Chapter 3
The Planck Distribution, a Necessary Consequence of the Fluctuating Zero-Point Field

With this chapter we initiate our analysis of the implications of considering the fluctuating zero-point radiation field (ZPF) as a fundamental constituent of an otherwise classical system. As announced in the introductory chapter, our journey starts with a fresh look at a simple though physically (and historically) relevant system, namely the electromagnetic radiation field in equilibrium with matter at temperature T . The blackbody problem, the one that gave birth to quantum mechanics, is thus revisited, taking into account the ZPF. The mere existence of this nonthermal field is shown to have far-reaching consequences. In particular, by performing a thermodynamic *and* statistical analysis of an ensemble of harmonic oscillators of frequency ω representing the modes of the radiation field of the respective frequency, we find that Planck's law, as well as irreducible (quantum) fluctuations, arise as *necessary* consequences of allowing for the presence of the pervarsive ZPF, without any assumption of discreteness.

3.1 Thermodynamics of the Harmonic Oscillator

Let us start by considering a one-dimensional harmonic oscillator of frequency ω, with the Hamiltonian given by[1]

$$H = (p^2 + \omega^2 q^2)/2. \tag{3.1}$$

For a material oscillator of mass $m = 1$, q and p stand for the oscillator's position and momentum, respectively. Now, of relevance for our purposes is that a monochromatic mode of frequency ω of the radiation field is equivalent to a harmonic oscillator of that same frequency. In this case H refers to the energy of such mode, and q and p represent its quadratures.

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[1] This first part of the exposition borrows from the work of Boyer (1969b, 2003), who has contributed substantially to the analysis of the Planck distribution from a perspective akin to the one developed here. See also Boyer (1969a, 1976, 1983, 1984, 2010a, b, 2012), Marshall (1965), and Theimer (1971).

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Several basic properties of the harmonic oscillator can be derived from the structure of (3.1), and thus hold irrespective of the oscillator's nature. In particular, for a given constant energy U, the trajectory in phase space is the ellipse

$$p^2 + \omega^2 q^2 = 2U, \tag{3.2}$$

and its area gives the action

$$J = \frac{1}{2\pi} \oint p dq = \frac{1}{2\pi\omega} \oint \sqrt{2U - \omega^2 q^2}\, d\left(\omega q\right) = \frac{U}{\omega}. \tag{3.3}$$

The action J is an adiabatic invariant of the harmonic oscillator (see e.g. Landau and Lifshitz (1976), Sect. 49; José and Saletan (1998), Sect. 6.4), which means that it remains constant under a slow change of the frequency. Therefore, the change dU in the energy concomitant with the slow change $d\omega$ is given by

$$dU = J d\omega = \frac{U}{\omega} d\omega, \tag{3.4}$$

so that the work dW done *by* the system on the external device effecting the change of frequency is

$$dW = -\frac{U}{\omega} d\omega. \tag{3.5}$$

From here it follows that if $S(T, \omega)$ stands for the entropy of the system when this latter is in thermodynamic equilibrium at temperature T, for a reversible process one may write

$$T dS(T, \omega) = dU(T, \omega) + dW = dU(T, \omega) - \frac{U}{\omega} d\omega, \tag{3.6}$$

consequently

$$T \left(\frac{\partial S}{\partial T}\right)_{\omega} dT + T \left(\frac{\partial S}{\partial \omega}\right)_{T} d\omega = \left(\frac{\partial U}{\partial T}\right)_{\omega} dT + \left[\left(\frac{\partial U}{\partial \omega}\right)_{T} - \frac{U}{\omega}\right] d\omega. \tag{3.7}$$

Since the changes in the variables T and ω are independent, this relation naturally splits into the pair of equations

$$T \left(\frac{\partial S}{\partial \omega}\right)_{T} = \left(\frac{\partial U}{\partial \omega}\right)_{T} - \frac{U}{\omega}, \tag{3.8}$$

$$T \left(\frac{\partial S}{\partial T}\right)_{\omega} = \left(\frac{\partial U}{\partial T}\right)_{\omega}. \tag{3.9}$$

We utilize these two relations by taking the partial derivative of the first one with respect to T and of the second one with respect to ω, and combine the results to get

$$\left(\frac{\partial S}{\partial \omega}\right)_T = -\frac{1}{\omega}\left(\frac{\partial U}{\partial T}\right)_\omega . \tag{3.10}$$

Substitution into Eq. (3.8) gives

$$\left(\frac{\partial U}{\partial \omega}\right)_T - \frac{U}{\omega} = -\frac{T}{\omega}\left(\frac{\partial U}{\partial T}\right)_\omega . \tag{3.11}$$

The solution of this equation can be found by writing $U = \omega f(T, \omega)$ to cancel the term U/ω, whence

$$\frac{\omega}{T}\left(\frac{\partial f}{\partial \omega}\right)_T = -\left(\frac{\partial f}{\partial T}\right)_\omega . \tag{3.12}$$

This equation holds for any function f of the single variable ω/T, as can be easily verified; hence Eq. (3.11) admits the general solution

$$U = \omega f(\omega/T). \tag{3.13}$$

Equation (3.13) is indeed a very important result: it is Wien's law, which establishes the general form of the mean energy U of any harmonic oscillator as a function of its frequency ω and the temperature T. This law will be at the basis of our considerations below.[2,3]

We now present some additional results concerning the thermodynamics of the harmonic oscillator that will be useful below. The Helmholtz free energy F takes the form

$$F(T, \omega) = -k_B T \phi(\omega/T), \tag{3.14}$$

where k_B is Boltzmann's constant and ϕ is a thermodynamic potential from which the thermodynamic functions of the oscillator can be determined. In particular the mean equilibrium energy becomes

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[2] Wien's law is a fundamental law of physics, since only simple and very general principles are required for its derivation. It is valid in classical as well as in quantum physics, and is even consistent with relativity, so it was the appropriate law to herald the 20th century. To get a better feeling of its fundamental nature, a derivation based solely on dimensionality arguments can be found in Sommerfeld's classical book on Thermodynamics (Sommerfeld 1956). Simple clear discussions of Wien's law can be seen in two highly pedagogical papers: Piña and de la Selva (2010), and del Río-Correa (2010).

[3] We recall that for the derivation of his law, Wien studied the Doppler effect of the modes of an adiabatically disturbed *radiation* field in thermal equilibrium (see e.g. Milonni 1994).

$$U(T, \omega) = k_B T^2 \left(\frac{\partial \phi}{\partial T} \right)_\omega = -k_B \omega \frac{d\phi(z)}{dz}, \tag{3.15}$$

with $z = \omega/T$. Comparison with Wien's law gives

$$f(z) = -k_B \frac{d\phi}{dz}. \tag{3.16}$$

Finally, the entropy is also a function of the variable z,

$$S(z) = k_B \phi(z) + z f(z). \tag{3.17}$$

These results suffice for our purposes.

3.1.1 Unfolding the Zero-Point Energy

In the low-temperature limit $T \to 0$, Eqs. (3.13) and (3.15) give for the mean energy

$$\mathcal{E}_0 \equiv U(0, \omega) = \omega f(\infty) = -k_B \omega \frac{d\phi}{dz}(\infty) = A\omega, \tag{3.18}$$

so that the zero-point energy $\mathcal{E}_0$—the mean energy of the oscillator at absolute temperature $T = 0$—is determined by the value that the function $f(z)$ (or $d\phi/dz$) attains at infinity.[4] In the usual thermodynamic analysis the value of the constant $A = f(\infty)$ is arbitrarily chosen as zero, so there is no athermal energy. However, the more general (and more natural) solution corresponds to a nonnull value of A. In the case of the radiation field oscillators, this represents a physically more reasonable choice than a vacuum that is completely devoid of electromagnetic phenomena. By taking A to be nonzero we attest the existence of a zero-point energy that fills the whole space and is proportional to the frequency of the oscillator,[5,6]

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[4] Some textbook demonstrations of Wien's law cast doubt about extending its validity to the limit $T = 0$. That Eq. (3.13) holds also at $T = 0$ is explicitly demonstrated in Cole (1990).

