6.69}$$

Qualitatively, this result is in line with the well-known proposal in Welton 1948 to identify the Lamb shift as a consequence of the fluctuations of the position of the electron due to the interaction with the vacuum field (Milonni 1994). We see that the fluctuations of $\boldsymbol{x}$ that give rise to the Lamb shift are smaller even in the mean than the Compton wavelength, which is by itself a small quantity in comparison with the Bohr radius $a_B = \hbar^2/me^2$, $\lambda_C = \alpha a_B$ ($\sim 10^{-9}$ cm for the electron). The Lamb shift is indeed a very small correction, so small that it requires the use of particularly refined spectroscopy to be detected.[6]

[6] There are several trembling motions of the electron, which should not be confused with each other. On one hand there is the zitterbewegung, a relativistic trembling of the order of λ_C, with a relatively well-defined frequency $\sim 2mc^2/\hbar$. This leads to the notion that an electron cannot be

It is interesting to observe that although the initial calculation of the Lamb shift on the basis of Eq. (6.45) exhibits it as an additional kinetic energy, the expression (6.64) describes it as the result of an extra potential energy. However, in both cases it is the fluctuations of $\boldsymbol{x}$ which give rise to the correction. The reason for this is that the fluctuations of the position variable generate both diffusion and extra potential energy. The two descriptions are therefore equivalent forms of accounting for the same process.

6.1.6 External Effects on the Radiative Corrections

By now it is clear that certain basic properties of the vacuum field—such as the intensity of its fluctuations or its spectral distribution—are directly reflected in the radiative corrections studied above. This means that a change in the properties of the vacuum should lead in principle to a corresponding modification of these corrections. The vacuum can be altered, for instance, by introducing material objects that modify the boundary conditions and hence affect the distribution of the normal modes of the field. Additionally, the background field can be modified by raising the temperature of the system or by introducing external radiation. These changes lead to observable effects on both the radiative lifetimes and the energy levels.

Such 'environmental' effects have been studied for more than 60 years, normally within the framework of quantum theory. However, some calculations have been made also within the framework of SED, in particular for the harmonic oscillator, leading to comparable results (see e.g. Cetto and de la Peña 1988a, b).[7] The formulas derived in the previous sections provide an opportunity to study the more general case, instead of restricting the calculations to the harmonic oscillator. The task is facilitated and becomes transparent by the use of the present theory, because the presence of the background radiation field is clear from the beginning.

6.1.6.1 External Effects on Atomic Lifetimes

In Sect. 6.1.3 we have already come across one observable effect of a change in the background field: according to Eq. (6.19) the rates of stimulated atomic transitions are directly proportional to the spectral distribution of the external (or additional) background field, be it a thermal field or otherwise. In the case of a thermal field at temperature T, in particular, with $g_a(\omega_{nk})$ given by Eq. (6.30), the (induced) transition rate from state n to state k becomes (with the help of Eqs. (6.4), (6.24) and (6.26))

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confined to a region in space smaller than a Compton wavelength. As just discussed, the fluctuations of $\boldsymbol{x}$ associated with the Lamb shift are even smaller than λ_C.

[7] Related topics in which SED has been very successful are the Casimir effect and the van der Waals forces. The literature on this subject was iniciated with the old papers by Marshall (1965) and Boyer (1968, 1969), and followed by an extensive series of works by these authors and several others. A review of Boyer's work on the subject is Boyer (1980a); a more extensive list of references is given in *The Dice*. See also the literature at the end of the chapter.

$$\frac{dN_{nk}}{dt} = \rho_0(\omega_{nk}) g_a(\omega_{nk}) B_{nk}$$
$$= \frac{4e^2 |\omega_{nk}|^3 |x_{nk}|^2}{3\hbar c^3} \frac{1}{e^{\hbar|\omega_{nk}|/k_B T} - 1}. \tag{6.70}$$

This result shows that no single Hamiltonian eigenstate is stable at $T > 0$ (as is well known), because the thermal field induces both upward and downward transitions. For downward transitions ($\omega_{nk} > 0$) one can rewrite Eq. (6.70) for comparison purposes in terms of A_{nk} as given by (6.25),

$$\frac{dN_{nk}}{dt} = \frac{A_{nk}}{e^{\hbar\omega_{nk}/k_B T} - 1}. \tag{6.71}$$

This expression indicates that the effect of the thermal field on the decay rate is barely noticeable at room temperature ($k_B T \simeq 0.025$ eV), since for typical atomic frequencies, the denominator in Eq. (6.71) ranges between $\exp(-40)$ and $\exp(-400)$. In fact, the temperature would have to be raised by several orders of magnitude to obtain a noticeable effect on the lifetimes—but then other effects on atomic stability due to such high temperatures would be dominant. On the other hand, the simple addition of a high-intensity monochromatic field of frequency ω_{nk}—such as that produced by a laser—can have a visible effect on the emission rates, as is well known. Such effect in fact lies at the basis of the functioning of the laser.

Also when the geometry of the system or the spectral distribution of the background field are modified by the presence of nearby conducting objects, the transition rates are affected accordingly. Assume, for simplicity, that the modified field is still isotropic, with the density of modes of a given frequency ω_{nk} simply reduced by a (geometrical) factor $g(\omega_{nk}) < 1$. Then according to the results of Sect. 6.1.3 the corresponding spontaneous and induced transition rates are reduced by this factor, since both A and ρB are proportional to the density of modes. By enclosing the atoms in a high-quality cavity that excludes the modes of this frequency, one can therefore virtually inhibit the corresponding transition. For the more general, anisotropic case the calculations are somewhat more complicated, without however leading to a substantial difference from a physical point of view. These cavity effects have been the subject of a large number of experimental tests since the early works of Kleppner (1981), Goy et al. (1983), and others. In those cases where observable effects were predicted, the experiments have served to confirm the theoretical predictions.

Considering the essential role played by the background field in determining the basic structure and quantum behavior of the atom, according to SED, one may well wonder whether a geometric modification of this field (i.e., a change in the distribution of single modes of certain frequencies) would not have an impact on the basic behavior of the atom at the level of quantum mechanics. To respond to this question, let us look back at the energy-balance condition (6.5),

$$\tau \langle \ddot{\boldsymbol{x}} \cdot \boldsymbol{p} \rangle_n = \frac{e^2}{m} \left\langle \boldsymbol{p} \cdot \hat{\boldsymbol{\mathcal{D}}} \right\rangle_n . \tag{6.72}$$

We observe that any such alteration of the field modes affects *both* the radiation reaction field and the ZPF on an equal footing. The reason is that on each side of this equation the field enters through its correlation function only (to lowest significative order in the approximation). Therefore, both sides of the equation are identically affected, and the equality continues to hold. This leads to an important conclusion, namely that the (quantum-mechanical) stationary solutions of the Schrödinger equation (which must comply with the balance equation) are immune to such environmental modifications of the background field.[8] This can be considered a sign of the robustness of the stationary quantum states.

6.1.6.2 External Effects on the Energy Levels

Let us now show how Eqs. (6.57) and (6.58) can be used to calculate the changes in the energy shift produced by the addition of an (external or thermal) background field. First we determine the shift $\delta\mathcal{E}_n(\rho)$ produced by the total field $\rho = \rho_0 + \rho_a$ (see Eq. (6.18)), following the same procedure that led to Eq. (6.53). Then, to the corresponding free-particle contribution $\delta\mathcal{E}_{\rm fp}(\rho)$ and the corresponding Lamb shift $\delta\mathcal{E}_{{\rm L}n}(\rho)$ we subtract the original shifts ($\delta\mathcal{E}_{\rm fp}(\rho_0)$ and $\delta\mathcal{E}_{{\rm L}n}(\rho_0)$) produced by the ZPF, thus obtaining the formulas for the variations of the (first-order) corrections. The calculation is straightforward using the cited equations, and the results are

$$\Delta\left(\delta\mathcal{E}_{\rm fp}\right) = \frac{4\pi e^2}{3\hbar}\sum_k |\boldsymbol{x}_{nk}|^2\,\omega_{kn}\int_0^\infty d\omega\,\frac{\rho_a}{\omega^2} = \frac{e^2\hbar}{\pi mc^3}\int_0^\infty d\omega\,\frac{\rho_a}{\rho_0}\omega, \tag{6.73}$$

$$\Delta\left(\delta\mathcal{E}_{{\rm L}n}\right) = -\frac{2e^2}{3\pi c^3}\sum_k |\boldsymbol{x}_{nk}|^2\,\omega_{kn}^3\int_0^\infty d\omega\,\frac{\rho_a}{\rho_0}\frac{\omega}{\omega_{kn}^2-\omega^2}, \tag{6.74}$$

for a homogeneous field. If, for instance, the additional field represents blackbody radiation at a temperature $T > 0$, i.e., if $\rho_a(\omega, T) = \rho_0(\omega)g_a(\omega, T)$ with $g_a(\omega, T)$ given by (6.30), then Eq. (6.73) gives

$$\Delta_T\left(\delta\mathcal{E}_{\rm fp}\right) = \frac{2\alpha}{\pi mc^2}(k_BT)^2\int_0^\infty dy\,\frac{y}{\exp y - 1}. \tag{6.75}$$

[8] A similar conclusion was arrived at in Cetto and de la Peña (1988a, b), where it was formulated in terms of a fluctuation-dissipation relation for SED.

