Another Look at the Shoelace TSP: The Case of Very Old Shoes

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Abstract. What is the most efficient way of lacing a shoe? Mathematically speaking, this question concerns the structure of certain special cases of the bipartite travelling salesman problem (BTSP).

We show that techniques developed for the analysis of the (standard) TSP may be applied successfully to characterize well-solvable cases of the BTSP and the shoelace problem. In particular, we present a polynomial time algorithm that decides whether there exists a renumbering of the cities such that the resulting distance matrix carries a benevolent combinatorial structure that allows one to write down the optimal solution without further analysis of input data. Our results generalize previously published well-solvable cases of the shoelace problem.

Keywords: Bipartite travelling salesman problem, shoelace problem, polynomially solvable case, relaxed Monge matrix, pick-and-place robot.

1 The Art of Shoelacing

In Europe, shoelaces are usually threaded in alternating zigzags, such that (when viewed from above) the eyes of the shoes seem to be joined horizontally by the shoelaces. In the USA, shoelaces are typically threaded in opposing zigzags, and when seen from above they seem to be crossed. A third standard method is the so-called shoe shop method, in which the shoelace makes a continuous zigzag from top to bottom and then returns to the top in a diagonal line.

To the non-expert it would appear that there are only three or four accepted methods of lacing our shoes. However, this is far, far, far from the truth! Experts in the area of shoelacing are familiar with dozens of methods, as for instance army lacing, bow-tie lacing, criss-cross lacing, double-helix lacing, gap lacing, hash lacing, hexagram lacing, hidden-knot lacing, ladder lacing, lattice lacing, left-right lacing, lightning lacing, over-under lacing, pentagram lacing, Roman lacing, sawtooth lacing, spider-web lacing, star lacing, train-track lacing, zigzag lacing, or zipper lacing.

Now a burning question arises: Which of these dozens of shoelacing methods is the most efficient one? Or, in a more scientific formulation: Which lacing method needs the smallest amount of shoelace? The mathematical literature contains several studies on this theme. There are the short technical papers by Halton [13], Misiurewicz [18] and Polster [19], and there also is a beautiful

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booklet [20] by Polster with the title *"The shoelace book: a mathematical guide to the best (and worst) ways to lace your shoes"*. In this paper, we will add some new insights to this research branch by exhibiting certain connections between the shoelace problem and the travelling salesman problem.

2 Technical Introduction

The travelling salesman problem (TSP). In the TSP, the objective is to find for a given $n \times n$ distance matrix $C = (c_{ij})$ a cyclic permutation τ of the set $\{1, 2, \ldots, n\}$ that minimizes the sum $c(\tau) = \sum_{i=1}^{n} c_{i\tau(i)}$. In TSP slang, the elements of $\{1, 2, \ldots, n\}$ are usually called *cities* or *points*, the cyclic permutations are called *tours*, and the value $c(\tau)$ is the *length* of permutation τ. The set of all permutations over set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{S}_n$. For $\tau \in \mathcal{S}_n$, we denote by τ^{-1} the *inverse* of τ, that is, the permutation for which $\tau^{-1}(i)$ is the predecessor of i in the tour τ, for $i = 1, \ldots, n$. We will also use a cyclic representation of cyclic permutations τ in the form

$$\tau \;=\; \langle i, \tau(i), \tau(\tau(i)), \ldots, \tau^{-1}(\tau^{-1}(i)), \tau^{-1}(i), i\rangle.$$

