A Longitudinal Study of the Development of Rational Number Concepts and Strategies in the Middle Grades

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Introduction

Research in the area of rational number knowledge and proportional reasoning has produced many important findings on how students think about and operate with rational numbers (Behr, Harel, Post, & Lesh, 1992; Behr, Lesh, Post, & Silver, 1983; Empson, Junk, Dominguez, & Turner, 2006; Kieren, 1976). The complex nature of this research has yet to discover a clear picture or model of how rational number knowledge develops over time. Some conjectures have been made concerning rational number development from cross-sectional studies, but without longitudinal evidence such trajectories are difficult to confirm. Defining a framework for interpreting students' understanding along a

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developmental path, if one exists, is a desired goal. Without such a framework, the research base will remain fragmented and primarily focused on further examination of understandings of a particular subconstruct's origin and phenomenology instead of the essential transitions among contexts and conceptions that should mark a more mature rational number understanding among constructs (Streefland, 1993).

Investigations into the way that young children are introduced to whole number operations have revealed certain barriers to rational number learning due to the inconsistencies between the mathematics of whole numbers and the mathematics of fractions (Bransford, Brown, & Cocking, 1999; Mack, 1993; Middleton, van den Heuvel-Panhuizen, & Shew, 1998). For example, the rules of thumb *multiplying makes larger* and *dividing makes smaller* when working with whole numbers become problematic when students must consider cases involving multiplication or division by proper fractions (Kieren, 1993). Recent research has led to a belief that the common part–whole introduction of fractions is not as effective in removing early-knowledge barriers to the mathematics of fractions as an approach emphasizing the ideas of partitioning and unit which are more closely related to thinking about fractions as quotients (Empson, 1999; Lamon, 2006; Mack, 1993; Streefland, 1993). But how might this partitioning approach assist students in gaining conceptual knowledge in other subconstructs such as measurement? Understanding the transitional paths from one subconstruct to another across the field of rational number concepts is paramount to building a meaningful model of rational number learning. It is clear, moreover, that this sought-after developmental trajectory is complex and will not follow a simple one-dimensional path, moving in an orderly, linear fashion from one subconstruct to another. Rather, it depends upon content and representations emphasized in instruction as well as contextual referents that give rise to initial conceptions of multiplicative quantities (Lamberg & Middleton, 2002, 2009). In fact, middle-school children who traverse this complex path will no doubt face barriers and perhaps take detours that divert, prolong, or even stall their progress toward rational number understanding in the course of compulsory instruction.

Longitudinal Analysis

Rational number understanding has been termed a "watershed concept" (Kieren, 1976). Fractions, ratios, and proportional reasoning are key underpinnings of algebra, calculus, statistics, and other higher mathematics that are becoming more and more critical for the development of workplace skills (Oksuz & Middleton, 2005). Cross-sectional studies of students at different ages are the norms for the field in examining students' reasoning and development (see for example, the work of Empson et al., 2006). This body of work has aided in the development of new curricular tasks and sequences aimed at providing a more theoretically defensible and psychologically connected approach to the teaching and learning of rational number (Carpenter, Fennema, & Romberg, 2012; Lamberg & Middleton, 2009; Lesh, Post, & Behr, 1988; Streefland, 1993; Toluk & Middleton, 2004).

However, due to their cross-sectional design, this body of work does not provide a coherent developmental picture of rational number knowledge as students move across several grade levels (Carraher, 1996). More recent studies, however, give us a glimpse of how this knowledge might develop, beginning with the ideas of unit and equivalence, then gradually developing the five interconnected interpretations or "subconstructs" that predominate the language of the field: Part–whole, measure, operator, quotient, and ratio (Lamon, 2006).[1] The work reported here, supported through a grant from the National Science Foundation, has allowed us to trace these changes in understanding related to learning rational number concepts as they developed over the middle-school years where this content is most heavily stressed. The results of the study are intended to contribute theoretically to the understanding of numbers and operations and pragmatically to the further design of curriculum materials and pedagogical strategies that will positively impact students' ability to think, represent, and communicate their understanding of rational number concepts and procedures over time.

The importance of knowing how rational number knowledge and proportional reasoning develop through the middle grade levels is prompted in part by the fact that such knowledge forms the foundation for the study of higher mathematics. This need is further evidenced by the fact that students in the United States have demonstrated weaknesses in these topic areas in comparative studies with other international student populations such as the Trends in International Mathematics and Science Study (Kelly, Mullis, & Martin, 2000; Mullis, Martin, Gonzalez, & Chrostowski, 2004). Some investigators have shown that even postsecondary students have difficulty representing fraction magnitudes (Bonato, Fabbri, Umiltà, & Zorzi, 2007).

Besides these reasons that pertain specifically to academic progress and global competitiveness, fundamental understanding of rational number is necessary for a well-informed citizenry which includes but is not limited to interpreting graphs and other data displays, projecting trends and forecasts, comparing quantities multiplicatively, and basic consumer and home skills.

Issues in Mapping Students' Growing Knowledge

In this study, we traced individual students' development of each of the rational number subconstructs through a constructivist lens. On the individual level, we utilized individual interviews, following a target sample of students from the sixth grade through the eighth grade to assess their growth individually. Yet we also recognize that the development of rational number knowledge in a classroom is distributed across members of student groups or the class, coordinated between internal and external structures, and across time where results of earlier tasks and events transform

[1] It must be noted that these five subconstructs are not the only way to parse student reasoning or mathematical manifestations of these concepts. Confrey, Maloney, Nguyen, Mojica, and Myers (2009), for example, provide a rich alternative framework.

the nature of later events (Hollan, Hutchins, & Kirsh, 2000; Roth & McGinn, 1998). We therefore observed students' mathematics classes twice per week, coordinating our understanding of their individual growth with their classroom experiences.

The inscriptions or representational tools recorded and analyzed in student interviews and in class observations provided a way to describe the propagation of rational number knowledge across classroom participants and within a single student's mind over time (e.g., Lamberg & Middleton, 2002). Examining student inscriptions was essential in our study due as they documented the form of knowledge at the moment of instruction and developmental sequence in which the knowledge arose. Inscriptions also served as the object of collective negotiations of meaning between the student and class, student and teacher, and student and researcher, and were appropriated (transported from one person to another) allowing us to trace the diffusion of knowledge across the 3 years of the study, when they appeared spontaneously in interview sessions.