With

$$\int_0^\infty dy\, \frac{y}{\exp y - 1} = \frac{\pi^2}{6} \tag{6.76}$$

equation (6.75) gives for the change of the free-particle energy the amount

$$\Delta_T \left(\delta \mathcal{E}_{\rm fp}\right) = \frac{\pi\alpha}{3mc^2}(k_B T)^2. \tag{6.77}$$

Further, the formula for the change of the Lamb shift proper is given according to Eq. (6.74) by

$$\Delta \left(\delta \mathcal{E}_{{\rm L}n}\right) = -\frac{4e^2}{3\pi c^3} \sum_k |\boldsymbol{x}_{nk}|^2\, \omega_{kn}^3 \int_0^\infty d\omega\, \frac{\omega}{\omega_{kn}^2 - \omega^2} \left(\frac{1}{\exp(\hbar\omega/k_B T) - 1}\right). \tag{6.78}$$

These results coincide with those obtained within QED (Knight 1972; Zhou and Yu 2010), and the corresponding thermal shifts have been experimentally observed (see e.g. Hollberg and Hall 1984). From the point of view of SED (or QED) their interpretation is clear: they represent additional contributions to the kinetic energy impressed on the particle by the thermal field, according to the discussion at the beginning of Sect. 6.1.5.

6.2 The Spin of the Electron

A most fundamental problem in quantum theory relates to the origin and nature of the spin of the electron. Since this is a purely quantum phenomenon (albeit normally considered of relativistic origin), the present theory should be expected to provide an explanation for it, instead of merely taking it as one more intrinsic property of the particle such as its mass or its electric charge.

Despite its importance for QM, the question about the origin and nature of spin has received relatively little attention in SED. Indeed, during the initial period of SED the electron was considered mainly as a spinless particle; to our knowledge the only exception to this was the work of Braffort and Taroni (1967), showing the existence of some effects due to spin. The 1980s saw the publication of a series of phenomenological discussions by Moore and Ramírez (1982), Moore (1984), Cavalleri (1985) and Rueda (1993), focusing on the importance for quantum theory of the appearance of (helical) vibrations that might be identified with the zitterbewegung. By using an (otherwise classical) harmonic oscillator model for the electron and separating the ZPF into components of circular polarization, Jáuregui and de la Peña (1981), de la Peña and Jáuregui (1982), derived the mean squared electron spin angular momentum and its projections, within a numerical factor of order 1; see also Sachidanandam(1983). Similarly, Barranco and coworkers 1989 studied the spin and the magnetic moment of a (classical) particle subject to the ZPF, using a simple

composite particle model with two constituents bound by a harmonic force. More recently, Muralidhar (2011) published a suggestive derivation of spin by assuming that the zero-point energy of the (free) electron, considered as a classical particle, is an energy of rotation within the region of space surrounding the particle. Specifically, by expressing the ZPF fluctuations as rotations on a complex plane, the spin angular momentum appears in this model connected with the imaginary part of the rotations.

The various SED calculations, though based on classical models, have led in one way or another to a result of order $\hbar^2$ for the mean square value of the spin and of order $\hbar$ for the spin projections. This strongly suggests the possibility of finding a reasonable explanation for the electron spin within the present approach. Our analysis of this possibility, presented in the following sections, exhibits the electron spin as one more emergent property arising from the interaction of the electron with the ZPF.

6.2.1 Unravelling the Spin

Let us again recall the equations for the averaged dynamical variables that ensue from the generalized Fokker-Planck equation, derived in Sect. 4.2.1. We are interested, in particular, in the balance equation for the angular momentum components

$$L_{ij} = x_i p_j - x_j p_i , \tag{6.79}$$

obtained from Eq. (4.28), namely

$$\frac{d}{dt}\left\langle L_{ij}\right\rangle_{\mathrm{Q}} = \left\langle M_{ij}\right\rangle_{\mathrm{Q}} + m\tau\left\langle x_i \dddot{x}_j - x_j \dddot{x}_i\right\rangle_{\mathrm{Q}} - e^2\left\langle x_i \hat{\mathcal{D}}_j - x_j \hat{\mathcal{D}}_i\right\rangle_{\mathrm{Q}} . \tag{6.80}$$

Here $M_{ij} = x_i f_j - x_j f_i$ is a component of the momentum of the external force (or torque), and the additional terms are due to radiation reaction and diffusion. For simplicity let us consider that only the ZPF is present and that the particle is in its ground state, $n = 0$. Then, since the state is (truly) stationary, $d\left\langle L_{ij}\right\rangle/dt = 0$, and Eq. (6.80) gives

$$\left\langle M_{ij}\right\rangle_0 + m\tau\left\langle x_i \dddot{x}_j - x_j \dddot{x}_i\right\rangle_0 - e^2\left\langle x_i \hat{\mathcal{D}}_j - x_j \hat{\mathcal{D}}_i\right\rangle_0 = 0. \tag{6.81}$$

To lowest order in τ one may take

$$m\left\langle x_i \dddot{x}_j - x_j \dddot{x}_i\right\rangle_0 = \frac{d}{dt}\left\langle M_{ij}\right\rangle_0 - \left\langle \dot{x}_i f_j - \dot{x}_j f_i\right\rangle_0 , \tag{6.82}$$

which under stationarity reduces to

$$m\tau\left\langle x_i \dddot{x}_j - x_j \dddot{x}_i\right\rangle_0 = -\tau\left\langle \dot{x}_i f_j - \dot{x}_j f_i\right\rangle_0 , \tag{6.83}$$

and Eq. (6.81) becomes thus

$$\langle M_{ij}\rangle_0 - \tau\langle \dot{x}_i f_j - \dot{x}_j f_i\rangle_0 = e^2\left\langle x_i\hat{\mathcal{D}}_j - x_j\hat{\mathcal{D}}_i\right\rangle_0 . \tag{6.84}$$

For central forces, $M_{ij} = 0$ and one may write $f_i = g(r)x_i$ with $g(r)$ a radial function, whence

$$\tau\langle \dot{x}_i f_j - \dot{x}_j f_i\rangle_0 = \tau\langle g(r)\left(\dot{x}_i x_j - \dot{x}_j x_i\right)\rangle_0 = -\frac{\tau}{m}\langle g(r)L_{ij}\rangle_0 , \tag{6.85}$$

which introduced in (6.84) leads to the balance equation for the angular momentum,

$$\tau\langle g(r)L_{ij}\rangle_0 = me^2\left\langle x_i\hat{\mathcal{D}}_j - x_j\hat{\mathcal{D}}_i\right\rangle_0 . \tag{6.86}$$

Alternatively, in the Markovian limit equation (4.45) can be applied directly to $\mathcal{G} = L_{ij}$, thus obtaining

$$m\tau\langle x_i\dddot{x}_j - x_j\dddot{x}_i\rangle_0 = \left\langle D_{ij}^{px} - D_{ji}^{px}\right\rangle_0 . \tag{6.87}$$

By combining this with Eqs. (6.83) and (6.85), we get a balance condition for the angular momentum components,

$$\tau\langle g(r)L_{ij}\rangle_0 = -m\langle D_{ji}^{px} - D_{ij}^{px}\rangle_0 . \tag{6.88}$$

Alternatively, one may take directly the (stochastic) equation of motion for a central-force problem,

$$\dot{p}_i = f_i + m\tau\dddot{x}_i + eE_i(t), \tag{6.89}$$

where $p_i = m\dot{x}_i$, $E_i(t)$ is the electric component of the ZPF, and $f_i = g(r)x_i$. By multiplying this equation by x_j and anti-symmetrizing one gets

$$\dot{p}_i x_j - \dot{p}_j x_i = m\tau\left(\dddot{x}_i x_j - \dddot{x}_j x_i\right) - e\left(x_i E_j - x_j E_i\right) . \tag{6.90}$$

Under stationarity, i.e. for the system in its ground state, one can perform the substitutions $\dot{p}_i \to f_i$ and $mx_i\dddot{x}_j \to -\dot{x}_i f_j$ (valid to zero order in τ), thus arriving at

$$\tau\overline{g(r)L_{ij}}^{(i)} = -me\overline{\left(x_i E_j - x_j E_i\right)}^{(i)} = -me\overline{(\boldsymbol{x}\times\boldsymbol{E})}_{ij}^{(i)} . \tag{6.91}$$

This expression, which is equivalent to (6.86), shows that there is an angular momentum $\boldsymbol{L}$ that results from the instantaneous torque exerted by the Lorentz force on the particle. Since only the fluctuating component of $\boldsymbol{x}$ can contribute to the average $\overline{(\boldsymbol{x}\times\boldsymbol{E})}^{(i)}$ (because $E_i(t)$ is purely random), it is clear that all the angular momentum thus generated is due to the random motion around the mean trajectory

followed by the particle; thus, it is independent of the system of coordinates, and has an 'internal' (intrinsic) nature. This angular momentum will turn out to be a crucial phenomenon in our proposal for the origin of the electron spin.

That the Lorentz force due to the background field exerts indeed a torque on the particle, is explained by the following argument. It is experimentally observed that the interaction of the electron with the radiation field takes place via the circular polarized modes of the field (or modes of a certain helicity). This is known to be the case for the photonic field, which, from the present perspective, is the excited state of the radiation field, additional to the zero-point component. It is therefore natural to assume that the same is true for the interaction of the electron with the modes of the field in its ground state, i.e., the ZPF. To analyze the effect of such interaction, one should therefore consider the ZPF as composed of modes of both right- and left-handed circular polarization. Taken separately, these modes will indeed induce, through the Lorentz force, rotational (viz right- and left-handed) motions on the particle. The two effects will tend to conceal each other when the ensemble of modes and of particles is considered in its entirety; yet by focusing on one of the two subensembles of a given polarization of the field modes, the effective rotation induced on the corresponding particles should be disclosed. It is just this mean rotation present in each one of the two separate subensembles what will be identified below with the two degrees of freedom proper to each of the corresponding spin-1/2 states of the particle.