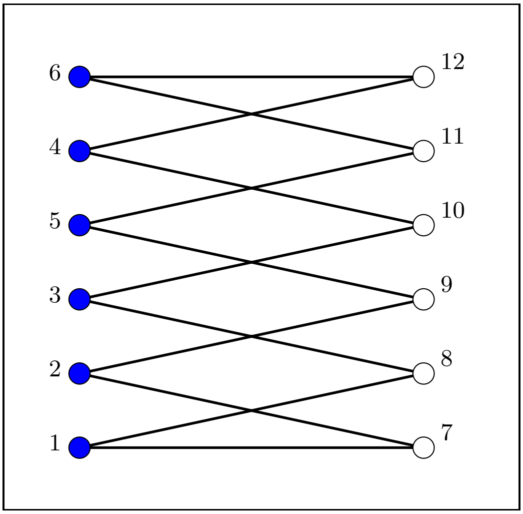
In the *maximization* version of the TSP (MaxTSP), one is interested in finding the *longest* tour. The characterization of polynomially solvable cases is one of the standard directions for research on NP-hard problems. For surveys on well-solvable cases of the TSP, we refer the reader to Gilmore, Lawler & Shmoys [12] and to Burkard & al [5].

The bipartite travelling salesman problem (BTSP). In the BTSP, there is an even number $n = 2k$ of cities which are partitioned into two classes: the class $K_1 = \{1, 2, \ldots, k\}$ of blue cities and the class $K_2 = \{k+1, k+2, \ldots, n\}$ of white cities. Any feasible tour in the BTSP has to alternate between blue and white cities. The objective is to find the shortest tour with this special structure. The set $\mathcal{T}_n$ of all feasible tours for the BTSP may formally be defined as

$$\mathcal{T}_n \;=\; \{\tau \in \mathcal{S}_n | \tau^{-1}(i), \tau(i) \in K_2 \text{ if } i \in K_1;\ \tau^{-1}(i), \tau(i) \in K_1 \text{ if } i \in K_2\}. \quad (1)$$

By $C[K_1, K_2]$ we denote the $k \times k$ matrix which is obtained from matrix C by deleting the rows with numbers from K_2 and by deleting the columns with numbers from K_1. Note that the length $c(\tau)$ of any feasible BTSP tour is calculated by using elements from $C[K_1, K_2]$ only.

The BTSP is NP-hard, and there is no constant factor approximation algorithm for it unless $P = NP$; see Frank, Korte, Triesch & Vygen [11]. The BTSP has also been investigated by Baltz [3], Baltz & Srivastav [4], Chalasani, Motwani & Rao [8], and Frank, Korte, Triesch & Vygen [11]. Its relevance for pick-and-place robots has been pointed out in Anily & Hassin [1], Atallah & Kosaraju [2], Leipälä & Nevalainen [15], and Michel, Schroeter & Srivastav [17].



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	5	5	5	5	5	5	40	40	40	40	40	40
Y coordinate	10	17	24	31	38	45	10	17	24	31	38	45

Fig. 1. An illustration to Halton's [13] optimal lacing for new shoes with neat and tidy rows of eyelets: the points with their coordinates, and an optimal BTSP tour

The shoelace problem. Halton [13] interprets the BTSP as a shoelacing problem: the cities represent the eyelets of a shoe, and the objective is to find an optimal shoe lacing strategy that minimizes the length of the shoelace. In Halton's model the eyelets are points in the Euclidean plane: the blue points lie on a straight line and have coordinates $(0, d), (0, 2d), \ldots, (0, kd)$, and the white points lie on some parallel line and have coordinates $(a, d), (a, 2d), \ldots, (a, kd)$. Halton proved that in his special case the tour

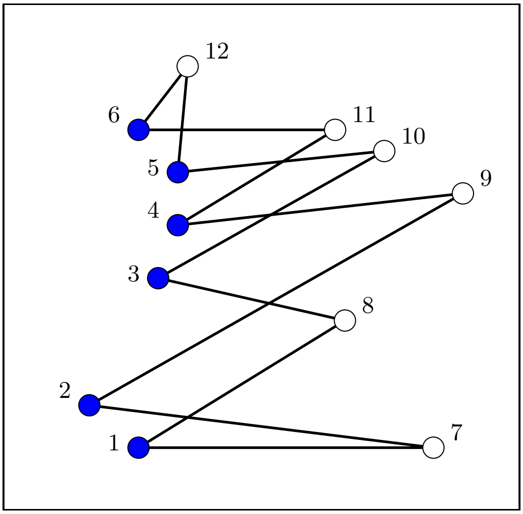
$$\tau^* \;=\; \langle 1, k+1, 2, k+3, 4, k+5, 6 \ldots, 7, k+6, 5, k+4, 3, k+2, 1 \rangle \qquad (2)$$

is the shortest tour in $\mathcal{T}_n$. Figure 1 illustrates Halton's case of brand-new shoes with two neat and tidy rows of eyelets.