In summary, this study is aimed at understanding the intellectual resources individual children bring to bear in developing rational number understanding and the classroom norms and practices that constrain and enable individual development *longitudinally*. Specifically, the scope of work is intended to advance the field of rational number learning by:

1. Uncovering patterns and mechanisms of individual development in students' understanding of rational numbers and proportional reasoning
2. Integrating the current piecemeal body of research on rational number into a coherent developmental model by examining how understanding of rational number subconstructs evolve concurrently and interactively.
3. Developing insight into the ways in which classroom instruction, especially the use of and talk around inscriptions impact students' ability to think about, represent, and communicate their understanding of rational number concepts and operations as it develops over time.
4. Generating transportable models of rational number development that can be factored into teacher pre- and in-service staff development to promote quality instructional practices in the future.

Method

Setting and Participants

This study analyzes data collected over a 20-month period in a longitudinal study conducted in an urban K-8 school located in the southwestern United States. The approximately 850 students enrolled in the school were predominately from a Hispanic lower-middle-class background. Over 90 % of the students received free or reduced lunch. Sixth-, seventh- and eighth-grade students participated in the study. Their classrooms were equipped with whiteboards on two walls, lined with low bookshelves and were furnished with round and rectangular tables at which students were typically seated in groups of four to six students. Students attended mathematics class daily. Each class lasted for 70 min except Wednesdays, when classes were shortened to 55 min to accommodate after-school teachers' meetings. The District-selected mathematics curriculum consisted of the NSF-sponsored, *Mathematics in Context* (2003) series supplemented with Arizona Instrument to Measure Standards (AIMS) test preparation materials, which the teachers used on an alternating basis. Some teachers favored drill and practice more than others, and these sessions lasted from 10 to 45 min in a typical 70-min class period. A significant number of the students in all three classes were English Language Learners (ELL). As a school norm, teachers tried to seat the ELL students with classmates whose English was sufficient to assist them as needed. Participating teachers often used overhead projectors during instruction.

Although the exact enrollment in each class varied over the 3 years of the study, the average ratio of teacher to students in the sixth-, seventh-, and eighth-grade classes was 1–30. The sixth-grade class was self-contained, where a single teacher conducted instruction in all subjects. The seventh- and eighth-grade classes followed a middle-school format where students traveled to different classrooms for subject instruction. Additionally, some seventh- and eighth-grade students were given the opportunity to attend a resource class for extended mathematics instruction. In this special resource class (held twice per week), students worked in small groups on challenging problems outside of the regular mathematics curriculum. As a part of the classroom norms in the resource class, students were expected to work together and present group solutions to the whole class.

Data Collection Procedures: Interviews and Classroom Observations

Interviews

To make comparisons across students possible, we designed parallel interview protocols to assess rational number knowledge across all five subconstructs (Behr et al., 1992; Lamon, 2006). These protocols were administered to all students enrolled in the study in the first and last two interview cycles of the school year, regardless of the grade level. The tasks in the first pair of these parallel protocols were the same in terms of context and level of difficulty, and they covered the subconstructs of operator, quotient, and part–whole. The other parallel pair involved the subconstructs of measurement and ratio. Both pairs of parallel protocols were administered in the fall and spring semesters to assess individual growth over time, which included both ability to correctly solve problems, and also, changes in preferred strategies for solving problems.

In addition to these parallel protocols administered to all interviewees, we captured the impact of curricular tasks and instruction using class-specific individual interview protocols with prompts adapted from tasks in the district-adopted *Mathematics in Context* (2003) curriculum. Like the parallel protocols described above, these additional tasks focused on one or more of the five subconstructs of rational number, but utilized the inscriptions and language that we observed being developed in students' classes.

Interview Procedures and Coding

All interviews (common, parallel protocols, and grade-specific protocols) were videotaped. Special attention was given to recording the students' written inscriptions and their verbal "think aloud" responses. Interviewers attempted to capture students' intuitive, procedural, and conceptual knowledge of rational numbers and track their change over time. Interviewers were trained to listen closely and carefully prompt students for additional thinking without commenting on the appropriateness of any of their solution strategies. Students who spoke little English were interviewed by interviewers fluent in both English and Spanish.

Each protocol was coded across five dimensions: (1) Problem subconstruct (the anticipated conception of rational number we hypothesized the problem would elicit); (2) Students' solution strategies (Convert to common fractions; Use of equivalent ratios; Measurement division; Multiply by a scale factor (operator); Part/Whole; Proportional Reasoning; Relating to a similar problem; or No Strategy observed/Strategy not code-abled); (3) Whether the strategy utilized was developed ad hoc, or if it had been previously observed in the student's class; (4) Whether the problem was solved correctly; and (5) Whether the problem strategy led to a sensible answer mathematically even if the answer was technically incorrect.

Analyses traced the proportion of strategies utilized across each of the interviews as students moved from early sixth grade, through the seventh grade, and finally, as they prepared to finish the eighth grade, comparing differences in strategy use for each of the four other variables.

Table 1 displays the number of students in cohorts who were individually interviewed by grade and by year. Arrows represent student groups followed up through successive grade levels. During the first year of the study, 53 sixth graders and 11 seventh graders participated. Eleven new sixth graders, four new seventh graders, and seven eighth graders entered the interview process in the second year, while 38 of the previous sixth graders and 8 of the previous seventh graders continued into the seventh grade and eighth grade, respectively.

Table 1 Participants involved in individual interviews

	6th grade	7th grade	8th grade	Total
1st year	53	11	--	64
2nd year	11	38+4	8+7	68
3rd year	12	9	32+4+4	61
Total	76	62	55	102

In the third year, 12 new sixth graders and 4 eighth graders entered the interview process, while 9 of the previous seventh graders and 38 of the previous eighth graders remained in the study. Among these 38 students, 32 were retained from the sixth grade across the 3 years. As a result, a total of 102 students took part in individual interviews during the 3-year study, and the 32 students who were followed over 3 years became our focus group in this paper.

