Playing Dominoes Is Hard, Except by Yourself

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Abstract. Dominoes is a popular and well-known game possibly dating back three millennia. Players are given a set of domino tiles, each with two labeled square faces, and take turns connecting them into a growing chain of dominoes by matching identical faces. We show that single-player dominoes is in P, while multiplayer dominoes is hard: when players cooperate, the game is NP-complete, and when players compete, the game is PSPACE-complete. In addition, we show that these hardness results easily extend to games involving team play.

Keywords: algorithmic combinatorial game theory, mathematical games and puzzles, computational complexity.

1 Introduction

Dominoes are 1×2 rectangular tiles with each 1×1 square marked with spots indicating a number, typically between 0 and 6. The precise origin of dominoes remains a mystery, though the earliest domino set dates back to an Egyptian tomb from around 1355 BC [1]. Even less clear is when and where various games involving dominoes arose.

Today, the game of Dominoes is immensely popular around the world, with annual tournaments including the World Domino Tournament, World Championship Domino Tournament, Domino USA, and Federación Internacional de Dominó. The game is typically played by two or four players who take turns laying down dominoes in a chain with the numbers matching at each adjacency. A traditional set of dominoes consists of all 28 unordered pairs of numbers between 0 and 6 (a "double-6" set). Larger domino sets include all numbers between 0 and 9 (double-9s) and all numbers between 0 and 12 (double-12s).

In this paper, we consider a generalized version of this game, where the numbers on each side of a domino can take on any value, and the number of tiles is unrestricted. We formalize this generalized domino game in Section 2, for one or more players under both cooperative and competitive play. In Section 3, we explore the cooperative version of the game. In particular, we show that single-player (cooperative) dominoes is in P, while p-player cooperative dominoes is NP-complete for any $p \geq 2$. In Section 4, we show that competitive dominoes is PSPACE-complete, as well as a competitive team variant. Some of our proofs are similar to those for the related card game of UNO; see Section 2.4. Finally, we list some open problems in Section 5.

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2 Game Definitions

2.1 Classic Dominoes

The typical domino set consists of $\binom{7}{2} + 7 = 28$ dominoes. Each domino is a rectangular tile made up of two faces, where each face contains an integer value from 0 to 6. (Typically, the dominoes faces are marked by dots similar to the faces of a die.) A complete set of dominoes consists of all possible pairs of numbers, including doubles, without repetition.

The game of dominoes[1] is normally played with two or four players. In the four-player game, the players sit around a table, play in clockwise order, and players sitting across from each other form a team. In the two-player game, each player forms his/her own team.

At the beginning, the dominoes are randomly and evenly distributed, and the first player places any single domino. On each subsequent turn, a player plays exactly one domino by matching one of its faces to one of the two ends of the current chain of dominoes, extending the chain by one domino. Figure 2 shows an example of a domino chain. Matching requires identically labeled domino faces. If a player cannot match any of his/her dominoes to the current chain, s/he can pass his/her turn.

The game ends when one player runs out of dominoes, in which case the team of that player wins, or when both ends of the chain become "blocked". An end is *blocked* if none of the remaining dominoes can be matched with that end. When both ends are blocked and thus no more moves are possible, the winning team is the one with the smallest sum of all face values on their unplayed tiles.

2.2 Generalized Dominoes

We consider a generalized version of dominoes in which each domino can have any integer (or symbol) on each end. Otherwise play is identical: the first play can be any tile, and each subsequent tile can be played if it matches an end of the chain. A player wins when s/he either plays all of his/her dominoes, and a player loses when s/he cannot place down any dominoes. Importantly, we do not allow players to pass (an assumption also made of cooperative UNO [2]). We also do not assume that each player has the same number of tiles at the beginning of the game.

Formally, a *domino* consists of two numbers, one on either end of the rectangular piece. A domino can be described as a multiset, $\{a, b\} \in \mathcal{P}(S)$, where $S = \{0, 1, 2, \ldots, c\}$ is the set of symbols in the current game. We refer to individual domino pieces with a single uppercase letter such as $X = \{a, b\}$.

At the beginning of the game, player i receives a multiset of dominoes T_i. In particular, we allow each player to receive more than one copy of the same domino; we will only use this flexibility to build "null" players when we have more than two players.

[1] In fact, there are many games played with domino pieces. The most popular game, described here, is also called block dominoes.