In a follow-up paper, Misiurewicz [18] argues that Halton's model is only a crude approximation of reality: as shoes get older and worn-out, the eyelets move out of place and will no longer form tidy rows. Misiurewicz observes that for proving optimality of permutation τ^*, one actually does not need to have the eyelets on two parallel lines; it is sufficient to require that the inequalities

$$c_{ij} + c_{\ell m} \;\le\; c_{im} + c_{\ell j} \qquad (3)$$

hold for all indices i and j with $1 \le i \le \ell \le k$ and $k+1 \le j \le m \le n$. In other words, Halton's tour τ^* also solves the shoelace problem for older and somewhat worn-out shoes; see Figure 2 for an illustration of Misiurewicz's case.



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	11	6	13	15	15	11	41	32	44	36	31	16
Y coordinate	10	14	26	31	36	40	10	22	34	38	40	46

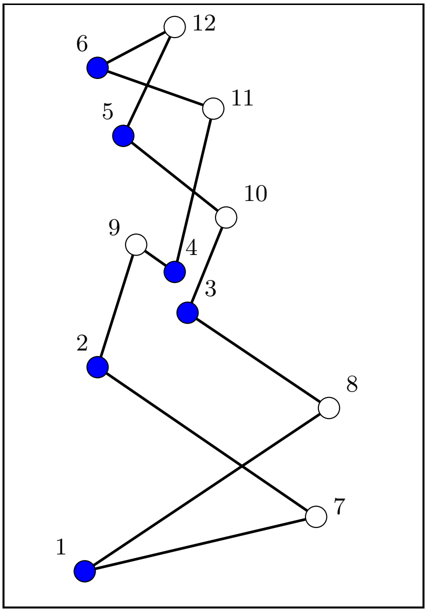
Fig. 2. An illustration to Misiurewicz's [18] optimal lacing for older and somewhat worn-out shoes: an instance with a Euclidean distance matrix

Results and organization of this paper. We show that the techniques developed for the analysis of the classical TSP can also be applied successfully to the shoelace problem. In Section 3, we review some of the well-solvable cases of the TSP which are relevant for the shoelace problem. We generalize the results of Halton [13] and Misiurewicz [18], and we characterize a new polynomially solvable case of the BTSP. In our case, the eyelets may indeed have *very* peculiar locations, so that the old shoes of Misiurewicz now turn into *very* old, deformed and mutilated shoes; see Figures 3 and 4 for an illustration (we hope that this justifies the title of the paper!). In Section 4, we present an algorithm for recognizing our new special case independently of the initial numbering of the points/eyelets.

3 Polynomially Solvable TSP Cases and the BTSP

We start by reviewing some known results on specially structured distance matrices. Readers who are familiar with the combinatorial optimization literature will already have recognized that the inequalities in (3) are the notorious Monge inequalities; see Burkard, Klinz & Rudolf [6] for further references. An $n \times n$ matrix $C = (c_{ij})$ is called a *Monge matrix*, if it satisfies the following conditions for all indices $i, j, m, \ell \in \{1, \ldots, n\}$ with $i < \ell$ and $j < m$:

$$c_{ij} + c_{\ell m} \ \le \ c_{im} + c_{\ell j}. \tag{4}$$



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	5	6	13	12	8	6	23	24	9	16	15	12
Y coordinate	6	21	25	28	38	43	10	18	30	32	40	46

Fig. 3. Instance 1 of the Euclidean BTSP with a relaxed Monge structure

As the inequality system (3) imposes the Monge inequalities only for the entries in $C[K_1, K_2]$, the system (3) is a relaxation of system (4).