Classroom Observations

In addition to individual interviews, the mathematics classes of students participating in the study were videotaped twice weekly. These 70-min observations were conducted to provide a contextual reference within which we embedded individual interviews and analyses. Interviewers were able to see their student interviewees engaging in mathematical activities within a social setting, to see what inscriptions occurred in the classroom, which were favored, and to look for clues to the origins of the problem solving and reasoning strategies students used in interview settings. While targeted students interacted in groups or whole class situations, our cameras recorded their development of mathematical notations and representations within the sociolinguistic structure of the classroom.

Assessment of Students' Rational Number Performance

There were two major purposes for collecting performance data: (1) to compare performance of our sample to a national/international sample; and (2) to describe student growth over time quantitatively. Quantitative assessment data were gathered at four time points: at the end of the fall semester in year 1, the beginning and end of year 2, and the beginning and end of year 3. Questions were drawn from released items from national/international mathematics assessments, the Trends in International Mathematics and Science Study (Martin & Kelly, 1998; International Association for the Evaluation of Educational Achievement, 2001; International Association for the Evaluation of Educational Achievement, 2005) and the National Assessment Educational Progress (NAEP). Utilizing questions from TIMSS and NAEP tests also allowed for comparisons of these students with students of similar age throughout the country and around the world.

To determine the rational number constructs the test items represented, the original form was piloted using a separate sample to ensure appropriate content and discrimination across the three grades. Three items were excluded due to the students' extremely low percentage of correct responses. As a result, the assessment consisted of 27 items, assessing 11 categories of rational number including: Ordering fractions, part–whole, ratio, relationships between fraction and decimal, proportion, linear measurement, rates, percent, equivalent fraction, operator, and decimal notation. Among the 27 items, 4 were free-response (item 1, 9, 18, and 21), while the remaining items were multiple-choice (see Table 2).

Table 2 Test item information

Item #	Source	Type	Item content
1	TIMSS 95	Ordering fractions	Write a fraction that is larger than 2/7
2	TIMSS 99	Part–whole	Which shows 2/3 of the square shaded?
3	NAEP 98	Part–whole	What fraction of the rectangle ABCD is shaded?
4	TIMSS 99	Ordering fractions	Given two common fractions. Which of these fractions is smallest?
5	NAEP 03	Ratio	Given two ratios. Which of the following ratios is equivalent to the ratio of 6:4?
6	NAEP 92	Relation between fraction and decimal	Given a common fraction, which is closest in value to 0.52?
7	TIMSS 99	Part–whole	Given a picture. What fraction of the circle is shaded?
8	TIMSS 99	Part–whole	Given a picture. Robin and Jim took *X* cherries from a basket. What fraction of the cherries remained in the basket?
9	TIMSS 99	Proportion	John and Mark sold *X* magazines. Knowing the total amount of money, how much money did Mark receive?
10	TIMSS 99	Part–whole	Penny had a bag of marbles. How many marbles were in the bag to start with?
11	TIMSS 95	Ratio	Given a picture with numbers, what is the ratio of red paint to the total amount of paint?
12	NAEP 03	Linear measure	Given a picture, the distance from Bay City to Exton is 60 miles, what is the distance from Bay City to Yardville?
13	TIMSS 99	Rates	A runner ran 3,000 m in exactly 8 min. What was his average speed in meters per second?
14	TIMSS 95	Percent	From 60 cents to 75 cents, what is the percent increase in the price?
15	TIMSS 03	Ordering fractions	Given two common fractions. In which of these pairs of # is 2.25 larger than the first number but smaller than the second number?
16	TIMSS 99	Proportion	If there are 300 calories in 100 g, how many calories are there in a 30 g portion of this food?
17	TIMSS 95	Ratio	3/5 of the students are girls. Add 5 girls and 5 boys, which statement is true of the class?
18	NAEP 03	Linear measure	Given a picture, a dot shows where 1/2 is. Use another dot to show where 3/4 is
19	TIMSS 99	Equivalent fraction	In which list of fractions is all of the fractions equivalent?
20	NAEP 03	Linear measure	3/4 of a yard of string is divided into pieces; they are 1/8 yard long each. How many pieces?

(continued)

Table 2 (continued)

Item #	Source	Type	Item content
21	TIMSS 95	Operator	Luis runs 5 km each day, the course is 1/4 km long. How many times through the course does he run each day?
22	TIMSS 99	Decimal	Which of these is the smallest number?
23	TIMSS 95	Ordering fractions	Which list shows the numbers from smallest to largest?
24	TIMSS 95	Ratio	The ratio of girls to boys is 4:3. How many girls are in the class
25	TIMSS 99	Ratio	The tables show some values of x and y, what are the values of P and Q?
26	TIMSS 03	Decimal	Divide a number by 100. By mistake multiplying it by 100, obtained an answer of 450. What was the right answer?
27	TIMSS 03	Decimal	45 L of fuel; consumer 8.5 L per 100 km. After traveling 350 km, how much remained?

Each test had two forms, A and B, which differed only in item order. Assessments were given to the entire sixth, seventh, and eighth grades, and students sitting next to each other received different test forms to prevent cheating. Since students included in the study were predominately Hispanic, a Spanish version of the test was created, translated by faculty and graduate assistants fluent in Spanish. Students were asked for their test language preference, and although most students were of Hispanic decent, only a few students preferred to take the Spanish version. Data were collected based on the students' original responses to test items, and were coded according to their correct (1) or incorrect (0) answers to items. Summing the number of correct responses formed a student's total score. Scores were also computed for items within each rational number subconstruct represented in the NAEP and TIMSS items.

Results

Comparison of Performance of Sample to a National/International Sample

Table 3 displays the number of students and gender distribution in each grade tested. Numbers in parentheses represent the number of classes involved in the testing at each grade level.