The game of dominoes progresses as players add dominoes to the current chain of dominoes on the board. The current chain of dominoes has two open ends, a left end ℓ and a right end r. If Player 1 has a domino $X = \{a, b\} \in T_1$, then Player 1 can match X to the left end of the current chain if and only if either $\ell = a$ or $\ell = b$ (or both). In this case, ℓ becomes b or a, respectively. Likewise, X can be matched on the right if and only if either $r = a$ or $r = b$, in which case r becomes b or a respectively.

In our figures, we follow the tradition of dominoes of the form $\{a, a\}$ (*doubles*) being oriented perpendicular to the chain, to highlight that they did not change the value of the end, but this unusual orientation does not change the behavior of these dominoes. (In some variants of the game, doubles can branch the chain of dominoes into a tree, but this is uncommon and not considered here.)

One key difference from traditional dominoes is that we assume perfect information, instead of hidden dominoes, to make the optimal strategy well-defined. For cooperative games, this assumption makes sense, as players should share all of their knowledge. For competitive two-player games, this assumption is also practical, assuming that the multiset of all dominoes is known, as that minus one player's hand is the other player's hand.

2.3 Variants

We now define two versions of perfect-information dominoes: cooperative and competitive.

Cooperative Dominoes

Instance: There are one or more players in the game. Each player has a multiset of dominoes. All dominoes (in the chain and in the players' hands) are visible to all players.
Question: Can all the players cooperate to help Player 1 play all of his/her dominoes (and win the game)?

Competitive Dominoes

Instance: There are two or more players in the game. Each player has a multiset of dominoes. Again, all dominoes are visible to all players.
Question: Does Player 1 have a winning strategy regardless of what the other players do?
Team version: In addition, the players are partitioned into teams. Can Player 1's team win?

In all variants we consider, we assume that players begin play in order of increasing index. Thus, Player 1 always makes the opening move.

2.4 UNO®

Our study of dominoes is similar to the recent study of the card game UNO[2] [2]. Dominoes and UNO both involve players attempting to play all their pieces by connecting them to pieces that have been played previously; the main difference is that UNO has two dimensions (color and number) for pieces to match, while dominoes has two chain ends for a piece to match. In particular, the multiplayer cooperative variant of dominoes roughly corresponds to the multiplayer cooperative version of UNO, and our proof that this dominoes game is NP-complete is similar to the proof for UNO. Single-player UNO is NP-complete in general, but polynomial when one of the two dimensions is constant, and this proof is similar to our result that single-player dominoes is polynomial. On the other hand, some complexities differ: two-player competitive UNO is polynomial, while we show that two-player competitive dominoes is PSPACE-complete.

3 Cooperative Dominoes

In this section, we consider the cooperative game of dominoes and analyze its complexity. For the multiplayer case, we first analyze the two-player case and then generalize the result.

3.1 Two-Player Cooperative Dominoes

Theorem 1. *Two-player cooperative dominoes is NP-complete.*

Proof. First, two-player cooperative dominoes is in NP: a certificate is a move sequence that causes Player 1 to win.

To show that two-player cooperative dominoes is NP-hard, we reduce from Hamiltonian Path; refer to Figure 1. Suppose $G = (V, E)$ is a graph and assume that G is simple and connected. Then construct a dominoes instance as follows:

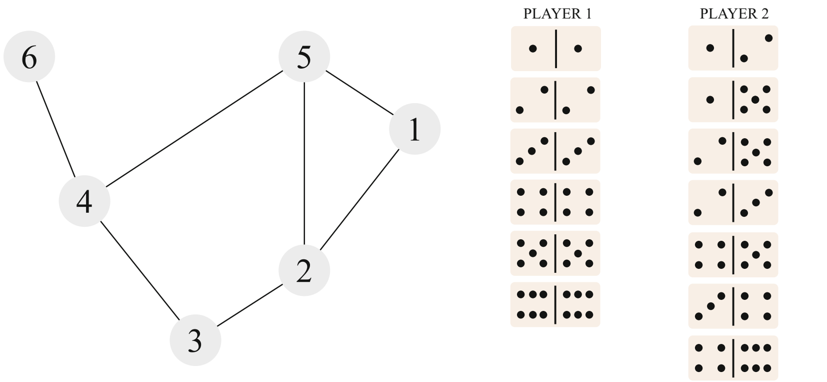
$$T_1 = \{\{i, i\} \mid i \in V\} \quad \text{and} \quad T_2 = \{\{i, j\} \mid ij \in E\} \cup \{\{*, *\}\}. \tag{1}$$

Here we give Player 2 an extra domino $\{*, *\}$ so that Player 2 can never win, as the domino matches nothing and Player 2 does not start. (If we did not give Player 2 this extra domino, s/he can win in the case where G is the path graph.) Thus, if there is a winning strategy, then Player 1 wins.