6.2.2 The Isotropic Harmonic Oscillator

As an example of application of the results of the previous section, let us consider the simplest case, namely a spherical isotropic harmonic oscillator of natural frequency ω_0, in its ground state. In this case $f_i = -m\omega_0^2 x_i$ so that $g(r) = -m\omega_0^2$, and Eq. (6.88) becomes

$$\left\langle L_{ij} \right\rangle_0 = \frac{1}{\tau\omega_0^2} \langle D_{ji}^{px} - D_{ij}^{px} \rangle_0 . \tag{6.92}$$

The calculation of the *right-hand* side is carried out in appendix B. The result, in terms of the Cartesian components of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$, is

$$\left\langle L_{ij} \right\rangle_0 = \langle 0 | \left(\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i \right) | 0 \rangle . \tag{6.93}$$

This convergence of results shows that indeed the fluctuations generate the angular momentum L_{ij}, as predicted by Eq. (6.92). Now, the numerical value $\left\langle L_{ij} \right\rangle_0 = 0$ for the ground state represents the net (average) angular momentum induced on the isotropic harmonic oscillator by the full ZPF. According to the discussion following Eq. (6.91), however, we should analyze separately the contributions arising from each of the two circular polarizations, characterized by the (circularly polarized) vectors

$$\epsilon_{k\pm} = \frac{1}{\sqrt{2}} \left(\epsilon_{ki} \pm i \epsilon_{kj} \right), \tag{6.94}$$

with ϵ_{ki}, ϵ_{kj} unit Cartesian vectors orthogonal to some axis $\boldsymbol{k}$, assumed to be fixed.

We will therefore now consider a situation in which the particle (the oscillator) is under the action of the background field of a given circular polarization with respect to an axis $\boldsymbol{k}$. Since according to the results in chapter 5 the response of the particle to the field is linear, when the particle is acted upon the (circularly) polarized field the appropriate variables to describe the system are the spherical variables (x^+, x^-, x_k), which are given, in accordance with (6.94), by,

$$x^{\pm} = \tfrac{1}{\sqrt{2}} \left(x_i \mp i x_j \right), \tag{6.95a}$$

$$x_i = \tfrac{1}{\sqrt{2}} \left(x^+ + x^- \right), \quad x_j = i \tfrac{1}{\sqrt{2}} \left(x^+ - x^- \right). \tag{6.95b}$$

The nonzero matrix elements of the oscillator are $x_{i01} = \left(x_{01}^+ + x_{01}^- \right) / \sqrt{2}$, and so on. Further, since $x_{10}^{\pm} = \left(x_{01}^{\mp} \right)^*$, Eq. (6.93) becomes

$$\left\langle L_{ij} \right\rangle_0 = m\omega_0 \left(x_{01}^+ x_{10}^- - x_{01}^- x_{10}^+ \right) = m\omega_0 \left(\left| x_{01}^+ \right|^2 - \left| x_{01}^- \right|^2 \right). \tag{6.96}$$

In the ground state, $\left\langle L_{ij} \right\rangle_0 = 0$; hence the two terms, $\left| x_{01}^+ \right|^2$ and $\left| x_{01}^- \right|^2$, contribute with equal magnitude and opposite sign to the k-th component of the total oscillator's angular momentum, as should be the case for a nonpolarized vacuum. These separate contributions are

$$\left\langle L_{ij} \right\rangle_0^+ = m\omega_0 \left| x_{01}^+ \right|^2, \quad \left\langle L_{ij} \right\rangle_0^- = -m\omega_0 \left| x_{01}^- \right|^2. \tag{6.97}$$

Using $x_{i01} = x_{j01} = \sqrt{\hbar/(2m\omega_0)}$ for the harmonic oscillator, one gets

$$m\omega_0 \left| x_{01}^{\pm} \right|^2 = \frac{\hbar}{2}, \tag{6.98}$$

whence the size of each separate contribution to the angular momentum in (6.96) is just $\hbar/2$. In order to distinguish this contribution from the (orbital) component of the angular momentum we write $\left\langle S_{ij} \right\rangle^{\pm}$ instead of $\left\langle L_{ij} \right\rangle_0^{\pm}$, so that

$$\left\langle S_{ij} \right\rangle^{\pm} = \pm \frac{\hbar}{2}. \tag{6.99}$$

Further, to study the square of the angular momentum we take into account that $\boldsymbol{L}^2$ corresponds to a 'classical' integral of motion for central forces; thus from equation (6.31) with $\xi = \boldsymbol{L}^2$, we get that for the (stationary) ground state, $\boldsymbol{L}^2$ satisfies the equation

$$m\tau\left\langle \dddot{x}_i \frac{\partial \boldsymbol{L}^2}{\partial p_i}\right\rangle_0 = e^2\left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i}\hat{\mathcal{D}}_i\right\rangle_0 . \tag{6.100}$$

The approximation $\dot{p}_k = f_k$ allows us to write $m\ddot{x}_k = -\omega_0^2 p_k$ for the harmonic oscillator. Further, since $\boldsymbol{L}^2 = \boldsymbol{r}^2\boldsymbol{p}^2 - (\boldsymbol{r}\cdot\boldsymbol{p})^2$ is a homogeneous function of $\boldsymbol{p}$ of degree 2, we have

$$p_i \frac{\partial \boldsymbol{L}^2}{\partial p_i} = 2\boldsymbol{L}^2. \tag{6.101}$$

With these results Eq. (6.100) reduces to

$$\left\langle \boldsymbol{L}^2\right\rangle_0 = -\frac{3mc^3}{4\omega_0^2}\left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i}\hat{\mathcal{D}}_i\right\rangle_0 . \tag{6.102}$$

The right-hand side of this equation is calculated in appendix C, and gives a result different from zero—even for s-states. Dividing again the full ensemble into two subensembles corresponding to different circular polarizations, one obtains

$$\left\langle \boldsymbol{L}^2\right\rangle_0 = \left\langle \boldsymbol{L}^2\right\rangle_0^+ + \left\langle \boldsymbol{L}^2\right\rangle_0^- , \tag{6.103}$$

where each separate contribution to the mean square angular angular momentum is given by

$$\left\langle \boldsymbol{L}^2\right\rangle_0^+ = \left\langle \boldsymbol{L}^2\right\rangle_0^- = \frac{3}{4}\hbar^2. \tag{6.104}$$

In terms of the notation introduced above (Eq. (6.99))—that distinguishes the (mean) orbital angular momentum from the (mean) angular momentum induced by the circular polarizations of the ZPF—Eq. (6.104) rewrites as

$$\left\langle \boldsymbol{S}^2\right\rangle^+ = \left\langle \boldsymbol{S}^2\right\rangle^- = \frac{3}{4}\hbar^2. \tag{6.105}$$

The fact that the results in Eqs. (6.99) and (6.105) do not depend on the oscillator's frequency ω_0, suggests that they hold in the general case, and for the free particle in particular (see Sect. 6.2.3). Therefore, we can conclude that when the transformation (6.95a, 6.95b) possesses physical meaning, so that the decompositions $\langle L_{ij}\rangle_0 = \langle L_{ij}\rangle_0^+ + \langle L_{ij}\rangle_0^-$ and $\langle \boldsymbol{L}^2\rangle_0^+ = \langle \boldsymbol{L}^2\rangle_0^+ + \langle \boldsymbol{L}^2\rangle_0^-$ make sense, equations (6.99) and (6.105) tell us that there exists an angular-momentum component that does *not* correspond to an orbital motion of the particle. It represents the angular momentum derived from the interaction of the particle with a given circular polarized mode of the ZPF. For an electron, which (as stated above) interacts with the radiation field via its circular polarized modes, the transformation (6.95) is indeed physically meaningful; thus the angular momentum induced by the ZPF can be identified with the electron's spin. The term 'intrinsic' usually attached to it, points

to the permanence of this effect (the ZPF is always and everywhere present), although here appears as induced on (or acquired by) the particle.

We have thus disclosed the existence of the vector $\boldsymbol{S}$ (with k-component $\epsilon_{ijk}S_{ij}$) that stands for the intrinsic spin-angular momentum of a charged particle for which the circularly polarized modes of the ZPF are physically relevant. In this form we see that the theory does contain the elements to account for the spin of the electron. The extension of these results to the general case (arbitrary potential), as well as the construction of the quantum operator corresponding to the vector $\boldsymbol{S}$, are left for the following section.

One additional point merits consideration. According to these results a charged particle acquires spin 1/2; so what can be said about scalar bosons? A possible answer to this question is that the spin 1/2 is acquired by elementary particles, such as the electron, whereas composite particles may acquire it or not, depending on their structure and their specific interaction with the radiation field. Thus, bosons come out to be composite structures, with an even number of elements (if of fermion type).