To benchmark our students against (inter)national norms, we compared mean performance and proportion correct for each of the 24 comparable items on the performance assessment. In terms of overall performance, students in our sample students scored at or just below the level of middle schoolers around the nation (for NAEP items) and the world (for TIMSS items). Only 10 of the 24 comparable items

Table 3 Number of students participating in each test administration broken out by gender

		Sixth grade *N* (*n*)	Seventh grade *N* (*n*)	Eighth grade *N* (*n*)	Total *N*
Test 1		74 (3)	27 (1)	–	101
	Female/male	33/41	14/13	–	47/54
Test 2		22 (1)	84 (3)	62 (2)	168
	Female/male	12/16	35/49	29/33	76/92
Test 3		27 (1)	74 (3)	51 (2)	152
	Female/male	16/11	32/42	26/25	74/78
Test 4		28 (1)	65 (3)	80 (3)	173
	Female/male	16/12	35/30	34/46	85/88
Test 5		26 (1)	61 (3)	85 (3)	172
	Female/male	15/11	32/29	43/42	90/172

Table 4 Mean and standard deviation of student test scores

Test	Sixth grade	Seventh grade	Eighth grade
1 (Fall, year 1)	**9.39 (3.16)**	8.56 (3.33)	–
2 (Fall, year 2)	6.73 (3.15)	**9.71 (3.70)**	12.81 (5.52)
3 (Spr, year 2)	8.48 (2.62)	**10.46 (4.05)**	14.73 (6.46)
4 (Fall, year 3)	7.89 (2.62)	8.88 (3.36)	**10.88 (4.45)**
5 (Spr, year 3)	8.96 (3.23)	9.51 (4.52)	**13.22 (4.96)**

Note: The bold items show the trajectory of sixth graders in year 1 as they matriculated through seventh and eighth grade

showed statistically significant differences in percent, correct. These differences centered around the predominant focus on Part/Whole fraction instruction in our observed classes. We propose that this instructional bias, which is typical of fraction instruction in the United States, resulted in a predominance of the use of Part/Whole strategies to the exclusion of other learned strategies—strategies which ultimately are more efficient, conceptually meaningful, and that are useful for more sophisticated ratio and proportional reasoning problems.

Comparison of Performance at Different Grade Levels

Table 4 presents students' average score and standard deviation by grade level for each administration. One way Analysis of Variance was performed on percent correct using grade as an independent variable. Post hoc Scheffe tests show that, eighth-grade students outperformed sixth and seventh graders for all administrations ($p<0.05$). Seventh graders outperformed sixth graders on administration 2 only. Sixth graders scored on average, higher than seventh graders on the first administration, but the difference is not statistically significant ($p>0.05$). Students grew significantly over time, with greatest gains appearing, not surprisingly during the academic years, with very little, but some growth occurring over the summer periods.

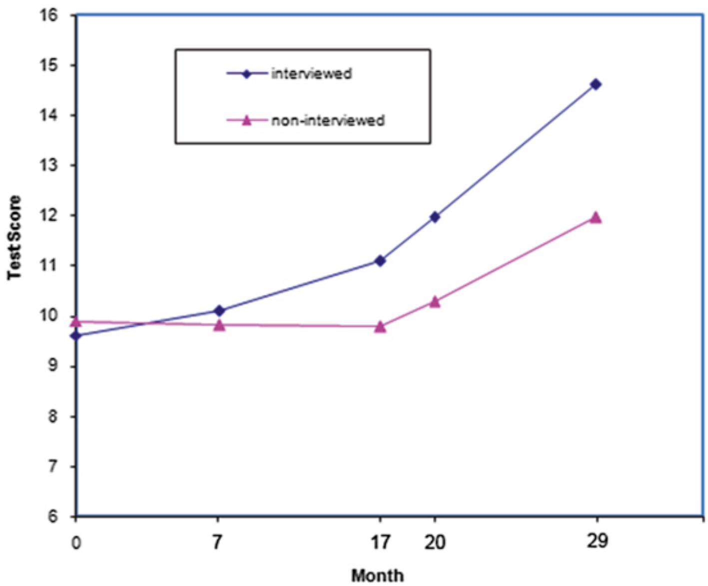

Fig. 1 Growth in rational number performance for students interviewed in the study versus non-interviewed students

Describing Students' Mathematics Achievement over Time

As we studied the results of individual interviews (see below), it became apparent that the students we were interviewing displayed more capability for solving ratio and proportion problems than the larger sample of students in the school that did not receive interviewing. To determine if a Hawthorne effect explained this difference in student abilities, average test scores for students who were interviewed in the study and peers who were never interviewed were separately computed and plotted in Fig. 1. The number of students in the former group was 33 and latter group was 56. Figure 1 presents a mean plot for these two groups of students across five test points.

It is obvious from the figure that interviewed students' mathematics performance increased steadily, and even accelerated over time. While non-interviewed students' mathematics performance did not increase from the beginning of study till the time when the third test was given, they linearly increased starting at about the third test (after 15 months of school time had elapsed). Although separate hierarchical linear/nonlinear models could be specified for each of these two groups to examine and compare student's growth on mathematics achievement over time, we decided to apply a two-level linear model to only the interviewed group, with the following justification:

1. Thirty-three students were target students in this study, and we had a large body of qualitative data for each of these 33 students. This made it possible to combine both qualitative and quantitative data outcomes to describe students' learning trajectories.

2. Among the 33 students in the interviewed group, only 4 data points were missing.
3. Among 56 students in the non-interviewed group, 50 % of data points were missing, and only 9 students completed all 5 tests. It would not have satisfactory power to apply linear model to this group.

The following two-level linear model was specified to interviewed group as following:

$$\text{Level 1: } \text{Total}_{ti} = \beta_{0i} + \beta_{1i}\left(\text{month}_{ti}\right) + r_{ti}$$

$$\text{Level 2: } \begin{aligned} \beta_{0i} &= \beta_{00} + \gamma_{0i} \\ \beta_{1i} &= \beta_{10} + \gamma_{1i} \end{aligned}$$

where:

Total_{ti}: the observed math achievement score of individual i at month level t
β_{0i}: the estimated status when month = 0
β_{1i}: the estimated growth rate for individual i per month_{ti} where month is a time-related variable
r_{ti}: the residual of individual i at month level t, which was assumed to have a mean of zero and equal variance of σ^2 across grades
β_{00}: the average true status when month = 0
β_{10}: the average slope for the population
γ_{0i}: the difference between the individual intercept and the average true status when month = 0
γ_{1i}: the difference between individual slope and average slope
γ_{0i} and γ_{1i} are assumed to have MVN with a mean of zero and equal variance

We estimated fixed effects: β_{00}, β_{10} and random effects: $e_{ti}, \gamma_{0i}, \gamma_{1i}$

In this model, predictors in level 1 (i.e., β_{0i} and β_{1i}) became criterion variables in Level 2, allowing students to have different starting points and growth rates. This model assumed that a straight line adequately represented each person's true change over time and that any deviations from linearity observed in the sample data resulted from random measurement error r_{ti}. The model was examined by using HLM 6.0 software. Table 5 presents the results.