Player 1 cannot place dominoes next to other dominoes that belonged to Player 1 because the dominoes, by construction, cannot match. Hence, whenever Player 2 plays a domino, Player 1 must place a domino adjacent to it. Therefore, if this game has a winner, Player 1 wins with a chain of dominoes that alternates between Player 1 and Player 2.

The sequence generated by the chain of dominoes describes a Hamiltonian path in G; refer to Figure 2. Because Player 1 gets one and only one domino per

[2] UNO is a registered trademark of Mattel Corporation.



(a) An instance of Hamiltonian path

(b) Corresponding instance of 2-player cooperative dominoes

Fig. 1. Reduction from Hamiltonian path to 2-player cooperative dominoes

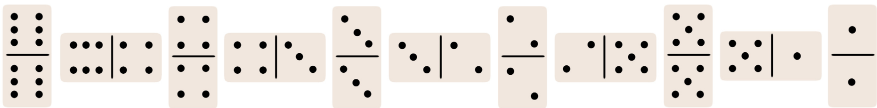


Fig. 2. Hamiltonian path represented as domino chain

vertex, each vertex will appear only once as a domino in the chain. If Player 1 wins, then Player 1 played all of his/her dominoes, and so all vertices will appear. The tiles that Player 2 plays are the edges that connect the vertices in the Hamiltonian path. Likewise, if there is a Hamiltonian path in G, then Player 1 and Player 2 can play the sequence of dominoes corresponding to the Hamiltonian path to make Player 1 win.

Because Hamiltonian Path is NP-hard [3], two-player cooperative dominoes is NP-hard, and so it is NP-complete. $\square$

3.2 p-Player Cooperative Dominoes

We can extend Theorem 1 to the p-player case for any fixed $p \geq 2$. Note that p must be fixed, because otherwise the statement follows trivially from the fact that the $p = 2$ case is NP-complete.

Corollary 2. *p-player cooperative dominoes is NP-complete for any fixed $p \geq 2$.*

Proof. As in Theorem 1, the game is in NP because any winning sequence for Player 1 is a certificate.

We show that p-player dominoes is NP-hard because we can simulate any two-player cooperative game. Players 1 and 2 correspond to Players 1 and 2 from the original reduction, and all other players serve as "null" players.

Each null player is given a domino $\{a, a\}$ for each face a that appears in $T_1 \cup T_2$, the set of dominoes owned by Player 1 and Player 2. (This construction requires T_i for $i > 2$ to be a multiset.) This construction ensures that players who are not Player 1 or 2 cannot change the faces at the end of the current chain, so they serve only to "communicate" Player 2's moves to Player 1. Note that the null players have at least twice as many dominoes as Players 1 and 2, so none of them can win. In addition, each null player has two dominoes for any domino played by Player 1 or 2, so these players can never lose by getting stuck. □

3.3 Single-Player Dominoes

Theorem 3. *One-player dominoes is in P.*

Proof. Construct the multigraph G where vertices correspond to numbers and edges correspond to dominoes. That is, if we have a domino $\{a, b\}$, then the multigraph G will have vertices a and b with an edge connecting them.

Now we claim the problem reduces to finding an Eulerian path in the multigraph. If we find an Eulerian path, then in the order of its edges, we can place the corresponding dominoes. Because two successive edges in the path share a vertex between them, the ends of those corresponding dominoes will have an end in common, so the chain will be valid. Conversely, if there is a way to place the dominoes, the resulting chain will correspond to an Eulerian path in the multigraph.

We can determine in polynomial time whether the multigraph has an Eulerian path: the classic theorem of Euler says that we just need to check that all nodes have even degree except for at most two nodes, which can have odd degree [4]. □

4 Competitive Dominoes

In this section we discuss competitive dominoes between two or more players. We do this by first analyzing the two-player case, and then we extend the result to the general multiplayer and team variants.

Lemma 4. *Two-player competitive dominoes is in PSPACE.*

Proof. Each instance of dominoes can easily be transformed into the formula game problem which is in PSPACE [5]. We can specify the possible moves as variables indexed by domino, direction, position, and turn. If there are n total dominoes in the game, then there are two directions, n positions, and n turns. This gives us $O(n^3)$ variables.