6.2.3 General Derivation of the Electron Spin

Let us now present a general procedure to derive the spin of the electron, based on the Heisenberg formalism of QM developed in Chap. 5. According to the above discussion, our ansatz is that the electron spin is an angular momentum of the particle generated by its interaction with the ZPF, specifically with one of the circular polarizations of the field. In contrast with the previous section, here the electron is subject to an arbitrary external central force, and in any given stationary state. Denoting with the index n the set of quantum numbers that characterize the state of the particle, including the orbital angular momentum and its projection along the z axis, we have (for simplicity in the writing we use $x_i = x$, $x_j = y$, and $x_k = z$)

$$\begin{aligned}\left\langle \hat{L}_z \right\rangle_n &= \langle n|\, \hat{L}_z\, |n\rangle = \sum\nolimits_k \left(x_{nk}\, p_{ykn} - y_{nk}\, p_{xkn}\right) \\ &= im \sum\nolimits_k \omega_{kn}\, (x_{nk}\, y_{kn} - y_{nk} x_{kn})\,. \end{aligned} \tag{6.106}$$

To take into account that the electron responds to modes of a given circular polarization of the ZPF, we transform again to the variables defined in Eqs. (6.95). Following the same procedure that led to (6.96), Eq. (6.106) transforms into

$$\left\langle \hat{L}_z \right\rangle_n = m \sum\nolimits_k \omega_{kn} \left(\left|x_{nk}^{+}\right|^2 - \left|x_{nk}^{-}\right|^2 \right). \tag{6.107}$$

This expression can be rewritten as

$$\left\langle \hat{L}_z \right\rangle_n = \langle O_z \rangle_n^{+} + \langle O_z \rangle_n^{-}\,, \tag{6.108}$$

with $\langle O_z\rangle_n^{\pm}$ given by ($\sigma = \pm$)

$$\langle O_z\rangle_n^{\sigma} = \sigma m \sum\nolimits_k \omega_{kn} \left|x_{nk}^{\pm}\right|^2 . \tag{6.109}$$

On the other hand, from the commutator $\left[\hat{x},\, \hat{p}_x\right] = i\hbar$ and $x_{kn} = x_{nk}^*$ it follows that

$$m \sum\nolimits_k \omega_{kn} \left|x_{nk}\right|^2 = \tfrac{1}{2}\hbar \tag{6.110}$$

and the same for $|y_{nk}|^2$. From (6.106) we obtain therefore

$$\hbar = m \sum\nolimits_k \omega_{kn} \left(\left|x_{nk}^{+}\right|^2 + \left|x_{nk}^{-}\right|^2\right) = \langle O_z\rangle_n^{+} - \langle O_z\rangle_n^{-} , \tag{6.111}$$

which combined with (6.108) gives

$$\langle O_z\rangle_n^{\sigma} = \tfrac{1}{2}\left\langle \hat{L}_z \right\rangle_n + \sigma \tfrac{1}{2}\hbar. \tag{6.112}$$

This result helps to avoid a possible confusion about the meaning of equation (6.108). It is simply

$$\left\langle \hat{L}_z \right\rangle_n = \tfrac{1}{2}\left\langle \hat{L}_z \right\rangle_n + \tfrac{1}{2}\hbar + \tfrac{1}{2}\left\langle \hat{L}_z \right\rangle_n - \tfrac{1}{2}\hbar. \tag{6.113}$$

Thus the spin projection is contained in the orbital angular momentum $\langle \hat{L}_z\rangle_n$ in a *dormant* form, so to say. It is the quantity $\langle O_z\rangle_n^{\sigma}$ what contains both a part of the orbital angular momentum and the spin associated with one or the other of the polarization states, $\sigma = \pm$.

To construct the (quantum) operator associated with the vector $\boldsymbol{S}$, we proceed as follows. Clearly the mean value $\langle \hat{L}_z\rangle_n$ does not depend on σ, whereas the term $\sigma\hbar/2$ does not depend on the set n; this shows that the operator $\hat{L}_z$ and the operator to be associated with $\sigma\hbar/2$ (which we shall call $\hat{\Sigma}_z$) belong to different Hilbert spaces. Therefore, in order to express $\langle O_z\rangle_n^{\sigma}$ in (6.112) as the average of an operator, we must extend the Hilbert space to include the dichotomous variable σ in addition to the quantum index n. This we do by resorting to the product space $\mathcal{H} = \mathcal{H}_n \otimes \mathcal{H}_2$, with $\mathcal{H}_2$ a bidimensional vector space spanned by an orthonormal basis having as elements the vectors $\{|\sigma\rangle\} = \{|+\rangle\, , |-\rangle\}$. In terms of $|n\sigma\rangle = |n\rangle \otimes |\sigma\rangle$, Eq. (6.112) rewrites as

$$\langle O_z\rangle_n^{\sigma} = \tfrac{1}{2}\, \langle n\sigma|\, \hat{L}_z\, |n\sigma\rangle + \tfrac{1}{2}\hbar\, \langle n\sigma|\, \hat{\Sigma}_z\, |n\sigma\rangle\, , \tag{6.114}$$

with $\hat{\Sigma}_z$ an operator that has $|\sigma\rangle$ as eigenvector,

$$\langle n\sigma|\, \hat{\Sigma}_z\, |n\sigma\rangle = \langle \sigma|\, \hat{\Sigma}_z\, |\sigma\rangle = \sigma. \tag{6.115}$$

Expressing $\hat{\Sigma}_z$ in the general form in terms of the Pauli matrices gives

$$\hat{\boldsymbol{\Sigma}}_z = a_0 \mathbb{I} + a_z \hat{\sigma}_z + a^+ \hat{\sigma}_+ + a^- \hat{\sigma}_-, \tag{6.116}$$

where $a^{\pm} = \left(a_x \mp i a_y\right)/\sqrt{2}$, and $\hat{\sigma}_+ = \sqrt{2}\,|+\rangle\langle -|$, $\hat{\sigma}_- = \sqrt{2}\,|-\rangle\langle +|$ are ladder operators. Condition (6.115) imposed on $\hat{\boldsymbol{\Sigma}}_z$ gives $a_0 = 0$, $a_z = 1$. Further, since we are here considering the variables (x^+, x^-, z), the polarization vectors (6.94) fix $\hat{z}$ as the preferred axis, whence $a^{\pm} = 0$, $\hat{\boldsymbol{\Sigma}}_z = \hat{\sigma}_z$, and Eq. (6.114) becomes

$$\langle O_z \rangle_n^{\sigma} = \langle n\sigma | \left(\tfrac{1}{2} \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}} \right) \cdot \hat{\boldsymbol{z}} \, | n\sigma \rangle \,, \tag{6.117}$$

with $\hat{\boldsymbol{S}}$ the vector operator defined as $\hat{\boldsymbol{S}}_z = \hbar \frac{1}{2} \hat{\Sigma}_z$, i.e.,

$$\hat{\boldsymbol{S}} = \tfrac{1}{2} \hbar \hat{\boldsymbol{\sigma}}. \tag{6.118}$$

The identification of the operator $\hat{\boldsymbol{S}}$ with the spin of the electron is thus justified. The independence of $\langle \hat{L}_z \rangle_n$ from σ and of $\langle \hat{\Sigma}_z \rangle$ from n, indicates that under the present conditions, the fluctuations associated with the spin are not correlated with those that characterize the kinematics of the particle in the configuration space: $\boldsymbol{L}$ and $\boldsymbol{S}$ constitute independent dynamical variables. Of course the spaces of the spin and of the orbital angular momentum may become connected by the presence of magnetic fields, which here have been omitted.

It should be stressed that even if the orbital momentum $\hat{\boldsymbol{L}}$ and the spin $\hat{\boldsymbol{S}}$ are both contained in the same expression for the angular motions, Eq. (6.113), this does not mean that the spin is treated here as an orbital angular momentum. Indeed, as is well known, there are fundamental differences between $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{S}}$. In particular, the mean value of $\hat{L}_z$, say, can be freely determined by adjusting external parameters, and may acquire a whole spectrum of values. However, only the sign of the projection $\hat{S}_z$ can be subject to external adjustment; its absolute value is determined by the fundamental commutator through (6.110), which in its turn is fixed by the ZPF. It is because of the *universal* value of the commutator that the spin of the electron is the same for all electrons under all circumstances, which reinforces its apparent 'intrinsic' nature.

The connection of the commutator $\left[\hat{x}, \hat{p}_x\right] = i\hbar$ with the spin of the electron deserves a couple of additional comments. As pointed out in chapter 4, Planck's constant $\hbar$ is a direct measure of the size of the fluctuations, both those of the ZPF and those impressed by it on the particle. Specifically, since the commutator implies that the fluctuations of x and p_x have a minimum value adjusted to the rule $\sigma_x^2 \sigma_{p_x}^2 \Big|_{\min} = \hbar^2/4$, one may write the *numerical* relation

$$\left| \langle \pm | \, \hat{S}_z \, | \pm \rangle \right| = \tfrac{1}{2} \hbar = \sigma_x \sigma_{p_x} \big|_{\min} \,, \tag{6.119}$$

which emphasizes the fact that the *value* of the electron spin is determined by the irreducible fluctuations of the phase-space variables x, p_x around the instantaneous position of the particle.

In QED the interaction of the electron with the vacuum leads to an effective radius of the order of Compton's wavelength λ_C. Similarly, in Chap. 9 we argue that the fluctuating motion of the electron assigns to it an effective structure, a result similar to that referred to in the discussion at the end of Sect. 6.1.5 above. From this point of view, the electron can still be considered as 'essentially' pointlike, but simultaneously possessing an 'effective' (measurable) size. The small, rapid trembling of the electron, taken as a (here nonrelativistic) zitterbewegung, suggests that it is tightly linked with the spin of the electron.[9] The appearance of an effective structure helps also to get an understanding of another important quality of the electron, namely its magnetic moment, a matter that is succinctly addressed in Sect. 6.2.5.