[5] That the only spectrum consistent with relativity (and hence with electromagnetic theory) corresponds to $\mathcal{E}_0(\omega) \sim \omega$, has been demonstrated independently by several authors. The earliest of such demonstrations are those in Marshall (1963), Santos (1968), and Boyer (1969b). See also Cole (1990), Milonni (1994), Chap. 2; and *The Dice*, Chap. 4. The present thermodynamic calculation leads to the same expression, Eq. (3.18). Further, the Schrödinger equation provides a similar prediction for the ground-state energy of a particle in a harmonic oscillator potential. Here we have a vivid example of the intrinsic unity of physics, reinforcing the idea that it refers to different aspects of a single reality.

[6] Taking $A = 0$ is equivalent to putting the boundary condition for the solutions of Maxwell's equations at infinity in the past equal to zero, i. e. no radiation. The choice $A \neq 0$ replaces this unnatural boundary condition by a zero-point field at infinity, simultaneously restoring time-reversal symmetry in electrodynamics.

$$\mathcal{E}_0 = A\omega = \tfrac{1}{2}\hbar\omega. \tag{3.19}$$

The value of A (with dimensions of action) must be universal because it determines the equilibrium spectrum at $T = 0$, which, according to Kirchhoff's law, has a universal character. We have put it equal to $\hbar/2$ in order to establish contact with present-day knowledge. However, it must be stressed that the presence of the Planck constant here does not imply any quantum connotation. In addition, it should be noticed that many of the results to be obtained in the present chapter do not depend on the precise value of $\mathcal{E}_0$, the only requirement being in such instances that it be different from zero.

A nonnull value of A means a violation of energy equipartition among the oscillators, since the equilibrium energy becomes now a function of the oscillator frequency. Though at this stage such violation can strictly be assured only at $T = 0$, the result suggests that the physics ensuing from the existence of $\mathcal{E}_0 \neq 0$ necessarily transcends classical physics. This opens up interesting possibilities that will be explored along this chapter.

In concluding this section, let us note that the existence of a zero-point energy provides a natural energy scale, which, along with $k_B T$, suggests to introduce the dimensionless quantity

$$\mathring{z} = \frac{2\mathcal{E}_0}{k_B T} = \frac{\hbar}{k_B} z = \frac{\hbar\omega}{k_B T}. \tag{3.20}$$

This will be the natural dimensionless variable of the thermodynamic functions, since the potential ϕ in Eq. (3.14) is a dimensionless function of z and can therefore be expressed as a function of $\mathring{z}$.

3.2 General Thermodynamic Equilibrium Distribution

Our aim is to find the average energy U per oscillator in an ensemble of such systems when equilibrium has been reached at a fixed temperature T. For this purpose we first follow the standard description of a canonical ensemble (Pathria 1996). In this case, the probability that a member of the ensemble is in a state with energy between $\mathcal{E}$ and $\mathcal{E} + d\mathcal{E}$ can be written in the general form

$$W_g(\mathcal{E})d\mathcal{E} = \frac{1}{Z_g(\beta)} g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E}, \tag{3.21a}$$

$$Z_g(\beta) = \int g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E}, \tag{3.21b}$$

where $\beta = 1/(k_B T)$, $Z_g(\beta)$ is the partition function that normalizes $W_g(\mathcal{E})$ to unity, and $g(\mathcal{E})$ is a weight function representing the density of states with energy $\mathcal{E}$. The mean value $\langle f(\mathcal{E})\rangle$ of any function $f(\mathcal{E})$ is thus

$$\langle f(\mathcal{E})\rangle = \int W_g(\mathcal{E}) f(\mathcal{E}) d\mathcal{E}. \tag{3.22}$$

For $f(\mathcal{E}) = \mathcal{E}$, (3.22) gives the mean energy

$$U = \langle \mathcal{E}\rangle = \int \mathcal{E} W_g(\mathcal{E}) d\mathcal{E}. \tag{3.23}$$

Equation (3.21a) constitutes the general form of a Boltzmann distribution.[7] In particular, the corresponding classical distribution for the harmonic oscillator is obtained from (3.21a) with $g(\mathcal{E})$ given by Pathria (1996)

$$g_{\text{classic}}(\mathcal{E}) = \frac{1}{s\omega}, \tag{3.24}$$

where s is a constant with dimensions of action, so g has the dimension of $(\text{energy})^{-1}$. In this case one gets from the above equations

$$W_{\text{cl}}(\mathcal{E}) = W_{g_{\text{cl}}}(\mathcal{E}) = \frac{e^{-\beta\mathcal{E}}}{\int e^{-\beta\mathcal{E}} d\mathcal{E}}; \tag{3.25a}$$

$$Z_{\text{cl}}(\beta) = \int g_{\text{cl}}(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E} = \frac{1}{s\beta\omega}; \tag{3.25b}$$

$$\langle \mathcal{E}\rangle = U = -\frac{1}{Z_{\text{cl}}}\frac{dZ_{\text{cl}}}{d\beta} = \frac{1}{\beta} = k_B T. \tag{3.25c}$$

From the last equation it follows that $U(T = 0) = 0$. This means that to allow for a zero-point energy, a form for $g(\mathcal{E})$ different from that given by Eq. (3.24) must be used. The specific structure of this $g(\mathcal{E})$ consistent with a zero-point energy for the harmonic oscillator will be determined below.

3.2.1 Thermal Fluctuations of the Energy

Equations (3.21a, 3.21b) and (3.22) lead to a series of important and general results. With $f(\mathcal{E}) = \mathcal{E}^r$, r a positive integer, it follows that (the prime indicates derivative with respect to β)[8]

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[7] This form of writing $W_g(\mathcal{E})$ was used, for example, by Einstein (1907) in his early work on the specific heat of solids. He considered the distribution in (3.21a) assuming from the start a form for the function $g(\mathcal{E})$ equivalent to (3.82) below, as was dictated by the quantization discovered by Planck. Here we proceed in the opposite sense, by allowing the theory to determine $g(\mathcal{E})$.

[8] The present discussion draws closely from de la Peña and Cetto (2002), de la Peña et al. (2008, 2010a, b), Valdés-Hernández et al. (2010), Valdés-Hernández (2010).

$$\left\langle \mathcal{E}^r \right\rangle' = -\frac{Z_g'}{Z_g}\left\langle \mathcal{E}^r \right\rangle - \frac{1}{Z_g}\int \mathcal{E}^{r+1} g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E} = -\frac{Z_g'}{Z_g}\left\langle \mathcal{E}^r \right\rangle - \left\langle \mathcal{E}^{r+1} \right\rangle, \tag{3.26}$$

and further, from (3.21b),

$$\langle \mathcal{E} \rangle = U = \frac{1}{Z_g}\int \mathcal{E} g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E} = -\frac{Z_g'}{Z_g}. \tag{3.27}$$

These two expressions combined give the recurrence relation

$$\left\langle \mathcal{E}^{r+1} \right\rangle = U\left\langle \mathcal{E}^r \right\rangle - \left\langle \mathcal{E}^r \right\rangle', \tag{3.28}$$

which can be extended to any continuous function $h(\mathcal{E})$ to obtain

$$-\left\langle h(\mathcal{E}) \right\rangle' = \left\langle \mathcal{E} h(\mathcal{E}) \right\rangle - U\left\langle h(\mathcal{E}) \right\rangle. \tag{3.29}$$

Thus $-\left\langle h(\mathcal{E}) \right\rangle'$ is given in general by the covariance of $h(\mathcal{E})$ and $\mathcal{E}$.