The formula starts with alternating quantifiers: for Player 1 to have a winning strategy, there must exist a move by Player 1 in turn 1 such that, for any move by Player 2 in turn 2, there exists a move by Player 1 in turn 3, and so on. The formula then has a predicate with four parts. The first part specifies that

Player 1 moves correctly, by checking that there is only one move per turn. The second specifies that Player 2 moves correctly, by checking that there is at most one move per turn. The third part specifies that the chain is correct, by checking all pairs of dominoes with each other, and ignoring the ones that do not apply. Finally, the last part specifies that Player 1 won by checking that Player 1 got rid of all of his/her pieces, or Player 2 was stuck at some point.

There are polynomially many variables and all checks require checking a single variable or a pair of variables. Because there are polynomially many pairs of variables, we can write the formula down in polynomial space. □

Theorem 5. *Two-player competitive dominoes is PSPACE-complete.*

Proof. By Lemma 4, it remains to show that competitive dominoes is PSPACE-hard. We do so by reducing from directed edge bipartite generalized geography (BIPARTITE-GG). An instance of BIPARTITE-GG consists of a directed bipartite graph $G = (A \cup B, E)$ where $E \subseteq A \times B$, and a start vertex $a^* \in A$. A token starts at vertex a, and two players alternate moving the token along edges, where Player 1 may only use edges directed from A to B, and Player 2 may only use edges directed from B to A. Each edge can be played at most once; a player loses if s/he has no possible moves. It is PSPACE-complete to determine whether Player 1 has a winning strategy [6].

Given any such graph G and start vertex a^*, we construct an instance of two-player competitive dominoes as follows; refer to Figure 3. For each directed edge (a, b) where $a \in A$ and $b \in B$, give Player 1 the domino $\{a, b\}$. Similarly, for each directed edge $(b, a) \in B \times A$, give Player 2 the domino $\{a, b\}$. We call these dominoes *edge dominoes*. An edge domino itself does not encode information about the direction of the original edge: the direction can be recovered by looking at which player owns the edge, because Player 1 receives dominoes only for edges pointing from nodes in A to nodes in B, and Player 2 receives dominoes only for edges pointing from nodes in B to nodes in A.

Each player also receives one nonsense domino $\{\#, \#\}$ that can be connected only to the other player's nonsense domino $\{\#, \#\}$. The purpose of these dominoes is to eliminate the option of a player winning by getting rid of all their dominoes, and instead focus on blocking the opponent. It never makes sense to play the nonsense domino first (as the other player would immediately win), and the nonsense domino can never be played if a player starts with a different domino, so it is impossible for a player to win by playing all dominoes.

BIPARTITE-GG requires Player 1 to start from some specified node $a^* \in A$. To reproduce this in our instance of two-player competitive dominoes, we define Player 2 to move first, and give Player 2 the *start domino* $\{a^*, \text{:-)}\}$, where :-) is a unique value. In addition, for each node $b \in B$, we give Player 1 the *garbage dominoes* $\{b, b'\}, \{b, b''\}, \{b', b'\}$, and $\{b'', b''\}$. The "garbage" values b' and b'' (drawn in figures as squares or triangles) appear only in dominoes belonging to Player 1, so they will effectively block Player 2.

We claim that this construction forces Player 2 to play the start domino first (assuming s/he moves first). As argued above, Player 2 cannot start with the