6.2.4 Angular Momentum of the Zero-Point Field

As a heuristic aid to our explanation of the spin of the electron, let us recall some of the most basic properties of the angular momentum of the radiation field. Since the detailed calculation is rather long, we present here a sketch of it (borrowing from Mandel and Wolf 1995, Sect. 10.6).

The total angular momentum operator of the electromagnetic field is usually defined as

$$\hat{\boldsymbol{J}} = \int_V \left(\boldsymbol{r} \times \hat{\boldsymbol{P}} \right) d^3 r, \tag{6.120}$$

where $\hat{\boldsymbol{P}}$ stands for the linear momentum density operator (ε_0 is the electric permeability of the vacuum)

$$\hat{\boldsymbol{P}} = \tfrac{1}{2}\varepsilon_0 \left[\hat{\boldsymbol{E}}(\boldsymbol{r}, t) \times \hat{\boldsymbol{B}}(\boldsymbol{r}, t) - \hat{\boldsymbol{B}}(\boldsymbol{r}, t) \times \hat{\boldsymbol{E}}(\boldsymbol{r}, t) \right]. \tag{6.121}$$

The expression (6.120) for $\hat{\boldsymbol{J}}$ can be decomposed into a term $\hat{\boldsymbol{J}}_L$ that depends on $\boldsymbol{r}$ and can therefore be identified with the orbital angular momentum of the field, plus a second term, $\hat{\boldsymbol{J}}_S$, independent of $\boldsymbol{r}$ and thus interpreted as an intrinsic angular momentum. Specifically,

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[9] The zitterbewegung is a phenomenon predicted by the Dirac equation for the electron. It consists of an oscillation (a trembling) around the relativistic motion of amplitude of order λ_C and frequency of order $2mc^2/\hbar$. The nonrelativistic 'zitterbewegung' discussed here differs from the relativistic jitter in that it involves nonrelativistic velocities and contains a wide spectrum of frequencies.

The often conjectured connection between spin and zitterbewegung was proposed for the first time by Schrödinger (1930) and investigated more deeply by Dirac (1958) (see also Maddox (1987)). Examples of related works, which include specific models of varying fortune, are Bhabha and Corben (1941), Huang (1952), Corben (1968), Barut and Zhangi (1984), Hestenes (1985), Hestenes (1990), Pavšič et al. (1993), Rodrigues et al. (1993), Rodrigues et al. (1998). Other mechanisms to generate zitterbewegung-like oscillations have been explored using relativistic models of the electron. For instance, by considering within SED that the structure of the particle is related to the difference between the centers of inertia and charge of the particle, it has been shown that the electron responds to the random field by performing a zitterbewegung (see e.g. Rueda 1993; also Cavalleri 1985, Cavalleri et al. 2010).

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_L + \hat{\boldsymbol{J}}_S, \tag{6.122}$$

where

$$\begin{aligned}\hat{\boldsymbol{J}}_L &= \frac{\varepsilon_0}{2}\int_V \sum_{i=1}^{3}\left\{\hat{E}_i\,(\boldsymbol{r}\times\boldsymbol{\nabla})\,\hat{A}_i + \left[(\boldsymbol{r}\times\boldsymbol{\nabla})\,\hat{A}_i\right]\hat{E}_i\right\} d^3r,\\ \hat{\boldsymbol{J}}_S &= \frac{\varepsilon_0}{2}\int_V \left(\hat{\boldsymbol{E}}\times\hat{\boldsymbol{A}} - \hat{\boldsymbol{A}}\times\hat{\boldsymbol{E}}\right) d^3r,\end{aligned} \tag{6.123}$$

and $\hat{\boldsymbol{A}}$ stands for the electromagnetic vector potential operator. Equation (6.123) shows that $\hat{\boldsymbol{J}}_S$ can give a result different from zero because the electromagnetic field is a vector field (i.e., because it has more than one spatial component); this endows the components $\hat{\boldsymbol{J}}_L$ and $\hat{\boldsymbol{J}}_S$ with very different properties and meaning. We use a discrete expansion in terms of plane waves, as is usual in QED (in the full three-dimensional reciprocal space $\boldsymbol{k}$, and with $\hat{a}^\dagger_{\boldsymbol{k}\sigma}$, $\hat{a}_{\boldsymbol{k}\sigma}$ creation and annihilation operators)

$$\hat{\boldsymbol{A}}\,(\boldsymbol{r},t) = \frac{1}{V^{1/2}}\sum_{\boldsymbol{k},\sigma}\left(\frac{\hbar}{2\omega\varepsilon_0}\right)^{1/2}\left[\boldsymbol{\epsilon}_{\boldsymbol{k}\sigma}\hat{a}_{\boldsymbol{k}\sigma}\,(0)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} + \text{h.c.}\right], \tag{6.124}$$

with $\boldsymbol{\epsilon}_{\boldsymbol{k}\sigma}$ the orthogonal circular polarization vectors given by (6.94). Once the integration in (6.123) is carried out, $\hat{\boldsymbol{J}}_S$ reduces to

$$\hat{\boldsymbol{J}}_S = \sum_{\boldsymbol{k},\sigma=\pm1}\hbar\hat{\boldsymbol{k}}\sigma\left(\hat{n}_{\boldsymbol{k}\sigma} + \tfrac{1}{2}\right), \tag{6.125}$$

with $\hat{n}_{\boldsymbol{k}\sigma} = \hat{a}^\dagger_{\boldsymbol{k}\sigma}\hat{a}_{\boldsymbol{k}\sigma}$ the photon number operator in the basis of circular polarization, and $\hat{\boldsymbol{k}}$ a unit vector in the direction of the wave vector $\boldsymbol{k}$. Equation (6.125) is an expansion in states of definite helicity, which assigns to individual photons a spin angular momentum projection of value $\sigma\hbar = \pm\hbar$ along the direction $\hat{\boldsymbol{k}}$. Of major importance for what follows is that in the absence of photons, with only the ZPF present, each mode of the vacuum state still contains a component of angular momentum, with mean value given by

$$\left\langle\hat{\boldsymbol{J}}_S\right\rangle_{\text{vac}} = \sum_{\boldsymbol{k},\sigma=\pm1}\frac{\hbar}{2}\hat{\boldsymbol{k}}\sigma = \sum_{\boldsymbol{k}}\tfrac{1}{2}\hbar\hat{\boldsymbol{k}} - \sum_{\boldsymbol{k}}\tfrac{1}{2}\hbar\hat{\boldsymbol{k}}. \tag{6.126}$$

Therefore, a nonzero contribution to $\langle\hat{\boldsymbol{J}}_S\rangle_{\text{vac}} = \pm\frac{1}{2}\hbar\hat{\boldsymbol{k}}$ is associated with every mode $(\boldsymbol{k},\sigma)$ of the ZPF. For the unpolarized field the contributions of the right- and left-hand polarizations compensate each other (for each $\boldsymbol{k}$), and $\langle\hat{\boldsymbol{J}}_S\rangle_{\text{vac}}$ vanishes. This is the reason why the term $1/2$ in Eq. (6.125) is frequently omitted.

Consider now particles that couple with the right or left circularly polarized portions of the vacuum field. Equation (6.126) suggests that they acquire a com-

ponent of angular momentum, just as they acquire energy or linear momentum, as a result of the coupling.

The above decomposition (6.122) into orbital and spinorial components of the radiation field possesses only a relative value. A detailed relativistic treatment of the field (recall that the radiation field *is* relativistic) shows that only the total angular momentum $\boldsymbol{J}$ satisfies a conservation law, but not its separate parts. The reason is that such decomposition is in general neither covariant nor gauge invariant, so it lacks a well-defined physical meaning. A detailed discussion of these matters can be seen in Rohrlich (1965) .

6.2.5 Gyromagnetic Factor for the Electron

It was discovered experimentally that the g-factor associated with the spin magnetic moment of the electron has an approximate value $g_S = 2$, whereas for the orbital magnetic moment the g-factor is $g_L = 1$. This characteristic value of g_S is incorporated into nonrelativistic quantum theory by hand, usually without further elaboration (it must be remarked that from the theory of Lande's factor it follows that for $L = 0$, $g = 2$; see Greiner 1998, Sect. 11.9). The issue is normally solved by resorting to the Dirac equation, which predicts the value $g_S = 2$ (plus corrections arising from QED). Since the present theory produces the electron spin, it becomes of interest to investigate the value predicted by it for the factor g_S.

Traditionally the gyromagnetic ratio of the electron has not been a subject for SED, due to the fact that the theory has paid little attention to the spin itself, as mentioned earlier. An exception to this is the (quite elaborate) calculation made in de la Peña and Jáuregui (1982) (see also *The Dice*, Sect. 8.3.4), using as a starting point the Fokker-Planck equation in the Markovian approximation. The problem studied was a spherical harmonic oscillator of natural frequency ω_0, subject to an external homogeneous magnetic field $\boldsymbol{B}$ in the z -direction. In terms of the Larmor frequency $\omega_L = |e|\, B/2mc$, the procedure led in the weak-field limit, when $\omega_L \ll \omega_0$, to a total average energy given (in the present notation) by

$$\mathcal{E}_\sigma = \frac{1}{2}\left[\frac{3}{2}\hbar\omega_0 + \omega_L\left(\langle L_z\rangle + 2S_z^\sigma\right)\right], \tag{6.127}$$

where the overall factor $1/2$ comes from the fact that the average is taken over half the ensemble, for a given value of σ. This expression contains already the correct result $g_S = 2$ in front of the term S_z^σ. To be precise, one should add that the referred calculations were made using the methods characteristically employed in SED during the eighties. Such methods led to the occurrence of some erroneous coefficients, although the correct Eq. (6.127) was derived.