Equation (3.28) with $r = 1$ gives a most important expression for the energy variance,

$$\sigma_{\mathcal{E}}^2 \equiv \left\langle (\mathcal{E} - U)^2 \right\rangle = \left\langle \mathcal{E}^2 \right\rangle - U^2 = -\frac{dU}{d\beta}, \tag{3.30}$$

which can be rewritten as the well-known relation (Mandl 1988)

$$\sigma_{\mathcal{E}}^2 = -\frac{dU}{d\beta} = k_B T^2 \left(\frac{\partial U}{\partial T}\right)_{\omega} = k_B T^2 C_{\omega} \tag{3.31}$$

in terms of the specific heat (or heat capacity) C_{ω}.[9] Because C_{ω} is surely finite at low temperatures, the right-hand side of this expression is zero at $T = 0$, whence

$$\sigma_{\mathcal{E}}^2 (T = 0) = 0, \tag{3.32}$$

which shows that the description provided by the distribution W_g does not allow for the dispersion of the energy at zero temperature. The fact that W_g offers a thermodynamic description that admits thermal fluctuations only, and has no room for temperature-independent fluctuations, is an important shortcoming, as is clear when we consider a collection of harmonic oscillators (such as those of the electromagnetic field in equilibrium inside a cavity) which are endowed with a zero-point energy given by (3.19). Indeed, for such system the distribution W_g leaves out the fluctuations of the nonzero nonthermal component of the energy. We continue to work here

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[9] C_{ω} coincides with the specific heat at constant volume, so the usual notation in this context is C_V. Still, we employ the subindex ω since we are considering ω to be a fixed parameter.

with the thermodynamic description, but later on we shall introduce a full-fledged statistical description that overcomes this limitation.

3.2.2 Some Consequences of the Recurrence Relation

The recurrence relation (3.28) and the Wien law can be recast into other interesting forms as follows. First we observe that the equation

$$\langle \mathcal{E}^r \rangle = \frac{1}{Z_g(\beta)} \int \mathcal{E}^r g(\mathcal{E}) e^{-\beta \mathcal{E}} d\mathcal{E}, \tag{3.33}$$

with the substitutions $\mathcal{E} = \mathcal{E}_0 \epsilon$ (ϵ dimensionless), and $\mathring{z} = 2\mathcal{E}_0 \beta$, gives

$$\langle \mathcal{E}^r \rangle = \mathcal{E}_0^r f_r(\mathcal{E}_0, \mathring{z}), \tag{3.34}$$

where $f_r(\mathcal{E}_0, \mathring{z})$ is defined as

$$f_r(\mathcal{E}_0, \mathring{z}) = \frac{\int \epsilon^r g(\mathcal{E}_0 \epsilon) e^{-\mathring{z}\epsilon/2} d\epsilon}{\int g(\mathcal{E}_0 \epsilon) e^{-\mathring{z}\epsilon/2} d\epsilon}.$$

As follows from Eq. (3.34), f_r is an adimensional function, hence it can be expressed as a function of the adimensional parameter $\mathring{z}$ only. For the harmonic oscillator we use Wien's law to write $\mathcal{E}_0 = A\omega$, so that Eq. (3.34) reads

$$\langle \mathcal{E}^r \rangle = \omega^r A^r f_r(\mathring{z}), \tag{3.35}$$

which is a generalization of Wien's law for any power r.

On the other hand, the general recurrence relation between the moments of the energy, Eq. (3.28), can be rewritten as follows, using Eq. (3.31),

$$\left\langle \mathcal{E}^{r+1} \right\rangle = U \langle \mathcal{E}^r \rangle - \frac{d}{d\beta} \langle \mathcal{E}^r \rangle = U \langle \mathcal{E}^r \rangle + \sigma_{\mathcal{E}}^2 \frac{d}{dU} \langle \mathcal{E}^r \rangle, \tag{3.36}$$

$$\text{or} \quad \left\langle \mathcal{E}^{r+1} \right\rangle = \left(U + \sigma_{\mathcal{E}}^2 \frac{d}{dU} \right) \langle \mathcal{E}^r \rangle. \tag{3.37}$$

Successive iterations of this equation yield

$$\langle \mathcal{E}^r \rangle = \left(U + \sigma_{\mathcal{E}}^2 \frac{d}{dU} \right)^{r-1} U. \tag{3.38}$$

This reveals $U + \sigma_{\mathcal{E}}^2\,(d/dU)$ as a kind of 'raising' operator for the higher moments of the energy, beginning with the first moment $\langle\mathcal{E}\rangle = U$. It is clear that for $\sigma_{\mathcal{E}}^2(U)$ even in U, the moments $\langle\mathcal{E}^r\rangle\,(U)$ have the parity of r.

The second centered moment of the energy is $\sigma_{\mathcal{E}}^2 = \left\langle(\mathcal{E}-U)^2\right\rangle$; for the third one we obtain

$$\begin{aligned}\left\langle(\mathcal{E}-U)^3\right\rangle &= \left\langle\mathcal{E}^3\right\rangle - 3U\sigma_{\mathcal{E}}^2 - U^3\\ &= \sigma_{\mathcal{E}}^2\frac{d}{dU}\left\langle(\mathcal{E}-U)^2\right\rangle,\end{aligned}\tag{3.39}$$

and by induction it can be seen that this last result generalizes into

$$\left\langle(\mathcal{E}-U)^r\right\rangle = \sigma_{\mathcal{E}}^2\frac{d}{dU}\left\langle(\mathcal{E}-U)^{r-1}\right\rangle\tag{3.40}$$

for any integer $r \geq 1$. This equation shows that at $T = 0$, all centered moments are zero because of (3.32); hence the energy is exactly $\mathcal{E}_0$, and its distribution function reduces to $\delta(\mathcal{E} - \mathcal{E}_0)$ in this thermodynamic analysis.

3.3 Planck's Law from the Thermostatistics of the Harmonic Oscillator

3.3.1 General Statistical Equilibrium Distribution

It now becomes necessary to extend our description so as to allow for nonthermal fluctuations of the zero-point energy of the field, which are excluded by W_g. This can be achieved by paying attention to the *statistical* distribution of the energy $W_s(\mathcal{E})$. Since for every frequency the field contains a huge number of modes, the central limit theorem applies (Grimmett and Stirzaker 1983; Papoulis 1991) and hence the field amplitude of frequency ω follows a normal distribution. This means that the energy distribution follows the simple law

$$W_s(\mathcal{E}) = \frac{1}{U}\,e^{-\mathcal{E}/U},\tag{3.41}$$

with

$$\int W_s(\mathcal{E})d\mathcal{E} = 1, \qquad \int \mathcal{E}W_s(\mathcal{E})d\mathcal{E} = U,\tag{3.42}$$

and the corresponding energy dispersion is given by (the subscript s denotes averages taken with respect to W_s, to be distinguished from those calculated with W_g)[10]

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[10] This is a well-known statistical result, established for the first time by Lorentz for the *thermal* radiation field. A simple demonstration is given in Vedral (2005). Inclusion of the zero-point

$$\left(\sigma_{\mathcal{E}}^2\right)_s = U^2. \tag{3.43}$$

This (exponential) distribution of the energy [subject to the constraints (3.42)] has the property of maximizing the statistical entropy S_s, defined as

$$S_s = -k_B \int W_s(\mathcal{E}) \ln c_s W_s(\mathcal{E}) d\mathcal{E}, \tag{3.44}$$

where c_s is an appropriate constant with dimension of energy. Since the entropy is usually interpreted as a measure of the disorder present in the system (see e.g. Callen 1985; Mandl 1988), the maximal entropy property means maximum disorder, which is the natural demand for a system constituted by a huge number of independent components once equilibrium has been reached.