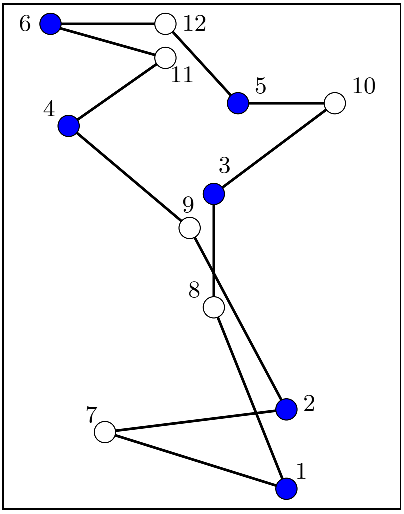
Supnick [21] proved that the TSP with a symmetric Monge distance matrix is always solved to optimality by the tour $\pi_1^* = \langle 1, 3, 5, 7, \ldots, 8, 6, 4, 2, 1\rangle$, and that the MaxTSP on symmetric Monge matrices is always solved by the tour $\sigma^* = \langle 1, n, 2, n-2, 4, n-4, \ldots, n-3, 3, n-1, 1\rangle$. Note that if the white points in the shoelace problem were numbered in the reverse order, that is, if points $i \in K_2$ were renumbered by $n + k + 1 - i$, then Halton's permutation τ^* in (2) would become the Supnick permutation σ^*. We mention this fact here to stress that the BTSP seems to have something in common with the MaxTSP.

Another well-known polynomially solvable case is the TSP with Kalmanson distance matrices. A symmetric $n \times n$ matrix C is a *Kalmanson* matrix if it fulfills the *Kalmanson conditions*

$$c_{ij} + c_{\ell m} \leq c_{i\ell} + c_{jm} \tag{5}$$

$$c_{im} + c_{j\ell} \leq c_{i\ell} + c_{jm}, \quad \text{for all } 1 \leq i < j < \ell < m \leq n. \tag{6}$$

Kalmanson [14] showed that the TSP with a Kalmanson matrix is solved by the tour $\pi_2^* = \langle 1, 2, 3, 4, 5, 6 \ldots, n-1, n, 1\rangle$. Furthermore, an optimal tour for the



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	22	22	16	4	18	2	7	16	14	26	12	12
Y coordinate	5	12	31	37	39	46	10	21	28	39	43	46

Fig. 4. Instance 2 of the Euclidean BTSP with a relaxed Monge structure

MaxTSP can always be found among $n/2$ specially structured tours containing among them Halton's tour τ^*.

Demidenko matrices form a common generalization of Supnick and Kalmanson matrices. A symmetric matrix $C = (c_{ij})$ is a Demidenko matrix if

$$c_{ij} + c_{\ell m} \quad \leq \quad c_{i\ell} + c_{jm}, \qquad \text{for all } 1 \leq i < j < \ell < m \leq n. \tag{7}$$

Demidenko [10] showed that an optimal tour for the TSP with an $n \times n$ Demidenko distance matrix can be found in $O(n^2)$ time. Deineko & Woeginger [9] proved that the MaxTSP with a Demidenko matrix remains NP-hard. However, for a subclass of Demidenko matrices the longest tour can be found in the set $\mathcal{T}_n$ of feasible BTSP tours as introduced in (1).

Proposition 1. *(Deineko & Woeginger [9]). Let C be a symmetric $n \times n$ Demidenko matrix with $n = 2k$, that additionally fulfills the conditions*

$$c_{ik} + c_{k+1,j} \quad \leq \quad c_{k+1,k} + c_{ij}, \qquad \textit{for } i \in K_1 \setminus \{k\},\ j \in K_2 \setminus \{k+1\}. \tag{8}$$

Then there exists an optimal MaxTSP tour which belongs to the set $\mathcal{T}_n$.

The problem of finding an optimal MaxTSP tour in $\mathcal{T}_n$ remains NP-hard. The following proposition identifies an almost trivial special case.