With the tools developed so far one can now make a straightforward calculation of g_S. For this purpose consider the electron acted on, in addition to the external force $\boldsymbol{f}(\boldsymbol{x})$, by a static uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The contribution of the orbital

angular momentum $\boldsymbol{L}$ to the Hamiltonian is given by

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B} = -\mu_z B, \tag{6.128}$$

where $\hat{\boldsymbol{\mu}} = -(g\mu_0 \hat{\boldsymbol{L}})/\hbar$ is the magnetic moment due to $\hat{\boldsymbol{L}}$, $\mu_0 = |e|\,\hbar/(2mc)$ is the Bohr magneton (with $-e = |e|$), and $g_L = 1$. Therefore the mean energy is

$$\mathcal{E} = -\frac{e}{2mc} B \langle \hat{L}_z \rangle = \frac{\mu_0}{\hbar} B \langle \hat{L}_z \rangle. \tag{6.129}$$

Consider a situation in which the spin projection along $\hat{\boldsymbol{z}}$ has a well-defined value, say $\langle \hat{S}_z \rangle = +\hbar/2$. This means that one should take into account only the action of the subensemble of the ZPF that corresponds to $\sigma = +$. Resorting to Eq. (6.112) to write the contribution to $\langle L_z \rangle$ from the subensemble with $\sigma = +$ as $(\langle \hat{L}_z \rangle + \hbar)/2$, the component of $\mathcal{E}$ of interest is

$$\mathcal{E}^+ = \frac{\mu_0}{\hbar} B \left(\frac{1}{2} \langle \hat{L}_z \rangle + \frac{\hbar}{2} \right) = \frac{\mu_0}{2\hbar} B \left(\langle \hat{L}_z \rangle + 2 \langle \hat{S}_z \rangle^+ \right). \tag{6.130}$$

An analogous result holds for the subensemble with $\sigma = -$, for which $\langle \hat{S}_z \rangle = -\hbar/2$,

$$\mathcal{E}^- = \frac{\mu_0}{\hbar} B \left(\frac{1}{2} \langle \hat{L}_z \rangle - \frac{\hbar}{2} \right) = \frac{\mu_0}{2\hbar} B \left(\langle \hat{L}_z \rangle + 2 \langle \hat{S}_z \rangle^- \right). \tag{6.131}$$

The corresponding Hamiltonians describing each part of the magnetic interaction of the electron are therefore $\hat{H}^+_{LS}$, $\hat{H}^-_{LS}$, with

$$\hat{H}^+_{LS} = \frac{\mu_0}{2\hbar} B \left(\hat{L}_z + 2\hat{S}_z \right) = \hat{H}^-_{LS}; \tag{6.132}$$

thus the complete Hamiltonian (which includes both polarizations) reads

$$\hat{H}_{LS} = \hat{H}^+_{LS} + \hat{H}^-_{LS} = \frac{\mu_0}{\hbar} B \left(\hat{L}_z + 2\hat{S}_z \right). \tag{6.133}$$

This contains the correct g-factor of 2 for the spin of the electron. It is clear that such value derives from the two degrees of freedom associated with the polarization of the ZPF.

The result (6.133) gives a precise meaning to the operator appearing in Eq. (6.117). Indeed, from this latter equation one can write $\hat{\boldsymbol{O}} = (\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}})/2$, whence

$$\hat{H}_{LS} = \frac{\mu_0}{\hbar} \boldsymbol{B} \cdot \left(\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}} \right) = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B}, \tag{6.134}$$

with

$$\hat{\boldsymbol{\mu}} = -\frac{2\mu_0}{\hbar} \hat{\boldsymbol{O}}. \tag{6.135}$$

This directly relates $\hat{\boldsymbol{O}}$ with the total magnetic moment operator of the atomic electron.

Since with the present results we have at hand the usual theory of the electron spin, it is straightforward to incorporate it as usual to the Schrödinger equation and thus arrive at the Pauli equation.

6.3 Concluding Comments

We have found that the theory predicts radiative corrections to the results derived with the Schrödinger equation, which to lowest-order coincide with the corresponding ones of nonrelativistic QED. This is not a coincidence, since both theories are essentially equivalent in their physical content (to the order of approximation here studied), although very different in their conceptual perspective. In addition, just as the quantum-mechanical behavior at the Schrödinger (or Heisenberg) level ensues from the interaction with the ZPF, also the spin of the electron emerges as a result of such interaction. This is a most noteworthy outcome, since in quantum mechanics the spin is considered to be an innate property of the electron.

Appendix A

Contribution of Diffusion to the Energy Shift

In Sect. 6.1.5, Eq. (6.51) is obtained for the radiative shift of the energy level n. To calculate its value in the Markovian approximation (which amounts to taking $e^2\hat{\mathcal{D}}(t)Q$ to lowest order in e^2, as explained in Chap. 4) we start from the expression for $\hat{\mathcal{D}}$ in terms of the diffusion operators, Eq. (4.17),

$$e^2\hat{\mathcal{D}}_i = D_{ij}^{pp}\frac{\partial}{\partial p_j} + D_{ij}^{px}\frac{\partial}{\partial x_j}, \tag{A.1}$$

with

$$D_{ij}^{pp} = e^2\int_{-\infty}^{t} dt'\varphi(t-t')\frac{\partial p_j}{\partial p_i'}, \quad D_{ij}^{px} = e^2\int_{-\infty}^{t} dt'\varphi(t-t')\frac{\partial x_j}{\partial p_i'}. \tag{A.2}$$

We thus have (in one-dimensional notation)

$$e^2x\hat{\mathcal{D}}Q = D^{pp}x\frac{\partial Q}{\partial p} + D^{px}x\frac{\partial Q}{\partial x}. \tag{A.3}$$

Performing the integration over phase space we get

$$e^2\left\langle x\hat{\mathcal{D}}\right\rangle = e^2\int x\hat{\mathcal{D}}Q dx dp = \int\left(D^{pp}x\frac{\partial Q}{\partial p} + D^{px}x\frac{\partial Q}{\partial x}\right)dx dp. \tag{A.4}$$

Upon an integration by parts this becomes

$$e^2\left\langle x\hat{\mathcal{D}}\right\rangle = -\left\langle D^{px} + x\left(\frac{\partial D^{pp}}{\partial p} + \frac{\partial D^{pp}}{\partial x}\right)\right\rangle = -\left\langle D^{px}\right\rangle, \tag{A.5}$$

where the last equality follows from Eq. (B.15) in Appendix 4B, namely

$$\frac{\partial D_{ij}^{pp}}{\partial p_j} + \frac{\partial D_{ij}^{px}}{\partial x_j} = 0. \tag{A.6}$$

For the calculation of $\langle D^{px}\rangle_n$ we resort to equation (B.14b) with $2\eta = \hbar$, which gives

$$\left\langle D^{px}\right\rangle_n = \frac{ie^2}{\hbar}\int_{-\infty}^{t} dt'\varphi(t-t')\left\langle\left[\hat{x}(t),\hat{x}(t')\right]\right\rangle_n, \tag{A.7}$$

so that

$$e^2\left\langle x\hat{\mathcal{D}}\right\rangle_n = -\frac{ie^2}{\hbar}\int_{-\infty}^{t} dt'\varphi(t-t')\left\langle\left[\hat{x}(t),\hat{x}(t')\right]\right\rangle_n. \tag{A.8}$$

The mean value of the commutator is

$$\left\langle\left[\hat{x}(t),\hat{x}(t')\right]\right\rangle_n = -2i\sum_k |x_{nk}|^2 \sin\omega_{kn}(t-t').$$

Thus, with $\varphi(t-t')$ given by Eq. (4.10), i.e.,

$$\varphi(t-t') = \frac{4\pi}{3}\int_0^\infty \rho_0(\omega)\cos\omega(t-t')d\omega, \tag{A.9}$$

and ρ_0 given by (6.4), Eq. (A.8) gives

$$e^2\left\langle x\hat{\mathcal{D}}\right\rangle_n = -\frac{4e^2}{3\pi c^3}\sum_k |x_{nk}|^2\int_0^\infty d\omega\,\omega^3\int_{-\infty}^{t} dt'\cos\omega(t-t')\sin\omega_{kn}(t-t'). \tag{A.10}$$

Making the change of variable $t - t' = s$ and introducing

$$\int_0^\infty ds \cos\omega s \sin\omega_{kn} s = \tfrac{1}{2}\int_0^\infty ds\,[\sin(\omega_{kn}+\omega)s + \sin(\omega_{kn}-\omega)s] \qquad (A.11)$$
$$= \frac{\omega_{kn}}{\omega_{kn}^2 - \omega^2},$$

in Eq. (A.10) gives, finally,

$$\frac{e^2}{2}\left\langle x\hat{D}\right\rangle_n = -\frac{2e^2}{3\pi c^3}\sum_k |x_{nk}|^2\,\omega_{kn}\int_0^\infty d\omega\,\frac{\omega^3}{\omega_{kn}^2-\omega^2}. \qquad (A.12)$$

Appendix B

Angular-Momentum Components for the Harmonic Oscillator

In this appendix we calculate the right-hand side of Eq. (6.92) for the harmonic oscillator in its ground state,

$$\left\langle L_{ij}\right\rangle_0 = \frac{1}{\tau\omega_0^2}\langle D_{ji}^{px} - D_{ij}^{px}\rangle. \qquad (B.1)$$