From Eq. (3.43) we see that W_s allows indeed for zero-point fluctuations, since at $T = 0$

$$(\sigma_{\mathcal{E}}^2)_s \Big|_0 = U^2(T = 0) = \mathcal{E}_0^2, \tag{3.45}$$

which means that there is a nonthermal contribution to the energy fluctuations, with variance $\mathcal{E}_0^2$. The thermal contribution $\sigma_{\mathcal{E}_T}^2$ to the energy fluctuations at any temperature is obtained by subtracting from the total ones this nonthermal term $\mathcal{E}_0^2$. This is true because the thermal and nonthermal fluctuations have an entirely different source, so they are statistically independent, with a null correlation [see the discussion following Eq. (3.90)]. That is,

$$\sigma_{\mathcal{E}_T}^2 = (\sigma_{\mathcal{E}}^2)_s - \mathcal{E}_0^2, \tag{3.46}$$

whence

$$\sigma_{\mathcal{E}_T}^2 = U^2 - \mathcal{E}_0^2. \tag{3.47}$$

Recalling that $\sigma_{\mathcal{E}}^2$ in Eq. (3.31) stands for the *thermal* fluctuations of the energy, we can combine this latter with (3.47) and write (omitting the subindex T)

$$\sigma_{\mathcal{E}}^2 = U^2 - \mathcal{E}_0^2 = -\frac{dU}{d\beta}. \tag{3.48}$$

Before studying the consequences of this relation we observe that the distribution (3.41) leads to recurrence relations incorporating the nonthermal fluctuations. Indeed (3.41) gives for the moments of the energy

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(Footnote 10 continued)
component does not modify this statistical property, since the argument to establish it remains in force.

$$\left\langle \mathcal{E}^r \right\rangle_s = r!U^r, \tag{3.49}$$

and making reiterative use of this equation one obtains

$$\left\langle \mathcal{E}^r \right\rangle_s = U \left\langle \mathcal{E}^{r-1} \right\rangle_s + (\sigma_{\mathcal{E}}^2)_s \frac{d}{dU} \left\langle \mathcal{E}^{r-1} \right\rangle_s . \tag{3.50}$$

Thus a sophisticated form of writing (3.49) in terms of a raising operator is

$$\left\langle \mathcal{E}^r \right\rangle_s = \left(U + (\sigma_{\mathcal{E}}^2)_s \frac{d}{dU} \right) \left\langle \mathcal{E}^{r-1} \right\rangle_s , \tag{3.51}$$

a result analogous to the previous recurrence relation (3.38), but now including the zero-point fluctuations. A much simpler, alternative form of this relation is

$$\left\langle \mathcal{E}^r \right\rangle_s = rU \left\langle \mathcal{E}^{r-1} \right\rangle_s . \tag{3.52}$$

3.3.2 *Mean Energy as Function of Temperature; Planck's Formula*

We note from Eq. (3.48) that knowledge of the variance $\sigma_{\mathcal{E}}^2$ as a function of U is enough to determine $U(\beta)$. Indeed, an integration of this equation—which articulates both thermodynamic and statistical information via Eqs. (3.31) and (3.46), respectively—

$$\frac{dU}{d\beta} = \mathcal{E}_0^2 - U^2(\beta) \tag{3.53}$$

gives the function $U(\beta)$. Subject to the condition $U \to \infty$ as $T \to \infty$, the result is

$$U(\beta) = \begin{cases} \frac{1}{\beta}, & \text{for } \mathcal{E}_0 = 0; \\ \mathcal{E}_0 \coth \mathcal{E}_0 \beta, & \text{for } \mathcal{E}_0 \neq 0. \end{cases} \tag{3.54}$$

Although the case $\mathcal{E}_0 = 0$ can of course be obtained from the last expression in the limit $\mathcal{E}_0 \to 0$, it is more illustrative to treat the two cases separately. As seen from Eq. (3.54), the mean energy as a function of the temperature depends critically on the presence of $\mathcal{E}_0$. For $\mathcal{E}_0 = 0$ the classical energy equipartition is recovered,

$$U_{\mathrm{cl}} = \beta^{-1} = k_B T, \tag{3.55}$$

whereas for $\mathcal{E}_0 = \hbar\omega/2$ Planck's law is obtained,

$$U_{\text{Planck}}(\omega, T) = \tfrac{1}{2}\hbar\omega \coth \tfrac{1}{2}\hbar\omega\beta. \tag{3.56}$$

By taking the limit $T \to 0$, we verify that U_{Planck} includes the zero-point energy,[11]

$$U_{\text{Planck}}(\beta \to \infty) = \tfrac{1}{2}\hbar\omega = \mathcal{E}_0. \tag{3.57}$$

This establishes Planck's law as a physical result whose ultimate meaning—or cause—is the existence of a fluctuating zero-point energy of the field oscillators.

It is important to stress that Planck's law has been obtained without the introduction of any explicit quantum or discontinuity requirement. Equation (3.53) results from a thermostatistical analysis of the field modes, based on the properties of W_g and W_s, together with Wien's law, which opens the door to their zero-point energy $A\omega$. This leads us to conclude that Wien's law with $A \neq 0$ in Eq. (3.18) constitutes an extension of classical physics into the quantum domain—as evidenced by the quantum properties of the harmonic oscillator that ensue from Planck's law (see below). Thus, strictly speaking, Wien's law stands as a precursor of Planck's, and should be considered historically to contain the first quantum law.

The demonstration that the law that gave rise to quantum theory stems from the existence of a fluctuating zero-point energy, brings to the fore the crucial importance of this nonthermal energy for the understanding of quantum mechanics or, more generally, of quantum theory.

A brief comment on the thermal fluctuations of the energy seems in place before ending this section. We have seen that for $\mathcal{E}_0 \neq 0$ the thermal energy dispersion is given by

$$\sigma^2_{\mathcal{E}_T}(U) = U^2 - \mathcal{E}_0{}^2 \quad (U = U_{\text{Planck}}), \tag{3.58}$$

whereas in the classical case ($\mathcal{E}_0 = 0$),

$$\sigma^2_{\mathcal{E}_T}(U) = U^2 \quad (U = U_{\text{cl}}). \tag{3.59}$$

Whilst in the latter case the thermal fluctuations of the oscillator's energy depend solely on its (purely) thermal mean energy, U_{cl}, in the former case Eq. (3.58) relates the thermal fluctuations with the *total* mean energy U_{Planck}, which includes the temperature-independent contribution. The statistical description initiated in Sect. 3.3.1 will be resumed below, in Sect. 3.6.

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[11] Planck's law without zero-point energy (the first relation derived by Planck) is obtained by fixing the constant of integration precisely as $-\mathcal{E}_0$, so that $U(\beta) = \mathcal{E}_0 \coth \mathcal{E}_0\beta - \mathcal{E}_0$. The existence of the zero-point energy remains hidden with this choice.

An additional comment is in place here. At first sight it would seem plausible to take the constant of integration in the first line of Eq. (3.54) as $\mathcal{E}_0$, so that the resulting function, $U(\omega, T) = k_B T + \mathcal{E}_0$ is apparently consistent with both the existence of a nonthermal energy and Wien's law. However, such choice must be discarded since this U cannot be obtained as a limit case of $\mathcal{E}_0 \neq 0$.