Table 5 Linear model of growth in math achievement (unconditional model)

Fixed effect	Coefficient	SE	*t* ratio	*p* value
Mean status at month = 0, β_{00}	9.03	0.57	15.52	0.000
Mean slope, β_{10}	0.16	0.02	7.93	0.000
Random effect	Variance component	df	χ^2	*p* value
Status at month = 0, γ_{0i}	8.21	32	120.99	0.000
Slope, γ_{1i}	0.0055	32	50.94	0.018
Level-1 error, e_{ti}	4.63			
Correlation between γ_{0i} and γ_{1i}	0.74			

The estimated mean intercept, $\hat{\beta}_{00}$, and mean growth rate, $\hat{\beta}_{10}$, for the math achievement data was 9.03 and 0.16, respectively. This means that the average math achievement score at month = 0 was estimated to be 9.03 and students were gaining an average 0.16 of a score per month. Both the mean intercept and growth rate have large *t* statistics indicating that both parameters are necessary for describing the mean growth trajectory of math achievement.

The estimates for the variances of individual growth parameters β_{0i} and β_{1i} were 8.21 and 0.0055, respectively. The χ^2 statistics for γ_{0i} was 120.99 (df = 32, $p < .05$), leading us to reject the null hypotheses and conclude that students vary significantly in month = 0. The χ^2 statistics for γ_{1i} was 0.0055 (df = 32, $p < .05$), leading us to reject the null hypotheses and conclude that there is also significant variation in students' math achievement growth rates. The variance of $\gamma_{1i} = 0.0055$ implied an estimated standard deviation of 0.074. Thus, a student whose growth was one standard deviation above average was expected to grow at the rate of $0.16 + 0.074 = 0.234$ scores per month. The correlation between mean and slope was 0.74, suggesting that students with a higher score at the starting point tended to learn faster.

In other words, interviewed students showed slightly, but significantly lower initial performance than non-interviewed students, but over time, they learned more, and at a faster rate, resulting in a set of learners with markedly different capabilities than the uninterviewed students in the school. Recall that there were nonsignificant differences overall in the performance of our sample with the (inter)national norms. Some kind of Hawthorne effect, therefore, must have occurred as a function of the student interview process. The reasons for this will be discussed following the rest of the results.

Interview Results

We present two cases to illustrate key transitional points in students' development for two of the subconstructs distinguishing our sample's performance from that of the (inter)national sample: Part–whole and ratio. These cases do not capture all students' developmental details, not even all of the details for the two students chosen. However, they illustrate common cognitive challenges and dilemmas students faced, and they show common realizations that moved students towards a deeper and more useful understanding in the two primary rational number subconstructs where sample students differed from their (inter)national peers. As such they can be thought of as representative of the larger sample of student growth patterns in these two areas for sampled students, but illustrative of the differences in international curriculum and learning. We are developing a full account of students' individual trajectories in a follow-up paper.

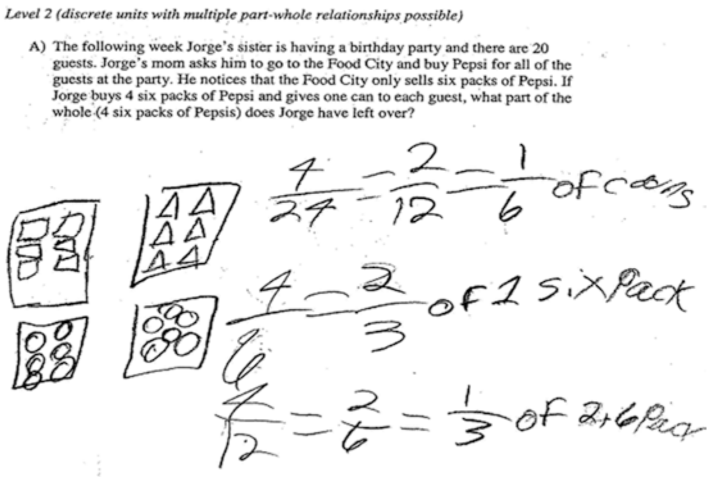

Fig. 2 Elias' flexible unitization

Elias: Part–Whole and Unitization

The case of Elias presents an example of how a student can extend a well-developed understanding of part–whole concepts and unitization to navigate through other rational number subconstructs, employing this knowledge to guess and check solutions in less familiar contexts (Fig. 2). Elias, like most of the interviewed samples, reflected a well-developed notion of fractions as part–whole concepts. When asked to express part–whole responses to contextualized questions, he responded with fraction notation, languages, and labels indicating an understanding of units and what each portion or unit represented. His was flexible, moving among suggested units, appropriately representing new, equivalent part–whole ratios correctly.

In describing the different units, for example, in a case of 4 cans out of a case of 24 cans of soda, Elias' was able to flexibly change the unit from 24 cans to one 6-pack and then to two 6-packs. With each new given unit, Elias correctly calculated the correct fraction and labeled his answer in terms of the appropriate unit. Thus the 4 cans became one-sixth of the 24 cans, two-thirds of a 6-pack, or one-third of two 6-packs.

Within the other rational number subconstructs, Elias' intuitive knowledge appeared to lack the depth necessary to transition smoothly into formal. For example, although he had an implicit understanding of ratio and could correctly solve simple ratio problems, he was not able to use this implicit understanding to explain his reasoning and computation in ratio terms (e.g., a to b, a per b, a for b, etc.). The following vignette illustrates his difficulty when he had to alter a recipe that called for 2 cups of flour and 1 cup of sugar because the cook only had ½ cup flour. In this particular context, the relationship between flour and sugar is a fairly simple part–part ratio (two parts flour to one part sugar). Elias immediately identified the correct numerical answer, but his explanation emphasized the partitioning of two cups of flour into four ½ cups. He then described a process of partitioning one cup of sugar until he finally revealed a method of taking away three ¼ cups, leaving one ¼ cup as the amount of sugar needed.