With D_{ij}^{px} given by Eq. (B.14b) in appendix 4B (and $2\eta = \hbar$), we get

$$\langle D_{ji}^{px} - D_{ij}^{px}\rangle_{\mathrm{Q}} = \frac{-2ie^2}{3\pi c^3}\int_0^\infty d\omega\,\omega^3\int_{-\infty}^t dt'\cos\omega(t-t')\left\langle\left[\hat{x}_j', \hat{x}_i\right] - \left[\hat{x}_i', \hat{x}_j\right]\right\rangle. \qquad (B.2)$$

For the ground state we have

$$\left\langle\left[\hat{x}_j(t'), \hat{x}_i(t)\right] - \left[\hat{x}_i(t'), \hat{x}_j(t)\right]\right\rangle_0 \qquad (B.3)$$
$$= -2\sum_k \left(x_{i0k}x_{jk0} - x_{j0k}x_{ik0}\right)\cos\omega_{k0}(t-t'),$$

where the summation is performed over all possible excited states $k > 0$ connected to the ground state via the matrix elements x_{i0k}. Inserting this expression into (B.2) and resorting to Eq. (D.8) in appendix 4D, namely

$$\int_{-\infty}^t dt'\cos\omega(t-t')\cos\omega_{k0}(t-t') = \frac{\pi}{2}[\delta(\omega-\omega_{k0}) + \delta(\omega+\omega_{k0})], \qquad (B.4)$$

we obtain (with $\omega_{k0} > 0$)

$$\langle D_{ji}^{px} - D_{ij}^{px} \rangle_0 = \frac{2e^2}{3mc^3} \sum_k im\omega_{k0}^3 \left(x_{i0k} x_{jk0} - x_{j0k} x_{ik0} \right)$$
$$= \tau \sum_k \omega_{k0}^2 \left(x_{i0k} p_{jk0} - x_{j0k} p_{ik0} \right). \tag{B.5}$$

For the harmonic oscillator the only term that contributes to the sum is $k = 1$, with $\omega_{k0} = \omega_0$, whence Eq. (B.5) reads

$$\langle D_{ji}^{px} - D_{ij}^{px} \rangle_0 = \tau \omega_0^2 \left(x_{i01} p_{j10} - x_{j01} p_{i10} \right) = \tau \omega_0^2 \left\langle L_{ij} \right\rangle_0. \tag{B.6}$$

Thus Eq. (B.1) reduces to an apparent tautology on account of the dynamics.

Appendix C

Calculation of $\langle S^2 \rangle$

To find $\left\langle \boldsymbol{L}^2 \right\rangle^{\pm}$ for the harmonic oscillator in its ground state we use Eq. (6.102), namely

$$-2\omega_0^2 \tau \left\langle \boldsymbol{L}^2 \right\rangle_0 = e^2 \left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle_0. \tag{C.1}$$

In order to calculate the right-hand side of this equation we proceed as in appendix A. Specifically, we resort to Eq. (A.1) to write

$$e^2 \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i Q = \frac{\partial \boldsymbol{L}^2}{\partial p_i} D_{ij}^{pp} \frac{\partial Q}{\partial p_j} + \frac{\partial \boldsymbol{L}^2}{\partial p_i} D_{ij}^{px} \frac{\partial Q}{\partial x_j}. \tag{C.2}$$

Integration of this expression over phase space gives (after an integration by parts),

$$e^2 \left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle = \int \left(\frac{\partial \boldsymbol{L}^2}{\partial p_i} D_{ij}^{pp} \frac{\partial Q}{\partial p_j} + \frac{\partial \boldsymbol{L}^2}{\partial p_i} D_{ij}^{px} \frac{\partial Q}{\partial x_j} \right) dx dp \tag{C.3}$$
$$= -\left\langle \frac{\partial}{\partial p_j} \left(\frac{\partial \boldsymbol{L}^2}{\partial p_i} D_{ij}^{pp} \right) + \frac{\partial}{\partial x_j} \left(\frac{\partial \boldsymbol{L}^2}{\partial p_i} D_{ij}^{px} \right) \right\rangle$$
$$= -\left\langle D_{ij}^{pp} \frac{\partial^2 \boldsymbol{L}^2}{\partial p_j \partial p_i} + D_{ij}^{px} \frac{\partial^2 \boldsymbol{L}^2}{\partial x_j \partial p_i} \right\rangle,$$

where in the third line we used Eq. (A.6). Taking into account that for the isotropic harmonic oscillator

$$\frac{\partial p_j}{\partial p_i'} = \delta_{ij} \cos \omega_0 (t - t'), \quad \frac{\partial x_j}{\partial p_i'} = \delta_{ij} \frac{1}{m\omega_0} \sin \omega_0 (t - t'), \tag{C.4}$$

equation (A.2) shows that the diffusion coefficients are diagonal,

$$D_{ij}^{pp} = \delta_{ij} e^2 \int_{-\infty}^{t} dt' \varphi(t-t') \cos\omega_0 (t-t') = \delta_{ij} D^{pp},$$
$$D_{ij}^{px} = \delta_{ij} \frac{e^2}{m\omega_0} \int_{-\infty}^{t} dt' \varphi(t-t') \sin\omega_0 (t-t') = \delta_{ij} D^{px},$$

whence Eq. (C.3) reduces to

$$e^2 \left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle = -\left\langle D^{pp} \frac{\partial^2 \boldsymbol{L}^2}{\partial p_i \partial p_i} + D^{px} \frac{\partial^2 \boldsymbol{L}^2}{\partial x_i \partial p_i} \right\rangle, \tag{C.5}$$

where summation over i is understood.

Now, since $\boldsymbol{L}^2 = \boldsymbol{r}^2 \boldsymbol{p}^2 - (\boldsymbol{r} \cdot \boldsymbol{p})^2$, we have that

$$\frac{\partial^2 \boldsymbol{L}^2}{\partial p_i \partial p_i} = 4\boldsymbol{r}^2, \quad \frac{\partial^2 \boldsymbol{L}^2}{\partial x_i \partial p_i} = 0, \tag{C.6}$$

and (C.5) reads

$$e^2 \left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle = -4e^2 \left\langle \boldsymbol{r}^2 \right\rangle \int_{-\infty}^{t} dt' \varphi(t-t') \cos\omega_0 (t-t'). \tag{C.7}$$

With $\varphi(t-t')$ given by (A.9), Eq. (C.7) becomes, for the ground state,

$$e^2 \left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle_0 = -\frac{8\hbar e^2}{3\pi c^3} \left\langle \boldsymbol{r}^2 \right\rangle_0 \int_0^{\infty} d\omega\, \omega^3 \int_{-\infty}^{t} dt' \cos\omega (t-t') \cos\omega_0 (t-t'). \tag{C.8}$$

For the integral over t' we use Eq. (B.4), so that

$$e^2 \left\langle \frac{\partial \boldsymbol{L}^2}{\partial p_i} \hat{\mathcal{D}}_i \right\rangle_0 = -\frac{4\hbar e^2}{3c^3} \omega_0^3 \left\langle \boldsymbol{r}^2 \right\rangle_0. \tag{C.9}$$

The factor $\langle \boldsymbol{r}^2 \rangle_0$ is calculated in accordance with the quantum methods, giving

$$\langle \boldsymbol{r}^2 \rangle_0 = 3\langle \hat{x}^2 \rangle_0 = 3x_{01}x_{10} = \frac{3\hbar}{2m\omega_0}.$$

Equation (C.1) becomes finally (Marshall 1965; de la Peña and Jáuregui 1982)

$$\left\langle \boldsymbol{L}^2 \right\rangle_0 = m\hbar\omega_0 \left\langle \boldsymbol{r}^2 \right\rangle_0 = \frac{3}{2}\hbar^2. \tag{C.10}$$

This value of $\langle \boldsymbol{L}^2 \rangle_0$ takes into account the action of the whole ZPF, with both states of circular polarization active. Considering the action of a single state of polarization $\sigma = \pm$, we get

$$\left\langle \boldsymbol{S}^2 \right\rangle^{\sigma} \equiv \left\langle \boldsymbol{L}^2 \right\rangle_0^{\sigma} = \frac{3}{4}\hbar^2. \tag{C.11}$$

References

*Barranco, A.V., Brunini, S.A., Franca, H.M.: Spin and paramagnetism in classical stochastic electrodynamics. Phys. Rev. A **39**, 5492 (1989)

Barut, A.O., Zhangi, N.: Classical model of the Dirac electron. Phys. Rev. Lett. **52**, 2009 (1984)

Bethe, H.A.: The electromagnetic shift of energy levels. Phys. Rev. **72**, 339 (1947)

Bhabha, H.J., Corben, H.C.: General classical theory of spinning particles in a Maxwell field. Proc. Roy. Soc. A **178**, 273 (1941)

Bordag, M., Klimchitskaya, G.L., Mohideen, U., Mostepanenko, V.M.: Advances in the Casimir Effect. Oxford University Press, Oxford (2009)

*Boyer, T.H.: Quantum electromagnetic zero-point energy and retarded dispersion forces. Phys. Rev. **174**, 1631 (1968)

Boyer, T.H.: Derivation of the blackbody radiation spectrum without quantum assumptions. Phys. Rev. **182**, 1374 (1969)

Boyer,T.H.: A brief survey of stochastic electrodynamics. In: Barut, A.O. (ed.) Foundations of Radiation Theory and Quantum Electrodynamics, p. 49. Plenum Press, London (1980a)

Boyer, T.H.: Thermal effects of acceleration through random classical radiation. Phys. Rev. D **21**, 2137 (1980b)