3.4 Planck, Einstein and the Zero-Point Energy

The previous discussion suggests separating the average energy U_{Planck} (which as of now will be denoted simply by U) into a thermal contribution U_T and a temperature-independent part $\mathcal{E}_0$,

$$U = U_T + \mathcal{E}_0, \tag{3.60}$$

so that Eq. (3.58) becomes

$$\sigma_{\mathcal{E}}^2 = U_T^2 + 2\mathcal{E}_0 U_T. \tag{3.61}$$

The first term in Eq. (3.60)

$$U_T = \mathcal{E}_0 \coth \mathcal{E}_0\beta - \mathcal{E}_0 = \frac{2\mathcal{E}_0}{e^{2\mathcal{E}_0\beta} - 1}, \tag{3.62}$$

with $\mathcal{E}_0 = \hbar\omega/2$, is Planck's law without the zero-point energy. At sufficiently low temperatures U_T takes the form

$$U_T(\beta \to \infty) = 2\mathcal{E}_0 e^{-2\mathcal{E}_0\beta}. \tag{3.63}$$

This is the (approximate) expression suggested by Wien at the end of the 19th century, and considered for some time to be the exact law for the blackbody spectral distribution. Equations (3.63) and (3.61) represent the germ of quantum theory, since it is precisely on their basis that Planck and Einstein advanced the notion of the quantum (for the material oscillators and for the radiation field, respectively). The following pages contain a discussion of their respective points of view and of the relations between these and our present notions based on the reality of the zero-point energy. A remarkable relationship will thus be disclosed.

3.4.1 Comments on Planck's Original Analysis

In his initial studies on the radiation field in equilibrium with matter, Planck (1900a, b) used as point of departure the expression for the derivative of the entropy[12]

$$\frac{\partial S}{\partial U} = \frac{1}{T}. \tag{3.64}$$

In line with the views and knowledge of his time, Planck recognized only the thermal energy, so U should be replaced here by U_T. In the high-temperature limit the relation (3.64) led him to write (putting $U_T(T \to \infty) = k_B T$)

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[12] An early account of the material in this and the following two subsections is presented in de la Peña and Cetto (2002), and de la Peña et al. (2010a, b).

$$\frac{\partial^2 S}{\partial U_T^2} = \frac{\partial}{\partial U_T}\left(\frac{k_B}{U_T}\right) = -\frac{k_B}{U_T^2}. \tag{3.65}$$

For low temperatures Planck used Wien's result (3.63), assuming it to afford an exact description of the properties of the equilibrium field. He thus wrote

$$U_T = 2\mathcal{E}_0 e^{-2\mathcal{E}_0\beta} = 2\mathcal{E}_0 e^{-2\mathcal{E}_0/k_B T} = 2\mathcal{E}_0 e^{-2(\mathcal{E}_0/k_B)(\partial S/\partial U_T)}, \tag{3.66}$$

whence

$$\frac{\partial S}{\partial U_T} = -\frac{k_B}{2\mathcal{E}_0}\ln\frac{U_T}{2\mathcal{E}_0}, \tag{3.67a}$$

$$\frac{\partial^2 S}{\partial U_T^2} = -\frac{k_B}{2\mathcal{E}_0 U_T}. \tag{3.67b}$$

Not surprisingly, Eqs. (3.65) and (3.67b) give different results, since different temperature regimes were used in each case. As the simplest possibility Planck assumed that the description for arbitrary temperatures could be obtained by interpolating Eqs. (3.65) and (3.67b) and consequently he proposed the relation

$$\frac{\partial^2 S}{\partial U_T^2} = -\frac{k_B}{U_T^2 + 2\mathcal{E}_0 U_T}. \tag{3.68}$$

This equation leads directly to Planck's law without the zero-point term [Eqs. (3.62)], a result that Planck (against his will) interpreted, as is well known, as due to the quantization of the energy exchanged between the material oscillators of the cavity and the equilibrium radiation field.[13]

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[13] The rationale behind Planck's reading of his formula is the following. If the system composed by the walls of the cavity (represented by a collection of material oscillators) and the enclosed radiation field exchanges energy not continuously but by lumps (which he called quanta) of value $n\hbar\omega$ ($n = 1, 2, 3, \ldots$), then the mean equilibrium energy is

$$U = \frac{\sum_{n=0}^{\infty} n\hbar\omega e^{-\beta n\hbar\omega}}{\sum_{n=0}^{\infty} e^{-\beta n\hbar\omega}}.$$

Performing the summations with the aid of the relation $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, one gets

$$U = \frac{\hbar\omega}{e^{\hbar\omega\beta} - 1},$$

which is just the U_T in Planck's theory. If by contrast a continuous exchange of energy is assumed instead of a discrete one, the sum above must be replaced by an integral from 0 to ∞. The reader can easily check that in this case the result is the classical formula $U_T = 1/\beta = k_B T$.

3.4.2 Einstein's Revolutionary Step

A few years later, Einstein argued that even though Eq. (3.68) was empirically confirmed (through Planck's law), its full meaning remained to be clarified. For this purpose Einstein chose also to take Eq. (3.64) as a safe point of departure, whence he wrote

$$\frac{\partial^2 S}{\partial U_T^2} = \frac{\partial}{\partial U_T}\frac{1}{T} = -\frac{1}{T^2 C_\omega}, \tag{3.69}$$

or

$$k_B T^2 C_\omega = -k_B \left(\frac{\partial^2 S}{\partial U_T^2}\right)^{-1}. \tag{3.70}$$

Equation (3.70) combined with (3.68) and (3.31) gives

$$k_B T^2 C_\omega = -\frac{dU_T}{d\beta} = \sigma_{\mathcal{E}}^2 = U_T^2 + 2\mathcal{E}_0 U_T, \tag{3.71}$$

which is the same as (3.61). Einstein recognized the disagreement between this result and the classical expression $\sigma_{\mathcal{E}}^2 = U_T^2$. As is well known, it is here where he made his most—according to him (Rigden 2005), his *only*—revolutionary step in physics. He interpreted the first term on the right-hand side of (3.71) as due to the fluctuations of the thermal field produced by the interference among its modes of a given frequency. This interpretation follows from considering the limit of (3.71) at high temperatures, at which $U_T \gg \mathcal{E}_0$ and therefore $\sigma_{\mathcal{E}}^2 = U_T^2$, as was predicted by Lorentz on the basis of Maxwell's equations and is discussed in relation with Eq. (3.43). Einstein thus saw in this term a direct manifestation of the wavelike nature of light.