Elias: Hm…sugar… you would need ¼.
Interviewer: How did you get that?
Elias: Cause if you cut 1 into half, wait…if you cut 2 into half it would equal 1, and if you cut 1 into half you cut…I am getting myself confused. I'm gonna do it another way. If you take 2 minus ¼ it would be ¼ . .. ½ I mean would equal 1 ½, take away ½ again, and it would equal 1 and it would equal ¼ of a cut, so it would be 1, 2, 3, 4, so it would be 4.
Interviewer: Draw a picture if you need to.
Elias: Oh yeah, if you have ¼ + ¼ + ¼ and how much sugar would you need there are 3 of these and take away to get ¼. This is my strategy, but you won't get it.

Elias was able to solve this problem quickly, without visible calculations, yet, as we have seen, when encouraged to reveal his thinking process, he expressed frustration in making himself clear to the interviewer and never explicitly described the proportional relationship between the flour and sugar quantities given in the original recipe. If Elias' understanding of the ratio subconstruct was developed beyond familiar part–whole relationships to part–part or part–part–whole, we would expect him to better attend to and express the multiplicative relationship involved in changing the quantities of flour and sugar (i.e., the amount of sugar is ½ the amount of flour). What we see here is a reliance on Part–whole reasoning, with a fallback on a Measure conception as evidenced by Elias's iteration of a ¼ unit. Like the majority of our sample, Elias used these two conceptions approximately 60 % of the time in his interviews. Rarely did he utilize equivalent ratios, proportional reasoning, or multiplication by a scale factor (operator conception) to solve rate and proportion problems.

Inez: Ratio Subconstruct

One of the most dramatic examples of growth in the ratio subconstruct was seen in the test scores and protocol work of Inez. During her mid-sixth grade year Inez was only able to correctly answer 4 out of the 13 ratio problems on the common test drawn from TIMMS and NAEP questions. By the fall semester of her seventh-grade year, she was able to answer 10 of the 13 correctly, dipping slightly to 8 at the end of that year, but coming back strongly in her eighth-grade year to a score of 11. What made this development interesting was her admitted lack of familiarity with ratio vocabulary and instructor-initiated inscriptions. In several conversations with her interviewer, Inez expressed a limited knowledge of the word "ratio" and with the ratio table method, which was used extensively by her seventh-grade teacher. Comments by Inez such as "What is the ratio?" and "I heard about ratio table but I

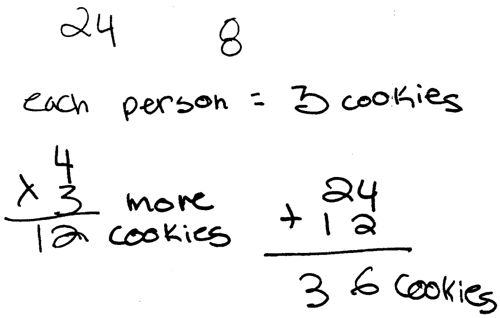

Fig. 3 Inez's use of a unit ratio

don't know about ratio." seemed to indicate that she wasn't aware of the formal language or notations typically associated with ratio problems.

In one protocol session during her sixth-grade year, Inez was given a problem to find the amount of calories in 30 g of ice cream given the fact that 450 calories were in 100 g. She tried to divide 450 by 30 and stated "We already have 30 g, so we can ignore the 100." It wasn't until the interviewer prompted her to determine the amount of calories in 1 g of ice cream, that she seemed to recognize the relationship between the original quantities of calories and grams and was able to find a specific ratio, a unit rate. By mid-year Inez started to show her own usage of a unit rate. Figure 3 shows her work in determining how many more cookies must be added to a given amount of cookies to maintain the initial ratio of cookies to guests.

Interviewer: You are shopping for a party and you buy 24 cookies for 8 people. Your cell phone rings and you are told that four more people are coming to the party. How many more cookies will you have to buy to keep the ratio the same? How many total cookies will you need?

Inez: We have 24 cookies and only 8 people. So each person will get three cookies. So when 4 more people are coming, we have to multiply by 4, so 12 more cookies and all together we need 36 cookies.

Despite her lack of familiarity with formal ratio symbols and operations, Inez, like many of our sample students, was able to solve a variety of contextual ratio problems by *using her own personal notation* for assigning correspondence between ratio quantities. From the fall semester of her sixth-grade year to protocols throughout her seventh-grade and eighth-grade year, Inez used an "=" to pair ratio quantities and then worked efficiently with this pairing to build up or down to a desired solution. In Fig. 4, for example, she established a relationship between 90 lions in the zoo with 1,800 kgs of food. Once she wrote this "equality" on her paper, she then divided or multiplied both sides as needed to create other equivalent ratios, often also adding corresponding parts of these pairs to solve given problems. Eventually she found the unit rate of 20 kg for one lion and then demonstrated her knowledge of how to use this rate to determine the number of kilograms for any given number of lions.

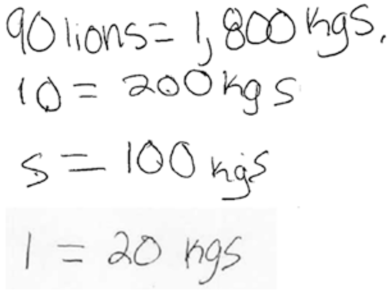

Fig. 4 Inez's equals sign used as "colon" ratio symbol

During the last year of the study, Inez was confronted by the interviewer about her use of the "equal sign" inscription for a ratio problem. Inez was quick to say that she knew that the two numbers were not really "equal," but that this was her own way of organizing the information in the problem. It was clear from her work that this method of organization provided a structure within which she could move easily to create equal ratios as needed.

Inez is also indicative of our sample students in that, informal, in-the-moment notations were used extensively, along with very few teacher-sanctioned inscriptions (like the ratio table, for example). These ad hoc inscriptions had meaning for each individual student, but were not capitalized on by the teachers in an attempt to systematically make them more formal and precise.

Summary of Interview Data

Interview protocols were coded based on the type of problem presented (Part–Whole, Measure, Quotient, Ratio, or Operator), strategies employed to solve the problem (including the use of heuristics, super-strategies, and taught procedures), and the sensibility of students' strategies and the correctness of their answers. Emulating the wonderful interpretive method of Carpenter and Moser (1984) for young children's arithmetic strategy development, we represent the development of children's strategies as graphs showing the proportion of each coded strategy over time. The following four figures show demonstrably that students *enter* into rational number instruction with a *variety* of strategies, both informal and formal for solving a wide variety of problem types. These strategies echo the general research on rational number development in that the predominant way of approaching problems appears to be conceptualizing them as Part–Whole, with smaller proportions of strategies focusing on using benchmark fractions, common denominator strategies, and even some proportional reasoning, though this was very rare (<5 % of total strategies). Through instruction, certain strategies became preferred in the participating students' classrooms. In particular, the use of fraction bars and ratio tables favored the development of benchmark fractions, measure strategies, and some use of equivalent ratios and proportional reasoning.