Proposition 2. *(Deineko & Woeginger [9]). Let C be a symmetric $n \times n$ matrix with $n = 2k$, that fulfills the conditions*

$$c_{1,k+1} + c_{ij} \geq c_{1j} + c_{i,k+1}, \quad i = 2, \ldots, k,\ j = k+2, \ldots, n \tag{9}$$
$$c_{p+1,k+p} + c_{ij} \geq c_{p+1,j} + c_{i,k+p}, \quad i = p+2, \ldots, k,\ j = k+p+1, \ldots, n \tag{10}$$
$$c_{p,k+p+1} + c_{ij} \geq c_{pj} + c_{i,k+p+1}, \quad i = p+1, \ldots, k,\ j = k+p+2, \ldots, n \tag{11}$$
$$p = 1, \ldots, k-2.$$

Then Halton's tour τ^* is a tour of maximum length in $\mathcal{T}_n$.

It is easy to see that conditions (9)–(11) form a relaxation of the Kalmanson conditions (6). Therefore, the TSP with a Kalmanson matrix that also fulfills (8) has τ^* as a tour of maximum length. Any Supnick matrix fulfills the inequalities in (8). Furthermore, a Supnick matrix satisfies the reverse inequalities of (9)–(11), where the $\geq$ signs are replaced by $\leq$. Therefore, if the points $i \in K_2$ are renumbered by $n + k + 1 - i$, then by Propositions 1 and 2, the permutation σ^* (which is obtained from τ^* by the same renumbering) constitutes an optimal solution to the MaxTSP with a Supnick matrix; we stress that the renumbering does not affect the inequalities in (8). This comment explains the relationship between the TSP and the MaxTSP with a Supnick matrix.

In the proof of Proposition 2 in [9], the well-known tour-improvement technique is used: starting from an arbitrary tour τ, a sequence of tours $\tau_1, \tau_2, \ldots, \tau_T$ is constructed, with $\tau_1 = \tau$ and $\tau_T = \tau^*$ such that

$$c(\tau_1) \ \leq \ c(\tau_2) \ \leq \ \cdots \ \leq \ c(\tau_T).$$

The inequalities (9)–(11) are used to establish the relationship $c(\tau_i) \leq c(\tau_{i+1})$. If inequalities (9)–(11) are all reversed, then it can be proved in a similar fashion that the tour τ^* is the shortest tour in $\mathcal{T}_n$. We summarize this result in the following theorem.

Theorem 3. *Let C be a symmetric $n \times n$ matrix with $n = 2k$, that fulfills the conditions*

$$c_{1,k+1} + c_{ij} \leq c_{1j} + c_{i,k+1}, \quad i = 2, \ldots, k,\ j = k+2, \ldots, n \tag{12}$$
$$c_{p+1,k+p} + c_{ij} \leq c_{p+1,j} + c_{i,k+p}, \quad i = p+2, \ldots, k,\ j = k+p+1, \ldots, n \tag{13}$$
$$c_{p,k+p+1} + c_{ij} \leq c_{pj} + c_{i,k+p+1}, \quad i = p+1, \ldots, k,\ j = k+p+2, \ldots, n \tag{14}$$
$$p = 1, \ldots, k-2.$$

Then the tour τ^* is a tour of minimum length for the BTSP.

Of course the system (12)–(14) is just a further relaxation of the Monge inequalities (4) and their relaxation (3). Figures 3 and 4 show two instances of the BTSP with the Euclidean distance matrices that satisfy (12)–(14) but violate some of the inequalities (3) of Misiurewicz.

The system (12)–(14) altogether contains $\Theta(n^3)$ inequalities. The following proposition shows that one needs only $O(n^2)$ time to verify these conditions.