As for the second term in (3.71), which in the context of classical thermodynamics is completely unexpected, the fact that it leads to the quantum theory of Planck led Einstein to interpret it in terms of light quanta (Einstein 1905a, b), seeing in the expression $2\mathcal{E}_0 U_T$ a manifestation of discrete properties of the radiation field, as follows. According to Planck, the average energy exchanged between n material oscillators (representing the walls of the cavity) of frequency ω and the radiation field is $\Delta U = \hbar\omega\langle n\rangle$, and contributes with $\sigma_{\Delta U}^2 = 2\mathcal{E}_0 \Delta U = \hbar^2\omega^2\langle n\rangle$ to the fluctuations of the field, as follows from (3.71). For Einstein, the linearity of the variance in $\langle n\rangle$ suggested a Poisson distribution of n independent events, each corresponding to an exchange of energy equal to $\hbar\omega = 2\mathcal{E}_0$.[14] It is the interpretation by Einstein of the linear term as representing a discrete or 'corpuscular' contribution, with each

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[14] A Poisson distribution refers to the probability of n independent discrete events taking place simultaneously, and has the form

$$P_a(n) = e^{-a}\frac{a^n}{n!}.$$

It is easy to verify that for this distribution the mean of n is $\langle n\rangle = a$ and its variance is precisely $\sigma_n^2 = \langle n\rangle$.

corpuscle being an independent packet of energy $\hbar\omega$, what gave birth to the notion of the photon (see Vedral 2005 for a simple derivation). It is clear from Eq. (3.71) that the discrete structure of the field will manifest itself only at very low temperatures, when the linear term dominates over the quadratic, wavelike one. However, it is important to stress, as Einstein did as of 1909, that the two terms coexist at all temperatures, and thus, both particle and wave manifestations of light coexist at all temperatures (Einstein 1909). This observation is sometimes ignored to argue that they are mutually exclusive, although there exist both theoretical arguments and experiments that demonstrate the possible coexistence of the two aspects of the behaviour of light.[15]

3.4.3 Disclosing the Zero-Point Field

It is important to note that no zero-point energy was considered by either Planck or Einstein in their analysis of Eqs. (3.68) and (3.71), respectively. Instead, as stated above, Planck interpreted the term $2\mathcal{E}_0 U_T$ in Eq. (3.68) as a result of the discontinuities in the processess of energy exchange between matter and field (more specifically in the emissions, as of 1912). Einstein in his turn saw in $2\mathcal{E}_0 U_T$ a manifestation of the corpuscular nature of the field, and thus pointed to it as the key to Planck's law. Now, from the point of view proposed here the consideration of the zero-point energy gives rise to a third understanding of Eq. (3.71) that does not depend on the notion of quanta. The elucidation of U_T^2 as the result of the interference of the modes of frequency ω of the thermal field suggests to interpret $2\mathcal{E}_0 U_T$ as due to additional interferences, now between the thermal field and a *zero-point radiation field* of mean energy $\mathcal{E}_0$ (per mode of frequency ω) that is present at all temperatures. As is by now clear, Eq. (3.71) lacks the extra term $\mathcal{E}_0^2$ representing the nonthermal fluctuations, just because the thermodynamic description has no room for them; this shortcoming has been overcome with the introduction of the distribution W_s, Eq. (3.41).

From this new perspective the notion of intrinsic discontinuities in the energy exchange or in the field itself is unnecessary to explain either Planck's law or the linear term in Eq. (3.71); it is the existence of a (fluctuating) zero-point radiation field (ZPF) what accounts for that law. This could of course not be Planck's or Einstein's interpretation because the zero-point energy (and more so the zero-point field) was still unknown at that time, even though their results were consistent with its existence.

The concept of a zero-point energy of the radiation field appeared for the first time in 1912, in a work where Planck attempted another derivation of his law, motivated by his well-known uneasiness with the idea of introducing discontinuities in

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[15] Graded realizations of complementarity relations (wave-like or particle-like behavior) have been under close scrutiny during the last decades; see e.g. Jaeger et al. (1995), Englert (1996), Engert and Bergou (2000), Liu et al. (2009), Flores and de Tata (2010) (see also Ghose and Home 1996). The general validity of Einstein's fluctuation formula (3.71) had been verified experimentally since earlier times; see Aldemade et al. (1966), Kattke and van der Ziel (1970). The authors are grateful to M. D. Godfrey for drawing their attention to these references.

our theoretical descriptions (Planck 1912). Some time thereafter Einstein and Stern (1913) used the idea of a zero-point energy, although applied to molecules, i.e., to mechanical oscillators. Unfortunately the authors were obliged to use the (incorrect) value $\hbar\omega$ for this energy; this along with other difficulties led Einstein to abandon such line of research.[16] Shortly thereafter the notion of a zero-point field was born anew, when Nernst made his visionary proposal (Nernst 1916), as briefly mentioned in the preface.

3.5 Continuous Versus Discrete

We have just seen how three alternative approaches provide three quite different readings of the same quantity, $U_T^2 + 2\mathcal{E}_0 U_T$. In these approaches, either the zero-point energy (of a continuous field) or the energy quantization is identified as the notion underlying the Planck spectral energy distribution. Therefore the next logical step is to inquire about the relation between the zero-point energy and quantization. Is quantization inevitably linked to Planck's law, or is it merely the result of a point of view, of a voluntary but dispensable choice?

3.5.1 The Partition Function

An answer to the above question is found from an analysis of the partition function obtained from (3.54). As follows from Eq. (3.27), $Z_g(\beta)$ can be determined by direct integration of

$$U = -\frac{d \ln Z_g(\beta)}{d\beta}, \tag{3.72}$$

with $U(\beta)$ given by the second of Eq. (3.54). The result is

$$Z_g = \frac{C}{\sin h\mathcal{E}_0\beta}, \tag{3.73}$$

where C is a numerical constant whose value is determined by requiring the classical result $Z_g = (s\beta\omega)^{-1}$ [Eq. (3.25b)] to be recovered in the limit $T \to \infty$. This leads to $C = \mathcal{E}_0/s\omega = \hbar/2s$, so that

$$Z_g(\beta) = \frac{\mathcal{E}_0}{s\omega \sinh \mathcal{E}_0\beta}. \tag{3.74}$$

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[16] More detailed discussions of these points, from a modern perspective, are presented in Milonni (1994); see also Boyer (1969a) and Jiménez et al. (1980).

On the other hand, from Eqs. (3.15) and (3.72) the thermodynamic potential ϕ can be written in the form

$$\phi = \ln Z_g. \tag{3.75}$$

This along with Eq. (3.17) gives for the entropy (up to an additive constant, and writing $S = S_g$)

$$\begin{aligned} \S_g &= k_B \ln Z_g + \frac{U}{T} \\ &= k_B \ln \tfrac{\hbar}{s} - k_B \ln(2 \sinh \mathcal{E}_0 \beta) + k_B \beta U, \end{aligned} \tag{3.76}$$

which in the zero-temperature limit reduces to

$$S_g(\beta \to \infty) = k_B \ln \frac{\hbar}{s}. \tag{3.77}$$

To set the origin of the entropy at $T = 0$ one must take $s = \hbar$,[17] hence the partition function takes the form

$$Z_g(\beta) = \frac{1}{2 \sin h \mathcal{E}_0 \beta}. \tag{3.78}$$

3.5.2 *The Origin of Discreteness*

Once we have determined the partition function Z_g we are in position to discuss the discontinuities characteristic of the quantum theory, which are hidden in the continuous description given by the distribution W_g. To this end we expand Eq. (3.78) and write (see Santos 1975; Theimer 1976; Landsberg 1981 for related discussions)

$$Z_g = \frac{1}{2 \sinh \mathcal{E}_0 \beta} = \frac{e^{-\beta \mathcal{E}_0}}{1 - e^{-2\beta \mathcal{E}_0}} = \sum_{n=0}^{\infty} e^{-\beta \mathcal{E}_0 (2n+1)} = \sum_{n=0}^{\infty} e^{-\beta \mathcal{E}_n}, \tag{3.79}$$

where

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[17] This is a most significant quantum result. In the quantum statistical description the finite quantity $\hbar^3$ plays the role of a minimal element of volume in phase space. This idea was introduced formally for the first time by Planck in his early studies of the blackbody spectrum (Planck 1900a, b). Later, in 1924, Bose assumed that two or more distributions of microstates that differ only in the permutation of phase points within a subregion of phase space of volume $\hbar^3$, are to be regarded as identical, which already corresponds to the Bose-Einstein statistics. In the classical description the volume of such elementary cells is taken to tend to zero in order to recover the continuity of the phase space. It is remarkable that, already in his classical statistical studies, Boltzmann introduced formally the idea of a discrete phase space (see e.g., Jones 2008, Chap. 3).