*Braffort, P., Surdin, M., Taroni, T.: L'energie moyenne d'un oscillateur harmonique non relativiste en électrodynamique aléatoire. C. R. Acad. Sci. Paris **261**, 4339 (1965)

*Braffort, P., Taroni, A.: Mean energy of a free electron in a uniform magnetic field in aleatory electrodynamics. Compt. Rend. B **264**, 1437 (1967)

*Cavalleri, G.: Schrödinger's equation as a consequence of zitterbewegung. Lett. Nouvo Cim. **43**, 285 (1985)

*Cavalleri, G., Barbero, F., Bertazzi, G., Cesaroni, E., Tonni, E., Bosi, L., Spavieri, G., Gillies, G.T.: A quantitative assessment of stochastic electrodynamics with spin (SEDS). Front. Phys. China **5**(1), 107–122 (2010)

*Cetto, A.M., de la Peña, L.: Environmental effects on the Lamb shift according to stochastic electrodynamics. Phys. Rev. A **37**, 1952 (1988a)

*Cetto, A.M., de la Peña, L.: Environmental effects on radiative lifetime and mass renormalization according to SED. Phys. Rev. A **37**, 1960 (1988b)

*Cetto, A.M., de la Peña, L.: Environmental effects on spontaneous emission and Lamb shift according to stochastic electrodynamics. Phys. Scr. T. **21**, 27 (1988c)

*Cetto, A.M., de la Peña, L.: Radiative corrections for the matter-zeropoint field system: establishing contact with quantum electrodynamics. Phys. Scr. **T151**(014009), 1–7 (2012)

*Cetto, A.M., de la Peña, L., Valdés-Hernandez, A.: Quantization as an emergent phenomenon due to matter-zeropoint field interaction. J. Phys. CPCS. **361**, 012013 (2012).http://iopscience.iop.org/1742-6596/361/1/012013

*Cetto, A.M., de la Peña, L., Valdés-Hernandez, A.: Atomic radiative corrections without QED: role of the zero-point field. Rev. Mex. Fis. **59**, 433 (2013)

*Cetto, A.M., de la Peña, L., Valdés-Hernández, A.: Emergence of quantization: the spin of the electron. J. Phys.: JPCS. **504**, 012007 (2014)

Corben, H.C.: Classical and Quantum Theories of Spinning Particles. Holden-Day, San Francisco (1968)

Dalibard, J., Dupont-Roc, J., Cohen-Tannoudji, C.: Vacuum fluctuations and radiation reaction: identification of their respective contributions. J. Phys. (Paris) **43**, 1617 (1982)
Davies, B.: Note on the Lamb shift. Am. J. Phys. **50**, 331 (1982)
Davydov, A.S.: Quantum Mechanics. Addison-Wesley, Reading (1965) (English translation from the Russian edition with additions by ter Haar D., Moscow, 1963)
*de la Peña, L., Cetto, A.M.: The harmonic oscillator in a random electromagnetic field. Rev. Mex. Fís. **25**, 1 (1976)
*de la Peña, L., Cetto, A.M.: Derivation of quantum mechanics from stochastic electrodynamics. J. Math. Phys. **18**, 1612 (1977)
*de la Peña, L., Cetto, A.M.: The quantum harmonic oscillator revisited: a new look from stochastic electrodynamics. J. Math. Phys. **20**, 469 (1979)
de la Peña, L., Cetto, A.M.: The Quantum Dice. An Introduction to Stochastic Electrodynamics. Kluwer Academic, Dordrecht (1996) (Referred to in this book as *The Dice*)
*de la Peña, L., Cetto, A.M., Valdés-Hernández, A.: Atomic radiative corrections without QED: role of the zero-point field. arXiv:quant-ph:1301.6200v1 (2010)
*de la Peña, L., Cetto, A.M., Valdés-Hernández, A.: The emerging quantum. An invitation, advanced school on quantum foundations and open quantum systems. http://www.fisica.ufpb.br/asqf2012/index.php/invited-lectures 2012
*de la Peña, L., Jáuregui, A.: The spin of the electron according to stochastic electrodynamics. Found. Phys. **12**, 441 (1982)
Dirac, P.A.M.: The Principles of Quantum Mechanics. Clarendon, Oxford 1930 (1947, 1958)
Einstein, A.: Zur Quantentheorie der Strahlung. Mitteil. d. Phys. Ges. Zürich **18**, 47 (1916)
Einstein, A.: Zur Quantentheorie der Strahlung. Zeit. Phys. **18**, 121 (1917)
Fain, V.M.: Sov. Phys. JETP **23**, 882 (1966)
Fain, V.M., Khanin, Y.L.: Quantum Electronics: Basic Theory. Pergamon, Oxford (1969)
Feynman, R.P.: In: Stoops, R. (ed.), The Quantum Theory of Fields, Wiley, NY (1961)
Goy, P., Raimond, J.M., Gross, M., Haroche, S.: Observation of cavity-enhanced single-atom spontaneous emission. Phys. Rev. Lett. **50**, 1903 (1983)
Greiner, W.: Quantum Mechanics: Special Chapters. Springer, Berlin (1998)
Hestenes, D.: Quantum mechanics from self-interaction. Found. Phys. **15**, 63 (1985)
Hestenes, D.: The zitterbewegung interpretation of quantum mechanics. Found. Phys. **20**, 1213 (1990)
Hollberg, L., Hall, J.L.: Measurement of the shift of Rydberg energy levels induced by blackbody radiation. Phys. Rev. Lett. **53**, 230 (1984)
Huang, K.: On the zitterbewegung of the Dirac electron. Am. J. Phys. **20**, 479 (1952)
*Jáuregui, A., de la Peña, L.: The spin and the anomalous magnetic moment of the electron in stochastic electrodynamics. Phys. Lett. A **86**, 280 (1981)
Kalitsin, N.St.: On the interaction of an electron with the fluctuating electromagnetic field of the vacuum. Sov. Phys. JETP **25**, 407 (1953). (in Russian)
Kleppner, D.: Inhibited spontaneous emission. Phys. Rev. Lett. **47**, 233 (1981)
Knight, P.L.: Effects of external fields on the Lamb shift. J. Phys. A **5**, 417 (1972)
Louisell, W.H.: Quantum Statistical Properties of Radiation. Wiley, New York (1973)
Maddox, J.: Where zitterbewegung may lead. Nature **325**, 306 (1987)
Mandel, L., Wolf, E.: Optical coherence and quantum optics. Cambridge University Press, Cambridge (1995)
Marshall, T.W.: A classical treatment of blackbody radiation. Nuovo Cim. **38**, 206 (1965)
Milonni, P.W.: The Quantum Vacuum. An Introduction to Quantum Electrodynamics. Academic Press, Boston (1994)
*Moore, S.M.: Stochastic physics and the Abraham-Lorentz equation. Lett. Nuovo Cim. **20**, 676 (1977)
*Moore, S.M., Ramírez, J.A.: Electron spin in classical stochastic electrodynamics. Lett. Nuovo Cimento **33**, 87 (1982)

*Moore, S.M.: Spin relaxation and electromagnetic vacuum fluctuations. Lett. Nuovo Cim. **40**, 385 (1984)
Muralidhar, K.: Classical origin of quantum spin. Apeiron **18**, 146 (2011)
Pavšič, M., Recami, E., Rodrigues Jr, W.A., Maccarrone, G.D., Raciti, F., Salesi, G.: Spin and electron structure. Phys. Lett. B **318**, 481 (1993)
Power, E.A.: Zero-point energy and the Lamb shift. Am. J. Phys. **34**, 516 (1966)
Rodrigues Jr., W.A., Vaz, J., Recami, E., Salesi, G.: About zitterbewegung and electron structure. arXiv:quant-ph/9803037v1 (1998)
Rodrigues Jr, W.A., Vaz, J., Recami, E., Salesi, G.: About zitterbewegung and electron structure. Phys. Lett. B **318**, 623 (1993)
Rohrlich, F.: Classical Charged Particles. Addison-Wesley, Reading (1965)
*Rueda, A.: Stochastic electrodynamics with particle structure. Part I: zero-point Induced Brownian behavior. Found. Phys. Lett. **6**, 139 (1993)
Sachidanandam, S.: A derivation of intrinsic spin one-half from random electrodynamics. Phys. Lett. A **97**, 323 (1983)
*Santos, E.: The harmonic oscillator in stochastic electrodynamics. Nuovo Cim. B **19**, 57 (1974)
Schrödinger, E.: Über die kräftefreie Bewegung in der relativistischen Quantenmechanik. Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. **24**, 418 (1930)
Sokolov, A.A., Tumanov, V.S.: The uncertainty relation and fluctuation theory. Soviet Phys. JETP **30**, 802. (Russian version); 3, 958, 1957 (English translation) (1956)
*Surdin, M.: L'état fondamental de l'oscillateur harmonique est-il un cycle limite? Ann. Inst. Henri Poincaré **13**, 363 (1970)
*Surdin, M.: The natural line-breadth in stochastic electrodynamics. Int. J. Theor. Phys. **9**, 185 (1974)
van der Waerden, B.L.: Exclusion principle and spin. In: Fierz, M., Weisskopf, V.F. (eds.) Theoretical Physics in the Twentieth Century, p. 202. Interscience, New York (1960)
Welton, T.A.: Some observable effects of the quantum-mechanical fluctuations of the electromagnetic field. Phys. Rev. **74**, 1157 (1948)
Zhou, W., Yu, H.: The Lamb shift in de Sitter spacetime, Phys. Rev. D **82**, 124067. arXiv:1012.4055v1 [hep-th], Dec 2010