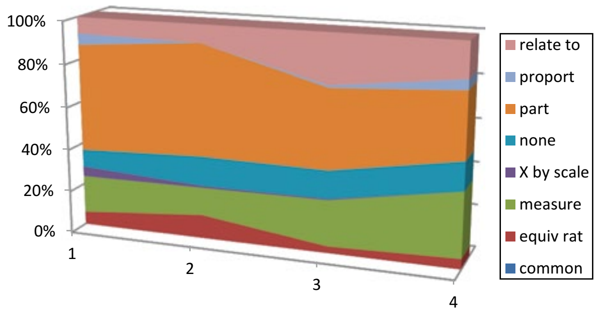

Fig. 5 Strategy type leading to sensible answers across four parallel administrations of interview protocols 2005–2007

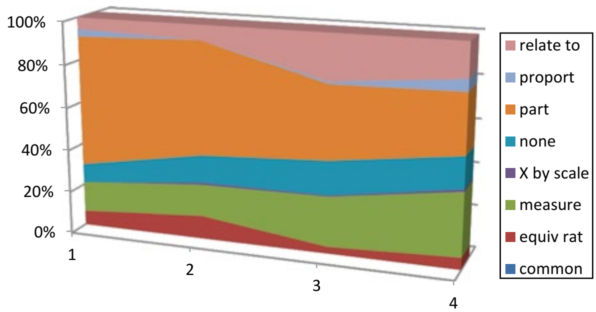

Fig. 6 Strategy type leading to technically correct answers across four parallel administrations

Findings show that sensible approaches to problem solution tended to lead to generation of correct answers (see Figs. 5 and 6). The proportions of strategies coded as sensible were nearly identical to the proportions of correct answers. The trend for the development of sensible and correct strategies shows that the proportion of problems solved using Part/Whole reasoning decreased over time, being supplanted by measure (quotitive division) strategies, relating problem quantities to benchmark fractions, and also by a variety of ad hoc strategies made up on the spot to solve the problem. Part–Whole, however, remained the dominant strategy preferred by students, even by the end of the eighth grade.

Examining errors, we show that, ad hoc strategies, were the most prominent strategies chosen when the strategies did not make sense for the problem situation (see Fig. 7). This indicates that for a large number of interview problems, students neither learned nor were able to generate, a meaningful method of solution, and instead relied on trial-and-error and other means–end solution methods. Fully 40 % of students' responses were idiosyncratic, and this trend remained relatively constant over the entire course of the longitudinal study. Part–Whole methods were the second-most prominent strategies used in ways that did not make sense for the problem context. Moreover, Part–Whole strategies were used *most* in cases where students' strategies yielded incorrect solutions. This echoes our comparisons with the (inter)national sample.

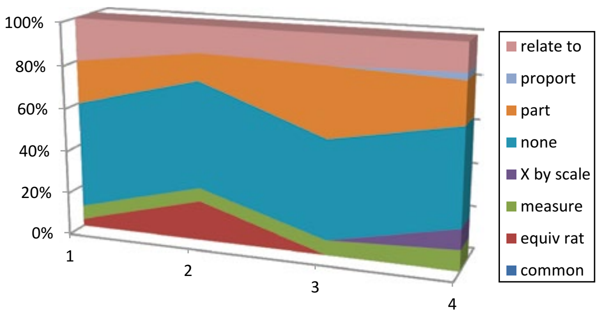

Fig. 7 Strategy type leading to non-sensible answers across four parallel administrations of interview protocols 2005–2007

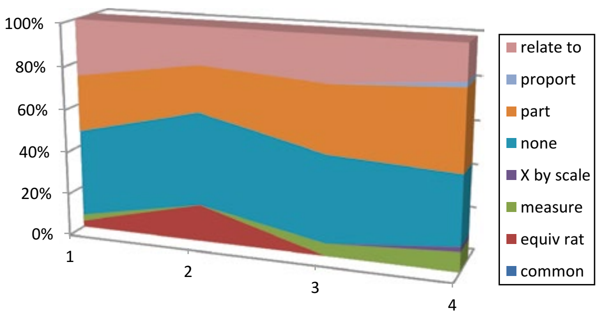

Fig. 8 Strategy type leading to technically incorrect answers across four parallel administrations of interview protocols 2005–2007

The most disappointing trend in our data suggests that potentially powerful methods of solution to rational number problems, such as proportional reasoning, the use of equivalent ratios, and common denominator strategies were neither stressed in students' classes (until well into the eighth-grade year), nor evident in their acquisition of strategies across the middle grades. Instead, the teachers focused class time on the use of robust-but-inefficient conceptual strategies such as the use of the fraction bar and ratio table. These strategies were heavily used in the students' textbooks, but were not exclusively emphasized there, indicating considerable teacher preference in the kinds of strategies legitimized in their instructional practices (Fig. 8).

Because our data are drawn from primarily poor, urban, largely Latino schools, we want to be careful generalizing the exact developmental trajectory of students' strategies to the rest of the United States. However, inasmuch as other middle-school curricula continue to overemphasize the use of part–whole conceptualizations of fractions, underemphasize the notion of fractions as indicated division, and underemphasize methods of computation that build strong understanding of factors and multiples, units, and partitions (e.g., Lamon, 2002; Ni & Zhou, 2005; Sophian, 2007; Thompson & Saldanha, 2003), it is likely that similar trends are occurring across many schools. Data on NAEP and TIMSS (National Science Foundation, 2002; NCES, 1999) show that US students generally lack understanding and skills in these and other areas related to proportional reasoning.