$$\mathcal{E}_n \equiv (2n+1)\mathcal{E}_0 = \hbar\omega n + \tfrac{1}{2}\hbar\omega. \tag{3.80}$$

Equation (3.79) allows now the determination of the function $g(\mathcal{E})$ by means of (3.21b),

$$Z_g(\beta) = \int g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E} = \sum_{n=0}^{\infty} e^{-\beta\mathcal{E}_n} = \int_0^{\infty} \sum_{n=0}^{\infty} \delta(\mathcal{E}-\mathcal{E}_n) e^{-\beta\mathcal{E}} d\mathcal{E}, \tag{3.81}$$

whence

$$g(\mathcal{E}) = \sum_{n=0}^{\infty} \delta(\mathcal{E}-\mathcal{E}_n). \tag{3.82}$$

The substitution of (3.82) into Eq. (3.21a) finally determines the probability density $W_g(\mathcal{E})$,

$$W_g(\mathcal{E}) = \frac{1}{Z_g} \sum_{n=0}^{\infty} \delta(\mathcal{E}-\mathcal{E}_n) e^{-\beta\mathcal{E}}. \tag{3.83}$$

This distribution gives for the mean value of any function $f(\mathcal{E})$

$$\langle f(\mathcal{E}) \rangle = \int W_g(\mathcal{E}) f(\mathcal{E}) d\mathcal{E} = \frac{1}{Z_g} \sum_{n=0}^{\infty} f(\mathcal{E}_n) e^{-\beta\mathcal{E}_n} = \sum_{n=0}^{\infty} w_n f(\mathcal{E}_n), \tag{3.84}$$

with the weights w_n given by

$$w_n = \frac{e^{-\beta\mathcal{E}_n}}{Z_g} = \frac{e^{-\beta\mathcal{E}_n}}{\sum_{n=0}^{\infty} e^{-\beta\mathcal{E}_n}}. \tag{3.85}$$

The final form of $W_g(\mathcal{E})$, Eq. (3.83), identifies $\{\mathcal{E}_n = \hbar\omega(n+1/2)\}$ with the set of discrete energy levels accesible to the oscillators. Such discreteness, seemingly excluding all other values of the energy, is due to the highly pathological distribution $g(\mathcal{E})$, Eq. (3.82). As a result, (3.84) shows that the mean value of a function of the *continuous* variable $\mathcal{E}$ calculated with the distribution $W_g(\mathcal{E})$, can be obtained equivalently by averaging over the set of *discrete* indices (or *states*) n, with respective weights w_n. Thus, although both averages are formally equivalent, their descriptions are essentially different: one refers to the continuous energy $\mathcal{E}$, the other one to discrete states (levels) with energy $\mathcal{E}_n$. As this latter is completely characterized by the state n, it is natural to interpret the last equality in Eq. (3.84) as a manifestation of the discrete (quantized) nature of the energy. Indeed, the last equality in Eq. (3.84) can be recognized as the description afforded by the density matrix for a canonical ensemble of quantum oscillators at temperature T, with the weights w_n given by (3.85) (see e.g. Cohen-Tannoudji et al. 1977).

The above discussion points to the fundamental role played by the zero-point energy in explaining quantization, by putting it at the root of Eq. (3.79) and hence of Eq. (3.82). From the present point of view, and contrary to the usual credo, the radiation field is not intrinsically quantized, but it becomes so when attaining equilibrium through its interaction with matter. In other words, quantization is here exhibited as an emergent property of matter and field in interaction, an idea that is closely examined from several angles in the following chapters, becoming thus the leitmotiv of the book.

3.6 A Quantum Statistical Distribution

The thermostatistical analysis of a canonical ensemble of oscillators has led to the conclusion that although $\mathcal{E}$ is a continuous variable, its equilibrium distribution possesses extremely peaked values. In other words, the energies that conform to the thermal equilibrium state described by the distribution W_g belong, roughly speaking, to a discrete spectrum. This explains why the mean value $\langle f(\mathcal{E})\rangle$, which corresponds to an equilibrium state, involves only the discrete set $\mathcal{E}_n$. However, the energy still fluctuates and in doing so tends to fill the interspaces between its discrete values.[18] Thus we find that temperature-independent fluctuations appear as a characteristic trait of quantum systems. A closer study of this property allows to establish contact with one of the most frequently used distributions in quantum statistics.

3.6.1 *Total Energy Fluctuations*

The appropriate statistical distribution that includes all (thermal as well as nonthermal) fluctuations is given by Eq. (3.41),

$$W_s(\mathcal{E}) = \frac{1}{U}\, e^{-\mathcal{E}/U}, \tag{3.86}$$

and the variance of the energy at all temperatures (including $T = 0$) is $(\sigma_{\mathcal{E}}^2)_s = U^2$. Using the decomposition (3.60) we may write for the total energy fluctuations

$$(\sigma_{\mathcal{E}}^2)_s = U^2 = (U_T + \mathcal{E}_0)^2 = U_T^2 + 2\mathcal{E}_0 U_T + \mathcal{E}_0^2. \tag{3.87}$$

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[18] The existence of energy fluctuations associated with the natural linewidth and other processes (see e.g. Schiff 1955; Louisell 1973), effectively dilutes this discrete distribution of energies into a somewhat smoothened-out distribution acquiring a more continuous shape. Thus $g(\mathcal{E})$ should be seen as a theoretical limiting distribution.

This result generalizes Eq. (3.59) to include both thermal and nonthermal energy fluctuations. In conformity with the present discussion, the total energy can be written in terms of its thermal and nonthermal *fluctuating* parts,

$$\mathcal{E} = \mathcal{E}_T + \mathcal{E}_0. \tag{3.88}$$

The total energy fluctuations are then given by

$$(\sigma^2_{\mathcal{E}})_s = \sigma^2_{\mathcal{E}_T} + \sigma^2_{\mathcal{E}_0} + 2\Gamma(\mathcal{E}_T, \mathcal{E}_0), \tag{3.89}$$

where $\Gamma(\mathcal{E}_T, \mathcal{E}_0)$ is the covariance

$$\Gamma(\mathcal{E}_T, \mathcal{E}_0) = \langle \mathcal{E}_T \mathcal{E}_0 \rangle - \langle \mathcal{E}_T \rangle \langle \mathcal{E}_0 \rangle. \tag{3.90}$$

Comparing Eqs. (3.89) and (3.87), and identifying the temperature-dependent part of the fluctuations of the whole field $U_T^2 + 2\mathcal{E}_0 U_T$ with $\sigma^2_{\mathcal{E}_T}$ and $\mathcal{E}_0^2$ with $\sigma^2_{\mathcal{E}_0}$, we verify that $\Gamma(\mathcal{E}_T, \mathcal{E}_0) = 0$, as was expected considering that the fluctuations of $\mathcal{E}_T$ and $\mathcal{E}_0$ are statistically independent, due to the independence of their sources.

The entropy S_s follows from Eqs. (3.44) and (3.86),

$$S_s = -k_B \int W_s(\mathcal{E}) \ln c_s W_s(\mathcal{E}) d\mathcal{E} = k_B \ln c_s^{-1} U + k_B, \tag{3.91}$$

whence

$$\frac{\partial S_s}{\partial U} = \frac{k_B}{U}. \tag{3.92}$$

A comparison with the thermodynamic entropy, which satisfies

$$\frac{\partial S_g}{\partial U} = \frac{1}{T}, \tag{3.93}$$

shows that these two entropies coincide only when $\mathcal{E}_0 = 0$, i.e., for $U = k_B T$.