Proposition 4. *The inequalities (12)–(14) can be verified in $O(n^2)$ time.*

Proof. Let $n = 2k$ throughout. To simplify notation, we consider an *asymmetric* $k \times k$ submatrix $A = C[K_1, K_2]$ of the $n \times n$ matrix C. The system (12)–(14) can then be rewritten as

$$\begin{aligned}
a_{11} + a_{st} &\le a_{1t} + a_{s1}, && 1 < s,t \le k && (15)\\
a_{p,p-1} + a_{st} &\le a_{pt} + a_{s,p-1}, && s = p+1,\dots,k; t = p,\dots,k; && (16)\\
a_{p-1,p} + a_{st} &\le a_{p-1,t} + a_{sp}, && s = p,\dots,k; t = p+1,\dots,k; && (17)\\
& && p = 2,3\dots,k-1. &&
\end{aligned}$$

We claim that the system (15)–(17) above is equivalent to the following system with $2(k-1)(k-2)+1$ inequalities:

$$\begin{aligned}
a_{11} + a_{22} &\le a_{12} + a_{21}; && && (18)\\
a_{p,p-1} + a_{sp} &\le a_{p,p} + a_{s,p-1}, && && (19)\\
a_{p,p-1} + a_{s,p+1} &\le a_{p,p+1} + a_{s,p-1}, && s = p+1,\dots,k; && (20)\\
a_{p-1,p} + a_{pt} &\le a_{pp} + a_{p-1,t}, && && (21)\\
a_{p-1,p} + a_{p+1,t} &\le a_{p+1,p} + a_{p-1,t}, && t = p+1,\dots,k; && (22)\\
& && p = 2,3,\dots,k-1. &&
\end{aligned}$$

Indeed, it can be seen easily that the inequalities (18)–(22) form a proper subset of the system (15)–(17). In particular, inequalities (16) and (17) with $p = k-1$ are contained in (18)–(22). So what remains to be shown is that the inequalities (15)–(17) with $p \le k-2$ follow from (18)–(22).

Consider $p^* \le k-1$, and assume that (16)–(17) are satisfied for all $p \ge p^*$. Then the inequalities (16) with $s = p^*$ and $s = p^*+1$, and the inequalities (17) with $t = p^*$ and $t = p^*+1$ are contained in (18)–(22). The inequalities for $s > p^*+1$ and $t > p^*+1$ follow immediately from (18)–(22) and from the following straightforward algebraic rearrangements:

$$\begin{aligned}
& a_{p^*,p^*-1} + a_{st} - a_{p^*t} - a_{s,p^*-1} \quad = \\
& \quad (a_{p^*,p^*-1} + a_{s,p^*+1} - a_{p^*,p^*+1} - a_{s,p^*-1}) + (a_{p^*,p^*+1} + a_{st} - a_{p^*t} - a_{s,p^*+1})
\end{aligned}$$

$$\begin{aligned}
& a_{p^*-1,p^*} + a_{st} - a_{p^*-1,t} - a_{s,p^*} \quad = \\
& \quad (a_{p^*-1,p^*} + a_{p^*+1,t} - a_{p^*-1,t} - a_{p^*+1,p^*}) + (a_{p^*+1,p^*} + a_{st} - a_{p^*+1,t} - a_{s,p^*})
\end{aligned}$$

Finally, the inequalities (15) follow from (16), (17) and (18), and from the following simple transformation:

$$\begin{aligned}
& a_{11} + a_{st} - a_{1t} - a_{s1} \quad = \\
& \quad (a_{11} + a_{22} - a_{12} - a_{21}) + (a_{12} + a_{st} - a_{1t} - a_{s2}) + (a_{21} + a_{s2} - a_{22} - a_{s1}).
\end{aligned}$$

This completes the proof of the proposition. □

4 The Recognition of Specially Structured Matrices

The combinatorial structure of the distance matrix C in Theorem 3 does heavily depend on the numbering of its rows and columns. Hence it is natural to formulate the following *recognition* problem:

> Given an $n \times n$ distance matrix $C = (c_{ij})$, does there exist a renumbering of the cities, that is, a permutation α of the rows and columns of C, such that the resulting matrix $(c_{\alpha(i)\alpha(j)})$ satisfies conditions (12)–(14)?

If we consider the submatrix $A = C[K_1, K_2]$, then the recognition problem above boils down to the problem of finding two permutations: one permutation for permuting the rows and one permutation for permuting the columns in the *asymmetric* matrix A:

> Given a $k \times k$ matrix $A = (a_{ij})$, does there exist a permutation γ of the rows and a permutation δ of the columns, such that the resulting permuted matrix $(c_{\gamma(i)\delta(j)})$ satisfies the conditions (15)–(17)?