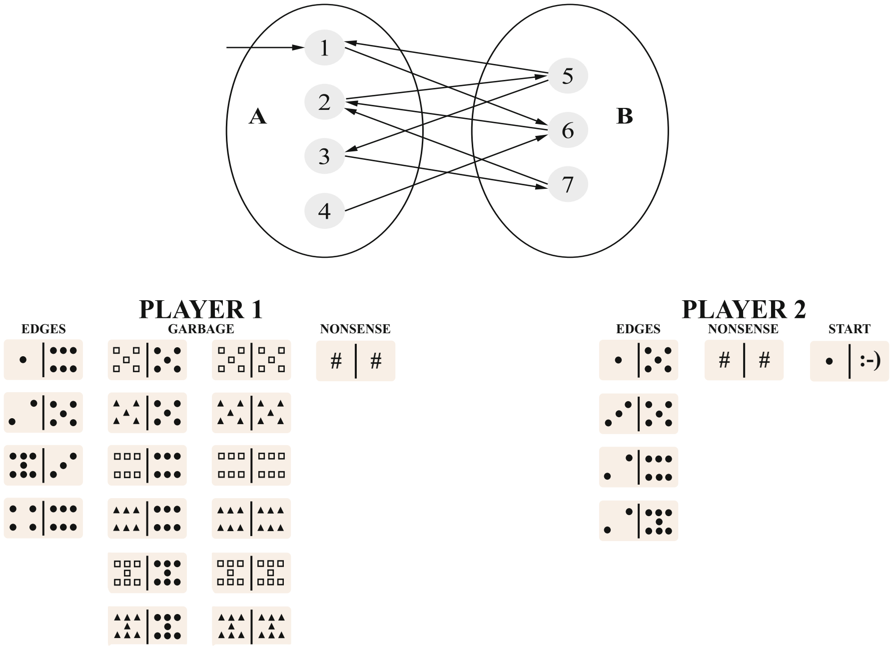


Fig. 3. Reduction from BIPARTITE-GG to two-player competitive dominoes

nonsense domino $\{\#, \#\}$. To see why Player 2 cannot start with an edge domino, consider two cases; refer to Figure 4.

(a) Case 1 (b) Case 2

Fig. 4. When Player 2 does not start with the start domino, in the example of Figure 3

Case 1: Player 2 starts with an edge domino $\{a, b\}$ where $a^* \neq a \in A$ and $b \in B$. Player 1 can play the garbage tile $\{b, b'\}$. Then Player 2's next move must be some edge domino $\{a, c\}$ where $c \in B$. But then Player 1 can play the garbage tile $\{c, c''\}$. This leaves Player 2 with no moves, which wins the game for Player 1.

Case 2: Player 2 starts with an edge domino $\{a^*, b\}$ where $b \in B$. Player 1 can play the garbage tile $\{b, b'\}$. Then Player 2's next move is forced to be $\{a^*, \text{:-)}\}$, as any other move would result in Player 1 winning as shown in Case 1. Then Player 1 can play the garbage tile $\{b', b'\}$. Again this leaves Player 2 with no moves.

Therefore Player 2 starts with the start domino $\{a, \text{:-)}\}$. Because the value :-) is unique, the domino chain can extend only on one end. We can think of this one-way chain as moves in BIPARTITE-GG. The domino ends keep track of the current vertex that the BIPARTITE-GG is on, and playing a domino changes the current vertex in BIPARTITE-GG. Additionally, placing a domino means that that domino cannot be used again, which models that BIPARTITE-GG removes directed edges as the players pick them.

If Player 1 has a winning strategy in the two-player competitive dominoes case with this arrangement, then the placement of the dominoes corresponds to the directed edges that Player 1 picks in response to Player 2. Likewise, if there is a winning strategy in BIPARTITE-GG, then Player 1 can use that strategy for the placement of dominoes in the two-player competitive dominoes game.

One minor point is that the above construction switches the starting player, but our definition of dominoes required Player 1 to move first. To fix this, simply reverse the roles of Players 1 and 2 in the BIPARTITE-GG instance before applying the reduction. □

Corollary 6. *p-player competitive dominoes is PSPACE-complete for any fixed* $p \geq 2$.

Proof. Similar to Lemma 4, this game is in PSPACE (Players 2 through p act as a collective opponent to Player 1). PSPACE-hardness follows from the fact that a game with p players can simulate any game with two players: all players besides Player 1 and Player 2 act as null players, as in the proof of Theorem 2. □

Corollary 7. *The team version of competitive dominoes is PSPACE-complete.*

Proof. Similar to Lemma 4, this game is in PSPACE (quantifiers switch whenever the team switches). PSPACE-hardness follows from the fact that a team game can simulate any game with two players: any player not on Player 1's team can serve the role of Player 2, and all other players can function as null players, as in the proof of Theorem 2. □

While Corollary 7 is simple, it confirms that competitive dominoes becomes only harder when players are grouped into teams, which is often how the real game is played.

5 Conclusion

In this paper, we determined the computational complexity of the game of dominoes under different variants: single-player, multiplayer cooperative, multiplayer competitive, and team competitive. All variants of multiplayer dominoes were intractable, while the single-player variant was easy.

Some details of our model deserve further study. First, we forbade players from passing, but the classic game allows passing exactly when a player has no feasible move. Second, we allowed the initial "hands" to have an unequal number of dominoes between the players, but the classic game distributes dominoes evenly. Third, we allowed arbitrary (multi)sets of dominoes for each player, but the classic four-player game distributes a "double-n" set of dominoes (exactly one of each possible domino $\{a, b\}$ with $a, b \in \{0, 1, \ldots, n\}$). Do our results extend to these models?

A final direction for future work would be to consider the competitive multiplayer game with imperfect information. Bounded team games with imperfect information are potentially as hard as NEXPTIME (see [7]). Analyzing dominoes in this setting seems much more difficult, however.

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