3.6.2 Quantum Fluctuations and Zero-Point Fluctuations

Let us now investigate how the nonthermal fluctuations become manifest in the statistical properties of the ensemble of oscillators. The value of the energy of the harmonic oscillator [(cf. Eq. (3.1)]

$$\mathcal{E} = (p^2 + \omega^2 q^2)/2 \tag{3.94}$$

can be used as a starting point to perform a transformation from the energy distribution $W_s(\mathcal{E})$ to a distribution $w_s(p, q)$ defined in the oscillator's phase space (p, q). To this end we introduce the pair of variables $(\mathcal{E}, \theta)$ related to the couple (p, q) by[19]

$$p = \sqrt{2\mathcal{E}} \cos \theta, \tag{3.95a}$$

$$q = \sqrt{\frac{2\mathcal{E}}{\omega^2}} \sin \theta, \tag{3.95b}$$

so that $w_s(p, q)$ is given by (Papoulis 1991; Birnbaum 1961)[20]

$$w_s(p, q) = W_s(\mathcal{E}(p, q), \theta(p, q)) \left| \frac{\partial(\mathcal{E}, \theta)}{\partial(p, q)} \right|, \tag{3.96}$$

with the Jacobian of the transformation

$$\frac{\partial(p, q)}{\partial(\mathcal{E}, \theta)} = \left| \frac{\partial(\mathcal{E}, \theta)}{\partial(p, q)} \right|^{-1} = \frac{1}{\omega}. \tag{3.97}$$

Now, $W_s(\mathcal{E})$ is a marginal probability density that can be obtained from $W_s(\mathcal{E}, \theta)$ by integrating over the variable θ, so that

$$W_s(\mathcal{E}) = \int_0^{2\pi} W_s(\mathcal{E}, \theta) d\theta. \tag{3.98}$$

For a system of harmonic oscillators in equilibrium, the trajectories (in general, the surfaces) of constant energy do not depend on θ, so all values of θ are equally probable, which means that

$$W_s(\mathcal{E}, \theta) = \frac{1}{2\pi} W_s(\mathcal{E}). \tag{3.99}$$

Using Eqs. (3.86), (3.94) and (3.96) we thus obtain for the distribution in phase space:

$$w_s(p, q) = \frac{\omega}{2\pi} W_s(\mathcal{E}(p, q)) = \frac{\omega}{2\pi U} \exp\left(- \frac{p^2 + \omega^2 q^2}{2U} \right). \tag{3.100}$$

This expression, which is known in quantum theory as the Wigner function for the harmonic oscillators (Hillery et al. 1984), can be factorized as a product of two normal distributions,

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[19] The transformation defined by (3.95a) and (3.95b) is an extended canonical transformation (Goldstein 1980), which differs from a canonical one—from the action and angle variables (J, θ), with $J = \mathcal{E}/\omega$, to the phase space variables (p, q)—only by a constant factor ω. Of course $\theta = \omega t$.

[20] When a probability $P(x)$ is expressed in terms of a new variable $y(x)$ as $W(y)$, the equality $P(x)dx = W(y)dy$ holds. Equation (3.96) is simply the generalization of this result to a two-dimensional space. See Papoulis (1991), Chap. 6 for a detailed derivation.

$$w_s(p,q) = w(p)w(q) = \frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-p^2/2\sigma_p^2} \cdot \frac{1}{\sqrt{2\pi\sigma_q^2}} e^{-q^2/2\sigma_q^2}, \tag{3.101}$$

where $\sigma_p^2 = U$ and $\sigma_q^2 = U/\omega^2$. The product of these dispersions gives

$$\sigma_q^2\sigma_p^2 = \frac{U^2}{\omega^2} = \frac{\mathcal{E}_0^2}{\omega^2} + \frac{\sigma_{\mathcal{E}_T}^2}{\omega^2} \geq \frac{\mathcal{E}_0^2}{\omega^2} = \frac{\hbar^2}{4}, \tag{3.102}$$

where Eq. (3.58) was used to write the second equality and the value $\mathcal{E}_0 = \hbar\omega/2$ was introduced into the last one.

Equation (3.102) points to the fluctuating zero-point energy as the ultimate (and irreducible) source of the so-called quantum fluctuations. Indeed, the magnitude of $\sigma_q^2\sigma_p^2$ is bounded from below because of the nonthermal energy fluctuations; the minimum value $\hbar^2/4$ is reached when all thermal fluctuations have been suppressed, which means $T = 0$. Therefore, descriptions afforded by purely thermal distributions such as W_g cannot account for the meaning of these inequalities. This result stresses again the fact that once a zero-point energy has been introduced into the theory, new distributions (specifically statistical rather than thermodynamic) are needed to include its fluctuations and to obtain the corresponding quantum statistical properties. Though here we have arrived at the Heisenberg inequality (3.102) by considering a system of harmonic oscillators, later on (particularly in Chap. 5) we will derive it for an arbitrary system, and again the presence of the ZPF will turn out to be decisive in reaching the result. Finally, note that the Heisenberg inequalities should be understood as referring to statistical variances, due to the statistical nature of (3.102).

3.6.3 Comments on the Reality of the Zero-Point Fluctuations

As mentioned earlier, the concept of a zero-point energy of the radiation field entered into scene as early as 1912, with Planck's second derivation of the blackbody spectrum. Yet further to the frustrated attempt by Einstein and Stern (1913), and despite the suggestive proposal made by Nernst (1916) to consider the ZPF as responsible for atomic stability, little or no attention was paid to its existence as a real physical entity that could have a role in the newly developing quantum mechanics.[21] Interestingly, it was the crystallographers who, prompted by Debye's theoretical work, set out to measure the spectroscopic effects of the zero-point energy through X-ray analysis and thereby seemengly verified its existence (James et al. 1928; Wollan 1931).

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[21] The value of the deep insight of Nernst will be substantiated in Chaps. 4–7. This conceptualization is in vivid contrast with the notion of 'virtual' usually applied to the fluctuating vacuum field.

As mentioned in Sect. 1.4.1, today it is well accepted that the fluctuations of the electromagnetic vacuum are responsible for important observable physical phenomena. Perhaps their best known manifestations, within the atomic domain, are the Lamb shift of energy levels (see e.g. Milonni 1994) and their contribution to the spontaneous transitions of the excited states to the ground state. They are known to contribute one half of the Einstein A-coefficient for 'spontaneous' transitions, the other half being due to radiation reaction (see e.g. Milonni 1994; Davydov 1965). [22] By far the most accepted evidence of the reality of the ZPF is the Casimir effect, that is, the force between two parallel neutral metallic plates resulting from the modification of the field by the boundaries (see e.g. Boyer 1970; Bordag et al. 2009). The existence of the ZPF vcan therefore be considered a reasonably well established physical fact.[23] In the following chapters we will have occasion to study in depth the essential role played more broadly by this random field in its interaction with matter at the atomic level.

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[22] This point is discussed in Chap. 6, in connection with the derivation of the Einstein A and B coefficients.

[23] There is an extended stance against the assumption of the reality of the fluctuating ZPF, based mainly on the argument that it does not activate photon detectors. Another frequent argument refers to the unobserved tremendous gravitational effects that such field should produce. The first objection has been answered by offering models of photon counters compatible with the reality of the zero-point fluctuations (Santos 2002a, b). The gravitational puzzle is related to the problem of the cosmological constant, and represents an age-old unsolved fundamental problem that besets a broad parcel of physics (see e.g. Weinberg 1989).

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