The following recognition algorithm is based on the technique developed by Burkard & Deineko [7] for the recognition of a similar relaxed Monge structure in a *symmetric* distance matrix.

Theorem 5. *For a given $k \times k$ matrix $A = (a_{ij})$, it can be decided in $O(k^4)$ time whether there exist permutations γ and δ such that the permuted matrix $(a_{\gamma(i)\delta(j)})$ satisfies conditions (15)–(17). If the permutations γ and δ exist, then they can be determined explicitly within this time bound.*

Proof. First, we try all k indices as candidates for the first position in permutation γ. Without loss of generality let $\gamma(1) = 1$. Then an index i can be placed in the first position of permutation δ if and only if the following inequalities are satisfied:

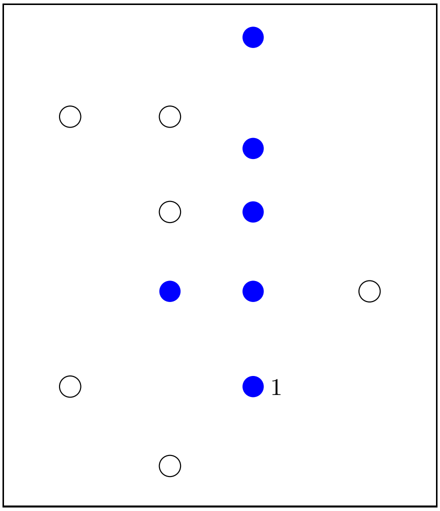
$$a_{1i} + a_{st} \;\leq\; a_{si} + a_{1t} \qquad \text{for all } s \neq 1,\, t \neq i. \tag{23}$$

If there is another candidate j with the same property, then it follows immediately from (23) that $a_{1i} + a_{sj} = a_{si} + a_{1j}$; in other words, we then have $a_{sj} = a_{si} + d$ for all s, where $d = a_{1i} - a_{1j}$ is the constant for fixed i and j. Since adding a constant to a row or a column of matrix A does not affect the inequalities (15)–(17), in this case any of the indices i or j may be placed in the first position of permutation σ.

We claim that an appropriate candidate i can be picked in $O(k^2)$ time. Indeed, the transformation $a'_{st} = a_{st} - a_{1t}$ for $s = 1, \ldots, k$ and $t = 1, \ldots, k$ transforms matrix A into a matrix $A' = (a'_{i,j})$ with zeroes in the first row. The inequalities (23) for matrix A are equivalent to the inequalities $a'_{st} \leq a'_{si}$ for matrix A', for all s, t and i. Therefore, an appropriate index i can be found in $O(k^2)$ time by looking through the indices of the maximal elements in the rows of matrix A'.

The indices for the second position in permutations δ and γ can be chosen by applying an analogous procedure to the submatrix $A[\{1, \ldots, k\}, \{1, \ldots, k\} \setminus \{i\}]$

with the first row fixed to be 1, and to submatrix $A[\{2,\ldots,k\},\{1,\ldots,k\}]$ with the first column fixed to be i. This yields an overall time complexity of $O(k^3)$ for each candidate on the position $\gamma(1)$, and therefore an $O(k^4)$ overall time complexity for the entire algorithm. $\square$



Point number	1	2	3	4	5	6	7	8	9	10	11	12
X coordinate	26	21	26	26	26	26	21	15	21	33	15	21
Y coordinate	12	18	18	23	27	34	7	12	23	18	29	29

Fig. 5. Recognizing a rectilinear instance of the BTSP. The first point has been fixed