Discussion

A major objective of this longitudinal study was to contribute significantly to the research base on rational number learning by exposing patterns and mechanisms of development in students' understanding of rational numbers and by reorganizing the current fragmented body of research into a more coherent developmental model; illustrating how rational number subconstructs evolve concurrently and interactively along the road to a more profound knowledge of rational number concepts. The former goal is realized in this manuscript, but the latter is still a major challenge for the field. In particular, the coherence of instruction for teaching rational number, and especially the problematic concepts of ratio and proportion is still lacking, resulting in continued fragmentation of knowledge in the US children, favoring less sophisticated conceptualizations in this subject matter, than evidenced by students in the TIMSS 1999 and 2003 samples. Curriculum and teaching appear to be the key levers here (e.g., Saxe, Diakow, & Gearhart, 2012; Saxe, Gearhart, & Seltzer, 1999) as we were able to show that instructional strategies that favored Part–Whole conceptions predominated in our sample classrooms, leading to an overreliance on Part–Whole conceptualizations by students, yielding performance deficits in comparison to the international norm on more powerful concepts of rate, ratio, and proportional reasoning.

Despite the narrow demographics of our studied samples, we see our results as transportable to the US educational system in general. In the United States, research clearly shows that instruction in rational number tends to favor Part–Whole interpretations far more than other interpretations of fractions (Ni & Zhou, 2005; Sophian, 2007; Thompson & Saldanha, 2003). Use of measure, quotient, and ratio subconstructs are much less evident. The unfortunate point of this is that a Part–Whole understanding of fractions does not allow the student to deal with units other than one without tremendous difficulty. As a result, improper fractions become confusing (Mack, 1993). Fraction division, in particular becomes conceptually impossible. In countries like Japan and China, who traditionally perform better on international assessments of fractions and algebra, rational numbers are explained in terms of measurement models like the number line or area models, as the result of any division problem, and as a multiplicative comparison of dividend and divisor, numerator, and denominator (Moseley, Okamoto, & Ishida, 2007).

Our study reinforces earlier work that suggests that in rational number development, students tend to utilize a small number of robust-but-inefficient strategies which are applicable across a variety of situations. In the reported project, for example, we found that students who did not have ready access to procedures for determining factors and multiples of whole numbers were greatly hampered in their capacity to solve complex problems involving fractions, particularly fractions in proportional relationships. Conversely, students who DO have ready access to efficient procedures are able to solve problem subgoals in real time and progress towards successful problem resolution much more readily (Kim et al., 2007).

Moreover, in our data, overall, we have seen children using powerful iterative methods, such as the repeated halving strategy, far beyond their proficiency with other methods of computing fractions (e.g., finding common denominators, dividing numerator and denominator by a common factor). Students persisted in the use of these iterative strategies even though they had earlier demonstrated the ability to conceptualize fractions as indicated division, knowledge of and the ability to use factors and multiples, and the ability to solve complex problems using a division procedure.

Siegler, Thompson, and Schneider (2011) show that sixth graders show great variability of strategy use. They found that selection of strategies depended upon students' familiarity with solving problems with some arithmetic operations but not others. They found, however, that strategy use was highly variable within arithmetic operations. The sixth grade in the United States appears to be a key transitional grade, where students struggle to consolidate learned strategies for whole number arithmetic, and reconcile these with new rational number strategies they are currently learning.

Empson, Levi, and Carpenter (2011) show "there is a broad class of children's strategies for fraction problems motivated by the same mathematical relationships that are essential to understanding high-school algebra and that these relationships cannot be presented to children as discrete skills or learned as isolated rules. The authors refer to the thinking that guides such strategies as Relational thinking." What we found in our current study is that our studied children came into the sixth grade armed with a number of fine strategies for thinking about fraction problems. What failed to happen for many of our studied students is that over the course of their 3 years in middle school, they were not able to develop relational thinking for fractions much beyond Part–Whole and Measure conceptions.

Conclusions

In conclusion, we found that:

1. Children come to the middle grades with many useful ways of thinking about and solving rational number problems.
2. Children leave middle school with only a slightly expanded set of skills. They tend to rely on ad hoc, means–end reasoning and reliance on simple Part/Whole conceptions of fractions as opposed to developing more efficient and powerful methods of computation.

3. Teaching of fractions overemphasizes conceptual strategies using inscriptions like the fraction bar and ratio table, leaving little time to develop proportional reasoning, common denominator, and other equivalent fraction methods of solution.
4. The very act of interviewing children, only once every 3 weeks, is an intervention that leads them to learn more and achieve better than their matched counterparts. Even though teaching the children was not a goal of this study, interviewed children demonstrated significantly higher gains on TIMSS and NAEP items than their peers.

Commentary on the Issue of Scale in Intensive Interview and Observational Methods

At first glance, the scale of our study, comprising 204 students—of which all 204 were administered quantitative assessments of their rational number knowledge, 102 were interviewed, and 32 remained in the study for the full 3 years of the project—would generally not be considered *large* when compared to the samples reported in other studies in this book. However, as pointed out in the introductory chapter of this volume, *scale* depends on a variety of factors, not just the size of the sample. In our case, the scale is determined by two factors: (1) methods utilized; and (2) characteristics of the measurement.

Individual constructivist teaching experiments of approximately 45 min took roughly 2¼ hours to transcribe. Analysis of each interview took an additional 1.5 h on average. Multiplying these factors by 102 students, interviewed 9 times per year for 3 years, we get a total experimenter time of roughly 12,500 h for our qualitative work. For the quantitative assessments, administration of tests to all 204 students pre- and post- each year, coding responses and analysis of the quantitative data took roughly 300 additional experimenter hours. A bit more can be added to account for cleaning up TIMSS and NAEP data to bring our rough estimate close to 13,000 h of work (we do not count reading, writing, meetings, and other preparatory/reflective work in these estimates, nor do we count the 2 h per week of classroom observations, plus transcription and analysis). Clearly, the qualitative methods employed to uncover students' thinking constituted the vast majority of our researcher time. Every additional student added to our sample added an additional week (40.5 researcher hours equivalent) of effort. Given restrictions of funding, relative to the sensitivity of measurement we needed to track students' development of strategies over time, 102 students as ongoing informants was at the upper limits of scale possible.

So *scale,* as a construct in mathematics education must be thought of in terms of the complexity of the questions asked, the intensity of the data collection process, and the density of the data record. We benefited from this understanding of scale in that our interview protocols and performance assessments shared a common scheme by which problems could be coded. By utilizing and combining sensitive idiographic techniques such as interviews and observations, with (inter)nationally validated tasks on the performance assessment, we were able to identify a key weakness in the instruction of our sample, and tie this weakness to inadequate development of proportional reasoning.

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