To illustrate the way our algorithm works, we consider the BTSP with a rectilinear distance matrix where the distances between points i and j are calculated as $c_{ij} = |x_i - x_j| + |y_i - y_j|$; see Figure 5 for an illustration. We assume here that the first entry of permutation γ is fixed as $\gamma(1) = 1$. The corresponding submatrix A of the distance matrix C and its transformed matrix A' then look as follows:

$$A_{6\times 6} = \begin{array}{cc} & \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} \mathbf{1} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 10 & 11 & 16 & 13 & 28 & 22 \\ 11 & 12 & 5 & 12 & 17 & 11 \\ 16 & 17 & 10 & 7 & 22 & 16 \\ 21 & 22 & 5 & 12 & 17 & 11 \\ 25 & 26 & 9 & 16 & 13 & 7 \\ 32 & 33 & 16 & 23 & 16 & 10 \end{pmatrix} \end{array} \qquad A'_{6\times 6} = \begin{array}{cc} & \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} \mathbf{1} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -11 & -1 & -11 & -11 \\ 6 & 6 & -6 & -6 & -6 & -6 \\ 11 & 11 & -11 & -1 & -11 & -11 \\ 15 & 15 & -7 & 3 & -15 & -15 \\ 22 & 22 & 0 & 10 & -12 & -12 \end{pmatrix} \end{array}$$

Note that in all rows of matrix A' the indices of the maximal elements are $\{7, 8\}$. Hence either of these two columns may be picked as the first column, and we will pick $\delta(1) = 7$. For choosing an appropriate row to be placed in the second position of permutation γ, we next consider the following 5×6 submatrix of the distance matrix:

$$A_{5\times 6} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{7} & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{pmatrix} 11 & 12 & 5 & 12 & 17 & 11 \\ 16 & 17 & 10 & 7 & 22 & 16 \\ 21 & 22 & 5 & 12 & 17 & 11 \\ 25 & 26 & 9 & 16 & 13 & 7 \\ 32 & 33 & 16 & 23 & 16 & 10 \end{pmatrix} \end{array} \qquad A'_{5\times 6} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{7} & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{pmatrix} 0 & 1 & -6 & 1 & 6 & 0 \\ 0 & 1 & -6 & -9 & 6 & 0 \\ 0 & 1 & -16 & -9 & -4 & -10 \\ 0 & 1 & -16 & -9 & -12 & -18 \\ 0 & 1 & -16 & -9 & -16 & -22 \end{pmatrix} \end{array}$$

Now the indices of maximal elements in columns 7 through 12 of matrix A' are: $\{2,3,4,5,6\}$; $\{2,3\}$; $\{2\}$; $\{2,3\}$; and $\{2,3\}$. The only index that belongs to all these sets is 2; hence $\gamma(2) = 2$. (If the intersection of these sets had been empty, the choice of $\gamma(1) = 1$ as the first entry in permutation γ had failed and would have to be reconsidered.)

We proceed with the following (analogous) steps and eventually find two permutations $\gamma = \langle 1,2,3,4,5,6 \rangle$ and $\delta = \langle \underline{7,8}, 9, 10, \underline{11,12} \rangle$ for the numbering of the points. In permutation δ, the points 7 and 8 as well as the points 11 and 12 may be permuted, so that under the choice $\gamma(1) = 1$ there altogether exist four pairs of permutations for feasibly renumbering the points. The corresponding numbering and the optimal BTSP solution are reported in Figure 6.

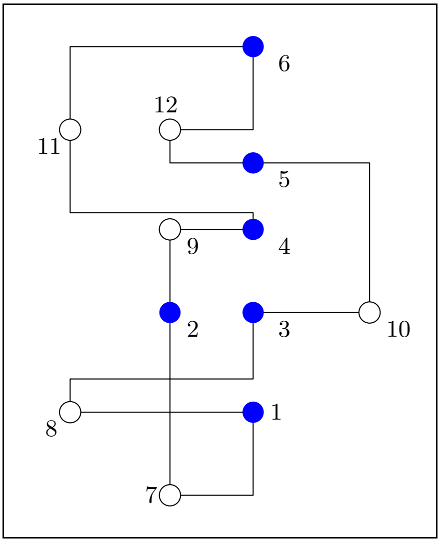


Fig. 6. Recognizing a rectilinear instance of the BTSP: The final numbering

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